A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I.   THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right) = H_S\left(t\right) + H_I + H_B, \tag{1}$$

$$H_S\left(t\right) = \varepsilon_0\left(t\right)|0\rangle\langle0| + \varepsilon_1\left(t\right)|1\rangle\langle1| + V_{10}\left(t\right)|1\rangle\langle0| + V_{01}\left(t\right)|0\rangle\langle1|, \tag{2}$$

$$H_I = |0\rangle\langle0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij} \tag{5}$$

II.   UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H\left(t\right)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}} \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V = |0\rangle\langle0|\hat{\varphi}_0 + |1\rangle\langle1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6)will give us the following result:

$$e^{\pm V} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1)} \tag{10}$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + \dots \tag{11}$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + \dots \tag{12}$$

$$= |0\rangle\langle 0| \left(1 \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm \dots\right) + |1\rangle\langle 1| \left(1 \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm \dots\right) \tag{13}$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} \tag{14}$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} - \alpha_{0\mathbf{k}}^* b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} - \alpha_{1\mathbf{k}}^* b_{\mathbf{k}}\right)} \tag{15}$$

$$= |0\rangle\langle 0|B_0^{\pm} + |1\rangle\langle 1|B_1^{\pm}, \tag{16}$$

$$B_i^{\pm} \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{17}$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX}\, e^{tY}\, e^{-\frac{t^2}{2}[X,Y]}\, e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \dots \tag{18}$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i, j we can show making $t = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} = e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]} \dots \tag{19}$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\,0} \dots \tag{20}$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} \tag{21}$$

By induction of this result we can write expresion of $B_i^{\pm}$ as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}})$:

$$B_i^{\pm} = \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{22}$$

$$D(\pm v_{i\mathbf{k}}) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{23}$$

$$B_i^{\pm} = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \tag{24}$$

this will help us to write operators O in the variational frame:

$$\overline{O} \equiv e^V O e^{-V}. \tag{25}$$

We use the following identities:

$$\overline{|0\rangle\langle 0|} = \mathrm{e}^V |0\rangle\langle 0| \mathrm{e}^{-V} \tag{26}$$

$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{27}$$

$$= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{28}$$

$$= |0\rangle\langle 0|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{29}$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+B_1^- \tag{30}$$

$$= |0\rangle\langle 0|, \tag{31}$$

$$\overline{|1\rangle\langle 1|} = \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{32}$$

$$= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+ + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{33}$$

$$= |1\rangle\langle 1|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{34}$$

$$= |1\rangle\langle 1|0\rangle\langle 0|B_1^+B_0^- + B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- \tag{35}$$

$$= B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- \tag{36}$$

$$= |1\rangle\langle 1|, \tag{37}$$

$$\overline{|0\rangle\langle 1|} = \mathrm{e}^V |0\rangle\langle 1| \mathrm{e}^{-V} \tag{38}$$

$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{39}$$

$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|B_1^+|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{40}$$

$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{41}$$

$$= |0\rangle\langle 1|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{42}$$

$$= |0\rangle\langle 1|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 1|1\rangle\langle 1|B_0^+B_1^- \tag{43}$$

$$= |0\rangle\langle 1|B_0^+B_1^-, \tag{44}$$

$$\overline{|1\rangle\langle 0|} = \mathrm{e}^V |1\rangle\langle 0| \mathrm{e}^{-V} \tag{45}$$

$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{46}$$

$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{47}$$

$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{48}$$

$$= |1\rangle\langle 0|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{49}$$

$$= |1\rangle\langle 0|B_1^+|0\rangle\langle 0|B_0^- + |1\rangle\langle 0|B_1^+|1\rangle\langle 1|B_1^- \tag{50}$$

$$= |1\rangle\langle 0|B_1^+B_0^- + |1\rangle\langle 0|1\rangle\langle 1|B_1^+B_1^- \tag{51}$$

$$= |1\rangle\langle 0|B_1^+B_0^-, \tag{52}$$

$$\overline{b_{\mathbf{k}}} = \mathrm{e}^V b_{\mathbf{k}} \mathrm{e}^{-V} \tag{53}$$

$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right) b_{\mathbf{k}} \left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{54}$$

$$= |0\rangle\langle 0|B_0^+ b_{\mathbf{k}} B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+ b_{\mathbf{k}}|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+ b_{\mathbf{k}}|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+ b_{\mathbf{k}} B_1^-|1\rangle\langle 1| \tag{55}$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+ b_{\mathbf{k}} B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+ b_{\mathbf{k}} B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+ b_{\mathbf{k}} B_0^- + |1\rangle\langle 1|B_1^+ b_{\mathbf{k}} B_1^- \tag{56}$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{57}$$

$$= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{58}$$

$$= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{59}$$

$$\overline{b_{\mathbf{k}}}^\dagger = \mathrm{e}^V b_{\mathbf{k}}^\dagger \mathrm{e}^{-V} \tag{60}$$

$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right) b_{\mathbf{k}}^\dagger \left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{61}$$

$$= |0\rangle\langle 0|B_0^+ b_{\mathbf{k}}^\dagger B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+ b_{\mathbf{k}}^\dagger|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+ b_{\mathbf{k}}^\dagger|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+ b_{\mathbf{k}}^\dagger B_1^-|1\rangle\langle 1| \tag{62}$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+ b_{\mathbf{k}}^\dagger B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+ b_{\mathbf{k}}^\dagger B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+ b_{\mathbf{k}}^\dagger B_0^- + |1\rangle\langle 1|1\rangle\langle 1|B_1^+ b_{\mathbf{k}}^\dagger B_1^- \tag{63}$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) \tag{64}$$

$$= b_{\mathbf{k}}^\dagger - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \tag{65}$$

We have used the following:

$$B_i^+ b_{\mathbf{k}} B_i^- = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{66}$$

$$B_i^+ b_{\mathbf{k}}^\dagger B_i^- = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \tag{67}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0\left(t\right)\left|0\rangle\langle0\right|} = \varepsilon_0\left(t\right)\left|0\rangle\langle0\right|, \tag{68}$$

$$\overline{\varepsilon_1\left(t\right)\left|1\rangle\langle1\right|} = \varepsilon_1\left(t\right)\left|1\rangle\langle1\right|, \tag{69}$$

$$\overline{V_{10}\left(t\right)\left|1\rangle\langle0\right|} = V_{10}\left(t\right)\left|1\rangle\langle0\right|B_1^+ B_0^-, \tag{70}$$

$$\overline{V_{01}\left(t\right)\left|0\rangle\langle1\right|} = V_{01}\left(t\right)\left|0\rangle\langle1\right|B_0^+ B_1^-, \tag{71}$$

$$\overline{g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}}} = g_{i\mathbf{k}}\left(\left|0\rangle\langle0\right|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + \left|1\rangle\langle1\right|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\right) + g_{i\mathbf{k}}^*\left(\left|0\rangle\langle0\right|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \left|1\rangle\langle1\right|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{72}$$

$$= g_{i\mathbf{k}}\left(\left(\left|0\rangle\langle0\right|+\left|1\rangle\langle1\right|\right)b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\left|1\rangle\langle1\right| - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\left|0\rangle\langle0\right|\right) + g_{i\mathbf{k}}^*\left(\left(\left|0\rangle\langle0\right|+\left|1\rangle\langle1\right|\right)b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\left|1\rangle\langle1\right| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\left|0\rangle\langle0\right|\right) \tag{73}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\left|0\rangle\langle0\right| - g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\left|0\rangle\langle0\right| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\left|1\rangle\langle1\right| - g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\left|1\rangle\langle1\right| \tag{74}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left|0\rangle\langle0\right| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left|1\rangle\langle1\right|, \tag{75}$$

$$\overline{\left|0\rangle\langle0\right|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)} = \left(\left|0\rangle\langle0\right|B_0^+ + \left|1\rangle\langle1\right|B_1^+\right)\left|0\rangle\langle0\right|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)\left(\left|0\rangle\langle0\right|B_0^- + \left|1\rangle\langle1\right|B_1^-\right) \tag{76}$$

$$= \left|0\rangle\langle0\right|B_0^+\left|0\rangle\langle0\right|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)\left|0\rangle\langle0\right|B_0^- \tag{77}$$

$$= \left|0\rangle\langle0\right|B_0^+\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)B_0^- = \left|0\rangle\langle0\right|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^*\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{78}$$

$$\overline{\left|1\rangle\langle1\right|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)} = \left(\left|0\rangle\langle0\right|B_0^+ + \left|1\rangle\langle1\right|B_1^+\right)\left|1\rangle\langle1\right|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)\left(\left|0\rangle\langle0\right|B_0^- + \left|1\rangle\langle1\right|B_1^-\right) \tag{79}$$

$$= \left|1\rangle\langle1\right|B_1^+\left|1\rangle\langle1\right|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)\left|1\rangle\langle1\right|B_1^- \tag{80}$$

$$= \left|1\rangle\langle1\right|B_1^+\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)B_1^- \tag{81}$$

$$= \left|1\rangle\langle1\right|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^*\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{82}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \omega_{\mathbf{k}}\left(\left|0\rangle\langle0\right|B_0^+ + \left|1\rangle\langle1\right|B_1^+\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(\left|0\rangle\langle0\right|B_0^- + \left|1\rangle\langle1\right|B_1^-\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(\left|0\rangle\langle0\right|B_0^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}}B_0^- + \left|1\rangle\langle1\right|B_1^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}}B_1^-\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\left(\left|0\rangle\langle0\right|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)+\left|1\rangle\langle1\right|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(\left|0\rangle\langle0\right|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)+\left|1\rangle\langle1\right|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(\left|0\rangle\langle0\right|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + \left|1\rangle\langle1\right|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(\left|0\rangle\langle0\right|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I} + \left|1\rangle\langle1\right|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{87}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{88}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger + \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger + \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) \right) \tag{89}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{90}$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{91}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{92}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \tag{93}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right). \tag{94}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t) |0\rangle\langle 0|} + \overline{\varepsilon_1(t) |1\rangle\langle 1|} + \overline{V_{10}(t) |1\rangle\langle 0|} + \overline{V_{01}(t) |0\rangle\langle 1|} \tag{95}$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_1^+ B_0^- + V_{01}(t) |0\rangle\langle 1| B_0^+ B_1^-, \tag{96}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{97}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \overline{\left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \sum_{\mathbf{k}} |1\rangle\langle 1| \overline{\left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{98}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{99}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{100}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} \tag{101}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right) \tag{102}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right). \tag{103}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j} V_{jj}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{104}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}} \tag{105}$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{106}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \tag{107}$$

$$\chi_{ij} \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \tag{108}$$

χ_{ij} is an imaginary number so $e^{\chi_{ij}}$ is the phase associated to B_{ij}. With the following definitions that we will proof and use from now:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_{ij} \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{\chi_{ij}} \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2i} & \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix} \tag{109}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot) \tag{110}$$
$$(\cdot)^{\Im} \equiv \Im(\cdot) \tag{111}$$

We assume that the bath is at equilibrium with inverse temperature $\beta = 1/k_{\mathrm{B}}T$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{113}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{114}$$

So the product of the matrix representation of $b^\dagger$ and b is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & 1 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & 2 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ldots \\ 0 & 0 & 0 & \ldots & n & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \, |j\rangle\langle j|, \tag{116}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \tag{117}$$

$$\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{118}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{119}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{120}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{121}$$

$$= \frac{1}{1 - \exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{122}$$

$$\equiv f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{123}$$

$$\mathrm{Tr}\left(\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{124}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{125}$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{126}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{127}$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{128}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{129}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}} \mid j_{\mathbf{k}} - 1\rangle, \tag{130}$$

$$b_{\mathbf{k}}^\dagger \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} \mid j_{\mathbf{k}} + 1\rangle. \tag{131}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (130)-(131):

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = \operatorname{Tr}\left(\rho_B B_{iz}\right) = \operatorname{Tr}\left(B_{iz}\rho_B\right) \tag{132}$$

$$= \operatorname{Tr}\left(\left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \, b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{133}$$

$$= \sum_{\mathbf{k}} \operatorname{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^\dagger \rho_B\right) + \sum_{\mathbf{k}} \operatorname{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \, b_{\mathbf{k}}\rho_B\right) \tag{134}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})\operatorname{Tr}\left(b_{\mathbf{k}}^\dagger \rho_B\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \operatorname{Tr}\left(b_{\mathbf{k}}\rho_B\right) \tag{135}$$

$$= \sum_{\mathbf{k}} \operatorname{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}} \operatorname{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) \tag{136}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})\operatorname{Tr}\left(b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \operatorname{Tr}\left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), \tag{137}$$

$$\operatorname{Tr}\left(b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right) b_{\mathbf{k}}^\dagger |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}^\dagger) \tag{138}$$

$$= \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right) \sqrt{j_{\mathbf{k}} + 1}\, |j_{\mathbf{k}} + 1\rangle\langle j_{\mathbf{k}}|\right) \tag{139}$$

$$= 0, \tag{140}$$

$$\operatorname{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right) b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}) \tag{141}$$

$$= \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right) \sqrt{j_{\mathbf{k}}}\, |j_{\mathbf{k}} - 1\rangle\langle j_{\mathbf{k}}|\right) \tag{142}$$

$$= 0. \tag{143}$$

we therefore find that:

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0 \tag{144}$$

Another important expected value is $B = \langle B^\pm\rangle_{\overline{H_{\bar{B}}}}$, where $B^\pm = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)}$ is given by:

$$\langle B^\pm\rangle_{H_B} = \operatorname{Tr}\left(\rho_B B_\pm\right) = \operatorname{Tr}\left(B_\pm \rho_B\right) \tag{145}$$

$$= \operatorname{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)} \rho_B\right) \tag{146}$$

$$= \prod_{\mathbf{k}} \operatorname{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\right)\rho_B\right) \tag{147}$$

$$= \prod_{\mathbf{k}} \langle D\left(\pm\alpha_{\mathbf{k}}\right)\rangle. \tag{148}$$

Given that we can write a density operator as:

$$\rho = \int P\left(\alpha\right)\left|\alpha\right\rangle\!\left\langle\alpha\right|\mathrm{d}^2\alpha \tag{149}$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right)\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P\left(\alpha\right)$ is given by:

$$\left\langle A\right\rangle = \mathrm{Tr}\left(A\rho\right) \tag{150}$$

$$= \int P\left(\alpha\right)\left\langle\alpha\left|A\right|\alpha\right\rangle\mathrm{d}^2\alpha \tag{151}$$

We are typically interested in thermal state density operators, for which it can be shown that $P\left(\alpha\right) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (151) we could obtain that the expected value for the displacement operator $D\left(h\right)$ with $h \in \mathbb{C}$ is equal to:

$$\left\langle D\left(h\right)\right\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle\alpha\left|D\left(h\right)\right|\alpha\right\rangle\mathrm{d}^2\alpha \tag{152}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)\right|0\right\rangle\mathrm{d}^2\alpha \tag{153}$$

$$D\left(h\right)D\left(\alpha\right) = D\left(h + \alpha\right)e^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{154}$$

$$D\left(-\alpha\right)\left(D\left(h\right)D\left(\alpha\right)\right) = D\left(-\alpha\right)D\left(h + \alpha\right)e^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{155}$$

$$= D\left(h\right)e^{\frac{1}{2}\left(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha)\right)}e^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{156}$$

$$= D\left(\alpha\right)e^{\frac{1}{2}\left(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2\right)}e^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{157}$$

$$= D\left(\alpha\right)e^{\left(h\alpha^* - h^*\alpha\right)}, \tag{158}$$

$$\left\langle D\left(h\right)\right\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle 0\left|D\left(h\right)\exp\left(h\alpha^* - h^*\alpha\right)\right|0\right\rangle\mathrm{d}^2\alpha \tag{159}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^* - h^*\alpha\right)\left\langle 0\left|D\left(h\right)\right|0\right\rangle\mathrm{d}^2\alpha \tag{160}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^* - h^*\alpha\right)\left\langle 0|h\right\rangle\mathrm{d}^2\alpha \tag{161}$$

$$\left|\alpha\right\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle, \tag{162}$$

$$\left\langle D\left(h\right)\right\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^* - h^*\alpha\right)\left\langle 0\right|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}\left|n\right\rangle\mathrm{d}^2\alpha \tag{163}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^* - h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{164}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha\right)\mathrm{d}^2\alpha, \tag{165}$$

$$\alpha = x + iy, \tag{166}$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 + y^2}{N} + h(x - iy) - h^*(x + iy)\right) dxdy \tag{167}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{N} + hx - h^*x\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N} - ihy - ih^*y\right) dy, \tag{168}$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N}\left(x^2 - Nhx + Nh^*x\right) \tag{169}$$

$$= -\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 + \frac{N(h^* - h)^2}{4}, \tag{170}$$

$$\frac{y^2}{N} - ihy - ih^*y = -\frac{1}{N}\left(y^2 + iNhy + iNh^*y\right) \tag{171}$$

$$= -\frac{1}{N}\left(y^2 + \frac{iN(h + h^*)}{2}\right) - \frac{N(h + h^*)^2}{4}, \tag{172}$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h+h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 - \frac{1}{N}\left(y^2 + \frac{iN(h+h^*)}{2}\right)\right) dxdy, \tag{173}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx, \tag{174}$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h+h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x + \frac{(Nh^* - Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{\left(y^2 + \frac{iN(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dy \tag{175}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h+h^*)^2}{4}\right)}{\pi N} \left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{176}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h+h^*)^2}{4}\right) \tag{177}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^{*2} - 2hh^* + h^2\right) - N\left(h^2 + 2hh^* + h^{*2}\right)}{4}\right) \tag{178}$$

$$= \exp\left(-|h|^2\left(N + \frac{1}{2}\right)\right) \tag{179}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2}\right)\right) \tag{180}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1}\right)\right) \tag{181}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{182}$$

In the last line we used $\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (147) using (182) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{183}$$

We will now force $\left\langle \overline{H_I} \right\rangle_{\overline{H_{\tilde{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B^+\sigma^+$ and $B^-\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\left\langle \overline{H_I} \right\rangle_{\overline{H_{\tilde{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_j^+B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j|\left((g_{j\mathbf{k}} - v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)$$ (184)

$$= \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'} + \sum_j |j\rangle\langle j|B_{jz} + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+B_{j'}^- - B_{jj'}\right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$$ (185)

$$\equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}.$$ (186)

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_1^+ B_0^- \rangle = B_{10}$$ (187)

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle$$ (188)

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle$$ (189)

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right) \right\rangle$$ (190)

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}$$ (191)

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}$$ (192)

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}.$$ (193)

From the definition $B_{01} = \langle B_0^+ B_1^- \rangle$ using the displacement operator we have:

$$\langle B_0^+ B_1^- \rangle = B_{01}$$ (194)

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle$$ (195)

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle$$ (196)

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right) \right\rangle$$ (197)

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}$$ (198)

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}$$ (199)

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}$$ (200)

We can check:

$$\langle B_0^+ B_1^- \rangle = B_{01} \tag{201}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{202}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)^*} \tag{203}$$

$$= \langle B_1^+ B_0^- \rangle^* \tag{204}$$

$$= B_{10}^*. \tag{205}$$

The parts of the Hamiltonian splitted are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\sigma^+ + V_{01}(t) B_{01}\sigma^-, \tag{206}$$

$$\overline{H_{\bar{I}}} \equiv V_{10}(t)\left(B_1^+ B_0^- - B_{10}\right)\sigma^+ + V_{01}(t)\left(B_0^+ B_1^- - B_{01}\right)\sigma^- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, \tag{207}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{208}$$

$$= H_B. \tag{209}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (207) we can verify the condition $\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\right)|n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+ B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{210}$$

$$= \left\langle \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+ B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{211}$$

$$= \sum_{n\mathbf{k}}\left(\left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}(t) B_j^+ B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle V_{jj'}(t) B_{jj'}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{212}$$

$$= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(\left\langle B_j^+ B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}} - \langle B_{jj'}\rangle_{\overline{H_{\bar{B}}}}\right) \tag{213}$$

$$= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(B_{jj'} - B_{jj'}\right) \tag{214}$$

$$= 0. \tag{215}$$

We used (144) and (193) to evaluate the expected values.

Let's consider the following Hermitian combinations:

$$B_x = B_x^\dagger \tag{216}$$

$$= \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}, \tag{217}$$

$$B_y = B_y^\dagger \tag{218}$$

$$= \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2i}, \tag{219}$$

$$B_{iz} = B_{iz}^\dagger \tag{220}$$

$$= \sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}}\right). \tag{221}$$

Writing the equations (206) and (207) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}(t) = (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\sigma^+ + V_{01}(t) B_{01}\sigma^- \tag{222}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) B_{01}\frac{\sigma_x - \mathrm{i}\sigma_y}{2} \tag{223}$$

$$= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+V_{10}(t)\left(B_{10}^{\Re}(t)+\mathrm{i}B_{10}^{\Im}(t)\right)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\left(B_{10}^{\Re}(t)-\mathrm{i}B_{10}^{\Im}(t)\right)\frac{\sigma_x-\mathrm{i}\sigma_y}{2} \tag{224}$$

$$= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+B_{10}^{\Re}(t)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}-V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right) \tag{225}$$

$$= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+B_{10}^{\Re}(t)\left(\sigma_x\frac{V_{10}(t)+V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)-V_{01}(t)}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\sigma_x\frac{V_{10}(t)-V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)+V_{01}(t)}{2}\right) \tag{226}$$

$$= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+B_{10}^{\Re}(t)\left(\sigma_x\frac{V_{10}(t)+V_{10}^*(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)-V_{10}^*(t)}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\sigma_x\frac{V_{10}(t)-V_{10}^*(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)+V_{10}^*(t)}{2}\right) \tag{227}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + B_{10}^{\Re}(t)\left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t)\right) + \mathrm{i}B_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x V_{10}^{\Im}(t) + \mathrm{i}\sigma_y V_{10}^{\Re}(t)\right) \tag{228}$$

$$= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\left(\sigma_x B_{10}^{\Re}(t)V_{10}^{\Re}(t)-\sigma_y B_{10}^{\Re}(t)V_{10}^{\Im}(t)\right)-\left(\sigma_x B_{10}^{\Im}(t)V_{10}^{\Im}(t)+\sigma_y B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{229}$$

$$= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{230}$$

$$= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right). \tag{231}$$

$$\overline{H_{\bar{I}}} = V_{10}(t)\left(\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10}\right) + V_{01}(t)\left(\sigma^- B_0^+ B_1^- - \sigma^- B_{01}\right) + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} \tag{232}$$

$$= |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} + \left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10}\right) + \left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^- B_0^+ B_1^- - \sigma^- B_{01}\right) \tag{233}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10} + \sigma^- B_0^+ B_1^- - \sigma^- B_{01}\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10} - \sigma^- B_0^+ B_1^- + \sigma^- B_{01}\right) \tag{234}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+ B_0^- - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10} + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+ B_1^- - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\right) \tag{235}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+ B_0^- - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10} + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+ B_1^- - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+ B_0^- - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10} - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+ B_1^- \right. \tag{236}$$

$$\left. + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\right) \tag{237}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} + \mathrm{i}\sigma_y\frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2}\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma_x\frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2}\right. \tag{238}$$

$$\left. + \mathrm{i}\sigma_y\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}\right) \tag{239}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x\frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2} - \sigma_y\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}\right) \tag{240}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2\mathrm{i}} - \sigma_y\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}\right) \tag{241}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2\mathrm{i}} - \sigma_y\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}\right) \tag{242}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x(-B_y) - \sigma_y B_x\right) \tag{243}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y - \sigma_y B_x\right). \tag{244}$$

III.   FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \le A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right) + \left\langle \overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} + O\left(\left\langle \overline{H_{\bar{I}}}^2\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{245}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle \overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = 0. \tag{246}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\left(t\right),\overline{H_{\bar{B}}}\right]=0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}=e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{247}$$

Then using the fact that $\overline{H_{\bar{S}}}\left(t\right)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right)=\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{248}$$

So Eq. (246) becomes:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}}=-\frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}} \tag{249}$$

$$=-\frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} \tag{250}$$

$$=-\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}} \tag{251}$$

$$=-\frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}}-\frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} \tag{252}$$

$$=0\quad\text{(by Eq. (246))}. \tag{253}$$

But since $\bar{H}_{\bar{B}}=H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}}=\frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} \tag{254}$$

$$=0. \tag{255}$$

This means we need to impose:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}}=0. \tag{256}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}\left(t\right)=-\beta\left(\left(\varepsilon_0\left(t\right)+R_0\right)\left|0\right\rangle\!\left\langle0\right|+\left(\varepsilon_1\left(t\right)+R_1\right)\left|1\right\rangle\!\left\langle1\right|+V_{10}\left(t\right)B_{10}\sigma^++V_{01}\left(t\right)B_{01}\sigma^-\right). \tag{257}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}\left(t\right)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2-\mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)+\mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)=0. \tag{258}$$

Let's define:

$$\varepsilon\left(t\right)\equiv\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), \tag{259}$$

$$\eta\equiv\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}. \tag{260}$$

The solutions of the equation (258) are:

$$\lambda = \beta \frac{-\operatorname{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right) \pm \sqrt{\left(\operatorname{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^2 - 4\operatorname{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}}{2} \tag{261}$$

$$= \beta \frac{-\varepsilon\left(t\right) \pm \eta\left(t\right)}{2} \tag{262}$$

$$= -\beta \frac{\varepsilon\left(t\right) \mp \eta\left(t\right)}{2}. \tag{263}$$

The value of $\operatorname{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\operatorname{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right) + \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) \tag{264}$$

$$= 2\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{265}$$

Given that $v_{i\mathbf{k}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\operatorname{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A\left(\varepsilon\left(t\right), \eta\left(t\right)\right)$ to calculate $\frac{\partial\operatorname{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}$ can lead to:

$$\frac{\partial\operatorname{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{266}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) + 2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) \tag{267}$$

$$= -\beta\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{268}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) = 0. \tag{269}$$

The derivates included in the expression given are related to:

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{270}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right)^* \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{271}$$

$$= \langle B_1^+ B_0^- \rangle^*, \tag{272}$$

$$R_i = \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{273}$$

$$= \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{274}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{275}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2\omega_{\mathbf{k}}^2} \left(v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* \right) \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \tag{276}$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = \left(v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im} \right)\left(v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im} \right) - \left(v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im} \right)\left(v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im} \right) \tag{277}$$

$$= \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im} \right) - \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} + i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im} \right) \tag{278}$$

$$= 2i \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} \right), \tag{279}$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}} - v_{0\mathbf{k}})(v_{1\mathbf{k}} - v_{0\mathbf{k}})^* \tag{280}$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - (v_{1\mathbf{k}} v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}}) \tag{281}$$

$$= \left(v_{1\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} \right)^2 + \left(v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{0\mathbf{k}}^{\Im} \right)^2 - \left(\left(v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im} \right)\left(v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im} \right) + \left(v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im} \right)\left(v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im} \right) \right) \tag{282}$$

$$= \left(v_{1\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} \right)^2 + \left(v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{0\mathbf{k}}^{\Im} \right)^2 - 2 \left(v_{1\mathbf{k}}^{\Re} v_{0\mathbf{k}}^{\Re} + v_{1\mathbf{k}}^{\Im} v_{0\mathbf{k}}^{\Im} \right) \tag{283}$$

$$= \left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} \right)^2. \tag{284}$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re} \right)^2 + \left(v_{i\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re} - i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re} + i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \right) \right) \tag{285}$$

$$= \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re} \right)^2 + \left(v_{i\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{286}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} \exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{287}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{2i\left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} \right)}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{288}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{i\left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} \right)}{\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \tag{289}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial \varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(\varepsilon_1\left(t\right) + R_1 + \varepsilon_0\left(t\right) + R_0\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{290}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - \mathrm{i}v_{i\mathbf{k}}^{\Im}\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{291}$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{292}$$

$$\frac{\partial\left|B_{10}\right|^2}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{293}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{294}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}\left|B_{10}\right|^2, \tag{295}$$

$$\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{296}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} - 4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} \tag{297}$$

$$= \frac{\varepsilon\left(t\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left(\left(\varepsilon_1\left(t\right) + R_1\right)\left(\varepsilon_0\left(t\right) + R_0\right) - \left|V_{10}\left(t\right)\right|^2\left|B_{10}\left(t\right)\right|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta\left(t\right)} \tag{298}$$

$$= \frac{\varepsilon\left(t\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon\left(t\right) - \varepsilon_i\left(t\right) - R_i\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{299}$$

$$= \frac{\varepsilon\left(t\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon\left(t\right) - \varepsilon_i\left(t\right) - R_i\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{i\mathbf{k}}^{\Re} - v_{i'\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{300}$$

$$= \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon\left(t\right) - 4\left(\varepsilon\left(t\right) - \varepsilon_i\left(t\right) - R_i\right) - \frac{4}{\omega_{\mathbf{k}}}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\left(t\right)}\right) + \frac{1}{\eta\left(t\right)}\left(-\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\varepsilon\left(t\right) + 2\left(\varepsilon\left(t\right) - \varepsilon_i\left(t\right) - R_i\right)\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right. \tag{301}$$

$$\left. + 4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{302}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right) = \frac{\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}} \tag{303}$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon\left(t\right) - 2\left(\varepsilon\left(t\right) - \varepsilon_i\left(t\right) - R_i\right) - \frac{2}{\omega_{\mathbf{k}}}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\left(t\right)}\right) + 2\frac{\left(\varepsilon\left(t\right) - \varepsilon_i\left(t\right) - R_i\right)\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} + 2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\varepsilon\left(t\right)}{\eta\left(t\right)}} \tag{304}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}-g_{i\mathbf{k}}^*\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g_{i\mathbf{k}}^*\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$=\frac{\left(v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{308}$$

Separating (307) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{309}$$

$$v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}^{\Re}=v_{i\mathbf{k}}^{\Re}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right) \tag{310}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{311}$$

$$v_{i\mathbf{k}}^{\Re}=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{312}$$

$$v_{i\mathbf{k}}^{\Re}=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{313}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}}=\frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{314}$$

$$=\frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2+\left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-iv_{i\mathbf{k}}^{\Im}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{315}$$

$$=2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-i\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{316}$$

$$\frac{\partial|B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Im}}=\frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^2+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{317}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^2+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{318}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}|B_{10}|^2 \tag{319}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{320}$$

$$= \frac{2\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}}-4\frac{\partial\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{2\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{321}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{\eta(t)} \tag{322}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{323}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}-v_{i'\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{324}$$

$$= \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2(\varepsilon(t)-\varepsilon_i(t)-R_i)\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{325}$$

$$\left.+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{326}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\dfrac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}}}{\dfrac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}}} \tag{327}$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\left(g_{i\mathbf{k}}^*\right)^{\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{\left(g_{i\mathbf{k}}^*\right)^{\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)} \tag{328}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v_{i\mathbf{k}}^{\Im}-\mathrm{i}\left(g_{i\mathbf{k}}^*-g_{i\mathbf{k}}\right)\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^*-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{330}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{331}$$

$$= \frac{\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{332}$$

Separating (332) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{333}$$

$$v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}=v_{i\mathbf{k}}^{\Im}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \tag{334}$$

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Im}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{335}$$

$$v_{i\mathbf{k}}^{\Im}=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{336}$$

$$v_{i\mathbf{k}}^{\Im}=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}})=v_{i\mathbf{k}}^{\Re}(\omega_{\mathbf{k}})+iv_{i\mathbf{k}}^{\Im}(\omega_{\mathbf{k}}) \tag{338}$$

$$=\frac{g_{i\mathbf{k}}^{\Re}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{339}$$

$$+i\frac{g_{i\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{340}$$

$$=\frac{g_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{341}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0)=\rho_S(0)\otimes\rho_B^{\text{Thermal}}$, the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T(0) e^{-V} \tag{342}$$

$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right) \left(\rho_S(0) \otimes \rho_B^{\text{Thermal}}\right) \left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{343}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: |0\rangle\langle 0|0\rangle B_0^+\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_0^- \tag{344}$$

$$= |0\rangle B_0^+\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_0^- \tag{345}$$

$$= |0\rangle\langle 0| \otimes B_0^+ \rho_B^{\text{Thermal}} B_0^- \tag{346}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: |1\rangle\langle 1|B_1^+|1\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_1^- \tag{347}$$

$$= |1\rangle\langle 1|B_1^+ \rho_B^{\text{Thermal}} B_1^- \tag{348}$$

$$= |1\rangle\langle 1| \otimes B_1^+ \rho_B^{\text{Thermal}} B_1^- \tag{349}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1|: |0\rangle\langle 0|B_0^+|0\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_1^- \tag{350}$$

$$= |0\rangle\langle 1|B_0^+ \rho_B^{\text{Thermal}}|1\rangle\langle 1|B_1^- \tag{351}$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_0^+ \rho_B^{\text{Thermal}} B_1^- \tag{352}$$

$$= |0\rangle\langle 1| \otimes B_0^+ \rho_B^{\text{Thermal}} B_1^- \tag{353}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0|: |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_0^- \tag{354}$$

$$= |1\rangle\langle 0| \otimes B_1^+ \rho_B^{\text{Thermal}} B_0^- \tag{355}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^\dagger(t) O(t) U(t) \tag{356}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\tilde{S}}}(t')\right). \tag{357}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\overline{\rho_S}(t)U(t), \text{ where} \tag{358}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\overline{\rho_T}(t)\right) \tag{359}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ V_{10}^\Re(t) & V_{10}^\Re(t) & 1 & V_{10}^\Im(t) & -V_{10}^\Im(t) & 1 \end{pmatrix} \tag{360}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\tilde{I}}}(t)$ as pointed in the equation (244):

$$\overline{H_{\tilde{I}}}(t) = \sum_i B_{iz}|i\rangle\langle i| + V_{10}^\Re(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^\Im(t)\left(\sigma_x B_y - \sigma_y B_x\right) \tag{361}$$

$$= B_{0z}|0\rangle\langle 0| + B_{1z}|1\rangle\langle 1| + V_{10}^\Re(t)\sigma_x B_x + V_{10}^\Re(t)\sigma_y B_y + V_{10}^\Im(t)\sigma_x B_y - V_{10}^\Im(t)\sigma_y B_x \tag{362}$$

$$= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right) \tag{363}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{364}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) \text{-i}\int_0^t [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]\mathrm{d}s. \tag{365}$$

Replacing the equation (365) in the equation (364) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]]\mathrm{d}s. \tag{366}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty}(-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2 ... \int_0^{t_{n-1}}\mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdots] \tag{367}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty}(-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2 ... \int_0^{t_{n-1}}\mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B^{\text{Thermal}}]] \cdots] \tag{368}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B^{\text{Thermal}}$. Consider the following notation:

$$\widetilde{\overline{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\overline{\rho_S}(0) \tag{369}$$
$$= W(t)\overline{\rho_S}(0) \tag{370}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2 ... \int_0^{t_{n-1}}\mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B^{\text{Thermal}}]] \cdots] \tag{371}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{372}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}W(t)\overline{\rho_S}(0) \tag{373}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}\widetilde{\overline{\rho_S}}(t) \tag{374}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B^{\text{Thermal}}] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (372) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\widetilde{\overline{\rho_S}}(t) \tag{375}$$

$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}}\right]\right] \tag{376}$$

Replacing $t_1 \to t - \tau$ and moving back into the Schrödinger picture gives:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(-\tau), \overline{\rho_S}(t)\rho_B^{\mathrm{Thermal}}\right]\right] \tag{377}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}(t)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{378}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = \sum_i C_i(t)\left(\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)\right) \tag{379}$$

$$\widetilde{A_i}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{380}$$

$$= U^\dagger(t)A_iU(t)\,\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} \tag{381}$$

$$= U^\dagger(t)A_iU(t)\,\mathbb{I} \tag{382}$$

$$= U^\dagger(t)A_iU(t) \tag{383}$$

$$\widetilde{B_i}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{384}$$

$$= U^\dagger(t)U(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{385}$$

$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{386}$$

$$= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{387}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.

Using the expression (379) to replace it in the equation (376)

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right]\mathrm{d}s \tag{388}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right), \left[\sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right), \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right]\mathrm{d}s \tag{389}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}} - \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right)\right]\mathrm{d}s \tag{390}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right)\sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}} - \sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right)\right. \tag{391}$$

$$\left. - \sum_{i \in J} C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right) + \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right)\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right)\right)\mathrm{d}s \tag{392}$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \left\langle\widetilde{B_j}(t)\widetilde{B_i}(s)\right\rangle_B \tag{393}$$

$$= \left\langle\widetilde{B_j}(\tau)\widetilde{B_i}(0)\right\rangle_B \tag{394}$$

The correlation functions relevant that appear in the equation (392) are:

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B^{\mathrm{Thermal}}\right) = \left\langle \widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B \tag{395}$$

$$= \left\langle \widetilde{B_j}(\tau)\,\widetilde{B_i}(0)\right\rangle_B \tag{396}$$

$$= \Lambda_{ji}(\tau) \tag{397}$$

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\rho_B^{\mathrm{Thermal}}\widetilde{B_i}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B^{\mathrm{Thermal}}\right) \tag{398}$$

$$= \left\langle \widetilde{B_i}(s)\,\widetilde{B_j}(t)\right\rangle_B \tag{399}$$

$$= \left\langle \widetilde{B_i}(-\tau)\,\widetilde{B_j}(0)\right\rangle_B \tag{400}$$

$$= \Lambda_{ij}(-\tau) \tag{401}$$

$$\mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\rho_B^{\mathrm{Thermal}}\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B^{\mathrm{Thermal}}\right) \tag{402}$$

$$= \left\langle \widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B \tag{403}$$

$$= \left\langle \widetilde{B_j}(\tau)\,\widetilde{B_i}(0)\right\rangle_B \tag{404}$$

$$= \Lambda_{ji}(\tau) \tag{405}$$

$$\mathrm{Tr}_B\left(\rho_B^{\mathrm{Thermal}}\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B^{\mathrm{Thermal}}\right) \tag{406}$$

$$= \left\langle \widetilde{B_i}(s)\,\widetilde{B_j}(t)\right\rangle_B \tag{407}$$

$$= \left\langle \widetilde{B_i}(-\tau)\,\widetilde{B_j}(0)\right\rangle_B \tag{408}$$

$$= \Lambda_{ij}(-\tau) \tag{409}$$

The cyclic property of the trace was use widely in the development of equations (395) and (409). Replacing in (392)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)\widetilde{A_i}(t)\widetilde{A_j}(s)\widetilde{\rho_S}(t) - \Lambda_{ji}(-\tau)\widetilde{A_i}(t)\widetilde{\rho_S}(t)\widetilde{A_j}(s)\big) + C_i(t)C_j(s)\big(\Lambda_{ji}(-\tau)\widetilde{\rho_S}(t)\widetilde{A_j}(s)\widetilde{A_i}(t) - \Lambda_{ij}(\tau)\widetilde{A_j}(s)\widetilde{\rho_S}(t)\widetilde{A_i}(t)\big)\Big)ds \tag{410}$$

$$= -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)\big[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\rho_S}(t)\big] + \Lambda_{ji}(-\tau)\big[\widetilde{\rho_S}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)\big]\big)\Big)ds \tag{411}$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}(\tau)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t) - \Lambda_{ij}(\tau)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,\widetilde{A_i}(t) = \Lambda_{ij}(\tau)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\rho_s}(t)\right] \tag{412}$$

$$\Lambda_{ji}(-\tau)\,\widetilde{\rho_S}(t)\,\widetilde{A_j}(s)\,\widetilde{A_i}(t) - \Lambda_{ji}(-\tau)\,\widetilde{A_i}(t)\,\widetilde{\rho_S}(t)\,\widetilde{A_j}(s) = \Lambda_{ji}(-\tau)\left[\widetilde{\rho_S}(t)\widetilde{A_j},\widetilde{A_i}(t)\right] \tag{413}$$

Returning to the Schroedinger picture we have:

$$U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,U^\dagger(t) = U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\,U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\,U(t)\,\widetilde{\rho_S}(t)\,U^\dagger(t) \tag{414}$$

$$= \left(U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{\rho_S}(t)\,U^\dagger(t)\right) \tag{415}$$

$$= A_i\widetilde{A_j}(s,t)\,\overline{\rho_S}(t) \tag{416}$$

This procedure applying to the relevant commutators give us:

$$U(t)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\rho_S}(t)\right]U^\dagger(t) = \left(U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,U^\dagger(t) - U(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right) \tag{417}$$

$$= A_i\widetilde{A_j}(s,t)\,\overline{\rho_S}(t) - \widetilde{A_j}(s,t)\,\overline{\rho_S}(t)\,A_i \tag{418}$$

$$= \left[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\right] \tag{419}$$

Introducing this transformed commutators in the equation (411) allow us to obtain the master equation of the system

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \sum_{ij} \int_0^t \mathrm{d}\tau \left(C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[A_i, \widetilde{A_j}(t-\tau, t) \overline{\rho_S}(t)\right]\right. \tag{420}$$

$$\left. + C_j(t) C_i(t-\tau) \Lambda_{ji}(-\tau) \left[\overline{\rho_S}(t) \widetilde{A_j}(t-\tau, t), A_i\right]\right) \tag{421}$$

where $i, j \in \{1, 2, 3, 4, 5.6\}$.

Here $\widetilde{A_j}(s, t) = U(t) U^\dagger(s) A_j U(s) U^\dagger(t)$ where $U(t)$ is given by (357). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. The environmental correlation functions are given by:

$$\Lambda_{ij}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_i}(t) \widetilde{B_j}(s) \rho_B^{\mathrm{Thermal}}\right) \tag{422}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau) \widetilde{B_j}(0) \rho_B^{\mathrm{Thermal}}\right) \tag{423}$$

Calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rho_B^{\mathrm{Thermal}}\right) \tag{424}$$

$$= \int \mathrm{d}^2\alpha P(\alpha) \left\langle \alpha \left| \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0)\right| \alpha\right\rangle \tag{425}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \left\langle \alpha \left| \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0)\right| \alpha\right\rangle \mathrm{d}^2\alpha \tag{426}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \left\langle \alpha \left| \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0)\right| \alpha\right\rangle \mathrm{d}^2\alpha \tag{427}$$

$$\widetilde{B_{jz}}(\tau) = \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{428}$$

$$\widetilde{B_{jz}}(0) = \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'}\right) \tag{429}$$

$$\left\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rho_B\right) \tag{430}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \sum_{\mathbf{k}'}\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^* b_{\mathbf{k}'}\right)\rho_B\right) \tag{431}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^* b_{\mathbf{k}'}\right)\rho_B\right) \tag{432}$$

$$+ \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}\right)\rho_B\right) \tag{433}$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}} = p_{j\mathbf{k}} \tag{434}$$

$$\left\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}'}b_{\mathbf{k}'}^\dagger+p_{j\mathbf{k}'}^* b_{\mathbf{k}'}\right)\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger+p_{j\mathbf{k}}^* b_{\mathbf{k}}\right)\rho_B\right) \tag{435}$$

$$= 0 + \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger+p_{j\mathbf{k}}^* b_{\mathbf{k}}\right)\rho_B\right) \tag{436}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+|p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*2} b_{\mathbf{k}} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B\right) \tag{437}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^{*2} b_{\mathbf{k}} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{438}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{439}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\rho_B\right) + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\rho_B\right)\right) \tag{440}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{441}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle\alpha_{\mathbf{k}}|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle\alpha_{\mathbf{k}}|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|\alpha_{\mathbf{k}}\rangle d^2\alpha_{\mathbf{k}}\right) \tag{442}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}})|0\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{443}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}})|0\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{444}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}} D(\alpha_{\mathbf{k}})|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{445}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}} D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{446}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*)(b_{\mathbf{k}} + \alpha_{\mathbf{k}})|0\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|(b_{\mathbf{k}} + \alpha_{\mathbf{k}})(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*)|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{447}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{448}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) + \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^*|0\rangle d^2\alpha_{\mathbf{k}}\right)\right. \tag{449}$$

$$\left. + \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) + \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^*|0\rangle d^2\alpha_{\mathbf{k}}\right)\right) \tag{450}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{451}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0||\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{452}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|0\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0||\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{453}$$

$$1 = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} \tag{454}$$

$$b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |0\rangle = 0 \tag{455}$$

$$b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |0\rangle = |0\rangle \tag{456}$$

$$\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rangle_B = \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0||\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0||\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{457}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|0\rangle d^2\alpha_{\mathbf{k}}\right) \tag{458}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}}\right) \tag{459}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\left(e^{i\omega_{\mathbf{k}}\tau} + e^{-i\omega_{\mathbf{k}}\tau}\right) \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}}\right) \tag{460}$$

$$\frac{1}{\pi N} \int_0^{2\pi} \int_0^\infty r^2 \exp\left(-\frac{r^2}{N}\right) r\,dr\,d\theta = \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} \tag{461}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(2\cos(\omega_{\mathbf{k}}\tau) N\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 e^{-i\omega_{\mathbf{k}}\tau} \tag{462}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(2\cos(\omega_{\mathbf{k}}\tau) N + e^{-i\omega_{\mathbf{k}}\tau}\right) \tag{463}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} + e^{-i\omega_{\mathbf{k}}\tau}\right) \tag{464}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} + \cos(\omega_{\mathbf{k}}\tau) - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{465}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{(2 + e^{\beta\omega_{\mathbf{k}}} - 1)\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{466}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{(1 + e^{\beta\omega_{\mathbf{k}}})\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{467}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos(\omega_{\mathbf{k}}\tau)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{\frac{\beta\omega_{\mathbf{k}}}{2}}} - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{468}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau) - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{469}$$

$$= \sum_{\mathbf{k}} |g_{j\mathbf{k}} - v_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau) - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{470}$$

$$\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\rangle_B = \int d^2\alpha_{\mathbf{k}} P(\alpha_{\mathbf{k}}) \langle\alpha_{\mathbf{k}}|\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)|\alpha_{\mathbf{k}}\rangle \tag{471}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle\alpha_{\mathbf{k}}|\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)|\alpha_{\mathbf{k}}\rangle d^2\alpha_{\mathbf{k}} \tag{472}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| \Sigma_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}\right)\Sigma_{\mathbf{k}'}\left((g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})^* b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{473}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| \Sigma_{\mathbf{k}\neq\mathbf{k}'}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}\right)\left((g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})^* b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{474}$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| \Sigma_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}\right)\left((g_{j'\mathbf{k}}-v_{j'\mathbf{k}})b_{\mathbf{k}}^\dagger+(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{475}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| \Sigma_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}\right)\left((g_{j'\mathbf{k}}-v_{j'\mathbf{k}})b_{\mathbf{k}}^\dagger+(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{476}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| \Sigma_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau}|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| \Sigma_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau}|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{477}$$

$$= \Sigma_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}}^\dagger b_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \Sigma_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{478}$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}}^\dagger b_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}})|0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{479}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \left| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{480}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \left| \left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*\right)(b_{\mathbf{k}} + \alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{481}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}|^2 \, \mathrm{d}^2\alpha_{\mathbf{k}} \tag{482}$$

$$= N \tag{483}$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})|0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{484}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \left| (b_{\mathbf{k}} + \alpha_{\mathbf{k}})\left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*\right) \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{485}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{486}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{487}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{488}$$

$$= N + 1 \tag{489}$$

$$\left\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \right\rangle_B = \Sigma_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N + \Sigma_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau}(N+1) \tag{490}$$

$$= \Sigma_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} + N\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau}\right)\right) \tag{491}$$

$$D(h') D(h) = \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) D(h' + h) \tag{492}$$

$$\langle D(h') D(h)\rangle_B = \mathrm{Tr}_B\left(\exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) D(h' + h) \rho_B^{\mathrm{Thermal}}\right) \tag{493}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \mathrm{Tr}_B\left(D(h' + h) \rho_B^{\mathrm{Thermal}}\right) \tag{494}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \frac{1}{\pi N} \int \mathrm{d}^2\alpha P(\alpha) \langle \alpha| D(h' + h)|\alpha\rangle \tag{495}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{496}$$

$$h' = h\exp(i\omega\tau) \tag{497}$$

$$\langle D(h\exp(i\omega\tau)) D(h)\rangle_B = \exp\left(\frac{1}{2}\left(hh^*\exp(i\omega\tau) - h^*h\exp(-i\omega\tau)\right)\right) \exp\left(-\frac{|h + h\exp(i\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{498}$$

$$\frac{1}{2}|h|^2\left(\exp(i\omega\tau) - \exp(-i\omega\tau)\right) = \frac{1}{2}\left(hh^*\exp(i\omega\tau) - h^*h\exp(-i\omega\tau)\right) \tag{499}$$

$$= \frac{1}{2}|h|^2\left(\cos(\omega\tau) + i\sin(\omega\tau) - \cos(\omega\tau) + i\sin(\omega\tau)\right) \tag{500}$$

$$= \frac{1}{2}|h|^2\left(2i\sin(\omega\tau)\right) \tag{501}$$

$$= i|h|^2\sin(\omega\tau) \tag{502}$$

$$-\frac{|h + h\exp(i\omega\tau)|^2}{2} = -|h|^2\frac{|1 + \exp(i\omega\tau)|^2}{2} \tag{503}$$

$$= -|h|^2 \, \frac{|1 + \cos{(\omega\tau)} + \mathrm{i}\sin{(\omega\tau)}|^2}{2} \tag{504}$$

$$= -|h|^2 \, \frac{(1 + \cos{(\omega\tau)})^2 + \sin^2{(\omega\tau)}}{2} \tag{505}$$

$$= -|h|^2 \, \frac{(1 + 2\cos{(\omega\tau)} + \cos^2{(\omega\tau)}) + \sin^2{(\omega\tau)}}{2} \tag{506}$$

$$= -|h|^2 \, \frac{2 + 2\cos{(\omega\tau)}}{2} \tag{507}$$

$$= -|h|^2 \, (1 + \cos{(\omega\tau)}) \tag{508}$$

$$\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\rangle_B = \exp\!\big(\mathrm{i}|h|^2\sin(\omega\tau)\big)\exp\!\Big(-|h|^2(1+\cos(\omega\tau))\coth\big(\tfrac{\beta\omega}{2}\big)\Big) \tag{509}$$

$$= \exp\left(\mathrm{i}\,|h|^2\sin{(\omega\tau)} - |h|^2\,(1 + \cos{(\omega\tau)})\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{510}$$

$$= \exp\left(-|h|^2\left(-\mathrm{i}\sin{(\omega\tau)} + \cos{(\omega\tau)}\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-|h|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{511}$$

$$= \langle D(h)\rangle_B \exp{(-\phi{(\tau)})} \tag{512}$$

$$\exp{(-\phi{(\tau)})} = \exp\left(-|h|^2\left(\cos{(\omega\tau)}\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin{(\omega\tau)}\right)\right) \tag{513}$$

$$\phi{(\tau)} = |h|^2\left(\cos{(\omega\tau)}\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin{(\omega\tau)}\right) \tag{514}$$

$$\langle D{(h')}\,D{(h)}\rangle_B = \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right)\exp\left(-\frac{|h + h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{515}$$

$$h' = v\exp{(\mathrm{i}\omega\tau)} \tag{516}$$

$$\left\langle \widetilde{B_1^+ B_0^-}{(\tau)}\widetilde{B_1^+ B_0^-}{(0)}\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_1^+ B_0^-}{(\tau)}\,\widetilde{B_1^+ B_0^-}{(0)}\,\rho_B^{\mathrm{Thermal}}\right) \tag{517}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_1^+ B_0^-}{(\tau)}\,\widetilde{B_1^+ B_0^-}{(0)}\,\rho_B^{\mathrm{Thermal}}\right) \tag{518}$$

$$= \mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\Big)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{519}$$

$$= \mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\Big)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{520}$$

$$= \Pi_{\mathbf{k}}\left(\exp\!\Big(\tfrac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\right)\mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{521}$$

$$= \Pi_{\mathbf{k}}\!\left(\exp\!\Big(\tfrac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\right)\Pi_{\mathbf{k}}\!\left(\exp\!\Big(-\big|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\big(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\big)\Big)\exp\!\Big(-\big|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big)\right) \tag{522}$$

$$= \Pi_{\mathbf{k}}\!\left(\exp\!\Big(\tfrac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\exp\!\Big(-\big|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\big(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\big)\Big)\exp\!\Big(-\big|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big)\right) \tag{523}$$

$$\left\langle \widetilde{B_0^+ B_1^-}{(\tau)}\widetilde{B_0^+ B_1^-}{(0)}\right\rangle_B = \Pi_{\mathbf{k}}\!\left(\exp\!\Big(\tfrac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\exp\!\Big(-\big|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\big(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\big)\Big)\exp\!\Big(-\big|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big)\right) \tag{524}$$

$$\left\langle \widetilde{B_1^+ B_0^-}{(\tau)}\widetilde{B_0^+ B_1^-}{(0)}\right\rangle_B = \mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\Big)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{525}$$

$$= \mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\Big)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{526}$$

$$= \mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}\!\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\Big)\Pi_{\mathbf{k}}D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\rho_B^{\mathrm{Thermal}}\right) \tag{527}$$

$$= \mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\Big)\Pi_{\mathbf{k}}D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\rho_B^{\mathrm{Thermal}}\right) \tag{528}$$

$$= \Pi_{\mathbf{k}}\,\mathrm{Tr}_B\!\left(\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\Big)D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{529}$$

$$= \Pi_{\mathbf{k}}\,\mathrm{Tr}_B\!\left(\left(D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega\tau+\pi)}\Big)D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{530}$$

$$= \Pi_{\mathbf{k}}\exp\!\Big(-\big|\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\big(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\big)\Big)\exp\!\Big(-\big|\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big) \tag{531}$$

$$= \Pi_{\mathbf{k}}\exp\!\Big(-\big|\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\big(\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\big)\Big)\exp\!\Big(-\big|\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\big|^2\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big) \tag{532}$$

$$\left\langle \widetilde{B_0^+ B_1^-}{(\tau)}\widetilde{B_1^+ B_0^-}{(0)}\right\rangle_B = \mathrm{Tr}_B\!\left(\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\!\left(D\!\Big(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{533}$$

$$= \text{Tr}_B \left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \tag{534}$$

$$= \prod_{\mathbf{k}} \text{Tr}_B \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i(\omega_{\mathbf{k}}\tau + \pi)} \right) D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \tag{535}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i\sin(\omega\tau + \pi) + \cos(\omega\tau + \pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \exp\left(-\left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{536}$$

$$= \left\langle |\widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B \tag{537}$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_{jz}}(0) \right\rangle_B = \text{Tr}_B \left(\prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B^{\text{Thermal}} \right) \tag{538}$$

$$\langle D(h) b \rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | D(h) b | \alpha \rangle \tag{539}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | D(-\alpha) D(h) b D(\alpha) | \alpha \rangle \tag{540}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b D(\alpha) | 0 \rangle \tag{541}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \tag{542}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \tag{543}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) (b + \alpha) | 0 \rangle \tag{544}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha | 0 \rangle \tag{545}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha | 0 \rangle \tag{546}$$

$$= \frac{1}{\pi N} \int \alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|h|^2}{2} \right) d^2\alpha \tag{547}$$

$$= h N \langle D(h) \rangle_B \tag{548}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle \alpha \left| D(h) b^\dagger \right| \alpha \right\rangle \tag{549}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{550}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{551}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{552}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) D(\alpha) \left(b^\dagger + \alpha^* \right) \right| 0 \right\rangle \tag{553}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \left\langle 0 \left| D(h) \left(b^\dagger + \alpha^* \right) \right| 0 \right\rangle \tag{554}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \tag{555}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) | 1 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \tag{556}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle -h | 1 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \tag{557}$$

$$\langle -h | = \exp\left(-\frac{|-h^*|^2}{2} \right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n | \tag{558}$$

$$\langle -h | 1 \rangle = \exp\left(-\frac{|-h^*|^2}{2} \right) (-h^*) \tag{559}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|-h^*|^2}{2} \right) (-h^*) + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha^* \exp\left(-\frac{|-h^*|^2}{2} \right) \tag{560}$$

$$= -h^* \langle D(h) \rangle_B (N + 1) \tag{561}$$

$$\langle b D(h) \rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | b D(h) | \alpha \rangle \tag{562}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha \exp\left(-\frac{|h|^2}{2} \right) \tag{563}$$

$$= h \langle D(h) \rangle_B (N+1) \tag{564}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| b^\dagger D(h) \right| \alpha \right\rangle \tag{565}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha \exp\left(-\frac{|h|^2}{2}\right) \tag{566}$$

$$= -h^* \langle D(h) \rangle_B N \tag{567}$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle_B \tag{568}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{569}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{570}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \right) \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{571}$$

$$= B_{10} \tag{572}$$

The correlation functions can be found readily as:

$$\widetilde{B_1^+ B_0^-}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau}\right) \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \right) \tag{573}$$

$$\widetilde{B_0^+ B_1^-}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau}\right) \exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \right) \tag{574}$$

$$\widetilde{B_x}(0) = \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \tag{575}$$

$$\widetilde{B_y}(0) = \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2i} \tag{576}$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right) \left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega}{2}\right)\right)\right) \tag{577}$$

$$B_{iz} = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \tag{578}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \right\rangle_B \tag{579}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})(g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \tag{580}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0) \right\rangle_B = \left\langle \frac{B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01}}{2} \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right\rangle_B \tag{581}$$

$$= \frac{1}{4} \left\langle \left(B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01} \right) \left(B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01} \right) \right\rangle_B \tag{582}$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- \right. \tag{583}$$

$$\left. - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_{10} B_1^+ B_0^- - B_{10} B_0^+ B_1^- + B_{10} B_{10} + B_{10} B_{01} - B_{01} B_1^+ B_0^- - B_{01} B_0^+ B_1^- + B_{01} B_{10} + B_{01} B_{01} \right\rangle \tag{584}$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \tag{585}$$

$$\left. + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right\rangle \tag{586}$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B = \Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{587}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{588}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{589}$$

$$S = \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{590}$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B = U\exp(-\phi(\tau))S \tag{591}$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau)\widetilde{B_1^+ B_0^-}(0)\right\rangle_B = U^*\exp(-\phi(\tau))S \tag{592}$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B = \exp(\phi(\tau))S \tag{593}$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau)\widetilde{B_1^+ B_0^-}(0)\right\rangle_B = \left\langle \widetilde{B_1^+ B_0^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B \tag{594}$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{595}$$

$$= U^{*1/2}S^{1/2} \tag{596}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\left\langle B_1^+ B_0^-(\tau)B_1^+ B_0^- + B_1^+ B_0^-(\tau)B_0^+ B_1^- - B_1^+ B_0^-(\tau)B_{10} - B_1^+ B_0^-(\tau)B_{01} + B_0^+ B_1^-(\tau)B_1^+ B_0^- \right. \tag{597}$$

$$\left. + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_{10} - B_0^+ B_1^-(\tau)B_{01}\right\rangle \tag{598}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\left\langle B_1^+ B_0^-(\tau)B_1^+ B_0^- + B_1^+ B_0^-(\tau)B_0^+ B_1^- - B_1^+ B_0^-(\tau)B_{10} - B_1^+ B_0^-(\tau)B_{01} + B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_0^+ B_1^- \right. \tag{599}$$

$$\left. - B_0^+ B_1^-(\tau)B_{10} - B_0^+ B_1^-(\tau)B_{01}\right\rangle \tag{600}$$

$$= \frac{1}{4}\left(U^*\exp(-\phi(\tau))S + \exp(\phi(\tau))S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau))S + U\exp(-\phi(\tau))S - B_{10}^{*2} - |B_{10}|^2\right) \tag{601}$$

$$= \frac{1}{4}\left(2U^\Re\exp(-\phi(\tau))S + 2\exp(\phi(\tau))S - 2\left(B_{10}^2\right)^\Re - 2|B_{10}|^2\right) \tag{602}$$

$$= \frac{1}{4}\left(2U^\Re\exp(-\phi(\tau))S + 2\exp(\phi(\tau))S - 2\left(U^*\right)^\Re S - 2S\right) \tag{603}$$

$$= \frac{S}{2}\left(U^\Re\exp(-\phi(\tau)) + \exp(\phi(\tau)) - \left(U^*\right)^\Re - 1\right) \tag{604}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle \frac{B_0^+ B_1^-(\tau)-B_1^+ B_0^-(\tau)+B_{10}-B_{01}}{2\mathrm{i}}\frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10}-B_{01}}{2\mathrm{i}}\right\rangle_B \tag{605}$$

$$= -\frac{1}{4}\left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}\right)\left(B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}\right)\right\rangle_B \tag{606}$$

$$= -\frac{1}{4}\left\langle B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_{10} - B_0^+ B_1^-(\tau)B_{01} - B_1^+ B_0^-(\tau)B_0^+ B_1^- + B_1^+ B_0^-(\tau)B_1^+ B_0^- \right. \tag{607}$$

$$\left. - B_1^+ B_0^-(\tau)B_{10} + B_1^+ B_0^-(\tau)B_{01} + B_{10}B_0^+ B_1^- - B_{10}B_1^+ B_0^- + B_{10}B_{10} - B_{10}B_{01} - B_{01}B_0^+ B_1^- + B_{01}B_1^+ B_0^- - B_{01}B_{10} + B_{01}B_{01}\right\rangle \tag{608}$$

$$= -\frac{1}{4}\left\langle B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_{10} - B_0^+ B_1^-(\tau)B_{01} \right. \tag{609}$$

$$\left. - B_1^+ B_0^-(\tau)B_0^+ B_1^- + B_1^+ B_0^-(\tau)B_1^+ B_0^- - B_1^+ B_0^-(\tau)B_{10} + B_1^+ B_0^-(\tau)B_{01}\right\rangle \tag{610}$$

$$= -\frac{1}{4}\left\langle B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_{01}B_{10} - B_{01}B_{01} - B_1^+ B_0^-(\tau)B_0^+ B_1^- + B_1^+ B_0^-(\tau)B_1^+ B_0^- - B_{10}B_{10} + B_{10}B_{01}\right\rangle \tag{611}$$

$$= -\frac{1}{4}\left(U\exp(-\phi(\tau))S - \exp(\phi(\tau))S - \exp(\phi(\tau))S + U^*\exp(-\phi(\tau))S + 2S - 2\left(U^*\right)^\Re S\right) \tag{612}$$

$$= -\frac{S}{4}\left(2U^\Re\exp(-\phi(\tau)) - 2\exp(\phi(\tau)) + 2 - 2U^\Re\right) \tag{613}$$

$$= \frac{S}{2}\left(\exp(\phi(\tau)) - U^\Re\exp(-\phi(\tau)) - 1 + U^\Re\right) \tag{614}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle \frac{B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01}}{2}\frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B \tag{615}$$

$$= \frac{1}{4\mathrm{i}}\left\langle \left(B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01}\right)\left(B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}\right)\right\rangle_B \tag{616}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_1^+ B_0^-(\tau)B_0^+ B_1^- - B_1^+ B_0^-(\tau)B_1^+ B_0^- + B_1^+ B_0^-(\tau)B_{10} - B_1^+ B_0^-(\tau)B_{01} + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_1^+ B_0^- \right. \tag{617}$$

$$\left. + B_0^+ B_1^-(\tau)B_{10} - B_0^+ B_1^-(\tau)B_{01} - B_{10}B_0^+ B_1^- + B_{10}B_1^+ B_0^- - B_{10}B_{10} + B_{10}B_{01} - B_{01}B_0^+ B_1^- + B_{01}B_1^+ B_0^- - B_{01}B_{10} + B_{01}B_{01}\right\rangle \tag{618}$$

$$= \tfrac{1}{4i}\big\langle B_1^+ B_0^-(\tau)B_0^+ B_1^- - B_1^+ B_0^-(\tau)B_1^+ B_0^- + B_1^+ B_0^-(\tau)B_{10} - B_1^+ B_0^-(\tau)B_{01} + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_{10}$$

$$- B_0^+ B_1^-(\tau)B_{01}\big\rangle \tag{620}$$

$$= \tfrac{1}{4i}\big\langle B_1^+ B_0^-(\tau)B_0^+ B_1^- - B_1^+ B_0^-(\tau)B_1^+ B_0^- + B_{10}B_{10} - B_{10}B_{01} + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_{01}B_{10} - B_{01}B_{01}\big\rangle \tag{621}$$

$$= \tfrac{1}{4i}\big\langle B_1^+ B_0^-(\tau)B_0^+ B_1^- - B_1^+ B_0^-(\tau)B_1^+ B_0^- + B_{10}B_{10} + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_1^+ B_0^- - B_{01}B_{01}\big\rangle \tag{622}$$

$$= \frac{1}{4i}\left(\exp\left(\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^*S - US\right) \tag{623}$$

$$= \frac{1}{4i}\left(-U^*\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S + U^*S - US\right) \tag{624}$$

$$= \frac{S}{4i}\left(-U^*\exp\left(-\phi\left(\tau\right)\right) + U\exp\left(-\phi\left(\tau\right)\right) + U^* - U\right) \tag{625}$$

$$= \frac{S\left(U - U^*\right)}{4i}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{626}$$

$$= \frac{2iU^{\Im}S}{4i}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{627}$$

$$= \frac{U^{\Im}S}{2}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{628}$$

$$\big\langle \widetilde{B_y}(\tau)\widetilde{B_x}(0)\big\rangle_B = \left\langle \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i}\,\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}\right\rangle_B \tag{629}$$

$$= \frac{1}{4i}\left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}\right)\left(B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}\right)\right\rangle_B \tag{630}$$

$$= \tfrac{1}{4i}\big\langle B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_{10} - B_0^+ B_1^-(\tau)B_{01} - B_1^+ B_0^-(\tau)B_1^+ B_0^- - B_1^+ B_0^-(\tau)B_0^+ B_1^-$$

$$+ B_1^+ B_0^-(\tau)B_{10} + B_1^+ B_0^-(\tau)B_{01} + B_{10}B_1^+ B_0^- + B_{10}B_0^+ B_1^- - B_{10}B_{10} - B_{10}B_{01} - B_{01}B_1^+ B_0^- - B_{01}B_0^+ B_1^- + B_{01}B_{10} + B_{01}B_{01}\big\rangle \tag{632}$$

$$= \tfrac{1}{4i}\big\langle B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_0^+ B_1^-(\tau)B_{10} - B_0^+ B_1^-(\tau)B_{01} - B_1^+ B_0^-(\tau)B_1^+ B_0^- - B_1^+ B_0^-(\tau)B_0^+ B_1^-$$

$$+ B_1^+ B_0^-(\tau)B_{10} + B_1^+ B_0^-(\tau)B_{01}\big\rangle \tag{634}$$

$$= \tfrac{1}{4i}\big\langle B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_{01}B_{10} - B_{01}B_{01} - B_1^+ B_0^-(\tau)B_1^+ B_0^- - B_1^+ B_0^-(\tau)B_0^+ B_1^- + B_{10}B_{10} + B_{10}B_{01}\big\rangle \tag{635}$$

$$= \tfrac{1}{4i}\big\langle B_0^+ B_1^-(\tau)B_1^+ B_0^- + B_0^+ B_1^-(\tau)B_0^+ B_1^- - B_{01}B_{01} - B_1^+ B_0^-(\tau)B_1^+ B_0^- - B_1^+ B_0^-(\tau)B_0^+ B_1^- + B_{10}B_{10}\big\rangle \tag{636}$$

$$= \frac{1}{4i}\left(U\exp\left(-\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + B_{10}^2 - B_{10}^{*2}\right) \tag{637}$$

$$= \frac{1}{4i}\left(U\exp\left(-\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + U^*S - US\right) \tag{638}$$

$$= \frac{S\left(U - U^*\right)}{4i}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{639}$$

$$= \frac{2iU^{\Im}S}{4i}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{640}$$

$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{10} \tag{641}$$

$$\big\langle B_1^+ B_0^-(\tau\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\big\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \prod_{\mathbf{k}} \exp\left(\tfrac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}\left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle \tag{642}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} \tag{643}$$

$$\big\langle B_0^+ B_1^-(\tau\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger\big\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{01} \tag{644}$$

$$\big\langle B_0^+ B_1^-(\tau\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\big\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01} \tag{645}$$

$$\big\langle \widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\big\rangle_B = \tfrac{1}{2}\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{01} \tag{646}$$

$$+ \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\Big) \tag{647}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{01} \tag{648}$$

$$+ \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\Big) \tag{649}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^* B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^* B_{01}\right) \tag{650}$$

$$+ \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)B_{01}\right)\Big) \tag{651}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{01} \right) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) B_{01} \right) \right) \tag{652}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (B_{10} - B_{01}) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) (B_{10} - B_{01}) \right) \tag{653}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} 2i B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* \right) \tag{654}$$

$$= i \sum_{\mathbf{k}'} B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* \right) \tag{655}$$

$$= i \sum_{\mathbf{k}'} B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) e^{i\omega_{\mathbf{k}'}\tau} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* e^{-i\omega_{\mathbf{k}'}\tau} \right) \tag{656}$$

$$\left\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \right\rangle_B \tag{657}$$

$$= \sum_{\mathbf{k}'} \left\langle \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \right\rangle_B \tag{658}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01} \right) \right\rangle_B \tag{659}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(B_1^+ B_0^- + B_0^+ B_1^- \right) \right\rangle_B \tag{660}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- \right. \tag{661}$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- \right\rangle \tag{662}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- \right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- \right\rangle_B \tag{663}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \right\rangle_B \tag{664}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} \left(D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}'}^* v_{0\mathbf{k}'} - v_{1\mathbf{k}'} v_{0\mathbf{k}'}^*}{\omega_{\mathbf{k}'}^2} \right) \right) \right) \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \right\rangle_B \tag{665}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right\rangle_B \tag{666}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right\rangle_B \tag{667}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right\rangle_B e^{i\omega_{\mathbf{k}'}\tau} \tag{668}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle_B \left(- \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* \left\langle D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right\rangle_B N_{\mathbf{k}'} \right) e^{i\omega_{\mathbf{k}'}\tau} \tag{669}$$

$$= - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle \prod_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle_B N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \tag{670}$$

$$= - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{10} N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \tag{671}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- \right\rangle_B = - \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{01} N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \tag{672}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- \right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} B_1^+ B_0^- \right\rangle_B \tag{673}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \right\rangle_B \tag{674}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle b_{\mathbf{k}'} D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_B \tag{675}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'} + 1) \left\langle D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_B \tag{676}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'} + 1) \left\langle D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_B \tag{677}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'} + 1) B_{10} \tag{678}$$

$$\left\langle \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}{}_{B_0+B_1^-}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-i\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{01} \tag{679}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{10}N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{01}N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\right. \tag{680}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-i\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{10}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-i\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{01}\right) \tag{681}$$

$$= \tfrac{1}{2}\textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{01}-B_{10}\right)+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-i\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\left(B_{10}-B_{01}\right)\right) \tag{682}$$

$$= \tfrac{1}{2}\textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{01}-B_{10}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-i\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\left(B_{01}-B_{10}\right)\right) \tag{683}$$

$$= i\textstyle\sum_{\mathbf{k}'}{}^{B_{10}\Im}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-i\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\right) \tag{684}$$

$$\left\langle \widetilde{B_y}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_B = \left\langle \left(\frac{B_0^{+}B_1^{-}\left(\tau\right)-B_1^{+}B_0^{-}\left(\tau\right)+B_{10}-B_{01}}{2i}\right)\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right)\right\rangle_B \tag{685}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left\langle \left(B_0^{+}B_1^{-}\left(\tau\right)-B_1^{+}B_0^{-}\left(\tau\right)+B_{10}-B_{01}\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right)\right\rangle_B \tag{686}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left\langle \left(B_0^{+}B_1^{-}\left(\tau\right)-B_1^{+}B_0^{-}\left(\tau\right)\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right)\right\rangle_B \tag{687}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left\langle B_0^{+}B_1^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}-B_1^{+}B_0^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+B_0^{+}B_1^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}-B_1^{+}B_0^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right\rangle \tag{688}$$

$$\left\langle B_0^{+}B_1^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_B = -\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01} \tag{689}$$

$$\left\langle B_0^{+}B_1^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01} \tag{690}$$

$$\left\langle B_1^{+}B_0^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_B = -\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{691}$$

$$\left\langle B_1^{+}B_0^{-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} \tag{692}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\right. \tag{693}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{694}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\right. \tag{695}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{696}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}+B_{01}\right)+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(-B_{10}-B_{01}\right)\right) \tag{697}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}+B_{01}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(B_{10}+B_{01}\right)\right) \tag{698}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle \textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^{+}B_1^{-}-B_1^{+}B_0^{-}+B_{10}-B_{01}}{2i}\right)\right\rangle_B \tag{699}$$

$$= \tfrac{1}{2i}\left\langle \textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(B_0^{+}B_1^{-}-B_1^{+}B_0^{-}+B_{10}-B_{01}\right)\right\rangle_B \tag{700}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left\langle \left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(B_0^{+}B_1^{-}-B_1^{+}B_0^{-}\right)\right\rangle_B \tag{701}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left\langle \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}\right\rangle \tag{702}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_0^{+}B_1^{-}\right\rangle-e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_1^{+}B_0^{-}\right\rangle+e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_0^{+}B_1^{-}\right\rangle-e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_1^{+}B_0^{-}\right\rangle\right) \tag{703}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_0^{+}B_1^{-}\right\rangle-e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_1^{+}B_0^{-}\right\rangle+e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_0^{+}B_1^{-}\right\rangle-e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_1^{+}B_0^{-}\right\rangle\right) \tag{704}$$

$$= \tfrac{1}{2i}\textstyle\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_0^{+}B_1^{-}\right\rangle-e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_1^{+}B_0^{-}\right\rangle+e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_0^{+}B_1^{-}\right\rangle-e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_1^{+}B_0^{-}\right\rangle\right) \tag{705}$$

$$\left\langle b_{\mathbf{k}'}^{\dagger}B_1^{+}B_0^{-}\right\rangle_B = -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}N_{\mathbf{k}'} \tag{706}$$

$$\left\langle b_{\mathbf{k}'}^{\dagger}B_0^{+}B_1^{-}\right\rangle_B = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{01}N_{\mathbf{k}'} \tag{707}$$

$$\left\langle b_{\mathbf{k}'}B_1^{+}B_0^{-}\right\rangle_B = \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{708}$$

$$\left\langle b_{\mathbf{k}'}B_0^{+}B_1^{-}\right\rangle_B = \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{01} \tag{709}$$

$$\left\langle \widetilde{B_{iz}(\tau)\widetilde{B_y}(0)} \right\rangle_B = \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(-\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} \right) - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(-\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right.$$ (710)

$$\left. + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right)$$ (711)

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right.$$ (712)

$$\left. + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right)$$ (713)

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left((-g_{i\mathbf{k}'} + v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right.$$ (714)

$$\left. + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right)$$ (715)

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10}+) N_{\mathbf{k}'} - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (B_{10}+) (N_{\mathbf{k}'} + 1) \right)$$ (716)

$$= \frac{1}{\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^\Re N_{\mathbf{k}'} - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^\Re (N_{\mathbf{k}'} + 1) \right)$$ (717)

$$= \mathrm{i} \sum_{\mathbf{k}'} \left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^\Re (N_{\mathbf{k}'} + 1) - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^\Re N_{\mathbf{k}'} \right)$$ (718)

$$= \mathrm{i} \sum_{\mathbf{k}'} \left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^\Re (N_{\mathbf{k}'} + 1) - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^\Re N_{\mathbf{k}'} \right)$$ (719)

$$= \mathrm{i} B_{10}^\Re \sum_{\mathbf{k}'} \left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* N_{\mathbf{k}'} \right)$$ (720)

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}(\tau)\, \widetilde{B_{jz}}(0) \right\rangle_B = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})(g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right)$$ (721)

$$U = \prod_{\mathbf{k}} \left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right)$$ (722)

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right)$$ (723)

$$\left\langle \widetilde{B_x}(\tau)\, \widetilde{B_x}(0) \right\rangle_B = \frac{|B_{10}|^2}{2} \left(U^\Re \exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^\Re - 1 \right)$$ (724)

$$\left\langle \widetilde{B_y}(\tau)\, \widetilde{B_y}(0) \right\rangle_B = \frac{|B_{10}|^2}{2} \left(\exp(\phi(\tau)) - U^\Re \exp(-\phi(\tau)) - 1 + U^\Re \right)$$ (725)

$$\left\langle \widetilde{B_x}(\tau)\, \widetilde{B_y}(0) \right\rangle_B = \frac{U^\Im |B_{10}|^2}{2} \left(\exp(-\phi(\tau)) - 1 \right)$$ (726)

$$\left\langle \widetilde{B_y}(\tau)\, \widetilde{B_x}(0) \right\rangle_B = \frac{U^\Im |B_{10}|^2}{2} \left(\exp(-\phi(\tau)) - 1 \right)$$ (727)

$$\left\langle \widetilde{B_{iz}}(\tau)\, \widetilde{B_x}(0) \right\rangle_B = \mathrm{i} B_{10}^\Im \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right)$$ (728)

$$\left\langle \widetilde{B_x}(\tau)\, \widetilde{B_{iz}}(0) \right\rangle_B = \mathrm{i} B_{10}^\Im \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right)$$ (729)

$$\left\langle \widetilde{B_{iz}}(\tau)\, \widetilde{B_y}(0) \right\rangle_B = \mathrm{i} B_{10}^\Re \sum_{\mathbf{k}} \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{\mathrm{i}\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right)$$ (730)

$$\left\langle \widetilde{B_y}(\tau)\, \widetilde{B_{iz}}(0) \right\rangle_B = \mathrm{i} B_{10}^\Re \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right)$$ (731)

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}})$$ (732)

$$v_{i\mathbf{k}} = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}})$$ (733)

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i(\omega)}{\omega}\, \mathrm{d}\omega$$ (734)

The integral version of the correlation functions are given by:

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + (g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}})e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{735}$$

$$= \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + (g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{736}$$

$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))g_{j\mathbf{k}}^*(1-F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^* g_{j\mathbf{k}}(1-F_j(\omega_{\mathbf{k}}))e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{737}$$

$$\approx \int_0^\infty \left(\sqrt{J_i(\omega)J_j^*(\omega)}(1-F_i(\omega))\left(1-F_j^*(\omega)\right)e^{i\omega\tau}N(\omega) + \sqrt{J_i^*(\omega)J_j(\omega)}\left(1-F_i^*(\omega)\right)\left(1-F_j(\omega)\right)e^{-i\omega\tau}(N(\omega)+1)\right)d\omega \tag{738}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{739}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right) \tag{740}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{741}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{742}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^* g_{1\mathbf{k}}F_0^*(\omega_{\mathbf{k}})F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}g_{1\mathbf{k}}^* F_0(\omega_{\mathbf{k}})F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{743}$$

$$\approx \exp\left(\int_0^\infty \frac{\sqrt{J_0^*(\omega)J_1(\omega)}F_0^*(\omega)F_1(\omega) - \sqrt{J_0(\omega)J_1^*(\omega)}F_0(\omega)F_1^*(\omega)}{\omega^2}d\omega\right) \tag{744}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-i\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{745}$$

$$= \sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\left(-i\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{746}$$

$$\approx \int_0^\infty \left|\frac{\sqrt{J_1(\omega)}F_1(\omega) - \sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\left(-i\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)d\omega \tag{747}$$

$$B_{10} = \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{748}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{749}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}) - g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right)\right) \tag{750}$$

$$\approx \exp\left(-\frac{1}{2}\int_0^\infty\left|\frac{\sqrt{J_1(\omega)}F_1(\omega) - \sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)d\omega\right)\exp\left(\int_0^\infty\frac{1}{2}\left(\frac{\sqrt{J_0(\omega)J_1^*(\omega)}F_0(\omega)F_1^*(\omega) - \sqrt{J_0^*(\omega)J_1(\omega)}F_0^*(\omega)F_1(\omega)}{\omega^2}\right)d\omega\right) \tag{751}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(U^{\Re}\exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^{\Re} - 1\right) \tag{752}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(\exp(\phi(\tau)) - U^{\Re}\exp(-\phi(\tau)) - 1 + U^{\Re}\right) \tag{753}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp(-\phi(\tau)) - 1\right) \tag{754}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp(-\phi(\tau)) - 1\right) \tag{755}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B = iB_{10}^{\Im}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* - (g_{i\mathbf{k}}-v_{i\mathbf{k}})^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{756}$$

$$= iB_{10}^{\Im}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^* - (g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{757}$$

$$= iB_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^* - g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{758}$$

$$Q(\omega) = \sqrt{J_i(\omega)}(1-F_i(\omega))\left(\frac{\sqrt{J_1(\omega)}F_1(\omega) - \sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right)^* \tag{759}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B \approx iB_{10}^{\Im}\int_0^\infty\left(Q(\omega)N(\omega)e^{i\omega\tau} - Q^*(\omega)(N(\omega)+1)e^{-i\omega\tau}\right)d\omega \tag{760}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = iB_{10}^{\Im}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})^* N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - (g_{i\mathbf{k}}-v_{i\mathbf{k}})\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{761}$$

$$= iB_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^*\left(1-F_i^*(\omega_{\mathbf{k}})\right)\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau} - g_{i\mathbf{k}}(1-F_i(\omega))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{762}$$

$$\approx iB_{10}^{\Im} \int_0^{\infty} \left(Q^*\left(\omega\right) N\left(\omega\right) e^{i\omega\tau} - Q\left(\omega\right) \left(N\left(\omega\right)+1\right) e^{-i\omega\tau}\right) d\omega \tag{763}$$

$$\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\rangle_B = iB_{10}^{\Re} \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(N_{\mathbf{k}}+1\right) - e^{i\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* N_{\mathbf{k}}\right) \tag{764}$$

$$= iB_{10}^{\Re} \Sigma_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}}^* \left(1 - F_i^*\left(\omega_{\mathbf{k}}\right)\right)\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right) - e^{i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}}\left(1 - F_i\left(\omega_{\mathbf{k}}\right)\right)\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* N_{\mathbf{k}}\right) \tag{765}$$

$$\approx iB_{10}^{\Re} \int_0^{\infty} \left(e^{-i\omega\tau} Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right) - e^{i\omega\tau} Q\left(\omega\right) N\left(\omega\right)\right) d\omega \tag{766}$$

$$\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\rangle_B = iB_{10}^{\Re} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right) e^{-i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{767}$$

$$= iB_{10}^{\Re} \Sigma_{\mathbf{k}} \left(g_{i\mathbf{k}}^* \left(1 - F_i^*\left(\omega_{\mathbf{k}}\right)\right) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - g_{i\mathbf{k}}\left(1 - F_i\left(\omega_{\mathbf{k}}\right)\right)\left(N_{\mathbf{k}}+1\right) e^{-i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{768}$$

$$= iB_{10}^{\Re} \int_0^{\infty} \left(e^{i\omega\tau} Q^*\left(\omega\right) N\left(\omega\right) - e^{-i\omega\tau} Q\left(\omega\right)\left(N\left(\omega\right)+1\right)\right) d\omega \tag{769}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda + \text{Det}\left(\overline{H_{\bar{S}}}\right) = 0 \tag{770}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S |\pm\rangle = \lambda_{\pm} |\pm\rangle$. Using this notation is possible to write $H_{\bar{S}} = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$.

The time-dependence of the system operators $\widetilde{A_i}(t)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_i}\left(\tau\right) = e^{i\overline{H_{\bar{S}}}\tau} A_i e^{-i\overline{H_{\bar{S}}}\tau} \tag{771}$$

$$= \sum_w e^{-iw\tau} A_i\left(w\right) \tag{772}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.

In order to use the equation (772) to descompose the equation (357) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U\left(t\right) \equiv \mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}\left(t'\right)\right)$ like:

$$U\left(t\right) \equiv \mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}\left(t'\right)\right) \tag{773}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} \left(-i\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n H\left(t_1\right) H\left(t_2\right)...H\left(t_n\right) \tag{774}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E\left(t\right)$ such that $\mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}\left(t'\right)\right) \equiv \exp\left(-it H_E\left(t\right)\right)$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E\left(t\right) = H_E^{(0)}\left(t\right) + H_E^{(1)}\left(t\right) + H_E^{(2)}\left(t\right) + ...$ so we can write:

$$U\left(t\right) = \exp\left(-it\left(H_E^{(0)}\left(t\right) + H_E^{(1)}\left(t\right) + H_E^{(2)}\left(t\right) + ...\right)\right) \tag{775}$$

The terms can be found expanding $\mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}\left(t'\right)\right)$ and $U\left(t\right)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_{\bar{S}}}(t')\,dt' \tag{776}$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'') \right] \tag{777}$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'') \right], \overline{H_{\bar{S}}}(t''') \right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'') \right], \overline{H_{\bar{S}}}(t') \right] \right) \tag{778}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}(t) = e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \tag{779}$$

$$= \sum_{w(t)} e^{-iw(t)t} A_i(w(t)) \tag{780}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the eigenvectors are time dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau, t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau, t) = U(t-\tau) U^\dagger(t) A_j(t) U(t) U^\dagger(t-\tau) \tag{781}$$

$$= e^{-i(t-\tau)H_E(t-\tau)} e^{iH_E(t)t} A_j(t) e^{-iH_E(t)t} e^{i(t-\tau)H_E(t-\tau)} \tag{782}$$

$$= e^{-i(t-\tau)H_E(t-\tau)} \sum_{w(t)} e^{-iw(t)t} A_j(w(t)) e^{i(t-\tau)H_E(t-\tau)} \tag{783}$$

$$= \sum_{w(t),w'(t-\tau)} e^{-iw(t)t} e^{iw'(t-\tau)} A_j'(w(t), w'(t-\tau)) \tag{784}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}(t-\tau)$ and $\overline{H_{\bar{S}}}(t)$ respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (772) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $V = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta \in V} \langle \alpha | A_i | \beta \rangle |\alpha\rangle\langle\beta| \tag{785}$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{i\overline{H_{\bar{S}}}\tau} = e^{i(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{786}$$

$$= e^{i\lambda_+|+\rangle\langle+|\tau} e^{i\lambda_-|-\rangle\langle-|\tau} \tag{787}$$

$$= \left(|-\rangle\langle-| + e^{i\lambda_+\tau}|+\rangle\langle+| \right) \left(|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-| \right) \tag{788}$$

$$= e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-| \tag{789}$$

Calculating the transformation (772) directly using the previous relationship we find that:

$$\widetilde{A_i}(\tau) = \left(e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-| \right) \left(\sum_{\alpha,\beta \in V} \langle \alpha | A_i | \beta \rangle |\alpha\rangle\langle\beta| \right) \left(e^{-i\lambda_+\tau}|+\rangle\langle+| + e^{-i\lambda_-\tau}|-\rangle\langle-| \right) \tag{790}$$

$$= \langle +|A_i|+\rangle|+\rangle\langle+| + e^{i\eta\tau}\langle +|A_i|-\rangle|+\rangle\langle-| + e^{-i\eta\tau}\langle -|A_i|+\rangle|-\rangle\langle+| + \langle -|A_i|-\rangle|-\rangle\langle-| \tag{791}$$

Here $\eta = \lambda_+ - \lambda-$. Comparing the RHS of the equations (772) and the explicit expression for $\widetilde{A_i}(\tau)$ and we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$A_i(0) = \langle +|A_i|+\rangle |+\rangle\langle +| + \langle -|A_i|-\rangle |-\rangle\langle -| \tag{792}$$

$$A_i(w) = \langle +|A_i|-\rangle |+\rangle\langle -| \tag{793}$$

$$A_i(-w) = \langle -|A_i|+\rangle |-\rangle\langle +| \tag{794}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A}_i(\tau) = \widetilde{A}_i^{\dagger}(\tau)$ and $\widetilde{B}_i(\tau) = \widetilde{B}_i^{\dagger}(\tau)$ we can use the equation (772) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}[\overline{H}_{\bar{S}}(t), \overline{\rho_S}(t)] - \frac{1}{2}\sum_{ij}\sum_{w,w'}\gamma_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho_S}(t) - \overline{\rho_S}(t)A_j^{\dagger}(w,w')\right] - \mathrm{i}\sum_{ij}\sum_w S_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho_S}(t) + \overline{\rho_S}(t)A_j^{\dagger}(w,w')\right] \tag{795}$$

where $A_j^{\dagger}(w) = A(-w)$ as expected from the equations (793) and (794). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}(w,w',t) = 2K_{ij}^{\Re}(w,w',t)$ and $S_{ij}(w,w',t) = K_{ij}^{\Im}(w,w',t)$, respectively, defined in terms of the response functions

$$K_{ij}(w,w',t) = \int_0^t C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}w\tau}e^{-\mathrm{i}t(w-w')}\mathrm{d}\tau \tag{796}$$

$$= K_{ijww'}(t) \tag{797}$$

If we extend the upper limit of integration to ∞ in the equation (796) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{\mathrm{d}\overline{\rho}_S}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)\right] - \sum_{ijww'}K_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho}_S(t) - \overline{\rho}_S(t)A_{jww'}^{\dagger}\right] \tag{798}$$

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_S}{\mathrm{d}t}e^V = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^V\right)}{\mathrm{d}t} \tag{799}$$

$$= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} \tag{800}$$

$$= -\mathrm{i}e^{-V}\left[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)\right]e^V - \sum_{ijww'}K_{ijww'}(t)\,e^{-V}\left[A_i, A_{jww'}\overline{\rho}_S(t) - \overline{\rho}_S(t)A_{jww'}^{\dagger}\right]e^V \tag{801}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^V = e^{-V}\overline{A\mathbb{I}B}e^V \tag{802}$$

$$= e^{-V}\overline{A}e^V e^{-V}\overline{B}e^V \tag{803}$$

$$= \left(e^{-V}\overline{A}e^V\right)\left(e^{-V}\overline{B}e^V\right) \tag{804}$$

$$= AB \tag{805}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^V = e^{-V}\overline{(AB - BA)}e^V \tag{806}$$

$$= e^{-V}\overline{AB}e^V - e^{-V}\overline{BA}e^V \tag{807}$$

$$= AB - BA \tag{808}$$

$$= [A, B] \tag{809}$$

So we will obtain that

$$\frac{d\rho_S}{dt} =- i e^{-V} \left[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)\right] e^V - \sum_{ijww'} K_{ijww'}(t) e^{-V} \left[A_i, A_{jww'} \overline{\rho}_S(t) - \overline{\rho}_S(t) A_{jww'}^\dagger\right] e^V \tag{810}$$

$$=- i\left[H_{\bar{S}}(t), \rho_S(t)\right] - \sum_{ijww'} K_{ijww'}(t) \left[e^{-V} A_i e^V, e^{-V} A_{jww'} \overline{\rho}_S(t) e^V - e^{-V} \overline{\rho}_S(t) A_{jww'}^\dagger e^V\right] \tag{811}$$

$$=- i\left[H_{\bar{S}}(t), \rho_S(t)\right] - \sum_{ijww'} K_{ijww'}(t) \left[e^{-V} A_i e^V, e^{-V} A_{jww'} e^V e^{-V} \overline{\rho}_S(t) e^V - e^{-V} \overline{\rho}_S(t) e^V e^{-V} A_{jww'}^\dagger e^V\right] \tag{812}$$

$$=- i\left[H_{\bar{S}}(t), \rho_S(t)\right] - \sum_{ijww'} K_{ijww'}(t) \left[e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_S(t) - \rho_S(t) e^{-V} A_{jww'}^\dagger e^V\right] \tag{813}$$

$$=- i\left[H_{\bar{S}}(t), \rho_S(t)\right] - \left(\sum_{ijww'} K_{ijww'}(t) \left(\left[e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_S(t)\right] - \left[e^{-V} A_i e^V, \rho_S(t) e^{-V} A_{jww'}^\dagger e^V\right]\right)\right) \tag{814}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (795) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (795) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1, 2, 3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right) |1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{815}$$

After performing the transformation (25) on the Hamiltonian (815) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R) |1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x \tag{816}$$

$$\overline{H_I} = B_z |1\rangle\langle 1| + \frac{\Omega}{2}(B_x \sigma_x + B_y \sigma_y) \tag{817}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{818}$$

The Hamiltonian (816) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R) |1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right) |1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{819}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{820}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z \tag{821}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I + \sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix

$\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\text{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\text{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4} \tag{822}$$

$$\lambda_\pm = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} \tag{823}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{824}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{825}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2} \tag{826}$$

$$\lambda_\pm = \frac{R \pm \eta}{2} \tag{827}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix} \tag{828}$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.

For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix} \tag{829}$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$. Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon + \eta}{2} \tag{830}$$

$$\lambda_- = \frac{\epsilon - \eta}{2} \tag{831}$$

$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle \tag{832}$$

$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle \tag{833}$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}} \tag{834}$$

$$\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}} \tag{835}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right) \tag{836}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1} \tag{837}$$

So the value of $\tan(\theta)$ is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}+1} \tag{838}$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{839}$$

$$= \frac{\Omega_r}{\epsilon+\eta} \tag{840}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (793)-(794) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{841}$$

$$= 2\sin(\theta)\cos(\theta) \tag{842}$$

$$= \sin(2\theta) \tag{843}$$

$$\langle -|\sigma_x|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \tag{844}$$

$$= -2\sin(\theta)\cos(\theta) \tag{845}$$

$$= -\sin(2\theta) \tag{846}$$

$$\langle -|\sigma_x|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{847}$$

$$= \cos^2(\theta) - \sin^2(\theta) \tag{848}$$

$$= \cos(2\theta) \tag{849}$$

$$\langle +|\sigma_y|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{850}$$

$$= i\sin(\theta)\cos(\theta) - i\sin(\theta)\cos(\theta) \tag{851}$$

$$= 0 \tag{852}$$

$$\langle -|\sigma_y|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \tag{853}$$

$$= i\sin(\theta)\cos(\theta) - i\sin(\theta)\cos(\theta) \tag{854}$$

$$= 0 \tag{855}$$

$$\langle -|\sigma_y|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{856}$$

$$= i\cos^2(\theta) + i\sin^2(\theta) \tag{857}$$

$$= i \tag{858}$$

$$\langle+|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{859}$$

$$= \cos(\theta)\cos(\theta) \tag{860}$$

$$= \cos^2(\theta) \tag{861}$$

$$\langle-|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \tag{862}$$

$$= \sin(\theta)\sin(\theta) \tag{863}$$

$$= \sin^2(\theta) \tag{864}$$

$$\langle-|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{865}$$

$$= -\sin(\theta)\cos(\theta) \tag{866}$$

$$= -\sin(\theta)\cos(\theta) \tag{867}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)(|+\rangle\langle+| - |-\rangle\langle-|) \tag{868}$$

$$A_1(\eta) = \cos(2\theta)|-\rangle\langle+| \tag{869}$$

$$A_2(0) = 0 \tag{870}$$

$$A_2(\eta) = i|-\rangle\langle+| \tag{871}$$

$$A_3(0) = \cos^2(\theta)|+\rangle\langle+| + \sin^2(\theta)|-\rangle\langle-| \tag{872}$$

$$A_3(\eta) = -\sin(\theta)\cos(\theta)|-\rangle\langle+| \tag{873}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (423) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)C_j(t-\tau)\Lambda_{ij}(\tau) \tag{874}$$

Using the notation of the master equation (795), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(721)-(731) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (423) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{875}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{876}$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{877}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)}\right) \tag{878}$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega\, J(\omega)(1 - F(\omega))^2 G_+(\tau) \tag{879}$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty d\omega \frac{J(\omega)}{\omega} F(\omega)(1 - F(\omega))\, iG_-(\tau) \tag{880}$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau) \tag{881}$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 \tag{882}$$

Finally taking the Hamiltonian (815) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\text{Det}(\overline{H_S}) = -\frac{\Omega_r^2}{4}$, $\text{Tr}(\overline{H_S}) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (338) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{883}$$

$$= \frac{g_k\left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)} \tag{884}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (421) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (795) and using the fact that

$$\widetilde{A_j}(t-\tau, t) = U(\tau)A_j U(-\tau) \tag{885}$$

$$= \sum_w e^{iw\tau} A_i(-w) \tag{886}$$

$$= \sum_w e^{-iw\tau} A_i(w) \tag{887}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \frac{1}{2}\sum_{ij}\sum_w \gamma_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)A_j^\dagger(w)\right] \tag{888}$$

$$- \sum_{ij}\sum_w S_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)A_j^\dagger(w)\right] \tag{889}$$

where $A_j^\dagger(w) = A(-w)$, as we can see the equation (889) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2K_{ij}^\Re(w,t)$ and $S_{ij}(w,t) = K_{ij}^\Im(w,t)$, respectively, defined in terms of the response functions

$$K_{ij}(w,t) = \int_0^t \Lambda'_{ij}(\tau)e^{iw\tau}d\tau \tag{890}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_k\left(g_k b_k^\dagger + g_k^* b_k\right)\right)|1\rangle\langle1| + \frac{\Omega(t)}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k \tag{891}$$

If $v_k \approx g_k$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (891) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that

$v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (225) and (244) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)\,|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t) \tag{892}$$

$$\overline{H_I} = \frac{\Omega(t)}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{893}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (875)-(882) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}G_+(\tau)\,d\omega \tag{894}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (894) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)d\omega \tag{895}$$

Writing the interaction Hamiltonian (893) in the similar way to the equation (244) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (225) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$ is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t) \tag{896}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \text{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{897}$$

$$= \frac{B^2}{2}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{898}$$

$$\Lambda_{22}(\tau) = \text{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{899}$$

$$= \frac{B^2}{2}\left(e^{\phi(\tau)} + e^{-\phi(\tau)}\right) \tag{900}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x,y\}$ respectively. The master equation for this section based on the equation (421) is:

$$\frac{d\rho_S(t)}{dt} = -i\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^2\int_0^t d\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{901}$$

$$\left. +C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right) \tag{902}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (897) and (900) on the equation (902) we obtain that:

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t d\tau\,\Omega(t-\tau)\left([\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)]\Lambda_x(\tau)\right. \tag{903}$$

$$\left. + [\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)]\Lambda_y(\tau) + [\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x]\Lambda_x(\tau) + [\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y]\Lambda_y(\tau)\right) \tag{904}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C.   Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (423) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle1| + \frac{\Omega(t)}{2}(|1\rangle\langle0| + |0\rangle\langle1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*} b_{\mathbf{k}} \right) \tag{905}$$

Using the equation (795), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)), \rho_S(t)\right] - \frac{1}{2}\sum_{w}\gamma_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) - \rho_S(t)A_3^{\dagger}(w)\right] \tag{906}$$

$$- \sum_{w} S_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) + \rho_S(t)A_3^{\dagger}(w)\right]\Big) \tag{907}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty \mathrm{d}\omega\, J(\omega)\, G_+(\tau) \tag{908}$$

$$\Lambda_{33}(-\tau) = \int_0^\infty \mathrm{d}\omega\, J(\omega)\, G_+(-\tau) \tag{909}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (907) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)\right] - \sum_{w}\left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^*(w,t)\left[\rho_S(t)A_3(w), A_3\right]\right) \tag{910}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle0| + |0\rangle\langle1|)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2} \tag{911}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \tag{912}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2} \tag{913}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{914}$$

The elements of the decomposition matrices are:

$$\langle+|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{915}$$

$$= \frac{1}{2} \tag{916}$$

$$\langle-|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} \tag{917}$$

$$= \frac{1}{2} \tag{918}$$

$$\langle-|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{919}$$

$$= -\frac{1}{2} \tag{920}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{921}$$

$$= \frac{\mathbb{I}}{2} \tag{922}$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle+| \tag{923}$$

$$= \frac{1}{4}(\sigma_z + i\sigma_y) \tag{924}$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle-| \tag{925}$$

$$= \frac{1}{4}(\sigma_z - i\sigma_y) \tag{926}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\widetilde{\Omega}}{2}[\sigma_x, \rho_S(t)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + i\sigma_y)\rho_S(t)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - i\sigma_y)\rho_S(t)\right] \tag{927}$$

$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z + i\sigma_y), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z - i\sigma_y), \frac{\sigma_z}{2}\right] \tag{928}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty\int_0^\infty J(\omega)G_+(\tau)e^{i\widetilde{\Omega}\tau}d\tau d\omega \tag{929}$$

$$= \int_0^\infty\int_0^\infty J(\omega)e^{i\widetilde{\Omega}\tau}\left((n(\omega)+1)e^{-i\tau\omega} + n(\omega)e^{i\tau\omega}\right)d\tau d\omega \tag{930}$$

$$= \int_0^\infty\int_0^\infty J(\omega)e^{i\widetilde{\Omega}\tau}(n(\omega)+1)e^{-i\tau\omega}d\tau d\omega \tag{931}$$

$$= \int_0^\infty\int_0^\infty J(\omega)(n(\omega)+1)e^{i\widetilde{\Omega}\tau-i\tau\omega}d\tau d\omega \tag{932}$$

$$= \int_0^\infty J(\omega)(n(\omega)+1)\pi\delta\left(\widetilde{\Omega}-\omega\right)d\omega \tag{933}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right) \tag{934}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right) G_+\left(\tau\right) e^{-i\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{935}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-i\widetilde{\Omega}\tau} \left(\left(n\left(\omega\right)+1\right) e^{-i\tau\omega} + n\left(\omega\right) e^{i\tau\omega}\right) \mathrm{d}\tau \mathrm{d}\omega \tag{936}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-i\widetilde{\Omega}\tau} n\left(\omega\right) e^{i\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{937}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) n\left(\omega\right) e^{-i\widetilde{\Omega}\tau+i\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{938}$$

$$= \int_0^\infty J\left(\omega\right) n\left(\omega\right) \pi\delta\left(-\widetilde{\Omega}+\omega\right) \mathrm{d}\omega \tag{939}$$

$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right) \tag{940}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s} = \pi\delta\left(\varepsilon\right) \pm i\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3\left(0\right)$ because the spectral density $J\left(\omega\right)$ is equal to zero when $\omega = 0$. Replacing in the equation (927) lead us to obtain:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t} = -i\frac{\widetilde{\Omega}}{2} \left[\sigma_x, \rho_S\left(t\right)\right] - \frac{\pi}{8} J\left(\widetilde{\Omega}\right) \left(\left(n\left(\widetilde{\Omega}\right)+1\right) \left[\sigma_z, \left(\sigma_z+i\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z-i\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{941}$$

$$- \frac{\pi}{8} J\left(\widetilde{\Omega}\right) \left(\left(n\left(\widetilde{\Omega}\right)+1\right) \left[\rho_S\left(t\right)\left(\sigma_z+i\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S\left(t\right)\left(\sigma_z-i\sigma_y\right), \sigma_z\right]\right) \tag{942}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t} = -i\frac{\Omega\left(t\right)}{2} \left[\sigma_x, \rho_S\left(t\right)\right] - \frac{\pi}{8} J\left(\Omega\left(t\right)\right) \left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\sigma_z, \left(\sigma_z+i\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\Omega\left(t\right)\right)\left[\sigma_z, \left(\sigma_z-i\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{943}$$

$$- \frac{\pi}{8} J\left(\Omega\left(t\right)\right) \left(\left(n\left(\Omega\left(t\right)\right)+1\right) \left[\rho_S\left(t\right)\left(\sigma_z+i\sigma_y\right), \sigma_z\right] + n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z-i\sigma_y\right), \sigma_z\right]\right) \tag{944}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right) = H_S\left(t\right) + H_I + H_B, \tag{945}$$

$$H_S\left(t\right) = \sum_n \varepsilon_n\left(t\right) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right) |n\rangle\langle m|, \tag{946}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right), \tag{947}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \tag{948}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right) \tag{949}$$

At first let's obtain $e^{\pm V}$ under the transformation (949), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)$, so the equation (949) can be written as $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \tag{950}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n| \hat{\varphi}_n \right)^2}{2!} + ... \tag{951}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + ... \tag{952}$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + ... \tag{953}$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + ... \right) \tag{954}$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \tag{955}$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*} b_{u'\mathbf{k}'} \right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}} \right) = e^{\pm \left(\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} \tag{956}$$

$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} \right) \tag{957}$$

$$= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}} \right) \right) \tag{958}$$

$$= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}} \right) \tag{959}$$

$$= \prod_u B_{nu\pm} \tag{960}$$

$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}} \right) \tag{961}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V} A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \tag{962}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (945):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{963}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{964}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{965}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+}B_{0u-} \tag{966}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{967}$$

$$= |0\rangle\langle 0|. \tag{968}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{969}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{970}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{971}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+}B_{nu-}\right), \, m \neq n, \tag{972}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{973}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{974}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{975}$$

$$\prod_u \left(B_{mu+}B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{976}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{977}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + ...\right)\left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + ...\right), \tag{978}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + ..., \tag{979}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + ...\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + ...\right) \prod_u B_{1u-} + ... \tag{980}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + ...\right) \tag{981}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + ...\right) + ... \tag{982}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}^*}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}^*}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + ...\right) \tag{983}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}^*}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}^*}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + ...\right) + ... \tag{984}$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + ... \tag{985}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{986}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{987}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right) \tag{988}$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{989}$$

The transformed Hamiltonians of the equations (946) to (948) written in terms of (963) to (987) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{990}$$

$$= \overline{\sum_n \varepsilon_n(t)\,|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{991}$$

$$= \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|\prod_u (B_{mu+}B_{nu-}) \tag{992}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) \tag{993}$$

$$= \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}\right) + ...\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) \tag{994}$$

$$= \Pi_u\,B_{0u+}\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}\right)\Pi_u\,B_{0u-} + \Pi_u\,B_{1u+}\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{1u\mathbf{k}}^* b_{u\mathbf{k}}\right)\Pi_u\,B_{1u-} + ... \tag{995}$$

$$= \sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\Pi_u\,B_{0u+}b_{u\mathbf{k}}^\dagger\Pi_u\,B_{0u-} + g_{0u\mathbf{k}}^*\Pi_u\,B_{0u+}b_{u\mathbf{k}}\Pi_u\,B_{0u-}\right) + \sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\Pi_u\,B_{1u+}b_{u\mathbf{k}}^\dagger\Pi_u\,B_{1u-} + g_{1u\mathbf{k}}^*\Pi_u\,B_{1u+}b_{u\mathbf{k}}\Pi_u\,B_{1u-}\right) + ... \tag{996}$$

$$= \sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{0u\mathbf{k}}^*\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{1u\mathbf{k}}^*\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + ... \tag{997}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{nu\mathbf{k}}^*\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{998}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{999}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{1000}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|\prod_u(B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{1001}$$

$$+ \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1002}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1003}$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}}\left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}})\,b_{u\mathbf{k}}^\dagger + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^*\,b_{u\mathbf{k}}\right) \tag{1004}$$

Using the previous functions we have that (1001) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|\prod_u(B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n R_n(t)|n\rangle\langle n| + \sum_n B_{z,n}(t)|n\rangle\langle n| \tag{1005}$$

$$\tag{1006}$$

Now in order to separate the elements of the hamiltonian (1006) let's follow the references of the equations (225) and (244) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \prod_u (B_{mu+}B_{nu-}) \right\rangle_{\overline{H_0}} = \left\langle \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}}) \exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}) \right) \right) \right\rangle_{\overline{H_0}} \tag{1007}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}) \right) \right) \left\langle \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}}) \right\rangle_{\overline{H_0}} \tag{1008}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left(-\tfrac{1}{2}\sum_\mathbf{k} \frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2} \right) \right) \tag{1009}$$

$$\equiv B_{nm} \tag{1010}$$

$$\left\langle \prod_u (B_{nu+}B_{mu-}) \right\rangle_{\overline{H_0}} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left(-\tfrac{1}{2}\sum_\mathbf{k} \frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2} \right) \right) \tag{1011}$$

$$= B^*_{nm} \tag{1012}$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+}B_{nu-}) - B_{nm} \tag{1013}$$

As we can see:

$$J^\dagger_{nm} = \left(\prod_u (B_{mu+}B_{nu-}) - B_{nm} \right)^\dagger \tag{1014}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B^*_{nm} \tag{1015}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{mn} \tag{1016}$$

$$= J_{mn} \tag{1017}$$

We can separate the Hamiltonian (1006) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)} = \sum_n (\varepsilon_n(t)+R_n)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|B_{nm} \tag{1018}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|J_{nm} + \sum_n B_{z,n}(t)|n\rangle\langle n|, \tag{1019}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \tag{1020}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_\mathrm{B} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_{\bar{S}}(t)}+H_{\bar{B}})} \right) \right) + \left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{S}}(t)}+H_{\bar{B}}} + O\left(\left\langle \overline{H^2_{\bar{I}}} \right\rangle_{\overline{H_{\bar{S}}(t)}+H_B} \right). \tag{1021}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}} \right) = C\left(R_0, R_1, ..., R_{d-1}, B_{01}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)} \right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v^\Re_{nu\mathbf{k}} + iv^\Im_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^\Re_{nu\mathbf{k}}$ and $v^\Im_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \right) |n\rangle\langle n| \tag{1022}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1023}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}} v_{nu\mathbf{k}}^* + g_{nu\mathbf{k}}^* v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1024}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1025}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} + i v_{nu\mathbf{k}}^{\Im} (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1026}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1027}$$

$$v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^* = \left(v_{mu\mathbf{k}}^{\Re} - i v_{mu\mathbf{k}}^{\Im} \right) \left(v_{nu\mathbf{k}}^{\Re} + i v_{nu\mathbf{k}}^{\Im} \right) - \left(v_{mu\mathbf{k}}^{\Re} + i v_{mu\mathbf{k}}^{\Im} \right) \left(v_{nu\mathbf{k}}^{\Re} - i v_{nu\mathbf{k}}^{\Im} \right) \tag{1028}$$

$$= \left(v_{mu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} + i v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - i v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} + v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im} \right) \tag{1029}$$

$$- \left(v_{mu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - i v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} + i v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} + v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im} \right) \tag{1030}$$

$$= 2i \left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} \right) \tag{1031}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} + i v_{nu\mathbf{k}}^{\Im} (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1032}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{i \left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} \right)}{\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1033}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1034}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}} v_{mu\mathbf{k}}^* + v_{nu\mathbf{k}}^* v_{mu\mathbf{k}}) \tag{1035}$$

$$= \left(v_{mu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im}\right)^2 + \left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - \left(v_{nu\mathbf{k}}^{\Re} + i v_{nu\mathbf{k}}^{\Im} \right) \left(v_{mu\mathbf{k}}^{\Re} - i v_{mu\mathbf{k}}^{\Im} \right) \tag{1036}$$

$$- \left(v_{nu\mathbf{k}}^{\Re} - i v_{nu\mathbf{k}}^{\Im} \right) \left(v_{mu\mathbf{k}}^{\Re} + i v_{mu\mathbf{k}}^{\Im} \right) \tag{1037}$$

$$= \left(v_{mu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im}\right)^2 + \left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2 \left(v_{nu\mathbf{k}}^{\Re} v_{mu\mathbf{k}}^{\Re} + v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Im} \right) \tag{1038}$$

$$= \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re} \right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im} \right)^2 \tag{1039}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{1040}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - (g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} - i v_{nu\mathbf{k}}^{\Im} (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \tag{1041}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2 g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2 g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1042}$$

$$B_{mn} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1043}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{i \left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} \right)}{\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re} \right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im} \right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1044}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Re}} \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1045}$$

$$= \frac{2v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1046}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Re} - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1047}$$

$$\frac{\partial R_{n'}}{\partial v_{nu\mathbf{k}}^{\Im}} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Im}} \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1048}$$

$$= \frac{2v_{nu\mathbf{k}}^{\Im} - 2g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1049}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Im} - g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1050}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\Pi_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \Pi_u \exp\left(-\frac{1}{2} \Sigma_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \right) \tag{1051}$$

$$= \Sigma_{u\mathbf{k}} \ln \exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) + \Sigma_u \ln \exp\left(-\frac{1}{2} \Sigma_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \tag{1052}$$

$$= \Sigma_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) + \Sigma_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \tag{1053}$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} - \left(v_{nu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1054}$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} = \frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} - \left(v_{nu\mathbf{k}}^{\Im} - v_{mu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1055}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \tag{1056}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \tag{1057}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^\dagger}{\partial a} \tag{1058}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} \tag{1059}$$

$$= B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} - \left(v_{nu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1060}$$

$$= B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1061}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Re}} = \left(\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} \right)^{\dagger} \tag{1062}$$

$$= \left(B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \tag{1063}$$

$$= B_{nm} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1064}$$

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} \tag{1065}$$

$$= B_{mn} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} - \left(v_{nu\mathbf{k}}^{\Im} - v_{mu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1066}$$

$$= B_{mn} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1067}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Im}} = \left(\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} \right)^{\dagger} \tag{1068}$$

$$= \left(B_{mn} \right)^{\dagger} \tag{1069}$$

$$= B_{nm} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1070}$$

Introducing this derivates in the equation (1045) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n} \left(2 \frac{v_{nu\mathbf{k}}^{\Re} - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n<m} \left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}} B_{nm} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right. \tag{1071}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}} B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \tag{1072}$$

$$= 0 \tag{1073}$$

We can obtain the variational parameters:

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1074}$$

$$=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1075}$$

$$v_{nu\mathbf{k}}^{\Re}=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1076}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1077}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{nu\mathbf{k}}^{\Im}}=\frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v_{nu\mathbf{k}}^{\Im}-g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}\right)+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}-\left(v_{nu\mathbf{k}}^{\Im}-v_{mu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1078}$$

$$\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}-\left(v_{nu\mathbf{k}}^{\Im}-v_{mu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1079}$$

$$=0 \tag{1080}$$

Rearranging we obtain

$$-2\frac{\partial A_{\rm B}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1081}$$

$$=-2\frac{\partial A_{\rm B}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(\frac{-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(\frac{iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1082}$$

$$v_{nu\mathbf{k}}^{\Im}=\frac{2\frac{\partial A_{\rm B}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(\frac{-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(\frac{iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\rm B}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1083}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1084}$$

$$v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+iv_{nu\mathbf{k}}^{\Im} \tag{1085}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1086}$$

$$i\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1087}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}+2ig_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1088}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1089}$$

$$-i\frac{\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1090}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1091}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1092}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1093}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^V\rho(0)e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle0|\otimes\rho_B\right)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1094}$$

$$0=\left(B_0^+|0\rangle\langle0|B_0^-\right)\otimes\rho_B \tag{1095}$$

$$0=\rho(0) \tag{1096}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t) O U(t) \tag{1097}$$

$$U(t) \equiv \mathcal{T} \exp\left(-\mathrm{i} \int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1098}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \tag{1099}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1100}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1101}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1102}$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1103}$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1104}$$

We can proof that $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+}B_{n-} - B_m B_n)^\dagger \tag{1105}$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \tag{1106}$$

$$= B_{n+}B_{m-} - B_n B_m \tag{1107}$$

$$= B_{nm} \tag{1108}$$

So we can say that the set of matrices (1101) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1101) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \tag{1109}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}\right) \tag{1110}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(\Re(V_{nm}(t)) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^\Im(t) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1111}$$

$$\left. +\Re(V_{nm}(t)) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^\Im(t) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1112}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(\Re(V_{nm}(t)) \sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re(V_{nm}(t)) \sigma_{nm,y}\frac{\mathrm{i}(B_{mn} - B_{nm})}{2}\right. \tag{1113}$$

$$\left. +\mathrm{i}\Im(V_{nm}(t)) \sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im(V_{nm}(t)) \sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1114}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(\Re(V_{nm}(t)) \sigma_{nm,x}B_{nm,x} - \Im(V_{nm}(t)) \sigma_{nm,x}B_{nm,y} + \Re(V_{nm}(t)) \sigma_{nm,y}B_{nm,y}\right. \tag{1115}$$

$$\left. +\Im(V_{nm}(t)) \sigma_{nm,y}B_{nm,x}\right) \tag{1116}$$

Let's define the set

$$P = \left\{(n, m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\right\} \tag{1117}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1-\delta_{mn}) \tag{1118}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1-\delta_{mn}) \tag{1119}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1120}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1121}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1122}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1123}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1124}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1125}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1126}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1127}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1128}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1129}$$
$$C_{3,nm}(t) = 1 \tag{1130}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1131}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1132}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J, p\in P} C_{jp}(t)\left(A_{jp}\otimes B_{jp}(t)\right) \tag{1133}$$

Here $J = \{1,2,3,4,5\}$ and P the set defined in (1117).

We write the interaction Hamiltonian transformed under (1097) as:

$$\widetilde{H_I}(t) = \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1134}$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t)A_{jp}U(t) \tag{1135}$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t}B_{jp}(t)(t)e^{-iH_B t} \tag{1136}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\rho_B\right]\right]ds \tag{1137}$$

Replacing the equation (1134) in (1137) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] \mathrm{d}s \tag{1138}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \left[\sum_{j'\in J, p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right), \widetilde{\rho_S}(t)\rho_B\right]\right] \mathrm{d}s \tag{1139}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \sum_{j'\in J, p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \right. \tag{1140}$$

$$\left. -\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right] \mathrm{d}s \tag{1141}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J, p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \right. \tag{1142}$$

$$-\sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1143}$$

$$-\sum_{j'\in J, p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1144}$$

$$\left. +\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right) \mathrm{d}s \tag{1145}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1146}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1147}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1145) are:

$$\text{Tr}_B \left(\widetilde{B_{jp}} (t) \widetilde{B_{j'p'}} (s) \rho_B \right) = \left\langle \widetilde{B_{jp}} (t) \widetilde{B_{j'p'}} (s) \right\rangle_B \tag{1148}$$

$$= \left\langle \widetilde{B_{jp}} (0) \widetilde{B_{j'p'}} (0) \right\rangle_B \tag{1149}$$

$$= \Lambda_{jpj'p'} (\tau) \tag{1150}$$

$$\text{Tr}_B \left(\widetilde{B_{jp}} (t) \rho_B \widetilde{B_{j'p'}} (s) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}} (s) \widetilde{B_{jp}} (t) \rho_B \right) \tag{1151}$$

$$= \left\langle \widetilde{B_{j'p'}} (s) \widetilde{B_{jp}} (t) \right\rangle_B \tag{1152}$$

$$= \left\langle \widetilde{B_{j'p'}} (-\tau) \widetilde{B_{jp}} (0) \right\rangle_B \tag{1153}$$

$$= \Lambda_{j'p'jp} (-\tau) \tag{1154}$$

$$\text{Tr}_B \left(\widetilde{B_{j'p'}} (s) \rho_B \widetilde{B_{jp}} (t) \right) = \text{Tr}_B \left(\widetilde{B_{jp}} (t) \widetilde{B_{j'p'}} (s) \rho_B \right) \tag{1155}$$

$$= \left\langle \widetilde{B_{jp}} (t) \widetilde{B_{j'p'}} (s) \right\rangle_B \tag{1156}$$

$$= \left\langle \widetilde{B_{jp}} (\tau) \widetilde{B_{j'p'}} (0) \right\rangle_B \tag{1157}$$

$$= \Lambda_{jpj'p'} (\tau) \tag{1158}$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'p'}} (s) \widetilde{B_{jp}} (t) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}} (s) \widetilde{B_{jp}} (t) \rho_B \right) \tag{1159}$$

$$= \left\langle \widetilde{B_{j'p'}} (s) \widetilde{B_{jp}} (t) \right\rangle_B \tag{1160}$$

$$= \left\langle \widetilde{B_{j'p'}} (-\tau) \widetilde{B_{jp}} (0) \right\rangle_B \tag{1161}$$

$$= \Lambda_{j'p'jp} (-\tau) \tag{1162}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1138)and (1145) and using the equations (1148)-(1162) we can re-write:

$$\frac{d\widetilde{\overline{\rho_S}} (t)}{dt} = - \int_0^t \sum_{j,j',p,p'} \left(C_{jp} (t) C_{j'p'} (s) \left(\Lambda_{jpj'p'} (\tau) \widetilde{A_{jp}} (t) \widetilde{A_{j'p'}} (s) \widetilde{\overline{\rho_S}} (t) - \Lambda_{j'p'jp} (-\tau) \widetilde{A_{jp}} (t) \widetilde{\overline{\rho_S}} (t) \widetilde{A_{j'p'}} (s) \right) \right. \tag{1163}$$

$$\left. + C_{jp} (t) C_{j'p'} (s) \left(\Lambda_{j'p'jp} (-\tau) \widetilde{\overline{\rho_S}} (t) \widetilde{A_{j'p'}} (s) \widetilde{A_{jp}} (t) - \Lambda_{jpj'p'} (\tau) \widetilde{A_{j'p'}} (s) \widetilde{\overline{\rho_S}} (t) \widetilde{A_{jp}} (t) \right) \right) ds \tag{1164}$$

$$= - \int_0^t \sum_{jj'pp'} \left(C_{jp} (t) C_{j'p'} (s) \left(\Lambda_{jpj'p'} (\tau) \left[\widetilde{A_{jp}} (t), \widetilde{A_{j'p'}} (s) \widetilde{\overline{\rho_S}} (t) \right] + \Lambda_{j'p'jp} (-\tau) \left[\widetilde{\overline{\rho_S}} (t) \widetilde{A_{j'p'}} (s), \widetilde{A_{jp}} (t) \right] \right) \right) \tag{1165}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{d\overline{\rho_S} (t)}{dt} = - \int_0^t \sum_{jj'pp'} \left(C_{jp} (t) C_{j'p'} (t-\tau) \left(\Lambda_{jpj'p'} (\tau) \left[A_{jp} (t), A_{j'p'} (t-\tau, t) \overline{\rho_S} (t) \right] + \Lambda_{j'p'jp} (-\tau) \left[\overline{\rho_S} (t) A_{j'p'} (t-\tau, t), A_{jp} (t) \right] \right) \right) d\tau - i \left[H_S (t), \overline{\rho_S} (t) \right] \tag{1166}$$

For this case we used that $A_{jp} (t-\tau, t) = U (t) U^\dagger (t-\tau) A_{jp} (t) U (t-\tau) U^\dagger (t)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1167}$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1168}$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right), \tag{1169}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{1170}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right) \right) \tag{1171}$$

At first let's obtain e^V under the transformation (1171), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \tag{1172}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n| \hat{\varphi} \right)^2}{2!} + \dots \tag{1173}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \tag{1174}$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \tag{1175}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1176}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \tag{1177}$$

Given that $\left[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} \tag{1178}$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{1179}$$

$$= B_\pm \tag{1180}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-$ because this form imposes that $e^{-V} e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \tag{1181}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1167):

$$\overline{|0\rangle\langle0|} = \left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^+\right)|0\rangle\langle0|\left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^-\right), \tag{1182}$$

$$= |0\rangle\langle0|, \tag{1183}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^+\right)|m\rangle\langle n|\left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^-\right), \tag{1184}$$

$$= |m\rangle\langle m|B^+|m\rangle\langle n|n\rangle\langle n|B^-, \tag{1185}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1186}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^+\right)|0\rangle\langle m|\left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^-\right), \tag{1187}$$

$$= |0\rangle\langle m|B^-\ m \neq 0, \tag{1188}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^+\right)|m\rangle\langle 0|\left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^-\right) \tag{1189}$$

$$= |0\rangle\langle m|B^+\ m \neq 0, \tag{1190}$$

$$\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^+\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle0| + \sum_{n=1}|n\rangle\langle n|B^-\right) \tag{1191}$$

$$= |0\rangle\langle0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B^+b_{\mathbf{k}}^\dagger b_{\mathbf{k}}B^- \tag{1192}$$

$$= |0\rangle\langle0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B^+b_{\mathbf{k}}^\dagger B^-\right)\left(B^+b_{\mathbf{k}}B^-\right) \tag{1193}$$

$$= |0\rangle\langle0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1194}$$

$$= |0\rangle\langle0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1195}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right) \tag{1196}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1197}$$

The transformed Hamiltonians of the equations (1168) to (1170) written in terms of (1182) to (1197) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1198}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1199}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \overline{\sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|)} + \overline{\sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n|} \tag{1200}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|} \right) + \sum_{m.n\neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \tag{1201}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) B^- |0\rangle\langle n| + V_{n0}(t) B^+ |n\rangle\langle 0| \right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1202}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+ \right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1203}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \left(\left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \tag{1204}$$

$$= \left(\mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B^+ \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \tag{1205}$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B^+ \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) B^- \tag{1206}$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{1207}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1208}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+ \right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1209}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1210}$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2 g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1211}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+ \right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1212}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1213}$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1214}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{1215}$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \right) \tag{1216}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1217}$$

Using the previous functions we have that (1214) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n\left(t\right) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}\left(t\right) |0\rangle\langle n|B^- + V_{n0}\left(t\right) |n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0} V_{mn}\left(t\right) |m\rangle\langle n| \tag{1218}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0\left(t\right) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1219}$$

Now in order to separate the elements of the hamiltonian (1219) let's follow the references of the equations (244) and (225) to separate the hamiltonian like:

$$\overline{H_S\left(t\right)} = \sum_{n=0} \varepsilon_n\left(t\right) |n\rangle\langle n| + B\sum_{n=1} \left(V_{0n}\left(t\right) |0\rangle\langle n| + V_{n0}\left(t\right) |n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}\left(t\right) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \tag{1220}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0\left(t\right) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \left(V_{0n}\left(t\right) |0\rangle\langle n| \left(B^- - B\right) + V_{n0}\left(t\right) |n\rangle\langle 0| \left(B^+ - B\right)\right), \tag{1221}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{1222}$$

Here B is given by:

$$B = \langle B^+\rangle$$
$$= \langle B^-\rangle$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1223}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1224}$$
$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1225}$$
$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1226}$$

Using this set of hermitian operators to write the Hamiltonians (1168)-(1170)

$$\overline{H_S(t)} = \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1227}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)\,|m\rangle\langle n| + V_{nm}(t)\,|n\rangle\langle m|) \tag{1228}$$

$$+ \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{1229}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))\,|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))\,|n\rangle\langle m|) + \varepsilon_0(t)\,|0\rangle\langle 0| \tag{1230}$$

$$+ B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{1231}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1232}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{1233}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\,\sigma_{nm,x} + \Im(V_{nm}(t))\,\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\,\sigma_{0n,x} + \Im(V_{mn}(t))\,\sigma_{0n,y}) \tag{1234}$$

$$+ \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{1235}$$

$$\overline{H_I(t)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n|\,(B^- - B) + V_{n0}(t)\,|n\rangle\langle 0|\,(B^+ - B)\right) \tag{1236}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\,(B^- - B)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\,(B^+ - B)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1237}$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1238}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left((B^- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + (B^+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\right)\right. \tag{1239}$$

$$+ \frac{\mathrm{i}\sigma_{0n,y}}{2}\left((B^+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - (B^- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\right)\Bigg) \tag{1240}$$

$$+ \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1241}$$

$$= \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}(B^+ + B^- - 2B)\Re(V_{0n}(t)) + \mathrm{i}(B^- - B - B^+ + B)\Im(V_{0n}(t))\right. \tag{1242}$$

$$+ \frac{\mathrm{i}\sigma_{0n,y}}{2}\left((B^+ - B - B^- + B)\Re(V_{0n}(t)) + \mathrm{i}(B - B^- + B - B^+)\Im(V_{0n}(t))\right)\Bigg) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1243}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t)))) \tag{1244}$$

$$+ \sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1245}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right) + \langle\overline{H_I}\rangle_{H_S+H_B} + O\left(\langle\overline{H_I^2}\rangle_{H_S+H_B}\right). \tag{1246}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (254) and using the chain rule we obtain that:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1247}$$

$$= 0 \tag{1248}$$

Let's recall the equations (1215) and (1217), we can write them in terms of the variational parameters

$$B = \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1249}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1250}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1251}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1252}$$

Introducing this derivates in the equation (1247) give us:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1253}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1254}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}} \tag{1255}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}} \tag{1256}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}}. \tag{1257}$$

C.   Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle0| + \sum_{n=1} |n\rangle\langle n|B^+\right)(|0\rangle\langle0| \otimes \rho_B)\left(|0\rangle\langle0| + \sum_{n=1} |n\rangle\langle n|B^-\right) \tag{1258}$$

$$0 = |0\rangle\langle0| \otimes \rho_B \tag{1259}$$

$$0 = \rho(0) \tag{1260}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t) O U(t) \tag{1261}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1262}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\,\overline{\rho_S}(t)\,U(t)\,,\text{ where} \tag{1263}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar\rho(t)) \tag{1264}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle0| + \sum_{n=1}\left(\Re(V_{0n}(t))B_x\sigma_{0n,x} + \Re(V_{0n}(t))B_y\sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right) \tag{1265}$$

$$+\Im(V_{0n}(t))B_x\sigma_{0n,y} - \Im(V_{0n}(t))B_y\sigma_{0n,x}) \tag{1266}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}}\mu_0(t)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1267}$$

$$B_{z,n} = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\text{ if}n \neq 0 \tag{1268}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1269}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1270}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1271}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1272}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1273}$$
$$B_{1n}(t) = B_x \tag{1274}$$
$$B_{2n}(t) = B_y \tag{1275}$$
$$B_{3n}(t) = B_{z,n} \tag{1276}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1277}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1278}$$
$$C_{10}(t) = 0 \tag{1279}$$
$$C_{20}(t) = 0 \tag{1280}$$
$$C_{40}(t) = 0 \tag{1281}$$
$$C_{50}(t) = 0 \tag{1282}$$
$$C_{30}(t) = 1 \tag{1283}$$
$$C_{1n}(t) = \Re(V_{0n}(t)) \tag{1284}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1285}$$
$$C_{3n}(t) = 1 \tag{1286}$$
$$C_{4n}(t) = \Im(V_{0n}(t)) \tag{1287}$$
$$C_{5n}(t) = -\Im(V_{0n}(t)) \tag{1288}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J} \sum_{n=1} C_{jn}(t)(A_{jn} \otimes B_{jn}(t)) \tag{1289}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1261) as:

$$\widetilde{H_I}(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1290}$$
$$\widetilde{A_i}(t) = U^\dagger(t) A_i U(t) \tag{1291}$$
$$\widetilde{B_i}(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \tag{1292}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] ds \tag{1293}$$

Replacing the equation (1290)in (1293)we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1294}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\left[\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1295}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \right. \tag{1296}$$

$$\left. -\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1297}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \right. \tag{1298}$$

$$-\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1299}$$

$$-\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1300}$$

$$\left. +\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1301}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1302}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1303}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1301) are:

$$\text{Tr}_B \left(\widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \, \rho_B \right) = \left\langle \widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \right\rangle_B \tag{1304}$$

$$= \left\langle \widetilde{B_{jn}} (0) \, \widetilde{B_{j'n'}} (0) \right\rangle_B \tag{1305}$$

$$= \Lambda_{jnj'n'} (\tau) \tag{1306}$$

$$\text{Tr}_B \left(\widetilde{B_{jn}} (t) \, \rho_B \, \widetilde{B_{j'n'}} (s) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \, \rho_B \right) \tag{1307}$$

$$= \left\langle \widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \right\rangle_B \tag{1308}$$

$$= \left\langle \widetilde{B_{j'n'}} (-\tau) \, \widetilde{B_{jn}} (0) \right\rangle_B \tag{1309}$$

$$= \Lambda_{j'n'jn} (-\tau) \tag{1310}$$

$$\text{Tr}_B \left(\widetilde{B_{j'n'}} (s) \, \rho_B \, \widetilde{B_{jn}} (t) \right) = \text{Tr}_B \left(\widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \, \rho_B \right) \tag{1311}$$

$$= \left\langle \widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \right\rangle_B \tag{1312}$$

$$= \left\langle \widetilde{B_{jn}} (\tau) \, \widetilde{B_{j'n'}} (0) \right\rangle_B \tag{1313}$$

$$= \Lambda_{jnj'n'} (\tau) \tag{1314}$$

$$\text{Tr}_B \left(\rho_B \, \widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \, \rho_B \right) \tag{1315}$$

$$= \left\langle \widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \right\rangle_B \tag{1316}$$

$$= \left\langle \widetilde{B_{j'n'}} (-\tau) \, \widetilde{B_{jn}} (0) \right\rangle_B \tag{1317}$$

$$= \Lambda_{j'n'jn} (-\tau) \tag{1318}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1294) and (1301) and using the equations (1304)-(1318) we can re-write:

$$\frac{\mathrm{d}\widetilde{\rho_S} (t)}{\mathrm{d}t} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn} (t) \, C_{j'n'} (s) \left(\Lambda_{jnj'n'} (\tau) \, \widetilde{A_{jn}} (t) \, \widetilde{A_{j'n'}} (s) \, \widetilde{\rho_S} (t) - \Lambda_{j'n'jn} (-\tau) \, \widetilde{A_{jn}} (t) \, \widetilde{\rho_S} (t) \, \widetilde{A_{j'n'}} (s) \right) \right.$$
$$\tag{1319}$$

$$\left. + C_{jn} (t) \, C_{j'n'} (s) \left(\Lambda_{j'n'jn} (-\tau) \, \widetilde{\rho_S} (t) \, \widetilde{A_{j'n'}} (s) \, \widetilde{A_{jn}} (t) - \Lambda_{jnj'n'} (\tau) \, \widetilde{A_{j'n'}} (s) \, \widetilde{\rho_S} (t) \, \widetilde{A_{jn}} (t) \right) \right) \mathrm{d}s \tag{1320}$$

$$= - \int_0^t \sum_{j,j',n,n'} \left(C_{jn} (t) \, C_{j'n'} (s) \left(\Lambda_{jnj'n'} (\tau) \left[\widetilde{A_{jn}} (t) , \widetilde{A_{j'n'}} (s) \, \widetilde{\rho_S} (t) \right] + \Lambda_{j'n'jn} (-\tau) \left[\widetilde{\rho_S} (t) \, \widetilde{A_{j'n'}} (s) , \widetilde{A_{jn}} (t) \right] \right) \right)$$
$$\tag{1321}$$

$$\frac{\mathrm{d}\overline{\rho_S} (t)}{\mathrm{d}t} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn} (t) \, C_{j'n'} (t-\tau) \left(\Lambda_{jnj'n'} (\tau) \left[A_{jn} (t) , A_{j'n'} (t-\tau,t) \overline{\rho_S} (t) \right] + \Lambda_{j'n'jn} (-\tau) \left[\overline{\rho_S} (t) A_{j'n'} (t-\tau,t) , A_{jn} (t) \right] \right) \right) \mathrm{d}\tau - \mathrm{i} \left[H_S (t) , \overline{\rho_S} (t) \right] \tag{1322}$$

For this case we used that $A_{jn} (t - \tau, t) = U (t) U^\dagger (t - \tau) A_{jn} (t) U (t - \tau) U^\dagger (t)$. This is a non-Markovian equation and if we take $n = 2$ (two sites), $\mu_0 (t) = 0$, $\mu_1 (t) = 1$ then we can reproduce a similar expression to (421) as expected.

VIII.   BIBLIOGRAPHY

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