A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

Nike Dattani*
Harvard-Smithsonian Center for Astrophysics

Camilo Chaparro Sogamoso†
National University of Colombia

I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \varepsilon_0(t)\,|0\rangle\langle0| + \varepsilon_1(t)\,|1\rangle\langle1| + V_{10}(t)\,|1\rangle\langle0| + V_{01}(t)\,|0\rangle\langle1|, \tag{2}$$

$$H_I = |0\rangle\langle0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij} \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}} \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V = |0\rangle\langle0|\hat{\varphi}_0 + |1\rangle\langle1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6)will give us the following result:

$$e^{\pm V} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1)} \tag{10}$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + \ldots \tag{11}$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + \ldots \tag{12}$$

$$= |0\rangle\langle 0|\left(1 \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm \ldots\right) + |1\rangle\langle 1|\left(1 \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm \ldots\right) \tag{13}$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} \tag{14}$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*b_{\mathbf{k}}\right)} \tag{15}$$

$$= |0\rangle\langle 0|B_{0\pm} + |1\rangle\langle 1|B_{1\pm}, \tag{16}$$

$$B_{i\pm} \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{17}$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX}\, e^{tY}\, e^{-\frac{t^2}{2}[X,Y]}\, e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \ldots \tag{18}$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i,j we can show making $t = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} = e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]} \ldots \tag{19}$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}0} \ldots \tag{20}$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} \tag{21}$$

By induction of this result we can write expresion of $B_{i\pm}$ as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}})$:

$$B_{i\pm} = \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{22}$$

$$D(\pm v_{i\mathbf{k}}) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{23}$$

this will help us to write operators O in the variational frame :

$$\overline{O} \equiv e^V O e^{-V}. \tag{24}$$

We use the following identities:

$$\overline{|0\rangle\langle0|} = e^V |0\rangle\langle0| e^{-V} \tag{25}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|0\rangle\langle0|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{26}$$

$$= \left(|0\rangle\langle0|0\rangle\langle0|B_{0+} + |1\rangle\langle1|0\rangle\langle0|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{27}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{28}$$

$$= |0\rangle\langle0|B_{0+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{29}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}B_{0-} + |0\rangle\langle0|1\rangle\langle1|B_{0+}B_{1-} \tag{30}$$

$$= |0\rangle\langle0|, \tag{31}$$

$$\overline{|1\rangle\langle1|} = \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|1\rangle\langle1|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{32}$$

$$= \left(|0\rangle\langle0|1\rangle\langle1|B_{0+} + |1\rangle\langle1|1\rangle\langle1|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{33}$$

$$= |1\rangle\langle1|B_{1+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{34}$$

$$= |1\rangle\langle1|0\rangle\langle0|B_{1+}B_{0-} + B_{1+}|1\rangle\langle1|1\rangle\langle1|B_{1-} \tag{35}$$

$$= B_{1+}|1\rangle\langle1|1\rangle\langle1|B_{1-} \tag{36}$$

$$= |1\rangle\langle1|B_{1+}B_{1-} \tag{37}$$

$$= |1\rangle\langle1|, \tag{38}$$

$$\overline{|0\rangle\langle1|} = e^V |0\rangle\langle1| e^{-V} \tag{39}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|0\rangle\langle1|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{40}$$

$$= \left(|0\rangle\langle0|0\rangle\langle1|B_{0+} + |1\rangle\langle1|B_{1+}|0\rangle\langle1|\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{41}$$

$$= \left(|0\rangle\langle0|0\rangle\langle1|B_{0+} + |1\rangle\langle1|0\rangle\langle1|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{42}$$

$$= |0\rangle\langle1|B_{0+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{43}$$

$$= |0\rangle\langle1|0\rangle\langle0|B_{0+}B_{0-} + |0\rangle\langle1|1\rangle\langle1|B_{0+}B_{1-} \tag{44}$$

$$= |0\rangle\langle1|B_{0+}B_{1-}, \tag{45}$$

$$\overline{|1\rangle\langle0|} = e^V |1\rangle\langle0| e^{-V} \tag{46}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|1\rangle\langle0|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{47}$$

$$= \left(|0\rangle\langle0|1\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}|1\rangle\langle0|\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{48}$$

$$= \left(|0\rangle\langle0|1\rangle\langle0|B_{0+} + |1\rangle\langle1|1\rangle\langle0|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{49}$$

$$= |1\rangle\langle0|B_{1+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{50}$$

$$= |1\rangle\langle0|B_{1+}|0\rangle\langle0|B_{0-} + |1\rangle\langle0|B_{1+}|1\rangle\langle1|B_{1-} \tag{51}$$

$$= |1\rangle\langle0|B_{1+}B_{0-} + |1\rangle\langle0|1\rangle\langle1|B_{1+}B_{1-} \tag{52}$$

$$= |1\rangle\langle0|B_{1+}B_{0-}, \tag{53}$$

$$\overline{b_{\mathbf{k}}} = e^V b_{\mathbf{k}} e^{-V} \tag{54}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)b_{\mathbf{k}}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{55}$$

$$= |0\rangle\langle0|B_{0+}b_{\mathbf{k}}B_{0-}|0\rangle\langle0| + |0\rangle\langle0|B_{0+}b_{\mathbf{k}}|1\rangle\langle1|B_{1-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}B_{1-}|1\rangle\langle1| \tag{56}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}b_{\mathbf{k}}B_{0-} + |0\rangle\langle0|1\rangle\langle1|B_{0+}b_{\mathbf{k}}B_{1-} + |1\rangle\langle1|0\rangle\langle0|B_{1+}b_{\mathbf{k}}B_{0-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}B_{1-} \tag{57}$$

$$= |0\rangle\langle0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{58}$$

$$= \left(|0\rangle\langle0| + |1\rangle\langle1|\right)b_{\mathbf{k}} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{59}$$

$$= b_{\mathbf{k}} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{60}$$

$$\overline{b_{\mathbf{k}}}^{\dagger} = e^V b_{\mathbf{k}}^{\dagger} e^{-V} \tag{61}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{62}$$

$$= |0\rangle\langle0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-}|0\rangle\langle0| + |0\rangle\langle0|B_{0+}b_{\mathbf{k}}^{\dagger}|1\rangle\langle1|B_{1-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}^{\dagger}|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-}|1\rangle\langle1| \tag{63}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-} + |0\rangle\langle0|1\rangle\langle1|B_{0+}b_{\mathbf{k}}^{\dagger}B_{1-} + |1\rangle\langle1|0\rangle\langle0|B_{1+}b_{\mathbf{k}}^{\dagger}B_{0-} + |1\rangle\langle1|1\rangle\langle1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-} \tag{64}$$

$$= |0\rangle\langle0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) \tag{65}$$

$$= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \tag{66}$$

We have used the following:

$$B_{i+} b_{\mathbf{k}} B_{i-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{67}$$

$$B_{i+} b_{\mathbf{k}}^\dagger B_{i-} = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \tag{68}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{69}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{70}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|} = V_{10}(t)\,|1\rangle\langle 0|\,B_{1+}B_{0-}, \tag{71}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|} = V_{01}(t)\,|0\rangle\langle 1|\,B_{0+}B_{1-}, \tag{72}$$

$$\overline{g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}}} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \tfrac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \tfrac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\right) + g_{i\mathbf{k}}^*\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{73}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}}^\dagger - \tfrac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \tfrac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^*\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}}^\dagger - \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{74}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{75}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{76}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{77}$$

$$= |0\rangle\langle 0|B_{0+}|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)|0\rangle\langle 0|B_{0-} \tag{78}$$

$$= |0\rangle\langle 0|B_{0+}\left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)B_{0-} \tag{79}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \tfrac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^*\left(b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{80}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{81}$$

$$= |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}\right)|1\rangle\langle 1|B_{1-} \tag{82}$$

$$= |1\rangle\langle 1|B_{1+}\left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}\right)B_{1-} \tag{83}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \tfrac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^*\left(b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{84}$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0+} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_{0-} + |1\rangle\langle 1|B_{1+} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_{1-}\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\Pi_{\mathbf{k}'}\,D\!\left(\tfrac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|\Pi_{\mathbf{k}'}\,D\!\left(\tfrac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0|\Pi_{\mathbf{k}'}\,D\!\left(-\tfrac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|\Pi_{\mathbf{k}'}\,D\!\left(-\tfrac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\!\left(\tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\!\left(-\tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\Pi_{\mathbf{k}'\neq\mathbf{k}}\,D\!\left(\tfrac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\!\left(-\tfrac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|D\!\left(\tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\!\left(-\tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\Pi_{\mathbf{k}'\neq\mathbf{k}}\,D\!\left(\tfrac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\!\left(-\tfrac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\!\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\!\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1|D\!\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\!\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{90}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right)\right) \tag{91}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{92}$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{93}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{94}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \tag{95}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) . \tag{96}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)|0\rangle\langle 0|} + \overline{\varepsilon_1(t)|1\rangle\langle 1|} + \overline{V_{10}(t)|1\rangle\langle 0|} + \overline{V_{01}(t)|0\rangle\langle 1|} \tag{97}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0|B_{1+}B_{0-} + V_{01}(t)|0\rangle\langle 1|B_{0+}B_{1-}, \tag{98}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{99}$$

$$= \sum_{\mathbf{k}} \overline{|0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \sum_{\mathbf{k}} \overline{|1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{100}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{101}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0|(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) + \sum_{\mathbf{k}} |1\rangle\langle 1|(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{102}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} \tag{103}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right) \tag{104}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right) . \tag{105}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{j+}B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{106}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}} \tag{107}$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{108}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) . \tag{109}$$

We assume that the bath is at equilibrium with inverse temperature $\beta = 1/k_\mathrm{B}T$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{110}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & \sqrt{n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{111}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{112}$$

So the product of the matrix representation of $b^\dagger$ and b is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 2 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{113}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\rangle\langle j\right|, \tag{114}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_\mathbf{k} b_\mathbf{k}^\dagger b_\mathbf{k}\right) = \sum_{j_\mathbf{k}} \exp\left(-j_\mathbf{k}\beta\omega_\mathbf{k}\right) |j_\mathbf{k}\rangle\langle j_\mathbf{k}|, \tag{115}$$

$$\exp\left(-\beta\sum_\mathbf{k}\omega_\mathbf{k} b_\mathbf{k}^\dagger b_\mathbf{k}\right) = \prod_\mathbf{k}\sum_{j_\mathbf{k}} \exp\left(-j_\mathbf{k}\beta\omega_\mathbf{k}\right) |j_\mathbf{k}\rangle\langle j_\mathbf{k}|. \tag{116}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_\mathbf{k} b_\mathbf{k}^\dagger b_\mathbf{k}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{117}$$

$$= \sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{118}$$

$$= \sum_{j_{\mathbf{k}}}\exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{119}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{120}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{121}$$

$$\mathrm{Tr}\left(\exp\left(-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{122}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{123}$$

$$= \prod_{\mathbf{k}}f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{124}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{125}$$

$$= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}}f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{126}$$

$$= \prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{127}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}}\,|\,j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{128}$$

$$b_{\mathbf{k}}^{\dagger}\,|\,j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1}\,|\,j_{\mathbf{k}}+1\rangle. \tag{129}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (128)-(129):

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Tr}\left(\rho_B B_{iz}\right) = \mathrm{Tr}\left(B_{iz}\rho_B\right) \tag{130}$$

$$= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}})^{*}\,b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{131}$$

$$= \sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})\,b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})^{*}\,b_{\mathbf{k}}\rho_B\right) \tag{132}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \operatorname{Tr}\left(b_{\mathbf{k}}^\dagger \rho_B\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \operatorname{Tr}\left(b_{\mathbf{k}} \rho_B\right) \tag{133}$$

$$= \sum_{\mathbf{k}} \operatorname{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} \operatorname{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})}\right) \tag{134}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \operatorname{Tr}\left(b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \operatorname{Tr}\left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})}\right), \tag{135}$$

$$\operatorname{Tr}\left(b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}})\right) b_{\mathbf{k}}^\dagger |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}^\dagger\text{)} \tag{136}$$

$$= \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right)\right) \sqrt{j_{\mathbf{k}}+1}\, |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{137}$$

$$= 0, \tag{138}$$

$$\operatorname{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}})\right) b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}\text{)} \tag{139}$$

$$= \operatorname{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right)\right) \sqrt{j_{\mathbf{k}}}\, |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) \tag{140}$$

$$= 0. \tag{141}$$

we therefore find that:

$$\langle B_{iz}\rangle_{\overline{H_{\tilde{B}}}} = 0 \tag{142}$$

Another important expected value is $B = \langle B_\pm\rangle_{\overline{H_{\tilde{B}}}}$, where $B_\pm = e^{\pm \sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)}$ is given by:

$$\langle B_\pm\rangle_{H_B} = \operatorname{Tr}\left(\rho_B B_\pm\right) = \operatorname{Tr}\left(B_\pm \rho_B\right) \tag{143}$$

$$= \operatorname{Tr}\left(e^{\pm \sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)} \rho_B\right) \tag{144}$$

$$= \prod_{\mathbf{k}} \operatorname{Tr}\left(D\left(\pm \alpha_{\mathbf{k}}\right) \rho_B\right) \tag{145}$$

$$= \prod_{\mathbf{k}} \langle D\left(\pm \alpha_{\mathbf{k}}\right)\rangle. \tag{146}$$

Given that we can write a density operator as:

$$\rho = \int P\left(\alpha\right) |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha \tag{147}$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right) \mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P\left(\alpha\right)$ is given by:

$$\langle A\rangle = \operatorname{Tr}\left(A\rho\right) \tag{148}$$

$$= \int P\left(\alpha\right) \langle\alpha|A|\alpha\rangle \mathrm{d}^2\alpha \tag{149}$$

We are typically interested in thermal state density operators, for which it can be shown that $P\left(\alpha\right) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta \omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | D(h) | \alpha \rangle \mathrm{d}^2\alpha \tag{150}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle \mathrm{d}^2\alpha \tag{151}$$

$$D(h) D(\alpha) = D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{152}$$

$$D(-\alpha)(D(h) D(\alpha)) = D(-\alpha) D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{153}$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{154}$$

$$= D\left(\alpha\right) e^{\frac{1}{2}\left(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2\right)} e^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{155}$$

$$= D\left(\alpha\right) e^{\left(h\alpha^* - h^*\alpha\right)}, \tag{156}$$

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0|D\left(h\right) \exp\left(h\alpha^* - h^*\alpha\right)|0\rangle \mathrm{d}^2\alpha \tag{157}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp\left(h\alpha^* - h^*\alpha\right) \langle 0|D\left(h\right)|0\rangle \mathrm{d}^2\alpha \tag{158}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp\left(h\alpha^* - h^*\alpha\right) \langle 0|h\rangle \mathrm{d}^2\alpha \tag{159}$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{160}$$

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp\left(h\alpha^* - h^*\alpha\right) \langle 0|\exp\left(-\frac{|h|^2}{2}\right) \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle \mathrm{d}^2\alpha \tag{161}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) \mathrm{d}^2\alpha \tag{162}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha\right) \mathrm{d}^2\alpha, \tag{163}$$

$$\alpha = x + \mathrm{i}y, \tag{164}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 + y^2}{N} + h\left(x - \mathrm{i}y\right) - h^*\left(x + \mathrm{i}y\right)\right) \mathrm{d}x\mathrm{d}y \tag{165}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{N} + hx - h^*x\right) \mathrm{d}x \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^*y\right) \mathrm{d}y, \tag{166}$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N}\left(x^2 - Nhx + Nh^*x\right) \tag{167}$$

$$= -\frac{1}{N}\left(x + \frac{\left(Nh^* - Nh\right)}{2}\right)^2 + \frac{N\left(h^* - h\right)^2}{4}, \tag{168}$$

$$-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^*y = -\frac{1}{N}\left(y^2 + \mathrm{i}Nhy + \mathrm{i}Nh^*y\right) \tag{169}$$

$$= -\frac{1}{N}\left(y^2 + \frac{\mathrm{i}N\left(h + h^*\right)}{2}\right) - \frac{N\left(h + h^*\right)^2}{4}, \tag{170}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 - \frac{1}{N}\left(y^2 + \frac{\mathrm{i}N(h + h^*)}{2}\right)\right) \mathrm{d}x\mathrm{d}y, \tag{171}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x - \mu\right)^2}{2\sigma^2}\right) \mathrm{d}x, \tag{172}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x + \frac{(Nh^* - Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) \mathrm{d}x \int_{-\infty}^{\infty} \exp\left(-\frac{\left(y^2 + \frac{\mathrm{i}N(h + h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) \mathrm{d}y \tag{173}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{174}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^* - h\right)^2}{4} - \frac{N\left(h + h^*\right)^2}{4}\right) \tag{175}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^{*2} - 2hh^* + h^2\right) - N\left(h^2 + 2hh^* + h^{*2}\right)}{4}\right) \tag{176}$$

$$= \exp\left(-|h|^2\left(N + \frac{1}{2}\right)\right) \tag{177}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega}-1} + \frac{1}{2}\right)\right) \tag{178}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)\right) \tag{179}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{180}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (145) using (180) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{181}$$

We will now force $\langle \overline{H_I} \rangle_{\overline{H_{\tilde{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_+\sigma_+$ and $B_-\sigma_-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle \overline{H_I} \rangle_{\overline{H_{\tilde{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{j+}B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j|\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{182}$$

$$= \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'} + \sum_j |j\rangle\langle j|B_{jz} + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-} - B_{jj'}\right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{183}$$

$$\equiv \overline{H_{\tilde{S}}(t)} + \overline{H_I} + \overline{H_{\tilde{B}}}. \tag{184}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_{1+}B_{0-}\rangle = B_{10} \tag{185}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \tag{186}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle \tag{187}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right) \right\rangle \tag{188}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{189}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{190}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}. \tag{191}$$

From the definition $B_{01} = \langle B_{0+}B_{1-}\rangle$ using the displacement operator we have:

$$\langle B_{0+}B_{1-}\rangle = B_{01} \tag{192}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle \tag{193}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle \tag{194}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\right\rangle \tag{195}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{196}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{197}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{198}$$

This can be checked in the following way:

$$\langle B_{0+}B_{1-}\rangle = B_{01} \tag{199}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{200}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)^*} \tag{201}$$

$$= \langle B_{1+}B_{0-}\rangle^* \tag{202}$$

$$= B_{10}^*. \tag{203}$$

The parts of the Hamiltonian splitted are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t)B_{10}\sigma_+ + V_{01}(t)B_{01}\sigma_-, \tag{204}$$

$$\overline{H_{\bar{I}}} \equiv V_{10}(t)(B_{1+}B_{0-} - B_{10})\sigma_+ + V_{01}(t)(B_{0+}B_{1-} - B_{01})\sigma_- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, \tag{205}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{206}$$

$$= H_B. \tag{207}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (205) we can verify the condition $\langle \overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\langle \overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\right)|n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-} - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{208}$$

$$= \left\langle \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-} - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{209}$$

$$= \sum_{n\mathbf{k}}\left(\langle(g_{n\mathbf{k}} - v_{n\mathbf{k}})b_{\mathbf{k}}^\dagger\rangle_{\overline{H_{\bar{B}}}} + \langle(g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|\left(\langle V_{jj'}(t)B_{j+}B_{j'-}\rangle_{\overline{H_{\bar{B}}}} - \langle V_{jj'}(t)B_{jj'}\rangle_{\overline{H_{\bar{B}}}}\right) \tag{210}$$

$$= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})\langle b_{\mathbf{k}}^\dagger\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(\langle B_{j+}B_{j'-}\rangle_{\overline{H_{\bar{B}}}} - \langle B_{jj'}\rangle_{\overline{H_{\bar{B}}}}\right) \tag{211}$$

$$= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})\langle b_{\mathbf{k}}^\dagger\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(B_{jj'} - B_{jj'}\right). \tag{212}$$

$$= 0. \tag{213}$$

We used (142) and (191) to evaluate the expected values.

Let's consider the following Hermitian combinations:

$$B_x = B_x^\dagger \tag{214}$$

$$= \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2}, \tag{215}$$

$$B_y = B_y^\dagger \tag{216}$$

$$= \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{01}}{2i}, \tag{217}$$

$$B_{iz} = B_{iz}^\dagger \tag{218}$$

$$= \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \tag{219}$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$\overline{H_S(t)} = (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\sigma_+ + V_{01}(t) B_{01}\sigma_- \tag{220}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01}\frac{\sigma_x - i\sigma_y}{2} \tag{221}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) (\Re(B_{10}(t)) + i\Im(B_{10}(t)))\frac{\sigma_x + i\sigma_y}{2} + V_{01}(t)(\Re(B_{10}(t)) - i\Im(B_{10}(t)))\frac{\sigma_x - i\sigma_y}{2} \tag{222}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))\left(V_{10}(t)\frac{\sigma_x + i\sigma_y}{2} + V_{01}(t)\frac{\sigma_x - i\sigma_y}{2}\right) + i\Im(B_{10}(t))\left(V_{10}(t)\frac{\sigma_x + i\sigma_y}{2} - V_{01}(t)\frac{\sigma_x - i\sigma_y}{2}\right) \tag{223}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y\frac{V_{10}(t) - V_{01}(t)}{2}\right) + i\Im(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y\frac{V_{10}(t) + V_{01}(t)}{2}\right) \tag{224}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) + V_{10}^*(t)}{2} + i\sigma_y\frac{V_{10}(t) - V_{10}^*(t)}{2}\right) + i\Im(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) - V_{10}^*(t)}{2} + i\sigma_y\frac{V_{10}(t) + V_{10}^*(t)}{2}\right) \tag{225}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))(\sigma_x\Re(V_{10}(t)) - \sigma_y\Im(V_{10}(t))) + i\Im(B_{10}(t))(i\sigma_x\Im(V_{10}(t)) + i\sigma_y\Re(V_{10}(t))) \tag{226}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + (\sigma_x\Re(B_{10}(t))\Re(V_{10}(t)) - \sigma_y\Re(B_{10}(t))\Im(V_{10}(t)) - (\sigma_x\Im(B_{10}(t))\Im(V_{10}(t)) + \sigma_y\Im(B_{10}(t))\Re(V_{10}(t)) \tag{227}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \sigma_x(\Re(B_{10}(t))\Re(V_{10}(t)) - \Im(B_{10}(t))\Im(V_{10}(t))) - \sigma_y(\Re(B_{10}(t))\Im(V_{10}(t)) + \Im(B_{10}(t))\Re(V_{10}(t))) \tag{228}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \sigma_x\left(B_{10}^\Re(t) V_{10}^\Re(t) - B_{10}^\Im(t) V_{10}^\Im(t)\right) - \sigma_y\left(B_{10}^\Re(t) V_{10}^\Im(t) + B_{10}^\Im(t) V_{10}^\Re(t)\right). \tag{229}$$

$$\overline{H_I} = V_{10}(t)(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}) + V_{01}(t)(\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} \tag{230}$$

$$= |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} + (\Re(V_{10}(t)) + i\Im(V_{10}(t)))(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}) + (\Re(V_{10}(t)) - i\Im(V_{10}(t)))(\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) \tag{231}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} + \sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) + i\Im(V_{10}(t))(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} - \sigma_- B_{0+} B_{1-} + \sigma_- B_{01}) \tag{232}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - i\sigma_y}{2} B_{01}\right) \tag{233}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - i\sigma_y}{2} B_{01}\right) + i\Im(V_{10}(t))\left(\frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} - \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} + \frac{\sigma_x - i\sigma_y}{2} B_{01} \tag{234}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_x\frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} + i\sigma_y\frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2}\right) + i\Im(V_{10}(t))\left(\sigma_x\frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} + i\sigma_y\frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{235}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))(\sigma_x B_x + \sigma_y B_y) + \Im(V_{10}(t))\left(i\sigma_x\frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} - \sigma_y\frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{236}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))(\sigma_x B_x + \sigma_y B_y) + \Im(V_{10}(t))\left(i^2\sigma_x\frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2i} - \sigma_y\frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{237}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))(\sigma_x B_x + \sigma_y B_y) + \Im(V_{10}(t))\left(i^2\sigma_x\frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2i} - \sigma_y\frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{238}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))(\sigma_x B_x + \sigma_y B_y) + \Im(V_{10}(t))(i^2\sigma_x(-B_y) - \sigma_y B_x) \tag{239}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))(\sigma_x B_x + \sigma_y B_y) + \Im(V_{10}(t))(\sigma_x B_y - \sigma_y B_x). \tag{240}$$

III.   FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta} \ln \left(\mathrm{Tr} \left(e^{-\beta \left(\overline{H_{\tilde{S}}}(t) + \overline{H_{\tilde{B}}} \right)} \right) \right) + \left\langle \overline{H_{\tilde{I}}} \right\rangle_{\overline{H_{\tilde{S}}}(t) + \overline{H_{\tilde{B}}}} + O \left(\left\langle \overline{H_{\tilde{I}}}^2 \right\rangle_{\overline{H_{\tilde{S}}}(t) + \overline{H_{\tilde{B}}}} \right). \tag{241}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle \overline{H_{\tilde{I}}} \right\rangle_{\overline{H_{\tilde{S}}}(t) + \overline{H_{\tilde{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = 0. \tag{242}$$

Using this condition and given that $\left[\overline{H_{\tilde{S}}}(t), \overline{H_{\tilde{B}}} \right] = 0$, we have:

$$e^{-\beta \left(\overline{H_{\tilde{S}}}(t) + \overline{H_{\tilde{B}}} \right)} = e^{-\beta \overline{H_{\tilde{S}}}(t)} e^{-\beta \overline{H_{\tilde{B}}}}. \tag{243}$$

Then using the fact that $\overline{H_{\tilde{S}}}(t)$ and $\overline{H_{\tilde{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} e^{-\beta \overline{H_{\tilde{B}}}} \right) = \mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} \right) \mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{B}}}} \right). \tag{244}$$

So Eq. (242) becomes:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = -\frac{1}{\beta} \frac{\partial \ln \left(\mathrm{Tr} \left(e^{-\beta \left(\overline{H_{\tilde{S}}}(t) + \overline{H_{\tilde{B}}} \right)} \right) \right)}{\partial v_{i\mathbf{k}}} \tag{245}$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} \right) \mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{B}}}} \right) \right)}{\partial v_{i\mathbf{k}}} \tag{246}$$

$$= -\frac{1}{\beta} \frac{\partial \left(\ln \left(\mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} \right) \right) + \ln \left(\mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{B}}}} \right) \right) \right)}{\partial v_{i\mathbf{k}}} \tag{247}$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta} \frac{\partial \ln \left(\mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{B}}}} \right) \right)}{\partial v_{i\mathbf{k}}} \tag{248}$$

$$= 0 \quad \text{(by Eq. (242))}. \tag{249}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\frac{\partial \ln \left(\mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} = \frac{1}{e^{-\beta \overline{H_{\tilde{S}}}(t)}} \frac{\partial \mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} \right)}{\partial v_{i\mathbf{k}}} \tag{250}$$

$$= 0. \tag{251}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr} \left(e^{-\beta \overline{H_{\tilde{S}}}(t)} \right)}{\partial v_{i\mathbf{k}}} = 0. \tag{252}$$

First we look at:

$$-\beta \overline{H_{\tilde{S}}}(t) = -\beta \left((\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma_+ + V_{01}(t) B_{01} \sigma_- \right). \tag{253}$$

Then the eigenvalues of $-\beta \overline{H_{\tilde{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right) + \mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right) = 0. \tag{254}$$

Let's define:

$$\varepsilon\left(t\right) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), \tag{255}$$

$$\eta \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}. \tag{256}$$

The solutions of the equation (254) are:

$$\lambda = \beta\frac{-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right) \pm \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}}{2} \tag{257}$$

$$= \beta\frac{-\varepsilon\left(t\right) \pm \eta\left(t\right)}{2} \tag{258}$$

$$= -\beta\frac{\varepsilon\left(t\right) \mp \eta\left(t\right)}{2}. \tag{259}$$

The value of $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right) + \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) \tag{260}$$

$$= 2\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{261}$$

Given that $v_{i\mathbf{k}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A\left(\varepsilon\left(t\right), \eta\left(t\right)\right)$ to calculate $\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial\Re(v_{i\mathbf{k}})}$ can lead to:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)} = \frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)} \tag{262}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) + 2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) \tag{263}$$

$$= -\beta\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{264}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) = 0. \tag{265}$$

The derivates included in the expression given are related to:

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{266}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)^*\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{267}$$

$$= \langle B_{1+}B_{0-}\rangle^*, \tag{268}$$

$$R_i = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{269}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{270}$$

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{271}$$

$$= \left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2\omega_{\mathbf{k}}^2}(v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{272}$$

$$v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^* = (\Re(v_{0\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{1\mathbf{k}}))-(\Re(v_{0\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{1\mathbf{k}})) \tag{273}$$

$$= (\Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\mathrm{i}\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}))-(\Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})-\mathrm{i}\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})) \tag{274}$$

$$= 2\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})), \tag{275}$$

$$|v_{1\mathbf{k}}-v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}}-v_{0\mathbf{k}})(v_{1\mathbf{k}}-v_{0\mathbf{k}})^* \tag{276}$$

$$= |v_{1\mathbf{k}}|^2+|v_{0\mathbf{k}}|^2-(v_{1\mathbf{k}}v_{0\mathbf{k}}^*+v_{1\mathbf{k}}^*v_{0\mathbf{k}}) \tag{277}$$

$$= (\Re(v_{1\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}}))^2+(\Re(v_{0\mathbf{k}}))^2+(\Im(v_{0\mathbf{k}}))^2-((\Re(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}}))+(\Re(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}}))) \tag{278}$$

$$= (\Re(v_{1\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}}))^2+(\Re(v_{0\mathbf{k}}))^2+(\Im(v_{0\mathbf{k}}))^2-2(\Re(v_{1\mathbf{k}})\Re(v_{0\mathbf{k}})+\Im(v_{1\mathbf{k}})\Im(v_{0\mathbf{k}})) \tag{279}$$

$$= (\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2. \tag{280}$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}}\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{\Re(v_{i\mathbf{k}})-\mathrm{i}\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{\Re(v_{i\mathbf{k}})+\mathrm{i}\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\right)\right) \tag{281}$$

$$= \sum_{\mathbf{k}}\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\Re(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}\Im(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{282}$$

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}}\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{283}$$

$$= \left(\prod_{\mathbf{k}}\exp\left(\frac{2\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{284}$$

$$= \left(\prod_{\mathbf{k}}\exp\left(\frac{\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{285}$$

Calculating the derivates respect to $\Re(\alpha_{i\mathbf{k}})$ and $\Im(\alpha_{i\mathbf{k}})$ we have:

$$\frac{\partial \varepsilon(t)}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial \Re(v_{i\mathbf{k}})} \tag{286}$$

$$= \frac{\partial\left(\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\Re(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-i\Im(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial \Re(v_{i\mathbf{k}})} \tag{287}$$

$$= \frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{288}$$

$$\frac{\partial |B_{10}|^2}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial \Re(v_{i\mathbf{k}})} \tag{289}$$

$$= -\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial \Re(v_{i\mathbf{k}})}\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{290}$$

$$= -\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial \Re(v_{i\mathbf{k}})}|B_{10}|^2, \tag{291}$$

$$\frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\tilde{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\tilde{S}}(t)}\right)}}{\partial \Re(v_{i\mathbf{k}})} \tag{292}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\tilde{S}}(t)}\right)\frac{\partial \mathrm{Tr}\left(\overline{H_{\tilde{S}}(t)}\right)}{\partial \Re(v_{i\mathbf{k}})}-4\frac{\partial \mathrm{Det}\left(\overline{H_{\tilde{S}}(t)}\right)}{\partial \Re(v_{i\mathbf{k}})}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\tilde{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\tilde{S}}(t)}\right)}} \tag{293}$$

$$= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial \Re(v_{i\mathbf{k}})}}{\eta(t)} \tag{294}$$

$$= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)+\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial \Re(v_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{295}$$

$$= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)+\frac{2(\Re(v_{i\mathbf{k}})-\Re(v_{i'\mathbf{k}}))}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{296}$$

$$= \frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) \tag{297}$$

$$+\frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\varepsilon(t)+2(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{298}$$

From the equation (265) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\frac{\partial \varepsilon(t)}{\partial \Re(v_{i\mathbf{k}})}}{\frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})}} \tag{299}$$

$$= \frac{\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{2\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}}{\frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}+2\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}} \tag{300}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{(2\Re(v_{i\mathbf{k}})-g_{i\mathbf{k}}-g_{i\mathbf{k}}^*)\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}+g_{i\mathbf{k}}^*)(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{301}$$

$$= \frac{(2\Re(v_{i\mathbf{k}})-2\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Re(g_{i\mathbf{k}})(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{302}$$

$$= \frac{(2\Re(v_{i\mathbf{k}})-2\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{303}$$

$$= \frac{(\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{304}$$

Separating (303) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}}))\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=\Re(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)-\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)$$

(305)

$$\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}})=\Re(v_{i\mathbf{k}})\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)$$

(306)

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)$$

(307)

$$\Re(v_{i\mathbf{k}})=\frac{\Re(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{\Re(v_{i'\mathbf{k}})}{\Re(g_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}$$

(308)

$$\Re(v_{i\mathbf{k}})=\frac{\Re(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{\Re(v_{i'\mathbf{k}})}{\Re(g_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}$$

(309)

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon(t)}{\partial\Im(v_{i\mathbf{k}})}=\frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial\Im(v_{i\mathbf{k}})}$$

(310)

$$=\frac{\partial\left(\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\Re(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}\Im(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial\Im(v_{i\mathbf{k}})}$$

(311)

$$=2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}$$

(312)

$$\frac{\partial|B_{10}|^2}{\partial\Im(v_{i\mathbf{k}})}=\frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial\Im(v_{i\mathbf{k}})}$$

(313)

$$=-\frac{2(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))}{\partial\Im(v_{i\mathbf{k}})}\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)$$

(314)

$$=-\frac{2(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))}{\partial\Im(v_{i\mathbf{k}})}|B_{10}|^2$$

(315)

$$\frac{\partial\eta(t)}{\partial\Re(v_{i\mathbf{k}})}=\frac{\partial\sqrt{\left(\mathrm{Tr}(\overline{H_{\bar{S}}(t)})\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial\Re(v_{i\mathbf{k}})}$$

(316)

$$=\frac{2\mathrm{Tr}(\overline{H_{\bar{S}}(t)})\frac{\partial\mathrm{Tr}(\overline{H_{\bar{S}}(t)})}{\partial\Im(v_{i\mathbf{k}})}-4\frac{\partial\mathrm{Det}(\overline{H_{\bar{S}}(t)})}{\partial\Im(v_{i\mathbf{k}})}}{2\sqrt{\left(\mathrm{Tr}(\overline{H_{\bar{S}}(t)})\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}$$

(317)

$$=\frac{\varepsilon(t)\left(2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial\Im(v_{i\mathbf{k}})}}{\eta(t)}$$

(318)

$$=\frac{\varepsilon(t)\left(2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))}{\partial\Im(v_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)}$$

(319)

$$=\frac{\varepsilon(t)\left(2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2(\Im(v_{i\mathbf{k}})-\Im(v_{i'\mathbf{k}}))}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)}$$

(320)

$$=\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\eta(t)}\right)$$

(321)

$$+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2(\varepsilon(t)-\varepsilon_i(t)-R_i)\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)$$

(322)

From the equation (265) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial\Im(v_{i\mathbf{k}})}}{\frac{\partial\eta(t)}{\partial\Im(v_{i\mathbf{k}})}} \tag{323}$$

$$= \frac{2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(2\varepsilon(t) - 4\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \frac{2}{\eta(t)}\left(\frac{\Im(g_{i\mathbf{k}}^*)}{\omega_{\mathbf{k}}}\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big)\frac{\Im(g_{i\mathbf{k}}^*)}{\omega_{\mathbf{k}}} + 2\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)} \tag{324}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\big(2\Im(v_{i\mathbf{k}}) - \mathrm{i}\big(g_{i\mathbf{k}}^* - g_{i\mathbf{k}}\big)\big)\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t) - 4\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \mathrm{i}\big(g_{i\mathbf{k}}^* - g_{i\mathbf{k}}\big)\big(\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big)\big) + 4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{325}$$

$$= \frac{2\big(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}})\big)\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t) - 4\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{4|V_{10}(t)|^2 B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2\Im(g_{i\mathbf{k}})\big(\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big)\big) + 4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{326}$$

$$= \frac{2\big(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}})\big)\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t) - 4\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2\Im(g_{i\mathbf{k}})\big(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\big) + 4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$= \frac{\big(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}})\big)\eta(t)}{\Im(v_{i\mathbf{k}})\left(\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \Im(g_{i\mathbf{k}})\big(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\big) + 2\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

Separating (**??**) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\big(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}})\big)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = \Im(v_{i\mathbf{k}})\left(\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \Im(g_{i\mathbf{k}})\big(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\big) + 2\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{329}$$

$$\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}}) = \Im(v_{i\mathbf{k}})\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \tag{330}$$

$$- \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\Im(g_{i\mathbf{k}})\big(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\big) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{331}$$

$$\Im(v_{i\mathbf{k}}) = \frac{\Im(g_{i\mathbf{k}})\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\big(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\big)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{332}$$

$$\Im(v_{i\mathbf{k}}) = \frac{\Im(g_{i\mathbf{k}})\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\big(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\big)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\big(\varepsilon(t) - \varepsilon_i(t) - R_i\big) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{333}$$

The variational parameters are:

$$v_{i\mathbf{k}} = \Re\left(v_{i\mathbf{k}}\right) + \mathrm{i}\Im\left(v_{i\mathbf{k}}\right) \tag{334}$$

$$= \frac{\Re\left(g_{i\mathbf{k}}\right)\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{335}$$

$$+\mathrm{i}\frac{\Im\left(g_{i\mathbf{k}}\right)\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{336}$$

$$= \frac{g_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$, the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T(0) e^{-V} \tag{338}$$

$$= \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)\left(\rho_S(0) \otimes \rho_B^{\text{Thermal}}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{339}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: |0\rangle\langle 0|0\rangle B_{0+}\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \tag{340}$$

$$= |0\rangle B_{0+}\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \tag{341}$$

$$= |0\rangle\langle 0| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{0-} \tag{342}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \tag{343}$$

$$= |1\rangle\langle 1|B_{1+}\rho_B^{\text{Thermal}}B_{1-} \tag{344}$$

$$= |1\rangle\langle 1| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{1-} \tag{345}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1| : |0\rangle\langle 0|B_{0+}|0\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \tag{346}$$

$$= |0\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \tag{347}$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}B_{1-} \tag{348}$$

$$= |0\rangle\langle 1| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{1-} \tag{349}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0| : |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \tag{350}$$

$$= |1\rangle\langle 0| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{0-} \tag{351}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^\dagger(t)O(t)U(t) \tag{352}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right). \tag{353}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_{\bar{S}}}}(t) = U^\dagger(t)\overline{\rho_{\bar{S}}}(t)U(t), \text{ where} \tag{354}$$

$$\overline{\rho_{\bar{S}}}(t) = \text{Tr}_B\left(\bar{\rho}_T(t)\right) \tag{355}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda\left(t\right)$ such that $\Lambda_{1i}\left(t\right)=A_i$, $\Lambda_{2i}\left(t\right)=B_i$ and $\Lambda_{3i}\left(t\right)=C_i\left(t\right)$ written as:

$$\begin{pmatrix} A\left(t\right) \\ B\left(t\right) \\ C\left(t\right) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ \Re\left(V_{10}\left(t\right)\right) & \Re\left(V_{10}\left(t\right)\right) & 1 & \Im\left(V_{10}\left(t\right)\right) & -\Im\left(V_{10}\left(t\right)\right) & 1 \end{pmatrix} \tag{356}$$

In this case $|1\rangle\langle1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle0| - |1\rangle\langle1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\tilde{I}}}\left(t\right)$ as pointed in the equation (**??**):

$$\overline{H_{\tilde{I}}}\left(t\right) = \sum_i B_{iz}|i\rangle\langle i| + \Re\left(V_{10}\left(t\right)\right)\left(\sigma_x B_x + \sigma_y B_y\right) + \Im\left(V_{10}\left(t\right)\right)\left(\sigma_x B_y - \sigma_y B_x\right) \tag{357}$$

$$= B_{0z}|0\rangle\langle0| + B_{1z}|1\rangle\langle1| + \Re\left(V_{10}\left(t\right)\right)\sigma_x B_x + \Re\left(V_{10}\left(t\right)\right)\sigma_y B_y + \Im\left(V_{10}\left(t\right)\right)\sigma_x B_y - \Im\left(V_{10}\left(t\right)\right)\sigma_y B_x \tag{358}$$

$$= \sum_i C_i\left(t\right)\left(A_i \otimes B_i\left(t\right)\right) \tag{359}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\rho_T}\left(t\right)}{\mathrm{d}t} = \text{-i}[\widetilde{H}_I\left(t\right), \widetilde{\rho_T}\left(t\right)]. \tag{360}$$

This equation has the formal solution

$$\widetilde{\rho_T}(t) = \rho\left(0\right)\text{-i}\int_0^t [\widetilde{H}_I\left(s\right), \widetilde{\rho_T}\left(s\right)]\mathrm{d}s. \tag{361}$$

Replacing the equation (361) in the equation (360) give us:

$$\frac{\mathrm{d}\widetilde{\rho_T}\left(t\right)}{\mathrm{d}t} = -\text{i}[\widetilde{H}_I\left(t\right), \rho_T\left(0\right)]\text{-}\int_0^t [\widetilde{H}_I\left(t\right), [\widetilde{H}_I\left(s\right), \widetilde{\rho_T}\left(s\right)]]\mathrm{d}s. \tag{362}$$

This equation allow us to iterate and write in terms of a series expansion with $\rho_T\left(0\right)$ the solution as:

$$\widetilde{\rho_T}(t) = \rho_T\left(0\right) + \sum_{n=0}^{\infty}\left(-\text{i}\right)^n\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{H}_I\left(t_1\right), [\widetilde{H}_I\left(t_2\right), \cdots[\widetilde{H}_I\left(t_n\right), \rho_T\left(0\right)]]\cdots] \tag{363}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho_{\overline{S}}}(t) = \rho_{\overline{S}}\left(0\right) + \sum_{n=1}^{\infty}\left(-\text{i}\right)^n\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{H}_I\left(t_1\right), [\widetilde{H}_I\left(t_2\right), \cdots[\widetilde{H}_I\left(t_n\right), \rho_{\overline{S}}\left(0\right)\rho_B^{\text{Thermal}}]]\cdots] \tag{364}$$

here we have assumed that $\rho_T\left(0\right) = \rho_{\overline{S}}(0) \otimes \rho_B^{\text{Thermal}}$. Consider the following notation:

$$\widetilde{\rho_{\overline{S}}}(t) = \left(1 + W_1\left(t\right) + W_2\left(t\right) + ...\right)\rho_S\left(0\right) \tag{365}$$

$$= W\left(t\right)\rho_{\overline{S}}\left(0\right) \tag{366}$$

in this case

$$W_n\left(t\right) = \left(-\text{i}\right)^n\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{H}_I\left(t_1\right), [\widetilde{H}_I\left(t_2\right), \cdots[\widetilde{H}_I\left(t_n\right), \left(\cdot\right)\rho_B^{\text{Thermal}}]]\cdots] \tag{367}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\rho_{\overline{S}}}(t)}{\mathrm{d}t} = \left(\dot{W}_1\left(t\right) + \dot{W}_2\left(t\right) + ... \right) \rho_{\overline{S}}\left(0\right) \tag{368}$$

$$= \left(\dot{W}_1\left(t\right) + \dot{W}_2\left(t\right) + ... \right) W\left(t\right)^{-1} W\left(t\right) \rho_{\overline{S}}\left(0\right) \tag{369}$$

$$= \left(\dot{W}_1\left(t\right) + \dot{W}_2\left(t\right) + ... \right) W\left(t\right)^{-1} \widetilde{\rho_{\overline{S}}}(t) \tag{370}$$

where we assumed that $W\left(t\right)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{H}_I(t)\rho_B(0)] = 0$ so $W_1\left(t\right) = 0$. Thus, to second order and taking $W\left(t\right) \approx \mathbb{I}$ then the equation (368) becomes:

$$\frac{\mathrm{d}\rho_{\overline{S}}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[H_{\overline{S}}, \rho_{\overline{S}}\left(t\right)\right] - \int_0^t \mathrm{d}\tau \left[H_I, \left[\widetilde{H}_I(-\tau), \rho_{\overline{S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right] \tag{371}$$

Replacing $t_1 \to t - \tau$

$$W_n\left(t\right) = \left(-\mathrm{i}\right)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{H}_I\left(t_1\right), [\widetilde{H}_I\left(t_2\right), \cdots [\widetilde{H}_I\left(t_n\right), \left(\cdot\right) \rho_B^{\mathrm{Thermal}}]] \cdots] \tag{372}$$

Taking as reference state $\rho_B^{\mathrm{Thermal}}$ and truncating at second order in $\overline{H}_{\overline{I}}(t)$, we obtain our master equation in the interaction picture in the transformed frame:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_{\overline{S}}}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B \left[\widetilde{\overline{H_{\overline{I}}}}\left(t\right), \left[\widetilde{\overline{H_{\overline{I}}}}(s), \widetilde{\overline{\rho_{\overline{S}}}}(t)\rho_B^{\mathrm{Thermal}} \right] \right] \mathrm{d}s \tag{373}$$

From the interaction picture applied on $\overline{H_{\overline{I}}}\left(t\right)$ we find:

$$\widetilde{\overline{H_{\overline{I}}}}\left(t\right) = U^\dagger\left(t\right) \mathrm{e}^{\mathrm{i}H_B t} \overline{H_{\overline{I}}}\left(t\right) \mathrm{e}^{-\mathrm{i}H_B t} U\left(t\right) \tag{374}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\overline{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\overline{I}}}}\left(t\right) = \sum_i C_i\left(t\right) \left(\widetilde{\overline{A_i}}(t) \otimes \widetilde{B_i}\left(t\right) \right) \tag{375}$$

$$\widetilde{\overline{A_i}}(t) = U^\dagger\left(t\right) \mathrm{e}^{\mathrm{i}H_B t} A_i \mathrm{e}^{-\mathrm{i}H_B t} U\left(t\right) \tag{376}$$

$$= U^\dagger\left(t\right) A_i U\left(t\right) \mathrm{e}^{\mathrm{i}H_B t} \mathrm{e}^{-\mathrm{i}H_B t} \tag{377}$$

$$= U^\dagger\left(t\right) A_i U\left(t\right) \mathbb{I} \tag{378}$$

$$= U^\dagger\left(t\right) A_i U\left(t\right) \tag{379}$$

$$\widetilde{B_i}\left(t\right) = U^\dagger\left(t\right) \mathrm{e}^{\mathrm{i}H_B t} B_i\left(t\right) \mathrm{e}^{-\mathrm{i}H_B t} U\left(t\right) \tag{380}$$

$$= U^\dagger\left(t\right) U\left(t\right) \mathrm{e}^{\mathrm{i}H_B t} B_i\left(t\right) \mathrm{e}^{-\mathrm{i}H_B t} \tag{381}$$

$$= \mathbb{I} \mathrm{e}^{\mathrm{i}H_B t} B_i\left(t\right) \mathrm{e}^{-\mathrm{i}H_B t} \tag{382}$$

$$= \mathrm{e}^{\mathrm{i}H_B t} B_i\left(t\right) \mathrm{e}^{-\mathrm{i}H_B t} \tag{383}$$

Here we have used the fact that $\left[\overline{H_{\overline{S}}}\left(t\right), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.

Using the expression (375) to replace it in the equation (373)

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\widetilde{H_I}}(t), \left[\widetilde{\widetilde{H_I}}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right]\mathrm{d}s \tag{384}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \left[\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right]\mathrm{d}s \tag{385}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}} -\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\right]\mathrm{d}s \tag{386}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right. \tag{387}$$

$$-\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right) \tag{388}$$

$$-\sum_{i\in J} C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right) \tag{389}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\right)\mathrm{d}s \tag{390}$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \left\langle\widetilde{B_j}(t)\widetilde{B_i}(s)\right\rangle_B \tag{391}$$

$$= \left\langle\widetilde{B_j}(\tau)\widetilde{B_i}(0)\right\rangle_B \tag{392}$$

The correlation functions relevant that appear in the equation (384) are:

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B^{\mathrm{Thermal}}\right) = \left\langle\widetilde{B_j}(t)\widetilde{B_i}(s)\right\rangle_B \tag{393}$$

$$= \left\langle\widetilde{B_j}(\tau)\widetilde{B_i}(0)\right\rangle_B \tag{394}$$

$$= \Lambda_{ji}(\tau) \tag{395}$$

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\rho_B^{\mathrm{Thermal}}\widetilde{B_i}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\widetilde{B_j}(t)\rho_B^{\mathrm{Thermal}}\right) \tag{396}$$

$$= \left\langle\widetilde{B_i}(s)\widetilde{B_j}(t)\right\rangle_B \tag{397}$$

$$= \left\langle\widetilde{B_i}(-\tau)\widetilde{B_j}(0)\right\rangle_B \tag{398}$$

$$= \Lambda_{ij}(-\tau) \tag{399}$$

$$\mathrm{Tr}_B\left(\widetilde{B_i}(s)\rho_B^{\mathrm{Thermal}}\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B^{\mathrm{Thermal}}\right) \tag{400}$$

$$= \left\langle\widetilde{B_j}(t)\widetilde{B_i}(s)\right\rangle_B \tag{401}$$

$$= \left\langle\widetilde{B_j}(\tau)\widetilde{B_i}(0)\right\rangle_B \tag{402}$$

$$= \Lambda_{ji}(\tau) \tag{403}$$

$$\mathrm{Tr}_B\left(\rho_B^{\mathrm{Thermal}}\widetilde{B_i}(s)\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\widetilde{B_j}(t)\rho_B^{\mathrm{Thermal}}\right) \tag{404}$$

$$= \left\langle\widetilde{B_i}(s)\widetilde{B_j}(t)\right\rangle_B \tag{405}$$

$$= \left\langle\widetilde{B_i}(-\tau)\widetilde{B_j}(0)\right\rangle_B \tag{406}$$

$$= \Lambda_{ij}(-\tau) \tag{407}$$

The cyclic property of the trace was use widely in the development of equations (393) and (407). Replacing in (384)

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{ij} \Big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)\widetilde{A_i}(t)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)-\Lambda_{ji}(-\tau)\widetilde{A_i}(t)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\big)+C_i(t)C_j(s)\big(\Lambda_{ji}(-\tau)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\widetilde{A_i}(t)-\Lambda_{ij}(\tau)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t)\big)\Big)\mathrm{d}s$$

$$= -\int_0^t \sum_{ij} \left(C_i(t)\, C_j(s) \left(\Lambda_{ij}(\tau)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\right] + \Lambda_{ji}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)\right]\right)\right)\mathrm{d}s$$

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{ij} \left(C_i(t)\, C_j(s) \left(\Lambda_{ij}(\tau)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t) - \Lambda_{ji}(-\tau)\,\widetilde{A_i}(t)\,\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\right) \right. \tag{408}$$

$$\left. +C_i(t)\, C_j(s) \left(\Lambda_{ji}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\,\widetilde{A_i}(t) - \Lambda_{ij}(\tau)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t)\right) \right)\mathrm{d}s \tag{409}$$

$$= -\int_0^t \sum_{ij} \left(C_i(t)\, C_j(s) \left(\Lambda_{ij}(\tau)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\right] + \Lambda_{ji}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)\right]\right)\right)\mathrm{d}s \tag{410}$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}(\tau)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t) - \Lambda_{ij}(\tau)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t) = \Lambda_{ij}(\tau)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\right] \tag{411}$$

$$\Lambda_{ji}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\,\widetilde{A_i}(t) - \Lambda_{ji}(-\tau)\,\widetilde{A_i}(t)\,\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s) = \Lambda_{ji}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j},\widetilde{A_i}(t)\right] \tag{412}$$

Returning to the interaction picture we have:

$$U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)U^\dagger(t) = U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\,U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\,U(t)\,\widetilde{\overline{\rho_S}}(t)U^\dagger(t) \tag{413}$$

$$= \left(U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{\overline{\rho_S}}(t)U^\dagger(t)\right) \tag{414}$$

$$= A_i\widetilde{A_j}(s,t)\,\overline{\rho_S}(t) \tag{415}$$

This procedure applying to the relevant commutators give us:

$$U(t)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\right]U^\dagger(t) = \left(U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)U^\dagger(t) - U(t)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t)\,U^\dagger(t)\right) \tag{416}$$

$$= A_i\widetilde{A_j}(s,t)\,\overline{\rho_S}(t) - \widetilde{A_j}(s,t)\,\overline{\rho_S}(t)A_i \tag{417}$$

$$= \left[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\right] \tag{418}$$

Introducing this transformed commutators in the equation (410) allow us to obtain the master equation of the system

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau \left(C_i(t)\, C_j(t-\tau)\,\Lambda_{ij}(\tau)\left[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\right] \right. \tag{419}$$

$$\left. +C_j(t)\, C_i(t-\tau)\,\Lambda_{ji}(-\tau)\left[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\right]\right) \tag{420}$$

where $i,j \in \{1,2,3,4,5.6\}$.

Here $\widetilde{A_j}(s,t) = U(t)\,U^\dagger(s)\,A_j U(s)\,U^\dagger(t)$ where $U(t)$ is given by (353). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. The environmental correlation functions are given by:

$$\Lambda_{ij}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(s)\,\rho_B^{\mathrm{Thermal}}\right) \tag{421}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)\,\widetilde{B_j}(0)\,\rho_B^{\mathrm{Thermal}}\right) \tag{422}$$

$$\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rangle_B = \text{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B^{\text{Thermal}}\right) \tag{423}$$

$$= \int \mathrm{d}^2\alpha\, P(\alpha)\left\langle\alpha\left|\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right|\alpha\right\rangle \tag{424}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{425}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{426}$$

$$\widetilde{B_{jz}}(\tau) = \sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{427}$$

$$\widetilde{B_{jz}}(0) = \sum_{\mathbf{k}'}\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^*b_{\mathbf{k}'}\right) \tag{428}$$

$$\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rangle_B = \text{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B\right) \tag{429}$$

$$= \text{Tr}_B\Big(\sum_{\mathbf{k}}\big((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\big)\sum_{\mathbf{k}'}\big((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^*b_{\mathbf{k}'}\big)\rho_B\Big) \tag{430}$$

$$= \text{Tr}_B\Big(\sum_{\mathbf{k}\neq\mathbf{k}'}\big((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\big)\big((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^*b_{\mathbf{k}'}\big)\rho_B\Big) \tag{431}$$

$$+\text{Tr}_B\Big(\sum_{\mathbf{k}}\big((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\big)\big((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}\big)\rho_B\Big) \tag{432}$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}} = p_{j\mathbf{k}} \tag{433}$$

$$\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rangle_B = \text{Tr}_B\Big(\sum_{\mathbf{k}\neq\mathbf{k}'}\big(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\big)\big(p_{j\mathbf{k}'}b_{\mathbf{k}'}^{\dagger}+p_{j\mathbf{k}'}^*b_{\mathbf{k}'}\big)\rho_B\Big)+\text{Tr}_B\Big(\sum_{\mathbf{k}}\big(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\big)\big(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^*b_{\mathbf{k}}\big)\rho_B\Big) \tag{434}$$

$$= 0 + \text{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^*b_{\mathbf{k}}\right)\rho_B\right) \tag{435}$$

$$
= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^{*2} b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \rho_B \right) \tag{436}
$$

$$
= \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^{*2} b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{437}
$$

$$
= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \mathrm{Tr}_B \left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) \tag{438}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \mathrm{Tr}_B \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rho_B \right) + e^{-i\omega_{\mathbf{k}}\tau} \mathrm{Tr}_B \left(b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rho_B \right) \right) \tag{439}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{440}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}}| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{441}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{442}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{443}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{444}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{445}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{446}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{447}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| \left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*\right)\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{448}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*\right) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{449}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{450}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{451}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{452}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{453}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{454}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{455}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{456}
$$

$$
1 = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \tag{457}
$$

$$
b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |0\rangle = 0 \tag{458}
$$

$$
b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |0\rangle = |0\rangle \tag{459}
$$

$$
\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{460}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{461}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{462}
$$

$$
+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{463}
$$

$$
= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\left(e^{i\omega_{\mathbf{k}}\tau} + e^{-i\omega_{\mathbf{k}}\tau}\right) \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{464}
$$

$$\frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N} \int_0^{2\pi} \int_0^\infty r^2 \exp\left(-\frac{r^2}{N}\right) r \, \mathrm{d}r \, \mathrm{d}\theta \tag{465}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(2\cos\left(\omega_{\mathbf{k}}\tau\right) N\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{466}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(2\cos\left(\omega_{\mathbf{k}}\tau\right) N + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{467}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{468}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} + \cos\left(\omega_{\mathbf{k}}\tau\right) - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{469}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{\left(2 + e^{\beta\omega_{\mathbf{k}}} - 1\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{470}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{\left(1 + e^{\beta\omega_{\mathbf{k}}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{471}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{472}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right) - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{473}$$

$$= \sum_{\mathbf{k}} |g_{j\mathbf{k}} - v_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right) - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{474}$$

$$\left\langle \widetilde{B_{jz}}\left(\tau\right) \widetilde{B_{j'z}}\left(0\right)\right\rangle_B = \int \mathrm{d}^2\alpha_{\mathbf{k}} P\left(\alpha_{\mathbf{k}}\right) \left\langle \alpha_{\mathbf{k}} \left| \widetilde{B_{jz}}\left(\tau\right) \widetilde{B_{j'z}}\left(0\right) \right| \alpha_{\mathbf{k}}\right\rangle \tag{475}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \left| \widetilde{B_{jz}}\left(\tau\right) \widetilde{B_{j'z}}\left(0\right) \right| \alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{476}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle\alpha_{\mathbf{k}}| \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{477}$$

$$\times \sum_{\mathbf{k}'} \left(\left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right) |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{478}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle\alpha_{\mathbf{k}}| \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{479}$$

$$\times \left(\left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right) |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{480}$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle\alpha_{\mathbf{k}}| \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \left(\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* b_{\mathbf{k}}\right) |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{481}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle\alpha_{\mathbf{k}}| \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \left(\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* b_{\mathbf{k}}\right) |\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{482}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle\alpha_{\mathbf{k}}\left| \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{483}$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle\alpha_{\mathbf{k}}\left| \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{484}$$

$$= \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle\alpha_{\mathbf{k}}\left| b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{485}$$

$$+ \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle\alpha_{\mathbf{k}}\left| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{486}$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle\alpha_{\mathbf{k}}\left| b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0\left| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{487}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \left| D\left(-\alpha_{\mathbf{k}}\right) b_{\mathbf{k}}^\dagger D\left(\alpha_{\mathbf{k}}\right) D\left(-\alpha_{\mathbf{k}}\right) b_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right) \right| 0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{488}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \left| \left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*\right) \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right) \right| 0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{489}$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) |\alpha_{\mathbf{k}}|^2 \, \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{490}$$

$$= N \tag{491}$$

$$\frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \left\langle 0 \left| D\left(-\alpha_{\mathbf{k}}\right) b_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right) D\left(-\alpha_{\mathbf{k}}\right) b_{\mathbf{k}}^\dagger D\left(\alpha_{\mathbf{k}}\right) \right| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{492}$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \left\langle 0 \left| \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}} \right) \left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^* \right) \right| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{493}$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{494}$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{495}$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{496}$$

$$= N + 1 \tag{497}$$

$$\left\langle \widetilde{B_{jz}}\left(\tau\right) \widetilde{B_{j'z}}\left(0\right) \right\rangle_B = \Sigma_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}} \right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}} \right)^* e^{i\omega_{\mathbf{k}}\tau} N + \Sigma_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}} \right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}} \right) e^{-i\omega_{\mathbf{k}}\tau} (N+1) \tag{498}$$

$$= \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}} \right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}} \right) e^{-i\omega_{\mathbf{k}}\tau} \tag{499}$$

$$+ N \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}} \right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}} \right)^* e^{i\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}} \right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}} \right) e^{-i\omega_{\mathbf{k}}\tau} \right) \tag{500}$$

$$D\left(h'\right) D\left(h\right) = \exp \left(\frac{1}{2} \left(h'h^* - h'^*h \right) \right) D\left(h' + h\right) \tag{501}$$

$$\langle D\left(h'\right) D\left(h\right) \rangle_B = \mathrm{Tr}_B \left(\exp \left(\frac{1}{2} \left(h'h^* - h'^*h \right) \right) D\left(h' + h\right) \rho_B^{\mathrm{Thermal}} \right) \tag{502}$$

$$= \exp \left(\frac{1}{2} \left(h'h^* - h'^*h \right) \right) \mathrm{Tr}_B \left(D\left(h' + h\right) \rho_B^{\mathrm{Thermal}} \right) \tag{503}$$

$$= \exp \left(\frac{1}{2} \left(h'h^* - h'^*h \right) \right) \frac{1}{\pi N} \int \mathrm{d}^2 \alpha P\left(\alpha\right) \langle \alpha | D\left(h' + h\right) | \alpha \rangle \tag{504}$$

$$= \exp \left(\frac{1}{2} \left(h'h^* - h'^*h \right) \right) \exp \left(-\frac{|h + h'|^2}{2} \coth \left(\frac{\beta\omega}{2} \right) \right) \tag{505}$$

$$h' = h \exp\left(i\omega\tau\right) \tag{506}$$

$$\langle D\left(h\exp\left(i\omega\tau\right)\right) D\left(h\right) \rangle_B = \exp \left(\frac{1}{2} \left(hh^* \exp\left(i\omega\tau\right) - h^*h \exp\left(-i\omega\tau\right) \right) \right) \exp \left(-\frac{|h + h\exp\left(i\omega\tau\right)|^2}{2} \coth \left(\frac{\beta\omega}{2} \right) \right) \tag{507}$$

$$\frac{1}{2} \left(hh^* \exp\left(i\omega\tau\right) - h^*h \exp\left(-i\omega\tau\right) \right) = \frac{1}{2} |h|^2 \left(\exp\left(i\omega\tau\right) - \exp\left(-i\omega\tau\right) \right) \tag{508}$$

$$= \frac{1}{2} |h|^2 \left(\cos\left(\omega\tau\right) + i\sin\left(\omega\tau\right) - \cos\left(\omega\tau\right) + i\sin\left(\omega\tau\right) \right) \tag{509}$$

$$= \frac{1}{2} |h|^2 \left(2i\sin\left(\omega\tau\right) \right) \tag{510}$$

$$= i |h|^2 \sin\left(\omega\tau\right) \tag{511}$$

$$-\frac{|h + h\exp\left(i\omega\tau\right)|^2}{2} = -|h|^2 \frac{|1 + \exp\left(i\omega\tau\right)|^2}{2} \tag{512}$$

$$= -|h|^2 \frac{|1 + \cos\left(\omega\tau\right) + i\sin\left(\omega\tau\right)|^2}{2} \tag{513}$$

$$= -|h|^2 \frac{\left(1 + \cos\left(\omega\tau\right)\right)^2 + \sin^2\left(\omega\tau\right)}{2} \tag{514}$$

$$= -\left|h\right|^2 \frac{\left(1 + 2\cos\left(\omega\tau\right) + \cos^2\left(\omega\tau\right)\right) + \sin^2\left(\omega\tau\right)}{2} \tag{515}$$

$$= -\left|h\right|^2 \frac{2 + 2\cos\left(\omega\tau\right)}{2} \tag{516}$$

$$= -\left|h\right|^2 \left(1 + \cos\left(\omega\tau\right)\right) \tag{517}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\right\rangle_B = \exp\!\left(\mathrm{i}\left|h\right|^2\sin(\omega\tau)\right)\exp\!\left(-\left|h\right|^2(1+\cos(\omega\tau))\coth\left(\tfrac{\beta\omega}{2}\right)\right) \tag{518}$$

$$= \exp\left(\mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right) - \left|h\right|^2\left(1 + \cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{519}$$

$$= \exp\left(-\left|h\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right) + \cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-\left|h\right|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{520}$$

$$= \left\langle D\left(h\right)\right\rangle_B \exp\left(-\phi\left(\tau\right)\right) \tag{521}$$

$$\exp\left(-\phi\left(\tau\right)\right) = \exp\left(-\left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right)\right) \tag{522}$$

$$\phi\left(\tau\right) = \left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right) \tag{523}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right)\exp\left(-\frac{\left|h + h'\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{524}$$

$$h' = v\exp\left(\mathrm{i}\omega\tau\right) \tag{525}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}\left(\tau\right)\widetilde{B_{1+}B_{0-}}\left(0\right)\rho_B^{\mathrm{Thermal}}\right) \tag{526}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}\left(\tau\right)\widetilde{B_{1+}B_{0-}}\left(0\right)\rho_B^{\mathrm{Thermal}}\right) \tag{527}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{528}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{529}$$

$$= \Pi_{\mathbf{k}}\left(\exp\!\left(\tfrac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{530}$$

$$= \Pi_{\mathbf{k}}\left(\exp\!\left(\tfrac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\Pi_{\mathbf{k}}\left(\exp\!\left(-\left|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\!\left(-\left|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{531}$$

$$= \Pi_{\mathbf{k}}\left(\exp\!\left(\tfrac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\!\left(-\left|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\!\left(-\left|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{532}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \Pi_{\mathbf{k}}\left(\exp\!\left(\tfrac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\!\left(-\left|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\!\left(-\left|\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{533}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{534}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\left(D\!\left(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{535}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\Pi_{\mathbf{k}}D\!\left(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{536}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\Pi_{\mathbf{k}}D\!\left(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{537}$$

$$= \Pi_{\mathbf{k}}\,\mathrm{Tr}_B\left(\left(D\!\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\!\left(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{538}$$

$$= \Pi_{\mathbf{k}}\,\mathrm{Tr}_B\left(\left(D\!\left(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega\tau+\pi)}\right)D\!\left(\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{539}$$

$$= \Pi_{\mathbf{k}}\exp\!\left(-\left|\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\!\left(-\left|\tfrac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{540}$$

$$= \Pi_{\mathbf{k}} \exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \left(\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{541}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{542}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\Pi_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{543}$$

$$= \Pi_{\mathbf{k}}\,\mathrm{Tr}_B\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{544}$$

$$= \Pi_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{545}$$

$$= \left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B \tag{546}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Sigma_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{547}$$

$$\left\langle D(h)b\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|D(h)b|\alpha\rangle \tag{548}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|D(-\alpha)D(h)bD(\alpha)|\alpha\rangle \tag{549}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0|D(-\alpha)D(h)bD(\alpha)|0\rangle \tag{550}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0|D(-\alpha)D(h)D(\alpha)D(-\alpha)bD(\alpha)|0\rangle \tag{551}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0|D(-\alpha)D(h)D(\alpha)(b+\alpha)|0\rangle \tag{552}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)(b+\alpha)|0\rangle \tag{553}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)b|0\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \tag{554}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \tag{555}$$

$$= \frac{1}{\pi N}\int \alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{556}$$

$$= hN\left\langle D(h)\right\rangle_B \tag{557}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|D(h)b^\dagger|\alpha\rangle \tag{558}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0|D(-\alpha)D(h)b^\dagger D(\alpha)|0\rangle \tag{559}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0|D(-\alpha)D(h)b^\dagger D(\alpha)|0\rangle \tag{560}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0|D(-\alpha)D(h)D(\alpha)D(-\alpha)b^\dagger D(\alpha)|0\rangle \tag{561}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0|D(-\alpha)D(h)D(\alpha)\left(b^\dagger+\alpha^*\right)|0\rangle \tag{562}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\left(b^\dagger+\alpha^*\right)|0\rangle \tag{563}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle 0 \left| D(h) b^\dagger \right| 0 \right\rangle \tag{564}$$

$$+ \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle 0 \left| D(h) \alpha^* \right| 0 \right\rangle \tag{565}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle 0 \left| D(h) \right| 1 \right\rangle \tag{566}$$

$$+ \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha^* \left\langle 0 \left| D(h) \right| 0 \right\rangle \tag{567}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle -h | 1 \right\rangle \tag{568}$$

$$+ \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha^* \left\langle 0 \left| D(h) \right| 0 \right\rangle \tag{569}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n| \tag{570}$$

$$\langle -h| 1 \rangle = \exp\left(-\frac{|-h^*|^2}{2}\right) (-h^*) \tag{571}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*) \tag{572}$$

$$+ \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha^* \exp\left(-\frac{|-h^*|^2}{2}\right) \tag{573}$$

$$= -h^* \left\langle D(h) \right\rangle_B (N+1) \tag{574}$$

$$\left\langle b D(h) \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha | b D(h)| \alpha \right\rangle \tag{575}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h \tag{576}$$

$$+ \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha \exp\left(-\frac{|h|^2}{2}\right) \tag{577}$$

$$= h \left\langle D(h) \right\rangle_B (N+1) \tag{578}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| b^\dagger D(h) \right| \alpha \right\rangle \tag{579}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h \tag{580}$$

$$+ \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha)\alpha\exp\left(-\frac{|h|^2}{2}\right) \tag{581}$$

$$= -h^* \left\langle D(h) \right\rangle_B N \tag{582}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau) \right\rangle_B = \left\langle \Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\right\rangle_B \tag{583}$$

$$= \Pi_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\Pi_{\mathbf{k}}\left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)\right\rangle_B \tag{584}$$

$$= \Pi_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\Pi_{\mathbf{k}}\left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)\right\rangle_B \tag{585}$$

$$= \Pi_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\right)\right)\Pi_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{586}$$

$$= B_{10} \tag{587}$$

The correlation functions can be found readily as:

$$\widetilde{B_{1+}B_{0-}}(\tau) = \Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right) \tag{588}$$

$$\widetilde{B_{0+}B_{1-}}(\tau) = \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \tag{589}$$

$$\widetilde{B_x}(0) = \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2} \tag{590}$$

$$\widetilde{B_y}(0) = \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^*}{2i} \tag{591}$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega}{2} \right) \right) \right) \tag{592}$$

$$B_{iz} = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \tag{593}$$

$$\left\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \right\rangle_B \tag{594}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \tag{595}$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \right\rangle_B = \left\langle \frac{B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*}{2} \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2} \right\rangle_B \tag{596}$$

$$= \frac{1}{4} \left\langle (B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*) (B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*) \right\rangle_B \tag{597}$$

$$= \frac{1}{4} \left\langle B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{10} - B_{1+}B_{0-}(\tau) B_{10}^* \right. \tag{598}$$

$$+ B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{10} - B_{0+}B_{1-}(\tau) B_{10}^* \tag{599}$$

$$- B_{10}B_{1+}B_{0-} - B_{10}B_{0+}B_{1-} + B_{10}B_{10} + B_{10}B_{10}^* - B_{10}^*B_{1+}B_{0-} - B_{10}^*B_{0+}B_{1-} + B_{10}^*B_{10} + B_{10}^*B_{10}^* \rangle \tag{600}$$

$$= \frac{1}{4} \left\langle B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{10} - B_{1+}B_{0-}(\tau) B_{10}^* \right. \tag{601}$$

$$\left. + B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{10} - B_{0+}B_{1-}(\tau) B_{10}^* \right\rangle \tag{602}$$

$$\left\langle B_{0+}B_{1-}(\tau) \widetilde{B_{0+}B_{1-}}(0) \right\rangle_B = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \exp \left(-\left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(-\left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{603}$$

$$U = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \tag{604}$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{605}$$

$$S = \prod_{\mathbf{k}} \exp \left(-\left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{606}$$

$$\left\langle B_{0+}B_{1-}(\tau) \widetilde{B_{0+}B_{1-}}(0) \right\rangle_B = U \exp(-\phi(\tau)) S \tag{607}$$

$$\left\langle B_{1+}B_{0-}(\tau) \widetilde{B_{1+}B_{0-}}(0) \right\rangle_B = U^* \exp(-\phi(\tau)) S \tag{608}$$

$$\left\langle B_{1+}B_{0-}(\tau) \widetilde{B_{0+}B_{1-}}(0) \right\rangle_B = \exp(\phi(\tau)) S \tag{609}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau) \widetilde{B_{1+}B_{0-}}(0) \right\rangle_B = \left\langle \widetilde{B_{1+}B_{0-}}(\tau) \widetilde{B_{0+}B_{1-}}(0) \right\rangle_B \tag{610}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau) \right\rangle_B = \prod_{\mathbf{k}} \left(\exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \prod_{\mathbf{k}} \exp \left(-\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{611}$$

$$= U^{*1/2} S^{1/2} \tag{612}$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \right\rangle_B = \frac{1}{4} \left\langle B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{10} - B_{1+}B_{0-}(\tau) B_{10}^* \right. \tag{613}$$

$$\left. + B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{10} - B_{0+}B_{1-}(\tau) B_{10}^* \right\rangle \tag{614}$$

$$= \frac{1}{4} \left(U^* \exp(-\phi(\tau)) S + \exp(\phi(\tau)) S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau)) S + U \exp(-\phi(\tau)) S - B_{10}^{*2} - |B_{10}|^2 \right) \tag{615}$$

$$= \frac{1}{4} \left(2\Re(U) \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2\Re\left(B_{10}^2 \right) - 2|B_{10}|^2 \right) \tag{616}$$

$$= \frac{1}{4} \left(2\Re(U) \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2\Re(U^*) S - 2S \right) \tag{617}$$

$$= \frac{S}{2} \left(\Re(U) \exp(-\phi(\tau)) + \exp(\phi(\tau)) - \Re(U^*) - 1 \right) \tag{618}$$

$$\left\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \right\rangle_B = \left\langle \frac{B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^*}{2i} \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^*}{2i} \right\rangle_B \tag{619}$$

$$= -\frac{1}{4} \left\langle (B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^*) (B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^*) \right\rangle_B \tag{620}$$

$$= -\frac{1}{4} \left\langle B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{10} - B_{0+}B_{1-}(\tau) B_{10}^* \right. \tag{621}$$

$$- B_{1+}B_{0-}(\tau) B_{0+}B_{1-} + B_{1+}B_{0-}(\tau) B_{1+}B_{0-} - B_{1+}B_{0-}(\tau) B_{10} + B_{1+}B_{0-}(\tau) B_{10}^* \tag{622}$$

$$+ B_{10}B_{0+}B_{1-} - B_{10}B_{1+}B_{0-} + B_{10}B_{10} - B_{10}B_{10}^* - B_{10}^*B_{0+}B_{1-} + B_{10}^*B_{1+}B_{0-} - B_{10}^*B_{10} + B_{10}^*B_{10}^* \rangle \tag{623}$$

$$= -\frac{1}{4}\Big\langle B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^* \tag{624}$$

$$-B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{10} + B_{1+}B_{0-}(\tau)\,B_{10}^*\Big\rangle \tag{625}$$

$$= -\frac{1}{4}\Big\langle B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{10}^*B_{10} - B_{10}^*B_{10}^* \tag{626}$$

$$-B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{10}B_{10} + B_{10}B_{10}^*\Big\rangle \tag{627}$$

$$= -\frac{1}{4}\big(U\exp(-\phi(\tau))\,S - \exp(\phi(\tau))\,S - \exp(\phi(\tau))\,S + U^*\exp(-\phi(\tau))\,S \tag{628}$$

$$\big|+2S - 2\Re(U^*)\,S\big) \tag{629}$$

$$= -\frac{S}{4}\big(2\Re(U)\exp(-\phi(\tau)) - 2\exp(\phi(\tau)) + 2 - 2\Re(U)\big) \tag{630}$$

$$= \frac{S}{2}\big(\exp(\phi(\tau)) - \Re(U)\exp(-\phi(\tau)) - 1 + \Re(U)\big) \tag{631}$$

$$\Big\langle \widetilde{B_x}(\tau)\,\widetilde{B_y}(0)\Big\rangle_B = \Big\langle \frac{B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*}{2}\,\frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^*}{2i}\Big\rangle_B \tag{632}$$

$$= \frac{1}{4i}\Big\langle \big(B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*\big)\big(B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^*\big)\Big\rangle_B \tag{633}$$

$$= \frac{1}{4i}\Big\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)\,B_{10} - B_{1+}B_{0-}(\tau)\,B_{10}^* \tag{634}$$

$$+ B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^* \tag{635}$$

$$-B_{10}B_{0+}B_{1-} + B_{10}B_{1+}B_{0-} - B_{10}B_{10} + B_{10}B_{10}^* - B_{10}^*B_{0+}B_{1-} + B_{10}^*B_{1+}B_{0-} - B_{10}^*B_{10} + B_{10}^*B_{10}^*\Big\rangle \tag{636}$$

$$= \frac{1}{4i}\Big\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)\,B_{10} - B_{1+}B_{0-}(\tau)\,B_{10}^* \tag{637}$$

$$+B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^*\Big\rangle \tag{638}$$

$$= \frac{1}{4i}\Big\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{10}B_{10} - B_{10}B_{10}^* \tag{639}$$

$$+B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{10}^*B_{10} - B_{10}^*B_{10}^*\Big\rangle \tag{640}$$

$$= \frac{1}{4i}\Big\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{10}B_{10} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} - B_{10}^*B_{10}^*\Big\rangle \tag{641}$$

$$= \frac{1}{4i}\big(\exp(\phi(\tau))\,S - U^*\exp(-\phi(\tau))\,S + U\exp(-\phi(\tau))\,S - \exp(\phi(\tau))\,S + U^*S - US\big) \tag{642}$$

$$= \frac{1}{4i}\big(-U^*\exp(-\phi(\tau))\,S + U\exp(-\phi(\tau))\,S + U^*S - US\big) \tag{643}$$

$$= \frac{S}{4i}\big(-U^*\exp(-\phi(\tau)) + U\exp(-\phi(\tau)) + U^* - U\big) \tag{644}$$

$$= \frac{S(U - U^*)}{4i}\big(\exp(-\phi(\tau)) - 1\big) \tag{645}$$

$$= \frac{2i\Im(U)\,S}{4i}\big(\exp(-\phi(\tau)) - 1\big) \tag{646}$$

$$= \frac{\Im(U)\,S}{2}\big(\exp(-\phi(\tau)) - 1\big) \tag{647}$$

$$\Big\langle \widetilde{B_y}(\tau)\,\widetilde{B_x}(0)\Big\rangle_B = \Big\langle \frac{B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^*}{2i}\,\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2}\Big\rangle_B \tag{648}$$

$$= \frac{1}{4i}\Big\langle \big(B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^*\big)\big(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*\big)\Big\rangle_B \tag{649}$$

$$= \frac{1}{4i}\Big\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^* \tag{650}$$

$$-B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{10} + B_{1+}B_{0-}(\tau)\,B_{10}^* \tag{651}$$

$$+B_{10}B_{1+}B_{0-} + B_{10}B_{0+}B_{1-} - B_{10}B_{10} - B_{10}B_{10}^* - B_{10}^*B_{1+}B_{0-} - B_{10}^*B_{0+}B_{1-} + B_{10}^*B_{10} + B_{10}^*B_{10}^*\Big\rangle \tag{652}$$

$$= \frac{1}{4i}\Big\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^* \tag{653}$$

$$-B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{10} + B_{1+}B_{0-}(\tau)\,B_{10}^*\Big\rangle \tag{654}$$

$$= \frac{1}{4i}\Big\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{10}^*B_{10} - B_{10}^*B_{10}^* \tag{655}$$

$$-B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{10}B_{10} + B_{10}B_{10}^*\Big\rangle \tag{656}$$

$$= \frac{1}{4i}\Big\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{10}^*B_{10}^* - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{10}B_{10}\Big\rangle \tag{657}$$

$$= \frac{1}{4i}\Big(U\exp(-\phi(\tau))\,S - U^*\exp(-\phi(\tau))\,S + B_{10}^2 - B_{10}^{*2}\Big) \tag{658}$$

$$= \frac{1}{4i}\Big(U\exp(-\phi(\tau))\,S - U^*\exp(-\phi(\tau))\,S + U^*S - US\Big) \tag{659}$$

$$= \frac{S(U - U^*)}{4i}\big(\exp(-\phi(\tau)) - 1\big) \tag{660}$$

$$= \frac{2i\Im(U)\,S}{4i}\big(\exp(-\phi(\tau)) - 1\big) \tag{661}$$

$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'} + 1\right)B_{10} \tag{662}$$

$$\left\langle B_{1+}B_{0-}\left(\tau\right)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right\rangle_{B} = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}\left\langle\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle \tag{663}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} \tag{664}$$

$$\left\langle B_{0+}B_{1-}\left(\tau\right)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'} + 1\right)B_{10}^{*} \tag{665}$$

$$\left\langle B_{0+}B_{1-}\left(\tau\right)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right\rangle_{B} = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^{*} \tag{666}$$

$$\left\langle\widetilde{B_{x}}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_{B} = \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'} + 1\right)B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'} + 1\right)B_{10}^{*}\right. \tag{667}$$

$$\left.+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^{*}\right) \tag{668}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'} + 1\right)B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'} + 1\right)B_{10}^{*}\right. \tag{669}$$

$$\left.+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^{*}\right) \tag{670}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}^{*}\right)\right. \tag{671}$$

$$\left.+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)B_{10}^{*}\right)\right) \tag{672}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}^{*}\right)\right. \tag{673}$$

$$\left.+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)B_{10}^{*}\right)\right) \tag{674}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\left(B_{10} - B_{10}^{*}\right)\right. \tag{675}$$

$$\left.+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)\left(B_{10} - B_{10}^{*}\right)\right) \tag{676}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}2i\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{677}$$

$$= i\sum_{\mathbf{k}'}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{678}$$

$$= i\sum_{\mathbf{k}'}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)e^{i\omega_{\mathbf{k}'}\tau} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}e^{-i\omega_{\mathbf{k}'}\tau}\right) \tag{679}$$

$$\left\langle\widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^{*}}{2}\right)\right\rangle_{B} \tag{680}$$

$$= \sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^{*}}{2}\right)\right\rangle_{B} \tag{681}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^{*}\right)\right\rangle_{B} \tag{682}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-}\right)\right\rangle_{B} \tag{683}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}\right. \tag{684}$$

$$\left.+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}\right\rangle \tag{685}$$

$$\left\langle\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}\right\rangle_{B} = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}\right\rangle_{B} \tag{686}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{i\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{687}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left\langle b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^* v_{0\mathbf{k}'} - v_{1\mathbf{k}'} v_{0\mathbf{k}'}^*}{\omega_{\mathbf{k}'}^2}\right)\right)\right) \right\rangle_B \tag{688}$$

$$\times \left\langle \Pi_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right) \right\rangle_B \tag{689}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right\rangle_B \tag{690}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right\rangle_B \tag{691}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right\rangle_B e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{692}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \Pi_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \Pi_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle_B \left(-\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right\rangle_B N_{\mathbf{k}'}\right) e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{693}$$

$$= -\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle_B N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{694}$$

$$= -\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{695}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+} B_{1-} \right\rangle_B = -\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{10}^* N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{696}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+} B_{0-} \right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \langle b_{\mathbf{k}'} B_{1+} B_{0-} \rangle_B \tag{697}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right) \right\rangle_B \tag{698}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle b_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle_B \tag{699}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \Pi_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1) \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right\rangle_B \left\langle \Pi_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle_B \tag{700}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \Pi_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1) \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right\rangle_B \left\langle \Pi_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle_B \tag{701}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1) B_{10} \tag{702}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+} B_{1-} \right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1) B_{10}^* \tag{703}$$

$$\left\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \right\rangle_B = \frac{1}{2} \sum_{\mathbf{k}'} \left(-\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{10}^* N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right. \tag{704}$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1) B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1) B_{10}^* \right) \tag{705}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* (B_{10}^* - B_{10}) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (N_{\mathbf{k}'}+1)(B_{10} - B_{10}^*) \right) \tag{706}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* (B_{10}^* - B_{10}) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (N_{\mathbf{k}'}+1)(B_{10}^* - B_{10}) \right) \tag{707}$$

$$= \mathrm{i} \sum_{\mathbf{k}'} \Im(B_{10}) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (N_{\mathbf{k}'}+1) \right) \tag{708}$$

$$\left\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \right\rangle_B = \left\langle \left(\frac{B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10} - B_{10}^*}{2\mathrm{i}}\right) \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{709}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle \left(B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10} - B_{10}^* \right) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{710}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle (B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau)) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{711}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle B_{0+} B_{1-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger - B_{1+} B_{0-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger \right. \tag{712}$$

$$\left. + B_{0+} B_{1-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} - B_{1+} B_{0-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right\rangle \tag{713}$$

$$\left\langle B_{0+} B_{1-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger \right\rangle_B = -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{10}^* \tag{714}$$

$$\left\langle B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^*\tag{715}$$

$$\left\langle B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger\right\rangle_B = -(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\tag{716}$$

$$\left\langle B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\tag{717}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \frac{1}{2i}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}^*+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right.\tag{718}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^*-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right)\tag{719}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}^*+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right.\tag{720}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^*-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right)\tag{721}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{10}^*)\right.\tag{722}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(-B_{10}-B_{10}^*)\right)\tag{723}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{10}^*)\ =\right.\tag{724}$$

$$\left.-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(B_{10}+B_{10}^*)\right)\tag{725}$$

$$= \frac{1}{i}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re(B_{10})\ =\right.\tag{726}$$

$$\left.-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re(B_{10})\right)\tag{727}$$

$$= i\sum_{\mathbf{k}'}\Re(B_{10})\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}e^{i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-i\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\right)\tag{728}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^*}{2i}\right)\right\rangle_B\tag{729}$$

$$= \frac{1}{2i}\left\langle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^*)\right\rangle_B\tag{730}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left\langle\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}\right)(B_{0+}B_{1-}-B_{1+}B_{0-})\right\rangle_B\tag{731}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}\right.\tag{732}$$

$$\left.(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-i\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}\right\rangle\tag{733}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle-e^{i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle\right.\tag{734}$$

$$\left.+e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\langle b_{\mathbf{k}'}B_{0+}B_{1-}\rangle-e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\langle b_{\mathbf{k}'}B_{1+}B_{0-}\rangle\right)\tag{735}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle-e^{i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle\right.\tag{736}$$

$$\left.+e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\langle b_{\mathbf{k}'}B_{0+}B_{1-}\rangle-e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\langle b_{\mathbf{k}'}B_{1+}B_{0-}\rangle\right)\tag{737}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle-e^{i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle\right.\tag{738}$$

$$\left.+e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\langle b_{\mathbf{k}'}B_{0+}B_{1-}\rangle-e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\langle b_{\mathbf{k}'}B_{1+}B_{0-}\rangle\right)\tag{739}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle_B = -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}N_{\mathbf{k}'}\tag{740}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle_B = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}^*N_{\mathbf{k}'}\tag{741}$$

$$\langle b_{\mathbf{k}'}B_{1+}B_{0-}\rangle_B = \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{10}\tag{742}$$

$$\langle b_{\mathbf{k}'}B_{0+}B_{1-}\rangle_B = \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{10}^*\tag{743}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = \frac{1}{2i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{*}N_{\mathbf{k}'}\right) - e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}N_{\mathbf{k}'}\right)\right.$$
$$\tag{744}$$

$$\left.+e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^{*}\right) - e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\right)$$
$$\tag{745}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{*}N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}N_{\mathbf{k}'}\right)\right.$$
$$\tag{746}$$

$$\left.+e^{-i\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^{*}\right)\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\right)$$
$$\tag{747}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(\left(-g_{i\mathbf{k}'}+v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{*}N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}N_{\mathbf{k}'}\right)\right.$$
$$\tag{748}$$

$$\left.+e^{-i\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^{*}\right)\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\right)$$
$$\tag{749}$$

$$= \frac{1}{2i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}+B_{10}^{*}\right)N_{\mathbf{k}'}\right.$$
$$\tag{750}$$

$$\left.-e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(B_{10}+B_{10}^{*}\right)\left(N_{\mathbf{k}'}+1\right)\right)$$
$$\tag{751}$$

$$= \frac{1}{i}\sum_{\mathbf{k}'}\left(e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\Re\left(B_{10}\right)N_{\mathbf{k}'}-e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)\right)$$
$$\tag{752}$$

$$= i\sum_{\mathbf{k}'}\left(e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)-e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\Re\left(B_{10}\right)N_{\mathbf{k}'}\right)$$
$$\tag{753}$$

$$= i\sum_{\mathbf{k}'}\left(e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)-e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\Re\left(B_{10}\right)N_{\mathbf{k}'}\right)$$
$$\tag{754}$$

$$= i\Re\left(B_{10}\right)\sum_{\mathbf{k}'}\left(e^{-i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)-e^{i\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}N_{\mathbf{k}'}\right) \tag{755}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right\rangle_{B} = \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-i\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{756}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right) \tag{757}$$

$$\phi\left(\tau\right) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-i\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{758}$$

$$\left\langle \widetilde{B_{x}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \frac{|B_{10}|^{2}}{2}\left(\Re\left(U\right)\exp\left(-\phi\left(\tau\right)\right)+\exp\left(\phi\left(\tau\right)\right)-\Re\left(U\right)-1\right) \tag{759}$$

$$\left\langle \widetilde{B_{y}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = \frac{|B_{10}|^{2}}{2}\left(\exp\left(\phi\left(\tau\right)\right)-\Re\left(U\right)\exp\left(-\phi\left(\tau\right)\right)-1+\Re\left(U\right)\right) \tag{760}$$

$$\left\langle \widetilde{B_{x}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = \frac{\Im\left(U\right)|B_{10}|^{2}}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right) \tag{761}$$

$$\left\langle \widetilde{B_{y}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \frac{\Im\left(U\right)|B_{10}|^{2}}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right) \tag{762}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = i\sum_{\mathbf{k}}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{763}$$

$$\left\langle \widetilde{B_{x}}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_{B} = i\sum_{\mathbf{k}}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}e^{-i\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{764}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = i\Re\left(B_{10}\right)\sum_{\mathbf{k}}\left(e^{-i\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{i\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right) \tag{765}$$

$$\left\langle \widetilde{B_{y}}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_{B} = i\sum_{\mathbf{k}}\Re\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right)e^{-i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}\right) \tag{766}$$

$$\Lambda_{11}(\tau) = \text{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B^{\text{Thermal}}\right) \tag{767}$$

$$= \frac{B(\tau)\,B(0)}{2}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{768}$$

$$\Lambda_{22}(\tau) = \text{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B^{\text{Thermal}}\right) \tag{769}$$

$$= \frac{B(\tau)\,B(0)}{2}\left(e^{\phi(\tau)} + e^{-\phi(\tau)}\right) \tag{770}$$

$$\Lambda_{33}(\tau) = \int_0^\infty d\omega\,J(\omega)\left(1 - F(\omega)\right)^2 G_+(\tau) \tag{771}$$

$$\Lambda_{32}(\tau) = B(\tau)\int_0^\infty d\omega\,\frac{J(\omega)}{\omega}F(\omega)\left(1 - F(\omega)\right)iG_-(\tau) \tag{772}$$

$$\Lambda_{23}(\tau) = -B(0)\int_0^\infty d\omega\,\frac{J(\omega)}{\omega}F(\omega,\tau)\left(1 - F(\omega,\tau)\right)iG_-(\tau) \tag{773}$$

$$\Lambda_{12}(\tau) = \Lambda_{21}(\tau) = \Lambda_{13}(\tau) = \Lambda_{31}(\tau) = 0 \tag{774}$$

With the phonon propagator given by:

$$\phi(\tau) = \int_0^\infty d\omega\,\frac{J(\omega)}{\omega^2}F(\omega)^2 G_+(\tau) \tag{775}$$

defined in terms of $G_\pm(\tau) = (n(\omega) + 1)\,e^{-i\tau\omega} \pm n(\omega)\,e^{-i\tau\omega}$ with $n(\omega) = \left(e^{\beta\omega} - 1\right)^{-1}$ the occupation number.
The eigenvalues of the Hamiltonian $\overline{H_S}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}\left(\overline{H_S}\right)\lambda + \text{Det}\left(\overline{H_S}\right) = 0 \tag{776}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_\pm = \frac{\xi\pm\eta}{2}$ and they satisfy $H_S\,|\pm\rangle = \lambda_\pm\,|\pm\rangle$. Using this notation is possible to write $H_S = \lambda_+\,|+\rangle\langle+| + \lambda_-\,|-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_i}(t)$ may be made explicit using the Fourier decomposition:

$$\widetilde{A_i}(\tau) = e^{i\overline{H_S}\tau}A_i e^{-i\overline{H_S}\tau} \tag{777}$$

$$= \sum_w e^{-iw\tau}A_i(w) \tag{778}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.
In order to use the equation (778) to descompose the equation (353) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\exp\left(-i\int_0^t dt'\overline{H_S}(t')\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-i\int_0^t dt'\overline{H_S}(t')\right) \tag{779}$$

$$= \mathbb{I} + \sum_{n=1}^\infty (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n\,H(t_1)\,H(t_2)...H(t_n) \tag{780}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\exp\left(-i\int_0^t dt'\overline{H_S}(t')\right) \equiv \exp\left(-it H_E(t)\right)$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...\right)\right) \tag{781}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_S}(t')\,\mathrm{d}t' \tag{782}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\left[\overline{H_S}(t'),\overline{H_S}(t'')\right] \tag{783}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\int_0^{t''}\mathrm{d}t'''\left(\left[\left[\overline{H_S}(t'),\overline{H_S}(t'')\right],\overline{H_S}(t''')\right] + \left[\left[\overline{H_S}(t'''),\overline{H_S}(t'')\right],\overline{H_S}(t')\right]\right) \tag{784}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}(t) = e^{\mathrm{i}H_E(t)t}A_i(t)e^{-\mathrm{i}H_E(t)t} \tag{785}$$

$$= \sum_{w(t)} e^{-\mathrm{i}w(t)t}A_i(w(t)) \tag{786}$$

$w(t)$ belongs to the set of differences of eigenvalues that depends of the time. As we can see the eigenvectors are time dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau,t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau,t) = U(t-\tau)U^\dagger(t)A_j(t)U(t)U^\dagger(t-\tau) \tag{787}$$

$$= e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}e^{\mathrm{i}H_E(t)t}A_j(t)e^{-\mathrm{i}H_E(t)t}e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{788}$$

$$= e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}\sum_{w(t)}e^{-\mathrm{i}w(t)t}A_j(w(t))e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{789}$$

$$= \sum_{w(t),w'(t-\tau)}e^{-\mathrm{i}w(t)t}e^{\mathrm{i}w'(t-\tau)}A'_j(w(t),w'(t-\tau)) \tag{790}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $H_S(t-\tau)$ and $H_S(t)$ respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (778) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $V=\{|+\rangle,|-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta\in V}\langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta| \tag{791}$$

Given that $[|+\rangle\langle+|,|-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{\mathrm{i}\overline{H_S}\tau} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{792}$$

$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} \tag{793}$$

$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{794}$$

$$= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-| \tag{795}$$

Calculating the transformation (778) directly using the previous relationship we find that:

$$\widetilde{A_i}\left(\tau\right) = \left(e^{\mathrm{i}\lambda_+\tau}\left|+\right\rangle\left\langle+\right| + e^{\mathrm{i}\lambda_-\tau}\left|-\right\rangle\left\langle-\right|\right)\left(\sum_{\alpha,\beta\in V}\left\langle\alpha\right|A_i\left|\beta\right\rangle\left|\alpha\right\rangle\left\langle\beta\right|\right)\left(e^{-\mathrm{i}\lambda_+\tau}\left|+\right\rangle\left\langle+\right| + e^{-\mathrm{i}\lambda_-\tau}\left|-\right\rangle\left\langle-\right|\right) \tag{796}$$

$$= \left\langle+\right|A_i\left|+\right\rangle\left|+\right\rangle\left\langle+\right| + e^{\mathrm{i}\eta\tau}\left\langle+\right|A_i\left|-\right\rangle\left|+\right\rangle\left\langle-\right| + e^{-\mathrm{i}\eta\tau}\left\langle-\right|A_i\left|+\right\rangle\left|-\right\rangle\left\langle+\right| + \left\langle-\right|A_i\left|-\right\rangle\left|-\right\rangle\left\langle-\right| \tag{797}$$

Here $\eta = \lambda_+ - \lambda-$. Comparing the RHS of the equations (778) and the explicit expression for $\widetilde{A_i}\left(\tau\right)$ and we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$A_i\left(0\right) = \left\langle+\right|A_i\left|+\right\rangle\left|+\right\rangle\left\langle+\right| + \left\langle-\right|A_i\left|-\right\rangle\left|-\right\rangle\left\langle-\right| \tag{798}$$
$$A_i\left(w\right) = \left\langle+\right|A_i\left|-\right\rangle\left|+\right\rangle\left\langle-\right| \tag{799}$$
$$A_i\left(-w\right) = \left\langle-\right|A_i\left|+\right\rangle\left|-\right\rangle\left\langle+\right| \tag{800}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}\left(\tau\right) = \widetilde{A_i}^{\dagger}\left(\tau\right)$ and $\widetilde{B_i}\left(\tau\right) = \widetilde{B_i}^{\dagger}\left(\tau\right)$ we can use the equation (778) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho}_S}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\overline{S}}\left(t\right),\overline{\rho}_S\left(t\right)\right] - \frac{1}{2}\sum_{ij}\sum_{w,w'}\gamma_{ij}\left(w,w',t\right)\left[A_i, A_j\left(w,w'\right)\overline{\rho}_S\left(t\right) - \overline{\rho}_S\left(t\right)A_j^{\dagger}\left(w,w'\right)\right] - \mathrm{i}\sum_{ij}\sum_{w}S_{ij}\left(w,w',t\right)\left[A_i, A_j\left(w,w'\right)\overline{\rho}_S\left(t\right) + \overline{\rho}_S\left(t\right)A_j^{\dagger}\left(w,w'\right)\right] \tag{801}$$

where $A_j^{\dagger}\left(w\right) = A\left(-w\right)$ as expected from the equations (799) and (800). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}\left(w,w',t\right) = 2\Re\left(K_{ij}\left(w,w',t\right)\right)$ and $S_{ij}\left(w,w',t\right) = \Im\left(K_{ij}\left(w,w',t\right)\right)$, respectively, defined in terms of the response functions

$$K_{ij}\left(w,w',t\right) = \int_0^t C_i\left(t\right)C_j\left(t-\tau\right)\Lambda_{ij}\left(\tau\right)e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{802}$$

$$= K_{ijww'}\left(t\right) \tag{803}$$

If we extend the upper limit of integration to ∞ in the equation (802) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{\mathrm{d}\overline{\rho}_{\overline{S}}}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\overline{S}}\left(t\right),\overline{\rho}_{\overline{S}}\left(t\right)\right] - \sum_{ijww'}K_{ijww'}\left(t\right)\left[A_i, A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right) - \overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}\right] \tag{804}$$

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_{\overline{S}}}{\mathrm{d}t}e^{V} = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_{\overline{S}}e^{V}\right)}{\mathrm{d}t} \tag{805}$$

$$= \frac{\mathrm{d}\rho_{\overline{S}}}{\mathrm{d}t} \tag{806}$$

$$= -\mathrm{i}e^{-V}\left[\overline{H}_{\overline{S}}\left(t\right),\overline{\rho}_{\overline{S}}\left(t\right)\right]e^{V} - \sum_{ijww'}K_{ijww'}\left(t\right)e^{-V}\left[A_i, A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right) - \overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}\right]e^{V} \tag{807}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V} = e^{-V}\overline{A\mathbb{I}B}e^{V} \tag{808}$$

$$= e^{-V}\overline{A}e^{V}e^{-V}\overline{B}e^{V} \tag{809}$$

$$= \left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\overline{B}e^{V}\right) \tag{810}$$

$$= AB \tag{811}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V} = e^{-V}\overline{(AB-BA)}e^{V} \tag{812}$$

$$= e^{-V}\overline{AB}e^{V} - e^{-V}\overline{BA}e^{V} \tag{813}$$

$$= AB - BA \tag{814}$$

$$= [A,B] \tag{815}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_{\overline{S}}}{\mathrm{d}t} = -\,\mathrm{i}e^{-V}\left[\overline{H}_{\overline{S}}\left(t\right),\overline{\rho}_{\overline{S}}\left(t\right)\right]e^{V} - \sum_{ijww'}K_{ijww'}\left(t\right)e^{-V}\left[A_{i},A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right)-\overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}\right]e^{V} \tag{816}$$

$$= -\,\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right] - \sum_{ijww'}K_{ijww'}\left(t\right)\left[e^{-V}A_{i}e^{V},e^{-V}A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right)e^{V}-e^{-V}\overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}e^{V}\right] \tag{817}$$

$$= -\,\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right] - \sum_{ijww'}K_{ijww'}\left(t\right)\left[e^{-V}A_{i}e^{V},e^{-V}A_{jww'}e^{V}e^{-V}\overline{\rho}_{\overline{S}}\left(t\right)e^{V}-e^{-V}\overline{\rho}_{\overline{S}}\left(t\right)e^{V}e^{-V}A_{jww'}^{\dagger}e^{V}\right] \tag{818}$$

$$= -\,\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right] - \sum_{ijww'}K_{ijww'}\left(t\right)\left[e^{-V}A_{i}e^{V},e^{-V}A_{jww'}e^{V}\rho_{\overline{S}}\left(t\right)-\rho_{\overline{S}}\left(t\right)e^{-V}A_{jww'}^{\dagger}e^{V}\right] \tag{819}$$

$$= -\,\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right] - \left(\sum_{ijww'}K_{ijww'}\left(t\right)\left(\left[e^{-V}A_{i}e^{V},e^{-V}A_{jww'}e^{V}\rho_{\overline{S}}\left(t\right)\right]-\left[e^{-V}A_{i}e^{V},\rho_{\overline{S}}\left(t\right)e^{-V}A_{jww'}^{\dagger}e^{V}\right]\right)\right) \tag{820}$$

V.   LIMIT CASES

In order to show the plausibility of the master equation (801) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A.   Time-independent variational quantum master equation

At first let's show that the master equation (801) reproduces the results of the reference [1], for the latter case we have that $i,j \in \{1,2,3\}$ and $\omega \in (0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{j}g_{k}\left(b_{k}^{\dagger}+b_{k}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_{x} + \sum_{k}\omega_{k}b_{k}^{\dagger}b_{k} \tag{821}$$

After performing the transformation (24) on the Hamiltonian (821) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_{S}} = \left(\delta + R\right)|1\rangle\langle 1| + \frac{\Omega_{r}}{2}\sigma_{x} \tag{822}$$

$$\overline{H_{I}} = B_{z}|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_{x}\sigma_{x} + B_{y}\sigma_{y}\right) \tag{823}$$

$$H_{B} = \sum_{k}\omega_{k}b_{k}^{\dagger}b_{k} \tag{824}$$

The Hamiltonian (822) differs from the transformed Hamiltonian H_{S} of the reference written like $H_{S} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_{z} + \frac{\Omega_{r}}{2}\sigma_{x}$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\,|1\rangle\langle1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle1| - \frac{\delta}{2}|0\rangle\langle0| \tag{825}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{826}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z \tag{827}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2}$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_S}$.

$$\lambda_+ = \frac{\epsilon + \eta}{2} \tag{828}$$

$$\lambda_- = \frac{\epsilon - \eta}{2} \tag{829}$$

$$|+\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}\begin{pmatrix} \epsilon + \eta \\ \Omega_r \end{pmatrix} \tag{830}$$

$$|-\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}\begin{pmatrix} -\Omega_r \\ \epsilon + \eta \end{pmatrix} \tag{831}$$

Using this basis we can find the decomposition matrices using the equations (799)-(800) and the fact that $|+\rangle = \cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle$ and $|-\rangle = -\sin(\theta)\,|1\rangle + \cos(\theta)\,|0\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\langle +| \sigma_x |+\rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{832}$$

$$= 2\sin(\theta)\cos(\theta) \tag{833}$$

$$= \sin(2\theta) \tag{834}$$

$$\langle -| \sigma_x |-\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{835}$$

$$= -2\sin(\theta)\cos(\theta) \tag{836}$$

$$= -\sin(2\theta) \tag{837}$$

$$\langle -| \sigma_x |+\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{838}$$

$$= \cos^2(\theta) - \sin^2(\theta) \tag{839}$$

$$= \cos(2\theta) \tag{840}$$

$$\langle +| \sigma_y |+\rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{841}$$

$$= i\sin(\theta)\cos(\theta) - i\sin(\theta)\cos(\theta) \tag{842}$$

$$= 0 \tag{843}$$

$$\langle -| \sigma_y |-\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{844}$$

$$= -i\sin(\theta)\cos(\theta) + i\sin(\theta)\cos(\theta) \tag{845}$$

$$= 0 \tag{846}$$

$$\langle -| \sigma_y |+\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{847}$$

$$= i\cos^2(\theta) + i\sin^2(\theta) \tag{848}$$

$$= i \tag{849}$$

$$\langle +| \frac{1+\sigma_z}{2} |+\rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{850}$$

$$= \cos(\theta)\cos(\theta) \tag{851}$$

$$= \cos^2(\theta) \tag{852}$$

$$\langle -| \frac{1+\sigma_z}{2} |-\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{853}$$

$$= \sin(\theta)\sin(\theta) \tag{854}$$

$$= \sin^2(\theta) \tag{855}$$

$$\langle -| \frac{1+\sigma_z}{2} |+\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{856}$$

$$= -\sin(\theta)\cos(\theta) \tag{857}$$

$$= -\sin(\theta)\cos(\theta) \tag{858}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)(|+\rangle\langle +| - |-\rangle\langle -|) \tag{859}$$

$$A_1(\eta) = \cos(2\theta)|-\rangle\langle +| \tag{860}$$

$$A_2(0) = 0 \tag{861}$$

$$A_2(\eta) = i|-\rangle\langle +| \tag{862}$$

$$A_3(0) = \cos^2(\theta)|+\rangle\langle +| + \sin^2(\theta)|-\rangle\langle -| \tag{863}$$

$$A_3(\eta) = -\sin(\theta)\cos(\theta)|-\rangle\langle +| \tag{864}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (422) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \tag{865}$$

Using the notation of the master equation (801), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(767)-(774) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (422) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{866}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{867}$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{868}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)}\right) \tag{869}$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega\, J(\omega)(1 - F(\omega))^2 G_+(\tau) \tag{870}$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty d\omega\, \frac{J(\omega)}{\omega} F(\omega)(1 - F(\omega))\, iG_-(\tau) \tag{871}$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau) \tag{872}$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 \tag{873}$$

Finally taking the Hamiltonian (821) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (334) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{874}$$

$$= \frac{g_k\left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)} \tag{875}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (420) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (801) and using the fact that

$$\widetilde{A_j}(t-\tau,t) = U(\tau) A_j U(-\tau) \tag{876}$$

$$= \sum_w e^{iw\tau} A_i(-w) \tag{877}$$

$$= \sum_w e^{-iw\tau} A_i(w) \tag{878}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t),\overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_{w}\gamma_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)A_j^\dagger(w)\right] \tag{879}$$

$$- \sum_{ij}\sum_{w}S_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)A_j^\dagger(w)\right] \tag{880}$$

where $A_j^\dagger(w) = A(-w)$, as we can see the equation (880) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2\Re(K_{ij}(w,t))$ and $S_{ij}(w,t) = \Im(K_{ij}(w,t))$, respectively, defined in terms of the response functions

$$K_{ij}(w,t) = \int_0^t \Lambda'_{ij}(\tau)e^{\mathrm{i}w\tau}\mathrm{d}\tau \tag{881}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (109) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_k \omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{882}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (882) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (223) and (??) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t) \tag{883}$$

$$\overline{H_I} = \frac{\Omega(t)}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{884}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (767)-(774) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}G_+(\tau)\,\mathrm{d}\omega \tag{885}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (885) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega \tag{886}$$

Writing the interaction Hamiltonian (884) in the similar way to the equation (??) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (223) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$ is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t) \tag{887}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B\right) \tag{888}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{889}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B\right) \tag{890}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) \tag{891}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(420) is:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i(t)\,C_i(t-\tau)\,\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\,\rho_S(t)\right]\right. \tag{892}$$

$$\left. + C_i(t)\,C_i(t-\tau)\,\Lambda_{ii}(-\tau)\left[\rho_S(t)\,\widetilde{A_i}(t-\tau,t), A_i\right]\right) \tag{893}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (888) and (891) on the equation (893) we obtain that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\,\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\,\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{894}$$

$$\left. + \left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\,\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\,\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\,\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right) \tag{895}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\,\rho_S(t)\right]^\dagger = \left[\rho_S(t)\,\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (422) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right) \tag{896}$$

Using the equation (801), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)), \rho_S(t)\right] - \frac{1}{2}\sum_w \gamma_{33}(w,t)\left[A_3, A_3(w)\,\rho_S(t) - \rho_S(t)\,A_3^\dagger(w)\right] \tag{897}$$

$$- \sum_w S_{33}(w,t)\left[A_3, A_3(w)\,\rho_S(t) + \rho_S(t)\,A_3^\dagger(w)\right]\right) \tag{898}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty d\omega\, J(\omega)\, G_+(\tau) \tag{899}$$

$$\Lambda_{33}(-\tau) = \int_0^\infty d\omega\, J(\omega)\, G_+(-\tau) \tag{900}$$

In our case $A_3 = \frac{\mathbb{I} + \sigma_z}{2}$, the equation (898) can be transformed in

$$\frac{d\rho_S}{dt} = -i\left[H_S(t), \rho_S(t)\right] - \sum_w \left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^*(w,t)\left[\rho_S(t)A_3(w), A_3\right]\right) \tag{901}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$, this Hamiltonian have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2} \tag{902}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right) \tag{903}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2} \tag{904}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(-|1\rangle + |0\rangle\right) \tag{905}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{906}$$

$$= \frac{1}{2} \tag{907}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{908}$$

$$= \frac{1}{2} \tag{909}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{910}$$

$$= -\frac{1}{2} \tag{911}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \tag{912}$$

$$= \frac{\mathbb{I}}{2} \tag{913}$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle +| \tag{914}$$

$$= \frac{1}{4}\left(\sigma_z + i\sigma_y\right) \tag{915}$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle -| \tag{916}$$

$$= \frac{1}{4}\left(\sigma_z - i\sigma_y\right) \tag{917}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{918}$$

$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] \tag{919}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right)G_+\left(\tau\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{920}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)e^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{921}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)e^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{922}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)e^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{923}$$

$$= \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)\pi\delta\left(\widetilde{\Omega}-\omega\right)\mathrm{d}\omega \tag{924}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right) \tag{925}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right)G_+\left(\tau\right)e^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{926}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)e^{-\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)e^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{927}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)e^{-\mathrm{i}\widetilde{\Omega}\tau}n\left(\omega\right)e^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{928}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)n\left(\omega\right)e^{-\mathrm{i}\widetilde{\Omega}\tau + \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{929}$$

$$= \int_0^\infty J\left(\omega\right)n\left(\omega\right)\pi\delta\left(-\widetilde{\Omega}+\omega\right)\mathrm{d}\omega \tag{930}$$

$$= \pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right) \tag{931}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm\mathrm{i}\varepsilon s} = \pi\delta\left(\varepsilon\right)\pm\mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3\left(0\right)$ because the spectral density $J\left(\omega\right)$ is equal to zero when $\omega = 0$. Replacing in the equation (918) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{932}$$

$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S\left(t\right)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right) \tag{933}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega\left(\mathrm{t}\right)}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\Omega\left(t\right)\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right)$$

$$\tag{934}$$

$$- \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right) \tag{935}$$

VI.   TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{936}$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|, \tag{937}$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right), \tag{938}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{939}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A.   Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right) \right) \tag{940}$$

At first let's obtain e^V under the transformation (940), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{941}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi} \right)^2}{2!} + \ldots \tag{942}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + \ldots \tag{943}$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \ldots \right) \tag{944}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|e^{\hat{\varphi}} \tag{945}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+ \tag{946}$$

Given that $\left[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} \tag{947}$$

$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{948}$$

$$= B_\pm \tag{949}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-$ because this form imposes that $e^{-V}e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{950}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (936):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \tag{951}$$

$$= |0\rangle\langle 0|, \tag{952}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \tag{953}$$

$$= |m\rangle\langle m| B_+ |m\rangle\langle n| n\rangle\langle n| B_-, \tag{954}$$

$$= |m\rangle\langle n|, \; m \neq 0, \; n \neq 0, \tag{955}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \tag{956}$$

$$= |0\rangle\langle m| B_- \; m \neq 0, \tag{957}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{958}$$

$$= |0\rangle\langle m| B_+ \; m \neq 0, \tag{959}$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{960}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B_+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_- \tag{961}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_+ b_{\mathbf{k}}^\dagger B_- \right) \left(B_+ b_{\mathbf{k}} B_- \right) \tag{962}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{963}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{964}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) \tag{965}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{966}$$

The transformed Hamiltonians of the equations (937) to (939) written in terms of (951) to (966) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{967}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{968}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \overline{\sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0| \right)} + \overline{\sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n|} \tag{969}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,\overline{|0\rangle\langle n|} + V_{n0}(t)\,\overline{|n\rangle\langle 0|} \right) + \sum_{m.n\neq 0} V_{mn}(t)\,\overline{|m\rangle\langle n|} \tag{970}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,B_-\,|0\rangle\langle n| + V_{n0}(t)\,B_+\,|n\rangle\langle 0| \right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{971}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n|B_- + V_{n0}(t)\,|n\rangle\langle 0|B_+ \right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{972}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+ \right) \left(\left(\sum_{n=0} \mu_n(t)\,|n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_- \right) \tag{973}$$

$$= \left(\mu_0(t)\,|0\rangle\langle 0| + \sum_{n=1} \mu_n(t)\,|n\rangle\langle n|B_+ \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_- \right) \tag{974}$$

$$= \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \sum_{n=1} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B_+ \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) B_- \tag{975}$$

$$= \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \sum_{n=1} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{976}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n|\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{977}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n|B_- + V_{n0}(t)\,|n\rangle\langle 0|B_+ \right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{978}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n|\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{979}$$

$$+ \sum_{n=0} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) - \sum_{n=1} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{980}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n|B_- + V_{n0}(t)\,|n\rangle\langle 0|B_+ \right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{981}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)\,g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{982}$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{983}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)\,g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{984}$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)\,g_{\mathbf{k}} \right) \tag{985}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{986}$$

Using the previous functions we have that (983) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n\left(t\right) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}\left(t\right) |0\rangle\langle n|B_- + V_{n0}\left(t\right) |n\rangle\langle 0|B_+\right) + \sum_{m.n \neq 0} V_{mn}\left(t\right) |m\rangle\langle n| \tag{987}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0\left(t\right) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{988}$$

Now in order to separate the elements of the hamiltonian (988) let's follow the references of the equations (??) and (223) to separate the hamiltonian like:

$$\overline{H_S\left(t\right)} = \sum_{n=0} \varepsilon_n\left(t\right) |n\rangle\langle n| + B \sum_{n=1} \left(V_{0n}\left(t\right) |0\rangle\langle n| + V_{n0}\left(t\right) |n\rangle\langle 0|\right) + \sum_{m.n \neq 0} V_{mn}\left(t\right) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \tag{989}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0\left(t\right) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \left(V_{0n}\left(t\right) |0\rangle\langle n| \left(B_- - B\right) + V_{n0}\left(t\right) |n\rangle\langle 0| \left(B_+ - B\right)\right), \tag{990}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{991}$$

Here B is given by (??) The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{992}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{993}$$

$$B_x = \frac{B_+ + B_- - 2B}{2} \tag{994}$$

$$B_y = \frac{B_- - B_+}{2\mathrm{i}} \tag{995}$$

Using this set of hermitian operators to write the Hamiltonians (937)-(939)

$$\overline{H_S(t)} = \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{m,n\neq 0}V_{mn}(t)\,|m\rangle\langle n| \tag{996}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)\,|m\rangle\langle n| + V_{nm}(t)\,|n\rangle\langle m|) \tag{997}$$

$$+ \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{998}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))\,|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))\,|n\rangle\langle m|) + \varepsilon_0(t)\,|0\rangle\langle 0| \tag{999}$$

$$+ B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{1000}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1001}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{1002}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1003}$$

$$+ \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)\,|n\rangle\langle n| \tag{1004}$$

$$\overline{H_I(t)} = \sum_{n=1}B_{z,n}\,|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n|\,(B_- - B) + V_{n0}(t)\,|n\rangle\langle 0|\,(B_+ - B)) \tag{1005}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))(B_- - B)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))(B_+ - B)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1006}$$

$$+ \sum_{n=1}B_{z,n}\,|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1007}$$

$$= \sum_{n=1}B_{z,n}\,|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}((B_- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + (B_+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))))\right. \tag{1008}$$

$$\left.+ \frac{\mathrm{i}\sigma_{0n,y}}{2}((B_+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - (B_- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))))\right) \tag{1009}$$

$$+ \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1010}$$

$$= \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}(B_+ + B_- - 2B)\Re(V_{0n}(t)) + \mathrm{i}(B_- - B - B_+ + B)\Im(V_{0n}(t))\right. \tag{1011}$$

$$\left.+ \frac{\mathrm{i}\sigma_{0n,y}}{2}((B_+ - B - B_- + B)\Re(V_{0n}(t)) + \mathrm{i}(B - B_- + B - B_+)\Im(V_{0n}(t)))\right) + \sum_{n=1}B_{z,n}\,|n\rangle\langle n| \tag{1012}$$

$$= \sum_{n=1}B_{z,n}\,|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1013}$$

$$+ \sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1014}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right) + \langle\overline{H_I}\rangle_{H_S+H_B} + O\left(\langle\overline{H_I^2}\rangle_{H_S+H_B}\right). \tag{1015}$$

Taking the equations (242)-(250) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (250) and using the chain rule we obtain that:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1016}$$

$$= 0 \tag{1017}$$

Let's recall the equations (984) and (986), we can write them in terms of the variational parameters

$$B = \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1018}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1019}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1020}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1021}$$

Introducing this derivates in the equation (1016) give us:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1022}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1023}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1024}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1025}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1026}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\,(0) = |0\rangle\langle0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho\,(0)\,e^{-V} = \left(|0\rangle\langle0| + \sum_{n=1} |n\rangle\langle n|B_+\right)(|0\rangle\langle0| \otimes \rho_B)\left(|0\rangle\langle0| + \sum_{n=1} |n\rangle\langle n|B_-\right) \tag{1027}$$

$$0 = |0\rangle\langle0| \otimes \rho_B \tag{1028}$$

$$0 = \rho\,(0) \tag{1029}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t)OU(t) \tag{1030}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\overline{H_S}\,(t')\right). \tag{1031}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\overline{\rho_S}(t)U(t),\ \text{where} \tag{1032}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\,(\bar{\rho}(t)) \tag{1033}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I\,(t)} = B_{z,0}|0\rangle\langle0| + \sum_{n=1}\left(\Re\,(V_{0n}\,(t))\,B_x\sigma_{0n,x} + \Re\,(V_{0n}\,(t))\,B_y\sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right) \tag{1034}$$

$$+\Im\,(V_{0n}\,(t))\,B_x\sigma_{0n,y} - \Im\,(V_{0n}\,(t))\,B_y\sigma_{0n,x}) \tag{1035}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}}\mu_0\,(t)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1036}$$

$$B_{z,n} = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n\,(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\ \text{if}\,n \neq 0 \tag{1037}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1038}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1039}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1040}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1041}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1042}$$
$$B_{1n}(t) = B_x \tag{1043}$$
$$B_{2n}(t) = B_y \tag{1044}$$
$$B_{3n}(t) = B_{z,n} \tag{1045}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1046}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1047}$$
$$C_{10}(t) = 0 \tag{1048}$$
$$C_{20}(t) = 0 \tag{1049}$$
$$C_{40}(t) = 0 \tag{1050}$$
$$C_{50}(t) = 0 \tag{1051}$$
$$C_{30}(t) = 1 \tag{1052}$$
$$C_{1n}(t) = \Re(V_{0n}(t)) \tag{1053}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1054}$$
$$C_{3n}(t) = 1 \tag{1055}$$
$$C_{4n}(t) = \Im(V_{0n}(t)) \tag{1056}$$
$$C_{5n}(t) = -\Im(V_{0n}(t)) \tag{1057}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J} \sum_{n=1} C_{jn}(t)(A_{jn} \otimes B_{jn}(t)) \tag{1058}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1030) as:

$$\widetilde{H_I}(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1059}$$

$$\widetilde{A_i}(t) = U^\dagger(t) A_i U(t) \tag{1060}$$

$$\widetilde{B_i}(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \tag{1061}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] ds \tag{1062}$$

Replacing the equation (1059)in (1062)we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H}_I(t),\left[\widetilde{H}_I(s),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1063}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1064}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1065}$$

$$\left.-\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1066}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \right. \tag{1067}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1068}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1069}$$

$$\left.+\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1070}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle\widetilde{B_{jn}}(t)(t)\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1071}$$

$$= \left\langle\widetilde{B_{jn}}(\tau)\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1072}$$

Here $s\to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1070) are:

$$\text{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1073}$$

$$= \left\langle \widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1074}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1075}$$

$$\text{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\,\widetilde{B_{j'n'}}(s)\right) = \text{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1076}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1077}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1078}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1079}$$

$$\text{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\,\widetilde{B_{jn}}(t)\right) = \text{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1080}$$

$$= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1081}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1082}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1083}$$

$$\text{Tr}_B\left(\rho_B\,\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) = \text{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1084}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1085}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1086}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1087}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1063)and (1070) and using the equations (1073)-(1087) we can re-write:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\rho_S}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\rho_S}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1088}$$

$$\left. + C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\rho_S}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\rho_S}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1089}$$

$$= -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t),\,\widetilde{A_{j'n'}}(s)\,\widetilde{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\widetilde{\rho_S}(t)\,\widetilde{A_{j'n'}}(s),\,\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1090}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t),\,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t),\,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t),\,\overline{\rho_S}(t)\right] \tag{1091}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^\dagger(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^\dagger(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (420) as expected.

VII.   TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1092}$$

$$H_S(t) = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1093}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right), \tag{1094}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \tag{1095}$$

A.   Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right) \tag{1096}$$

At first let's obtain $e^{\pm V}$ under the transformation (1096), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right)$, so the equation (1096) can be written as $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \tag{1097}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n| \hat{\varphi}_n \right)^2}{2!} + \dots \tag{1098}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \tag{1099}$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \tag{1100}$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots \right) \tag{1101}$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \tag{1102}$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^\dagger - f_{nu'\mathbf{k}'}^* b_{u'\mathbf{k}'} \right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D(\pm\alpha_{nu\mathbf{k}}) = e^{\pm \left(\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - \alpha_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right)} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right)} \tag{1103}$$

$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right)} \right) \tag{1104}$$

$$= \prod_u \left(\prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}) \right) \tag{1105}$$

$$= \prod_{u\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}) \tag{1106}$$

$$= \prod_u B_{nu\pm} \tag{1107}$$

$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}) \tag{1108}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \tag{1109}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1092):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \tag{1110}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1111}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1112}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1113}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1114}$$

$$= |0\rangle\langle 0|. \tag{1115}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \tag{1116}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1117}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1118}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-} \right), \; m \neq n, \tag{1119}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}} \right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}} \right) \right), \tag{1120}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} \right) D\left(-\alpha_{nu\mathbf{k}} \right) \right), \tag{1121}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}} \right) \exp\left(\frac{1}{2} \left(-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}} \right) \right) \right). \tag{1122}$$

$$\prod_u \left(B_{mu+} B_{nu-} \right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}} \right) \exp\left(\frac{1}{2} \left(-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}} \right) \right) \right). \tag{1123}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \tag{1124}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + ... \right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \right) \left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + ... \right), \tag{1125}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + ..., \tag{1126}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + ... \right) \prod_u B_{0u-} \tag{1127}$$

$$+ |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + ... \right) \prod_u B_{1u-} + ... \tag{1128}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + ... \right) \tag{1129}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + ... \right) + ... \tag{1130}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + ... \right) \tag{1131}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + ... \right) + ... \tag{1132}$$

The transformed Hamiltonians of the equations (1093) to (1095) written in terms of (1110) to (1135) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{1138}$$

$$= \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \overline{\sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{1139}$$

$$= \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) \tag{1140}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1141}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{0u\mathbf{k}}b_{u\mathbf{k}}\right) + ...\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1142}$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{0u\mathbf{k}}b_{u\mathbf{k}}\right) \prod_u B_{0u-} \tag{1143}$$

$$+ \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{1u\mathbf{k}}b_{u\mathbf{k}}\right) \prod_u B_{1u-} + ... \tag{1144}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \prod_u B_{0u+}b^\dagger_{u\mathbf{k}} \prod_u B_{0u-} + g^*_{0u\mathbf{k}} \prod_u B_{0u+}b_{u\mathbf{k}} \prod_u B_{0u-}\right) \tag{1145}$$

$$+ \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \prod_u B_{1u+}b^\dagger_{u\mathbf{k}} \prod_u B_{1u-} + g^*_{1u\mathbf{k}} \prod_u B_{1u+}b_{u\mathbf{k}} \prod_u B_{1u-}\right) + ... \tag{1146}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{0u\mathbf{k}} \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1147}$$

$$+ \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{1u\mathbf{k}} \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + ... \tag{1148}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{nu\mathbf{k}} \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1149}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}}b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1150}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b^\dagger_{u\mathbf{k}}b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)\right) \tag{1151}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b^\dagger_{u\mathbf{k}}b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}\right. \tag{1152}$$

$$\left. - \left(v_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)\right) + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}}b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1153}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1154}$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}})\,b^\dagger_{u\mathbf{k}} + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^*\,b_{u\mathbf{k}}\right) \tag{1155}$$

Using the previous functions we have that (1152) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \tag{1156}$$

$$+ \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \tag{1157}$$

Now in order to separate the elements of the hamiltonian (1157) let's follow the references of the equations (223) and (??) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \prod_u (B_{mu+}B_{nu-}) \right\rangle_{\overline{H_0}} = \left\langle \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1158}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}})\right)\right)\left\langle \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1159}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1160}$$

$$\equiv B_{nm} \tag{1161}$$

$$\left\langle \prod_u (B_{nu+}B_{mu-}) \right\rangle_{\overline{H_0}} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{nu\mathbf{k}}^* v_{mu\mathbf{k}} - v_{nu\mathbf{k}} v_{mu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1162}$$

$$= B_{nm}^* \tag{1163}$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+}B_{nu-}) - B_{nm} \tag{1164}$$

As we can see:

$$J_{nm}^\dagger = \left(\prod_u (B_{mu+}B_{nu-}) - B_{nm}\right)^\dagger \tag{1165}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{nm}^* \tag{1166}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{mn} \tag{1167}$$

$$= J_{mn} \tag{1168}$$

We can separate the Hamiltonian (1157) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}}(t) = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \tag{1169}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \tag{1170}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \tag{1171}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_{\overline{S}}(t)}+H_B)}\right)\right) + \langle\overline{H_I}\rangle_{\overline{H_{\overline{S}}(t)}+H_B} + O\left(\langle\overline{H_I^2}\rangle_{\overline{H_{\overline{S}}(t)}+H_B}\right). \tag{1172}$$

Taking the equations (242)-(250) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}(t)}}\right) = C\left(R_0, R_1, R_2, ..., R_{d-1}, B_{01}, B_{02}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = \Re\left(v_{nu\mathbf{k}}\right) + i\Im\left(v_{nu\mathbf{k}}\right)$. So our approach will be based on the derivation respect to $\Re\left(v_{nu\mathbf{k}}\right)$ and $\Im\left(v_{nu\mathbf{k}}\right)$. The Hamiltonian $\overline{H_{\overline{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \right) |n\rangle\langle n| \tag{1173}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \tag{1174}$$

$$\prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1175}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}} v_{nu\mathbf{k}}^* + g_{nu\mathbf{k}}^* v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1176}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \tag{1177}$$

$$\prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1178}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*)\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1179}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \tag{1180}$$

$$\prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1181}$$

$$v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^* = (\Re(v_{mu\mathbf{k}}) - \mathrm{i}\Im(v_{mu\mathbf{k}}))(\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})) - (\Re(v_{mu\mathbf{k}}) + \mathrm{i}\Im(v_{mu\mathbf{k}}))(\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}})) \tag{1182}$$

$$= (\Re(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) - \mathrm{i}\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) + \Im(v_{mu\mathbf{k}})\Im(v_{nu\mathbf{k}})) \tag{1183}$$

$$- (\Re(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) + \mathrm{i}\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) + \Im(v_{mu\mathbf{k}})\Im(v_{nu\mathbf{k}})) \tag{1184}$$

$$= 2\mathrm{i}(\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) - \Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}})) \tag{1185}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*)\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1186}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}(\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) - \Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}))}{\omega_{u\mathbf{k}}^2} \right) \right) \tag{1187}$$

$$\prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1188}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1189}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}} v_{mu\mathbf{k}}^* + v_{nu\mathbf{k}}^* v_{mu\mathbf{k}}) \tag{1190}$$

$$= (\Re(v_{mu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}))^2 + (\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2 \tag{1191}$$

$$- ((\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}}))(\Re(v_{mu\mathbf{k}}) - \mathrm{i}\Im(v_{mu\mathbf{k}})) + (\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}}))(\Re(v_{mu\mathbf{k}}) + \mathrm{i}\Im(v_{mu\mathbf{k}}))) \tag{1192}$$

$$= (\Re(v_{mu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}))^2 + (\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2$$
$$- 2(\Re(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) + \Im(v_{nu\mathbf{k}})\Im(v_{mu\mathbf{k}})) \tag{1193}$$

$$= (\Re(v_{mu\mathbf{k}}) - \Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}) - \Im(v_{nu\mathbf{k}}))^2 \tag{1194}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{1195}$$

$$= \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2 - (g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*)\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \tag{1196}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \frac{\partial}{\partial \Re\left(v_{nu\mathbf{k}}\right)} \sum_{u\mathbf{k}} \left(\frac{\left(\Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{nu\mathbf{k}}\right)\right)^2 - 2\Re\left(g_{nu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} \right) \tag{1201}$$

$$= \frac{2\Re\left(v_{nu\mathbf{k}}\right) - 2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1202}$$

$$= 2\frac{\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1203}$$

$$\frac{\partial R_{n'}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \frac{\partial}{\partial \Im\left(v_{nu\mathbf{k}}\right)} \sum_{u\mathbf{k}} \left(\frac{\left(\Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{nu\mathbf{k}}\right)\right)^2 - 2\Re\left(g_{nu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} \right) \tag{1204}$$

$$= \frac{2\Im\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1205}$$

$$= 2\frac{\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1206}$$

Given that:

$$\ln B_{mn} = \ln \left(\left(\prod_{u\mathbf{k}} \exp \left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \right) \tag{1207}$$

$$\prod_{u} \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \right) \tag{1208}$$

$$= \sum_{u\mathbf{k}} \ln \exp \left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1209}$$

$$+ \sum_{u} \ln \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \tag{1210}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1211}$$

$$+ \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \tag{1212}$$

$$\frac{\partial \ln B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) - \left(\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1213}$$

$$\frac{\partial \ln B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) - \left(\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1214}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}}\frac{\partial B_{mn}}{\partial a} \tag{1215}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn}\frac{\partial \ln B_{mn}}{\partial a} \tag{1216}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} \tag{1217}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = B_{mn}\frac{\partial \ln B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} \tag{1218}$$

$$= B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) - \left(\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1219}$$

$$= B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1220}$$

$$\frac{\partial B_{nm}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \left(\frac{\partial B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)}\right)^\dagger \tag{1221}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)^\dagger \tag{1222}$$

$$= B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1223}$$

$$\frac{\partial B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = B_{mn}\frac{\partial \ln B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} \tag{1224}$$

$$= B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) - \left(\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1225}$$

$$= B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1226}$$

$$\frac{\partial B_{nm}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \left(\frac{\partial B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)}\right)^\dagger \tag{1227}$$

$$= \left(B_{mn}\right)^\dagger \tag{1228}$$

$$= B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1229}$$

Introducing this derivates in the equation (1201) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1230}$$

$$+ \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1231}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1232}$$

$$= 0 \tag{1233}$$

We can obtain the variational parameters:

$$- 2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Re\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\Re\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\Re\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1234}$$

$$= -\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1235}$$

$$\Re\left(v_{nu\mathbf{k}}\right) = \frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2\Re(g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im(v_{mu\mathbf{k}})+\Re(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im(v_{mu\mathbf{k}})+\Re(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1236}$$

$$= \frac{2\Re\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1237}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1238}$$

$$+ \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)-\left(\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1239}$$

$$+ \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)-\left(\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1240}$$

$$= 0 \tag{1241}$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Im(v_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\Im(v_{nu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\Im(v_{nu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1242}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Im(g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1243}$$

$$\Im(v_{nu\mathbf{k}})=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Im(g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1244}$$

$$=\frac{2\Im(g_{nu\mathbf{k}})\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1245}$$

$$v_{nu\mathbf{k}}=\Re(v_{nu\mathbf{k}})+\mathrm{i}\Im(v_{nu\mathbf{k}}) \tag{1246}$$

$$=\frac{2\Re(g_{nu\mathbf{k}})\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im(v_{mu\mathbf{k}})+\Re(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im(v_{mu\mathbf{k}})+\Re(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1247}$$

$$\mathrm{i}\frac{2\Im(g_{nu\mathbf{k}})\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1248}$$

$$=\frac{2\Re(g_{nu\mathbf{k}})\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}\Im(g_{nu\mathbf{k}})\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1249}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im(v_{mu\mathbf{k}})+\Re(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im(v_{mu\mathbf{k}})+\Re(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1250}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1251}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1252}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1253}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1254}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle0|\otimes\rho_B$, as we can see this state is indepen-dent of the variation transformation:

$$e^V\rho(0)e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)(|0\rangle\langle0|\otimes\rho_B)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1255}$$

$$0=(B_{0+}|0\rangle\langle0|B_{0-})\otimes\rho_B \tag{1256}$$

$$0=\rho(0) \tag{1257}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t)OU(t) \tag{1258}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1259}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\overline{\rho_S}(t)U(t), \text{ where} \tag{1260}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1261}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1262}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1263}$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1264}$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1265}$$

We can proof that $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+}B_{n-} - B_m B_n)^\dagger \tag{1266}$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \tag{1267}$$

$$= B_{n+}B_{m-} - B_n B_m \tag{1268}$$

$$= B_{nm} \tag{1269}$$

So we can say that the set of matrices (1262) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1262) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|B_{nm} + \sum_n B_{z,n}(t)|n\rangle\langle n|, \tag{1270}$$

$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)|n\rangle\langle m|B_{nm} + V_{mn}(t)|m\rangle\langle n|B_{mn}\right) \tag{1271}$$

$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}\Im\left(V_{nm}(t)\right)B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1272}$$

$$\left. + \Re\left(V_{nm}(t)\right)B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}\Im\left(V_{nm}(t)\right)B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1273}$$

$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1274}$$

$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1275}$$

$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1276}$$

$$\left. + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,x}\right) \tag{1277}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 \mid 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\right\} \tag{1278}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \tag{1279}$$

$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \tag{1280}$$

$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1281}$$

$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1282}$$

$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1283}$$

$$B_{1,nm}(t) = B_{nm,x} \tag{1284}$$

$$B_{2,nm}(t) = B_{nm,y} \tag{1285}$$

$$B_{3,nm}(t) = B_{z,n}(t) \tag{1286}$$

$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1287}$$

$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1288}$$

$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1289}$$

$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1290}$$

$$C_{3,nm}(t) = 1 \tag{1291}$$

$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1292}$$

$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1293}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t)\left(A_{jp} \otimes B_{jp}(t)\right) \tag{1294}$$

Here $J = \{1, 2, 3, 4, 5\}$ and P the set defined in (1278).

We write the interaction Hamiltonian transformed under (1258) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)\right) \tag{1295}$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \tag{1296}$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t)(t) e^{-iH_B t} \tag{1297}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] ds \tag{1298}$$

Replacing the equation (1295) in (1298) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1299}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1300}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1301}$$

$$\left.-\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1302}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1303}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1304}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1305}$$

$$\left.+\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1306}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1307}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1308}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1306) are:

$$\mathrm{Tr}_B \left(\widetilde{B_{jp}}(t) \, \widetilde{B_{j'p'}}(s) \, \rho_B \right) = \left\langle \widetilde{B_{jp}}(t) \, \widetilde{B_{j'p'}}(s) \right\rangle_B \tag{1309}$$

$$= \left\langle \widetilde{B_{jp}}(0) \, \widetilde{B_{j'p'}}(0) \right\rangle_B \tag{1310}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1311}$$

$$\mathrm{Tr}_B \left(\widetilde{B_{jp}}(t) \, \rho_B \, \widetilde{B_{j'p'}}(s) \right) = \mathrm{Tr}_B \left(\widetilde{B_{j'p'}}(s) \, \widetilde{B_{jp}}(t) \, \rho_B \right) \tag{1312}$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \, \widetilde{B_{jp}}(t) \right\rangle_B \tag{1313}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \, \widetilde{B_{jp}}(0) \right\rangle_B \tag{1314}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1315}$$

$$\mathrm{Tr}_B \left(\widetilde{B_{j'p'}}(s) \, \rho_B \, \widetilde{B_{jp}}(t) \right) = \mathrm{Tr}_B \left(\widetilde{B_{jp}}(t) \, \widetilde{B_{j'p'}}(s) \, \rho_B \right) \tag{1316}$$

$$= \left\langle \widetilde{B_{jp}}(t) \, \widetilde{B_{j'p'}}(s) \right\rangle_B \tag{1317}$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \, \widetilde{B_{j'p'}}(0) \right\rangle_B \tag{1318}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1319}$$

$$\mathrm{Tr}_B \left(\rho_B \, \widetilde{B_{j'p'}}(s) \, \widetilde{B_{jp}}(t) \right) = \mathrm{Tr}_B \left(\widetilde{B_{j'p'}}(s) \, \widetilde{B_{jp}}(t) \, \rho_B \right) \tag{1320}$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \, \widetilde{B_{jp}}(t) \right\rangle_B \tag{1321}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \, \widetilde{B_{jp}}(0) \right\rangle_B \tag{1322}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1323}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1299) and (1306) and using the equations (1309)-(1323) we can re-write:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t) \, C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \, \widetilde{A_{jp}}(t) \, \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau) \, \widetilde{A_{jp}}(t) \, \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \right) \right. \tag{1324}$$

$$\left. + C_{jp}(t) \, C_{j'p'}(s) \left(\Lambda_{j'p'jp}(-\tau) \, \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \, \widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau) \, \widetilde{A_{j'p'}}(s) \, \widetilde{\rho_S}(t) \widetilde{A_{jp}}(t) \right) \right) \mathrm{d}s \tag{1325}$$

$$= -\int_0^t \sum_{jj'pp'} \left(C_{jp}(t) \, C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \left[\widetilde{A_{jp}}(t) , \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) , \widetilde{A_{jp}}(t) \right] \right) \right) \tag{1326}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'} \left(C_{jp}(t) \, C_{j'p'}(t-\tau) \left(\Lambda_{jpj'p'}(\tau) \left[A_{jp}(t) , A_{j'p'}(t-\tau,t) \overline{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\overline{\rho_S}(t) A_{j'p'}(t-\tau,t) , A_{jp}(t) \right] \right) \right) \mathrm{d}\tau - \mathrm{i} \left[H_S(t) , \overline{\rho_S}(t) \right] \tag{1327}$$

For this case we used that $A_{jp}(t-\tau,t) = U(t) \, U^\dagger(t-\tau) \, A_{jp}(t) \, U(t-\tau) \, U^\dagger(t)$. This is a non-Markovian equation.

VIII. BIBLIOGRAPHY

[1] McCutcheon D P S, Dattani N S, Gauger E M, Lovett B W and Nazir A 2011 Phys. Rev. B 84 081305
[2] Dara P S McCutcheon and Ahsan Nazir 2010 New J. Phys. 12 113042
[3] Supplement: Theoretical model of phonon induced dephasing. A.J. Ramsay ey al 2009.
[4] Felix A Pollock et al 2013 New J. Phys. 15 075018

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* n.dattani@cfa.harvard.edu
† edcchaparroso@unal.edu.co