A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

Nike Dattani*
Harvard-Smithsonian Center for Astrophysics

Camilo Chaparro Sogamoso†
National University of Colombia

I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right) = H_S\left(t\right) + H_I + H_B, \tag{1}$$
$$H_S\left(t\right) = \varepsilon_0\left(t\right)|0\rangle\langle 0| + \varepsilon_1\left(t\right)|1\rangle\langle 1| + V_{10}\left(t\right)|1\rangle\langle 0| + V_{01}\left(t\right)|0\rangle\langle 1|, \tag{2}$$
$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H\left(t\right)$, the unitary transformation defined by $e^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V\left(t\right) \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}\left(t\right)$ defined as:

$$v_{i\mathbf{k}}\left(t\right) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}\left(t\right). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i\left(t\right) \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$
$$V\left(t\right) = |0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V(t)}$ using the notation (6) will give us the following result:

$$e^{\pm V(t)} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t))} \tag{10}$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)))^2}{2!} + \ldots \tag{11}$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2(t)}{2!} + \ldots \tag{12}$$

$$= |0\rangle\langle 0| \left(\mathbb{I} \pm \hat{\varphi}_0(t) + \frac{\hat{\varphi}_0^2(t)}{2!} \pm \ldots\right) + |1\rangle\langle 1| \left(\mathbb{I} \pm \hat{\varphi}_1(t) + \frac{\hat{\varphi}_1^2(t)}{2!} \pm \ldots\right) \tag{13}$$

$$= |0\rangle\langle 0| e^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1| e^{\pm\hat{\varphi}_1(t)} \tag{14}$$

$$= |0\rangle\langle 0| e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1| e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*(t)b_{\mathbf{k}}\right)} \tag{15}$$

$$= |0\rangle\langle 0| B_0^{\pm}(t) + |1\rangle\langle 1| B_1^{\pm}(t), \tag{16}$$

$$B_i^{\pm}(t) \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{17}$$

Let's recall the Zassenhaus formula:

$$e^{r(X+Y)} = e^{rX}\, e^{rY}\, e^{-\frac{r^2}{2}[X,Y]}\, e^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\cdots. \tag{18}$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i, j we can show making $r = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} = e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\cdots \tag{19}$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}0}\cdots \tag{20}$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. \tag{21}$$

By induction of this result we can write an expresion of $B_i^{\pm}(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$D(\pm v_{i\mathbf{k}}(t)) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, \tag{22}$$

$$B_i^{\pm}(t) = \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). \tag{23}$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$\overline{O(t)} \equiv \mathrm{e}^{V(t)} O(t)\, \mathrm{e}^{-V(t)}. \tag{24}$$

We will use the following identities:

$$
\begin{aligned}
\overline{|0\rangle\langle 0|\left(t\right)} &= \mathrm{e}^{V(t)}|0\rangle\langle 0|\mathrm{e}^{-V(t)} \tag{25}\\
&= \left(|0\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{26}\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\left(t\right)\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{27}\\
&= |0\rangle\langle 0|B_0^+\left(t\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{28}\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+\left(t\right)B_0^-\left(t\right) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+\left(t\right)B_1^-\left(t\right) \tag{29}\\
&= |0\rangle\langle 0|, \tag{30}\\
\overline{|1\rangle\langle 1|\left(t\right)} &= \left(|0\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{31}\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+\left(t\right) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\left(t\right)\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{32}\\
&= |1\rangle\langle 1|B_1^+\left(t\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{33}\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_1^+\left(t\right)B_0^-\left(t\right) + B_1^+\left(t\right)|1\rangle\langle 1|1\rangle\langle 1|B_1^-\left(t\right) \tag{34}\\
&= B_1^+\left(t\right)|1\rangle\langle 1|1\rangle\langle 1|B_1^-\left(t\right) \tag{35}\\
&= |1\rangle\langle 1|, \tag{36}\\
\overline{|0\rangle\langle 1|\left(t\right)} &= \mathrm{e}^{V(t)}|0\rangle\langle 1|\mathrm{e}^{-V(t)} \tag{37}\\
&= \left(|0\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{38}\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{39}\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\left(t\right) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\left(t\right)\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{40}\\
&= |0\rangle\langle 1|B_0^+\left(t\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{41}\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_0^+\left(t\right)B_0^-\left(t\right) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+\left(t\right)B_1^-\left(t\right) \tag{42}\\
&= |0\rangle\langle 1|B_0^+\left(t\right)B_1^-\left(t\right), \tag{43}\\
\overline{|1\rangle\langle 0|\left(t\right)} &= \mathrm{e}^{V(t)}|1\rangle\langle 0|\mathrm{e}^{-V(t)} \tag{44}\\
&= \left(|0\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{45}\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{46}\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\left(t\right)\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{47}\\
&= |1\rangle\langle 0|B_1^+\left(t\right)\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{48}\\
&= |1\rangle\langle 0|0\rangle\langle 0|B_1^+\left(t\right)B_0^-\left(t\right) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+\left(t\right)B_1^-\left(t\right) \tag{49}\\
&= |1\rangle\langle 0|B_1^+\left(t\right)B_0^-\left(t\right), \tag{50}\\
\overline{b_{\mathbf{k}}\left(t\right)} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}\mathrm{e}^{-V(t)} \tag{51}\\
&= \left(|0\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{52}\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| \tag{53}\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,b_{\mathbf{k}}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,b_{\mathbf{k}}B_0^-(t) + |1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}B_1^-(t) \tag{54}\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{55}\\
&= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{56}\\
&= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{57}\\
\overline{b_{\mathbf{k}}\left(t\right)}^{\dagger} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V(t)} \tag{58}\\
&= \left(|0\rangle\langle 0|B_0^+\left(t\right) + |1\rangle\langle 1|B_1^+\left(t\right)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^-\left(t\right) + |1\rangle\langle 1|B_1^-\left(t\right)\right) \tag{59}\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle 1| \tag{60}\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) \tag{61}\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right) \tag{62}\\
&= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}. \tag{63}
\end{aligned}
$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^+(t)\, b_{\mathbf{k}} B_i^-(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \tag{64}$$

$$B_i^+(t)\, b_{\mathbf{k}}^{\dagger} B_i^-(t) = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. \tag{65}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|\,(t)} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{66}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|\,(t)} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{67}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|\,(t)} = V_{10}(t)\,|1\rangle\langle 0|B_1^+(t)\,B_0^-(t), \tag{68}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|\,(t)} = V_{01}(t)\,|0\rangle\langle 1|B_0^+(t)\,B_1^-(t), \tag{69}$$

$$\overline{\left(g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^*b_{\mathbf{k}}\right)(t)} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)\right)+g_{i\mathbf{k}}^*\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{70}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^*\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^*b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{72}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^*b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{73}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^*b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^*b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) \tag{74}$$

$$= |0\rangle\langle 0|B_0^+(t)\,|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^*b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^-(t) \tag{75}$$

$$= |0\rangle\langle 0|B_0^+(t)\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^*b_{\mathbf{k}}\right)B_0^-(t) \tag{76}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)+g_{0\mathbf{k}}^*\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{77}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^*b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^*b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) \tag{78}$$

$$= |1\rangle\langle 1|B_1^+(t)\,|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^*b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^-(t) \tag{79}$$

$$= |1\rangle\langle 1|B_1^+(t)\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1\mathbf{k}}^*b_{\mathbf{k}}\right)B_1^-(t) \tag{80}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)+g_{1\mathbf{k}}^*\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{81}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\,(t)} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) \tag{82}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_0^-(t)+|1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_1^-(t)\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\Bigg(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{85}$$

$$+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\Bigg) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left|\tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left|\tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{90}$$

$$= \omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{91}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \omega_{\mathbf{k}}\left(|1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{92}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} - v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} - v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right) \tag{93}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right). \tag{94}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} \tag{95}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0|B_1^{+}(t)\,B_0^{-}(t) + V_{01}(t)\,|0\rangle\langle 1|B_0^{+}(t)\,B_1^{-}(t)\,, \tag{96}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} \tag{97}$$

$$= \overline{\sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} + \overline{\sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} \tag{98}$$

$$= \sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) + \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{99}$$

$$= \sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right) - \sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) - \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{100}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} \tag{101}$$

$$= \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right)\right) \tag{102}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{\mathbf{k}}\left(|1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right)\right). \tag{103}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \textstyle\sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^{+}(t)B_{j'}^{-}(t) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{j\mathbf{k}}|j\rangle\langle j|\left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^{*}b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^{*}\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right). \tag{104}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H\left(t\right)}=\overline{H_{\bar{S}}}+\overline{H_{\bar{I}}}+\overline{H_{\bar{B}}}. \tag{105}$$

Let's define:

$$R_{i}\left(t\right)\equiv\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right), \tag{106}$$

$$B_{iz}\left(t\right)\equiv\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right), \tag{107}$$

$$\chi_{ij}\left(t\right)\equiv\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}\left(t\right)v_{j\mathbf{k}}\left(t\right)-v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}^{2}}\right). \tag{108}$$

$\chi_{ij}\left(t\right)$ is an imaginary number so $e^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}\left(t\right)$ as we'll will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix}B_{iz}\left(t\right) & B_{i\pm}\left(t\right)\\ B_{x}\left(t\right) & B_{ij}\left(t\right)\\ B_{y}\left(t\right) & R_{i}\left(t\right)\end{pmatrix}\equiv\begin{pmatrix}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right) & e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\\ \frac{B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{10}^{*}(t)}{2} & e^{\chi_{ij}(t)}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\\ \frac{B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{10}^{*}(t)}{2\mathrm{i}} & \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\end{pmatrix}. \tag{109}$$

$$(\cdot)^{\Re}\equiv\Re(\cdot), \tag{110}$$

$$(\cdot)^{\Im}\equiv\Im(\cdot). \tag{111}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta=\frac{1}{k_{\mathrm{B}}T}$, considering the stationary bath state as reference written in the following way:

$$\rho_{B}=\frac{e^{-\beta H_{B}}}{\mathrm{Tr}\left(e^{-\beta H_{B}}\right)}. \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^{\dagger}=\begin{pmatrix}0 & 0 & 0 & \ldots & 0 & \ldots\\ \sqrt{1} & 0 & 0 & \ldots & 0 & \ldots\\ 0 & \sqrt{2} & 0 & \ldots & 0 & \ldots\\ 0 & 0 & \sqrt{3} & \ldots & 0 & \ldots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \ldots\\ 0 & 0 & 0 & \ldots & \sqrt{n} & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}, \tag{113}$$

$$b=\begin{pmatrix}0 & \sqrt{1} & 0 & 0 & \ldots & 0 & \ldots\\ 0 & 0 & \sqrt{2} & 0 & \ldots & 0 & \ldots\\ 0 & 0 & 0 & \sqrt{3} & \ldots & 0 & \ldots\\ 0 & 0 & 0 & 0 & \ddots & \vdots & \ldots\\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \ldots\\ 0 & 0 & 0 & 0 & \ldots & 0 & \ddots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}. \tag{114}$$

So the product of the matrix representation of $b^\dagger$ and b with $-\beta$ is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right| , \tag{116}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\!\langle j|, |i\rangle\!\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|, \tag{117}$$

$$\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|. \tag{118}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|\right) \tag{119}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{120}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{121}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{122}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), \tag{123}$$

$$\mathrm{Tr}\left(\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|\right) \tag{124}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|\right) \tag{125}$$

$$= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{126}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{127}$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{128}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{129}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}} - 1\rangle, \tag{130}$$
$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} \mid j_{\mathbf{k}} + 1\rangle. \tag{131}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (130)-(131):

$$\begin{aligned}
\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Tr}\left(\rho_B B_{iz}(t)\right) = \mathrm{Tr}\left(B_{iz}(t)\,\rho_B\right) && (132)\\
&= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right)\right)\rho_B\right) && (133)\\
&= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\rho_B\right) && (134)\\
&= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) && (135)\\
&= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) && (136)\\
&= \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), && (137)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}^{\dagger}\text{)} && (138)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) && (139)\\
&= 0, && (140)\\
\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} && (141)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) && (142)\\
&= 0. && (143)
\end{aligned}$$

we therefore find that:

$$\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = 0. \tag{144}$$

Another important expected value is $B(t) = \langle B^{\pm}(t)\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm}(t) = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$
\begin{aligned}
\left\langle B^{\pm}\right\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B^{\pm}\right) = \mathrm{Tr}\left(B^{\pm}\rho_B\right) && (145)\\
&= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) && (146)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\right)\rho_B\right) && (147)\\
&= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\right)\right\rangle. && (148)
\end{aligned}
$$

Given that we can write a density operator as:

$$
\rho = \int P\left(\alpha\right)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \tag{149}
$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right)\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P\left(\alpha\right)$ is given by:

$$
\begin{aligned}
\langle A\rangle =& \mathrm{Tr}\left(A\rho\right) && (150)\\
=& \int P\left(\alpha\right)\langle\alpha\left|A\right|\alpha\rangle\mathrm{d}^2\alpha. && (151)
\end{aligned}
$$

We are typically interested in thermal state density operators, for which it can be shown that $P\left(\alpha\right) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega}-1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_BT$.

Using the integral representation (151) we could obtain that the expected value for the displacement operator $D\left(h\right)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle\alpha|D(h)|\alpha\rangle \mathrm{d}^2\alpha \tag{152}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(-\alpha)D(h)D(\alpha)|0\rangle \mathrm{d}^2\alpha, \tag{153}$$

$$D(h)D(\alpha) = D(h+\alpha)e^{\frac{1}{2}(h\alpha^*-h^*\alpha)}, \tag{154}$$

$$D(-\alpha)(D(h)D(\alpha)) = D(-\alpha)D(h+\alpha)e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} \tag{155}$$

$$= D(h)e^{\frac{1}{2}(-\alpha(h+\alpha)^*+\alpha^*(h+\alpha))}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} \tag{156}$$

$$= D(\alpha)e^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^* h+|\alpha|^2\right)}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} \tag{157}$$

$$= D(\alpha)e^{(h\alpha^*-h^*\alpha)}, \tag{158}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(h)\exp(h\alpha^*-h^*\alpha)|0\rangle \mathrm{d}^2\alpha \tag{159}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|0\rangle \mathrm{d}^2\alpha \tag{160}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|h\rangle \mathrm{d}^2\alpha, \tag{161}$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{162}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha \tag{163}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{164}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha\right)\mathrm{d}^2\alpha, \tag{165}$$

$$\alpha = x + \mathrm{i}y, \tag{166}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{N}+h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)\right)\mathrm{d}x\mathrm{d}y \tag{167}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{N}+hx-h^*x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y\right)\mathrm{d}y, \tag{168}$$

$$-\frac{x^2}{N}+hx-h^*x = -\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) \tag{169}$$

$$= -\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2+\frac{N(h^*-h)^2}{4}, \tag{170}$$

$$-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y = -\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) \tag{171}$$

$$= -\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)-\frac{N(h+h^*)^2}{4}, \tag{172}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, \tag{173}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x, \tag{174}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(x+\frac{(Nh^*-Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}y \tag{175}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{176}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right) \tag{177}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}\right) \tag{178}$$

$$= \exp\left(-|h|^2\left(N+\frac{1}{2}\right)\right) \tag{179}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)\right) \tag{180}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)\right) \tag{181}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{182}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (147) using (182) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{183}$$

We will now force $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B^+\sigma^+$ and $B^-\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right.\right. \tag{184}$$

$$= \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'}(t) + \sum_j |j\rangle\langle j|B_{jz}(t) + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'| \left(B_j^+(t)B_{j'}^-(t) - B_{jj'}(t) \right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{185}$$

$$\equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{186}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_i^+(t)B_j^-(t)\rangle = B_{ij}(t) \tag{187}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right\rangle \tag{188}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \tag{189}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle \tag{190}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \tag{191}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \tag{192}$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}. \tag{193}$$

From the definition $B_{01}(t) = \langle B_0^+(t)B_1^-(t)\rangle$ using the displacement operator we have:

$$\langle B_0^+(t)B_1^-(t)\rangle = B_{01}(t) \tag{194}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right\rangle \tag{195}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \tag{196}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle \tag{197}$$

$$= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \right) \tag{198}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \right) \tag{199}$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}. \tag{200}$$

We can check:

$$\begin{aligned}
\left\langle B_0^+(t)\,B_1^-(t)\right\rangle &= B_{01}(t) && (201)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} && (202)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)^*} && (203)\\
&= \left\langle B_1^+(t)\,B_0^-(t)\right\rangle^* && (204)\\
&= B_{10}^*(t)\,. && (205)
\end{aligned}$$

The parts of the splitted Hamiltonian are:

$$\begin{aligned}
\overline{H_{\bar{S}}(t)} &\equiv (\varepsilon_0(t)+R_0(t))\,|0\rangle\!\langle 0|+(\varepsilon_1(t)+R_1(t))\,|1\rangle\!\langle 1|+V_{10}(t)\,B_{10}\sigma^+ + V_{01}(t)\,B_{01}\sigma^-, && (206)\\
\overline{H_{\bar{I}}} &\equiv V_{10}(t)\left(B_1^+(t)\,B_0^-(t)-B_{10}(t)\right)\sigma^+ + V_{01}(t)\left(B_0^+(t)\,B_1^-(t)-B_{01}(t)\right)\sigma^- + |0\rangle\!\langle 0|B_{0z}(t)+|1\rangle\!\langle 1|B_{1z}(t)\,, && (207)\\
\overline{H_{\bar{B}}} &\equiv \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (208)\\
&= H_B. && (209)
\end{aligned}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (207) we can verify the condition $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}}=0$ in the following way:

$$\begin{aligned}
\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} &= \left\langle\sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)|n\rangle\!\langle n|+\sum_{j\neq j'}V_{jj'}(t)\,|j\rangle\!\langle j'|\left(B_j^+(t)\,B_{j'}^-(t)-B_{jj'}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (210)\\
&= \left\langle\sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)|n\rangle\!\langle n|\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\sum_{j\neq j'}V_{jj'}(t)\,|j\rangle\!\langle j'|\left(B_j^+(t)\,B_{j'}^-(t)-B_{jj'}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (211)\\
&= \sum_{n\mathbf{k}}\left(\left\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))^*b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\!\langle n|+\sum_{j\neq j'}|j\rangle\!\langle j'|\left(\left\langle V_{jj'}(t)\,B_j^+(t)B_{j'}^-(t)\right\rangle_{\overline{H_{\bar{B}}}}-\langle V_{jj'}(t)\,B_{jj'}(t)\rangle_{\overline{H_{\bar{B}}}}\right) && (212)\\
&= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\!\langle n|+\sum_{j\neq j'}|j\rangle\!\langle j'|V_{jj'}(t)\left(\left\langle B_j^+(t)\,B_{j'}^-(t)\right\rangle_{\overline{H_{\bar{B}}}}-\langle B_{jj'}(t)\rangle_{\overline{H_{\bar{B}}}}\right) && (213)\\
&= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t))^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\!\langle n|+\sum_{j\neq j'}|j\rangle\!\langle j'|V_{jj'}(t)\left(B_{jj'}(t)-B_{jj'}(t)\right) && (214)\\
&= 0. && (215)
\end{aligned}$$

We used (144) and (193) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_x(t) = B_x^\dagger(t) \tag{216}$$

$$= \frac{B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}, \tag{217}$$

$$B_y(t) = B_y^\dagger(t) \tag{218}$$

$$= \frac{B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, \tag{219}$$

$$B_{iz}(t) = B_{iz}^\dagger(t) \tag{220}$$

$$= \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}} \right). \tag{221}$$

Writing the equations (206) and (207) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}(t) = \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right)|j\rangle\langle j| + V_{10}\left(t\right)B_{10}\left(t\right)\sigma^{+} + V_{01}\left(t\right)B_{01}\left(t\right)\sigma^{-} \tag{222}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right)|j\rangle\langle j| + V_{10}\left(t\right)B_{10}\left(t\right)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}\left(t\right)B_{01}\left(t\right)\frac{\sigma_x - \mathrm{i}\sigma_y}{2} \tag{223}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right)|j\rangle\langle j| + V_{10}\left(t\right)\left(B^{\Re}_{10}\left(t\right) + \mathrm{i}B^{\Im}_{10}\left(t\right)\right)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\left(B^{\Re}_{10}\left(t\right) - \mathrm{i}B^{\Im}_{10}\left(t\right)\right)\frac{\sigma_x - \mathrm{i}\sigma_y}{2} \tag{224}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right)|j\rangle\langle j| + B^{\Re}_{10}(t)\left(V_{10}(t)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) + \mathrm{i}B^{\Im}_{10}(t)\left(V_{10}(t)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} - V_{01}(t)\frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) \tag{225}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j\left(t\right)\right)|j\rangle\langle j| + B^{\Re}_{10}(t)\left(\sigma_x\frac{V_{10}(t) + V_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t) - V_{01}(t)}{2}\right) + \mathrm{i}B^{\Im}_{10}(t)\left(\sigma_x\frac{V_{10}(t) - V_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t) + V_{01}(t)}{2}\right) \tag{226}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j\left(t\right)\right)|j\rangle\langle j| + B^{\Re}_{10}(t)\left(\sigma_x V^{\Re}_{10}\left(t\right) - \sigma_y V^{\Im}_{10}\left(t\right)\right) + \mathrm{i}B^{\Im}_{10}(t)\left(\mathrm{i}\sigma_x V^{\Im}_{10}(t) + \mathrm{i}\sigma_y V^{\Re}_{10}(t)\right) \tag{227}$$

$$= \left(\varepsilon_0(t) + R_0\left(t\right)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1\left(t\right)\right)|1\rangle\langle 1| + B^{\Re}_{10}(t)\left(\sigma_x V^{\Re}_{10}\left(t\right) - \sigma_y V^{\Im}_{10}\left(t\right)\right) + \mathrm{i}B^{\Im}_{10}(t)\left(\mathrm{i}\sigma_x V^{\Im}_{10}(t) + \mathrm{i}\sigma_y V^{\Re}_{10}(t)\right) \tag{228}$$

$$= \left(\varepsilon_0(t) + R_0\left(t\right)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1\left(t\right)\right)|1\rangle\langle 1| + \left(\sigma_x B^{\Re}_{10}\left(t\right)V^{\Re}_{10}\left(t\right) - \sigma_y B^{\Re}_{10}\left(t\right)V^{\Im}_{10}\left(t\right)\right) - \left(\sigma_x B^{\Im}_{10}\left(t\right)V^{\Im}_{10}\left(t\right) + \sigma_y B^{\Im}_{10}\left(t\right)V^{\Re}_{10}\left(t\right)\right) \tag{229}$$

$$= \left(\varepsilon_0\left(t\right) + R_0\left(t\right)\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right) + R_1\left(t\right)\right)|1\rangle\langle 1| + \sigma_x\left(B^{\Re}_{10}\left(t\right)V^{\Re}_{10}\left(t\right) - B^{\Im}_{10}\left(t\right)V^{\Im}_{10}\left(t\right)\right) - \sigma_y\left(B^{\Re}_{10}\left(t\right)V^{\Im}_{10}\left(t\right) + B^{\Im}_{10}\left(t\right)V^{\Re}_{10}\left(t\right)\right) \tag{230}$$

$$= \left(\varepsilon_0(t) + R_0\left(t\right)\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right) + R_1\left(t\right)\right)|1\rangle\langle 1| + \sigma_x\left(B^{\Re}_{10}\left(t\right)V^{\Re}_{10}\left(t\right) - B^{\Im}_{10}\left(t\right)V^{\Im}_{10}\left(t\right)\right) - \sigma_y\left(B^{\Re}_{10}\left(t\right)V^{\Im}_{10}\left(t\right) + B^{\Im}_{10}\left(t\right)V^{\Re}_{10}\left(t\right)\right). \tag{231}$$

$$\overline{H_{\bar{I}}} = V_{10}\left(t\right)\left(\sigma^{+}B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - \sigma^{+}B_{10}\left(t\right)\right) + V_{01}\left(t\right)\left(\sigma^{-}B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - \sigma^{-}B_{01}\left(t\right)\right) + |0\rangle\langle 0|B_{0z}\left(t\right) + |1\rangle\langle 1|B_{1z}\left(t\right) \tag{232}$$

$$= |0\rangle\langle 0|B_{0z}\left(t\right) + |1\rangle\langle 1|B_{1z}\left(t\right) + \left(V^{\Re}_{10}\left(t\right) + \mathrm{i}V^{\Im}_{10}\left(t\right)\right)\left(\sigma^{+}B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - \sigma^{+}B_{10}\left(t\right)\right) + \left(V^{\Re}_{10}\left(t\right) - \mathrm{i}V^{\Im}_{10}\left(t\right)\right)\left(\sigma^{-}B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - \sigma^{-}B_{01}\left(t\right)\right) \tag{233}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}\left(t\right)\left(\sigma^{+}B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - \sigma^{+}B_{10}\left(t\right) + \sigma^{-}B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - \sigma^{-}B_{01}\left(t\right)\right) + \mathrm{i}V^{\Im}_{10}\left(t\right)\left(\sigma^{+}B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - \sigma^{+}B_{10}\left(t\right) - \sigma^{-}B \right. \tag{234}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}\left(t\right)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{235}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}\left(t\right)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{236}$$

$$+ \mathrm{i}V^{\Im}_{10}\left(t\right)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}\left(t\right) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{237}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}\left(t\right)\left(\sigma_x\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) + B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2} + \mathrm{i}\sigma_y\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - B_{10}\left(t\right.}{2}\right. \tag{238}$$

$$+ \mathrm{i}V^{\Im}_{10}\left(t\right)\left(\sigma_x\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2} + \mathrm{i}\sigma_y\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) + B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{239}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V^{\Im}_{10}(t)\left(\mathrm{i}\sigma_x\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2} - \sigma_y\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) + B^{+}_{0}}{}\right. \tag{240}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V^{\Im}_{10}(t)\left(\mathrm{i}^2\sigma_x\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) - B^{+}_{0}\left(t\right)B^{-}_{1}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2\mathrm{i}} - \sigma_y\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) + B^{+}_{0}}{}\right. \tag{241}$$

$$= \sum B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V^{\Im}_{10}(t)\left(\mathrm{i}^2\sigma_x\frac{B^{+}_{1}B^{-}_{0}\left(t\right) - B^{+}_{0}B^{-}_{1}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2\mathrm{i}} - \sigma_y\frac{B^{+}_{1}\left(t\right)B^{-}_{0}\left(t\right) + B^{+}_{0}\left(t\right)B_{1}}{2}\right.$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{245}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}(t)\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}(t)\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{246}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}(t), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{247}$$

Then using the fact that $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{248}$$

So Eq. (246) becomes:

$$\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}(t)} &= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (249)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (250)\\
&= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (251)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (252)\\
&= 0 \quad \text{(by Eq. (246))}. && (253)
\end{aligned}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\begin{aligned}
\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} &= \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} && (254)\\
&= 0. && (255)
\end{aligned}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{256}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left(\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1| + V_{10}(t)B_{10}(t)\sigma^+ + V_{01}(t)B_{01}(t)\sigma^-\right). \tag{257}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \text{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{258}$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{259}$$

$$\eta \equiv \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{260}$$

The solutions of the equation (258) are:

$$\lambda = \beta\frac{-\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{261}$$

$$= \beta\frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{262}$$

$$= -\beta\frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{263}$$

The value of $\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = \exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\exp\left(\frac{\eta(t)\beta}{2}\right) + \exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\exp\left(-\frac{\eta(t)\beta}{2}\right) \tag{264}$$

$$= 2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right). \tag{265}$$

Given that $v_{i\mathbf{k}}(t)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A\left(\varepsilon(t), \eta(t)\right)$ to calculate $\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{266}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right) + 2\left(\frac{\beta}{2}\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\sinh\left(\frac{\eta(t)\beta}{2}\right) \tag{267}$$

$$= -\beta\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\left(\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right)\right). \tag{268}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right) = 0. \tag{269}$$

The derivates included in the expression given are related to:

$$\begin{aligned}
\left\langle B_0^+\left(t\right)B_1^-\left(t\right)\right\rangle &= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (270)\\
&= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^*\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (271)\\
&= \left\langle B_1^+\left(t\right)B_0^-\left(t\right)\right\rangle^*, && (272)\\
R_i\left(t\right) &= \sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right) && (273)\\
&= \sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^2}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right), && (274)\\
\left\langle B_0^+\left(t\right)B_1^-\left(t\right)\right\rangle &= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (275)\\
&= \left(\prod_{\mathbf{k}}\exp\left(\frac{v_{0\mathbf{k}}^*\left(t\right)v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)v_{1\mathbf{k}}^*\left(t\right)}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), && (276)\\
v_{0\mathbf{k}}^*\left(t\right)v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)v_{1\mathbf{k}}^*\left(t\right) &= \left(v_{0\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{1\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{1\mathbf{k}}^{\Im}\left(t\right)\right)-\left(v_{0\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{1\mathbf{k}}^{\Im}\left(t\right)\right) && (277)\\
&= \left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)\right)-\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)+\mathrm{i}v_{0} && (278)\\
&= 2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)\right), && (279)\\
\left|v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right|^2 &= \left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)^* && (280)\\
&= \left|v_{1\mathbf{k}}\left(t\right)\right|^2+\left|v_{0\mathbf{k}}\left(t\right)\right|^2-\left(v_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}^*\left(t\right)+v_{1\mathbf{k}}^*\left(t\right)v_{0\mathbf{k}}\left(t\right)\right) && (281)\\
&= \left(v_{1\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{1\mathbf{k}}^{\Im}\left(t\right)\right)^2+\left(v_{0\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{0\mathbf{k}}^{\Im}\left(t\right)\right)^2-\left(\left(v_{1\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{1\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{0\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)\right)+\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v\right.\right. && (282)\\
&= \left(v_{1\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{1\mathbf{k}}^{\Im}\left(t\right)\right)^2+\left(v_{0\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{0\mathbf{k}}^{\Im}\left(t\right)\right)^2-2\left(v_{1\mathbf{k}}^{\Re}\left(t\right)v_{0\mathbf{k}}^{\Re}\left(t\right)+v_{1\mathbf{k}}^{\Im}\left(t\right)v_{0\mathbf{k}}^{\Im}\left(t\right)\right) && (283)\\
&= \left(v_{1\mathbf{k}}^{\Re}\left(t\right)-v_{0\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)^2. && (284)
\end{aligned}$$

Rewriting in terms of real and imaginary parts.

$$R_i\left(t\right)=\sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{i\mathbf{k}}^{\Im}\left(t\right)\right)^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{i\mathbf{k}}^{\Im}\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{i\mathbf{k}}^{\Im}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right) \tag{285}$$

$$=\sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{i\mathbf{k}}^{\Im}\left(t\right)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}\left(t\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}\left(t\right)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{286}$$

$$\left\langle B_0^{+}\left(t\right)B_1^{-}\left(t\right)\right\rangle=\left(\prod_{\mathbf{k}}\exp\left(\frac{v_{0\mathbf{k}}^{*}\left(t\right)v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)v_{1\mathbf{k}}^{*}\left(t\right)}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{287}$$

$$=\left(\prod_{\mathbf{k}}\exp\left(\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)\right)}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-v_{0\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\right.\right.\right. \tag{288}$$

$$=\left(\prod_{\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)\right)}{\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-v_{0\mathbf{k}}^{\Re}\left(t\right)\right)^2+\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\beta\right.\right.\right. \tag{289}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(\varepsilon_{1}(t)+R_{1}+\varepsilon_{0}(t)+R_{0}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{290}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{291}$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}, \tag{292}$$

$$\frac{\partial|B_{10}(t)|^{2}}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{293}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{294}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\left|B_{10}(t)\right|^{2}, \tag{295}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{296}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{297}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}(t)\right)\left(\varepsilon_{0}(t)+R_{0}(t)\right)-|V_{10}(t)|^{2}|B_{10}(t)|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \tag{298}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{299}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Re}(t)-v_{i'\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{300}$$

$$= \frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\varepsilon(t)\right. \tag{301}$$

$$\left.+2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{302}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}} \tag{303}$$

$$=\frac{\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{2g_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^2}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{304}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}-g_{i\mathbf{k}}^{*}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$=\frac{\left(v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{308}$$

Separating ([307]) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Re}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{309}$$

$$v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}^{\Re}=v_{i\mathbf{k}}^{\Re}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{310}$$

$$v_{i\mathbf{k}}^{\Re}\left(t\right)=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{311}$$

$$v_{i\mathbf{k}}^{\Re}\left(t\right)=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{312}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}=\frac{\partial\left(\varepsilon_{1}\left(t\right)+R_{1}\left(t\right)+\varepsilon_{0}\left(t\right)+R_{0}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{313}$$

$$=\frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\left(t\right)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}\left(t\right)\right)^{2}}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}\left(t\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}\left(t\right)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{314}$$

$$=2\frac{v_{i\mathbf{k}}^{\Im}\left(t\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{315}$$

$$\frac{\partial\left|B_{10}\left(t\right)\right|^{2}}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}=\frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{316}$$

$$=-\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-v_{0\mathbf{k}}^{\Re}\left(t\right)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{317}$$

$$=-\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\left|B_{10}\left(t\right)\right|^{2}, \tag{318}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}=\frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} \tag{319}$$

$$=\frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} \tag{320}$$

$$=\frac{\varepsilon\left(t\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}(t)\right)\left(\varepsilon_{0}(t)+R_{0}(t)\right)-\left|V_{10}(t)\right|^{2}\left|B_{10}(t)\right|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\eta\left(t\right)} \tag{321}$$

$$=\frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{322}$$

$$=\frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-v_{i'\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{323}$$

$$=\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{324}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}} \tag{325}$$

$$=\frac{2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\left(g_{i\mathbf{k}}^{*}\right)^{\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{\left(g_{i\mathbf{k}}^{*}\right)^{\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. \tag{326}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v_{i\mathbf{k}}^{\Im}(t)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$=\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

$$=\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$=\frac{\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{330}$$

Separating (330) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Im}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{331}$$

$$v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}=v_{i\mathbf{k}}^{\Im}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right) \tag{332}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{333}$$

$$v_{i\mathbf{k}}^{\Im}(t)=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{334}$$

$$v_{i\mathbf{k}}^{\Im}(t)=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{335}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t) = v_{i\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{i\mathbf{k}}^{\Im}(t) \tag{336}$$

$$= \frac{g_{i\mathbf{k}}^{\Re}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

$$+\mathrm{i}\frac{g_{i\mathbf{k}}^{\Im}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{338}$$

$$= \frac{g_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{339}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}$, where $\rho_B^{\mathrm{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^{V(0)}\rho_T(0)\,e^{-V(0)} \tag{340}$$
$$= \left(|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)\right)(\rho_S(0)\otimes\rho_B)\left(|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)\right), \tag{341}$$
$$\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } |0\rangle\langle 0|0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{342}$$
$$= |0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{343}$$
$$= |0\rangle\langle 0| \otimes B_0^+(0)\,\rho_B B_0^-(0)\,, \tag{344}$$
$$\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) \tag{345}$$
$$= |1\rangle\langle 1|B_1^+(0)\,\rho_B B_1^-(0) \tag{346}$$
$$= |1\rangle\langle 1| \otimes B_1^+(0)\,\rho_B B_1^-(0)\,, \tag{347}$$
$$\text{for } \rho_S(0) = |0\rangle\langle 1|\text{ : } |0\rangle\langle 0|B_0^+(0)\,|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) \tag{348}$$
$$= |0\rangle\langle 1|B_0^+(0)\,\rho_B|1\rangle\langle 1|B_1^-(0) \tag{349}$$
$$= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0)\,\rho_B B_1^-(0) \tag{350}$$
$$= |0\rangle\langle 1| \otimes B_0^+(0)\,\rho_B B_1^-(0)\,, \tag{351}$$
$$\text{for } \rho_S(0) = |1\rangle\langle 0|\text{ : } |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{352}$$
$$= |1\rangle\langle 0| \otimes B_1^+(0)\,\rho_B B_0^-(0)\,. \tag{353}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^\dagger(t)\,O(t)\,U(t)\,, \tag{354}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right). \tag{355}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\,\overline{\rho_S}(t)\,U(t)\,,\ \text{where} \tag{356}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}_T(t))\,. \tag{357}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \tag{358}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (236):

$$\begin{aligned} \overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) && (359) \\ &= B_{0z}(t)\,|0\rangle\langle 0| + B_{1z}(t)\,|1\rangle\langle 1| + V_{10}^{\Re}(t)\,\sigma_x B_x(t) + V_{10}^{\Re}(t)\,\sigma_y B_y(t) + V_{10}^{\Im}(t)\,\sigma_x B_y(t) - V_{10}^{\Im}(t)\,\sigma_y B_x(t) && (360) \\ &= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right). && (361) \end{aligned}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{362}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0)\,\text{-i}\int_0^t [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]\mathrm{d}s. \tag{363}$$

Replacing the equation (363) in the equation (362) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]]\mathrm{d}s. \tag{364}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdot \cdot]. \tag{365}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\,\rho_B]] \cdot \cdot]. \tag{366}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\begin{aligned} \widetilde{\overline{\rho_S}}(t) &= \left(1 + W_1(t) + W_2(t) + ...\right)\overline{\rho_S}(0) && (367) \\ &= W(t)\,\overline{\rho_S}(0). && (368) \end{aligned}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\,\rho_B]] \cdot \cdot]. \tag{369}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{370}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} W(t)\overline{\rho_S}(0) \tag{371}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1}\widetilde{\overline{\rho_S}}(t). \tag{372}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (370) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\widetilde{\overline{\rho_S}}(t) \tag{373}$$
$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]. \tag{374}$$

Replacing $t_1 \to t-\tau$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(-\tau), \overline{\rho_S}(t)\rho_B\right]\right]. \tag{375}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}(t)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = U^{\dagger}(t)\,\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t). \tag{376}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = \sum_i C_i(t)\left(\widetilde{A_i}(t)\otimes\widetilde{B_i}(t)\right), \tag{377}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\,\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{378}$$
$$= U^{\dagger}(t)\,A_i U(t)\,\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} \tag{379}$$
$$= U^{\dagger}(t)\,A_i U(t)\,\mathbb{I} \tag{380}$$
$$= U^{\dagger}(t)\,A_i U(t), \tag{381}$$
$$\widetilde{B_i}(t) = U^{\dagger}(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{382}$$
$$= U^{\dagger}(t)\,U(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{383}$$
$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{384}$$
$$= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}. \tag{385}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (377) to replace it in the equation (374)

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B \left[\widetilde{H_{\bar{I}}}(t), \left[\widetilde{H_{\bar{I}}}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] \mathrm{d}s \tag{386}$$

$$= -\int_0^t \mathrm{Tr}_B \left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \left[\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right), \widetilde{\rho_S}(t)\rho_B\right]\right] \mathrm{d}s \tag{387}$$

$$= -\int_0^t \mathrm{Tr}_B \left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\rho_S}(t)\rho_B - \widetilde{\rho_S}(t)\rho_B \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\right] \mathrm{d}s \tag{388}$$

$$= -\int_0^t \mathrm{Tr}_B \Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\rho_S}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\rho_S}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) \tag{389}$$

$$-\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\rho_S}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\rho_S}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}s. \tag{390}$$

In order to calculate the correlation functions we define:

$$\mathscr{B}_{ij}(t,s) = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(s)\,\rho_B\right) \tag{391}$$

An useful property

$$\mathscr{B}^*_{ji}(t,s) = \mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right)^\dagger \tag{392}$$

$$= \mathrm{Tr}_B\left(\rho_B^\dagger\,\widetilde{B_i}^\dagger(s)\,\widetilde{B_j}^\dagger(t)\right) \tag{393}$$

$$= \mathrm{Tr}_B\left(\rho_B\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right) \tag{394}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) \tag{395}$$

$$= \mathscr{B}_{ij}(s,t) \tag{396}$$

The correlation functions relevant that appear in the equation (390) are:

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right) = \left\langle\widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B \tag{397}$$

$$= \mathscr{B}_{ji}(t,s) \tag{398}$$

$$= \mathscr{B}^*_{ij}(s,t) \tag{399}$$

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\rho_B\widetilde{B_i}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) \tag{400}$$

$$= \mathscr{B}_{ij}(s,t) \tag{401}$$

$$\mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\rho_B\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right) \tag{402}$$

$$= \mathscr{B}^*_{ij}(s,t) \tag{403}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) \tag{404}$$

$$= \mathscr{B}_{ij}(s,t) \tag{405}$$

The cyclic property of the trace was use widely in the development of equations (397) and (405). Replacing in (390)

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) \tag{406}$$

$$-\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}s. \tag{407}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s)\big) \tag{408}$$

$$+\sum_{ij} C_i(s)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t)-\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{409}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s)\big) \tag{410}$$

$$+\sum_{ij} C_i(s)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t)-\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{411}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s)\big)\text{ (by permuting i and j because i,j}\in\text{J)} \tag{412}$$

$$+\sum_{ij} C_i(s)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t)-\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{413}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s) \tag{414}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t)-\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{415}$$

$$= -\int_0^t \Big(\sum_{ij} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\mathscr{B}_{ji}(t,s)-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\mathscr{B}_{ij}(s,t) \tag{416}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\mathscr{B}_{ij}(s,t)-\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\mathscr{B}_{ji}(t,s)\big)\Big)\mathrm{d}s \tag{417}$$

$$= -\int_0^t \left(\sum_{ij} C_j\left(t\right)C_i\left(s\right)\left(\mathscr{B}_{ji}\left(t,s\right)\left[\widetilde{A_j}\left(t\right),\widetilde{A_i}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]+\mathscr{B}_{ij}\left(s,t\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(s\right),\widetilde{A_j}\left(t\right)\right]\right)\right)\mathrm{d}s \tag{418}$$

$$= -\int_0^t \Big(\sum_{ij} C_i(t)C_j(s)\big(\mathscr{B}_{ij}(t,s)\big[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\widetilde{\rho_S}}(t)\big]+\mathscr{B}_{ji}(s,t)\big[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)\big]\big)\Big)\mathrm{d}s\text{ (exchanging i and j)} \tag{419}$$

$$= -\int_0^t \left(\sum_{ij} C_i\left(t\right)C_j\left(s\right)\left(\mathscr{B}_{ij}\left(t,s\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]+\mathscr{B}^*_{ij}\left(t,s\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}s \tag{420}$$

$$= -\int_0^t \left(\sum_{ij} C_i\left(t\right)C_j\left(s\right)\left(\mathscr{B}_{ij}\left(t,s\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]-\mathscr{B}^*_{ij}\left(t,s\right)\left[\widetilde{A_i}\left(t\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\right]\right)\right)\mathrm{d}s \tag{421}$$

We could identify the following commutators in the equation deduced:

$$\mathscr{B}_{ij}\left(t,s\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)-\mathscr{B}_{ij}\left(t,s\right)\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)=\mathscr{B}_{ij}\left(t,s\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_s}}\left(t\right)\right], \tag{422}$$

$$\mathscr{B}^*_{ij}\left(t,s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\mathscr{B}^*_{ij}\left(t,s\right)\widetilde{A_i}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)=\mathscr{B}^*_{ij}\left(t,s\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j},\widetilde{A_i}\left(t\right)\right]. \tag{423}$$

Returning to the Schroedinger picture we have:

$$U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)U^\dagger\left(t\right)=U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{A_j}\left(s\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)U^\dagger\left(t\right), \tag{424}$$

$$=\left(U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{A_j}\left(s\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)U^\dagger\left(t\right)\right), \tag{425}$$

$$=A_i\left(t\right)\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right). \tag{426}$$

This procedure applying to the relevant commutators give us:

$$U\left(t\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]U^\dagger\left(t\right)=\left(U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)U^\dagger\left(t\right)-U\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right) \tag{427}$$

$$=A_i\left(t\right)\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right)-\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right)A_i \tag{428}$$

$$=\left[A_i\left(t\right),\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right)\right]. \tag{429}$$

Introducing this transformed commutators in the equation (421) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathscr{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ij}\int_0^t \mathrm{d}s C_i(t)C_j(s)\Big(\mathscr{B}_{ij}(t,s)\left[A_i\left(t\right),\widetilde{A_j}(s,t)\,\overline{\rho_S}\left(t\right)\right] + \mathscr{B}^*_{ij}(t,s)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}(s,t),A_i\right]\Big) \tag{430}$$

$$s = t - \tau \text{ (Change of variables in the integration process)} \tag{431}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(t,t-\tau)\left[A_i\left(t\right),\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}\left(t\right)\right] + \mathscr{B}^*_{ij}(t,t-\tau)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}(t-\tau,t),A_i\left(t\right)\right]\Big) \tag{432}$$

where $i,j \in \{1,2,3,4,5.6\}$.

Here $\widetilde{A_j}\left(t-\tau,t\right) = U\left(t\right)U^{\dagger}\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$ where $U\left(t\right)$ is given by (355). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I\left(t\right)$. In order to write in a simplified way we define the following notation:

$$\mathscr{B}_{ij}\left(t,s\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B\right) \tag{433}$$

$$= \mathrm{Tr}_B\left(e^{iH_Bt}B_i\left(t\right)e^{-iH_Bt}e^{iH_Bs}B_j\left(s\right)e^{-iH_Bs}\rho_B\right) \tag{434}$$

$$= \mathrm{Tr}_B\left(e^{-iH_Bs}e^{iH_Bt}B_i\left(t\right)e^{-iH_Bt}e^{iH_Bs}B_j\left(s\right)\rho_B\right) \tag{435}$$

$$B_i\left(t,r\right) \equiv e^{iH_Br}B_i\left(t\right)e^{-iH_Br} \tag{436}$$

$$\mathscr{B}_{ij}\left(t,s\right) = \mathrm{Tr}_B\left(e^{iH_B(t-s)}B_i\left(t\right)e^{-iH_B(t-s)}B_j\left(s\right)\rho_B\right) \tag{437}$$

$$= \mathrm{Tr}_B\left(e^{iH_B\tau}B_i\left(t\right)e^{-iH_B\tau}B_j\left(s\right)\rho_B\right) \tag{438}$$

$$= \mathrm{Tr}_B\left(B_i\left(t,\tau\right)B_j\left(s,0\right)\rho_B\right) \tag{439}$$

Calculating the correlation functions allow us to obtain:

$$\tag{440}$$
$$\tag{441}$$
$$\tag{442}$$
$$\tag{443}$$
$$\tag{444}$$

$$\frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle \alpha_{\mathbf{k}}\middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}})\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{445}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{446}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} + \left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{447}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{448}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left|\alpha_{\mathbf{k}}\right|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{449}$$

$$= N+1, \tag{450}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\,\widetilde{B_{j'z}}(0)\right\rangle_B = \sum_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N + \sum_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})\,e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N+1) \tag{451}$$

$$= \textstyle\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} + N\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right), \tag{452}$$

$$D(h')\,D(h) = \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)D(h'+h), \tag{453}$$

$$\left\langle D(h')\,D(h)\right\rangle_B = \mathrm{Tr}_B\left(\exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)D(h'+h)\,\rho_B\right) \tag{454}$$

$$= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\mathrm{Tr}_B\left(D(h'+h)\,\rho_B\right) \tag{455}$$

$$= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\frac{1}{\pi N}\int \mathrm{d}^2\alpha P(\alpha)\left\langle \alpha\left|D(h'+h)\right|\alpha\right\rangle \tag{456}$$

$$= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\exp\left(-\frac{\left|h+h'\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{457}$$

$$h' = h\exp(\mathrm{i}\omega\tau), \tag{458}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))\,D(h)\right\rangle_B = \exp\left(\frac{1}{2}(hh^{*}\exp(\mathrm{i}\omega\tau)-h^{*}h\exp(-\mathrm{i}\omega\tau))\right)\exp\left(-\frac{\left|h+h\exp(\mathrm{i}\omega\tau)\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{459}$$

$$\frac{1}{2}|h|^2(\exp(\mathrm{i}\omega\tau)-\exp(-\mathrm{i}\omega\tau)) = \frac{1}{2}\left(hh^{*}\exp(\mathrm{i}\omega\tau)-h^{*}h\exp(-\mathrm{i}\omega\tau)\right) \tag{460}$$

$$= \frac{1}{2}\left|h\right|^2\left(\cos(\omega\tau)+\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)+\mathrm{i}\sin(\omega\tau)\right) \tag{461}$$

$$= \frac{1}{2}\left|h\right|^2\left(2\mathrm{i}\sin(\omega\tau)\right) \tag{462}$$

$$= \mathrm{i}\left|h\right|^2\sin(\omega\tau), \tag{463}$$

$$-\frac{\left|h+h\exp(\mathrm{i}\omega\tau)\right|^2}{2} = -\left|h\right|^2\frac{\left|1+\exp(\mathrm{i}\omega\tau)\right|^2}{2} \tag{464}$$

$$= -\left|h\right|^2\frac{\left(1+2\cos(\omega\tau)+\cos^2(\omega\tau)\right)+\sin^2(\omega\tau)}{2} \tag{465}$$

$$= - \left|h\right|^2 \frac{2 + 2\cos\left(\omega\tau\right)}{2} \tag{466}$$

$$= - \left|h\right|^2 \left(1 + \cos\left(\omega\tau\right)\right), \tag{467}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\right\rangle_B = \exp\left(\mathrm{i}|h|^2\sin(\omega\tau)\right)\exp\left(-|h|^2(1+\cos(\omega\tau))\coth\left(\tfrac{\beta\omega}{2}\right)\right) \tag{468}$$

$$= \exp\left(\mathrm{i}\left|h\right|^2 \sin\left(\omega\tau\right) - \left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{469}$$

$$= \exp\left(-\left|h\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right) + \cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-\left|h\right|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{470}$$

$$= \left\langle D\left(h\right)\right\rangle_B \exp\left(-\phi\left(\tau\right)\right), \tag{471}$$

$$\exp\left(-\phi\left(\tau\right)\right) = \exp\left(-\left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right)\right), \tag{472}$$

$$\phi\left(\tau\right) = \left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right), \tag{473}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{474}$$

$$h' = v\exp\left(i\omega\tau\right), \tag{475}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_1^+B_0^-}\left(\tau\right)\widetilde{B_1^+B_0^-}\left(0\right)\rho_B\right) \tag{476}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_1^+B_0^-}\left(\tau\right)\widetilde{B_1^+B_0^-}\left(0\right)\rho_B\right) \tag{477}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{478}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{479}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{480}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\prod_{\mathbf{k}}\left(\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{481}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{482}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}} - v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{483}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}} - v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{484}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}} - v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{485}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}} - v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{486}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{487}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{488}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{489}$$

$$= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau+\pi) + \cos(\omega_{\mathbf{k}}\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{490}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \left(\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)-\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \quad (491)$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \quad (492)$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \quad (493)$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}_B\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \quad (494)$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau+\pi)+\cos(\omega_{\mathbf{k}}\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (495)$$

$$= \left\langle |\widetilde{B_1^+B_0^-}(\tau)\,\widetilde{B_0^+B_1^-}(0)\right\rangle_B, \quad (496)$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\rho_B\right), \quad (497)$$

$$\langle D(h)\,b\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\,|D(h)\,b|\,\alpha\rangle \quad (498)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,\alpha\rangle \quad (499)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,0\rangle \quad (500)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha)|\,0\rangle \quad (501)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,(b+\alpha)|\,0\rangle \quad (502)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\,|D(h)\,(b+\alpha)|\,0\rangle \quad (503)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)b|0\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \quad (504)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\,|D(h)\,\alpha|\,0\rangle \quad (505)$$

$$= \frac{1}{\pi N}\int \alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \quad (506)$$

$$= hN\,\langle D(h)\rangle_B\,, \quad (507)$$

$$\left\langle D(h)\,b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle\alpha\,\middle|D(h)\,b^\dagger\middle|\,\alpha\right\rangle \quad (508)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\,\middle|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \quad (509)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\,\middle|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \quad (510)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\,\middle|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \quad (511)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{512}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\left\langle 0\left|D(h)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{513}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\left\langle 0\left|D(h)b^\dagger\right|0\right\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha^*|0\rangle \tag{514}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|1\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle \tag{515}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle -h|1\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle, \tag{516}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right)\sum_n \frac{(-h^*)^n}{\sqrt{n!}}\langle n|, \tag{517}$$

$$\langle -h|1\rangle = \exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*), \tag{518}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|-h^*|^2}{2}\right)(-h^*)+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\exp\left(-\tfrac{|-h^*|^2}{2}\right) \tag{519}$$

$$= -h^*\langle D(h)\rangle_B (N+1), \tag{520}$$

$$\langle bD(h)\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle \alpha|bD(h)|\alpha\rangle \tag{521}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{522}$$

$$= h\langle D(h)\rangle_B (N+1), \tag{523}$$

$$\left\langle b^\dagger D(h)\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle \alpha\left|b^\dagger D(h)\right|\alpha\right\rangle \tag{524}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{525}$$

$$= -h^*\langle D(h)\rangle_B N, \tag{526}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\right\rangle_B = \left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\right\rangle_B \tag{527}$$

$$= \prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle_B \tag{528}$$

$$= \prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle_B \tag{529}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{530}$$

$$= B_{10}. \tag{531}$$

The correlation functions can be found readily as:

$$\widetilde{B_1^+B_0^-}(\tau) = \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right), \tag{532}$$

$$\widetilde{B_0^+B_1^-}(\tau) = \prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right), \tag{533}$$

$$\widetilde{B_x}\left(0\right)=\frac{B_1^+B_0^-+B_0^+B_1^--B_{10}-B_{01}}{2}, \tag{534}$$

$$\widetilde{B_y}\left(0\right)=\frac{B_0^+B_1^--B_1^+B_0^-+B_{10}-B_{01}}{2\mathrm{i}}, \tag{535}$$

$$B_{10}=\left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{536}$$

$$B_{iz}=\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*b_{\mathbf{k}}\right), \tag{537}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B=\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j\mathbf{k}}-v_{jk}\right)^*b_{\mathbf{k}}\right)\right\rangle_B \tag{538}$$

$$=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{539}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B=\left\langle\frac{B_1^+B_0^-\left(\tau\right)+B_0^+B_1^-\left(\tau\right)-B_{10}-B_{01}}{2}\frac{B_1^+B_0^-+B_0^+B_1^--B_{10}-B_{01}}{2}\right\rangle_B \tag{540}$$

$$=\frac{1}{4}\left\langle\left(B_1^+B_0^-\left(\tau\right)+B_0^+B_1^-\left(\tau\right)-B_{10}-B_{01}\right)\left(B_1^+B_0^-+B_0^+B_1^--B_{10}-B_{01}\right)\right\rangle_B \tag{541}$$

$$=\tfrac{1}{4}\Big\langle B_1^+B_0^-(\tau)B_1^+B_0^-+B_1^+B_0^-(\tau)B_0^+B_1^--B_1^+B_0^-(\tau)B_{10}-B_1^+B_0^-(\tau)B_{01}+B_0^+B_1^-(\tau)B_1^+B_0^-+B_0^+B_1^-(\tau)B_0^+B_1^- \tag{542}$$

$$-B_0^+B_1^-(\tau)B_{10}-B_0^+B_1^-(\tau)B_{01}-B_{10}B_1^+B_0^--B_{10}B_0^+B_1^-+B_{10}B_{10}+B_{10}B_{01}-B_{01}B_1^+B_0^--B_{01}B_0^+B_1^-+B_{01}B_{10}+B_{01}B_{01}\Big\rangle \tag{543}$$

$$=\frac{1}{4}\langle B_1^+B_0^-(\tau)B_1^+B_0^-+B_1^+B_0^-(\tau)B_0^+B_1^--B_1^+B_0^-(\tau)B_{10}-B_1^+B_0^-(\tau)B_{01}+B_0^+B_1^-(\tau)B_1^+B_0^- \tag{544}$$

$$+B_0^+B_1^-\left(\tau\right)B_0^+B_1^--B_0^+B_1^-\left(\tau\right)B_{10}-B_0^+B_1^-\left(\tau\right)B_{01}\rangle, \tag{545}$$

$$\left\langle\widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B=\Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{546}$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right), \tag{547}$$

$$\phi\left(\tau\right)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{548}$$

$$S=\prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{549}$$

$$\left\langle\widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B=U\exp(-\phi(\tau))S, \tag{550}$$

$$\left\langle\widetilde{B_1^+B_0^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B=U^*\exp(-\phi(\tau))S, \tag{551}$$

$$\left\langle\widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B=\exp(\phi(\tau))S, \tag{552}$$

$$\left\langle\widetilde{B_0^+B_1^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B=\left\langle\widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B, \tag{553}$$

$$\left\langle\widetilde{B_1^+B_0^-}\left(\tau\right)\right\rangle_B=\prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{554}$$

$$=U^{*1/2}S^{1/2}, \tag{555}$$

$$\left\langle\widetilde{B_x}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B=\frac{1}{4}\langle B_1^+B_0^-(\tau)B_1^+B_0^-+B_1^+B_0^-(\tau)B_0^+B_1^--B_1^+B_0^-(\tau)B_{10}-B_1^+B_0^-(\tau)B_{01}+B_0^+B_1^-(\tau)B_1^+B_0^- \tag{556}$$

$$+B_0^+B_1^-\left(\tau\right)B_0^+B_1^--B_0^+B_1^-\left(\tau\right)B_{10}-B_0^+B_1^-\left(\tau\right)B_{01}\rangle, \tag{557}$$

$$\left\langle\widetilde{B_x}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B=\frac{1}{4}\langle B_1^+B_0^-\left(\tau\right)B_1^+B_0^-+B_1^+B_0^-\left(\tau\right)B_0^+B_1^--B_1^+B_0^-\left(\tau\right)B_{10}-B_1^+B_0^-\left(\tau\right)B_{01} \tag{558}$$

$$+B_0^+B_1^-\left(\tau\right)B_1^+B_0^-+B_0^+B_1^-\left(\tau\right)B_0^+B_1^--B_0^+B_1^-\left(\tau\right)B_{10}-B_0^+B_1^-\left(\tau\right)B_{01}\rangle \tag{559}$$

$$=\frac{1}{4}\Big(U^*\exp\left(-\phi\left(\tau\right)\right)S+\exp(\phi\left(\tau\right))S-B_{10}^2-\left|B_{10}\right|^2+\exp(\phi\left(\tau\right))S+U\exp(-\phi\left(\tau\right))S-B_{10}^{*2}-\left|B_{10}\right|^2\Big) \tag{560}$$

$$= \frac{1}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right)S + 2\exp\left(\phi\left(\tau\right)\right)S - 2\left(B_{10}^{2}\right)^{\Re} - 2\left|B_{10}\right|^{2}\right) \tag{561}$$

$$= \frac{1}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right)S + 2\exp\left(\phi\left(\tau\right)\right)S - 2\left(U^{*}\right)^{\Re}S - 2S\right) \tag{562}$$

$$= \frac{S}{2}\left(U^{\Re}\exp\left(-\phi\left(\tau\right)\right) + \exp\left(\phi\left(\tau\right)\right) - \left(U^{*}\right)^{\Re} - 1\right), \tag{563}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}}{2\mathrm{i}}\frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B \tag{564}$$

$$= -\frac{1}{4}\left\langle\left(B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}\right)\left(B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}\right)\right\rangle_B \tag{565}$$

$$= -\tfrac{1}{4}\left\langle B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_1^+B_0^-(\tau)B_0^+B_1^- + B_1^+B_0^-(\tau)B_1^+B_0^-\right. \tag{566}$$

$$\left. - B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01} + B_{10}B_0^+B_1^- - B_{10}B_1^+B_0^- + B_{10}B_{10} - B_{10}B_{01} - B_{01}B_0^+B_1^- + B_{01}B_1^+B_0^- - B_{01}B_{10} + B_{01}B_{01}\right\rangle \tag{567}$$

$$= -\frac{1}{4}\left\langle B_0^+B_1^-\left(\tau\right)B_0^+B_1^- - B_0^+B_1^-\left(\tau\right)B_1^+B_0^- + B_0^+B_1^-\left(\tau\right)B_{10} - B_0^+B_1^-\left(\tau\right)B_{01}\right. \tag{568}$$

$$\left. - B_1^+B_0^-\left(\tau\right)B_0^+B_1^- + B_1^+B_0^-\left(\tau\right)B_1^+B_0^- - B_1^+B_0^-\left(\tau\right)B_{10} + B_1^+B_0^-\left(\tau\right)B_{01}\right\rangle \tag{569}$$

$$= -\tfrac{1}{4}\left\langle B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_{01}B_{10} - B_{01}B_{01} - B_1^+B_0^-(\tau)B_0^+B_1^- + B_1^+B_0^-(\tau)B_1^+B_0^- - B_{10}B_{10} + B_{10}B_{01}\right\rangle \tag{570}$$

$$= -\frac{1}{4}\left(U\exp(-\phi\left(\tau\right))S - \exp\left(\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^{*}\exp\left(-\phi\left(\tau\right)\right)S + 2S - 2\left(U^{*}\right)^{\Re}S\right) \tag{571}$$

$$= -\frac{S}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right) - 2\exp\left(\phi\left(\tau\right)\right) + 2 - 2U^{\Re}\right) \tag{572}$$

$$= \frac{S}{2}\left(\exp\left(\phi\left(\tau\right)\right) - U^{\Re}\exp\left(-\phi\left(\tau\right)\right) - 1 + U^{\Re}\right), \tag{573}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_1^+B_0^-\left(\tau\right) + B_0^+B_1^-\left(\tau\right) - B_{10} - B_{01}}{2}\frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B \tag{574}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_1^+B_0^-\left(\tau\right) + B_0^+B_1^-\left(\tau\right) - B_{10} - B_{01}\right)\left(B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}\right)\right\rangle_B \tag{575}$$

$$= \tfrac{1}{4\mathrm{i}}\left\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10}\right. \tag{576}$$

$$\left. - B_0^+B_1^-(\tau)B_{01} - B_{10}B_0^+B_1^- + B_{10}B_1^+B_0^- - B_{10}B_{10} + B_{10}B_{01} - B_{01}B_0^+B_1^- + B_{01}B_1^+B_0^- - B_{01}B_{10} + B_{01}B_{01}\right\rangle \tag{577}$$

$$= \tfrac{1}{4\mathrm{i}}\left\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10}\right. \tag{578}$$

$$\left. - B_0^+B_1^-(\tau)B_{01}\right\rangle \tag{579}$$

$$= \tfrac{1}{4\mathrm{i}}\left\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_{10}B_{10} - B_{10}B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_{01}B_{10} - B_{01}B_{01}\right\rangle \tag{580}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_{10}B_{10} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- - B_{01}B_{01}\right\rangle \tag{581}$$

$$= \frac{1}{4\mathrm{i}}\left(\exp\left(\phi\left(\tau\right)\right)S - U^{*}\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^{*}S - US\right) \tag{582}$$

$$= \frac{1}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S + U^{*}S - US\right) \tag{583}$$

$$= \frac{S}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi\left(\tau\right)\right) + U\exp\left(-\phi\left(\tau\right)\right) + U^{*} - U\right) \tag{584}$$

$$= \frac{S\left(U - U^{*}\right)}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{585}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{586}$$

$$= \frac{U^{\Im}S}{2}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right), \tag{587}$$

$$\left\langle\widetilde{B_y}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \left\langle\frac{B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}}{2\mathrm{i}}\frac{B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}}{2}\right\rangle_B \tag{588}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}\right)\left(B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}\right)\right\rangle_B \tag{589}$$

$$= \tfrac{1}{4\mathrm{i}}\left\langle B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_0^+B_1^-\right. \tag{590}$$

$$\left. + B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01} + B_{10}B_1^+B_0^- + B_{10}B_0^+B_1^- - B_{10}B_{10} - B_{10}B_{01} - B_{01}B_1^+B_0^- - B_{01}B_0^+B_1^- + B_{01}B_{10} + B_{01}B_{01}\right\rangle \tag{591}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_0^+B_1^-\left(\tau\right)B_1^+B_0^- + B_0^+B_1^-\left(\tau\right)B_0^+B_1^- - B_0^+B_1^-\left(\tau\right)B_{10} - B_0^+B_1^-\left(\tau\right)B_{01}\right. \tag{592}$$

$$\left. -B_1^+B_0^-\left(\tau\right)B_1^+B_0^- - B_1^+B_0^-\left(\tau\right)B_0^+B_1^- + B_1^+B_0^-\left(\tau\right)B_{10} + B_1^+B_0^-\left(\tau\right)B_{01}\right\rangle \tag{593}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_0^+B_1^-\left(\tau\right)B_1^+B_0^- + B_0^+B_1^-\left(\tau\right)B_0^+B_1^- - B_{01}B_{10} - B_{01}B_{01} - B_1^+B_0^-\left(\tau\right)B_1^+B_0^-\right. \tag{594}$$

$$\left. -B_1^+B_0^-\left(\tau\right)B_0^+B_1^- + B_{10}B_{10} + B_{10}B_{01}\right\rangle \tag{595}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_0^+B_1^-\left(\tau\right)B_1^+B_0^- + B_0^+B_1^-\left(\tau\right)B_0^+B_1^- - B_{01}B_{10} - B_{01}B_{01} - B_1^+B_0^-\left(\tau\right)B_1^+B_0^-\right. \tag{596}$$

$$\left. -B_1^+B_0^-\left(\tau\right)B_0^+B_1^- + B_{10}B_{10} + B_{10}B_{01}\right\rangle \tag{597}$$

$$= \tfrac{1}{4\mathrm{i}}\left\langle B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- - B_{01}B_{01} - B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_0^+B_1^- + B_{10}B_{10}\right\rangle \tag{598}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + B_{10}^2 - B_{10}^{*2}\right) \tag{599}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + U^*S - US\right) \tag{600}$$

$$= \frac{S\left(U - U^*\right)}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{601}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{602}$$

$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{10}, \tag{603}$$

$$\left\langle B_1^+B_0^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\prod_{\mathbf{k}}\exp\left(\tfrac{1}{2}\left(\tfrac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}\left\langle\prod_{\mathbf{k}}\left(D\left(\tfrac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle \tag{604}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}, \tag{605}$$

$$\left\langle B_0^+B_1^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger\right\rangle_B = -(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}. \tag{606}$$

$$\left\langle B_0^+B_1^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}, \tag{607}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \tfrac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10} - (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}\right. \tag{608}$$

$$\left. +(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\right) \tag{609}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10} - (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}\right. \tag{610}$$

$$\left. +(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\right) \tag{611}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*B_{01}\right)\right. \tag{612}$$

$$\left. +\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{01}\right)\right) \tag{613}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'} + 1)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*B_{01}\right)\right. \tag{614}$$

$$\left. +\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{01}\right)\right) \tag{615}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(B_{10}-B_{01}) + (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)(B_{10}-B_{01})\right) \tag{616}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}2\mathrm{i}B_{10}^{\Im}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{617}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{618}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right), \tag{619}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \left\langle \sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_{B} \tag{620}$$

$$= \sum_{\mathbf{k}'}\left\langle\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_{B} \tag{621}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}\right)\right\rangle_{B} \tag{622}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\,b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}\right)\right\rangle_{B} \tag{623}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right. \tag{624}$$

$$\left. + (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\,b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\right\rangle, \tag{625}$$

$$\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B} = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B} \tag{626}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{627}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^{*}v_{0\mathbf{k}'}-v_{1\mathbf{k}'}v_{0\mathbf{k}'}^{*}}{\omega_{\mathbf{k}'}^{2}}\right)\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{628}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{629}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{630}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{631}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}N_{\mathbf{k}'}\right)e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{632}$$

$$= -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{633}$$

$$= -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{634}$$

$$\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\right\rangle_{B} = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})B_{01}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{635}$$

$$\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B} = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\right\rangle_{B} \tag{636}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{637}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle b_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{638}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{639}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{640}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{10}, \tag{641}$$

$$\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1}^{-}\right\rangle_{B} = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)B_{01}, \tag{642}$$

$$\left\langle\widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})B_{01}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right. \tag{643}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)B_{10}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)B_{01}\right) \tag{644}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(B_{01}-B_{10})+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)(B_{10}-B_{01})\right) \tag{645}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(B_{01}-B_{10})-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)(B_{01}-B_{10})\right) \tag{646}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)\right), \tag{647}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \left\langle\!\!\left(\frac{B_0^+B_1^-(\tau)-B_1^+B_0^-(\tau)+B_{10}-B_{01}}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\!\!\right\rangle_B \tag{648}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\!\!\left(B_0^+B_1^-(\tau)-B_1^+B_0^-(\tau)+B_{10}-B_{01}\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\!\!\right\rangle_B \tag{649}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-(\tau)-B_1^+B_0^-(\tau)\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\right\rangle_B \tag{650}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left\langle B_0^+B_1^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger-B_1^+B_0^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger+B_0^+B_1^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}-B_1^+B_0^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}\right\rangle, \tag{651}$$

$$\left\langle B_0^+B_1^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger\right\rangle_B = -(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}, \tag{652}$$

$$\left\langle B_0^+B_1^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}, \tag{653}$$

$$\left\langle B_1^+B_0^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger\right\rangle_B = -(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}, \tag{654}$$

$$\left\langle B_1^+B_0^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}, \tag{655}$$

$$\langle\widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\rangle_B = \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right. \tag{656}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{657}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right. \tag{658}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{659}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{01})+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(-B_{10}-B_{01})\right) \tag{660}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{01})-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(B_{10}+B_{01})\right), \tag{661}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^+B_1^--B_1^+B_0^-+B_{10}-B_{01}}{2\mathrm{i}}\right)\right\rangle_B \tag{662}$$

$$= \frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^--B_1^+B_0^-+B_{10}-B_{01}\right)\right\rangle_B \tag{663}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^--B_1^+B_0^-\right)\right\rangle_B \tag{664}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^--(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-\right. \tag{665}$$

$$\left.-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-\right\rangle \tag{666}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle\right. \tag{667}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right\rangle \tag{668}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle\right. \tag{669}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right\rangle \tag{670}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle\right. \tag{671}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right) \tag{672}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle_B = -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}N_{\mathbf{k}'}, \tag{673}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle_B = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{01}N_{\mathbf{k}'}, \tag{674}$$

$$\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle_B = \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}, \tag{675}$$

$$\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle_B = \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{01}, \tag{676}$$

$$\langle\widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\rangle_B = \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}^*N_{\mathbf{k}'}\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}N_{\mathbf{k}'}\right)\right) \tag{677}$$

$$+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^{*}\right)-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\Bigg) \quad (678)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{*}N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}N_{\mathbf{k}'}\right)\right. \quad (679)$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^{*}\right)\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\right) \quad (680)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}+B_{10}^{*}\right)N_{\mathbf{k}'}\right. \quad (681)$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(B_{10}+B_{10}^{*}\right)\left(N_{\mathbf{k}'}+1\right)\right) \quad (682)$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}\left(N_{\mathbf{k}'}+1\right)\right) \quad (683)$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}\right) \quad (684)$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}\right) \quad (685)$$

$$=\mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}N_{\mathbf{k}'}\right) \quad (686)$$

The correlation functions are equal to:

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_{B}=\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \quad (687)$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right), \quad (688)$$

$$\phi(\tau)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \quad (689)$$

$$\left\langle\widetilde{B_{x}}(\tau)\widetilde{B_{x}}(0)\right\rangle_{B}=\frac{\left|B_{10}\right|^{2}}{2}\left(U^{\Re}\exp\left(-\phi(\tau)\right)+\exp\left(\phi(\tau)\right)-U^{\Re}-1\right), \quad (690)$$

$$\left\langle\widetilde{B_{y}}(\tau)\widetilde{B_{y}}(0)\right\rangle_{B}=\frac{\left|B_{10}\right|^{2}}{2}\left(\exp\left(\phi(\tau)\right)-U^{\Re}\exp\left(-\phi(\tau)\right)-1+U^{\Re}\right), \quad (691)$$

$$\left\langle\widetilde{B_{x}}(\tau)\widetilde{B_{y}}(0)\right\rangle_{B}=\frac{U^{\Im}\left|B_{10}\right|^{2}}{2}\left(\exp\left(-\phi(\tau)\right)-1\right), \quad (692)$$

$$\left\langle\widetilde{B_{y}}(\tau)\widetilde{B_{x}}(0)\right\rangle_{B}=\frac{U^{\Im}\left|B_{10}\right|^{2}}{2}\left(\exp\left(-\phi(\tau)\right)-1\right), \quad (693)$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{x}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \quad (694)$$

$$\left\langle\widetilde{B_{x}}(\tau)\widetilde{B_{iz}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \quad (695)$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{y}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \quad (696)$$

$$\left\langle\widetilde{B_{y}}(\tau)\widetilde{B_{iz}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}\right). \quad (697)$$

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \tag{698}$$

$$v_{i\mathbf{k}} = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}). \tag{699}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^{\infty} \frac{J_i(\omega)}{\omega} \mathrm{d}\omega. \tag{700}$$

The integral version of the correlation functions are given by:

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}}) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{701}$$

$$= \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}})) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{702}$$

$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))\, g_{j\mathbf{k}}^*(1-F_j(\omega_{\mathbf{k}}))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^* g_{j\mathbf{k}}(1-F_j(\omega_{\mathbf{k}})) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{703}$$

$$\approx \int_0^{\infty} \left(\sqrt{J_i(\omega)J_j^*(\omega)}(1-F_i(\omega))\left(1-F_j^*(\omega)\right)e^{\mathrm{i}\omega\tau}N(\omega) + \sqrt{J_i^*(\omega)J_j(\omega)}(1-F_i^*(\omega))(1-F_j(\omega))e^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega, \tag{704}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{705}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right) \tag{706}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})\, g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})\, g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{707}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})\, g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})\, g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{708}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* g_{1\mathbf{k}} F_0^*(\omega_{\mathbf{k}})\, F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} g_{1\mathbf{k}}^* F_0(\omega_{\mathbf{k}})\, F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{709}$$

$$\approx \exp\left(\int_0^{\infty} \frac{\sqrt{J_0^*(\omega)\, J_1(\omega)} F_0^*(\omega)\, F_1(\omega) - \sqrt{J_0(\omega)\, J_1^*(\omega)} F_0(\omega)\, F_1^*(\omega)}{\omega^2} \mathrm{d}\omega\right), \tag{710}$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{711}$$

$$= \sum_{\mathbf{k}} \left|\frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2 \left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{712}$$

$$\approx \int_0^{\infty} \left|\frac{\sqrt{J_1(\omega)} F_1(\omega) - \sqrt{J_0(\omega)} F_0(\omega)}{\omega}\right|^2 \left(-\mathrm{i}\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega, \tag{713}$$

$$B_{10} = \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{714}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{715}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g^*_{1\mathbf{k}}F^*_1(\omega_{\mathbf{k}})-g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})g^*_{0\mathbf{k}}F^*_0(\omega_{\mathbf{k}})}{\omega^2_{\mathbf{k}}}\right)\right) \tag{716}$$

$$\approx \exp\left(-\frac{1}{2}\int_0^{\infty}\left|\frac{\sqrt{J_1(\omega)}F_1(\omega)-\sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega\right)\exp\left(\int_0^{\infty}\frac{1}{2}\left(\frac{\sqrt{J_0(\omega)J^*_1(\omega)}F_0(\omega)F^*_1(\omega)-\sqrt{J^*_0(\omega)J_1(\omega)}F^*_0(\omega)F_1(\omega)}{\omega^2}\right)\mathrm{d}\omega\right), \tag{717}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(U^{\Re}\exp\left(-\phi\left(\tau\right)\right)+\exp\left(\phi\left(\tau\right)\right)-U^{\Re}-1\right), \tag{718}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(\exp\left(\phi\left(\tau\right)\right)-U^{\Re}\exp\left(-\phi\left(\tau\right)\right)-1+U^{\Re}\right), \tag{719}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right), \tag{720}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right), \tag{721}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B = \mathrm{i}B^{\Im}_{10}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{722}$$

$$= \mathrm{i}B^{\Im}_{10}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}})\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}})\right)^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{723}$$

$$= \mathrm{i}B^{\Im}_{10}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-g^*_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{724}$$

$$Q\left(\omega\right) = \sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega\right)\right)\left(\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega\right)}{\omega}\right)^*, \tag{725}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B \approx \mathrm{i}B^{\Im}_{10}\int_0^{\infty}\left(Q\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega, \tag{726}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}B^{\Im}_{10}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{727}$$

$$= \mathrm{i}B^{\Im}_{10}\sum_{\mathbf{k}}\left(g^*_{i\mathbf{k}}(1-F^*_i(\omega_{\mathbf{k}}))\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}-g_{i\mathbf{k}}(1-F_i(\omega))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{728}$$

$$\approx \mathrm{i}B^{\Im}_{10}\int_0^{\infty}\left(Q^*\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega, \tag{729}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \mathrm{i}B^{\Re}_{10}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{730}$$

$$= \mathrm{i}B^{\Re}_{10}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}g^*_{i\mathbf{k}}(1-F^*_i(\omega_{\mathbf{k}}))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{731}$$

$$\approx \mathrm{i}B^{\Re}_{10}\int_0^{\infty}\left(e^{-\mathrm{i}\omega\tau}Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right)-e^{\mathrm{i}\omega\tau}Q\left(\omega\right)N\left(\omega\right)\right)\mathrm{d}\omega, \tag{732}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}B^{\Re}_{10}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}})(N_{\mathbf{k}}+1)\,e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{733}$$

$$= \mathrm{i}B^{\Re}_{10}\sum_{\mathbf{k}}\left(g^*_{i\mathbf{k}}(1-F^*_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))(N_{\mathbf{k}}+1)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{734}$$

$$= \mathrm{i}B^{\Re}_{10}\int_0^{\infty}\left(e^{\mathrm{i}\omega\tau}Q^*\left(\omega\right)N\left(\omega\right)-e^{-\mathrm{i}\omega\tau}Q\left(\omega\right)\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega. \tag{735}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda+\mathrm{Det}\left(\overline{H_{\bar{S}}}\right)=0. \tag{736}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm}=\frac{\xi\pm\eta}{2}$ and they satisfy $H_S\left|\pm\right\rangle=\lambda_{\pm}\left|\pm\right\rangle$. Using this notation is possible to write $H_{\bar{S}}=\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_i}(\tau) = e^{\mathrm{i}\overline{H_{\bar{S}}}\tau} A_i e^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} \tag{737}$$
$$= \sum_{w} e^{-\mathrm{i}w\tau} \mathscr{A}_i(w). \tag{738}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.
In order to use the equation (738) to descompose the equation (355) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \tag{739}$$
$$= \mathbb{I} + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1) H(t_2) ... H(t_n). \tag{740}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \equiv \exp(-\mathrm{i}tH_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...\right)\right). \tag{741}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \tag{742}$$
$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{743}$$
$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{744}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}(t) = U^{\dagger}(t) A_i(t) U(t) \tag{745}$$
$$= e^{\mathrm{i}H_E(t)t} A_i(t) e^{-\mathrm{i}H_E(t)t} \tag{746}$$
$$= \sum_{w(t)} e^{-\mathrm{i}w(t)t} \mathscr{A}_i(w(t)). \tag{747}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau, t)$ using the Magnus expansion generates:

$$\begin{aligned}
\widetilde{A_j}\left(t-\tau,t\right) &= U\left(t\right)U^{\dagger}\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right) && (748)\\
&= e^{-\mathrm{i}tH_E(t)}e^{\mathrm{i}(t-\tau)H_E(t-\tau)}A_j\left(t\right)e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}e^{\mathrm{i}tH_E(t)} && (749)\\
&= e^{-\mathrm{i}tH_E(t)}\left(\sum_{w'(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j\left(w\left(t-\tau\right)\right)\right)e^{\mathrm{i}tH_E(t)} && (750)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) && (751)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) && (752)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) && (753)
\end{aligned}$$

where $w'\left(t-\tau\right)$ and $w\left(t\right)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}\left(t-\tau\right)$ and $\overline{H_E}\left(t\right)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (738) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V}=\left\{\left|+\right\rangle,\left|-\right\rangle\right\}$ in the following way:

$$A_i = \sum_{\alpha,\beta\in\mathrm{V}} \langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|. \tag{754}$$

Given that $\left[|+\rangle\langle+|,|-\rangle\langle-|\right]=0$, then using the Zassenhaus formula we obtain:

$$\begin{aligned}
e^{\mathrm{i}\overline{H_E}\tau} &= e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} && (755)\\
&= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} && (756)\\
&= \left(|-\rangle\langle-|+e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) && (757)\\
&= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|. && (758)
\end{aligned}$$

Calculating the transformation (738) directly using the previous relationship we find that:

$$\begin{aligned}
U^{\dagger}\left(\tau\right)A_i\left(\tau\right)U\left(\tau\right) &= \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in\mathrm{V}}\langle\alpha|A_i\left(\tau\right)|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) && (759)\\
&= \langle+|A_i\left(\tau\right)|+\rangle|+\rangle\langle+|+e^{\mathrm{i}\eta\tau}\langle+|A_i\left(\tau\right)|-\rangle|+\rangle\langle-|+e^{-\mathrm{i}\eta\tau}\langle-|A_i\left(\tau\right)|+\rangle|-\rangle\langle+|+\langle-|A_i\left(\tau\right)|-\rangle|-\rangle\langle-|. && (760)\\
&= \mathscr{A}_i\left(0\right)+\mathscr{A}_i\left(-w\right)e^{\mathrm{i}w\tau}+\mathscr{A}_i\left(w\right)e^{-\mathrm{i}w\tau} && (761)
\end{aligned}$$

Here $w=\lambda_+-\lambda-$. Comparing the RHS of the equations (738) and the explicit expression for $\widetilde{A_i}\left(\tau\right)$ in (746), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$\begin{aligned}
\mathscr{A}_i\left(0\right) &= \langle+|A_i\left(\tau\right)|+\rangle|+\rangle\langle+|+\langle-|A_i\left(\tau\right)|-\rangle|-\rangle\langle-|, && (762)\\
\mathscr{A}_i\left(-w\right) &= \langle+|A_i\left(\tau\right)|-\rangle|+\rangle\langle-|, && (763)\\
\mathscr{A}_i\left(w\right) &= \langle-|A_i\left(\tau\right)|+\rangle|-\rangle\langle+|. && (764)
\end{aligned}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}\left(\tau\right)=\widetilde{A_i}^{\dagger}\left(\tau\right)$ and $\widetilde{B_i}\left(\tau\right)=\widetilde{B_i}^{\dagger}\left(\tau\right)$ we can use the equation (738) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big] + \mathscr{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \tag{765}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{766}$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\left[A_i, \overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right)\right]\Big) \tag{767}$$

Given that $\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) = \mathscr{A}_j^\dagger\left(-w\left(t-\tau\right),-w'\left(t\right)\right)$ from the Fourier decomposition (738) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\mathscr{B}_{ij}\left(\tau\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B\right) \tag{768}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B\right). \tag{769}$$

Let's define:

$$\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) = \mathscr{A}_{jww'}\left(t-\tau,t\right) \tag{770}$$

The master equation can be re-written in the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{771}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \tag{772}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{773}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \tag{774}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{775}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \tag{776}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{777}$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right]\Big) \tag{778}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \textstyle\sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathrm{Tr}_B\Big(\Big[A_i,\widetilde{B_i}(\tau)\widetilde{B_j}(0)\rho_B e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\Big] \tag{779}$$

$$-\Big[A_i,\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\Big]\Big) \tag{780}$$

Given that if we define:

$$D_{ijww'}\left(t-\tau,t\right) = C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right) \tag{781}$$

then

$$D^\dagger_{ijww'}\left(t-\tau,t\right) = \left(C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right)^\dagger \tag{782}$$

$$= \mathscr{B}^*_{ij}\left(\tau\right)C_i\left(t\right)C_j\left(t-\tau\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}^\dagger_{jww'}\left(t-\tau,t\right) \tag{783}$$

We used the fact that $C_i(t), C_j(t-\tau)$ are real. Now let's consider the following trace recalling that $\mathrm{Tr}(A)^* = \mathrm{Tr}\left(A^\dagger\right)$ so:

$$\mathrm{Tr}_B\left(\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\right) = \mathrm{Tr}_B\left(e^{-\mathrm{i}\tau H_B(\tau)}B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\right) \tag{784}$$
$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B e^{-\mathrm{i}\tau H_B(\tau)}\right) \text{ (by cyclic permutivity of trace)} \tag{785}$$
$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i e^{-\mathrm{i}\tau H_B(\tau)}\rho_B\right) \text{ (by commutativity of } e^{-\mathrm{i}\tau H_B(\tau)} \text{ and } \rho_B) \tag{786}$$
$$= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) \text{ (by definition of time evolution)} \tag{787}$$
$$= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) \tag{788}$$
$$= \mathrm{Tr}_B\left(\rho_B B_j\widetilde{B_i}(\tau)\right) \tag{789}$$
$$= \mathrm{Tr}_B\left(\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^\dagger\right) \text{ (by definition of adjoint)} \tag{790}$$
$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^* \tag{791}$$
$$= \mathscr{B}_{ij}^*(\tau) \tag{792}$$

So we can write the master equation like:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{793}$$
$$-\mathscr{B}_{ij}^*(\tau)\left[A_i,\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j^\dagger(w(t-\tau),w'(t))\right]\Big) \tag{794}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\left(\left[A_i, D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\right]\right) \tag{795}$$

Let's define the response matrix in the following way.

$$\mathscr{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau,t) \tag{796}$$

Then the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)\mathscr{D}_{ijww'}^\dagger(t)\right]\right) \tag{797}$$

If we extend the upper limit of integration to ∞ in the equation (796) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_S(t)}{\mathrm{d}t}e^V = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^V\right)}{\mathrm{d}t} \tag{798}$$
$$= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} \tag{799}$$
$$= -\mathrm{i}e^{-\mathrm{V}}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]e^V - \sum_{ijww'}\int_0^t \mathrm{d}\tau\Big(e^{-V}\left[A_i,D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right]e^V - e^{-V}\left[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\right]e^V\Big). \tag{800}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V} = e^{-V}\overline{A\mathbb{I}B}e^{V} \tag{801}$$
$$= e^{-V}\overline{A}e^{V}e^{-V}\bar{B}e^{V} \tag{802}$$
$$= \left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\bar{B}e^{V}\right) \tag{803}$$
$$= AB. \tag{804}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V} = e^{-V}\overline{(AB-BA)}e^{V} \tag{805}$$
$$= e^{-V}\overline{AB}e^{V} - e^{-V}\overline{BA}e^{V} \tag{806}$$
$$= AB - BA \tag{807}$$
$$= [A,B]. \tag{808}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]\right)e^{V} \tag{809}$$
$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^{V} - \sum_{ijww'}\left(e^{-V}\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]e^{V}\right) \tag{810}$$
$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)e^{V}\right]\right) \tag{811}$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}(t)\,e^{V}e^{-V}\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}(t)e^{V}e^{-V}\mathscr{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right) \tag{812}$$
$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}\left(t\right)e^{V}\rho_S\left(t\right)\right] - \left[e^{-V}A_ie^{V},\rho_S\left(t\right)e^{-V}\mathscr{D}^{\dagger}_{ijww'}\left(t\right)e^{V}\right]\right). \tag{813}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (797) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (797) reproduces the results of the reference [1], for the latter case we have that $i,j\in\{1,2,3\}$ and $\omega\in(0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{814}$$

After performing the transformation (24) on the Hamiltonian (814) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)\,|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, \tag{815}$$
$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), \tag{816}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{817}$$

The Hamiltonian (815) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\left|1\middle\rangle\middle\langle 1\right| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)\left|1\middle\rangle\middle\langle 1\right| - \frac{\delta}{2}|0\rangle\langle 0| \tag{818}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{819}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \tag{820}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \tag{821}$$

$$\lambda_\pm = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} \tag{822}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{823}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{824}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \tag{825}$$

$$\lambda_\pm = \frac{R \pm \eta}{2}. \tag{826}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{827}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle$.

For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{828}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle$. Summarizing these results we can write:

$$\lambda_{+} = \frac{\epsilon + \eta}{2}, \tag{829}$$
$$\lambda_{-} = \frac{\epsilon - \eta}{2}, \tag{830}$$
$$|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle\,, \tag{831}$$
$$|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle\,, \tag{832}$$
$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}, \tag{833}$$
$$\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}. \tag{834}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{835}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{836}$$

So the value of $\tan(\theta)$ using (836) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}} + 1} \tag{837}$$
$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{838}$$
$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{839}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (763)-(764) and the fact that $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\begin{aligned}
\langle +|\sigma_x|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (840)\\
&= 2\sin(\theta)\cos(\theta) && (841)\\
&= \sin(2\theta)\,, && (842)\\
\langle -|\sigma_x|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (843)\\
&= -2\sin(\theta)\cos(\theta) && (844)\\
&= -\sin(2\theta)\,, && (845)\\
\langle -|\sigma_x|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (846)\\
&= \cos^2(\theta) - \sin^2(\theta) && (847)\\
&= \cos(2\theta)\,, && (848)\\
\langle +|\sigma_y|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (849)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (850)\\
&= 0, && (851)\\
\langle -|\sigma_y|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (852)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (853)\\
&= 0, && (854)\\
\langle -|\sigma_y|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (855)\\
&= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) && (856)\\
&= \mathrm{i}. && (857)
\end{aligned}$$

$$\begin{aligned}
\langle +|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (858)\\
&= \cos(\theta)\cos(\theta) && (859)\\
&= \cos^2(\theta)\,, && (860)\\
\langle -|\frac{1+\sigma_z}{2}|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (861)\\
&= \sin(\theta)\sin(\theta) && (862)\\
&= \sin^2(\theta)\,, && (863)\\
\langle -|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (864)\\
&= -\sin(\theta)\cos(\theta) && (865)\\
&= -\sin(\theta)\cos(\theta)\,. && (866)
\end{aligned}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$
\begin{aligned}
A_1(0) &= \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), && (867)\\
A_1(\eta) &= \cos(2\theta)\,|-\rangle\langle+|, && (868)\\
A_2(0) &= 0, && (869)\\
A_2(\eta) &= \mathrm{i}|-\rangle\langle+|, && (870)\\
A_3(0) &= \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-|, && (871)\\
A_3(\eta) &= -\sin(\theta)\cos(\theta)\,|-\rangle\langle+|. && (872)
\end{aligned}
$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (800) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (781) is equivalent to:

$$
\begin{aligned}
\mathscr{D}_{ijww'}(t) &= \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau,t) && (873)\\
&= \int_0^t \mathrm{d}\tau C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w(t-\tau),w'(t)\right) && (874)\\
&= \int_0^t \mathrm{d}\tau C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}\tau w}e^{-\mathrm{i}t\left(w-w'\right)}\mathscr{A}_j(w,w'). && (875)
\end{aligned}
$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (396) in the following way:

$$
\Lambda'_{ij}(\tau) = C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau). \quad (876)
$$

So the response matrix can be rewritten as:

$$
\mathscr{D}_{ijww'}(t) = \left(\int_0^t \mathrm{d}\tau \Lambda'_{ij}(\tau)\,e^{\mathrm{i}\tau w}e^{-\mathrm{i}t\left(w-w'\right)}\right)\mathscr{A}_j(w,w') \quad (877)
$$

Let's define the response function like:

$$
\begin{aligned}
K_{ij}(w,w',t) &= \int_0^t C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau && (878)\\
&= \int_0^t \Lambda'_{ij}(\tau)\,e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau && (879)\\
&= K_{ijww'}(t). && (880)
\end{aligned}
$$

Then we have the following equivalence:

$$
\begin{aligned}
\mathscr{D}_{ijww'}(t) &= K_{ijww'}(t)\,\mathscr{A}_j(w,w') && (881)\\
&= K_{ijww'}(t)\,\mathscr{A}_{jww'} && (882)
\end{aligned}
$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, \mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\mathscr{D}^{\dagger}_{ijww'}(t)\right]\right) \quad (883)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, K_{ijww'}(t)\mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)K^{*}_{ijww'}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \quad (884)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(K_{ijww'}(t)\left[A_i, \mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - K^{*}_{ijww'}(t)\left[A_i, \overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \quad (885)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t)+\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - \left(K^{\Re}_{ijww'}(t)-\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \quad (886)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'} K^{\Re}_{ijww'}(t)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t) - \overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] - \mathrm{i}\sum_{ijww'} K^{\Im}_{ijww'}(t)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t) + \overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] \quad (887)$$

Using the notation of the master equation (797), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(687)-(697) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (396) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \quad (888)$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \quad (889)$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \quad (890)$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right), \quad (891)$$

$$\Lambda'_{33}(\tau) = \int_0^{\infty}\mathrm{d}\omega J(\omega)(1 - F(\omega))^2 G_+(\tau), \quad (892)$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^{\infty}\mathrm{d}\omega\frac{J(\omega)}{\omega}F(\omega)(1 - F(\omega))\,\mathrm{i}G_-(\tau), \quad (893)$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \quad (894)$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \quad (895)$$

$$= \Lambda'_{13}(\tau) \quad (896)$$

$$= \Lambda'_{31}(\tau) \quad (897)$$

$$= 0. \quad (898)$$

Finally taking the Hamiltonian (814) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that Det $\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, Tr $\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (336) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \quad (899)$$

$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \quad (900)$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (432) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)\,U^{\dagger}(t-\tau) = U(\tau)$. From the equation (797) and using the fact that

$$\widetilde{A_j}(t-\tau,t) = U(\tau)\,A_j U(-\tau) \tag{901}$$
$$= \sum_w e^{\mathrm{i}w\tau}\mathscr{A}_j(-w) \tag{902}$$
$$= \sum_w e^{-\mathrm{i}w\tau}\mathscr{A}_j(w)\,. \tag{903}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t),\overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_w \gamma_{ij}(w,t)\left[A_i,\mathscr{A}_j(w)\,\overline{\rho}_S(t) - \overline{\rho}_S(t)\,\mathscr{A}_j^{\dagger}(w)\right] \tag{904}$$
$$-\sum_{ij}\sum_w S_{ij}(w,t)\left[A_i,\mathscr{A}_j(w)\,\overline{\rho}_S(t) + \overline{\rho}_S(t)\,\mathscr{A}_j^{\dagger}(w)\right]. \tag{905}$$

where $\mathscr{A}_j^{\dagger}(w) = \mathscr{A}_j(-w)$, as we can see the equation (905) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2K_{ij}^{\Re}(w,t)$ and $S_{ij}(w,t) = K_{ij}^{\Im}(w,t)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}(w,t) = \int_0^t \Lambda'_{ij}(\tau)\,e^{\mathrm{i}w\tau}\mathrm{d}\tau.$$

The fact $\mathscr{A}_j^{\dagger}(w) = \mathscr{A}_j(-w)$ can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{\mathrm{i}\overline{H_S}(t-\tau)} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)(t-\tau)} \tag{906}$$
$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|(t-\tau)}e^{\mathrm{i}\lambda_-|-\rangle\langle-|(t-\tau)} \tag{907}$$
$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{908}$$
$$= e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|. \tag{909}$$

Where $\lambda_+,\lambda_-$are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (738) of (762)-(764) directly using the previous relationship we find that:

$$\widetilde{A_i(0)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{910}$$
$$= \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|, \tag{911}$$
$$\widetilde{A_i(w)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|-\rangle|+\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{912}$$
$$= \langle+|A_i|-\rangle|+\rangle\langle-|e^{\mathrm{i}w(t-\tau)}, \tag{913}$$
$$\widetilde{A_i(-w)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle-|A_i|+\rangle|-\rangle\langle+|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{914}$$
$$= \langle-|A_i|+\rangle|-\rangle\langle+|e^{-\mathrm{i}w(t-\tau)}. \tag{915}$$

Here $w = \lambda_+ - \lambda-$. So we can see that for the equation (748) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w \neq w'$ then $A'_j(w,w') = 0$ so:

$$\widetilde{A_j}\left(t-\tau,t\right)=U\left(t\right)U^{\dagger}\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right) \tag{916}$$
$$=U\left(t\right)\left(\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}A_j\left(w(t-\tau)\right)\right)U^{\dagger}\left(t\right) \tag{917}$$
$$=\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}U\left(t\right)A_j\left(w(t-\tau)\right)U^{\dagger}\left(t\right) \tag{918}$$
$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w(t-\tau),w'\left(t\right)\right) \tag{919}$$
$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_{jww'} \tag{920}$$
$$==\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w\right)\delta_{ww'} \tag{921}$$
$$=\sum_{w}e^{-\mathrm{i}(t-\tau)w}e^{\mathrm{i}tw}A_j\left(w\right) \tag{922}$$
$$=\sum_{w}e^{\mathrm{i}\tau w}A_j\left(w\right) \tag{923}$$
$$=U^{\dagger}\left(-\tau\right)A_jU\left(-\tau\right) \tag{924}$$

So using now as reference the equation (887) and $A_j'\left(w,w'\right)=0$ we can deduce that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijw}K_{ij}^{\Re}(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t)-\overline{\rho_S}(t)A_j^{\dagger}(w)\right]-\mathrm{i}\sum_{ijw}K_{ij}^{\Im}(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t)+\overline{\rho_S}(t)A_j^{\dagger}(w)\right] \tag{925}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k\ll\omega_k$ then $f_k\approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z=0$. The Hamiltonian studied is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega\left(t\right)}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{926}$$

If $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then $B\left(\tau\right)=B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (926) using the Hamiltonian of the equation (1) we can say that $\delta=\varepsilon_1\left(t\right)$, $\varepsilon_0\left(t\right)=0$, $V_{10}\left(t\right)=\frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then, in this case and using the equation (225) and (236) we obtain the following transformed Hamiltonians:

$$\overline{H_S}=\left(\delta+R_1\right)|1\rangle\langle 1|+\frac{B\sigma_x}{2}\Omega\left(t\right), \tag{927}$$
$$\overline{H_{\mathrm{I}}}=\frac{\Omega\left(t\right)}{2}\left(B_x\sigma_x+B_y\sigma_y\right). \tag{928}$$

In this case $R_1=\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}=g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right)=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta+R_1=\delta'$. If $F\left(\omega_{\mathbf{k}}\right)=1$ and using the equations (888)-(895) we can deduce that the only terms that survive are $\Lambda_{11}\left(\tau\right)$ and $\Lambda_{22}\left(\tau\right)$. The phonon propagator for this case is:

$$\phi\left(\tau\right)=\int_0^{\infty}\frac{J\left(\omega\right)}{\omega^2}G_{+}\left(\tau\right)\mathrm{d}\omega. \tag{929}$$

Writing $G_{+}\left(\tau\right)=\coth\left(\frac{\beta\omega}{2}\right)\cos\left(\omega\tau\right)-i\sin\left(\omega\tau\right)$ so (929) can be written as:

$$\phi(\tau)=\int_0^\infty \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau)-i\sin(\omega\tau)\right)\mathrm{d}\omega. \tag{930}$$

Writing the interaction Hamiltonian (928) in the similar way to the equation (236) allow us to to write $A_1=\sigma_x$, $A_2=\sigma_y$, $B_1(t)=B_x$, $B_2(t)=B_y$ and $C_1(t)=\frac{\Omega(t)}{2}=C_2(t)$. Now taking the equation (225) with $\delta'|1\rangle\langle 1|=\frac{\delta'}{2}\sigma_z+\frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S}=\frac{\delta'}{2}\sigma_z+\frac{B\sigma_x}{2}\Omega(t). \tag{931}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\begin{aligned}\Lambda_{11}(\tau)&=\mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) &(932)\\ &=\frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}-2\right), &(933)\\ \Lambda_{22}(\tau)&=\mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) &(934)\\ &=\frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}\right). &(935)\end{aligned}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau)=\Lambda_i(-\tau)$ for $i\in\{x,y\}$ respectively. The master equation for this section based on the equation(432) is:

$$\begin{aligned}\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}&=-\mathrm{i}\left[\frac{\delta'}{2}\sigma_z+\frac{\Omega_r(t)\sigma_x}{2},\rho_S(t)\right]-\sum_{i=1}^{2}\int_0^t\mathrm{d}\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i,\widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. &(936)\\ &\left.+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t),A_i\right]\right). &(937)\end{aligned}$$

Replacing $C_i(t)=\frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t)=\widetilde{\sigma_i}(t-\tau,t)$, also using the equations (932) and (935) on the equation (937) we obtain that:

$$\begin{aligned}\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}&=-\frac{\mathrm{i}}{2}\left[\delta'\sigma_z+\Omega_r(t)\sigma_x,\rho_S(t)\right]-\frac{\Omega(t)}{4}\int_0^t\mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x,\widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. &(938)\\ &\left.+\left[\sigma_y,\widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau)+\left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t),\sigma_x\right]\Lambda_x(\tau)+\left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t),\sigma_y\right]\Lambda_y(\tau)\right). &(939)\end{aligned}$$

As we can see $\left[A_j,\widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger=\left[\rho_S(t)\widetilde{A_i}(t-\tau,t),A_j\right]$, $\Lambda_x(\tau)=\Lambda_x(-\tau)$ and $\Lambda_y(\tau)=\Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega)=0$, so there is no transformation in this case. As we can see from the definition (396) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\hbar=1$ the Hamiltonian of the reference can be written in the form:

$$H=\Delta|1\rangle\langle 1|+\frac{\Omega(t)}{2}\left(|1\rangle\langle 0|+|0\rangle\langle 1|\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+|1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger+g_{\mathbf{k}}^*b_{\mathbf{k}}\right). \tag{940}$$

Using the equation (797), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)\right), \rho_S\left(t\right)] - \frac{1}{2}\sum_w \gamma_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) - \rho_S\left(t\right)A_3^{\dagger}\left(w\right)\right] \tag{941}$$

$$-\sum_w S_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) + \rho_S\left(t\right)A_3^{\dagger}\left(w\right)\right]\Bigg). \tag{942}$$

The correlation functions are relevant if $F\left(\omega\right) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}\left(\tau\right) = \int_0^{\infty} \mathrm{d}\omega J\left(\omega\right) G_+\left(\tau\right), \tag{943}$$

$$\Lambda_{33}\left(-\tau\right) = \int_0^{\infty} \mathrm{d}\omega J\left(\omega\right) G_+\left(-\tau\right). \tag{944}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (942) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)\right)] - \sum_w \left(K_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right)\right] + K_{33}^{*}\left(w,t\right)\left[\rho_S\left(t\right)A_3\left(w\right), A_3\right]\right). \tag{945}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3\left(w\right)$, we have to remember that $H_S = \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{946}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right), \tag{947}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{948}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{949}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{950}$$

$$= \frac{1}{2}, \tag{951}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{952}$$

$$= \frac{1}{2}, \tag{953}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{954}$$

$$= -\frac{1}{2}. \tag{955}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{956}$$
$$= \frac{\mathbb{I}}{2}, \tag{957}$$
$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle+| \tag{958}$$
$$= \frac{1}{4}(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}), \tag{959}$$
$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle-| \tag{960}$$
$$= \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y). \tag{961}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}[\sigma_x, \rho_S(t))] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)\right] \tag{962}$$
$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right]. \tag{963}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty\int_0^\infty J(\omega)\, G_+(\tau)\, e^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{964}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\, e^{\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega)\, e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{965}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\, e^{\mathrm{i}\widetilde{\Omega}\tau}(n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{966}$$
$$= \int_0^\infty\int_0^\infty J(\omega)(n(\omega)+1)\,\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{967}$$
$$= \int_0^\infty J(\omega)(n(\omega)+1)\,\pi\delta\left(\widetilde{\Omega} - \omega\right)\mathrm{d}\omega \tag{968}$$
$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right) + 1\right), \tag{969}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty\int_0^\infty J(\omega)\, G_+(\tau)\, e^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{970}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\, e^{-\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega)\, e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{971}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\, e^{-\mathrm{i}\widetilde{\Omega}\tau} n(\omega)\,\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{972}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\, n(\omega)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau + \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{973}$$
$$= \int_0^\infty J(\omega)\, n(\omega)\,\pi\delta\left(-\widetilde{\Omega} + \omega\right)\mathrm{d}\omega \tag{974}$$
$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right). \tag{975}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s} = \pi\delta(\varepsilon) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (962) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}[\sigma_x, \rho_S(t))] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)] + n\left(\widetilde{\Omega}\right)[\sigma_z, (\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)]\right) \tag{976}$$
$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)[\rho_S(t)(\sigma_z + \mathrm{i}\sigma_y), \sigma_z] + n\left(\widetilde{\Omega}\right)[\rho_S(t)(\sigma_z - \mathrm{i}\sigma_y), \sigma_z]\right). \tag{977}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}[\sigma_x, \rho_S(t))] - \frac{\pi}{8}J(\Omega(t))\left((n(\Omega(t))+1)[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)] + n(\Omega(t))[\sigma_z, (\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)]\right) \tag{978}$$
$$- \frac{\pi}{8}J(\Omega(t))\left((n(\Omega(t))+1)[\rho_S(t)(\sigma_z + \mathrm{i}\sigma_y), \sigma_z] + n(\Omega(t))[\rho_S(t)(\sigma_z - \mathrm{i}\sigma_y), \sigma_z]\right). \tag{979}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{980}$$
$$H_S(t) = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|, \tag{981}$$
$$H_I = \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right), \tag{982}$$
$$H_B = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}. \tag{983}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}}|n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{984}$$

At first let's obtain $e^{\pm V}$ under the transformation (984), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)$, so the equation (984) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$\begin{aligned}
e^{\pm V} &= e^{\pm \sum_n |n\rangle\langle n|\hat{\varphi}_n} && (985)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!} + ... && (986)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + ... && (987)\\
&= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + ... && (988)\\
&= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + ...\right) && (989)\\
&= \sum_n |n\rangle\langle n| e^{\pm\hat{\varphi}_n} && (990)
\end{aligned}$$

Given that $\left[f_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} - f^*_{nu\mathbf{k}}b_{u\mathbf{k}}, f_{nu'\mathbf{k}'}b^\dagger_{u'\mathbf{k}'} - f^*_{nu'\mathbf{k}'}b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right) = e^{\pm\left(\alpha_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} - \alpha^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$\begin{aligned}
e^{\pm\sum_{u\mathbf{k}}\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} - f^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)} &= \prod_u e^{\pm\sum_{\mathbf{k}}\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} - f^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)} && (991)\\
&= \prod_u \left(\prod_{\mathbf{k}} e^{\pm\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} - f^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)}\right) && (992)\\
&= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) && (993)\\
&= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) && (994)\\
&= \prod_u B_{nu\pm} && (995)\\
B_{nu\pm} &\equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) && (996)
\end{aligned}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \qquad (997)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (980):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{998}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{999}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1000}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1001}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1002}$$

$$= |0\rangle\langle 0|. \tag{1003}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1004}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1005}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1006}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right), \ m \neq n, \tag{1007}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1008}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1009}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1010}$$

$$\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1011}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1012}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right) \left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), \tag{1013}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, \tag{1014}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots \tag{1015}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) \tag{1016}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots \tag{1017}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) \tag{1018}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots \tag{1019}$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1020}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1021}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{1022}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) \tag{1023}$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{1024}$$

The transformed Hamiltonians of the equations (981) to (983) written in terms of (998) to (1022) are:

$$
\begin{aligned}
\overline{H_S\left(t\right)} &= \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} && (1025)\\
&= \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} && (1026)\\
&= \sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|\prod_u\left(B_{mu+}B_{nu-}\right) && (1027)\\
\overline{H_I} &= \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) && (1028)\\
&= \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)+...\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) && (1029)\\
&= \prod_u B_{0u+}\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{0u-}+\prod_u B_{1u+}\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{1u-}+\ldots && (1030)\\
&= \sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\prod_u B_{0u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{0u-}+g_{0u\mathbf{k}}^{*}\prod_u B_{0u+}b_{u\mathbf{k}}\prod_u B_{0u-}\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\prod_u B_{1u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{1u-}+g_{1u\mathbf{k}}^{*}\prod_u B_{1u+}b_{u\mathbf{k}}\prod_u B_{1u-}\right)+\ldots && (1031)\\
&= \sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\ldots && (1032)\\
&= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1033)\\
&= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1034)\\
\overline{H_B} &= \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) && (1035)
\end{aligned}
$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$
\begin{aligned}
\overline{H} &= \sum_n \varepsilon_n(t)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_u\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) && (1036)\\
&+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1037)
\end{aligned}
$$

Let's define the following functions:

$$
\begin{aligned}
R_n\left(t\right) &= \sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1038)\\
B_{z,n}\left(t\right) &= \sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)b_{u\mathbf{k}}^{\dagger}+\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)^{*}b_{u\mathbf{k}}\right) && (1039)
\end{aligned}
$$

Using the previous functions we have that (1036) can be re-written in the following way:

$$
\begin{aligned}
\overline{H} &= \sum_n \varepsilon_n(t)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_u\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_n R_n(t)|n\rangle\langle n|+\sum_n B_{z,n}(t)|n\rangle\langle n| && (1040)\\
& && (1041)
\end{aligned}
$$

Now in order to separate the elements of the hamiltonian (1041) let's follow the references of the equations (225) and (236) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle\prod_u (B_{mu+}B_{nu-})\right\rangle_{\overline{H_0}}=\left\langle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1042}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\left\langle\prod_{u\mathbf{k}}D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1043}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\tfrac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1044}$$

$$\equiv B_{nm} \tag{1045}$$

$$\left\langle\prod_u (B_{nu+}B_{mu-})\right\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\tfrac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1046}$$

$$=B^*_{nm} \tag{1047}$$

Following the reference [4] we define:

$$J_{nm}=\prod_u (B_{mu+}B_{nu-})-B_{nm} \tag{1048}$$

As we can see:

$$J^\dagger_{nm}=\left(\prod_u (B_{mu+}B_{nu-})-B_{nm}\right)^\dagger \tag{1049}$$

$$=\prod_u (B_{nu+}B_{mu-})-B^*_{nm} \tag{1050}$$

$$=\prod_u (B_{nu+}B_{mu-})-B_{mn} \tag{1051}$$

$$=J_{mn} \tag{1052}$$

We can separate the Hamiltonian (1041) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)}=\sum_n(\varepsilon_n(t)+R_n)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|B_{nm} \tag{1053}$$

$$\overline{H_{\bar{I}}}=\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|J_{nm}+\sum_n B_{z,n}(t)|n\rangle\langle n|, \tag{1054}$$

$$\overline{H_{\bar{B}}}=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b^\dagger_{u\mathbf{k}}b_{u\mathbf{k}} \tag{1055}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A\leq A_{\mathrm{B}}\equiv-\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right)+\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}+O\left(\left\langle\overline{H^2_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_B}}\right). \tag{1056}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right)=C\left(R_0,R_1,...,R_{d-1},B_{01},...,B_{0(d-1)},...,B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}}=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^{\Re}_{nu\mathbf{k}}$ and $v^{\Im}_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)}=\sum_n \left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1057}$$

$$+\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1058}$$

$$=\sum_n \left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\frac{g_{nu\mathbf{k}}v^*_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1059}$$

$$+\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1060}$$

$$=\sum_n \left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}\right)v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\left(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1061}$$

$$+\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1062}$$

$$v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}=\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)-\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right) \tag{1063}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1064}$$

$$-\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1065}$$

$$=2\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right) \tag{1066}$$

$$\overline{H_{\bar{S}}(t)}=\sum_n \left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}\right)v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\left(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1067}$$

$$+\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1068}$$

$$\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2=\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)^* \tag{1069}$$

$$=\left|v_{mu\mathbf{k}}\right|^2+\left|v_{nu\mathbf{k}}\right|^2-\left(v_{nu\mathbf{k}}v^*_{mu\mathbf{k}}+v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}\right) \tag{1070}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1071}$$

$$-\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1072}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2\left(v^{\Re}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{nu\mathbf{k}}v^{\Im}_{mu\mathbf{k}}\right) \tag{1073}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2 \tag{1074}$$

$$R_n(t)=\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1075}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-\left(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}\right)v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\left(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1076}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2g^{\Re}_{nu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-2g^{\Im}_{nu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \tag{1077}$$

$$B_{mn}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1078}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1079}$$

Then we can obtain using the chain rule that:

$$\begin{aligned}
\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1080)\\
&= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1081)\\
&= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1082)\\
\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1083)\\
&= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1084)\\
&= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1085)
\end{aligned}$$

Given that:

$$\begin{aligned}
\ln B_{mn} &= \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) && (1086)\\
&= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1087)\\
&= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1088)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1089)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1090)\\
\frac{\partial \ln B_{mn}}{\partial a} &= \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} && (1091)\\
\frac{\partial B_{mn}}{\partial a} &= B_{mn} \frac{\partial \ln B_{mn}}{\partial a} && (1092)\\
\frac{\partial B_{mn}}{\partial a} &= \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} && (1093)
\end{aligned}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1094}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1095}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1096}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1097}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1098}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1099}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1100}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1101}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1102}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1103}$$

$$= (B_{mn})^{\dagger} \tag{1104}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1105}$$

Introducing this derivates in the equation (1080) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1106}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1107}$$

$$= 0 \tag{1108}$$

We can obtain the variational parameters:

$$
-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1109}
$$

$$
=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1110}
$$

$$
v^{\Re}_{nu\mathbf{k}}=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1111}
$$

$$
=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1112}
$$

Let's consider the imaginary part of the variation parameters

$$
\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}}=\frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}}-g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1113}
$$

$$
\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1114}
$$

$$
=0 \tag{1115}
$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1116}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1117}$$

$$v^{\Im}_{nu\mathbf{k}}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1118}$$

$$=\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1119}$$

$$v_{nu\mathbf{k}}=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}} \tag{1120}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1121}$$

$$\mathrm{i}\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1122}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1123}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1124}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1125}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1126}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1127}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1128}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^{V}\rho(0)\,e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1129}$$

$$0=\left(B^{+}_0|0\rangle\langle 0|B^{-}_0\right)\otimes\rho_B \tag{1130}$$

$$0=\rho(0) \tag{1131}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1132}$$
$$U(t) \equiv \mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1133}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\,,\ \text{where} \tag{1134}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1135}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1136}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1137}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1138}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1139}$$

We can proof that $B_{nm} = B^{\dagger}_{mn}$

$$B^{\dagger}_{mn} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1140}$$
$$= B^{\dagger}_{n-}B^{\dagger}_{m+} - B_n B_m \tag{1141}$$
$$= B_{n+}B_{m-} - B_n B_m \tag{1142}$$
$$= B_{nm} \tag{1143}$$

So we can say that the set of matrices (1136) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1136) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t)\,|n\rangle\langle n|, \tag{1144}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)\,|n\rangle\langle m| B_{nm} + V_{mn}(t)\,|m\rangle\langle n| B_{mn}\right) \tag{1145}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V^{\Im}_{nm}(t)\,B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1146}$$
$$\left.+\Re\left(V_{nm}(t)\right)B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V^{\Im}_{nm}(t)\,B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1147}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1148}$$
$$\left.+\mathrm{i}\Im\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1149}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1150}$$
$$\left.+\Im\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,x}\right) \tag{1151}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1152}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1-\delta_{mn}) \tag{1153}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1-\delta_{mn}) \tag{1154}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1155}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1156}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1157}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1158}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1159}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1160}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1161}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1162}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1163}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1164}$$
$$C_{3,nm}(t) = 1 \tag{1165}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1166}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1167}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J, p\in P} C_{jp}(t)\left(A_{jp}\otimes B_{jp}(t)\right) \tag{1168}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1152).

We write the interaction Hamiltonian transformed under (1132) as:

$$\widetilde{H_I}(t) = \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes \widetilde{B_{jp}}(t)\right) \tag{1169}$$
$$\widetilde{A_{jp}}(t) = U^{\dagger}(t)\,A_{jp}U(t) \tag{1170}$$
$$\widetilde{B_{jp}}(t) = \mathrm{e}^{\mathrm{i}H_B t}B_{jp}(t)\,(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{1171}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1172}$$

Replacing the equation (1169) in (1172) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1173}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right), \widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1174}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1175}$$

$$\left. -\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1176}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1177}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1178}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1179}$$

$$\left. +\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1180}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1181}$$

$$= \left\langle \widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1182}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1180) are:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) &= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B && (1183)\\
&= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B && (1184)\\
&= \Lambda_{jpj'p'}\left(\tau\right) && (1185)\\
\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) && (1186)\\
&= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B && (1187)\\
&= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B && (1188)\\
&= \Lambda_{j'p'jp}\left(-\tau\right) && (1189)\\
\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) && (1190)\\
&= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B && (1191)\\
&= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B && (1192)\\
&= \Lambda_{jpj'p'}\left(\tau\right) && (1193)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) && (1194)\\
&= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B && (1195)\\
&= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B && (1196)\\
&= \Lambda_{j'p'jp}\left(-\tau\right) && (1197)
\end{aligned}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1173)and (1180) and using the equations (1183)-(1197) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}\left(t\right)}{\mathrm{d}t} = -\int_0^t\sum_{j,j',p,p'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)-\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\right)\right. \tag{1198}$$

$$\left.+C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right)-\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1199}$$

$$= -\int_0^t\sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[\widetilde{A_{jp}}\left(t\right),\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\right]+\Lambda_{j'p'jp}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right),\widetilde{A_{jp}}\left(t\right)\right]\right)\right) \tag{1200}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\int_0^t\sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right),A_{j'p'}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right]+\Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}\left(t\right)A_{j'p'}\left(t-\tau,t\right),A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau-\mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}\left(t\right)\right] \tag{1201}$$

For this case we used that $A_{jp}\left(t-\tau,t\right)=U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right)=H_{S}\left(t\right)+H_{I}+H_{B}, \tag{1202}$$

$$H_{S}\left(t\right)=\sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|, \tag{1203}$$

$$H_{I}=\left(\sum_{n=0}\mu_{n}\left(t\right)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right), \tag{1204}$$

$$H_{B}=\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{1205}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0}\mu_{n}\left(t\right)|n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V=\left(\sum_{n=1}|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}\right)\right) \tag{1206}$$

At first let's obtain e^{V} under the transformation (1206), consider $\hat{\varphi}=\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}\right)$:

$$e^{V}=e^{\sum_{n=1}|n\rangle\langle n|\hat{\varphi}} \tag{1207}$$

$$=\mathbb{I}+\sum_{n=1}|n\rangle\langle n|\hat{\varphi}+\frac{\left(\sum_{n=1}|n\rangle\langle n|\hat{\varphi}\right)^{2}}{2!}+... \tag{1208}$$

$$=\mathbb{I}+\sum_{n=1}|n\rangle\langle n|\hat{\varphi}+\frac{\sum_{n=1}|n\rangle\langle n|\hat{\varphi}^{2}}{2!}+... \tag{1209}$$

$$=\mathbb{I}-\sum_{n=1}|n\rangle\langle n|+\sum_{n=1}|n\rangle\langle n|\left(\mathbb{I}+\hat{\varphi}+\frac{\hat{\varphi}^{2}}{2!}+...\right) \tag{1210}$$

$$=|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|e^{\hat{\varphi}} \tag{1211}$$

$$=|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B^{+} \tag{1212}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger}-b_{\mathbf{k}'},b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}\right]=0$ if $\mathbf{k}^{'}\neq\mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)=e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}\right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}}\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}\right)}=\prod_{\mathbf{k}}e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}\right)} \tag{1213}$$

$$=\prod_{\mathbf{k}}D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1214}$$

$$=B_{\pm} \tag{1215}$$

As we can see $e^{-V}=|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B_{-}$ because this form imposes that $e^{-V}e^{V}=\mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V}=\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B^{+}\right)A\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1216}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1202):

$$\begin{aligned}
\overline{|0\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), && (1217)\\
&= |0\rangle\langle 0|, && (1218)\\
\overline{|m\rangle\langle n|} &= \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), && (1219)\\
&= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, && (1220)\\
&= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, && (1221)\\
\overline{|0\rangle\langle m|} &= \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), && (1222)\\
&= |0\rangle\langle m|B^{-}\ m \neq 0, && (1223)\\
\overline{|m\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) && (1224)\\
&= |0\rangle\langle m|B^{+}\ m \neq 0, && (1225)\\
\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}} &= \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) && (1226)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B^{+}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B^{-} && (1227)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B^{+}b^{\dagger}_{\mathbf{k}}B^{-}\right)\left(B^{+}b_{\mathbf{k}}B^{-}\right) && (1228)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1229)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1230)\\
&= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right) && (1231)\\
&= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right) && (1232)
\end{aligned}$$

The transformed Hamiltonians of the equations (1203) to (1205) written in terms of (1217) to (1232) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1233}$$

$$= \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1234}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \overline{\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n|} \tag{1235}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)\overline{|0\rangle\langle n|} + V_{n0}(t)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0}V_{mn}(t)\overline{|m\rangle\langle n|} \tag{1236}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)B^{-}|0\rangle\langle n| + V_{n0}(t)B^{+}|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1237}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1238}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(\left(\sum_{n=0}\mu_n(t)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1239}$$

$$= \left(\mu_0(t)|0\rangle\langle 0| + \sum_{n=1}\mu_n(t)|n\rangle\langle n|B^{+}\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1240}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}B^{+}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)B^{-} \tag{1241}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1242}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1243}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1244}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1245}$$

$$+\sum_{n=0}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1246}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1247}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1248}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1249}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1250}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\right) \tag{1251}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1252}$$

Using the previous functions we have that (1249) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n|B^- + V_{n0}(t)\,|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1253}$$

$$+\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n|n\rangle\langle n| + \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1254}$$

Now in order to separate the elements of the hamiltonian (1254) let's follow the references of the equations (236) and (225) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| + \sum_{n=1} R_n|n\rangle\langle n| \tag{1255}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)\,|n\rangle\langle 0|\left(B^+ - B\right)\right), \tag{1256}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{1257}$$

Here B is given by:

$$\begin{aligned} B &= \left\langle B^+ \right\rangle \\ &= \left\langle B^- \right\rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1258}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1259}$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1260}$$

$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1261}$$

Using this set of hermitian operators to write the Hamiltonians (1203)-(1205)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1262}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \tag{1263}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1264}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \tag{1265}$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1266}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1267}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x}-\mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x}+\mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1268}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1269}$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1270}$$

$$\overline{H_I(t)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^{-}-B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^{+}-B\right)\right) \tag{1271}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\left(B^{-}-B\right)\frac{\sigma_{0n,x}-\mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\left(B^{+}-B\right)\frac{\sigma_{0n,x}+\mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1272}$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1273}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^{-}-B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + \left(B^{+}-B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\right)\right. \tag{1274}$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^{+}-B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - \left(B^{-}-B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\right)\right) \tag{1275}$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1276}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^{+}+B^{-}-2B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B^{-}-B-B^{+}+B\right)\Im(V_{0n}(t))\right. \tag{1277}$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^{+}-B-B^{-}+B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B-B^{-}+B-B^{+}\right)\Im(V_{0n}(t))\right)\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1278}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1279}$$

$$+\sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1280}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right)+\left\langle\overline{H_I}\right\rangle_{\overline{H_S+H_B}}+O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1281}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)=C\left(R_1,R_2,...,R_{d-1},B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (254) and using the chain rule we obtain that:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}}+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1282}$$
$$= 0 \tag{1283}$$

Let's recall the equations (1250) and (1252), we can write them in terms of the variational parameters

$$B = \exp\left(-\left(1/2\right)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1284}$$
$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1285}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1286}$$
$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1287}$$

Introducing this derivates in the equation (1282) give us:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right)+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1288}$$
$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right)-\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1289}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1290}$$
$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1291}$$

Now taking $v_{\mathbf{k}}=g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1292}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_{B}$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho\left(0\right)e^{-V}=\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(|0\rangle\langle 0|\otimes\rho_{B}\right)\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1293}$$

$$0=|0\rangle\langle 0|\otimes\rho_{B} \tag{1294}$$

$$0=\rho\left(0\right) \tag{1295}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}\equiv U^{\dagger}\left(t\right)OU\left(t\right) \tag{1296}$$

$$U\left(t\right)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{S}}\left(t'\right)\right). \tag{1297}$$

Therefore:

$$\widetilde{\overline{\rho_{S}}}\left(t\right)=U^{\dagger}\left(t\right)\overline{\rho_{S}}\left(t\right)U\left(t\right),\text{ where} \tag{1298}$$

$$\overline{\rho_{S}}\left(t\right)=\mathrm{Tr}_{B}\left(\bar{\rho}\left(t\right)\right) \tag{1299}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_{I}\left(t\right)}=B_{z,0}|0\rangle\langle 0|+\sum_{n=1}\left(\Re\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,x}+\Re\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,y}+B_{z,n}|n\rangle\langle n|\right. \tag{1300}$$

$$\left.+\Im\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,y}-\Im\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,x}\right) \tag{1301}$$

where

$$B_{z,0}=\sum_{\mathbf{k}}g_{\mathbf{k}}\mu_{0}\left(t\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1302}$$

$$B_{z,n}=\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_{n}\left(t\right)-\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\text{ if}n\neq 0 \tag{1303}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1304}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1305}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1306}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1307}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1308}$$
$$B_{1n}(t) = B_x \tag{1309}$$
$$B_{2n}(t) = B_y \tag{1310}$$
$$B_{3n}(t) = B_{z,n} \tag{1311}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1312}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1313}$$
$$C_{10}(t) = 0 \tag{1314}$$
$$C_{20}(t) = 0 \tag{1315}$$
$$C_{40}(t) = 0 \tag{1316}$$
$$C_{50}(t) = 0 \tag{1317}$$
$$C_{30}(t) = 1 \tag{1318}$$
$$C_{1n}(t) = \Re\left(V_{0n}(t)\right) \tag{1319}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1320}$$
$$C_{3n}(t) = 1 \tag{1321}$$
$$C_{4n}(t) = \Im\left(V_{0n}(t)\right) \tag{1322}$$
$$C_{5n}(t) = -\Im\left(V_{0n}(t)\right) \tag{1323}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn} \otimes B_{jn}(t)\right) \tag{1324}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1296) as:

$$\widetilde{H_I}(t) = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1325}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\, A_i U(t) \tag{1326}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{1327}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t))$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1328}$$

Replacing the equation (1325)in (1328)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1329}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1330}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1331}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1332}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1333}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1334}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1335}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1336}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1337}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1338}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1336) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1339}$$

$$= \left\langle\widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1340}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1341}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1342}$$

$$= \left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1343}$$

$$= \left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1344}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1345}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1346}$$

$$= \left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1347}$$

$$= \left\langle\widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1348}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1349}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1350}$$

$$= \left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1351}$$

$$= \left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1352}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1353}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1329)and (1336) and using the equations (1339)-(1353) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1354}$$

$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1355}$$

$$= -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t)\,,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,,\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1356}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t)\,,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t)\,,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1357}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (432) as expected.

VIII. BIBLIOGRAPHY

[1] McCutcheon D P S, Dattani N S, Gauger E M, Lovett B W and Nazir A 2011 Phys. Rev. B 84 081305
[2] Dara P S McCutcheon and Ahsan Nazir 2010 New J. Phys. 12 113042
[3] Supplement: Theoretical model of phonon induced dephasing. A.J. Ramsay ey al 2009.
[4] Felix A Pollock et al 2013 New J. Phys. 15 075018

* n.dattani@cfa.harvard.edu
† edcchaparroso@unal.edu.co