A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \varepsilon_0(t)|0\rangle\langle0| + \varepsilon_1(t)|1\rangle\langle1| + V_{10}(t)|1\rangle\langle0| + V_{01}(t)|0\rangle\langle1|, \tag{2}$$

$$H_I = |0\rangle\langle0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^*b_{\mathbf{k}}\right) + |1\rangle\langle1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^*b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

I. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv |0\rangle\langle0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}(t)$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}(t). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i(t) \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V(t) = |0\rangle\langle0|\hat{\varphi}_0(t) + |1\rangle\langle1|\hat{\varphi}_1(t). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V(t)}$ using the notation (6) will give us the following result:

$$e^{\pm V(t)} = e^{\pm(|0\rangle\langle 0|\hat\varphi_0(t) + |1\rangle\langle 1|\hat\varphi_1(t))} \tag{10}$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat\varphi_0(t) + |1\rangle\langle 1|\hat\varphi_1(t)) + \frac{(\pm(|0\rangle\langle 0|\hat\varphi_0(t) + |1\rangle\langle 1|\hat\varphi_1(t)))^2}{2!} + ... \tag{11}$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat\varphi_0(t) + |1\rangle\langle 1|\hat\varphi_1(t)) + \frac{|0\rangle\langle 0|\hat\varphi_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat\varphi_1^2(t)}{2!} + ... \tag{12}$$

$$= |0\rangle\langle 0|\left(\mathbb{I} \pm \hat\varphi_0(t) + \frac{\hat\varphi_0^2(t)}{2!} \pm ...\right) + |1\rangle\langle 1|\left(\mathbb{I} \pm \hat\varphi_1(t) + \frac{\hat\varphi_1^2(t)}{2!} \pm ...\right) \tag{13}$$

$$= |0\rangle\langle 0|e^{\pm\hat\varphi_0(t)} + |1\rangle\langle 1|e^{\pm\hat\varphi_1(t)} \tag{14}$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*(t)b_{\mathbf{k}}\right)} \tag{15}$$

$$= |0\rangle\langle 0|B_0^\pm(t) + |1\rangle\langle 1|B_1^\pm(t), \tag{16}$$

$$B_i^\pm(t) \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{17}$$

Let's recall the Zassenhaus formula:

$$e^{r(X+Y)} = e^{rX} \, e^{rY} \, e^{-\frac{r^2}{2}[X,Y]} \, e^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])} \, e^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[X,Y],Y],Y])} \cdots \tag{18}$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i, j we can show making $r = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} = e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]} \cdots \tag{19}$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}0} \cdots \tag{20}$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. \tag{21}$$

By induction of this result we can write an expresion of $B_i^\pm(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$D(\pm v_{i\mathbf{k}}(t)) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, \tag{22}$$

$$B_i^\pm(t) = \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). \tag{23}$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$\overline{O(t)} \equiv e^{V(t)}O(t)e^{-V(t)}. \tag{24}$$

We will use the following identities:

$$\overline{|0\rangle\langle0|\,(t)} = e^{V(t)}|0\rangle\langle0|e^{-V(t)} \tag{25}$$

$$= \left(|0\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\right)|0\rangle\langle0|\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{26}$$

$$= \left(|0\rangle\langle0|0\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|0\rangle\langle0|B_1^+\,(t)\right)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{27}$$

$$= |0\rangle\langle0|B_0^+\,(t)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{28}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle0|1\rangle\langle1|B_0^+\,(t)\,B_1^-\,(t) \tag{29}$$

$$= |0\rangle\langle0|, \tag{30}$$

$$\overline{|1\rangle\langle1|\,(t)} = \left(|0\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\right)|1\rangle\langle1|\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{31}$$

$$= \left(|0\rangle\langle0|1\rangle\langle1|B_0^+\,(t) + |1\rangle\langle1|1\rangle\langle1|B_1^+\,(t)\right)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{32}$$

$$= |1\rangle\langle1|B_1^+\,(t)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{33}$$

$$= |1\rangle\langle1|0\rangle\langle0|B_1^+\,(t)\,B_0^-\,(t) + B_1^+\,(t)\,|1\rangle\langle1|1\rangle\langle1|B_1^-\,(t) \tag{34}$$

$$= B_1^+\,(t)\,|1\rangle\langle1|1\rangle\langle1|B_1^-\,(t) \tag{35}$$

$$= |1\rangle\langle1|, \tag{36}$$

$$\overline{|0\rangle\langle1|\,(t)} = e^{V(t)}|0\rangle\langle1|e^{-V(t)} \tag{37}$$

$$= \left(|0\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\right)|0\rangle\langle1|\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{38}$$

$$= \left(|0\rangle\langle0|0\rangle\langle1|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\,|0\rangle\langle1|\right)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{39}$$

$$= \left(|0\rangle\langle0|0\rangle\langle1|B_0^+\,(t) + |1\rangle\langle1|0\rangle\langle1|B_1^+\,(t)\right)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{40}$$

$$= |0\rangle\langle1|B_0^+\,(t)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{41}$$

$$= |0\rangle\langle1|0\rangle\langle0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle1|1\rangle\langle1|B_0^+\,(t)\,B_1^-\,(t) \tag{42}$$

$$= |0\rangle\langle1|B_0^+\,(t)\,B_1^-\,(t)\,, \tag{43}$$

$$\overline{|1\rangle\langle0|\,(t)} = e^{V(t)}|1\rangle\langle0|e^{-V(t)} \tag{44}$$

$$= \left(|0\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\right)|1\rangle\langle0|\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{45}$$

$$= \left(|0\rangle\langle0|1\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\,|1\rangle\langle0|\right)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{46}$$

$$= \left(|0\rangle\langle0|1\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|1\rangle\langle0|B_1^+\,(t)\right)\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{47}$$

$$= |1\rangle\langle0|0\rangle\langle0|B_1^+\,(t)\,B_0^-\,(t) + |1\rangle\langle0|1\rangle\langle1|B_1^+\,(t)\,B_1^-\,(t) \tag{48}$$

$$= |1\rangle\langle0|B_1^+\,(t)\,B_0^-\,(t)\,, \tag{49}$$

$$\overline{b_{\mathbf{k}}\,(t)} = e^{V(t)}b_{\mathbf{k}}e^{-V(t)} \tag{50}$$

$$= \left(|0\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\right)b_{\mathbf{k}}\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{51}$$

$$= |0\rangle\langle0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle0| + |0\rangle\langle0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle1|B_1^-(t) + |1\rangle\langle1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle0|B_0^-(t) + |1\rangle\langle1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle1| \tag{52}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_0^+\,(t)\,b_{\mathbf{k}}B_0^-\,(t) + |0\rangle\langle0|1\rangle\langle1|B_0^+\,(t)\,b_{\mathbf{k}}B_1^-\,(t) + |1\rangle\langle1|0\rangle\langle0|B_1^+\,(t)\,b_{\mathbf{k}}B_0^-\,(t) + |1\rangle\langle1|B_1^+\,(t)\,b_{\mathbf{k}}B_1^-\,(t) \tag{53}$$

$$= |0\rangle\langle0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{54}$$

$$= \left(|0\rangle\langle0| + |1\rangle\langle1|\right)b_{\mathbf{k}} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{55}$$

$$= b_{\mathbf{k}} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\,, \tag{56}$$

$$\overline{b_{\mathbf{k}}\,(t)}^{\dagger} = e^{V(t)}b_{\mathbf{k}}^{\dagger}e^{-V(t)} \tag{57}$$

$$= \left(|0\rangle\langle0|B_0^+\,(t) + |1\rangle\langle1|B_1^+\,(t)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle0|B_0^-\,(t) + |1\rangle\langle1|B_1^-\,(t)\right) \tag{58}$$

$$= |0\rangle\langle0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle0| + |0\rangle\langle0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle1|B_1^-(t) + |1\rangle\langle1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle0|B_0^-(t) + |1\rangle\langle1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle1| \tag{59}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_0^+\,(t)b_{\mathbf{k}}^{\dagger}B_0^-\,(t) + |0\rangle\langle0|1\rangle\langle1|B_0^+\,(t)b_{\mathbf{k}}^{\dagger}B_1^-\,(t) + |1\rangle\langle1|0\rangle\langle0|B_1^+\,(t)b_{\mathbf{k}}^{\dagger}B_0^-\,(t) + |1\rangle\langle1|1\rangle\langle1|B_1^+\,(t)b_{\mathbf{k}}^{\dagger}B_1^-\,(t) \tag{60}$$

$$= |0\rangle\langle0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) \tag{61}$$

$$= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\,. \tag{62}$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$:

$$B_i^+(t)\, b_{\mathbf{k}}\, B_i^-(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \tag{63}$$

$$B_i^+(t)\, b_{\mathbf{k}}^\dagger\, B_i^-(t) = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. \tag{64}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|}\,(t) = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{65}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|}\,(t) = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{66}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|}\,(t) = V_{10}(t)\,|1\rangle\langle 0|\,B_1^+(t)\,B_0^-(t), \tag{67}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|}\,(t) = V_{01}(t)\,|0\rangle\langle 1|\,B_0^+(t)\,B_1^-(t), \tag{68}$$

$$\overline{\left(g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}}\right)(t)} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)\right) + g_{i\mathbf{k}}^*\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{69}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^*\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{70}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^*\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{72}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{73}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)}\,(t) = \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) \tag{74}$$

$$= |0\rangle\langle 0|B_0^+(t)\,|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^-(t) \tag{75}$$

$$= |0\rangle\langle 0|B_0^+(t)\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)B_0^-(t) \tag{76}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^*\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{77}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)}\,(t) = \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) \tag{78}$$

$$= |1\rangle\langle 1|B_1^+(t)\,|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^-(t) \tag{79}$$

$$= |1\rangle\langle 1|B_1^+(t)\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)B_1^-(t) \tag{80}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^*\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{81}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\,(t) = \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) \tag{82}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\,B_0^-(t) + |1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\,B_1^-(t)\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\sum_j|j\rangle\langle j|D\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (89)$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \quad (92)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right). \quad (93)$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|} \quad (94)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_1^+(t) B_0^-(t) + V_{01}(t) |0\rangle\langle 1| B_0^+(t) B_1^-(t), \quad (95)$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \quad (96)$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \quad (97)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (98)$$

$$= \sum_{\mathbf{k},i} |i\rangle\langle i| \left(g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (99)$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} \quad (100)$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right) \quad (101)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right). \quad (102)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right). \quad (103)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}(t). \quad (104)$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (105)$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^* b_{\mathbf{k}} \right), \quad (106)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right). \quad (107)$$

$\chi_{ij}(t)$ is an imaginary number so $e^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$
\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} & e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \\ \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} & e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} e^{\chi_{ij}(t)} \end{pmatrix}, \tag{108}
$$

$$
(\cdot)^{\Re} \equiv \Re(\cdot), \tag{109}
$$

$$
(\cdot)^{\Im} \equiv \Im(\cdot). \tag{110}
$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_B T}$, considering the stationary bath state as reference written in the following way:

$$
\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B} \right)}. \tag{111}
$$

We can show using the coherence representation of the creation and annihilation operators that:

$$
b^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{112}
$$

$$
b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{113}
$$

So the product of the matrix representation of $b^{\dagger}$ and b with $-\beta$ is:

$$
-\beta \omega b^{\dagger} b = -\beta \omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{114}
$$

$$
= \sum_{j=0}^{\infty} -j\beta\omega\, |j\rangle\langle j|, \tag{115}
$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|,, \tag{116}$$

$$e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{117}$$

The value of $\mathrm{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right)$ is:

$$\mathrm{Tr}\left(e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{118}$$

$$= \sum_{j_{\mathbf{k}}} \left(e^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} \tag{119}$$

$$= \frac{1}{1 - e^{-\beta\omega_{\mathbf{k}}}} \quad \text{(by geometric series)} \tag{120}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), \tag{121}$$

$$\mathrm{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{122}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{123}$$

$$= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{124}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{125}$$

$$= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{126}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{127}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}}\,|\,j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}} - 1\rangle, \tag{128}$$

$$b_{\mathbf{k}}^\dagger\,|\,j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1}\,|\,j_{\mathbf{k}} + 1\rangle. \tag{129}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (128)-(129):

$$\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Tr}\left(B_{iz}(t)\rho_B\right) \tag{130}$$

$$= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{131}$$

$$= \sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger\rho_B\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\rho_B\right) \tag{132}$$

$$= \sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,\mathrm{Tr}\left(b_{\mathbf{k}}^\dagger\rho_B\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) \tag{133}$$

$$= \sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) \tag{134}$$

$$= \sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,\mathrm{Tr}\left(b_{\mathbf{k}}^\dagger\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right), \tag{135}$$

$$\mathrm{Tr}\left(b_{\mathbf{k}}^\dagger\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad\text{(by cyclic permutivity of trace, move }b_{\mathbf{k}}^\dagger) \tag{136}$$

$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{137}$$

$$= 0, \tag{138}$$

$$\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad\text{(by cyclic permutivity of trace, move }b_{\mathbf{k}}) \tag{139}$$

$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) \tag{140}$$

$$= 0. \tag{141}$$

we therefore find that:

$$\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = 0. \tag{142}$$

Another important expected value is $B(t)=\langle B^\pm(t)\rangle_{\overline{H_{\bar{B}}}}$, where $B^\pm(t)=e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\langle B^\pm(t)\rangle_{H_B} = \mathrm{Tr}\left(\rho_B B^\pm(t)\right) = \mathrm{Tr}\left(B^\pm(t)\rho_B\right) \tag{143}$$

$$= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) \tag{144}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}(t)\right)\rho_B\right) \tag{145}$$

$$= \prod_{\mathbf{k}}\langle D\left(\pm\alpha_{\mathbf{k}}(t)\right)\rangle. \tag{146}$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha. \tag{147}$$

where $P(\alpha)$ satisfies $\int P(\alpha)\,\mathrm{d}^2\alpha=1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P(\alpha)$ is given by:

$$\langle A\rangle = \mathrm{Tr}(A\rho) \tag{148}$$

$$= \int P(\alpha)\,\langle\alpha|A|\alpha\rangle\,\mathrm{d}^2\alpha. \tag{149}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}}$ where $N = (e^{\beta\omega} - 1)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | D(h) | \alpha \rangle \mathrm{d}^2\alpha \tag{150}$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle \mathrm{d}^2\alpha, \tag{151}$$

$$D(h) D(\alpha) = D(h+\alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)}, \tag{152}$$

$$D(-\alpha)(D(h) D(\alpha)) = D(-\alpha) D(h+\alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{153}$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{154}$$

$$= D(\alpha) e^{\frac{1}{2}(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2)} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{155}$$

$$= D(\alpha) e^{(h\alpha^* - h^*\alpha)}, \tag{156}$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) e^{(h\alpha^* - h^*\alpha)} | 0 \rangle \mathrm{d}^2\alpha \tag{157}$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} \langle 0 | D(h) | 0 \rangle \mathrm{d}^2\alpha \tag{158}$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} \langle 0 | h \rangle \mathrm{d}^2\alpha, \tag{159}$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{160}$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} \langle 0 | e^{-\frac{|h|^2}{2}} \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle \mathrm{d}^2\alpha \tag{161}$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} e^{-\frac{|h|^2}{2}} \mathrm{d}^2\alpha \tag{162}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int e^{-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha} \mathrm{d}^2\alpha, \tag{163}$$

$$\alpha = x + \mathrm{i}y, \tag{164}$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2 + y^2}{N} + h(x - \mathrm{i}y) - h^*(x + \mathrm{i}y)} \mathrm{d}x\mathrm{d}y \tag{165}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} e^{-\frac{x^2}{N} + hx - h^*x} \mathrm{d}x \int_{-\infty}^{\infty} e^{-\frac{x^2}{N} + hx - h^*x} \mathrm{d}y, \tag{166}$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N}(x^2 - Nhx + Nh^*x) \tag{167}$$

$$= -\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 + \frac{N(h^* - h)^2}{4}, \tag{168}$$

$$-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^*y = -\frac{1}{N}(y^2 + \mathrm{i}Nhy + \mathrm{i}Nh^*y) \tag{169}$$

$$= -\frac{1}{N}\left(y^2 + \frac{\mathrm{i}N(h + h^*)}{2}\right) - \frac{N(h + h^*)^2}{4}, \tag{170}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} e^{-\frac{(x - \mu)^2}{2\sigma^2}} \mathrm{d}x, \tag{171}$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}}}{\pi N} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{iN(h+h^*)}{2}\right)} \mathrm{d}x\mathrm{d}y, \tag{172}$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}}}{\pi N} \int_{-\infty}^{\infty} e^{-\frac{\left(x+\frac{(Nh^*-Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}} \mathrm{d}x \int_{-\infty}^{\infty} e^{-\frac{\left(y^2+\frac{iN(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}} \mathrm{d}y \tag{173}$$

$$= \frac{e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}}}{\pi N} \left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{174}$$

$$= e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}} \tag{175}$$

$$= e^{-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}} \tag{176}$$

$$= e^{-|h|^2\left(N+\frac{1}{2}\right)} \tag{177}$$

$$= e^{-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)} \tag{178}$$

$$= e^{-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)} \tag{179}$$

$$= e^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}. \tag{180}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (145) using (**??**) is given by:

$$B = e^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{181}$$

We will now force $\left\langle \overline{H_{\bar{I}}}(t) \right\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}(t)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+(t)\sigma^+$ and $B_i^-(t)\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}(t)$ and we subtract their expected value in order to satisfy $\left\langle \overline{H_{\bar{I}}}(t) \right\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}(t)$ is:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{182}$$

$$= \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'}(t) + \sum_j |j\rangle\langle j|B_{jz}(t) + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+(t)B_{j'}^-(t)-B_{jj'}(t)\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{183}$$

$$\equiv \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}. \tag{184}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_i^+(t)B_j^-(t)\rangle = B_{ij}(t) \tag{185}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right\rangle \tag{186}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \right\rangle \tag{187}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \tag{188}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{189}$$

$$= \prod_{\mathbf{k}} e^{-\frac{1}{2}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{190}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \tag{191}$$

From the definition $B_{01}(t) = \langle B_0^+(t) B_1^-(t) \rangle$ using the displacement operator we have:

$$\langle B_0^+(t) B_1^-(t) \rangle = B_{01}(t) \tag{192}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \tag{193}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \tag{194}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \tag{195}$$

$$= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \tag{196}$$

$$= \prod_{\mathbf{k}} \left(e^{-\frac{1}{2}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \tag{197}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \tag{198}$$

We can check:

$$\langle B_0^+(t) B_1^-(t) \rangle = B_{01}(t) \tag{199}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{200}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)^*} \tag{201}$$

$$= \langle B_1^+(t) B_0^-(t) \rangle^* \tag{202}$$

$$= B_{10}^*(t). \tag{203}$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + V_{10}(t) B_{10}\sigma^+ + V_{01}(t) B_{01}\sigma^-, \tag{204}$$

$$\overline{H_{\bar{I}}} \equiv V_{10}(t) \left(B_1^+(t) B_0^-(t) - B_{10}(t) \right) \sigma^+ + V_{01}(t) \left(B_0^+(t) B_1^-(t) - B_{01}(t) \right) \sigma^- + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t), \tag{205}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{206}$$

$$= H_B. \tag{207}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (205) we can verify the condition $\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+(t) B_{j'}^-(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{208}$$

$$= \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+(t) B_{j'}^-(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{209}$$

$$= \sum_{n\mathbf{k}} \left(\left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| \left(\left\langle V_{jj'}(t) B_j^+(t) B_{j'}^-(t) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle V_{jj'}(t) B_{jj'}(t) \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{210}$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(\left\langle B_j^+(t) B_{j'}^-(t) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_{jj'}(t) \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{211}$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(B_{jj'}(t) - B_{jj'}(t) \right) \tag{212}$$

$$= 0. \tag{213}$$

We used (142) and (**??**) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_x(t) = B_x^\dagger(t) \tag{214}$$

$$= \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}, \tag{215}$$

$$B_y(t) = B_y^\dagger(t) \tag{216}$$

$$= \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, \tag{217}$$

$$B_{iz}(t) = B_{iz}^\dagger(t) \tag{218}$$

$$= \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right). \tag{219}$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}(t) = \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \sigma^+ + V_{01}(t) B_{01}(t) \sigma^- \tag{220}$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) B_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \tag{221}$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) \left(B_{10}^{\Re}(t) + \mathrm{i}B_{10}^{\Im}(t) \right) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) \left(B_{10}^{\Re}(t) - \mathrm{i}B_{10}^{\Im}(t) \right) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \tag{222}$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \right) \tag{223}$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \tag{224}$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\mathrm{i}\sigma_x V_{10}^{\Im}(t) + \mathrm{i}\sigma_y V_{10}^{\Re}(t) \right) \tag{225}$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\mathrm{i}\sigma_x V_{10}^{\Im}(t) + \mathrm{i}\sigma_y V_{10}^{\Re}(t) \right) \tag{226}$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) - \left(\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \tag{227}$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \tag{228}$$

$$
= \left(\varepsilon_0\left(t\right) + R_0\left(t\right)\right)|0\rangle\langle0| + \left(\varepsilon_1\left(t\right) + R_1\left(t\right)\right)|1\rangle\langle1| + \sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right) - B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right) - \sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right) + B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right), \tag{229}
$$

$$
\overline{H_{\bar{I}}} = V_{10}\left(t\right)\left(\sigma^+ B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+ B_{10}\left(t\right)\right) + V_{01}\left(t\right)\left(\sigma^- B_0^+\left(t\right)B_1^-\left(t\right) - \sigma^- B_{01}\left(t\right)\right) + |0\rangle\langle0|B_{0z}\left(t\right) + |1\rangle\langle1|B_{1z}\left(t\right) \tag{230}
$$

$$
= |0\rangle\langle0|B_{0z}\left(t\right) + |1\rangle\langle1|B_{1z}\left(t\right) + \left(V_{10}^{\Re}\left(t\right) + iV_{10}^{\Im}\left(t\right)\right)\left(\sigma^+ B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+ B_{10}\left(t\right)\right) + \left(V_{10}^{\Re}\left(t\right) - iV_{10}^{\Im}\left(t\right)\right)\left(\sigma^- B_0^+\left(t\right)B_1^-\left(t\right) - \sigma^- B_{01}\left(t\right)\right) \tag{231}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma^+ B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+ B_{10}\left(t\right) + \sigma^- B_0^+\left(t\right)B_1^-\left(t\right) - \sigma^- B_{01}\left(t\right)\right) + iV_{10}^{\Im}\left(t\right)\left(\sigma^+ B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+ B_{10}\left(t\right) - \sigma^- B_0^+\left(t\right)B_1^-\left(t\right) + \sigma^- B_{01}\left(t\right)\right) \tag{232}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x + i\sigma_y}{2}B_1^+\left(t\right)B_0^-\left(t\right) - \frac{\sigma_x + i\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x - i\sigma_y}{2}B_0^+\left(t\right)B_1^-\left(t\right) - \frac{\sigma_x - i\sigma_y}{2}B_{01}\left(t\right)\right) \tag{233}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x + i\sigma_y}{2}B_1^+\left(t\right)B_0^-\left(t\right) - \frac{\sigma_x + i\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x - i\sigma_y}{2}B_0^+\left(t\right)B_1^-\left(t\right) - \frac{\sigma_x - i\sigma_y}{2}B_{01}\left(t\right)\right) \tag{234}
$$

$$
+ iV_{10}^{\Im}\left(t\right)\left(\frac{\sigma_x + i\sigma_y}{2}B_1^+\left(t\right)B_0^-\left(t\right) - \frac{\sigma_x + i\sigma_y}{2}B_{10}\left(t\right) - \frac{\sigma_x - i\sigma_y}{2}B_0^+\left(t\right)B_1^-\left(t\right) + \frac{\sigma_x - i\sigma_y}{2}B_{01}\left(t\right)\right) \tag{235}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2} + i\sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) - B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2}\right) \tag{236}
$$

$$
+ iV_{10}^{\Im}\left(t\right)\left(\sigma_x\frac{B_1^+\left(t\right)B_0^-\left(t\right) - B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2} + i\sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{237}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(i\sigma_x\frac{B_1^+\left(t\right)B_0^-\left(t\right) - B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2} - \sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{238}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(i^2\sigma_x\frac{B_1^+\left(t\right)B_0^-\left(t\right) - B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2i} - \sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{239}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(i^2\sigma_x\frac{B_1^+ B_0^-\left(t\right) - B_0^+ B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2i} - \sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{240}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(i^2\sigma_x\left(-B_y\left(t\right)\right) - \sigma_y B_x\left(t\right)\right) \tag{241}
$$

$$
= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y\left(t\right) - \sigma_y B_x\left(t\right)\right). \tag{242}
$$

II.   FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$
A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}\right)}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2\right\rangle_{\overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}}\right). \tag{243}
$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\left(t\right)\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\left(t\right)\}$:

$$
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}\left(t\right)} = 0. \tag{244}
$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\left(t\right), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$
e^{-\beta\left(\overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{245}
$$

Then using the fact that $\overline{H_{\bar{S}}}\left(t\right)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$
\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{246}
$$

So Eq. (244) becomes:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{247}$$

$$= -\frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{248}$$

$$= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{249}$$

$$= -\frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{250}$$

$$= 0 \quad\text{(by Eq. (244)).} \tag{251}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} \tag{252}$$

$$= 0. \tag{253}$$

This means we need to impose:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{254}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left((\varepsilon_0(t)+R_0(t))\,|0\rangle\langle0| + (\varepsilon_1(t)+R_1(t))\,|1\rangle\langle1| + V_{10}(t)\,B_{10}(t)\,\sigma^+ + V_{01}(t)\,B_{01}(t)\,\sigma^-\right). \tag{255}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{256}$$

Let's define:

$$\varepsilon(t) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{257}$$

$$\eta \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{258}$$

The solutions of the equation (256) are:

$$\lambda = \beta\frac{-\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{259}$$

$$= \beta\frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{260}$$

$$= -\beta\frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{261}$$

The value of $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = e^{-\frac{\varepsilon(t)\beta}{2}}e^{\frac{\eta(t)\beta}{2}} + e^{-\frac{\varepsilon(t)\beta}{2}}e^{-\frac{\eta(t)\beta}{2}} \tag{262}$$

$$= 2e^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right). \tag{263}$$

Given that $v_{i\mathbf{k}}(t)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A\left(\varepsilon(t),\eta(t)\right)$ to calculate $\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(2e^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{264}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)e^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)e^{-\frac{\varepsilon(t)\beta}{2}}\sinh\left(\frac{\eta(t)\beta}{2}\right) \tag{265}$$

$$= -\beta e^{-\frac{\varepsilon(t)\beta}{2}}\left(\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right)\right). \tag{266}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right) = 0. \tag{267}$$

The derivates included in the expression given are related to:

$$\langle B_0^+(t)B_1^-(t)\rangle = \left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{268}$$

$$= \left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^* e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{269}$$

$$= \langle B_1^+(t)B_0^-(t)\rangle^*, \tag{270}$$

$$R_i(t) = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{271}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{272}$$

$$\langle B_0^+(t)B_1^-(t)\rangle = \left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{273}$$

$$= \left(\prod_{\mathbf{k}}e^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{274}$$

$$v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t) = \left(v_{0\mathbf{k}}^{\Re}(t) - iv_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t) + iv_{1\mathbf{k}}^{\Im}(t)\right) - \left(v_{0\mathbf{k}}^{\Re}(t) + iv_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t) - iv_{1\mathbf{k}}^{\Im}(t)\right) \tag{275}$$

$$= \left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Re}(t) + iv_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) - iv_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Im}(t)\right) \tag{276}$$

$$- \left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Re}(t) - iv_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) + iv_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Im}(t)\right) \tag{277}$$

$$= 2i\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right), \tag{278}$$

$$|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2 = (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))^* \tag{279}$$

$$= |v_{1\mathbf{k}}(t)|^2 + |v_{0\mathbf{k}}(t)|^2 - (v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t) + v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)) \tag{280}$$

$$= \left(v_{1\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t)\right)^2 + \left(v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{0\mathbf{k}}^{\Im}(t)\right)^2 - 2\left(v_{1\mathbf{k}}^{\Re}(t)v_{0\mathbf{k}}^{\Re}(t) + v_{1\mathbf{k}}^{\Im}(t)v_{0\mathbf{k}}^{\Im}(t)\right) \tag{281}$$

$$= \left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\right)^2. \tag{282}$$

Rewriting in terms of real and imaginary parts.

$$R_i(t) = \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re}(t) - iv_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}^{\Re}(t) + iv_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{283}$$

$$= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - iv_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{284}$$

$$\langle B_0^+(t)B_1^-(t)\rangle = \left(\prod_{\mathbf{k}}e^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)\left(e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{285}$$

$$= \left(\prod_{\mathbf{k}}e^{\frac{2i\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{2\omega_{\mathbf{k}}^2}}\right)\left(e^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{286}$$

$$= \left(\prod_{\mathbf{k}}e^{\frac{i\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}}\right)\left(e^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{287}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{288}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - iv_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{289}$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{290}$$

$$\frac{\partial|B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(e^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{291}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}e^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{292}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}|B_{10}(t)|^2, \tag{293}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{294}$$

$$= \frac{2\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} - 4\frac{\partial\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{295}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left((\varepsilon_1(t) + R_1(t))(\varepsilon_0(t) + R_0(t)) - |V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \tag{296}$$

$$= \frac{\varepsilon\left(t\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon\left(t\right) - \varepsilon_{i}\left(t\right) - R_{i}\left(t\right)\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}} \frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} |B_{10}\left(t\right)|^{2} |V_{10}\left(t\right)|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{297}$$

$$= \frac{\varepsilon\left(t\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon\left(t\right) - \varepsilon_{i}\left(t\right) - R_{i}\left(t\right)\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{i\mathbf{k}}^{\Re}(t) - v_{i'\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}} |B_{10}\left(t\right)|^{2} |V_{10}\left(t\right)|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{298}$$

$$= \frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon\left(t\right) - 4\left(\varepsilon\left(t\right) - \varepsilon_{i}\left(t\right) - R_{i}\left(t\right)\right) - \frac{4}{\omega_{\mathbf{k}}}|B_{10}\left(t\right)|^{2}|V_{10}\left(t\right)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\left(t\right)}\right) + \frac{1}{\eta\left(t\right)}\left(-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\varepsilon\left(t\right) + 2\left(\varepsilon\left(t\right) - \varepsilon_{i}\left(t\right) - R_{i}\left(t\right)\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \tag{299}$$

$$+ 4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^{2}}|B_{10}\left(t\right)|^{2}|V_{10}\left(t\right)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{300}$$

From the equation (267) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}} \tag{301}$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + 2\frac{\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^{2}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{302}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}-g_{i\mathbf{k}}^{*}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \left(g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{303}$$

$$= \frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{304}$$

$$= \frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$= \frac{\left(v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{306}$$

Separating (305) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Re}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g_{i\mathbf{k}}^{\Re}\left(2\left(\varepsilon_{i}(t)+R_{i}(t)\right)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{307}$$

$$v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re} = v_{i\mathbf{k}}^{\Re}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right) \tag{308}$$

$$+ 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{309}$$

$$v_{i\mathbf{k}}^{\Re}(t) = \frac{g_{i\mathbf{k}}^{\Re}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{310}$$

$$v_{i\mathbf{k}}^{\Re}(t) = \frac{g_{i\mathbf{k}}^{\Re}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{311}$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} = \frac{\partial\left(\varepsilon_{1}\left(t\right)+R_{1}\left(t\right)+\varepsilon_{0}\left(t\right)+R_{0}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{312}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}\left(t\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} - \mathrm{i}v_{i\mathbf{k}}^{\Im}\left(t\right)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{313}$$

$$= 2\frac{v_{i\mathbf{k}}^{\Im}\left(t\right)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{314}$$

$$\frac{\partial\left|B_{10}(t)\right|^{2}}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} = \frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{315}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{316}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\left|B_{10}\left(t\right)\right|^{2}, \tag{317}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} \tag{318}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} - 4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} \tag{319}$$

$$= \frac{\varepsilon\left(t\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}(t)\right)\left(\varepsilon_{0}(t)+R_{0}(t)\right)-\left|V_{10}(t)\right|^{2}\left|B_{10}(t)\right|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}}{\eta\left(t\right)} \tag{320}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{321}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-v_{i'\mathbf{k}}^{\Im}(t)\right)\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^{2}}\right)}{\eta\left(t\right)} \tag{322}$$

$$= \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\frac{4\left(\varepsilon_{i}(t)+R_{i}(t)\right)-2\varepsilon(t)-\frac{4\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}}{\eta(t)} + \frac{1}{\eta(t)}\left(2\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Im}(t)\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^{2}}\right). \tag{323}$$

From the equation ([267](#)) and replacing the derivatives obtained we have:

$$\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}} = \tanh\left(\frac{\beta\eta(t)}{2}\right) \tag{324}$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}(t)\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\right)+\frac{2}{\eta(t)}\left(\frac{g_{i\mathbf{k}}^{*\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}}. \tag{325}$$

Rearranging this equation will lead to:

$$\tanh\left(\tfrac{\beta\eta(t)}{2}\right) = \frac{\left(2v_{i\mathbf{k}}^{\Im}(t) - \mathrm{i}\left(g_{i\mathbf{k}}^* - g_{i\mathbf{k}}\right)\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \mathrm{i}\left(g_{i\mathbf{k}}^* - g_{i\mathbf{k}}\right)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right)\right) + 4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{326}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}(t) - g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g_{i\mathbf{k}}^{\Im}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right)\right) + 4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}(t) - g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

$$= \frac{\left(v_{i\mathbf{k}}^{\Im}(t) - g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{329}$$

Separating (329) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Im}(t) - g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Im}(t)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{330}$$

$$v_{i\mathbf{k}}^{\Im} - g_{i\mathbf{k}}^{\Im} = v_{i\mathbf{k}}^{\Im}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) \tag{331}$$

$$+ 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{332}$$

$$v_{i\mathbf{k}}^{\Im}(t) = \frac{g_{i\mathbf{k}}^{\Im}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{333}$$

$$v_{i\mathbf{k}}^{\Im}(t) = \frac{g_{i\mathbf{k}}^{\Im}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{334}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t) = v_{i\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{i\mathbf{k}}^{\Im}(t) \tag{335}$$

$$= \frac{g_{i\mathbf{k}}^{\Re}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{336}$$

$$+ \mathrm{i}\frac{g_{i\mathbf{k}}^{\Im}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

$$= \frac{g_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{338}$$

Let's obtain the explicit form of $v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t)$ and $v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t)$, at first we have:

$$a_i\left(\omega_{\mathbf{k}},t\right)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{339}$$

$$b_i\left(\omega_{\mathbf{k}},t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}|B_{10}\left(t\right)|^2|V_{10}\left(t\right)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}.\tag{340}$$

So the equation (335) written in explicit form is:

$$v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right),\tag{341}$$

$$v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)=g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right).\tag{342}$$

This system of equations has the following solutions:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{343}$$

$$=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+\left(g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{344}$$

$$=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{345}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)(1-b_1(\omega_{\mathbf{k}},t)b_0(\omega_{\mathbf{k}},t))=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{346}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=\frac{g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{347}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+\frac{g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}b_1\left(\omega_{\mathbf{k}},t\right)\tag{348}$$

$$=\frac{g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{349}$$

For a shorter representation let's define:

$$r_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_i\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{350}$$

$$s_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_{(i+1)\bmod2}\left(\omega_{\mathbf{k}},t\right)b_{i\bmod2}\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{351}$$

So the variational parameter are:

$$\begin{pmatrix}v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\\v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\end{pmatrix}\equiv\begin{pmatrix}r_0\left(\omega_{\mathbf{k}},t\right)&s_0\left(\omega_{\mathbf{k}},t\right)\\r_1\left(\omega_{\mathbf{k}},t\right)&s_1\left(\omega_{\mathbf{k}},t\right)\end{pmatrix}\begin{pmatrix}g_0\left(\omega_{\mathbf{k}}\right)\\g_1\left(\omega_{\mathbf{k}}\right)\end{pmatrix}.\tag{352}$$

Given that $v_{i\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\equiv g_i\left(\omega_{\mathbf{k}}\right)F_i\left(\omega_{\mathbf{k}},t\right)$ then we can write:

$$F_0\left(\omega_{\mathbf{k}},t\right)=r_0\left(\omega_{\mathbf{k}},t\right)+\frac{g_1\left(\omega_{\mathbf{k}}\right)}{g_0\left(\omega_{\mathbf{k}}\right)}s_0\left(\omega_{\mathbf{k}},t\right)\tag{353}$$

$$F_1\left(\omega_{\mathbf{k}},t\right)=\frac{g_0\left(\omega_{\mathbf{k}}\right)}{g_1\left(\omega_{\mathbf{k}}\right)}r_1\left(\omega_{\mathbf{k}},t\right)+s_1\left(\omega_{\mathbf{k}},t\right)\tag{354}$$

In the continuous limit we have:

$$F_0\left(\omega,t\right)=r_0\left(\omega,t\right)+\frac{L_1\left(\omega\right)}{L_0\left(\omega\right)}s_0\left(\omega,t\right)\tag{355}$$

$$F_1\left(\omega,t\right)=\frac{L_0\left(\omega\right)}{L_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right)\tag{356}$$

III. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$, where $\rho_B^{\text{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^{V(0)}\rho_T(0)e^{-V(0)} \tag{357}$$

$$= \left(|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)\right)(\rho_S(0) \otimes \rho_B)\left(|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)\right), \tag{358}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: \quad |0\rangle\langle 0|0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{359}$$

$$= |0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{360}$$

$$= |0\rangle\langle 0| \otimes B_0^+(0)\rho_B B_0^-(0), \tag{361}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: \quad |1\rangle\langle 1|B_1^+(0)|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) \tag{362}$$

$$= |1\rangle\langle 1|B_1^+(0)\rho_B B_1^-(0) \tag{363}$$

$$= |1\rangle\langle 1| \otimes B_1^+(0)\rho_B B_1^-(0), \tag{364}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1|: \quad |0\rangle\langle 0|B_0^+(0)|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) \tag{365}$$

$$= |0\rangle\langle 1|B_0^+(0)\rho_B|1\rangle\langle 1|B_1^-(0) \tag{366}$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0)\rho_B B_1^-(0) \tag{367}$$

$$= |0\rangle\langle 1| \otimes B_0^+(0)\rho_B B_1^-(0), \tag{368}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0|: \quad |1\rangle\langle 1|B_1^+(0)|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{369}$$

$$= |1\rangle\langle 0| \otimes B_1^+(0)\rho_B B_0^-(0). \tag{370}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^\dagger(t)O(t)U(t), \tag{371}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right). \tag{372}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\overline{\rho_S}(t)U(t), \text{ where} \tag{373}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\overline{\rho}_T(t)). \tag{374}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \tag{375}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (**??**):

$$\overline{H_{\bar{I}}}(t) = \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \tag{376}$$

$$= B_{0z}(t)|0\rangle\langle 0| + B_{1z}(t)|1\rangle\langle 1| + V_{10}^{\Re}(t)\sigma_x B_x(t) + V_{10}^{\Re}(t)\sigma_y B_y(t) + V_{10}^{\Im}(t)\sigma_x B_y(t) - V_{10}^{\Im}(t)\sigma_y B_x(t) \tag{377}$$

$$= \sum_i C_i(t)(A_i \otimes B_i(t)). \tag{378}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{379}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) - \mathrm{i}\int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]\mathrm{d}t'. \tag{380}$$

Replacing the equation (380) in the equation (379) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]]\mathrm{d}t'. \tag{381}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty}(-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdots]. \tag{382}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty}(-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \operatorname{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]] \cdots]. \tag{383}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\overline{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\overline{\rho_S}(0) \tag{384}$$
$$= W(t)\overline{\rho_S}(0). \tag{385}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \operatorname{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]] \cdots]. \tag{386}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{387}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}W(t)\overline{\rho_S}(0) \tag{388}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}\widetilde{\overline{\rho_S}}(t). \tag{389}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\operatorname{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (387) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\widetilde{\overline{\rho_S}}(t) \tag{390}$$

$$= -\int_0^t \mathrm{d}t_1 \operatorname{Tr}_B \left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]. \tag{391}$$

Replacing $t_1 \to t - \tau$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \int_0^t \mathrm{d}\tau \operatorname{Tr}_B\left[\overline{H_{\bar{I}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(-\tau), \overline{\rho_S}(t)\rho_B\right]\right]. \tag{392}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}(t)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = U^\dagger(t)\, \mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}(t)\, \mathrm{e}^{-\mathrm{i}H_B t}U(t). \tag{393}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = \sum_i C_i(t)\left(\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)\right), \tag{394}$$

$$\widetilde{A_i}(t) = U^\dagger(t)\, \mathrm{e}^{\mathrm{i}H_B t}A_i \mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{395}$$

$$= U^\dagger(t)\, A_i U(t)\, \mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} \tag{396}$$

$$= U^\dagger(t)\, A_i U(t)\, \mathbb{I} \tag{397}$$

$$= U^\dagger(t)\, A_i U(t), \tag{398}$$

$$\widetilde{B_i}(t) = U^\dagger(t)\, \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{399}$$

$$= U^\dagger(t)\, U(t)\, \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{400}$$

$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{401}$$

$$= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t}. \tag{402}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.

Using the expression (394) to replace it in the equation (391)

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \operatorname{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}t' \tag{403}$$

$$= -\int_0^t \operatorname{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right), \left[\sum_i C_i(t')\left(\widetilde{A_i}(t') \otimes \widetilde{B_i}(t')\right), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}t' \tag{404}$$

$$= -\int_0^t \operatorname{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_S}}(t)\rho_B - \widetilde{\overline{\rho_S}}(t)\rho_B \sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\right]\mathrm{d}t' \tag{405}$$

$$= -\int_0^t \operatorname{Tr}_B\left(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_S}}(t)\rho_B - \sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right) \tag{406}$$

$$- \sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right) + \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\right)\mathrm{d}t'. \tag{407}$$

In order to calculate the correlation functions we define:

$$\mathcal{B}_{ij}(t,t') = \operatorname{Tr}_B\left(\widetilde{B_i}(t)\widetilde{B_j}(t')\rho_B\right). \tag{408}$$

An useful property is

$$\mathcal{B}_{ji}^*(t,t') = \operatorname{Tr}_B\left(\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B\right)^\dagger \tag{409}$$

$$= \operatorname{Tr}_B\left(\rho_B^\dagger \widetilde{B_i}^\dagger(t')\widetilde{B_j}^\dagger(t)\right) \tag{410}$$

$$= \operatorname{Tr}_B\left(\rho_B \widetilde{B_i}(t')\widetilde{B_j}(t)\right) \tag{411}$$

$$= \operatorname{Tr}_B\left(\widetilde{B_i}(t')\widetilde{B_j}(t)\rho_B\right) \tag{412}$$

$$= \mathcal{B}_{ij}(t',t). \tag{413}$$

The correlation functions relevant that appear in the equation (407) are:

$$\text{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(t')\,\rho_B\right) = \left\langle\widetilde{B_j}(t)\,\widetilde{B_i}(t')\right\rangle_B \tag{414}$$

$$= \mathcal{B}_{ji}(t,t') \tag{415}$$

$$= \mathcal{B}_{ij}^*(t',t) \tag{416}$$

$$\text{Tr}_B\left(\widetilde{B_j}(t)\,\rho_B\,\widetilde{B_i}(t')\right) = \text{Tr}_B\left(\widetilde{B_i}(t')\,\widetilde{B_j}(t)\,\rho_B\right) \tag{417}$$

$$= \mathcal{B}_{ij}(t',t) \tag{418}$$

$$\text{Tr}_B\left(\widetilde{B_i}(t')\,\rho_B\,\widetilde{B_j}(t)\right) = \text{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(t')\,\rho_B\right) \tag{419}$$

$$= \mathcal{B}_{ij}^*(t',t) \tag{420}$$

$$\text{Tr}_B\left(\rho_B\,\widetilde{B_i}(t')\,\widetilde{B_j}(t)\right) = \text{Tr}_B\left(\widetilde{B_i}(t')\,\widetilde{B_j}(t)\,\rho_B\right) \tag{421}$$

$$= \mathcal{B}_{ij}(t',t) \tag{422}$$

The cyclic property of the trace was use widely in the development of equations (414) and (422). Replacing in (407)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \text{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\rho_S}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\rho_S}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big) \tag{423}$$

$$- \sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\rho_S}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\rho_S}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)dt'. \tag{424}$$

$$= -\int_0^t \text{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \tag{425}$$

$$+ \sum_{ij} C_i(t')C_j(t)\big(\widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)dt' \tag{426}$$

$$= -\int_0^t \text{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \tag{427}$$

$$+ \sum_{ij} C_i(t')C_j(t)\big(\widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)dt' \tag{428}$$

$$= -\int_0^t \text{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \quad \text{(by permuting i and j because i,j}\in\text{J)} \tag{429}$$

$$+ \sum_{ij} C_i(t')C_j(t)\big(\widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)dt' \tag{430}$$

$$= -\int_0^t \text{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t') \tag{431}$$

$$+ \widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)dt' \tag{432}$$

$$= -\int_0^t\Big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\rho_S}(t)\mathcal{B}_{ji}(t,t') - \widetilde{A_j}(t)\widetilde{\rho_S}(t)\widetilde{A_i}(t')\mathcal{B}_{ij}(t',t) \tag{433}$$

$$+ \widetilde{\rho_S}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\mathcal{B}_{ij}(t',t) - \widetilde{A_i}(t')\widetilde{\rho_S}(t)\widetilde{A_j}(t)\mathcal{B}_{ji}(t,t')\big)\Big)dt' \tag{434}$$

$$= -\int_0^t\left(\sum_{ij} C_j(t)C_i(t')\left(\mathcal{B}_{ji}(t,t')\left[\widetilde{A_j}(t),\widetilde{A_i}(t')\widetilde{\rho_S}(t)\right] + \mathcal{B}_{ij}(t',t)\left[\widetilde{\rho_S}(t)\widetilde{A_i}(t'),\widetilde{A_j}(t)\right]\right)\right)dt' \tag{435}$$

$$= -\int_0^t\left(\sum_{ij} C_i(t)C_j(t')\big(\mathcal{B}_{ij}(t,t')[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\rho_S}(t)] + \mathcal{B}_{ji}(t',t)[\widetilde{\rho_S}(t)\widetilde{A_j}(t'),\widetilde{A_i}(t)]\big)\right)dt' \quad \text{(exchanging i and j)} \tag{436}$$

$$= -\int_0^t\left(\sum_{ij} C_i(t)C_j(t')\left(\mathcal{B}_{ij}(t,t')\left[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\rho_S}(t)\right] + \mathcal{B}_{ij}^*(t,t')\left[\widetilde{\rho_S}(t)\widetilde{A_j}(t'),\widetilde{A_i}(t)\right]\right)\right)dt' \tag{437}$$

$$= -\int_0^t\left(\sum_{ij} C_i(t)C_j(t')\left(\mathcal{B}_{ij}(t,t')\left[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\rho_S}(t)\right] - \mathcal{B}_{ij}^*(t,t')\left[\widetilde{A_i}(t),\widetilde{\rho_S}(t)\widetilde{A_j}(t')\right]\right)\right)dt' \tag{438}$$

We could identify the following commutators in the equation deduced:

$$\mathcal{B}_{ij}(t,t')\widetilde{A_i}(t)\widetilde{A_j}(t')\widetilde{\rho_S}(t) - \mathcal{B}_{ij}(t,t')\widetilde{A_j}(t')\widetilde{\rho_S}(t)\widetilde{A_i}(t) = \mathcal{B}_{ij}(t,t')\left[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\rho_s}(t)\right], \tag{439}$$

$$\mathcal{B}_{ij}^*(t,t')\widetilde{\rho_S}(t)\widetilde{A_j}(s)\widetilde{A_i}(t) - \mathcal{B}_{ij}^*(t,t')\widetilde{A_i}(t)\widetilde{\rho_S}(t)\widetilde{A_j}(s) = \mathcal{B}_{ij}^*(t,t')\left[\widetilde{\rho_S}(t)\widetilde{A_j}(t'),\widetilde{A_i}(t)\right]. \tag{440}$$

Returning to the Schroedinger picture we have:

$$U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right) = U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{A_j}\left(t'\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right),\tag{441}$$

$$= \left(U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{A_j}\left(t'\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right)\right),\tag{442}$$

$$= A_i\left(t\right)\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right).\tag{443}$$

This procedure applying to the relevant commutators give us:

$$U\left(t\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)\right]U^\dagger\left(t\right) = \left(U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right) - U\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right)\tag{444}$$

$$= A_i\left(t\right)\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right) - \widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)A_i\tag{445}$$

$$= \left[A_i\left(t\right),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right].\tag{446}$$

Introducing this transformed commutators in the equation (438) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathcal{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\tilde{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ij}\int_0^t \mathrm{d}s\, C_i(t)\, C_j(t')\left(\mathcal{B}_{ij}\left(t,t'\right)\left[A_i\left(t\right),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right] + \mathcal{B}_{ij}^*\left(t,t'\right)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}\left(t',t\right),A_i\right]\right),\tag{447}$$

$$t' = t - \tau \text{ (Change of variables in the integration process)},\tag{448}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\tilde{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau\, C_i(t)\, C_j(t')\left(\mathcal{B}_{ij}\left(t,t'\right)\left[A_i(t),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right] + \mathcal{B}_{ij}^*\left(t,t'\right)\left[\overline{\rho_S}(t)\widetilde{A_j}\left(t',t\right),A_i\left(t\right)\right]\right).\tag{449}$$

where $i,j \in \{1,2,3,4,5.6\}$ and $t' = t - \tau$.

Here $\widetilde{A_j}\left(t-\tau,t\right) = U\left(t\right)U^\dagger\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^\dagger\left(t\right)$ where $U\left(t\right)$ is given by (372). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I\left(t\right)$. In order to write in a simplified way we define the following notation:

$$\mathcal{B}_{ij}\left(t,t'\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(t'\right)\rho_B\right)\tag{450}$$

$$= \mathrm{Tr}_B\left(e^{iH_B t}B_i\left(t\right)e^{-iH_B t}e^{iH_B t'}B_j\left(t'\right)e^{-iH_B t'}\rho_B\right)\tag{451}$$

$$e^A = \sum_{k=0}^\infty \frac{A^k}{k!}\tag{452}$$

$$e^{-iH_B t'}e^{-\beta H_B} = \sum_{m=0}^\infty \frac{\left(-iH_B t'\right)^m}{m!}\sum_{n=0}^\infty \frac{\left(-\beta H_B\right)^n}{n!}\tag{453}$$

$$= \sum_{m,n}\frac{\left(-iH_B t'\right)^m}{m!}\frac{\left(-\beta H_B\right)^n}{n!}\tag{454}$$

$$= \sum_{m,n}\frac{\left(-it'\right)^m}{m!}\frac{\left(-\beta\right)^n}{n!}H_B^m H_B^n\tag{455}$$

$$= \sum_{m,n}\frac{\left(-it'\right)^m}{m!}\frac{\left(-\beta\right)^n}{n!}H_B^n H_B^m \text{ (because the powers of a matrix commute)}\tag{456}$$

$$= \sum_{m,n}\frac{\left(-\beta\right)^n}{n!}H_B^n\frac{\left(-it'\right)^m}{m!}H_B^m\tag{457}$$

$$= \sum_{m,n}\frac{\left(-\beta H_B\right)^n}{n!}\frac{\left(-it'H_B\right)^m}{m!}\tag{458}$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_Bt')^m}{m!} \tag{459}$$

$$= e^{-\beta H_B} e^{-iH_Bt'} \tag{460}$$

$$0 = e^{-iH_Bt'} e^{-\beta H_B} - e^{-\beta H_B} e^{-iH_Bt'} \text{ (then } e^{-iH_Bt'} \text{ and } \rho_B \text{ commute)} \tag{461}$$

$$\mathcal{B}_{ij}(t,t') = \text{Tr}_B \left(e^{iH_Bt} B_i(t) e^{-iH_Bt} e^{iH_Bt'} B_j(t') \rho_B e^{-iH_Bt'} \right) \text{ (by permuting } e^{-iH_Bt'} \text{ and } \rho_B) \tag{462}$$

$$= \text{Tr}_B \left(\left(e^{iH_Bt} B_i(t) e^{-iH_Bt} e^{iH_Bt'} B_j(t') \right) \rho_B e^{-iH_Bt'} \right) \text{ (by associative property)} \tag{463}$$

$$= \text{Tr}_B \left(e^{-iH_Bt'} \left(e^{iH_Bt} B_i(t) e^{-iH_Bt} e^{iH_Bt'} B_j(t') \right) \rho_B \right) \text{ (by cyclic property of the trace)} \tag{464}$$

$$= \text{Tr}_B \left(\left(e^{-iH_Bt'} e^{iH_Bt} \right) B_i(t) \left(e^{-iH_Bt} e^{iH_Bt'} \right) B_j(t') \rho_B \right) \text{ (by associative property)} \tag{465}$$

$$[iH_Bt, -iH_Bt'] = iH_Bt(-iH_Bt') - (-iH_Bt')iH_Bt \tag{466}$$

$$= tt'H_B^2 - tt'H_B^2 \tag{467}$$

$$= 0 \text{ (so } iH_Bt \text{ and } -iH_Bt' \text{ commute)} \tag{468}$$

$$e^{-iH_Bt'} e^{iH_Bt} = e^{iH_Bt - iH_Bt'} \text{ (by the Zassenhaus formula because } iH_Bt \text{ and } -iH_Bt' \text{ commute)} \tag{469}$$

$$= e^{iH_B(t-t')} \tag{470}$$

$$= e^{iH_B\tau} \tag{471}$$

$$e^{iH_Bt'} e^{-iH_Bt} = e^{-iH_Bt + iH_Bt'} \text{ (by the Zassenhaus formula because } -iH_Bt \text{ and } iH_Bt' \text{ commute)} \tag{472}$$

$$= e^{iH_B(-t+t')} \tag{473}$$

$$= e^{-iH_B\tau} \tag{474}$$

$$\mathcal{B}_{ij}(t,t') = \text{Tr}_B \left(e^{iH_B\tau} B_i(t) e^{-iH_B\tau} B_j(t') \rho_B \right) \tag{475}$$

$$B_i(t,\tau) \equiv e^{iH_B\tau} B_i(t) e^{-iH_B\tau} \tag{476}$$

$$\mathcal{B}_{ij}(t,t') = \text{Tr}_B \left(e^{iH_B(t-t')} B_i(t) e^{-iH_B(t-t')} B_j(t') \rho_B \right) \tag{477}$$

$$t' = t - \tau \tag{478}$$

$$\mathcal{B}_{ij}(t,t') = \text{Tr}_B \left(e^{iH_B\tau} B_i(t) e^{-iH_B\tau} B_j(t') \rho_B \right) \tag{479}$$

$$= \text{Tr}_B \left(B_i(t,\tau) B_j(t',0) \rho_B \right) \tag{480}$$

For the following results $i,j \in \{3,6\}$, calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{j\bmod 2z}}(t) \widetilde{B_{j\bmod 2z}}(t') \right\rangle_B = \text{Tr}_B \left(B_{j\bmod 2z}(t,\tau) B_{j\bmod 2z}(t',0) \rho_B \right) \tag{481}$$

$$= \int \mathrm{d}^2\alpha P(\alpha) \langle \alpha | B_{j\bmod 2z}(t,\tau) B_{j\bmod 2z}(t',0) | \alpha \rangle \tag{482}$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | B_{j\bmod 2z}(t,\tau) B_{j\bmod 2z}(t',0) | \alpha \rangle \, \mathrm{d}^2\alpha, \tag{483}$$

$$q_{j\mathbf{k}}(t) = g_{j\bmod 2\mathbf{k}} - v_{j\bmod 2\mathbf{k}}(t) \tag{484}$$

$$B_{j\bmod 2z}(t,\tau) = \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_\mathbf{k}\tau} \right), \tag{485}$$

$$B_{j\bmod 2z}(t',0) = \sum_{\mathbf{k'}} \left(q_{j\mathbf{k'}}(t') b_{\mathbf{k'}}^\dagger + q_{j\mathbf{k'}}^*(t') b_{\mathbf{k'}} \right), \tag{486}$$

$$\left\langle \widetilde{B_{j\bmod 2z}}(t) \widetilde{B_{j\bmod 2z}}(t') \right\rangle_B = \text{Tr}_B \left(B_{j\bmod 2z}(t,\tau) B_{j\bmod 2z}(t',0) \rho_B \right) \tag{487}$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_\mathbf{k}\tau} \right) \sum_{\mathbf{k'}} \left(q_{j\mathbf{k'}}(t') b_{\mathbf{k'}}^\dagger + q_{j\mathbf{k'}}^*(t') b_{\mathbf{k'}} \right) \rho_B \right) \tag{488}$$

$$= \mathrm{Tr}_B\left(\sum_\mathbf{k} \left(q_{j\mathbf{k}}(t) b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_\mathbf{k} e^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \sum_{\mathbf{k}'} \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \tag{489}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'} \left(q_{j\mathbf{k}}(t) b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_\mathbf{k} e^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \tag{490}$$

$$+ \mathrm{Tr}_B\left(\sum_\mathbf{k} \left(q_{j\mathbf{k}}(t) b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_\mathbf{k} e^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \left(q_{j\mathbf{k}}(t') b_\mathbf{k}^\dagger + q_{j\mathbf{k}}^*(t') b_\mathbf{k} \right) \rho_B \right), \tag{491}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'} \left(q_{j\mathbf{k}}(t) b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_\mathbf{k} e^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \tag{492}$$

$$+ \mathrm{Tr}_B\left(\sum_\mathbf{k} \left(q_{j\mathbf{k}}(t) b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_\mathbf{k} e^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \left(q_{j\mathbf{k}}(t') b_\mathbf{k}^\dagger + q_{j\mathbf{k}}^*(t') b_\mathbf{k} \right) \rho_B \right) \tag{493}$$

$$0 = \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'} \left(q_{j\mathbf{k}}(t) b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_\mathbf{k} e^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \tag{494}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = 0 + \mathrm{Tr}_B\left(\sum_\mathbf{k} \left(q_{j\mathbf{k}}(t) b_\mathbf{k}^\dagger \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) b_\mathbf{k} e^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \left(q_{j\mathbf{k}}(t') b_\mathbf{k}^\dagger + q_{j\mathbf{k}}^*(t') b_\mathbf{k} \right) \rho_B \right) \tag{495}$$

$$= \mathrm{Tr}_B \sum_\mathbf{k} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}(t') \left(b_\mathbf{k}^\dagger \right)^2 \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') b_\mathbf{k}^\dagger b_\mathbf{k} \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') b_\mathbf{k} b_\mathbf{k}^\dagger \mathrm{e}^{-\mathrm{i}\omega\mathbf{k}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') b_\mathbf{k}^2 \mathrm{e}^{-\mathrm{i}\omega\mathbf{k}\tau} \right) \rho_B \tag{496}$$

$$= \mathrm{Tr}_B\left(\sum_\mathbf{k} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') b_\mathbf{k}^\dagger b_\mathbf{k} \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \rho_B \right) + \mathrm{Tr}_B\left(\sum_\mathbf{k} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') b_\mathbf{k} b_\mathbf{k}^\dagger \mathrm{e}^{-\mathrm{i}\omega\mathbf{k}\tau} \rho_B \right) \tag{497}$$

$$= \sum_\mathbf{k} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \mathrm{Tr}_B\left(b_\mathbf{k}^\dagger b_\mathbf{k} \rho_B \right) + \sum_\mathbf{k} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega\mathbf{k}\tau} \mathrm{Tr}_B\left(b_\mathbf{k} b_\mathbf{k}^\dagger \rho_B \right) \tag{498}$$

$$= \sum_\mathbf{k} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \left\langle \alpha_\mathbf{k} \left| b_\mathbf{k}^\dagger b_\mathbf{k} \right| \alpha_\mathbf{k} \right\rangle + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega\mathbf{k}\tau} \left\langle \alpha_\mathbf{k} \left| b_\mathbf{k} b_\mathbf{k}^\dagger \right| \alpha_\mathbf{k} \right\rangle \right) \mathrm{d}^2\alpha_\mathbf{k} \tag{499}$$

$$= \sum_\mathbf{k} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \left(\mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| D(-\alpha_\mathbf{k}) b_\mathbf{k}^\dagger b_\mathbf{k} D(\alpha_\mathbf{k}) \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right) \tag{500}$$

$$+ \sum_\mathbf{k} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') \left(e^{-\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| D(-\alpha_\mathbf{k}) b_\mathbf{k} b_\mathbf{k}^\dagger D(\alpha_\mathbf{k}) \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right) \tag{501}$$

$$= \sum_\mathbf{k} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \left(\mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| D(-\alpha_\mathbf{k}) b_\mathbf{k}^\dagger D(\alpha_\mathbf{k}) D(-\alpha_\mathbf{k}) b_\mathbf{k} D(\alpha_\mathbf{k}) \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right) \tag{502}$$

$$+ \sum_\mathbf{k} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') \left(e^{-\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| D(-\alpha_\mathbf{k}) b_\mathbf{k} D(\alpha_\mathbf{k}) D(-\alpha_\mathbf{k}) b_\mathbf{k}^\dagger D(\alpha_\mathbf{k}) \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right) \tag{503}$$

$$= \sum_\mathbf{k} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \left(\mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| \left(b_\mathbf{k}^\dagger + \alpha_\mathbf{k}^* \right)(b_\mathbf{k} + \alpha_\mathbf{k}) \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right) \tag{504}$$

$$+ \sum_\mathbf{k} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') \left(e^{-\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| (b_\mathbf{k} + \alpha_\mathbf{k}) \left(b_\mathbf{k}^\dagger + \alpha_\mathbf{k}^* \right) \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right), \tag{505}$$

$$= \sum_\mathbf{k} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| b_\mathbf{k}^\dagger b_\mathbf{k} + b_\mathbf{k}^\dagger \alpha_\mathbf{k} + b_\mathbf{k} \alpha_\mathbf{k}^* + |\alpha_\mathbf{k}|^2 \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right. \tag{506}$$

$$\left. + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega\mathbf{k}\tau} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left\langle 0 \left| b_\mathbf{k} b_\mathbf{k}^\dagger + b_\mathbf{k}^\dagger \alpha_\mathbf{k} + b_\mathbf{k} \alpha_\mathbf{k}^* + |\alpha_\mathbf{k}|^2 \right| 0 \right\rangle \mathrm{d}^2\alpha_\mathbf{k} \right) \tag{507}$$

$$= \sum_\mathbf{k} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \left\langle 0 \left| b_\mathbf{k}^\dagger b_\mathbf{k} + |\alpha_\mathbf{k}|^2 \right| 0 \right\rangle + q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \left\langle 0 \left| b_\mathbf{k}^\dagger \alpha_\mathbf{k} + b_\mathbf{k} \alpha_\mathbf{k}^* \right| 0 \right\rangle \right. \tag{508}$$

$$\left. + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega\mathbf{k}\tau} \left\langle 0 \left| b_\mathbf{k} b_\mathbf{k}^\dagger + |\alpha_\mathbf{k}|^2 \right| 0 \right\rangle + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega\mathbf{k}\tau} \left\langle 0 \left| b_\mathbf{k}^\dagger \alpha_\mathbf{k} + b_\mathbf{k} \alpha_\mathbf{k}^* \right| 0 \right\rangle \right) \mathrm{d}^2\alpha_\mathbf{k} \tag{509}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \sum_\mathbf{k} \left(\left\langle 0 \left| q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \left(b_\mathbf{k}^\dagger b_\mathbf{k} + |\alpha_\mathbf{k}|^2 \right) + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega\mathbf{k}\tau} \left(b_\mathbf{k} b_\mathbf{k}^\dagger + |\alpha_\mathbf{k}|^2 \right) \right| 0 \right\rangle \right) \mathrm{d}^2\alpha_\mathbf{k} \tag{510}$$

$$= \sum_\mathbf{k} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega\mathbf{k}\tau} \left(\left\langle 0 \left| |\alpha_\mathbf{k}|^2 \right| 0 \right\rangle + \left\langle 0 \left| b_\mathbf{k}^\dagger b_\mathbf{k} \right| 0 \right\rangle \right) \mathrm{d}^2\alpha_\mathbf{k} \tag{511}$$

$$+ \sum_\mathbf{k} \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega\mathbf{k}\tau} \left(\left\langle 0 \left| b_\mathbf{k} b_\mathbf{k}^\dagger \right| 0 \right\rangle + \left\langle 0 \left| |\alpha_\mathbf{k}|^2 \right| 0 \right\rangle \right) \mathrm{d}^2\alpha_\mathbf{k}, \tag{512}$$

$$1 = \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_\mathbf{k}|^2}{N}} \mathrm{d}^2\alpha_\mathbf{k}, \tag{513}$$

$$b_\mathbf{k}^\dagger b_\mathbf{k} |0\rangle = 0, \tag{514}$$

$$b_\mathbf{k} b_\mathbf{k}^\dagger |0\rangle = |0\rangle, \tag{515}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(\left(q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^*(t')e^{i\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^*(t)\,q_{j\mathbf{k}}(t')\,e^{-i\omega_{\mathbf{k}}\tau}\right)\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\right. \tag{516}$$

$$\left.+q_{j\mathbf{k}}^*(t)\,q_{j\mathbf{k}}(t')\left(e^{-i\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right|0\right\rangle\right)\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{517}$$

$$=\frac{1}{\pi N}\int\sum_{\mathbf{k}}e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^*(t')e^{i\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^*(t)q_{j\mathbf{k}}(t')e^{-i\omega_{\mathbf{k}}\tau}\right)|\alpha_{\mathbf{k}}|^2+q_{j\mathbf{k}}^*(t)q_{j\mathbf{k}}(t')e^{-i\omega_{\mathbf{k}}\tau}\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{518}$$

$$\int_0^{2\pi}\int_0^{+\infty}r^2 e^{-\frac{r^2}{N}}r\mathrm{d}r\mathrm{d}\theta=\int|\alpha_{\mathbf{k}}|^2 e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\mathrm{d}^2\alpha_{\mathbf{k}} \tag{519}$$

$$=\pi N^2 \tag{520}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}}\left(\left(q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^*(t')e^{i\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^*(t)\,q_{j\mathbf{k}}(t')\,e^{-i\omega_{\mathbf{k}}\tau}\right)N+q_{j\mathbf{k}}^*(t)\,q_{j\mathbf{k}}(t')\,e^{-i\omega_{\mathbf{k}}\tau}\right) \tag{521}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j'\mathrm{mod}2z}}(t')\right\rangle_B = \mathrm{Tr}_B\left(B_{j\mathrm{mod}2z}(t,\tau)\,B_{j'\mathrm{mod}2z}(t',0)\,\rho_B\right) \tag{522}$$

$$=\int\mathrm{d}^2\alpha P(\alpha)\left\langle\alpha\left|B_{j\mathrm{mod}2z}(t,\tau)\,B_{j'\mathrm{mod}2z}(t',0)\right|\alpha\right\rangle \tag{523}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\left|B_{j\mathrm{mod}2z}(t,\tau)\,B_{j'\mathrm{mod}2z}(t',0)\right|\alpha\right\rangle\mathrm{d}^2\alpha \tag{524}$$

$$=\left\langle\alpha\right|\sum_{\mathbf{k}}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\,b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^*(t)b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j'\mathbf{k}'}(t')\,b_{\mathbf{k}'}^\dagger+q_{j'\mathbf{k}'}^*(t')\,b_{\mathbf{k}'}\right)\left|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{525}$$

$$=\left\langle\alpha\right|\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\,b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^*(t)b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\,b_{\mathbf{k}'}^\dagger+q_{j'\mathbf{k}'}^*(t')b_{\mathbf{k}'}\right)\left|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{526}$$

$$+\left\langle\alpha\right|\sum_{\mathbf{k}}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\,b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^*(t)b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\,b_{\mathbf{k}}^\dagger+q_{j'\mathbf{k}'}^*(t')b_{\mathbf{k}}\right)\left|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{527}$$

$$=\left\langle\alpha\right|\sum_{\mathbf{k}}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\,b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^*(t)b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}}(t')\,b_{\mathbf{k}}^\dagger+q_{j'\mathbf{k}}^*(t')b_{\mathbf{k}}\right)\left|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{528}$$

$$=\left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}(t)\,q_{j'\mathbf{k}}^*(t')\,b_{\mathbf{k}}^\dagger b_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{529}$$

$$+\left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}^*(t)\,q_{j'\mathbf{k}}(t')\,b_{\mathbf{k}}b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{530}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)\,q_{j'\mathbf{k}}^*(t')\,e^{i\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{531}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^*(t)\,q_{j'\mathbf{k}}(t')\,e^{-i\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{532}$$

$$\left\langle b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right\rangle_B = \frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D(-\alpha_{\mathbf{k}})\,b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\,D(-\alpha_{\mathbf{k}})\,b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{533}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\left(b_{\mathbf{k}}^\dagger+\alpha_{\mathbf{k}}^*\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{534}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger+\alpha_{\mathbf{k}}b_{\mathbf{k}}^\dagger+b_{\mathbf{k}}\alpha_{\mathbf{k}}^*+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{535}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{536}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{537}$$

$$=N+1, \tag{538}$$

$$\left\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right\rangle_B = \frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|\left(b_{\mathbf{k}}^\dagger+\alpha_{\mathbf{k}}^*\right)(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{539}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}|\alpha_{\mathbf{k}}|^2\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{540}$$

$$=N, \tag{541}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j'\mathrm{mod}2z}}\left(t'\right)\right\rangle_B = \sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}^*\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N + q_{j\mathbf{k}}^*\left(t\right)q_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N+1\right)\right) \tag{542}$$

$$= \sum_{\mathbf{k}}2N\left(q_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}^*\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Re} + \sum_{\mathbf{k}}q_{j\mathbf{k}}^*\left(t\right)q_{j'\mathbf{k}}\left(t'\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{543}$$

$$D\left(h'\right)D\left(h\right) = \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}D\left(h'+h\right), \tag{544}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \mathrm{Tr}_B\left(\mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}D\left(h'+h\right)\rho_B\right) \tag{545}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\mathrm{Tr}_B\left(D\left(h'+h\right)\rho_B\right) \tag{546}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\frac{1}{\pi N}\int \mathrm{d}^2\alpha P\left(\alpha\right)\left\langle \alpha\left|D\left(h'+h\right)\right|\alpha\right\rangle \tag{547}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\mathrm{e}^{-\frac{\left|h+h'\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{548}$$

$$h' = h\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{549}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_B = \mathrm{e}^{\frac{1}{2}\left(hh^*\mathrm{e}^{\mathrm{i}\omega\tau}-h^*h\mathrm{e}^{-\mathrm{i}\omega\tau}\right)}\mathrm{e}^{-\frac{\left|h+h\mathrm{e}^{\mathrm{i}\omega\tau}\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{550}$$

$$\frac{1}{2}\left|h\right|^2\left(\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\right) = \frac{1}{2}\left(hh^*\mathrm{e}^{\mathrm{i}\omega\tau}-h^*h\mathrm{e}^{-\mathrm{i}\omega\tau}\right) \tag{551}$$

$$= \frac{1}{2}\left|h\right|^2\left(\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)-\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{552}$$

$$= \frac{1}{2}\left|h\right|^2\left(2\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{553}$$

$$= \mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right), \tag{554}$$

$$-\frac{\left|h+h\mathrm{e}^{\mathrm{i}\omega\tau}\right|^2}{2} = -\left|h\right|^2\frac{\left|1+\mathrm{e}^{\mathrm{i}\omega\tau}\right|^2}{2} \tag{555}$$

$$= -\left|h\right|^2\frac{\left(1+2\cos\left(\omega\tau\right)+\cos^2\left(\omega\tau\right)\right)+\sin^2\left(\omega\tau\right)}{2} \tag{556}$$

$$= -\left|h\right|^2\frac{2+2\cos\left(\omega\tau\right)}{2} \tag{557}$$

$$= -\left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right), \tag{558}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_B = \mathrm{e}^{\mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right)}\mathrm{e}^{-\left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)} \tag{559}$$

$$= \mathrm{e}^{\mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right)-\left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)} \tag{560}$$

$$= \mathrm{e}^{-\left|h\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)}\mathrm{e}^{-\left|h\right|^2\coth\left(\frac{\beta\omega}{2}\right)} \tag{561}$$

$$= \left\langle D\left(h\right)\right\rangle_B\mathrm{e}^{-\phi\left(\tau\right)}, \tag{562}$$

$$\mathrm{e}^{-\phi\left(\tau\right)} = \mathrm{e}^{-\left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)}, \tag{563}$$

$$\phi\left(\tau\right) = \left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right), \tag{564}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\mathrm{e}^{-\frac{\left|h+h'\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{565}$$

$$h' = v\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{566}$$

$$m_{\mathbf{k}}\left(t\right) = \frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}, \tag{567}$$

$$\Gamma_{\mathbf{k}}\left(t\right) = \frac{v_{1\mathbf{k}}^*\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}^2} \tag{568}$$

$$\left\langle \widetilde{B_1^+B_0^-}\left(t\right)\widetilde{B_1^+B_0^-}\left(t'\right)\right\rangle_B = \left\langle B_1^+B_0^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)\right\rangle_B \tag{569}$$

$$= \left\langle B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\right\rangle_B \tag{570}$$

$$= \mathrm{Tr}_B\left(B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\rho_B\right) \tag{571}$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \text{Tr}_B \left(\prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t) \, e^{i\omega_{\mathbf{k}} \tau} \right) \right) \prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t') \right) \right) \rho_B \right) \tag{572}$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \text{Tr}_B \left(\prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t) \, e^{i\omega \tau} \right) D\left(m_{\mathbf{k}}(t') \right) \right) \rho_B \right) \tag{573}$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(m_{\mathbf{k}}(t) e^{i\omega \tau} m_{\mathbf{k}}^*(t') - \left(m_{\mathbf{k}}(t) e^{i\omega \tau} \right)^* m_{\mathbf{k}}(t') \right) - \frac{|m_{\mathbf{k}}(t) e^{i\omega \tau} + m_{\mathbf{k}}(t')|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \tag{574}$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} e^{i\left(m_{\mathbf{k}}(t) e^{i\omega \tau} m_{\mathbf{k}}^*(t') \right)^\Im - \frac{\left| \frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega \tau} + \frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \tag{575}$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} e^{i\left(m_{\mathbf{k}}(t) e^{i\omega \tau} m_{\mathbf{k}}^*(t') \right)^\Im - \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega \tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \tag{576}$$

$$\left\langle \widetilde{B_0^+ B_1^-}(t) \widetilde{B_0^+ B_1^-}(t') \right\rangle_B = e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} \left(e^{i\left(m_{\mathbf{k}}(t) e^{i\omega \tau} m_{\mathbf{k}}^*(t') \right)^\Im - \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega \tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) \tag{577}$$

$$\langle D(h) \, b \rangle_B = \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle \alpha | D(h) \, b | \alpha \rangle \tag{578}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle \alpha | D(-\alpha) D(h) \, b D(\alpha) | \alpha \rangle \tag{579}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) \, b D(\alpha) | 0 \rangle \tag{580}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) \, b D(\alpha) | 0 \rangle \tag{581}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \tag{582}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) (b + \alpha) | 0 \rangle \tag{583}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \, b | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \, \alpha | 0 \rangle \tag{584}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \, \alpha | 0 \rangle \tag{585}$$

$$= \frac{1}{\pi N} \int \alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} e^{-\frac{|h|^2}{2}} d^2\alpha \tag{586}$$

$$= hN \langle D(h) \rangle_B , \tag{587}$$

$$\langle D(h) \, b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle \alpha | D(h) \, b^\dagger | \alpha \rangle \tag{588}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) \, b^\dagger D(\alpha) | 0 \rangle \tag{589}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) \, b^\dagger D(\alpha) | 0 \rangle \tag{590}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) \, b^\dagger D(\alpha) | 0 \rangle \tag{591}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) \left(b^\dagger + \alpha^* \right) | 0 \rangle \tag{592}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \left(b^\dagger + \alpha^* \right) | 0 \rangle \tag{593}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \, b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \, \alpha^* | 0 \rangle \tag{594}$$

$$= \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) | 1 \rangle + \frac{1}{\pi N} \int d^2\alpha \, e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 | D(h) | 0 \rangle \tag{595}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \langle -h|1\rangle + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 | D(h) | 0\rangle, \tag{596}$$

$$\langle -h| = \mathrm{e}^{-\frac{|-h^*|^2}{2}} \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n| \tag{597}$$

$$\langle -h|1\rangle = \mathrm{e}^{-\frac{|-h^*|^2}{2}} \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n|1\rangle \tag{598}$$

$$\langle -h|1\rangle = \mathrm{e}^{-\frac{|-h^*|^2}{2}} (-h^*), \tag{599}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \mathrm{e}^{-\frac{|-h^*|^2}{2}} (-h^*) + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha^* \mathrm{e}^{-\frac{|-h^*|^2}{2}} \tag{600}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \mathrm{e}^{-\frac{|-h^*|^2}{2}} (-h^*) + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha^* \mathrm{e}^{-\frac{|-h^*|^2}{2}} \tag{601}$$

$$= -h^* \left\langle D(h) \right\rangle_B (N+1), \tag{602}$$

$$\langle b D(h) \rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \langle \alpha | b D(h) | \alpha\rangle \tag{603}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} h + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} \tag{604}$$

$$= h \left\langle D(h) \right\rangle_B (N+1), \tag{605}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \langle \alpha | b^\dagger D(h) | \alpha\rangle \tag{606}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} h + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^* - h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} \tag{607}$$

$$= -h^* \left\langle D(h) \right\rangle_B N. \tag{608}$$

The correlation functions can be found readily as:

$$B_1^+ B_0^- (t,\tau) = \prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right), \tag{609}$$

$$B_0^+ B_1^- (t,\tau) = \prod_{\mathbf{k}} \left(D\left(-m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right), \tag{610}$$

$$B_{10}(t) = \mathrm{e}^{\chi_{10}(t)} \left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} |m_{\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right), \tag{611}$$

$$B_x(t,\tau) = \frac{B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t)}{2}, \tag{612}$$

$$B_y(t,\tau) = \frac{B_0^+ B_1^- (t,\tau) - B_1^+ B_0^- (t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, \tag{613}$$

$$B_{i\mathrm{mod}2z}(t,\tau) = \sum_{\mathbf{k}} \left(q_{i\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{i\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), \tag{614}$$

$$\left\langle \widetilde{B_{i\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \left\langle B_{i\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \right\rangle_B \tag{615}$$

$$= \left\langle \sum_{\mathbf{k}} \left(q_{i\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{i\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t') b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*(t') b_{\mathbf{k}} \right) \right\rangle_B \tag{616}$$

$$= \sum_{\mathbf{k}} q_{i\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} q_{i\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \tag{617}$$

$$\left\langle \widetilde{B_x}(t) \widetilde{B_x}(t') \right\rangle_B = \left\langle B_x(t,\tau) B_x(t',0) \right\rangle_B \tag{618}$$

$$= \left\langle \left(\frac{B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left(\frac{B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t',0) - B_{10}(t') - B_{01}(t')}{2} \right) \right\rangle_B \tag{619}$$

$$= \frac{1}{4} \left\langle \left(B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t) \right) \left(B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t',0) - B_{10}(t') - B_{01}(t') \right) \right\rangle_B \tag{620}$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) + B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) - B_1^+ B_0^- (t,\tau) B_{10}(t') - B_1^+ B_0^- (\tau) B_{01}(t') \right. \tag{621}$$

$$+ B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) - B_0^+ B_1^- (t,\tau) B_{10}(t') - B_0^+ B_1^- (t,\tau) B_{01}(t') \tag{622}$$

$$- B_{10}(t) B_1^+ B_0^- (t',0) - B_{10}(t) B_0^+ B_1^- (t',0) + B_{10}(t) B_{10}(t') + B_{10}(t) B_{01}(t') - B_{01}(t) B_1^+ B_0^- (t',0) \tag{623}$$

$$\left. - B_{01}(t) B_0^+ B_1^- (t',0) + B_{01}(t) B_{10}(t') + B_{01}(t) B_{01}(t') \right\rangle \tag{624}$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) + B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) + B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) \right. \tag{625}$$

$$\left. + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) \right\rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(t') + B_{10}(t'))}{4}, \tag{626}$$

$$U_{10}(t,t') = \prod_{\mathbf{k}} e^{i(m_{\mathbf{k}}(t) m_{\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau})^{\Im}}, \tag{627}$$

$$\left\langle B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) e^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \prod_{\mathbf{k}} \left(D(-m_{\mathbf{k}}(t')) e^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \tag{628}$$

$$= e^{\chi_{10}(t) + \chi_{01}(t')} \left\langle \prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} \right) \right) \prod_{\mathbf{k}} \left(D\left(-m_{\mathbf{k}}(t') \right) \right) \right\rangle_B \tag{629}$$

$$= e^{\chi_{10}(t) + \chi_{01}(t')} \prod_{\mathbf{k}} \left\langle \left(D\left(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} \right) D\left(-m_{\mathbf{k}}(t') \right) \right) \right\rangle_B \tag{630}$$

$$= e^{\chi_{10}(t) + \chi_{01}(t')} U_{10}^* (t,t') \prod_{\mathbf{k}} e^{-\frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{631}$$

$$\left\langle B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(-m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} \right) e^{-\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t') \right) e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \tag{632}$$

$$= \prod_{\mathbf{k}} e^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle D\left(-m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} \right) D\left(m_{\mathbf{k}}(t') \right) \right\rangle_B \tag{633}$$

$$= e^{\chi_{01}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} \left\langle D\left(-m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} \right) D\left(m_{\mathbf{k}}(t') \right) \right\rangle_B \tag{634}$$

$$= e^{\chi_{01}(t) + \chi_{10}(t')} U_{10}^* (t,t') \prod_{\mathbf{k}} e^{-\frac{\left| (v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{635}$$

$$\left\langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) \right\rangle_B = e^{\chi_{10}(t) + \chi_{10}(t')} U_{10}(t,t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{636}$$

$$\left\langle B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) \right\rangle_B = e^{\chi_{01}(t) + \chi_{01}(t')} U_{10}(t,t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{637}$$

$$\left\langle \widetilde{B_x}(t) \widetilde{B_x}(t') \right\rangle_B = \frac{1}{4} \left\langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) + B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) + B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) \right. \tag{638}$$

$$\left. + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) \right\rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(t') + B_{10}(t'))}{4}, \tag{639}$$

$$= \frac{1}{4} \left(2 U_{10}(t,t') \left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right. \tag{640}$$

$$\left. + 2 U_{10}^* (t,t') \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} \prod_{\mathbf{k}} e^{-\frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) \tag{641}$$

$$- \left(e^{\chi_{10}(t)} e^{-\frac{1}{2} \sum_{\mathbf{k}} |m_{\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right)^{\Re} \left(e^{\chi_{01}(t')} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left(|m_{\mathbf{k}}(t')|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right)^{\Re} \tag{642}$$

$$= \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} U_{10}(t,t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right. \tag{643}$$

$$+ \left(e^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}^*\left(t,t'\right) \prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{644}$$

$$- \left(e^{\chi_{10}(t)}e^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \left(e^{\chi_{01}(t')}e^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{645}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B = \left\langle B_y\left(t,\tau\right)B_y\left(t',0\right)\right\rangle_B \tag{646}$$

$$= \frac{1}{2}\left(\left(e^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re} U_{10}\left(t,t'\right)\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \tag{647}$$

$$\left. + \left(e^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{648}$$

$$= \left\langle \left(\frac{B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2i}\right)\left(\frac{B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t',0) + B_{10}(t') - B_{01}(t')}{2i}\right)\right\rangle_B \tag{649}$$

$$= -\frac{1}{4}\left\langle \left(B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)\right)\left(B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t',0) + B_{10}(t') - B_{01}(t')\right)\right\rangle_B \tag{650}$$

$$= -\frac{1}{4}\left\langle B_0^+ B_1^-(t,\tau)B_0^+ B_1^-(t',0) - B_0^+ B_1^-(t,\tau)B_1^+ B_0^-(t',0) + B_0^+ B_1^-(t,\tau)B_{10}(t') - B_0^+ B_1^-(\tau)B_{01}(t') - B_1^+ B_0^-(t,\tau)B_0^+ B_1^-(t',0) \right. \tag{651}$$

$$+ B_1^+ B_0^-(t,\tau)B_1^+ B_0^-(t',0) - B_1^+ B_0^-(t,\tau)B_{10}(t') + B_1^+ B_0^-(t,\tau)B_{01}(t') + B_{10}(t)B_0^+ B_1^-(t',0) - B_{10}(t)B_1^+ B_0^-(t',0) \tag{652}$$

$$\left. + B_{10}(t)B_{10}(t') - B_{10}(t)B_{01}(t') - B_{01}(t)B_0^+ B_1^-(t',0) + B_{01}(t)B_1^+ B_0^-(t',0) - B_{01}(t)B_{10}(t') + B_{01}(t)B_{01}(t')\right\rangle \tag{653}$$

$$= -\frac{1}{4}\left\langle B_0^+ B_1^-(t,\tau)B_0^+ B_1^-(t',0) - B_0^+ B_1^-(t,\tau)B_1^+ B_0^-(t',0) - B_1^+ B_0^-(t,\tau)B_0^+ B_1^-(t',0) + B_1^+ B_0^-(t,\tau)B_1^+ B_0^-(t',0)\right\rangle \tag{654}$$

$$+ \left(B_{01}(t)\right)^{\Im} \left(B_{10}(t')\right)^{\Im} \tag{655}$$

$$= -\frac{1}{4}\left(2\left(e^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re} U_{10}\left(t,t'\right)\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \tag{656}$$

$$\left. -2\left(e^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Re} U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}} e^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{657}$$

$$+ \left(e^{\chi_{01}(t)}e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \left(e^{\chi_{10}(t')}e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{658}$$

$$= -\frac{1}{2}\left(\left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}\left(t,t'\right)\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \tag{659}$$

$$\left. - \left(e^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}} e^{\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{660}$$

$$+ \left(e^{\chi_{01}(t)}e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \left(e^{\chi_{10}(t')}e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{661}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B = \left\langle B_x(t,\tau)B_y\left(t',0\right)\right\rangle_B \tag{662}$$

$$= \left\langle \left(\frac{B_1^+ B_0^-(t,\tau) + B_0^+ B_1^-(t,\tau) - B_{10}(t) - B_{01}(t)}{2}\right)\left(\frac{B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t',0) + B_{10}(t') - B_{01}(t')}{2i}\right)\right\rangle_B \tag{663}$$

$$= \frac{1}{4i}\left\langle B_1^+ B_0^-(t,\tau)B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t,\tau)B_1^+ B_0^-(t',0) + B_1^+ B_0^-(t,\tau)B_{10}(t') - B_1^+ B_0^-(t,\tau)B_{01}(t') + B_{01}(t)B_1^+ B_0^-(t',0) \right. \tag{664}$$

$$+ B_0^+ B_1^-(t,\tau)B_0^+ B_1^-(t',0) - B_0^+ B_1^-(t,\tau)B_1^+ B_0^-(t',0) + B_0^+ B_1^-(t,\tau)B_{10}(t') - B_0^+ B_1^-(t,\tau)B_{01}(t') + B_{01}(t)B_{01}(t') \tag{665}$$

$$\left. - B_{10}(t)B_0^+ B_1^-(t',0) + B_{10}(t)B_1^+ B_0^-(t',0) - B_{10}(t)B_{10}(t') + B_{10}(t)B_{01}(t') - B_{01}(t)B_0^+ B_1^-(t',0) - B_{01}(t)B_{10}(t')\right\rangle_B \tag{666}$$

$$= \frac{1}{4i}\left\langle B_1^+ B_0^-(t,\tau)B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t,\tau)B_1^+ B_0^-(t',0) + B_1^+ B_0^-(t,\tau)B_{10}(t') - B_1^+ B_0^-(t,\tau)B_{01}(t') \right. \tag{667}$$

$$+ B_0^+ B_1^-(t,\tau)B_0^+ B_1^-(t',0) - B_0^+ B_1^-(t,\tau)B_1^+ B_0^-(t',0) + B_0^+ B_1^-(t,\tau)B_{10}(t') - B_0^+ B_1^-(t,\tau)B_{01}(t') \tag{668}$$

$$- B_{10}(t)B_0^+ B_1^-(t',0) + B_{10}(t)B_1^+ B_0^-(t',0) - B_{10}(t)B_{10}(t') + B_{10}(t)B_{01}(t') - B_{01}(t)B_0^+ B_1^-(t',0) \tag{669}$$

$$\left. + B_{01}(t)B_1^+ B_0^-(t',0) - B_{01}(t)B_{10}(t') + B_{01}(t)B_{01}(t')\right\rangle_B \tag{670}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) - B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) - B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) \right\rangle \tag{671}$$

$$+ \frac{1}{4i} \left(B_{10}(t) + B_{01}(t) \right) \left(B_{10}(t') - B_{01}(t') \right) \tag{672}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) - B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) \right. \tag{673}$$

$$\left. - B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) \right\rangle + \frac{1}{4i} \left(B_{10}(t) + B_{01}(t) \right) \left(B_{10}(t') - B_{01}(t') \right) \tag{674}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) - B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) \right. \tag{675}$$

$$\left. - B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) \right\rangle + \left(B_{10}(t) \right)^{\Re} \left(B_{10}(t') \right)^{\Im} \tag{676}$$

$$= \frac{1}{4i} \left(\left(e^{\chi_{10}(t) + \chi_{01}(t')} - e^{\chi_{01}(t) + \chi_{10}(t')} \right) U_{10}^* (t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right. \tag{677}$$

$$\left. + \left(e^{\chi_{01}(t) + \chi_{01}(t')} - e^{\chi_{10}(t) + \chi_{10}(t')} \right) U_{10}(t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t') \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) \tag{678}$$

$$+ \left(B_{10}(t) \right)^{\Re} \left(B_{10}(t') \right)^{\Im} \tag{679}$$

$$= \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}^* (t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right. \tag{680}$$

$$\left. + \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}(t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t') \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \left(B_{10}(t) \right)^{\Re} \left(B_{10}(t') \right)^{\Im} \tag{681}$$

$$\left\langle \widetilde{B}_y(t) \widetilde{B}_x(t') \right\rangle_B = \left\langle \left(\frac{B_0^+ B_1^- (t,\tau) - B_1^+ B_0^- (t,\tau) + B_{10}(t) - B_{01}(t)}{2i} \right) \left(\frac{B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t',0) - B_{10}(t') - B_{01}(t')}{2} \right) \right\rangle_B \tag{682}$$

$$= \frac{1}{4i} \left\langle \left(B_0^+ B_1^- (t,\tau) - B_1^+ B_0^- (t,\tau) + B_{10}(t) - B_{01}(t) \right) \left(B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t',0) - B_{10}(t') - B_{01}(t') \right) \right\rangle_B \tag{683}$$

$$= \frac{1}{4i} \left\langle B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) - B_0^+ B_1^- (t,\tau) B_{10}(t') - B_0^+ B_1^- (t,\tau) B_{01}(t') + B_{01}(t) B_{01}(t') \right. \tag{684}$$

$$- B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) - B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) + B_1^+ B_0^- (t,\tau) B_{10}(t') + B_1^+ B_0^- (t,\tau) B_{01}(t') \tag{685}$$

$$+ B_{10}(t) B_0^+ B_1^- (t',0) - B_{10}(t) B_{10}(t') - B_{10}(t) B_{01}(t') - B_{01}(t) B_1^+ B_0^- (t',0) - B_{01}(t) B_0^+ B_1^- (t',0) \tag{686}$$

$$\left. + B_{01}(t) B_{10}(t') + B_{10}(t) B_1^+ B_0^- (t',0) \right\rangle \tag{687}$$

$$= \frac{1}{4i} \left\langle B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) - B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) - B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) \right\rangle \tag{688}$$

$$+ \left(B_{10}(t) \right)^{\Im} \left(B_{10}(t') \right)^{\Re} \tag{689}$$

$$= \frac{1}{4i} \left(e^{\chi_{01}(t) + \chi_{10}(t')} U_{10}^* (t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right. \tag{690}$$

$$+ e^{\chi_{01}(t) + \chi_{01}(t')} U_{10}(t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t') \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{691}$$

$$- e^{\chi_{10}(t) + \chi_{10}(t')} U_{10}(t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t') \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{692}$$

$$\left. - e^{\chi_{10}(t) + \chi_{01}(t')} U_{10}^* (t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \left(B_{10}(t) \right)^{\Im} \left(B_{10}(t') \right)^{\Re} \tag{693}$$

$$= \frac{1}{4i} \left(2i \left(e^{\chi_{01}(t) + \chi_{10}(t')} \right)^{\Im} U_{10}^* (t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right. \tag{694}$$

$$\left. + 2i \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}(t,t') \prod_{\mathbf{k}} e^{- \frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t') \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \left(B_{10}(t) \right)^{\Im} \left(B_{10}(t') \right)^{\Re} \tag{695}$$

$$= (B_{10}(t))^{\Im}(B_{10}(t'))^{\Re} + \frac{1}{2}\left(\left(e^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im} U_{10}^*(t,t') \prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{696}$$

$$\left.+ \left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im} U_{10}(t,t') \prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{697}$$

$$\left\langle b^\dagger D(h)\right\rangle_B = -h^* \left\langle D(h)\right\rangle_B N \tag{698}$$

$$\left\langle bD(h)\right\rangle_B = h \left\langle D(h)\right\rangle_B (N+1) \tag{699}$$

$$\left\langle D(h) b^\dagger\right\rangle_B = -h^* \left\langle D(h)\right\rangle_B (N+1) \tag{700}$$

$$\left\langle D(h) b\right\rangle_B = h \left\langle D(h)\right\rangle_B N \tag{701}$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) = q_{j\mathbf{k}}(t) \tag{702}$$

$$\left\langle B_1^+ B_0^-(t,\tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger\right\rangle_B = \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \left\langle \prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)e^{i\omega_{\mathbf{k}}\tau}\right)\right)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t')) b_{\mathbf{k}'}^\dagger\right\rangle_B \tag{703}$$

$$= e^{\chi_{10}(t)} \left\langle D\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t')) b_{\mathbf{k}'}^\dagger\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}(t)e^{i\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_B \tag{704}$$

$$= e^{\chi_{10}(t)} q_{i\mathbf{k}'}(t') \left\langle D\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) b_{\mathbf{k}'}^\dagger\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}(t)e^{i\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_B \tag{705}$$

$$= e^{\chi_{10}(t)} q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) \left\langle \prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)e^{i\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_B \tag{706}$$

$$= q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) e^{\chi_{10}(t)} \left\langle \prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)e^{i\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_B \tag{707}$$

$$= -q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{10}(t) \tag{708}$$

$$\left\langle B_1^+ B_0^-(t,\tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t'))^* \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} \left\langle \prod_{\mathbf{k}} D\left(m_{\mathbf{k}}(t)e^{i\omega_{\mathbf{k}}\tau}\right)\right\rangle \tag{709}$$

$$= q_{i\mathbf{k}'}^*(t') \left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}(t), \tag{710}$$

$$\left\langle B_0^+ B_1^-(t,\tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger\right\rangle_B = -q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{01}(t), \tag{711}$$

$$\left\langle B_0^+ B_1^-(t,\tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'}\right\rangle_B = q_{i\mathbf{k}'}^*(t') \left(-m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t), \tag{712}$$

$$\left\langle B_x(t,\tau) B_{i\bmod 2z}(t',0)\right\rangle_B = \left\langle \left(\frac{B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*}{2}\right) \sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{713}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle \left(B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*\right)\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{714}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle \left(B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau)\right)\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{715}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right. \tag{716}$$

$$\left.+ B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right\rangle_B \tag{717}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}(t)\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{10}(t) + q_{i\mathbf{k}'}(t')\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{01}(t)\right. \tag{718}$$

$$\left.+ (q_{i\mathbf{k}'}(t'))^*\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}(t) + (q_{i\mathbf{k}'}(t'))^*\left(-m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t)\right) \tag{719}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}(t')\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{10}(t) + q_{i\mathbf{k}'}(t')\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{01}(t)\right. \tag{720}$$

$$\left.+ (q_{i\mathbf{k}'}(t'))^*\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}(t) + (q_{i\mathbf{k}'}(t'))^*\left(-m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t)\right) \tag{721}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* B_{10}(t) + \left(-m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* B_{01}(t)\right)\right. \tag{722}$$

$$\left.+ (q_{i\mathbf{k}'}(t'))^* N_{\mathbf{k}'}\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau} B_{10}(t) - m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau} B_{01}(t)\right)\right) \tag{723}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* B_{10}(t) - \left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* B_{01}(t)\right)\right. \tag{724}$$

$$\left.+ (q_{i\mathbf{k}'}(t'))^* N_{\mathbf{k}'}\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau} B_{10}(t) - m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau} B_{01}(t)\right)\right) \tag{725}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau}\right)^* (B_{10}(t) - B_{01}(t)) + (q_{i\mathbf{k}'}(t'))^* N_{\mathbf{k}'} m_{\mathbf{k}'}(t)e^{i\omega_{\mathbf{k}'}\tau} (B_{10}(t)\right. \tag{726}$$

$$-B_{01}\left(t\right)))\tag{727}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}2\mathrm{i}B_{10}^{\Im}\left(t\right)\left(\left(q_{i\mathbf{k}'}\left(t'\right)\right)^{*}N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right)\tag{728}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(\left(q_{i\mathbf{k}'}\left(t'\right)\right)^{*}N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right)\tag{729}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(\left(q_{i\mathbf{k}'}\left(t'\right)\right)^{*}N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right),\tag{730}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\tag{731}$$

$$\left\langle B_{i\bmod 2z}\left(t,\tau\right)B_{x}\left(t',0\right)\right\rangle_{B}=\left\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(q_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_{B}\tag{732}$$

$$=\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(q_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_{B}\tag{733}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(q_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{10}\left(t'\right)-B_{01}\left(t'\right)\right)\right\rangle_{B}\tag{734}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}\left(t\right)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(q_{i\mathbf{k}'}\left(t\right)\right)^{*}b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)\right)\right\rangle_{B}\tag{735}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)+q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)+\left(q_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right.\tag{736}$$

$$\left.+\left(q_{i\mathbf{k}'}\left(t\right)\right)^{*}b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)\right\rangle,\tag{737}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}\tag{738}$$

$$=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B}\tag{739}$$

$$=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(m_{\mathbf{k}'}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}'}\left(t'\right)}{2}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B}\tag{740}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\tag{741}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\tag{742}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\tag{743}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left(-\left(m_{\mathbf{k}'}\left(t'\right)\right)^{*}\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}N_{\mathbf{k}'}\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\tag{744}$$

$$=-\left(m_{\mathbf{k}'}\left(t'\right)\right)^{*}q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\tag{745}$$

$$=-\left(m_{\mathbf{k}'}\left(t'\right)\right)^{*}q_{i\mathbf{k}'}\left(t\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau},\tag{746}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)\right\rangle_{B}=\left(m_{\mathbf{k}'}\left(t'\right)\right)^{*}q_{i\mathbf{k}'}\left(t\right)B_{01}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau},\tag{747}$$

$$\left\langle q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}=\left(q_{i\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}\tag{748}$$

$$=\left(q_{i\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B}\tag{749}$$

$$=\left(q_{i\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle b_{\mathbf{k}'}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B}\tag{750}$$

$$=\left(q_{i\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B}\tag{751}$$

$$=\left(q_{i\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B}\tag{752}$$

$$= (q_{i\mathbf{k}'}(t))^* \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1) B_{10}(t'), \tag{753}$$

$$\left\langle (q_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+} B_1^-(t',0) \right\rangle_B = (q_{i\mathbf{k}'}(t))^* \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(-m_{\mathbf{k}'}(t')\right)(N_{\mathbf{k}'}+1) B_{01}(t'), \tag{754}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau) B_x(t',0) \right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'} \Big(-\left(m_{\mathbf{k}'}(t')\right)^* q_{i\mathbf{k}'}(t) B_{10}(t') N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(-m_{\mathbf{k}'}(t')\right)^* q_{i\mathbf{k}'}(t) B_{01}(t') N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{755}$$

$$+ q_{i\mathbf{k}'}^*(t) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1) B_{10}(t') + q_{i\mathbf{k}'}^*(t) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(-m_{\mathbf{k}'}(t')\right)(N_{\mathbf{k}'}+1) B_{01}(t') \Big) \tag{756}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}(t) N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t') \left(B_{01}(t') - B_{10}(t')\right) + q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (N_{\mathbf{k}'} \tag{757}$$

$$+1)\left(B_{10}(t') - B_{01}(t')\right)\Big) \tag{758}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}(t) N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t') \left(B_{01}(t') - B_{10}(t')\right) - q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} (N_{\mathbf{k}'} \tag{759}$$

$$+1)\left(B_{01}(t') - B_{10}(t')\right)\Big) \tag{760}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} B_{01}^{\Im}(t') \Big(q_{i\mathbf{k}'}(t) N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t') - q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)\Big), \tag{761}$$

$$\left\langle B_y(t,\tau) B_{i\mathrm{mod}2z}(t',0) \right\rangle_B = \left\langle \left(\frac{B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \right) \sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \Big) \right\rangle_B \tag{762}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \left\langle \left(B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)\right)\Big(q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \Big) \right\rangle_B \tag{763}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \left\langle \left(B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau)\right)\Big(q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + \left(q_{i\mathbf{k}'}(t')\right)^* b_{\mathbf{k}'} \Big) \right\rangle_B \tag{764}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \Big\langle B_0^+ B_1^-(t,\tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger - B_1^+ B_0^-(t,\tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + B_0^+ B_1^-(t,\tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \tag{765}$$

$$- B_1^+ B_0^-(t,\tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \Big\rangle, \tag{766}$$

$$\left\langle B_0^+ B_1^-(t,\tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \right\rangle_B = -q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{01}(t), \tag{767}$$

$$\left\langle B_0^+ B_1^-(t,\tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right\rangle_B = q_{i\mathbf{k}'}^*(t') \left(-m_{\mathbf{k}'}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t), \tag{768}$$

$$\left\langle B_1^+ B_0^-(t,\tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \right\rangle_B = -q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) B_{10}(t), \tag{769}$$

$$\left\langle B_1^+ B_0^-(t,\tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right\rangle_B = q_{i\mathbf{k}'}^*(t') m_{\mathbf{k}'}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} N_{\mathbf{k}'} B_{10}(t), \tag{770}$$

$$\left\langle B_y(t,\tau) B_{i\mathrm{mod}2z}(t',0) \right\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} (B_{01}(t) + B_{10}(t)) \Big(q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'}+1) - q_{i\mathbf{k}'}^*(t') m_{\mathbf{k}'}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} N_{\mathbf{k}'} \Big) \tag{771}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}^*(t') (N_{\mathbf{k}'}+1) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t) - q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t) \Big) (B_{10}(t) + B_{01}(t)), \tag{772}$$

$$= \frac{2}{2\mathrm{i}}\sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'}+1) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t) (B_{10}(t))^\Re - q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t) (B_{10}(t))^\Re \Big), \tag{773}$$

$$= \frac{(B_{10}(t))^\Re}{\mathrm{i}}\sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'}+1) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t) - q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t) \Big), \tag{774}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau) B_y(t',0) \right\rangle_B = \left\langle \sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \Big) \left(\frac{B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t',0) + B_{10}(t') - B_{01}(t')}{2\mathrm{i}} \right) \right\rangle_B \tag{775}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \sum_{\mathbf{k}'} \Big(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \Big) \left(B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t',0) + B_{10}(t') \right. \tag{776}$$

$$\left. - B_{01}(t') \right) \right\rangle_B \tag{777}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \left\langle \Big(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \Big) \left(B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t',0) \right) \right\rangle_B \tag{778}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \Big\langle q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_0^+ B_1^-(t',0) - q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-(t',0) \tag{779}$$

$$+q_{i\mathbf{k}'}^*\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-\left(t',0\right)-q_{i\mathbf{k}'}^*\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-\left(t',0\right)\Big\rangle \tag{780}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^\dagger,B_0^+B_1^-\left(t',0\right)\right\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^\dagger,B_1^+B_0^-\left(t',0\right)\right\rangle \tag{781}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)\left\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\right\rangle-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\right\rangle\Big\rangle \tag{782}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\left(t',0\right)\right\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\left(t',0\right)\right\rangle+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*(t)\left\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\right\rangle \tag{783}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\right\rangle \tag{784}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\left(t',0\right)\right\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\left(t',0\right)\right\rangle+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*(t)\left\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\right\rangle \tag{785}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\right\rangle\Big) \tag{786}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\left(t',0\right)\right\rangle_B=-m_{\mathbf{k}'}^*\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}, \tag{787}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\left(t',0\right)\right\rangle_B=m_{\mathbf{k}'}^*\left(t'\right)B_{01}\left(t'\right)N_{\mathbf{k}'}, \tag{788}$$

$$\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\right\rangle_B=m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right), \tag{789}$$

$$\left\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\right\rangle_B=-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right), \tag{790}$$

$$\left\langle B_{i\,\mathrm{mod}2z}(t,\tau)B_y\left(t',0\right)\right\rangle_B=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-\left(-m_{\mathbf{k}'}\left(t'\right)\right)^*B_{01}\left(t'\right)N_{\mathbf{k}'}\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}^*\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \tag{791}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)\right)-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\Big) \tag{792}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-q_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\right)^*B_{01}\left(t'\right)N_{\mathbf{k}'}+q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^*\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \tag{793}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(q_{i\mathbf{k}'}^*\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)-q_{i\mathbf{k}'}^*\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\right)\Big) \tag{794}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(B_{10}\left(t'\right)+B_{01}\left(t'\right)\right)\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^*\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\right) \tag{795}$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^*\left(t'\right)B_{10}^\Re\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^\Re\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\right) \tag{796}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^\Re\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^*\left(t'\right)B_{10}^\Re\left(t'\right)N_{\mathbf{k}'}\right) \tag{797}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^\Re\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^*\left(t'\right)B_{10}^\Re\left(t'\right)N_{\mathbf{k}'}\right) \tag{798}$$

$$=\mathrm{i}B_{10}^\Re\left(t'\right)\sum_{\mathbf{k}'}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^*\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^*\left(t'\right)N_{\mathbf{k}'}\right). \tag{799}$$

The correlation functions are equal to:

$$\left\langle\widetilde{B_{i\,\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j\,\mathrm{mod}2z}}\left(t'\right)\right\rangle_B=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)^*\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right),$$
$$\tag{800}$$

$$\left\langle\widetilde{B_x}\left(t\right)\widetilde{B_x}\left(t'\right)\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^\Re U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{801}$$

$$+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^\Re U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{802}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^\Re\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^\Re\right)$$
$$\tag{803}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_y}\left(t'\right)\right\rangle_B = -\frac{1}{2}\left(\left(\exp\left(\chi_{01}(t)+\chi_{01}\left(t'\right)\right)\right)^{\Re}U_{10}\left(t,t'\right)\Pi_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{804}$$

$$-\left(\exp\left(\chi_{10}(t)+\chi_{01}\left(t'\right)\right)\right)^{\Re}U_{10}^*\left(t,t'\right)\Pi_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{805}$$

$$+\left(\exp\left(\chi_{01}(t)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im}\left(\exp\left(\chi_{10}\left(t'\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im} \tag{806}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_y}\left(t'\right)\right\rangle_B = \frac{1}{2}\left(\left(\exp\left(\chi_{10}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^{\Im}U_{10}^*\left(t,t'\right)\Pi_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{807}$$

$$\left.+\left(\exp\left(\chi_{01}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^{\Im}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)+(B \tag{808}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_x}\left(t'\right)\right\rangle_B = \frac{1}{2}\left(\left(\exp\left(\chi_{01}\left(t\right)+\chi_{10}\left(t'\right)\right)\right)^{\Im}U_{10}^*\left(t,t'\right)\Pi_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{809}$$

$$\left.+\left(\exp\left(\chi_{01}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^{\Im}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)+(B \tag{810}$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_x}\left(t'\right)\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}}B_{01}^{\Im}\left(t'\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\frac{v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{811}$$

$$\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_{iz}}\left(t'\right)\right\rangle_B = \mathrm{i}B_{10}^{\Im}\left(t\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t'\right)\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t'\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{812}$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_y}\left(t'\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(t'\right)\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left(\frac{v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right), \tag{813}$$

$$\left\langle \widetilde{B_y}\left(t\right)\widetilde{B_{iz}}\left(t'\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(t\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t'\right)\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t'\right)\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*\right). \tag{814}$$

The spectral density is defined in the usual way:

$$J_i\left(\omega\right) \equiv \sum_{\mathbf{k}}\left|g_{i\mathbf{k}}\right|^2\delta\left(\omega-\omega_{\mathbf{k}}\right), \tag{815}$$

$$v_{i\mathbf{k}}\left(t\right) = g_{i\mathbf{k}}F_i\left(\omega_{\mathbf{k}},t\right). \tag{816}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i\left(\omega\right)}{\omega}\mathrm{d}\omega. \tag{817}$$

The integral version of the correlation functions are given by:

$$\chi_{10}\left(t\right)=\int_0^\infty \frac{\sqrt{J_1^*\left(\omega\right)J_0\left(\omega\right)}F_1^*\left(\omega,t\right)F_0\left(\omega,t\right)-\sqrt{J_1\left(\omega\right)J_0^*\left(\omega\right)}F_1\left(\omega,t\right)F_0^*\left(\omega,t\right)}{2\omega^2}\mathrm{d}\omega$$

$$U_{10}\left(t,t'\right)=\exp\left(i\left(\int_0^\infty \frac{\left(\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t\right)\right)\left(\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t'\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t'\right)\right)^*\exp\left(i\omega\tau\right)}{\omega^2}\mathrm{d}\right.\right.$$

$$B_{10}\left(t\right)=\exp\left(\chi_{10}\left(t\right)\right)\exp\left(-\frac{1}{2}\int_0^\infty \left|\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t\right)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega\right),$$

$$\xi^+\left(t,t'\right)=\exp\left(-\int_0^\infty \frac{|\left(\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t\right)\right)\exp\left(i\omega\tau\right)+\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t'\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t'\right)|^2}{2\omega^2}\right.$$

$$\xi^-\left(t,t'\right)=\exp\left(-\int_0^\infty \frac{|\left(\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t\right)\right)\exp\left(i\omega\tau\right)-\left(\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t'\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t'\right)\right)}{2\omega^2}\right.$$

$$Q\left(\omega,t\right)=\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t\right)}{\omega}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\exp\left(\chi_{10}\left(t\right)+\chi_{10}\left(t'\right)\right)\right)^\Re U_{10\left(t,t'\right)}\xi^+\left(t,t'\right)+\left(\exp\left(\chi_{10}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^\Re U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)\right)-(B_{10}(t))^\Re(B_0$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B=-\frac{1}{2}\left(\left(\exp\left(\chi_{01}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^\Re U_{10}\left(t,t'\right)\xi^+\left(t,t'\right)-\left(\exp\left(\chi_{10}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^\Re U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)\right)+(B_{01}(t))^\Im$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B=\frac{1}{2}\left(\left(\exp\left(\chi_{10}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^\Im U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)+\left(\exp\left(\chi_{01}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^\Im U_{10}\left(t,t'\right)\xi^+\left(t,t'\right)\right)+\left(B_{10}\left(t\right)\right)$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\exp\left(\chi_{01}\left(t\right)+\chi_{10}\left(t'\right)\right)\right)^\Im U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)+\left(\exp\left(\chi_{01}\left(t\right)+\chi_{01}\left(t'\right)\right)\right)^\Im U_{10}\left(t,t'\right)\xi^+\left(t,t'\right)\right)+\left(B_{10}\left(t\right)\right)$$

$$\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B=\int_0^\infty \left(\sqrt{J_i(\omega)J_j^*(\omega)}(1-F_i(\omega,t))\left(1-F_j^*\left(\omega,t'\right)\right)e^{i\omega\tau}N(\omega)+\sqrt{J_i^*(\omega)J_j(\omega)}(1-F_i^*(\omega,t))\left(1-F_j\left(\omega,t'\right)\right)e^{-i\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega,$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_x}\left(t'\right)\right\rangle_B=\mathrm{i}B_{01}^\Im\left(t'\right)\int_0^\infty \left(\sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega,t\right)\right)Q^*\left(\omega,t'\right)N\left(\omega\right)e^{i\omega\tau}-\sqrt{J_i^*\left(\omega\right)}\left(1-F_i^*\left(\omega,t\right)\right)Q\left(\omega,t'\right)e^{-i\omega\tau}\left(N\left(\omega\right)\right.\right.$$

$$\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_{iz}}\left(t'\right)\right\rangle_B=\mathrm{i}B_{01}^\Im\left(t\right)\int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)}\left(1-F_i^*\left(\omega,t'\right)\right)Q\left(\omega,t\right)N\left(\omega\right)e^{i\omega\tau}-\sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega,t'\right)\right)Q^*\left(\omega,t\right)e^{-i\omega\tau}\left(N\left(\omega\right)+\right.\right.$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_y}\left(t'\right)\right\rangle_B=iB_{10}^\Re\left(t'\right)\int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)}\left(1-F_i^*\left(\omega,t'\right)\right)Q\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)e^{-i\omega\tau}-\sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega,t'\right)\right)Q^*\left(\omega,t'\right)e^{i\omega\tau}N\right.$$

$$\left\langle \widetilde{B_y}\left(t\right)\widetilde{B_{iz}}\left(t'\right)\right\rangle_B=iB_{10}^\Re\left(t\right)\int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)}\left(1-F_i^*\left(\omega,t'\right)\right)Q\left(\omega,t\right)N\left(\omega\right)e^{i\omega\tau}-\sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega,t'\right)\right)Q^*\left(\omega,t\right)e^{-i\omega\tau}\left(N\left(\omega\right)+\right.\right.$$

$$\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t')) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1),$$

$$(818)$$

$$= \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}},t))(g_{j\mathbf{k}} - g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}},t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + (g_{i\mathbf{k}} - g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}},t))^*(g_{j\mathbf{k}} - g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}},t'))e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1))$$

$$(819)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}}(1 - F_i(\omega_{\mathbf{k}},t))\, g_{j\mathbf{k}}^*(1 - F_j(\omega_{\mathbf{k}},t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + g_{i\mathbf{k}}^*(1 - F_i(\omega_{\mathbf{k}},t))^* g_{j\mathbf{k}}(1 - F_j(\omega_{\mathbf{k}},t'))e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1))$$

$$(820)$$

$$\approx \int_0^\infty \left(\sqrt{J_i(\omega)J_j^*(\omega)}(1 - F_i(\omega,t))\left(1 - F_j^*(\omega,t')\right)e^{\mathrm{i}\omega\tau}N(\omega) + \sqrt{J_i^*(\omega)J_j(\omega)}(1 - F_i^*(\omega,t))\left(1 - F_j(\omega,t')\right)e^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega,$$

$$(821)$$

$$\chi_{10}(t) = \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)\,v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)\,v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right) \tag{822}$$

$$= \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}},t)\,g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t) - g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)\,g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}},t)}{\omega_{\mathbf{k}}^2}\right) \tag{823}$$

$$= \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{g_{1\mathbf{k}}^* g_{0\mathbf{k}}F_1^*(\omega_{\mathbf{k}},t)\,F_0(\omega_{\mathbf{k}},t) - g_{1\mathbf{k}}g_{0\mathbf{k}}^* F_1(\omega_{\mathbf{k}},t)\,F_0^*(\omega_{\mathbf{k}},t)}{\omega_{\mathbf{k}}^2}\right) \tag{824}$$

$$\approx \int_0^\infty \frac{\sqrt{J_1^*(\omega)\,J_0(\omega)}F_1^*(\omega,t)\,F_0(\omega,t) - \sqrt{J_1(\omega)\,J_0^*(\omega)}F_1(\omega,t)\,F_0^*(\omega,t)}{2\omega^2}\mathrm{d}\omega, \tag{825}$$

$$U_{10}(t,t') = \prod_{\mathbf{k}} \exp\left(i\left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{826}$$

$$= \exp\left(i\sum_{\mathbf{k}}\left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{827}$$

$$= \exp\left(i\left(\sum_{\mathbf{k}}\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{828}$$

$$= \exp\left(i\left(\sum_{\mathbf{k}}\frac{(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t') - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{829}$$

$$\approx \exp\left(i\left(\int_0^\infty \frac{\left(\sqrt{J_1(\omega)}F_1(\omega,t) - \sqrt{J_0(\omega)}F_0(\omega,t)\right)\left(\sqrt{J_1(\omega)}F_1(\omega,t') - \sqrt{J_0(\omega)}F_0(\omega,t')\right)^* \exp(i\omega\tau)}{\omega^2}\right.\right.$$

$$(830)$$

$$(831)$$

$$B_{10}(t) = \left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)\,v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)\,v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{832}$$

$$= \exp\left(\chi_{10}(t)\right)\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{833}$$

$$\approx \exp\left(\chi_{10}(t)\right)\exp\left(-\frac{1}{2}\int_0^\infty \left|\frac{\sqrt{J_1(\omega)}F_1(\omega,t) - \sqrt{J_0(\omega)}F_0(\omega,t)}{\omega}\right|^2 \coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega\right) \tag{834}$$

$$(835)$$

$$\xi^+(t,t') = \Pi_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{836}$$

$$= \exp\left(-\sum_{\mathbf{k}}\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{837}$$

$$\left(\frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t') - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')|^2}{\ }\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda + \mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = 0. \tag{876}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_\pm = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S |\pm\rangle = \lambda_\pm |\pm\rangle$. Using this notation is possible to write $H_{\bar{S}} = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$.

The time-dependence of the system operators $\widetilde{A}_i(t)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A}_i(\tau) = e^{i\overline{H_{\bar{S}}}\tau} A_i e^{-i\overline{H_{\bar{S}}}\tau} \tag{877}$$

$$= \sum_w e^{-iw\tau} \mathcal{A}_i(w). \tag{878}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.

In order to use the equation (878) to descompose the equation (372) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right) \tag{879}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) ... H(t_n). \tag{880}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right) \equiv \exp(-it H_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-it\left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...\right)\right). \tag{881}$$

The terms can be found expanding $\mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\, dt', \tag{882}$$

$$H_E^{(1)}(t) = -\frac{i}{2t}\int_0^t dt' \int_0^{t'} dt'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{883}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{884}$$

In this case the Fourier decomposition using the expansion of $H_E(t)$ is:

$$\widetilde{A}_i(t) = U^\dagger(t) A_i(t) U(t) \tag{885}$$

$$\widetilde{A}_i(t) = e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \tag{886}$$

$$= \sum_{w(t)} e^{-iw(t)t} \mathcal{A}_i(t, w(t)). \tag{887}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau,t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau,t) = U(t)U^\dagger(t-\tau)A_j(t)U(t-\tau)U^\dagger(t) \tag{888}$$

$$= e^{-itH_E(t)}e^{i(t-\tau)H_E(t-\tau)}A_j(t)e^{-i(t-\tau)H_E(t-\tau)}e^{itH_E(t)} \tag{889}$$

$$= e^{-itH_E(t)}\left(\sum_{w'(t-\tau)}e^{-i(t-\tau)w(t-\tau)}\mathcal{A}_j(t,w(t-\tau))\right)e^{itH_E(t)} \tag{890}$$

$$= \sum_{w(t),w'(t-\tau)}e^{iw'(t)t}e^{-i(t-\tau)w(t-\tau)}\mathcal{A}_j(t,w(t-\tau),w'(t)) \tag{891}$$

$$= \sum_{w(t),w'(t-\tau)}e^{iw'(t)t}e^{-i(t-\tau)w(t-\tau)}\mathcal{A}_j(t,w(t-\tau),w'(t)) \tag{892}$$

$$= \sum_{w(t),w'(t-\tau)}e^{i\tau w(t-\tau)}e^{-it\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_j(t,w(t-\tau),w'(t)) \tag{893}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}(t-\tau)$ and $\overline{H_E}(t)$ respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (878) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $V = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta \in V}\langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|. \tag{894}$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{i\overline{H_E}\tau} = e^{i(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{895}$$

$$= e^{i\lambda_+|+\rangle\langle+|\tau}e^{i\lambda_-|-\rangle\langle-|\tau} \tag{896}$$

$$= \left(|-\rangle\langle-| + e^{i\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|\right) \tag{897}$$

$$= e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|. \tag{898}$$

Calculating the transformation (878) directly using the previous relationship we find that:

$$U^\dagger(\tau)A_i(\tau)U(\tau) = \left(e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta \in V}\langle\alpha|A_i(\tau)|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-i\lambda_+\tau}|+\rangle\langle+| + e^{-i\lambda_-\tau}|-\rangle\langle-|\right) \tag{899}$$

$$= \langle+|A_i(\tau)|+\rangle|+\rangle\langle+| + e^{i\eta\tau}\langle+|A_i(\tau)|-\rangle|+\rangle\langle-| + e^{-i\eta\tau}\langle-|A_i(\tau)|+\rangle|-\rangle\langle+| + \langle-|A_i(\tau)|-\rangle|-\rangle\langle-|. \tag{900}$$

$$= \mathcal{A}_i(0) + \mathcal{A}_i(-w)e^{iw\tau} + \mathcal{A}_i(w)e^{-iw\tau} \tag{901}$$

Here $w = \lambda_+ - \lambda_-$. Comparing the RHS of the equations (878) and the explicit expression for $\widetilde{A_i}(\tau)$ in (886), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$\mathcal{A}_i(0) = \langle+|A_i(\tau)|+\rangle|+\rangle\langle+| + \langle-|A_i(\tau)|-\rangle|-\rangle\langle-|, \tag{902}$$

$$\mathcal{A}_i(-w) = \langle+|A_i(\tau)|-\rangle|+\rangle\langle-|, \tag{903}$$

$$\mathcal{A}_i(w) = \langle-|A_i(\tau)|+\rangle|-\rangle\langle+|. \tag{904}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau) = \widetilde{A_i}^\dagger(\tau)$ and $\widetilde{B_i}(\tau) = \widetilde{B_i}^\dagger(\tau)$ we can use the equation (878) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau\, C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\big] + \mathcal{B}_{ji}(-\tau)\big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\big]\Big) \quad (905)$$

$$= -\mathrm{i}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\, C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\big[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\big] \quad (906)$$

$$-\mathcal{B}_{ji}(-\tau)\left[A_i,\overline{\rho_S}(t)\,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_j\left(w(t-\tau),w'(t)\right)\right]\Big) \quad (907)$$

Given that $\mathcal{A}_j\left(w(t-\tau),w'(t)\right) = \mathcal{A}_j^\dagger\left(-w(t-\tau),-w'(t)\right)$ from the Fourier decomposition (878) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\mathcal{B}_{ij}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(s)\,\rho_B\right) \quad (908)$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)\,\widetilde{B_j}(0)\,\rho_B\right). \quad (909)$$

Let's define:

$$\mathcal{A}_j\left(w(t-\tau),w'(t)\right) = \mathcal{A}_{jww'}(t-\tau,t) \quad (910)$$

The master equation can be re-written in the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\, C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right] \quad (911)$$

$$+ \sum_{ijww'}\mathcal{B}_{ji}(-\tau)\left[A_i,\overline{\rho_S}(t)\,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\right] \quad (912)$$

$$= -\mathrm{i}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\, C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right] \quad (913)$$

$$+ \sum_{ijww'}\mathcal{B}_{ji}(-\tau)\left[A_i,\overline{\rho_S}(t)\,e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\right] \quad (914)$$

$$= -\mathrm{i}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\, C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right] \quad (915)$$

$$+ \sum_{ijww'}\mathcal{B}_{ji}(-\tau)\left[A_i,\overline{\rho_S}(t)\,e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\right] \quad (916)$$

$$= -\mathrm{i}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\, C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right] \quad (917)$$

$$-\mathcal{B}_{ji}(-\tau)\left[A_i,\overline{\rho_S}(t)\,e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\right]\Big) \quad (918)$$

$$= -\mathrm{i}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\, C_i(t)C_j(t-\tau)\mathrm{Tr}_B\Big(\left[A_i,\widetilde{B_i}(\tau)\widetilde{B_j}(0)\rho_B e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right] \quad (919)$$

$$-\left[A_i,\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\right]\Big) \quad (920)$$

Given that if we define:

$$D_{ijww'}(t-\tau,t) = C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t) \quad (921)$$

then

$$D_{ijww'}^\dagger(t-\tau,t) = \left(C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\right)^\dagger \quad (922)$$

$$= \mathcal{B}_{ij}^*(\tau)C_i(t)C_j(t-\tau)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}^\dagger(t-\tau,t) \quad (923)$$

We used the fact that $C_i(t), C_j(t-\tau)$ are real. Now let's consider the following trace recalling that $\text{Tr}(A)^* = \text{Tr}(A^\dagger)$ so:

$$\text{Tr}_B\left(\widetilde{B_j}(-\tau)\,\widetilde{B_i}(0)\,\rho_B\right) = \text{Tr}_B\left(e^{-i\tau H_B(\tau)}B_j e^{i\tau H_B(\tau)}B_i\rho_B\right) \tag{924}$$

$$= \text{Tr}_B\left(B_j e^{i\tau H_B(\tau)}B_i\rho_B e^{-i\tau H_B(\tau)}\right) \text{ (by cyclic permutivity of trace)} \tag{925}$$

$$= \text{Tr}_B\left(B_j e^{i\tau H_B(\tau)}B_i e^{-i\tau H_B(\tau)}\rho_B\right) \text{ (by commutativity of } e^{-i\tau H_B(\tau)} \text{ and } \rho_B\text{)} \tag{926}$$

$$= \text{Tr}_B\left(B_j \widetilde{B_i}(\tau)\,\rho_B\right) \text{ (by definition of time evolution)} \tag{927}$$

$$= \text{Tr}_B\left(B_j \widetilde{B_i}(\tau)\,\rho_B\right) \tag{928}$$

$$= \text{Tr}_B\left(\rho_B B_j \widetilde{B_i}(\tau)\right) \tag{929}$$

$$= \text{Tr}_B\left(\left(\widetilde{B_i}(\tau)\,B_j\rho_B\right)^\dagger\right) \text{ (by definition of adjoint)} \tag{930}$$

$$= \text{Tr}_B\left(\widetilde{B_i}(\tau)\,B_j\rho_B\right)^* \tag{931}$$

$$= \mathcal{B}_{ij}^*(\tau) \tag{932}$$

So we can write the master equation like:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t d\tau C_i(t)C_j(t-\tau)\left(\mathcal{B}_{ij}(\tau)\left[A_i,\,e^{i\tau w(t-\tau)}e^{-it\left(w(t-\tau)-w'(t)\right)}A_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\right]\right. \tag{933}$$

$$\left. -\mathcal{B}_{ij}^*(\tau)\left[A_i,\overline{\rho_S}(t)\,e^{-i\tau w(t-\tau)}e^{it\left(w(t-\tau)-w'(t)\right)}A_j^\dagger(w(t-\tau),w'(t))\right]\right) \tag{934}$$

$$= -i\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t d\tau\left([A_i,D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)] - \left[A_i,\overline{\rho_S}(t)\,D_{ijww'}^\dagger(t-\tau,t)\right]\right) \tag{935}$$

Let's define the response matrix in the following way.

$$\mathcal{D}_{ijww'}(t) = \int_0^t d\tau D_{ijww'}(t-\tau,t) \tag{936}$$

Then the master equation can be written as:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left([A_i,\mathcal{D}_{ijww'}(t)\overline{\rho_S}(t)] - \left[A_i,\overline{\rho_S}(t)\,\mathcal{D}_{ijww'}^\dagger(t)\right]\right) \tag{937}$$

If we extend the upper limit of integration to ∞ in the equation (936) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{d\overline{\rho}_S(t)}{dt}e^V = \frac{d\left(e^{-V}\overline{\rho}_S e^V\right)}{dt} \tag{938}$$

$$= \frac{d\rho_S}{dt} \tag{939}$$

$$= -ie^{-V}\left[\overline{H_S}(t),\overline{\rho_S}(t)\right]e^V - \sum_{ijww'}\int_0^t d\tau\left(e^{-V}[A_i,D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)]e^V - e^{-V}\left[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\right]e^V\right). \tag{940}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^V = e^{-V}\overline{A\mathbb{I}B}e^V \tag{941}$$

$$= e^{-V}\overline{A}e^V e^{-V}\bar{B}e^V \tag{942}$$

$$= \left(e^{-V}\overline{A}e^V\right)\left(e^{-V}\bar{B}e^V\right) \tag{943}$$

$$= AB. \tag{944}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^V = e^{-V}\overline{(AB-BA)}e^V \tag{945}$$

$$= e^{-V}\overline{AB}e^V - e^{-V}\overline{BA}e^V \tag{946}$$

$$= AB - BA \tag{947}$$

$$= [A,B]. \tag{948}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^V - e^{-V}\sum_{ijww'}\left(\left[A_i,\mathcal{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathcal{D}_{ijww'}^\dagger\left(t\right)\right]\right)e^V \tag{949}$$

$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^V - \sum_{ijww'}\left(e^{-V}\left[A_i,\mathcal{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right]e^V - e^{-V}\left[A_i,\overline{\rho_S}\left(t\right)\mathcal{D}_{ijww'}^\dagger\left(t\right)\right]e^V\right) \tag{950}$$

$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_i e^V, e^{-V}\mathcal{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)e^V\right] - \left[e^{-V}A_i e^V, e^{-V}\overline{\rho_S}\left(t\right)\mathcal{D}_{ijww'}^\dagger\left(t\right)e^V\right]\right) \tag{951}$$

$$= -\mathrm{i}\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_i e^V, e^{-V}\mathcal{D}_{ijww'}(t)e^V e^{-V}\overline{\rho_S}(t)e^V\right] - \left[e^{-V}A_i e^V, e^{-V}\overline{\rho_S}(t)e^V e^{-V}\mathcal{D}_{ijww'}^\dagger(t)e^V\right]\right) \tag{952}$$

$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_i e^V, e^{-V}\mathcal{D}_{ijww'}\left(t\right)e^V \rho_S\left(t\right)\right] - \left[e^{-V}A_i e^V, \rho_S\left(t\right)e^{-V}\mathcal{D}_{ijww'}^\dagger\left(t\right)e^V\right]\right). \tag{953}$$

IV.   LIMIT CASES

In order to show the plausibility of the master equation (937) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A.   Time-independent variational quantum master equation

At first let's show that the master equation (937) reproduces the results of the reference [1], for the latter case we have that $i,j \in \{1,2,3\}$ and $\omega \in (0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{954}$$

After performing the transformation (24) on the Hamiltonian (954) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, \tag{955}$$

$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), \tag{956}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{957}$$

The Hamiltonian (955) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{958}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{959}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \tag{960}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2 - \epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \tag{961}$$

$$\lambda_\pm = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2 - \epsilon^2 - \Omega_r^2}{4}\right)}}{2} \tag{962}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{963}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{964}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \tag{965}$$

$$\lambda_\pm = \frac{R \pm \eta}{2}. \tag{966}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{967}$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.

For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{968}$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$.
Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon + \eta}{2}, \tag{969}$$

$$\lambda_- = \frac{\epsilon - \eta}{2}, \tag{970}$$

$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \tag{971}$$

$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, \tag{972}$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}, \tag{973}$$

$$\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}. \tag{974}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{975}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x) + 1}. \tag{976}$$

So the value of $\tan(\theta)$ using (976) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}} + 1} \tag{977}$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}}{\frac{\epsilon + \sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}} \tag{978}$$

$$= \frac{\Omega_r}{\epsilon + \eta}. \tag{979}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (903)-(904) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ with$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{980}$$

$$= 2\sin(\theta)\cos(\theta) \tag{981}$$

$$= \sin(2\theta), \tag{982}$$

$$\langle -|\sigma_x|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \tag{983}$$

$$= -2\sin(\theta)\cos(\theta) \tag{984}$$

$$= -\sin(2\theta), \tag{985}$$

$$\langle -|\sigma_x|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{986}$$

$$= \cos^2(\theta) - \sin^2(\theta) \tag{987}$$

$$= \cos(2\theta), \tag{988}$$

$$\langle +|\sigma_y|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{989}$$

$$= i\sin(\theta)\cos(\theta) - i\sin(\theta)\cos(\theta) \tag{990}$$

$$= 0, \tag{991}$$

$$\langle -|\sigma_y|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \tag{992}$$

$$= i\sin(\theta)\cos(\theta) - i\sin(\theta)\cos(\theta) \tag{993}$$

$$= 0, \tag{994}$$

$$\langle -|\sigma_y|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{995}$$

$$= i\cos^2(\theta) + i\sin^2(\theta) \tag{996}$$

$$= i. \tag{997}$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{998}$$

$$= \cos(\theta)\cos(\theta) \tag{999}$$

$$= \cos^2(\theta), \tag{1000}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \tag{1001}$$

$$= \sin(\theta)\sin(\theta) \tag{1002}$$

$$= \sin^2(\theta), \tag{1003}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{1004}$$

$$= -\sin(\theta)\cos(\theta) \tag{1005}$$

$$= -\sin(\theta)\cos(\theta). \tag{1006}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1\left(0\right) = \sin\left(2\theta\right)\left(\left|+\right\rangle\!\left\langle+\right| - \left|-\right\rangle\!\left\langle-\right|\right), \tag{1007}$$

$$A_1\left(\eta\right) = \cos\left(2\theta\right)\left|-\right\rangle\!\left\langle+\right|, \tag{1008}$$

$$A_2\left(0\right) = 0, \tag{1009}$$

$$A_2\left(\eta\right) = \mathrm{i}\left|-\right\rangle\!\left\langle+\right|, \tag{1010}$$

$$A_3\left(0\right) = \cos^2\left(\theta\right)\left|+\right\rangle\!\left\langle+\right| + \sin^2\left(\theta\right)\left|-\right\rangle\!\left\langle-\right|, \tag{1011}$$

$$A_3\left(\eta\right) = -\sin\left(\theta\right)\cos\left(\theta\right)\left|-\right\rangle\!\left\langle+\right|. \tag{1012}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (940) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (921) is equivalent to:

$$\mathcal{D}_{ijww'}\left(t\right) = \int_0^t \mathrm{d}\tau D_{ijww'}\left(t-\tau,t\right) \tag{1013}$$

$$= \int_0^t \mathrm{d}\tau C_i\left(t\right) C_j\left(t-\tau\right) \Lambda_{ij}\left(\tau\right) e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathcal{A}_j\left(w\left(t-\tau\right), w'\left(t\right)\right) \tag{1014}$$

$$= \int_0^t \mathrm{d}\tau C_i\left(t\right) C_j\left(t-\tau\right) \Lambda_{ij}\left(\tau\right) e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)} \mathcal{A}_j\left(w, w'\right). \tag{1015}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}\left(\tau\right)$ relate with the correlation functions defined in the equation (413) in the following way:

$$\Lambda'_{ij}\left(\tau\right) = C_i\left(t\right) C_j\left(t-\tau\right) \Lambda_{ij}\left(\tau\right). \tag{1016}$$

So the response matrix can be rewritten as:

$$\mathcal{D}_{ijww'}\left(t\right) = \left(\int_0^t \mathrm{d}\tau \Lambda'_{ij}\left(\tau\right) e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)}\right) \mathcal{A}_j\left(w, w'\right) \tag{1017}$$

Let's define the response function like:

$$K_{ij}\left(w, w', t\right) = \int_0^t C_i\left(t\right) C_j\left(t-\tau\right) \Lambda_{ij}\left(\tau\right) e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1018}$$

$$= \int_0^t \Lambda'_{ij}\left(\tau\right) e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1019}$$

$$= K_{ijww'}\left(t\right). \tag{1020}$$

Then we have the following equivalence:

$$\mathcal{D}_{ijww'}\left(t\right) = K_{ijww'}\left(t\right) \mathcal{A}_j\left(w, w'\right) \tag{1021}$$

$$= K_{ijww'}\left(t\right) \mathcal{A}_{jww'} \tag{1022}$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\tilde{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left([A_i,\mathcal{D}_{ijww'}(t)\overline{\rho_S}(t)] - \left[A_i,\overline{\rho_S}(t)\mathcal{D}^{\dagger}_{ijww'}(t)\right]\right) \tag{1023}$$

$$= -\mathrm{i}\left[\overline{H_{\tilde{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left([A_i,K_{ijww'}(t)\mathcal{A}_{jww'}\overline{\rho_S}(t)] - \left[A_i,\overline{\rho_S}(t)K^*_{ijww'}(t)\mathcal{A}^{\dagger}_{jww'}\right]\right) \tag{1024}$$

$$= -\mathrm{i}\left[\overline{H_{\tilde{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(K_{ijww'}(t)[A_i,\mathcal{A}_{jww'}\overline{\rho_S}(t)] - K^*_{ijww'}(t)\left[A_i,\overline{\rho_S}(t)\mathcal{A}^{\dagger}_{jww'}\right]\right) \tag{1025}$$

$$= -\mathrm{i}\left[\overline{H_{\tilde{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t)+\mathrm{i}K^{\Im}_{ijww'}(t)\right)[A_i,\mathcal{A}_{jww'}\overline{\rho_S}(t)] - \left(K^{\Re}_{ijww'}(t)-\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\overline{\rho_S}(t)\mathcal{A}^{\dagger}_{jww'}\right]\right) \tag{1026}$$

$$= -\mathrm{i}\left[\overline{H_{\tilde{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}K^{\Re}_{ijww'}(t)\left[A_i,\mathcal{A}_{jww'}\overline{\rho_S}(t)-\overline{\rho_S}(t)\mathcal{A}^{\dagger}_{jww'}\right] - \mathrm{i}\sum_{ijww'}K^{\Im}_{ijww'}(t)\left[A_i,\mathcal{A}_{jww'}\overline{\rho_S}(t)+\overline{\rho_S}(t)\mathcal{A}^{\dagger}_{jww'}\right] \tag{1027}$$

Using the notation of the master equation (937), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(??)-(??) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (413) are equal to:

$$\left\langle \widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left((g_{1\mathbf{k}}-v_{1\mathbf{k}})(g_{1\mathbf{k}}-v_{1\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + (g_{1\mathbf{k}}-v_{1\mathbf{k}})^*(g_{1\mathbf{k}}-v_{1\mathbf{k}})e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{1028}$$

$$= \sum_{\mathbf{k}} |g_{1\mathbf{k}}-v_{1\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{1029}$$

$$\approx \int_0^\infty J_1(\omega)(1-F_1(\omega))^2 \left(e^{i\omega\tau}N(\omega) + e^{-i\omega\tau}(N(\omega)+1)\right) d\omega \tag{1030}$$

$$G_\pm(\omega,\tau) = e^{i\omega\tau}N(\omega) + e^{-i\omega\tau}(N(\omega)+1) \tag{1031}$$

$$\left\langle \widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B \approx \int_0^\infty J_1(\omega)(1-F_1(\omega))^2 G_+(\omega,t)\, d\omega \tag{1032}$$

$$\chi_{10}(t) = 0 \text{ (because } v_{0\mathbf{k}}(t)=0 \text{ for all } \mathbf{k}) \tag{1033}$$

$$U_{10}(t,s) = \prod_{\mathbf{k}} \exp\left(i\left(\frac{(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))^* e^{i\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^\Im\right) \tag{1034}$$

$$= \prod_{\mathbf{k}} \exp\left(i\left(\frac{v_{1\mathbf{k}}^2(t)e^{i\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^\Im\right) \tag{1035}$$

$$= \prod_{\mathbf{k}} \exp\left(i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\right) \tag{1036}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\left(\prod_{\mathbf{k}}\exp\left(i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\right)\prod_{\mathbf{k}}\exp\left(-\frac{|v_{1\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \prod_{\mathbf{k}}\exp\left(-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\right)\prod_{\mathbf{k}}\exp\left(-\frac{|v_{1\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1037}$$

$$- \left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1038}$$

$$= \tfrac{1}{2}\left(\prod_{\mathbf{k}}\exp\left(i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \prod_{\mathbf{k}}\exp\left(-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1039}$$

$$- \left(\exp\left(-\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1040}$$

$$|v_{1\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\pm v_{1\mathbf{k}}|^2 = v_{1\mathbf{k}}^2|e^{i\omega_{\mathbf{k}}\tau}\pm 1|^2 \tag{1041}$$

$$= v_{1\mathbf{k}}^2|\cos(\omega_{\mathbf{k}}\tau)+i\sin(\omega_{\mathbf{k}}\tau)\pm 1|^2 \tag{1042}$$

$$= v_{1\mathbf{k}}^2\left((1\pm\cos(\omega_{\mathbf{k}}\tau))^2 + \sin^2(\omega_{\mathbf{k}}\tau)\right) \tag{1043}$$

$$= 2v_{1\mathbf{k}}^2(1\pm\cos(\omega_{\mathbf{k}}\tau)) \tag{1044}$$

$$B \equiv \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{1045}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\left(\exp\left(\sum_{\mathbf{k}}i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(\sum_{\mathbf{k}}-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1046}$$

$$- \left(\exp\left(-\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1047}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\left(-i\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{1048}$$

$$\approx \int_0^\infty \frac{J_1(\omega)F_1^2(\omega)}{\omega^2}\left(-i\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right) d\omega \tag{1049}$$

$$= \int_0^\infty \frac{J_1(\omega)F_1^2(\omega)}{\omega^2}G_+(\omega,\tau)\, d\omega \tag{1050}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\left(\exp\left(\sum_{\mathbf{k}}i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{2v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(\sum_{\mathbf{k}}-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{2v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) - B^2 \tag{1051}$$

$$= \tfrac{1}{2}\left(\exp\left(\sum_{\mathbf{k}}i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(\sum_{\mathbf{k}}-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) - B^2 \tag{1052}$$

$$= \tfrac{1}{2}\left(\exp\left(-\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2\cos(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(-\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\right. \tag{1053}$$

$$\frac{B^2}{}\left(\quad \prime(\ )\quad \prime(\ )\quad \right)$$

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{1095}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{1096}$$

$$= \frac{\Omega_r^2}{4}\left(\cosh\left(\phi\left(\tau\right)\right) - 1\right) \tag{1097}$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{1098}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} - e^{-\phi(\tau)}\right), \tag{1099}$$

$$\Lambda'_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)\left(1 - F(\omega)\right)^2 G_+(\tau), \tag{1100}$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty \mathrm{d}\omega\frac{J(\omega)}{\omega}F(\omega)\left(1 - F(\omega)\right)\mathrm{i}G_-(\tau), \tag{1101}$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \tag{1102}$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \tag{1103}$$

$$= \Lambda'_{13}(\tau) \tag{1104}$$

$$= \Lambda'_{31}(\tau) \tag{1105}$$

$$= 0. \tag{1106}$$

Finally taking the Hamiltonian (954) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (335) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{1107}$$

$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{1108}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (449) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (937) and using the fact that

$$\widetilde{A_j}(t-\tau,t) = U(\tau)A_jU(-\tau) \tag{1109}$$

$$= \sum_w e^{\mathrm{i}w\tau}\mathcal{A}_j(-w) \tag{1110}$$

$$= \sum_w e^{-\mathrm{i}w\tau}\mathcal{A}_j(w). \tag{1111}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_{w}\gamma_{ij}(w,t)\left[A_i, \mathcal{A}_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)\mathcal{A}_j^\dagger(w)\right] \tag{1112}$$

$$- \sum_{ij}\sum_{w}S_{ij}(w,t)\left[A_i, \mathcal{A}_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)\mathcal{A}_j^\dagger(w)\right]. \tag{1113}$$

where $\mathcal{A}_j^\dagger(w) = \mathcal{A}_j(-w)$, as we can see the equation (1113) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2K_{ij}^{\Re}(w,t)$ and $S_{ij}(w,t) = K_{ij}^{\Im}(w,t)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}(w,t) = \int_0^t \Lambda'_{ij}(\tau)e^{\mathrm{i}w\tau}\mathrm{d}\tau.$$

The fact $\mathcal{A}_j^\dagger(w) = \mathcal{A}_j(-w)$ can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{\mathrm{i}\overline{H_S}(t-\tau)} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)(t-\tau)} \tag{1114}$$

$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|(t-\tau)}e^{\mathrm{i}\lambda_-|-\rangle\langle-|(t-\tau)} \tag{1115}$$

$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1116}$$

$$= e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|. \tag{1117}$$

Where λ_+, λ_- are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (878) of (902)-(904) directly using the previous relationship we find that:

$$\widetilde{A_i(0)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1118}$$

$$= \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|, \tag{1119}$$

$$\widetilde{A_i(w)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|-\rangle|+\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1120}$$

$$= \langle+|A_i|-\rangle|+\rangle\langle-|e^{\mathrm{i}w(t-\tau)}, \tag{1121}$$

$$\widetilde{A_i(-w)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle-|A_i|+\rangle|-\rangle\langle+|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1122}$$

$$= \langle-|A_i|+\rangle|-\rangle\langle+|e^{-\mathrm{i}w(t-\tau)}. \tag{1123}$$

Here $w = \lambda_+ - \lambda_-$. So we can see that for the equation (888) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w \neq w'$ then $A'_j(w, w') = 0$ so:

$$\widetilde{A_j}(t-\tau,t) = U(t)U^\dagger(t-\tau)A_j(t)U(t-\tau)U^\dagger(t) \tag{1124}$$

$$= U(t)\left(\sum_{w(t-\tau)}e^{-i(t-\tau)w(t-\tau)}A_j(w(t-\tau))\right)U^\dagger(t) \tag{1125}$$

$$= \sum_{w(t-\tau)}e^{-i(t-\tau)w(t-\tau)}U(t)A_j(w(t-\tau))U^\dagger(t) \tag{1126}$$

$$= \sum_{w'(t),w(t-\tau)}e^{-i(t-\tau)w(t-\tau)}e^{itw'(t)}A_j(w(t-\tau),w'(t)) \tag{1127}$$

$$= \sum_{w'(t),w(t-\tau)}e^{-i(t-\tau)w(t-\tau)}e^{itw'(t)}A_{jww'} \tag{1128}$$

$$== \sum_{w'(t),w(t-\tau)}e^{-i(t-\tau)w(t-\tau)}e^{itw'(t)}A_j(w)\delta_{ww'} \tag{1129}$$

$$= \sum_w e^{-i(t-\tau)w}e^{itw}A_j(w) \tag{1130}$$

$$= \sum_w e^{i\tau w}A_j(w) \tag{1131}$$

$$= U^\dagger(-\tau)A_jU(-\tau) \tag{1132}$$

So using now as reference the equation (1027) and $A'_j(w,w') = 0$ we can deduce that:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i\left[\overline{H_S}(t),\overline{\rho_S}(t)\right] - \sum_{ijw}K_{ij}^\Re(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t) - \overline{\rho_S}(t)A_j^\dagger(w)\right] - i\sum_{ijw}K_{ij}^\Im(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t) + \overline{\rho_S}(t)A_j^\dagger(w)\right] \tag{1133}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (106) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_k\left(g_k b_k^\dagger + g_k^* b_k\right)\right)|1\rangle\langle1| + \frac{\Omega(t)}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k. \tag{1134}$$

If $v_k \approx g_k$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (1134) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_k \approx g_k$ then, in this case and using the equation (??) and (??) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)|1\rangle\langle1| + \frac{B\sigma_x}{2}\Omega(t), \tag{1135}$$

$$\overline{H_I} = \frac{\Omega(t)}{2}(B_x\sigma_x + B_y\sigma_y). \tag{1136}$$

In this case $R_1 = \sum_k\left(\omega_k\frac{v_k}{\omega_k} - 2\frac{v_k}{\omega_k}g_k\right)$ from (27) and given that $v_k \approx g_k$ and $\frac{v_k}{\omega_k} = g_k/\omega_k$ then $R_1 = \sum_k\left(-\omega_k^{-1}|g_k|^2\right) = \sum_k\left(-\omega_k|\frac{v_k}{\omega_k}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_k) = 1$ and using the equations (1095)-(1103) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}G_+(\tau)\,d\omega. \tag{1137}$$

Writing $G_+ (\tau) = \coth \left(\frac{\beta\omega}{2} \right) \cos (\omega\tau) - i \sin (\omega\tau)$ so (1137) can be written as:

$$\phi (\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left(\coth \left(\frac{\beta\omega}{2} \right) \cos (\omega\tau) - i \sin (\omega\tau) \right) d\omega. \tag{1138}$$

Writing the interaction Hamiltonian (1136) in the similar way to the equation (**??**) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1 (t) = B_x$, $B_2 (t) = B_y$ and $C_1 (t) = \frac{\Omega(t)}{2} = C_2 (t)$. Now taking the equation (**??**) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$ is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega (t). \tag{1139}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11} (\tau) = \mathrm{Tr}_B \left(\widetilde{B_1} (\tau) \widetilde{B_1} (0) \rho_B \right) \tag{1140}$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \tag{1141}$$

$$\Lambda_{22} (\tau) = \mathrm{Tr}_B \left(\widetilde{B_2} (\tau) \widetilde{B_2} (0) \rho_B \right) \tag{1142}$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right). \tag{1143}$$

These functions match with the equations $\Lambda_x (\tau)$ and $\Lambda_y (\tau)$ of the reference [2] and $\Lambda_i (\tau) = \Lambda_i (-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(449) is:

$$\frac{d\rho_S (t)}{dt} = -i \left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r (t) \sigma_x}{2}, \rho_S (t) \right] - \sum_{i=1}^2 \int_0^t d\tau \left(C_i (t) C_i (t - \tau) \Lambda_{ii} (\tau) \left[A_i, \widetilde{A_i} (t - \tau, t) \rho_S (t) \right] \right) \tag{1144}$$

$$+ C_i (t) C_i (t - \tau) \Lambda_{ii} (-\tau) \left[\rho_S (t) \widetilde{A_i} (t - \tau, t), A_i \right] \bigg). \tag{1145}$$

Replacing $C_i (t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i} (t - \tau, t) = \widetilde{\sigma_i} (t - \tau, t)$, also using the equations (1140) and (1143) on the equation (1145) we obtain that:

$$\frac{d\rho_S (t)}{dt} = -\frac{i}{2} [\delta'\sigma_z + \Omega_r (t) \sigma_x, \rho_S (t)] - \frac{\Omega (t)}{4} \int_0^t d\tau \Omega (t - \tau) ([\sigma_x, \widetilde{\sigma_x} (t - \tau, t) \rho_S (t)] \Lambda_x (\tau) \tag{1146}$$

$$+ [\sigma_y, \widetilde{\sigma_y} (t - \tau, t) \rho_S (t)] \Lambda_y (\tau) + [\rho_S (t) \widetilde{\sigma_x} (t - \tau, t), \sigma_x] \Lambda_x (\tau) + [\rho_S (t) \widetilde{\sigma_y} (t - \tau, t), \sigma_y] \Lambda_y (\tau)). \tag{1147}$$

As we can see $\left[A_j, \widetilde{A_i} (t - \tau, t) \rho_S (t) \right]^\dagger = \left[\rho_S (t) \widetilde{A_i} (t - \tau, t), A_j \right]$, $\Lambda_x (\tau) = \Lambda_x (-\tau)$ and $\Lambda_y (\tau) = \Lambda_y (-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F (\omega) = 0$, so there is no transformation in this case. As we can see from the definition (413) the only term that survives is $\Lambda_{33} (\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega (t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right). \tag{1148}$$

Using the equation (937), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{d\rho_S}{dt} = -i\left[H_S\left(t\right), \rho_S\left(t\right)\right), \rho_S\left(t\right)\right] - \frac{1}{2}\sum_w \gamma_{33}\left(w, t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) - \rho_S\left(t\right)A_3^\dagger\left(w\right)\right] \tag{1149}$$

$$- \sum_w S_{33}\left(w, t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) + \rho_S\left(t\right)A_3^\dagger\left(w\right)\right]\right). \tag{1150}$$

The correlation functions are relevant if $F\left(\omega\right) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}\left(\tau\right) = \int_0^\infty d\omega\, J\left(\omega\right) G_+\left(\tau\right), \tag{1151}$$

$$\Lambda_{33}\left(-\tau\right) = \int_0^\infty d\omega\, J\left(\omega\right) G_+\left(-\tau\right). \tag{1152}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (1150) can be transformed in

$$\frac{d\rho_S}{dt} = -i\left[H_S\left(t\right), \rho_S\left(t\right)\right] - \sum_w \left(K_{33}\left(w, t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right)\right] + K_{33}^*\left(w, t\right)\left[\rho_S\left(t\right)A_3\left(w\right), A_3\right]\right). \tag{1153}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3\left(w\right)$, we have to remember that $H_S = \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{1154}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right), \tag{1155}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{1156}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{1157}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1158}$$

$$= \frac{1}{2}, \tag{1159}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1160}$$

$$= \frac{1}{2}, \tag{1161}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1162}$$

$$= -\frac{1}{2}. \tag{1163}$$

The decomposition matrices are

$$A_3\left(0\right) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{1164}$$

$$= \frac{\mathbb{I}}{2}, \tag{1165}$$

$$A_3\left(\eta\right) = -\frac{1}{2}|-\rangle\langle+| \tag{1166}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right), \tag{1167}$$

$$A_3\left(-\eta\right) = -\frac{1}{2}|+\rangle\langle-| \tag{1168}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} - \mathrm{i}\sigma_{\mathrm{y}}\right). \tag{1169}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{1170}$$

$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right]. \tag{1171}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right) G_+\left(\tau\right) e^{\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{1172}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right) + 1\right)e^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau \mathrm{d}\omega \tag{1173}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right) + 1\right)e^{-\mathrm{i}\tau\omega}\mathrm{d}\tau \mathrm{d}\omega \tag{1174}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right) + 1\right)e^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau \mathrm{d}\omega \tag{1175}$$

$$= \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right) + 1\right)\pi\delta\left(\widetilde{\Omega} - \omega\right)\mathrm{d}\omega \tag{1176}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right) + 1\right), \tag{1177}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right) G_+\left(\tau\right) e^{-\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{1178}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right) + 1\right)e^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau \mathrm{d}\omega \tag{1179}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-\mathrm{i}\widetilde{\Omega}\tau} n\left(\omega\right)e^{\mathrm{i}\tau\omega}\mathrm{d}\tau \mathrm{d}\omega \tag{1180}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) n\left(\omega\right) e^{-\mathrm{i}\widetilde{\Omega}\tau + \mathrm{i}\tau\omega}\mathrm{d}\tau \mathrm{d}\omega \tag{1181}$$

$$= \int_0^\infty J\left(\omega\right) n\left(\omega\right) \pi\delta\left(-\widetilde{\Omega} + \omega\right)\mathrm{d}\omega \tag{1182}$$

$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right). \tag{1183}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm \mathrm{i}\varepsilon s} = \pi\delta\left(\varepsilon\right) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take

account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1170) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S(t)\right] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right) + 1\right)\left[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, (\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)\right]\right) \tag{1184}$$

$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right) + 1\right)\left[\rho_S(t)(\sigma_z + \mathrm{i}\sigma_y), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S(t)(\sigma_z - \mathrm{i}\sigma_y), \sigma_z\right]\right). \tag{1185}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x, \rho_S(t)\right] - \frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right) + 1\right)\left[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)\right] + n\left(\Omega(t)\right)\left[\sigma_z, (\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)\right]\right) \tag{1186}$$

$$- \frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right) + 1\right)\left[\rho_S(t)(\sigma_z + \mathrm{i}\sigma_y), \sigma_z\right] + n\left(\Omega(t)\right)\left[\rho_S(t)(\sigma_z - \mathrm{i}\sigma_y), \sigma_z\right]\right). \tag{1187}$$

V. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1188}$$

$$H_S(t) = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{1189}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right), \tag{1190}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \tag{1191}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right) \tag{1192}$$

At first let's obtain $e^{\pm V}$ under the transformation (1192), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)$, so the equation (1192) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n|\hat{\varphi}_n} \tag{1193}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!} + \dots \tag{1194}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + \dots \tag{1195}$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + \dots \tag{1196}$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots\right) \tag{1197}$$

$$= \sum_n |n\rangle\langle n|e^{\pm\hat{\varphi}_n} \tag{1198}$$

Given that $\left[f_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}, f_{nu'\mathbf{k}'}b_{u'\mathbf{k}'}^\dagger - f_{nu'\mathbf{k}'}^* b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right) = e^{\pm\left(\alpha_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger - \alpha_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)} \tag{1199}$$

$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)}\right) \tag{1200}$$

$$= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) \tag{1201}$$

$$= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1202}$$

$$= \prod_u B_{nu\pm} \tag{1203}$$

$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1204}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1205}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1188):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1206}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1207}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1208}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1209}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1210}$$

$$= |0\rangle\langle 0|. \tag{1211}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1212}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1213}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1214}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right), \, m \neq n, \tag{1215}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1216}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1217}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1218}$$

$$\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1219}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1220}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \cdots\right)\left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \cdots\right), \tag{1221}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + \cdots, \tag{1222}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \cdots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \cdots\right) \prod_u B_{1u-} + \cdots \tag{1223}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \cdots\right) \tag{1224}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \cdots\right) + \cdots \tag{1225}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}^*}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}^*}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \cdots\right) \tag{1226}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}^*}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}^*}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \cdots\right) + \cdots \tag{1227}$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \cdots \tag{1228}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1229}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{1230}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right) \tag{1231}$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{1232}$$

The transformed Hamiltonians of the equations (1189) to (1191) written in terms of (1206) to (1230) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1233}$$

$$= \overline{\sum_n \varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1234}$$

$$= \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) \tag{1235}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1236}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}\right) + ...\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1237}$$

$$= \Pi_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}\right) \Pi_u B_{0u-} + \Pi_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{1u\mathbf{k}}^* b_{u\mathbf{k}}\right) \Pi_u B_{1u-} + ... \tag{1238}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \Pi_u B_{0u+} b_{u\mathbf{k}}^\dagger \Pi_u B_{0u-} + g_{0u\mathbf{k}}^* \Pi_u B_{0u+} b_{u\mathbf{k}} \Pi_u B_{0u-}\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \Pi_u B_{1u+} b_{u\mathbf{k}}^\dagger \Pi_u B_{1u-} + g_{1u\mathbf{k}}^* \Pi_u B_{1u+} b_{u\mathbf{k}} \Pi_u B_{1u-}\right) + ... \tag{1239}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \tfrac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{0u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \tfrac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \tfrac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{1u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \tfrac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + ... \tag{1240}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{nu\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1241}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1242}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{1243}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{1244}$$

$$+ \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1245}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1246}$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}}) b_{u\mathbf{k}}^\dagger + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^* b_{u\mathbf{k}}\right) \tag{1247}$$

Using the previous functions we have that (1244) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n R_n(t)|n\rangle\langle n| + \sum_n B_{z,n}(t)|n\rangle\langle n| \tag{1248}$$

$$\tag{1249}$$

Now in order to separate the elements of the hamiltonian (1249) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \textstyle\prod_u (B_{mu+}B_{nu-})\right\rangle_{\overline{H_0}} = \left\langle \textstyle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1250}$$

$$= \left(\textstyle\prod_{u\mathbf{k}}\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\left\langle \textstyle\prod_{u\mathbf{k}}D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1251}$$

$$= \left(\textstyle\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\textstyle\prod_u\exp\left(-\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1252}$$

$$\equiv B_{nm} \tag{1253}$$

$$\left\langle \textstyle\prod_u (B_{nu+}B_{mu-})\right\rangle_{\overline{H_0}} = \left(\textstyle\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\textstyle\prod_u\exp\left(-\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1254}$$

$$= B^*_{nm} \tag{1255}$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+}B_{nu-}) - B_{nm} \tag{1256}$$

As we can see:

$$J^\dagger_{nm} = \left(\prod_u (B_{mu+}B_{nu-}) - B_{nm}\right)^\dagger \tag{1257}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B^*_{nm} \tag{1258}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{mn} \tag{1259}$$

$$= J_{mn} \tag{1260}$$

We can separate the Hamiltonian (1249) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)} = \sum_n (\varepsilon_n(t) + R_n)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|B_{nm} \tag{1261}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|J_{nm} + \sum_n B_{z,n}(t)\,|n\rangle\langle n|, \tag{1262}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \tag{1263}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \le A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S(t)}+H_{\bar{B}})}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_S(t)}+H_{\bar{B}}} + O\left(\left\langle\overline{H^2_{\bar{I}}}\right\rangle_{\overline{H_S(t)}+H_B}\right). \tag{1264}$$

Taking the equations (244)-(252) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right) = C\left(R_0, R_1, ..., R_{d-1}, B_{01}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v^{\Re}_{nu\mathbf{k}} + iv^{\Im}_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^{\Re}_{nu\mathbf{k}}$ and $v^{\Im}_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \right) |n\rangle\langle n| \tag{1265}$$

$$+ \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1266}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}} v_{nu\mathbf{k}}^* + g_{nu\mathbf{k}}^* v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1267}$$

$$+ \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1268}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} + iv_{nu\mathbf{k}}^{\Im}(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1269}$$

$$+ \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1270}$$

$$v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^* = \left(v_{mu\mathbf{k}}^{\Re} - iv_{mu\mathbf{k}}^{\Im} \right) \left(v_{nu\mathbf{k}}^{\Re} + iv_{nu\mathbf{k}}^{\Im} \right) - \left(v_{mu\mathbf{k}}^{\Re} + iv_{mu\mathbf{k}}^{\Im} \right) \left(v_{nu\mathbf{k}}^{\Re} - iv_{nu\mathbf{k}}^{\Im} \right) \tag{1271}$$

$$= \left(v_{mu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} + iv_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - iv_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} + v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im} \right) \tag{1272}$$

$$- \left(v_{mu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - iv_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} + iv_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} + v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im} \right) \tag{1273}$$

$$= 2i \left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} \right) \tag{1274}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} + iv_{nu\mathbf{k}}^{\Im}(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1275}$$

$$+ \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{i\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} \right)}{\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1276}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1277}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}} v_{mu\mathbf{k}}^* + v_{nu\mathbf{k}}^* v_{mu\mathbf{k}}) \tag{1278}$$

$$= \left(v_{mu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im}\right)^2 + \left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - \left(v_{nu\mathbf{k}}^{\Re} + iv_{nu\mathbf{k}}^{\Im} \right) \left(v_{mu\mathbf{k}}^{\Re} - iv_{mu\mathbf{k}}^{\Im} \right) \tag{1279}$$

$$- \left(v_{nu\mathbf{k}}^{\Re} - iv_{nu\mathbf{k}}^{\Im} \right) \left(v_{mu\mathbf{k}}^{\Re} + iv_{mu\mathbf{k}}^{\Im} \right) \tag{1280}$$

$$= \left(v_{mu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im}\right)^2 + \left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2 \left(v_{nu\mathbf{k}}^{\Re} v_{mu\mathbf{k}}^{\Re} + v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Im} \right) \tag{1281}$$

$$= \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re} \right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im} \right)^2 \tag{1282}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{1283}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - (g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} - iv_{nu\mathbf{k}}^{\Im} (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \tag{1284}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1285}$$

$$B_{mn} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1286}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{i\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} \right)}{\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re} \right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im} \right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1287}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Re}} \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1288}$$

$$= \frac{2v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1289}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Re} - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1290}$$

$$\frac{\partial R_{n'}}{\partial v_{nu\mathbf{k}}^{\Im}} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Im}} \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1291}$$

$$= \frac{2v_{nu\mathbf{k}}^{\Im} - 2g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1292}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Im} - g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1293}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\Pi_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \Pi_u \exp\left(-\frac{1}{2} \Sigma_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \right) \tag{1294}$$

$$= \Sigma_{u\mathbf{k}} \ln \exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) + \Sigma_u \ln \exp\left(-\frac{1}{2} \Sigma_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1295}$$

$$= \Sigma_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) + \Sigma_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1296}$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} - \left(v_{nu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right)}{\omega_{u\mathbf{k}}^2} \tag{1297}$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} = \frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} - \left(v_{nu\mathbf{k}}^{\Im} - v_{mu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right)}{\omega_{u\mathbf{k}}^2} \tag{1298}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \tag{1299}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \tag{1300}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^\dagger}{\partial a} \tag{1301}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} \tag{1302}$$

$$= B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} - \left(v_{nu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1303}$$

$$= B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1304}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Re}} = \left(\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} \right)^{\dagger} \tag{1305}$$

$$= \left(B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \tag{1306}$$

$$= B_{nm} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1307}$$

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} \tag{1308}$$

$$= B_{mn} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} - \left(v_{nu\mathbf{k}}^{\Im} - v_{mu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1309}$$

$$= B_{mn} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1310}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Im}} = \left(\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} \right)^{\dagger} \tag{1311}$$

$$= \left(B_{mn} \right)^{\dagger} \tag{1312}$$

$$= B_{nm} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1313}$$

Introducing this derivates in the equation (1288) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n} \left(2 \frac{v_{nu\mathbf{k}}^{\Re} - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n<m} \left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}} B_{nm} \left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right. \tag{1314}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}} B_{mn} \left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \tag{1315}$$

$$= 0 \tag{1316}$$

We can obtain the variational parameters:

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1317}$$

$$= -\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)) \tag{1318}$$

$$v^{\Re}_{nu\mathbf{k}} = \frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1319}$$

$$= \frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1320}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}}-g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1321}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)) \tag{1322}$$

$$= 0 \tag{1323}$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1324}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1325}$$

$$v_{nu\mathbf{k}}^{\Im}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1326}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1327}$$

$$v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+iv_{nu\mathbf{k}}^{\Im} \tag{1328}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1329}$$

$$i\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1330}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2ig_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1331}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1332}$$

$$-i\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1333}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1334}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1335}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1336}$$

C.   Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^V\rho(0)e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)(|0\rangle\langle0|\otimes\rho_B)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1337}$$

$$0=\left(B_0^+|0\rangle\langle0|B_0^-\right)\otimes\rho_B \tag{1338}$$

$$0=\rho(0) \tag{1339}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t) O U(t) \tag{1340}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1341}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\,\overline{\rho_S}(t)\,U(t)\,, \text{ where} \tag{1342}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1343}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1344}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1345}$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1346}$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1347}$$

We can proof that $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+}B_{n-} - B_m B_n)^\dagger \tag{1348}$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \tag{1349}$$

$$= B_{n+}B_{m-} - B_n B_m \tag{1350}$$

$$= B_{nm} \tag{1351}$$

So we can say that the set of matrices (1344) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1344) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t)\,|n\rangle\langle m|B_{nm} + \sum_n B_{z,n}(t)\,|n\rangle\langle n|, \tag{1352}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m} \left(V_{nm}(t)\,|n\rangle\langle m|B_{nm} + V_{mn}(t)\,|m\rangle\langle n|B_{mn}\right) \tag{1353}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m} \left(\Re(V_{nm}(t))\,B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^\Im(t)\,B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1354}$$

$$\left. +\Re(V_{nm}(t))\,B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^\Im(t)\,B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1355}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re(V_{nm}(t))\,\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re(V_{nm}(t))\,\sigma_{nm,y}\frac{\mathrm{i}(B_{mn} - B_{nm})}{2}\right. \tag{1356}$$

$$\left. +\mathrm{i}\Im(V_{nm}(t))\,\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im(V_{nm}(t))\,\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1357}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re(V_{nm}(t))\,\sigma_{nm,x}B_{nm,x} - \Im(V_{nm}(t))\,\sigma_{nm,x}B_{nm,y} + \Re(V_{nm}(t))\,\sigma_{nm,y}B_{nm,y}\right. \tag{1358}$$

$$\left. +\Im(V_{nm}(t))\,\sigma_{nm,y}B_{nm,x}\right) \tag{1359}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 \,\middle|\, 0 \leq n,m \leq d-1 \wedge (n = m \vee n < m)\right\} \tag{1360}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \tag{1361}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \tag{1362}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1363}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1364}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1365}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1366}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1367}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1368}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1369}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1370}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1371}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1372}$$
$$C_{3,nm}(t) = 1 \tag{1373}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1374}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1375}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t)(A_{jp} \otimes B_{jp}(t)) \tag{1376}$$

Here $J = \{1, 2, 3, 4, 5\}$ and P the set defined in (1360).

We write the interaction Hamiltonian transformed under (1340) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)\right) \tag{1377}$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \tag{1378}$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t)(t) e^{-iH_B t} \tag{1379}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \text{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] ds \tag{1380}$$

Replacing the equation (1377) in (1380) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1381}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P}C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\left[\sum_{j'\in J,p'\in P}C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1382}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P}C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\sum_{j'\in J,p'\in P}C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \right.\tag{1383}$$

$$\left.-\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J,p'\in P}C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1384}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P}C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P}C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1385}$$

$$-\sum_{j\in J,p\in P}C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J,p'\in P}C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1386}$$

$$-\sum_{j'\in J,p'\in P}C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j\in J,p\in P}C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1387}$$

$$\left.+\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J,p'\in P}C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P}C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1388}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1389}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1390}$$

Here $s\to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1388) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1391}$$

$$= \left\langle \widetilde{B_{jp}}(0)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1392}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1393}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\rho_B\,\widetilde{B_{j'p'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1394}$$

$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1395}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1396}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1397}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\rho_B\,\widetilde{B_{jp}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) \tag{1398}$$

$$= \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1399}$$

$$= \left\langle \widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1400}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1401}$$

$$\mathrm{Tr}_B\left(\rho_B\,\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1402}$$

$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1403}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1404}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1405}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1381)and (1388) and using the equations (1391)-(1405) we can re-write:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{jpj'p'}(\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{\rho_S}(t)\,\widetilde{A_{j'p'}}(s)\right)\right. \tag{1406}$$

$$\left. + C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{j'p'jp}(-\tau)\,\widetilde{\rho_S}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\rho_S}(t)\,\widetilde{A_{jp}}(t)\right)\right)\,\mathrm{d}s \tag{1407}$$

$$= -\int_0^t \sum_{jj'pp'} \left(C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{jpj'p'}(\tau)\left[\widetilde{A_{jp}}(t), \widetilde{A_{j'p'}}(s)\,\widetilde{\rho_S}(t)\right] + \Lambda_{j'p'jp}(-\tau)\left[\widetilde{\rho_S}(t)\,\widetilde{A_{j'p'}}(s), \widetilde{A_{jp}}(t)\right]\right)\right) \tag{1408}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'} \left(C_{jp}(t)\,C_{j'p'}(t-\tau)\left(\Lambda_{jpj'p'}(\tau)\left[A_{jp}(t), A_{j'p'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'p'jp}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'p'}(t-\tau,t), A_{jp}(t)\right]\right)\right)\,\mathrm{d}\tau - \mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] \tag{1409}$$

For this case we used that $A_{jp}(t-\tau,t) = U(t)\,U^\dagger(t-\tau)\,A_{jp}(t)\,U(t-\tau)\,U^\dagger(t)$. This is a non-Markovian equation.

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1410}$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1411}$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right), \tag{1412}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{1413}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right) \right) \tag{1414}$$

At first let's obtain e^V under the transformation (1414), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{1415}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi} \right)^2}{2!} + \dots \tag{1416}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + \dots \tag{1417}$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \tag{1418}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|e^{\hat{\varphi}} \tag{1419}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^+ \tag{1420}$$

Given that $\left[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} \tag{1421}$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{1422}$$

$$= B_\pm \tag{1423}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-$ because this form imposes that $e^{-V}e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^- \right) \tag{1424}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1410):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \tag{1425}$$

$$= |0\rangle\langle 0|, \tag{1426}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \tag{1427}$$

$$= |m\rangle\langle m| B^+ |m\rangle\langle n| n\rangle\langle n| B^-, \tag{1428}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1429}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \tag{1430}$$

$$= |0\rangle\langle m| B^-\ m \neq 0, \tag{1431}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \tag{1432}$$

$$= |0\rangle\langle m| B^+\ m \neq 0, \tag{1433}$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \tag{1434}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B^- \tag{1435}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B^+ b_{\mathbf{k}}^\dagger B^- \right) \left(B^+ b_{\mathbf{k}} B^- \right) \tag{1436}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{1437}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{1438}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) \tag{1439}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1440}$$

The transformed Hamiltonians of the equations (1411) to (1413) written in terms of (1425) to (1440) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1441}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1442}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \overline{\sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|)} + \overline{\sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n|} \tag{1443}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \tag{1444}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) B^- |0\rangle\langle n| + V_{n0}(t) B^+ |n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1445}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1446}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+\right) \left(\left(\sum_{n=0} \mu_n(t) |n\rangle\langle n|\right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^-\right) \tag{1447}$$

$$= \left(\mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B^+\right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^-\right) \tag{1448}$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B^+ \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) B^- \tag{1449}$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1450}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1451}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1452}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1453}$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2 g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1454}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1455}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1456}$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1457}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1458}$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}}\right) \tag{1459}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1460}$$

Using the previous functions we have that (1457) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1461}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1462}$$

Now in order to separate the elements of the hamiltonian (1462) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \tag{1463}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| \left(B^- - B\right) + V_{n0}(t) |n\rangle\langle 0| \left(B^+ - B\right)\right), \tag{1464}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{1465}$$

Here B is given by:

$$B = \langle B^+ \rangle$$
$$= \langle B^- \rangle$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1466}$$

$$\sigma_{nm,y} = i\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1467}$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1468}$$

$$B_y = \frac{B^- - B^+}{2i} \tag{1469}$$

Using this set of hermitian operators to write the Hamiltonians (1411)-(1413)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1470}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \tag{1471}$$

$$+ \sum_{n=1}(\varepsilon_n(t) + R_n)|n\rangle\langle n| \tag{1472}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + i\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - i\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \tag{1473}$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t) + R_n)|n\rangle\langle n| \tag{1474}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + i\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - i\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2}\right) \tag{1475}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)|n\rangle\langle n| \tag{1476}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1477}$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t) + R_n)|n\rangle\langle n| \tag{1478}$$

$$\overline{H_I(t)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(V_{0n}(t)|0\rangle\langle n|(B^- - B) + V_{n0}(t)|n\rangle\langle 0|(B^+ - B)) \tag{1479}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))(B^- - B)\frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))(B^+ - B)\frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2}\right) \tag{1480}$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1481}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left((B^- - B)(\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) + (B^+ - B)(\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))\right)\right) \tag{1482}$$

$$+ \frac{i\sigma_{0n,y}}{2}\left((B^+ - B)(\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) - (B^- - B)(\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))\right)\right) \tag{1483}$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1484}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}(B^+ + B^- - 2B)\Re(V_{0n}(t)) + i(B^- - B - B^+ + B)\Im(V_{0n}(t))\right) \tag{1485}$$

$$+ \frac{i\sigma_{0n,y}}{2}\left((B^+ - B - B^- + B)\Re(V_{0n}(t)) + i(B - B^- + B - B^+)\Im(V_{0n}(t))\right)\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1486}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t)))) \tag{1487}$$

$$+ \sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1488}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right) + \langle\overline{H_I}\rangle_{H_S+H_B} + O\left(\langle\overline{H_I^2}\rangle_{H_S+H_B}\right). \tag{1489}$$

Taking the equations (244)-(252) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (252) and using the chain rule we obtain that:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1490}$$

$$= 0 \tag{1491}$$

Let's recall the equations (1458) and (1460), we can write them in terms of the variational parameters

$$B = \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1492}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1493}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1494}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1495}$$

Introducing this derivates in the equation (1490) give us:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1496}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1497}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}} \tag{1498}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}} \tag{1499}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}}. \tag{1500}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) (|0\rangle\langle 0| \otimes \rho_B) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \tag{1501}$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \tag{1502}$$

$$0 = \rho(0) \tag{1503}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t) O U(t) \tag{1504}$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \tag{1505}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \tag{1506}$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\bar{\rho}(t)) \tag{1507}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle 0| + \sum_{n=1} \left(\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n}|n\rangle\langle n| \right) \tag{1508}$$

$$+ \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x}) \tag{1509}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1510}$$

$$B_{z,n} = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \text{ if } n \neq 0 \tag{1511}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1512}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1513}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1514}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1515}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1516}$$
$$B_{1n}(t) = B_x \tag{1517}$$
$$B_{2n}(t) = B_y \tag{1518}$$
$$B_{3n}(t) = B_{z,n} \tag{1519}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1520}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1521}$$
$$C_{10}(t) = 0 \tag{1522}$$
$$C_{20}(t) = 0 \tag{1523}$$
$$C_{40}(t) = 0 \tag{1524}$$
$$C_{50}(t) = 0 \tag{1525}$$
$$C_{30}(t) = 1 \tag{1526}$$
$$C_{1n}(t) = \Re(V_{0n}(t)) \tag{1527}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1528}$$
$$C_{3n}(t) = 1 \tag{1529}$$
$$C_{4n}(t) = \Im(V_{0n}(t)) \tag{1530}$$
$$C_{5n}(t) = -\Im(V_{0n}(t)) \tag{1531}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J} \sum_{n=1} C_{jn}(t) (A_{jn}(t) \otimes B_{jn}(t)) \tag{1532}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1504) as:

$$\widetilde{H_I}(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t) \right) \tag{1533}$$

$$\widetilde{A_i}(t) = U^{\dagger}(t) A_i U(t) \tag{1534}$$

$$\widetilde{B_i}(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \tag{1535}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \text{Tr}_B \left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \tag{1536}$$

Replacing the equation (1533) in (1536) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1537}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\left[\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right),\widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1538}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1539}$$

$$\left.-\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1540}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1541}$$

$$-\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1542}$$

$$-\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1543}$$

$$\left.+\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1}C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1544}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle\widetilde{B_{jn}}(t)(t)\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1545}$$

$$= \left\langle\widetilde{B_{jn}}(\tau)\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1546}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1544) are:

$$\text{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1547}$$

$$= \left\langle \widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1548}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1549}$$

$$\text{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\,\widetilde{B_{j'n'}}(s)\right) = \text{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1550}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1551}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1552}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1553}$$

$$\text{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\,\widetilde{B_{jn}}(t)\right) = \text{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1554}$$

$$= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1555}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1556}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1557}$$

$$\text{Tr}_B\left(\rho_B\,\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) = \text{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1558}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1559}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1560}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1561}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1537) and (1544) and using the equations (1547)-(1561) we can re-write:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\rho_S}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\rho_S}(t)\,\widetilde{A_{j'n'}}(s)\right)\right.$$

$$\tag{1562}$$

$$\left. + C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\rho_S}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\rho_S}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1563}$$

$$= -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t),\widetilde{A_{j'n'}}(s)\,\widetilde{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\widetilde{\rho_S}(t)\,\widetilde{A_{j'n'}}(s),\widetilde{A_{jn}}(t)\right]\right)\right)$$

$$\tag{1564}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t),A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t),A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - i\left[H_S(t),\overline{\rho_S}(t)\right]$$

$$\tag{1565}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^\dagger(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^\dagger(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (449) as expected.

VII. BIBLIOGRAPHY

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