A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|, \tag{2}$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j \rangle = \delta_{ij} \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}} \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \tag{8}$$

$$V = |0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6)will give us the following result:

$$e^{\pm V} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1)} \tag{10}$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + \dots \tag{11}$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + \dots \tag{12}$$

$$= |0\rangle\langle 0|\left(1 \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm \dots\right) + |1\rangle\langle 1|\left(1 \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm \dots\right) \tag{13}$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} \tag{14}$$

$$= |0\rangle\langle 0|e^{\pm \sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} - \alpha_{0\mathbf{k}}^* b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm \sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} - \alpha_{1\mathbf{k}}^* b_{\mathbf{k}}\right)} \tag{15}$$

$$= |0\rangle\langle 0|B_{0\pm} + |1\rangle\langle 1|B_{1\pm}, \tag{16}$$

$$B_{i\pm} \equiv e^{\pm \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{17}$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX}\, e^{tY}\, e^{-\frac{t^2}{2}[X,Y]}\, e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \dots \tag{18}$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i, j we can show making $t = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} = e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]} \dots \tag{19}$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}0} \dots \tag{20}$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} \tag{21}$$

By induction of this result we can write expresion of $B_{i\pm}$ as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}})$:

$$B_{i\pm} = \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{22}$$

$$D(\pm v_{i\mathbf{k}}) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{23}$$

this will help us to write operators O in the variational frame :

$$\overline{O} \equiv e^{V}Oe^{-V}. \tag{24}$$

We use the following identities:

$$\overline{|0\rangle\langle0|} = \mathrm{e}^{V}|0\rangle\langle0|\mathrm{e}^{-V} \tag{25}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|0\rangle\langle0|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{26}$$

$$= \left(|0\rangle\langle0|0\rangle\langle0|B_{0+} + |1\rangle\langle1|0\rangle\langle0|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{27}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{28}$$

$$= |0\rangle\langle0|B_{0+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{29}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}B_{0-} + |0\rangle\langle0|1\rangle\langle1|B_{0+}B_{1-} \tag{30}$$

$$= |0\rangle\langle0|, \tag{31}$$

$$\overline{|1\rangle\langle1|} = \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|1\rangle\langle1|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{32}$$

$$= \left(|0\rangle\langle0|1\rangle\langle1|B_{0+} + |1\rangle\langle1|1\rangle\langle1|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{33}$$

$$= |1\rangle\langle1|B_{1+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{34}$$

$$= |1\rangle\langle1|0\rangle\langle0|B_{1+}B_{0-} + B_{1+}|1\rangle\langle1|1\rangle\langle1|B_{1-} \tag{35}$$

$$= B_{1+}|1\rangle\langle1|1\rangle\langle1|B_{1-} \tag{36}$$

$$= |1\rangle\langle1|B_{1+}B_{1-} \tag{37}$$

$$= |1\rangle\langle1|, \tag{38}$$

$$\overline{|0\rangle\langle1|} = \mathrm{e}^{V}|0\rangle\langle1|\mathrm{e}^{-V} \tag{39}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|0\rangle\langle1|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{40}$$

$$= \left(|0\rangle\langle0|0\rangle\langle1|B_{0+} + |1\rangle\langle1|B_{1+}|0\rangle\langle1|\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{41}$$

$$= \left(|0\rangle\langle0|0\rangle\langle1|B_{0+} + |1\rangle\langle1|0\rangle\langle1|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{42}$$

$$= |0\rangle\langle1|B_{0+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{43}$$

$$= |0\rangle\langle1|0\rangle\langle0|B_{0+}B_{0-} + |0\rangle\langle1|1\rangle\langle1|B_{0+}B_{1-} \tag{44}$$

$$= |0\rangle\langle1|B_{0+}B_{1-}, \tag{45}$$

$$\overline{|1\rangle\langle0|} = \mathrm{e}^{V}|1\rangle\langle0|\mathrm{e}^{-V} \tag{46}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)|1\rangle\langle0|\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{47}$$

$$= \left(|0\rangle\langle0|1\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}|1\rangle\langle0|\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{48}$$

$$= \left(|0\rangle\langle0|1\rangle\langle0|B_{0+} + |1\rangle\langle1|1\rangle\langle0|B_{1+}\right)\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{49}$$

$$= |1\rangle\langle0|B_{1+}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{50}$$

$$= |1\rangle\langle0|B_{1+}|0\rangle\langle0|B_{0-} + |1\rangle\langle0|B_{1+}|1\rangle\langle1|B_{1-} \tag{51}$$

$$= |1\rangle\langle0|B_{1+}B_{0-} + |1\rangle\langle0|1\rangle\langle1|B_{1+}B_{1-} \tag{52}$$

$$= |1\rangle\langle0|B_{1+}B_{0-}, \tag{53}$$

$$\overline{b_{\mathbf{k}}} = \mathrm{e}^{V}b_{\mathbf{k}}\mathrm{e}^{-V} \tag{54}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)b_{\mathbf{k}}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{55}$$

$$= |0\rangle\langle0|B_{0+}b_{\mathbf{k}}B_{0-}|0\rangle\langle0| + |0\rangle\langle0|B_{0+}b_{\mathbf{k}}|1\rangle\langle1|B_{1-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}B_{1-}|1\rangle\langle1| \tag{56}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}b_{\mathbf{k}}B_{0-} + |0\rangle\langle0|1\rangle\langle1|B_{0+}b_{\mathbf{k}}B_{1-} + |1\rangle\langle1|0\rangle\langle0|B_{1+}b_{\mathbf{k}}B_{0-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}B_{1-} \tag{57}$$

$$= |0\rangle\langle0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{58}$$

$$= \left(|0\rangle\langle0| + |1\rangle\langle1|\right)b_{\mathbf{k}} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{59}$$

$$= b_{\mathbf{k}} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{60}$$

$$\overline{b_{\mathbf{k}}}^{\dagger} = \mathrm{e}^{V}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V} \tag{61}$$

$$= \left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{62}$$

$$= |0\rangle\langle0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-}|0\rangle\langle0| + |0\rangle\langle0|B_{0+}b_{\mathbf{k}}^{\dagger}|1\rangle\langle1|B_{1-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}^{\dagger}|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-}|1\rangle\langle1| \tag{63}$$

$$= |0\rangle\langle0|0\rangle\langle0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-} + |0\rangle\langle0|1\rangle\langle1|B_{0+}b_{\mathbf{k}}^{\dagger}B_{1-} + |1\rangle\langle1|0\rangle\langle0|B_{1+}b_{\mathbf{k}}^{\dagger}B_{0-} + |1\rangle\langle1|1\rangle\langle1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-} \tag{64}$$

$$= |0\rangle\langle0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) \tag{65}$$

$$= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle1|\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} - |0\rangle\langle0|\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{66}$$

We have used the following:

$$B_{i+}b_{\mathbf{k}}B_{i-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{67}$$

$$B_{i+}b_{\mathbf{k}}^{\dagger}B_{i-} = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{68}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0\left(t\right)|0\rangle\langle0|} = \varepsilon_0\left(t\right)|0\rangle\langle0|, \tag{69}$$

$$\overline{\varepsilon_1\left(t\right)|1\rangle\langle1|} = \varepsilon_1\left(t\right)|1\rangle\langle1|, \tag{70}$$

$$\overline{V_{10}\left(t\right)|1\rangle\langle0|} = V_{10}\left(t\right)|1\rangle\langle0|B_{1+}B_{0-}, \tag{71}$$

$$\overline{V_{01}\left(t\right)|0\rangle\langle1|} = V_{01}\left(t\right)|0\rangle\langle1|B_{0+}B_{1-}, \tag{72}$$

$$\overline{g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}}} = g_{i\mathbf{k}}\left(|0\rangle\langle0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\right) + g_{i\mathbf{k}}^{*}\left(|0\rangle\langle0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{73}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle0|+|1\rangle\langle1|)b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle1| - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle0|+|1\rangle\langle1|)b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle0|\right) \tag{74}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle1| \tag{75}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|0\rangle\langle0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)|1\rangle\langle1|, \tag{76}$$

$$\overline{|0\rangle\langle0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} = (|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+})|0\rangle\langle0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}) \tag{77}$$

$$= |0\rangle\langle0|B_{0+}|0\rangle\langle0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)|0\rangle\langle0|B_{0-} \tag{78}$$

$$= |0\rangle\langle0|B_{0+}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_{0-} \tag{79}$$

$$= |0\rangle\langle0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{80}$$

$$\overline{|1\rangle\langle1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1k}^{*}b_{\mathbf{k}}\right)} = (|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+})|1\rangle\langle1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1k}^{*}b_{\mathbf{k}}\right)(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}) \tag{81}$$

$$= |1\rangle\langle1|B_{1+}|1\rangle\langle1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1k}^{*}b_{\mathbf{k}}\right)|1\rangle\langle1|B_{1-} \tag{82}$$

$$= |1\rangle\langle1|B_{1+}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_{1-} \tag{83}$$

$$= |1\rangle\langle1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{84}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \omega_{\mathbf{k}}\left(|0\rangle\langle0|B_{0+} + |1\rangle\langle1|B_{1+}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle0|B_{0-} + |1\rangle\langle1|B_{1-}\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle0|B_{0+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{0-} + |1\rangle\langle1|B_{1+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{1-}\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle0|\prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle1|\prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle0|\prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right. \tag{87}$$

$$\left. + |1\rangle\langle1|\prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\Pi_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\Pi_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{90}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{91}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle0|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\right) + |1\rangle\langle1|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{92}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{93}$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{94}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{95}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \tag{96}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right). \tag{97}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)} |0\rangle\langle 0| + \overline{\varepsilon_1(t)} |1\rangle\langle 1| + \overline{V_{10}(t)} |1\rangle\langle 0| + \overline{V_{01}(t)} |0\rangle\langle 1| \tag{98}$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_{1+} B_{0-} + V_{01}(t) |0\rangle\langle 1| B_{0+} B_{1-}, \tag{99}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{100}$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{101}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{102}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) + \sum_{\mathbf{k}} |1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{103}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} \tag{104}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right) \tag{105}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right). \tag{106}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{j+} B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{107}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}} \tag{108}$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{109}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \tag{110}$$

With the following definitions:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_i \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{-(1/2) \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^2 \coth(\beta \omega_{\mathbf{k}}/2)} \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2i} & \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix} \tag{111}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot) \tag{112}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot) \tag{113}$$

We assume that the bath is at equilibrium with inverse temperature $\beta = 1/k_{\mathrm{B}}T$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B} \right)} \tag{114}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{115}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{116}$$

So the product of the matrix representation of $b^\dagger$ and b is:

$$-\beta \omega b^\dagger b = -\beta \omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{117}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \, |j\rangle\langle j|, \tag{118}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \tag{119}$$

$$\exp\left(-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{120}$$

The value of $\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{121}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{122}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{123}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{124}$$

$$\equiv f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{125}$$

$$\text{Tr}\left(\exp\left(-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{126}$$

$$= \prod_{\mathbf{k}} \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{127}$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{128}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}\left(e^{-\beta H_B}\right)} \tag{129}$$

$$= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{130}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{131}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}} \mid j_{\mathbf{k}} - 1\rangle, \tag{132}$$

$$b_{\mathbf{k}}^\dagger \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} \mid j_{\mathbf{k}} + 1\rangle. \tag{133}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\tilde{B}}}} = 0$ using the following property based on (132)-(133):

$$\langle B_{iz}\rangle_{\overline{H_{\tilde{B}}}} = \text{Tr}\left(\rho_B B_{iz}\right) = \text{Tr}\left(B_{iz}\rho_B\right) \tag{134}$$

$$= \text{Tr}\left(\left(\sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{135}$$

$$= \sum_{\mathbf{k}} \text{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger \rho_B\right) + \sum_{\mathbf{k}} \text{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}}\rho_B\right) \tag{136}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}} \right) \operatorname{Tr} \left(b_{\mathbf{k}}^\dagger \rho_B \right) + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}} \right)^* \operatorname{Tr} \left(b_{\mathbf{k}} \rho_B \right) \tag{137}$$

$$= \sum_{\mathbf{k}} \operatorname{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} \operatorname{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) \tag{138}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \operatorname{Tr} \left(b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \operatorname{Tr} \left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right), \tag{139}$$

$$\operatorname{Tr} \left(b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) = \operatorname{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}}^\dagger |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}^\dagger) \tag{140}$$

$$= \operatorname{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) \right) \sqrt{j_{\mathbf{k}} + 1} \, |j_{\mathbf{k}} + 1\rangle\langle j_{\mathbf{k}}| \right) \tag{141}$$

$$= 0, \tag{142}$$

$$\operatorname{Tr} \left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) = \operatorname{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}) \tag{143}$$

$$= \operatorname{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) \right) \sqrt{j_{\mathbf{k}}} \, |j_{\mathbf{k}} - 1\rangle\langle j_{\mathbf{k}}| \right) \tag{144}$$

$$= 0. \tag{145}$$

we therefore find that:

$$\langle B_{iz} \rangle_{\overline{H_{\bar{B}}}} = 0 \tag{146}$$

Another important expected value is $B = \langle B_\pm \rangle_{\overline{H_{\bar{B}}}}$, where $B_\pm = e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}$ is given by:

$$\langle B_\pm \rangle_{H_B} = \operatorname{Tr} \left(\rho_B B_\pm \right) = \operatorname{Tr} \left(B_\pm \rho_B \right) \tag{147}$$

$$= \operatorname{Tr} \left(e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \rho_B \right) \tag{148}$$

$$= \prod_{\mathbf{k}} \operatorname{Tr} \left(D \left(\pm \alpha_{\mathbf{k}} \right) \rho_B \right) \tag{149}$$

$$= \prod_{\mathbf{k}} \langle D \left(\pm \alpha_{\mathbf{k}} \right) \rangle. \tag{150}$$

Given that we can write a density operator as:

$$\rho = \int P \left(\alpha \right) |\alpha\rangle\langle\alpha| \mathrm{d}^2 \alpha \tag{151}$$

where $P \left(\alpha \right)$ satisfies $\int P \left(\alpha \right) \mathrm{d}^2 \alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P \left(\alpha \right)$ is given by:

$$\langle A \rangle = \operatorname{Tr} \left(A \rho \right) \tag{152}$$

$$= \int P \left(\alpha \right) \langle \alpha | A | \alpha \rangle \mathrm{d}^2 \alpha \tag{153}$$

We are typically interested in thermal state density operators, for which it can be shown that $P \left(\alpha \right) = \frac{1}{\pi N} \exp \left(-\frac{|\alpha|^2}{N} \right)$ where $N = \left(e^{\beta \omega} - 1 \right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (153) we could obtain that the expected value for the displacement operator $D \left(h \right)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle\alpha|D\left(h\right)|\alpha\rangle\mathrm{d}^2\alpha \tag{154}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)|0\rangle\mathrm{d}^2\alpha \tag{155}$$

$$D\left(h\right)D\left(\alpha\right) = D\left(h+\alpha\right)e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)} \tag{156}$$

$$D\left(-\alpha\right)\left(D\left(h\right)D\left(\alpha\right)\right) = D\left(-\alpha\right)D\left(h+\alpha\right)e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)} \tag{157}$$

$$= D\left(h\right)e^{\frac{1}{2}\left(-\alpha(h+\alpha)^*+\alpha^*(h+\alpha)\right)}e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)} \tag{158}$$

$$= D\left(\alpha\right)e^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^*h+|\alpha|^2\right)}e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)} \tag{159}$$

$$= D\left(\alpha\right)e^{\left(h\alpha^*-h^*\alpha\right)}, \tag{160}$$

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D\left(h\right)\exp\left(h\alpha^*-h^*\alpha\right)|0\rangle\mathrm{d}^2\alpha \tag{161}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|D\left(h\right)|0\rangle\mathrm{d}^2\alpha \tag{162}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|h\rangle\mathrm{d}^2\alpha \tag{163}$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{164}$$

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle\mathrm{d}^2\alpha \tag{165}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{166}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha\right)\mathrm{d}^2\alpha, \tag{167}$$

$$\alpha = x + \mathrm{i}y, \tag{168}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{N}+h\left(x-\mathrm{i}y\right)-h^*\left(x+\mathrm{i}y\right)\right)\mathrm{d}x\mathrm{d}y \tag{169}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{N}+hx-h^*x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y\right)\mathrm{d}y, \tag{170}$$

$$-\frac{x^2}{N}+hx-h^*x = -\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) \tag{171}$$

$$= -\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2+\frac{N\left(h^*-h\right)^2}{4}, \tag{172}$$

$$-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y = -\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) \tag{173}$$

$$= -\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)-\frac{N\left(h+h^*\right)^2}{4}, \tag{174}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, \tag{175}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x, \tag{176}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}y \tag{177}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{178}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4} \right) \tag{179}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N(h^{*2} - 2hh^* + h^2) - N(h^2 + 2hh^* + h^{*2})}{4} \right) \tag{180}$$

$$= \exp\left(-|h|^2 \left(N + \frac{1}{2} \right) \right) \tag{181}$$

$$= \exp\left(-|h|^2 \left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2} \right) \right) \tag{182}$$

$$= \exp\left(-\frac{|h|^2}{2} \left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} \right) \right) \tag{183}$$

$$= \exp\left(-\frac{|h|^2}{2} \coth\left(\frac{\beta\omega}{2} \right) \right). \tag{184}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2} \right)$. So the value of (149) using (184) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{185}$$

We will now force $\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_+\sigma_+$ and $B_-\sigma_-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{j+}B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{186}$$

$$= \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'} + \sum_j |j\rangle\langle j|B_{jz} + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'| \left(B_{j+}B_{j'-} - B_{jj'} \right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{187}$$

$$\equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{188}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_{1+}B_{0-} \rangle = B_{10} \tag{189}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle \tag{190}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \tag{191}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \tag{192}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \tag{193}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \tag{194}$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)}. \tag{195}$$

From the definition $B_{01} = \langle B_{0+} B_{1-} \rangle$ using the displacement operator we have:

$$\langle B_{0+} B_{1-} \rangle = B_{01} \tag{196}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \tag{197}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \tag{198}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle \tag{199}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{200}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{201}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{202}$$

This can be checked in the following way:

$$\langle B_{0+} B_{1-} \rangle = B_{01} \tag{203}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{204}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)^*} \tag{205}$$

$$= \langle B_{1+} B_{0-} \rangle^* \tag{206}$$

$$= B_{10}^*. \tag{207}$$

The parts of the Hamiltonian splitted are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\sigma_+ + V_{01}(t) B_{01}\sigma_-, \tag{208}$$

$$\overline{H_{\bar{I}}} \equiv V_{10}(t)(B_{1+}B_{0-} - B_{10})\sigma_+ + V_{01}(t)(B_{0+}B_{1-} - B_{01})\sigma_- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, \tag{209}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{210}$$

$$= H_B. \tag{211}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (209) we can verify the condition $\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_{j+} B_{j'-} - B_{jj'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{212}$$

$$= \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_{j+} B_{j'-} - B_{jj'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{213}$$

$$= \sum_{n\mathbf{k}} \left(\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \rangle_{\overline{H_{\bar{B}}}} + \langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})^{*} b_{\mathbf{k}} \rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| \left(\langle V_{jj'}(t) B_{j+} B_{j'-} \rangle_{\overline{H_{\bar{B}}}} - \langle V_{jj'}(t) B_{jj'} \rangle_{\overline{H_{\bar{B}}}} \right) \tag{214}$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) \langle b_{\mathbf{k}}^{\dagger} \rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^{*} \langle b_{\mathbf{k}} \rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(\langle B_{j+} B_{j'-} \rangle_{\overline{H_{\bar{B}}}} - \langle B_{jj'} \rangle_{\overline{H_{\bar{B}}}} \right) \tag{215}$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^{*} \langle b_{\mathbf{k}} \rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(B_{jj'} - B_{jj'} \right) \tag{216}$$

$$= 0. \tag{217}$$

We used (146) and (195) to evaluate the expected values.

Let's consider the following Hermitian combinations:

$$B_x = B_x^{\dagger} \tag{218}$$

$$= \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}, \tag{219}$$

$$B_y = B_y^{\dagger} \tag{220}$$

$$= \frac{B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10} - B_{01}}{2i}, \tag{221}$$

$$B_{iz} = B_{iz}^{\dagger} \tag{222}$$

$$= \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^{*} b_{\mathbf{k}} \right). \tag{223}$$

Writing the equations (208) and (209) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}(t) = (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\sigma_+ + V_{01}(t) B_{01}\sigma_- \tag{224}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01}\frac{\sigma_x - i\sigma_y}{2} \tag{225}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t)\left(B_{10}^{\Re}(t) + iB_{10}^{\Im}(t)\right)\frac{\sigma_x + i\sigma_y}{2} + V_{01}(t)\left(B_{10}^{\Re}(t) - iB_{10}^{\Im}(t)\right)\frac{\sigma_x - i\sigma_y}{2} \tag{226}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + B_{10}^{\Re}(t)\left(V_{10}(t)\frac{\sigma_x + i\sigma_y}{2} + V_{01}(t)\frac{\sigma_x - i\sigma_y}{2}\right) + iB_{10}^{\Im}(t)\left(V_{10}(t)\frac{\sigma_x + i\sigma_y}{2} - V_{01}(t)\frac{\sigma_x - i\sigma_y}{2}\right) \tag{227}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + B_{10}^{\Re}(t)\left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2}\right) + iB_{10}^{\Im}(t)\left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2}\right) \tag{228}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + B_{10}^{\Re}(t)\left(\sigma_x \frac{V_{10}(t) + V_{10}^{*}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{10}^{*}(t)}{2}\right) + iB_{10}^{\Im}(t)\left(\sigma_x \frac{V_{10}(t) - V_{10}^{*}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{10}^{*}(t)}{2}\right) \tag{229}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + B_{10}^{\Re}(t)\left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t)\right) + iB_{10}^{\Im}(t)\left(i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t)\right) \tag{230}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t)\right) - \left(\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \tag{231}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \tag{232}$$

$$= (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right). \tag{233}$$

$$\overline{H_{\bar I}} = V_{10}(t)\left(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}\right) + V_{01}(t)\left(\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}\right) + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} \tag{234}$$

$$= |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} + \left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}\right) + \left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}\right) \tag{235}$$

$$= \textstyle\sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} + \sigma_- B_{0+} B_{1-} - \sigma_- B_{01}\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} - \sigma_- B_{0+} B_{1-} + \sigma_- B_{01}\right) \tag{236}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{10} + \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{01}\right) \tag{237}$$

$$= \textstyle\sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{10} + \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{01}\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{10} - \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{0+} B_{1-}\right. \tag{238}$$

$$\left. + \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{01}\right) \tag{239}$$

$$= \textstyle\sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} + \mathrm{i}\sigma_y \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2}\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2}\right. \tag{240}$$

$$\left. + \mathrm{i}\sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{241}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{242}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2\mathrm{i}} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{243}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2\mathrm{i}} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \tag{244}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\left(-B_y\right) - \sigma_y B_x\right) \tag{245}$$

$$= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y - \sigma_y B_x\right). \tag{246}$$

III.   FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar S}}(t) + \overline{H_{\bar B}}\right)}\right)\right) + \left\langle \overline{H_{\bar I}}\right\rangle_{\overline{H_{\bar S}}(t) + \overline{H_{\bar B}}} + O\left(\left\langle \overline{H_{\bar I}}^2\right\rangle_{\overline{H_{\bar S}}(t) + \overline{H_{\bar B}}}\right). \tag{247}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle \overline{H_{\bar I}}\right\rangle_{\overline{H_{\bar S}}(t) + \overline{H_{\bar B}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = 0. \tag{248}$$

Using this condition and given that $\left[\overline{H_{\bar S}}(t), \overline{H_{\bar B}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar S}}(t) + \overline{H_{\bar B}}\right)} = e^{-\beta\overline{H_{\bar S}}(t)} e^{-\beta\overline{H_{\bar B}}}. \tag{249}$$

Then using the fact that $\overline{H_{\bar S}}(t)$ and $\overline{H_{\bar B}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar S}}(t)} e^{-\beta\overline{H_{\bar B}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar S}}(t)}\right) \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar B}}}\right). \tag{250}$$

So Eq. (248) becomes:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = -\frac{1}{\beta} \frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}} \tag{251}$$

$$= -\frac{1}{\beta} \frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} \tag{252}$$

$$= -\frac{1}{\beta} \frac{\partial \left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right) + \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}} \tag{253}$$

$$= -\frac{1}{\beta} \frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta} \frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} \tag{254}$$

$$= 0 \quad \text{(by Eq. (248))}. \tag{255}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} = \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}} \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} \tag{256}$$

$$= 0. \tag{257}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} = 0. \tag{258}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left((\varepsilon_0(t) + R_0)\left|0\rangle\langle0\right| + (\varepsilon_1(t) + R_1)\left|1\rangle\langle1\right| + V_{10}(t) B_{10}\sigma_+ + V_{01}(t) B_{01}\sigma_-\right). \tag{259}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{260}$$

Let's define:

$$\varepsilon(t) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{261}$$

$$\eta \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{262}$$

The solutions of the equation (260) are:

$$\lambda = \beta \frac{-\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{263}$$

$$= \beta \frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{264}$$

$$= -\beta \frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{265}$$

The value of $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right) + \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) \tag{266}$$

$$= 2\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{267}$$

Given that $v_{i\mathbf{k}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A\left(\varepsilon\left(t\right),\eta\left(t\right)\right)$ to calculate $\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}$ can lead to:

$$\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{268}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) + 2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) \tag{269}$$

$$= -\beta\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{270}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) = 0. \tag{271}$$

The derivates included in the expression given are related to:

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{272}$$

$$= \left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)^{*}\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{273}$$

$$= \langle B_{1+}B_{0-}\rangle^{*}, \tag{274}$$

$$R_{i} = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{275}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{276}$$

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{277}$$

$$= \left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2\omega_{\mathbf{k}}^{2}}\left(v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{278}$$

$$v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*} = \left(v_{0\mathbf{k}}^{\Re}-iv_{0\mathbf{k}}^{\Im}\right)\left(v_{1\mathbf{k}}^{\Re}+iv_{1\mathbf{k}}^{\Im}\right) - \left(v_{0\mathbf{k}}^{\Re}+iv_{0\mathbf{k}}^{\Im}\right)\left(v_{1\mathbf{k}}^{\Re}-iv_{1\mathbf{k}}^{\Im}\right) \tag{279}$$

$$= \left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Re}+iv_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im}-iv_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re}+v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Im}\right) - \left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Re}-iv_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im}+iv_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re}+v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Im}\right) \tag{280}$$

$$= 2i\left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re}\right), \tag{281}$$

$$|v_{1\mathbf{k}}-v_{0\mathbf{k}}|^{2} = (v_{1\mathbf{k}}-v_{0\mathbf{k}})(v_{1\mathbf{k}}-v_{0\mathbf{k}})^{*} \tag{282}$$

$$= |v_{1\mathbf{k}}|^{2} + |v_{0\mathbf{k}}|^{2} - \left(v_{1\mathbf{k}}v_{0\mathbf{k}}^{*} + v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}\right) \tag{283}$$

$$= \left(v_{1\mathbf{k}}^{\Re}\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}\right)^{2}+\left(v_{0\mathbf{k}}^{\Re}\right)^{2}+\left(v_{0\mathbf{k}}^{\Im}\right)^{2}-\left(\left(v_{1\mathbf{k}}^{\Re}+iv_{1\mathbf{k}}^{\Im}\right)\left(v_{0\mathbf{k}}^{\Re}-iv_{0\mathbf{k}}^{\Im}\right)+\left(v_{1\mathbf{k}}^{\Re}-iv_{1\mathbf{k}}^{\Im}\right)\left(v_{0\mathbf{k}}^{\Re}+iv_{0\mathbf{k}}^{\Im}\right)\right) \tag{284}$$

$$= \left(v_{1\mathbf{k}}^{\Re}\right)^{2} + \left(v_{1\mathbf{k}}^{\Im}\right)^{2} + \left(v_{0\mathbf{k}}^{\Re}\right)^{2} + \left(v_{0\mathbf{k}}^{\Im}\right)^{2} - 2\left(v_{1\mathbf{k}}^{\Re}v_{0\mathbf{k}}^{\Re}+v_{1\mathbf{k}}^{\Im}v_{0\mathbf{k}}^{\Im}\right) \tag{285}$$

$$= \left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^{2} + \left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^{2}. \tag{286}$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re} - iv_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re} + iv_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \right) \right) \tag{287}$$

$$= \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - iv_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{288}$$

$$\langle B_{0+} B_{1-} \rangle = \left(\prod_{\mathbf{k}} \exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{289}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{2i\left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re}\right)}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{290}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{i\left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right), \tag{291}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial(\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{292}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - iv_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{293}$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{294}$$

$$\frac{\partial |B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{295}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2} \frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}} \exp\left(-\sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{296}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2} \frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2, \tag{297}$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{298}$$

$$= \frac{2\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right) \frac{\partial \text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} - 4\frac{\partial \text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{299}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2\frac{\partial\left((\varepsilon_1(t) + R_1)(\varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2 \right)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta(t)} \tag{300}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2\left((\varepsilon(t) - \varepsilon_i(t) - R_i)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2} \frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \tag{301}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2\left((\varepsilon(t) - \varepsilon_i(t) - R_i)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2\left(v_{i\mathbf{k}}^{\Re} - v_{i'\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \tag{302}$$

$$= \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(-\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right. \tag{303}$$

$$\left. + 4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{304}$$

From the equation (271) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2} \right) = \frac{\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \tag{305}$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left(2\frac{\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) - \frac{2}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + 2\frac{(\varepsilon(t) - \varepsilon_i(t) - R_i)\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} + 2\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) - \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \varepsilon(t)}{\eta(t)}} \tag{306}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}} - g_{i\mathbf{k}}^*\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t) - 4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right) - \frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \left(g_{i\mathbf{k}}+g_{i\mathbf{k}}^*\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right) + 4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$= \frac{\left(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t) - 4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right) - \frac{4|V_{10}(t)|^2 B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right) + 4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{308}$$

$$= \frac{\left(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t) - 4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right) - \frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right) + 4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{309}$$

$$= \frac{\left(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(\varepsilon(t) - 2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right) + 2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{310}$$

Separating (309) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Re}\left(\varepsilon(t) - 2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right) - g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right) + 2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{311}$$

$$v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re} = v_{i\mathbf{k}}^{\Re}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right) \tag{312}$$

$$+ 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{313}$$

$$v_{i\mathbf{k}}^{\Re} = \frac{g_{i\mathbf{k}}^{\Re}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t)-\varepsilon_i(t)-R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{314}$$

$$v_{i\mathbf{k}}^{\Re} = \frac{g_{i\mathbf{k}}^{\Re}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t)-\varepsilon_i(t)-R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{315}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{316}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2+\left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - iv_{i\mathbf{k}}^{\Im}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{317}$$

$$= 2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{318}$$

$$\frac{\partial|B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^2+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{319}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^2+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{320}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}|B_{10}|^2 \tag{321}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}=\frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{322}$$

$$=\frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}} \tag{323}$$

$$=\frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{\eta(t)} \tag{324}$$

$$=\frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{325}$$

$$=\frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}-v_{i'\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{326}$$

$$=\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2(\varepsilon(t)-\varepsilon_i(t)-R_i)\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{327}$$

$$\left.+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{328}$$

From the equation (271) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}}} \tag{329}$$

$$=\frac{2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\left(g_{i\mathbf{k}}^*\right)^{\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{\left(g_{i\mathbf{k}}^*\right)^{\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)} \tag{330}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v_{i\mathbf{k}}^{\Im}-\mathrm{i}\left(g_{i\mathbf{k}}^*-g_{i\mathbf{k}}\right)\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^*-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{331}$$

$$=\frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{332}$$

$$=\frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{333}$$

$$=\frac{\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{334}$$

Separating (334) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{\omega_\mathbf{k}}\right)-g_{i\mathbf{k}}^{\Im}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_\mathbf{k}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)\tag{335}$$

$$v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}=v_{i\mathbf{k}}^{\Im}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{\omega_\mathbf{k}}\right)\tag{336}$$

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Im}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_\mathbf{k}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)\tag{337}$$

$$v_{i\mathbf{k}}^{\Im}=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_\mathbf{k}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{\omega_\mathbf{k}}\right)}\tag{338}$$

$$v_{i\mathbf{k}}^{\Im}=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_\mathbf{k}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{\omega_\mathbf{k}}\right)}\tag{339}$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_\mathbf{k})=v_{i\mathbf{k}}^{\Re}(\omega_\mathbf{k})+\mathrm{i}v_{i\mathbf{k}}^{\Im}(\omega_\mathbf{k})\tag{340}$$

$$=\frac{g_{i\mathbf{k}}^{\Re}(\omega_\mathbf{k})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(\omega_\mathbf{k})}{\omega_\mathbf{k}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{\omega_\mathbf{k}}\right)}\tag{341}$$

$$+\mathrm{i}\frac{g_{i\mathbf{k}}^{\Im}(\omega_\mathbf{k})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(\omega_\mathbf{k})}{\omega_\mathbf{k}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{\omega_\mathbf{k}}\right)}\tag{342}$$

$$=\frac{g_{i\mathbf{k}}(\omega_\mathbf{k})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(\omega_\mathbf{k})}{\omega_\mathbf{k}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_\mathbf{k}}{2}\right)}{\omega_\mathbf{k}}\right)}\tag{343}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0)=\rho_S(0)\otimes\rho_B^{\mathrm{Thermal}}$, the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T(0) e^{-V} \tag{344}$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) \left(\rho_S(0) \otimes \rho_B^{\text{Thermal}}\right)(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \tag{345}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: |0\rangle\langle 0|0\rangle B_{0+}\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \tag{346}$$

$$= |0\rangle B_{0+}\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \tag{347}$$

$$= |0\rangle\langle 0| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{0-} \tag{348}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \tag{349}$$

$$= |1\rangle\langle 1|B_{1+}\rho_B^{\text{Thermal}}B_{1-} \tag{350}$$

$$= |1\rangle\langle 1| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{1-} \tag{351}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1| : |0\rangle\langle 0|B_{0+}|0\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \tag{352}$$

$$= |0\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \tag{353}$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}B_{1-} \tag{354}$$

$$= |0\rangle\langle 1| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{1-} \tag{355}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0| : |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \tag{356}$$

$$= |1\rangle\langle 0| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{0-} \tag{357}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^\dagger(t) O(t) U(t) \tag{358}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\tilde{S}}}(t')\right). \tag{359}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\overline{\rho_S}(t) U(t), \text{ where} \tag{360}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\overline{\rho}_T(t)\right) \tag{361}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ V_{10}^\Re(t) & V_{10}^\Re(t) & 1 & V_{10}^\Im(t) & -V_{10}^\Im(t) & 1 \end{pmatrix} \tag{362}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\tilde{I}}}(t)$ as pointed in the equation (246):

$$\overline{H_{\tilde{I}}}(t) = \sum_i B_{iz}|i\rangle\langle i| + V_{10}^\Re(t)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^\Im(t)\left(\sigma_x B_y - \sigma_y B_x\right) \tag{363}$$

$$= B_{0z}|0\rangle\langle 0| + B_{1z}|1\rangle\langle 1| + V_{10}^\Re(t)\sigma_x B_x + V_{10}^\Re(t)\sigma_y B_y + V_{10}^\Im(t)\sigma_x B_y - V_{10}^\Im(t)\sigma_y B_x \tag{364}$$

$$= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right) \tag{365}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{366}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) - \mathrm{i}\int_0^t [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]\mathrm{d}s. \tag{367}$$

Replacing the equation (367) in the equation (366) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]]\mathrm{d}s. \tag{368}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]]\cdots] \tag{369}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B^{\mathrm{Thermal}}]]\cdots] \tag{370}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B^{\mathrm{Thermal}}$. Consider the following notation:

$$\widetilde{\overline{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\overline{\rho_S}(0) \tag{371}$$
$$= W(t)\overline{\rho_S}(0) \tag{372}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B^{\mathrm{Thermal}}]]\cdots] \tag{373}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{374}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}W(t)\overline{\rho_S}(0) \tag{375}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}\widetilde{\overline{\rho_S}}(t) \tag{376}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B^{\mathrm{Thermal}}] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (374) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\widetilde{\overline{\rho_S}}(t) \tag{377}$$
$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right] \tag{378}$$

Replacing $t_1 \to t - \tau$ and moving back into the Schrödinger picture gives:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \int_0^t \mathrm{d}\tau \, \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(-\tau), \overline{\rho_S}(t)\rho_B^{\text{Thermal}}\right]\right] \tag{379}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}(t)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{380}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = \sum_i C_i(t)\left(\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)\right) \tag{381}$$

$$\widetilde{A_i}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{382}$$

$$= U^\dagger(t)A_iU(t)\,\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} \tag{383}$$

$$= U^\dagger(t)A_iU(t)\,\mathbb{I} \tag{384}$$

$$= U^\dagger(t)A_iU(t) \tag{385}$$

$$\widetilde{B_i}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{386}$$

$$= U^\dagger(t)U(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{387}$$

$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{388}$$

$$= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{389}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.

Using the expression (381) to replace it in the equation (378)

$$\tfrac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}}\right]\right]\mathrm{d}s \tag{390}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right), \left[\sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right), \widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}}\right]\right]\mathrm{d}s \tag{391}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)\right)\widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}} - \widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s) \otimes \widetilde{B_i}\right.\right. \tag{392}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}} - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) \right. \tag{393}$$

$$\left. - \sum_{i \in J} C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\overline{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\right)\mathrm{d}s \tag{394}$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \left\langle\widetilde{B_j}(t)\widetilde{B_i}(s)\right\rangle_B \tag{395}$$

$$= \left\langle\widetilde{B_j}(\tau)\widetilde{B_i}(0)\right\rangle_B \tag{396}$$

The correlation functions relevant that appear in the equation (394) are:

$$\text{Tr}_B \left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B^{\text{Thermal}} \right) = \left\langle \widetilde{B_j}(t)\,\widetilde{B_i}(s) \right\rangle_B \tag{397}$$

$$= \left\langle \widetilde{B_j}(\tau)\,\widetilde{B_i}(0) \right\rangle_B \tag{398}$$

$$= \Lambda_{ji}(\tau) \tag{399}$$

$$\text{Tr}_B \left(\widetilde{B_j}(t)\,\rho_B^{\text{Thermal}}\,\widetilde{B_i}(s) \right) = \text{Tr}_B \left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B^{\text{Thermal}} \right) \tag{400}$$

$$= \left\langle \widetilde{B_i}(s)\,\widetilde{B_j}(t) \right\rangle_B \tag{401}$$

$$= \left\langle \widetilde{B_i}(-\tau)\,\widetilde{B_j}(0) \right\rangle_B \tag{402}$$

$$= \Lambda_{ij}(-\tau) \tag{403}$$

$$\text{Tr}_B \left(\widetilde{B_i}(s)\,\rho_B^{\text{Thermal}}\,\widetilde{B_j}(t) \right) = \text{Tr}_B \left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B^{\text{Thermal}} \right) \tag{404}$$

$$= \left\langle \widetilde{B_j}(t)\,\widetilde{B_i}(s) \right\rangle_B \tag{405}$$

$$= \left\langle \widetilde{B_j}(\tau)\,\widetilde{B_i}(0) \right\rangle_B \tag{406}$$

$$= \Lambda_{ji}(\tau) \tag{407}$$

$$\text{Tr}_B \left(\rho_B^{\text{Thermal}}\,\widetilde{B_i}(s)\,\widetilde{B_j}(t) \right) = \text{Tr}_B \left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B^{\text{Thermal}} \right) \tag{408}$$

$$= \left\langle \widetilde{B_i}(s)\,\widetilde{B_j}(t) \right\rangle_B \tag{409}$$

$$= \left\langle \widetilde{B_i}(-\tau)\,\widetilde{B_j}(0) \right\rangle_B \tag{410}$$

$$= \Lambda_{ij}(-\tau) \tag{411}$$

The cyclic property of the trace was use widely in the development of equations (397) and (411). Replacing in (394)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \sum_{ij} \big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)\widetilde{A_i}(t)\widetilde{A_j}(s)\widetilde{\rho_S}(t) - \Lambda_{ji}(-\tau)\widetilde{A_i}(t)\widetilde{\rho_S}(t)\widetilde{A_j}(s)\big) + C_i(t)C_j(s)\big(\Lambda_{ji}(-\tau)\widetilde{\rho_S}(t)\widetilde{A_j}(s)\widetilde{A_i}(t) - \Lambda_{ij}(\tau)\widetilde{A_j}(s)\widetilde{\rho_S}(t)\widetilde{A_i}(t)\big)\big)\,ds \tag{412}$$

$$= -\int_0^t \sum_{ij} \big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\rho_S}(t)] + \Lambda_{ji}(-\tau)[\widetilde{\rho_S}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)]\big)\big)\,ds \tag{413}$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}(\tau)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t) - \Lambda_{ij}(\tau)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,\widetilde{A_i}(t) = \Lambda_{ij}(\tau)\left[\widetilde{A_i}(t),\,\widetilde{A_j}(s)\,\widetilde{\rho_s}(t) \right] \tag{414}$$

$$\Lambda_{ji}(-\tau)\,\widetilde{\rho_S}(t)\,\widetilde{A_j}(s)\,\widetilde{A_i}(t) - \Lambda_{ji}(-\tau)\,\widetilde{A_i}(t)\,\widetilde{\rho_S}(t)\,\widetilde{A_j}(s) = \Lambda_{ji}(-\tau)\left[\widetilde{\rho_S}(t)\,\widetilde{A_j},\,\widetilde{A_i}(t) \right] \tag{415}$$

Returning to the Schroedinger picture we have:

$$U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,U^\dagger(t) = U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\,U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\,U(t)\,\widetilde{\rho_S}(t)\,U^\dagger(t) \tag{416}$$

$$= \left(U(t)\,\widetilde{A_i}(t)\,U^\dagger(t) \right)\left(U(t)\,\widetilde{A_j}(s)\,U^\dagger(t) \right)\left(U(t)\,\widetilde{\rho_S}(t)\,U^\dagger(t) \right) \tag{417}$$

$$= A_i \widetilde{A_j}(s,t)\,\overline{\rho_S}(t) \tag{418}$$

This procedure applying to the relevant commutators give us:

$$U(t)\left[\widetilde{A_i}(t),\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t) \right]U^\dagger(t) = \left(U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,U^\dagger(t) - U(t)\,\widetilde{A_j}(s)\,\widetilde{\rho_S}(t)\,\widetilde{A_i}(t)\,U^\dagger(t) \right) \tag{419}$$

$$= A_i \widetilde{A_j}(s,t)\,\overline{\rho_S}(t) - \widetilde{A_j}(s,t)\,\overline{\rho_S}(t)\,A_i \tag{420}$$

$$= \left[A_i,\,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t) \right] \tag{421}$$

Introducing this transformed commutators in the equation (413) allow us to obtain the master equation of the system

$$\frac{d\overline{\rho_S}(t)}{dt} = -i\left[H_S(t), \overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t d\tau \left(C_i(t)C_j(t-\tau)\Lambda_{ij}(\tau)\left[A_i, \widetilde{A_j}(t-\tau,t)\overline{\rho_S}(t)\right]\right. \tag{422}$$

$$\left. + C_j(t)C_i(t-\tau)\Lambda_{ji}(-\tau)\left[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t), A_i\right]\right) \tag{423}$$

where $i, j \in \{1, 2, 3, 4, 5.6\}$.

Here $\widetilde{A_j}(s,t) = U(t)U^\dagger(s)A_jU(s)U^\dagger(t)$ where $U(t)$ is given by (359). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. The environmental correlation functions are given by:

$$\Lambda_{ij}(\tau) = \text{Tr}_B\left(\widetilde{B_i}(t)\widetilde{B_j}(s)\rho_B^{\text{Thermal}}\right) \tag{424}$$

$$= \text{Tr}_B\left(\widetilde{B_i}(\tau)\widetilde{B_j}(0)\rho_B^{\text{Thermal}}\right) \tag{425}$$

Calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \text{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B^{\text{Thermal}}\right) \tag{426}$$

$$= \int d^2\alpha P(\alpha)\left\langle \alpha \left| \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right| \alpha\right\rangle \tag{427}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle \alpha \left| \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right| \alpha\right\rangle d^2\alpha \tag{428}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle \alpha \left| \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right| \alpha\right\rangle d^2\alpha \tag{429}$$

$$\widetilde{B_{jz}}(\tau) = \sum_{\mathbf{k}}\left((g_{j\mathbf{k}} - v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right) \tag{430}$$

$$\widetilde{B_{jz}}(0) = \sum_{\mathbf{k}'}\left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'})b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'}\right) \tag{431}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \text{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B\right) \tag{432}$$

$$= \text{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^* b_{\mathbf{k}'}\right)\rho_B\right) \tag{433}$$

$$= \text{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^* b_{\mathbf{k}'}\right)\rho_B\right) \tag{434}$$

$$+ \text{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}\right)\rho_B\right) \tag{435}$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}} = p_{j\mathbf{k}} \tag{436}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \text{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^* b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}'}b_{\mathbf{k}'}^\dagger+p_{j\mathbf{k}'}^* b_{\mathbf{k}'}\right)\rho_B\right) + \text{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^* b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger+p_{j\mathbf{k}}^* b_{\mathbf{k}}\right)\rho_B\right) \tag{437}$$

$$= 0 + \text{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^* b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^\dagger+p_{j\mathbf{k}}^* b_{\mathbf{k}}\right)\rho_B\right) \tag{438}$$

$$= \text{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}+|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}+|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*2} b_{\mathbf{k}}b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\rho_B\right) \tag{439}$$

$$= \text{Tr}_B\left(\sum_{\mathbf{k}}p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau}\rho_B\right)+\text{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\rho_B\right)+\text{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau}\rho_B\right)+\text{Tr}_B\left(\sum_{\mathbf{k}}p_{j\mathbf{k}}^{*2} b_{\mathbf{k}}b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{440}$$

$$= \text{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\rho_B\right) + \text{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{441}$$

$$= \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{i\omega_{\mathbf{k}}\tau}\text{Tr}_B\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\rho_B\right)+e^{-i\omega_{\mathbf{k}}\tau}\text{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\rho_B\right)\right) \tag{442}$$

$$= \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{i\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle \alpha_{\mathbf{k}}|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}|\alpha_{\mathbf{k}}\right\rangle d^2\alpha_{\mathbf{k}}+e^{-i\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle \alpha_{\mathbf{k}}|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger|\alpha_{\mathbf{k}}\right\rangle d^2\alpha_{\mathbf{k}}\right) \tag{443}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{444}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{445}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{446}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{447}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{448}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*)(b_{\mathbf{k}} + \alpha_{\mathbf{k}}) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| (b_{\mathbf{k}} + \alpha_{\mathbf{k}})(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{449}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{450}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \right. \tag{451}$$

$$\left. + \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \right) \tag{452}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{453}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{454}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{455}$$

$$1 = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \tag{456}$$

$$b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |0\rangle = 0 \tag{457}$$

$$b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |0\rangle = |0\rangle \tag{458}$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| |\alpha_{\mathbf{k}}|^2 |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{459}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0| b_{\mathbf{k}} b_{\mathbf{k}}^\dagger |0\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{460}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{461}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\left(e^{i\omega_{\mathbf{k}}\tau} + e^{-i\omega_{\mathbf{k}}\tau} \right) \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{462}$$

$$\frac{1}{\pi N} \int_0^{2\pi} \int_0^\infty r^2 \exp\left(-\frac{r^2}{N}\right) r \, \mathrm{d}r \mathrm{d}\theta = \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \tag{463}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(2\cos(\omega_{\mathbf{k}}\tau) N\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 e^{-i\omega_{\mathbf{k}}\tau} \tag{464}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(2\cos(\omega_{\mathbf{k}}\tau) N + e^{-i\omega_{\mathbf{k}}\tau}\right) \tag{465}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} + e^{-i\omega_{\mathbf{k}}\tau}\right) \tag{466}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} + \cos(\omega_{\mathbf{k}}\tau) - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{467}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{(2 + e^{\beta\omega_{\mathbf{k}}} - 1)\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{468}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{(1 + e^{\beta\omega_{\mathbf{k}}})\cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{469}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos(\omega_{\mathbf{k}}\tau)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}} - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{470}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau) - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{471}$$

$$= \sum_{\mathbf{k}} |g_{j\mathbf{k}} - v_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau) - i\sin(\omega_{\mathbf{k}}\tau)\right) \tag{472}$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \rangle_B = \int \mathrm{d}^2\alpha_{\mathbf{k}} P(\alpha_{\mathbf{k}}) \langle \alpha_{\mathbf{k}} | \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) | \alpha_{\mathbf{k}} \rangle \tag{473}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{474}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right) \langle \alpha_\mathbf{k} | \Sigma_\mathbf{k}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_\mathbf{k}^\dagger e^{i\omega_\mathbf{k}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_\mathbf{k} e^{-i\omega_\mathbf{k}\tau}\right)\Sigma_{\mathbf{k}'}\left((g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})^* b_{\mathbf{k}'}\right)|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} \tag{475}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right) \langle \alpha_\mathbf{k}| \Sigma_{\mathbf{k}\neq\mathbf{k}'}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_\mathbf{k}^\dagger e^{i\omega_\mathbf{k}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_\mathbf{k} e^{-i\omega_\mathbf{k}\tau}\right)\left((g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})b_{\mathbf{k}'}^\dagger+(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'})^* b_{\mathbf{k}'}\right)|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} \tag{476}$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right) \langle \alpha_\mathbf{k}| \Sigma_\mathbf{k}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_\mathbf{k}^\dagger e^{i\omega_\mathbf{k}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_\mathbf{k} e^{-i\omega_\mathbf{k}\tau}\right)\left((g_{j'\mathbf{k}}-v_{j'\mathbf{k}})b_\mathbf{k}^\dagger+(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* b_\mathbf{k}\right)|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} \tag{477}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right) \langle \alpha_\mathbf{k}| \Sigma_\mathbf{k}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_\mathbf{k}^\dagger e^{i\omega_\mathbf{k}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_\mathbf{k} e^{-i\omega_\mathbf{k}\tau}\right)\left((g_{j'\mathbf{k}}-v_{j'\mathbf{k}})b_\mathbf{k}^\dagger+(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* b_\mathbf{k}\right)|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} \tag{478}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle \alpha_\mathbf{k}|\Sigma_\mathbf{k}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* b_\mathbf{k}^\dagger b_\mathbf{k} e^{i\omega_\mathbf{k}\tau}|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} + \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle \alpha_\mathbf{k}|\Sigma_\mathbf{k}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})b_\mathbf{k} b_\mathbf{k}^\dagger e^{-i\omega_\mathbf{k}\tau}|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} \tag{479}$$

$$= \Sigma_\mathbf{k}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* e^{i\omega_\mathbf{k}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle \alpha_\mathbf{k}|b_\mathbf{k}^\dagger b_\mathbf{k}|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} + \Sigma_\mathbf{k}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})e^{-i\omega_\mathbf{k}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle \alpha_\mathbf{k}|b_\mathbf{k} b_\mathbf{k}^\dagger|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} \tag{480}$$

$$\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle \alpha_\mathbf{k}|b_\mathbf{k}^\dagger b_\mathbf{k}|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle 0|D(-\alpha_\mathbf{k})b_\mathbf{k}^\dagger D(\alpha_\mathbf{k})D(-\alpha_\mathbf{k})b_\mathbf{k} D(\alpha_\mathbf{k})|0\rangle d^2\alpha_\mathbf{k} \tag{481}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\left\langle 0\left|D(-\alpha_\mathbf{k})b_\mathbf{k}^\dagger D(\alpha_\mathbf{k})D(-\alpha_\mathbf{k})b_\mathbf{k} D(\alpha_\mathbf{k})\right|0\right\rangle d^2\alpha_\mathbf{k} \tag{482}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\left\langle 0\left|\left(b_\mathbf{k}^\dagger+\alpha_\mathbf{k}^*\right)(b_\mathbf{k}+\alpha_\mathbf{k})\right|0\right\rangle d^2\alpha_\mathbf{k} \tag{483}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)|\alpha_\mathbf{k}|^2 d^2\alpha_\mathbf{k} \tag{484}$$

$$= N \tag{485}$$

$$\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle \alpha_\mathbf{k}|b_\mathbf{k} b_\mathbf{k}^\dagger|\alpha_\mathbf{k}\rangle d^2\alpha_\mathbf{k} = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle 0|D(-\alpha_\mathbf{k})b_\mathbf{k} D(\alpha_\mathbf{k})D(-\alpha_\mathbf{k})b_\mathbf{k}^\dagger D(\alpha_\mathbf{k})|0\rangle d^2\alpha_\mathbf{k} \tag{486}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\left\langle 0\left|(b_\mathbf{k}+\alpha_\mathbf{k})\left(b_\mathbf{k}^\dagger+\alpha_\mathbf{k}^*\right)\right|0\right\rangle d^2\alpha_\mathbf{k} \tag{487}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\left\langle 0\left|b_\mathbf{k} b_\mathbf{k}^\dagger + \alpha_\mathbf{k} b_\mathbf{k}^\dagger + b_\mathbf{k}\alpha_\mathbf{k}^* + |\alpha_\mathbf{k}|^2\right|0\right\rangle d^2\alpha_\mathbf{k} \tag{488}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\left\langle 0\left|b_\mathbf{k} b_\mathbf{k}^\dagger+|\alpha_\mathbf{k}|^2\right|0\right\rangle d^2\alpha_\mathbf{k} \tag{489}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle 0||\alpha_\mathbf{k}|^2|0\rangle d^2\alpha_\mathbf{k} + \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_\mathbf{k}|^2}{N}\right)\langle 0|b_\mathbf{k} b_\mathbf{k}^\dagger|0\rangle d^2\alpha_\mathbf{k} \tag{490}$$

$$= N+1 \tag{491}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\right\rangle_B = \Sigma_\mathbf{k}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* e^{i\omega_\mathbf{k}\tau}N + \Sigma_\mathbf{k}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})e^{-i\omega_\mathbf{k}\tau}(N+1) \tag{492}$$

$$= \Sigma_\mathbf{k}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})e^{-i\omega_\mathbf{k}\tau}+N\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^* e^{i\omega_\mathbf{k}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})e^{-i\omega_\mathbf{k}\tau}\right)\right) \tag{493}$$

$$D(h')D(h) = \exp\left(\frac{1}{2}(h'h^*-h'^*h)\right)D(h'+h) \tag{494}$$

$$\langle D(h')D(h)\rangle_B = \text{Tr}_B\left(\exp\left(\frac{1}{2}(h'h^*-h'^*h)\right)D(h'+h)\rho_B^{\text{Thermal}}\right) \tag{495}$$

$$= \exp\left(\frac{1}{2}(h'h^*-h'^*h)\right)\text{Tr}_B\left(D(h'+h)\rho_B^{\text{Thermal}}\right) \tag{496}$$

$$= \exp\left(\frac{1}{2}(h'h^*-h'^*h)\right)\frac{1}{\pi N}\int d^2\alpha P(\alpha)\langle \alpha|D(h'+h)|\alpha\rangle \tag{497}$$

$$= \exp\left(\frac{1}{2}(h'h^*-h'^*h)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{498}$$

$$h' = h\exp(i\omega\tau) \tag{499}$$

$$\langle D(h\exp(i\omega\tau))D(h)\rangle_B = \exp\left(\frac{1}{2}(hh^*\exp(i\omega\tau)-h^*h\exp(-i\omega\tau))\right)\exp\left(-\frac{|h+h\exp(i\omega\tau)|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{500}$$

$$\frac{1}{2}|h|^2(\exp(i\omega\tau)-\exp(-i\omega\tau)) = \frac{1}{2}(hh^*\exp(i\omega\tau)-h^*h\exp(-i\omega\tau)) \tag{501}$$

$$= \frac{1}{2}|h|^2(\cos(\omega\tau)+i\sin(\omega\tau)-\cos(\omega\tau)+i\sin(\omega\tau)) \tag{502}$$

$$= \frac{1}{2}|h|^2(2i\sin(\omega\tau)) \tag{503}$$

$$= i|h|^2\sin(\omega\tau) \tag{504}$$

$$-\frac{|h+h\exp(i\omega\tau)|^2}{2} = -|h|^2\frac{|1+\exp(i\omega\tau)|^2}{2} \tag{505}$$

$$= -|h|^2 \frac{|1 + \cos(\omega\tau) + \mathrm{i}\sin(\omega\tau)|^2}{2} \tag{506}$$

$$= -|h|^2 \frac{(1 + \cos(\omega\tau))^2 + \sin^2(\omega\tau)}{2} \tag{507}$$

$$= -|h|^2 \frac{(1 + 2\cos(\omega\tau) + \cos^2(\omega\tau)) + \sin^2(\omega\tau)}{2} \tag{508}$$

$$= -|h|^2 \frac{2 + 2\cos(\omega\tau)}{2} \tag{509}$$

$$= -|h|^2 (1 + \cos(\omega\tau)) \tag{510}$$

$$\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\rangle_B = \exp(\mathrm{i}|h|^2\sin(\omega\tau))\exp\left(-|h|^2(1+\cos(\omega\tau))\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{511}$$

$$= \exp\left(\mathrm{i}|h|^2\sin(\omega\tau) - |h|^2(1+\cos(\omega\tau))\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{512}$$

$$= \exp\left(-|h|^2\left(-\mathrm{i}\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-|h|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{513}$$

$$= \langle D(h)\rangle_B \exp(-\phi(\tau)) \tag{514}$$

$$\exp(-\phi(\tau)) = \exp\left(-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin(\omega\tau)\right)\right) \tag{515}$$

$$\phi(\tau) = |h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin(\omega\tau)\right) \tag{516}$$

$$\langle D(h')D(h)\rangle_B = \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{517}$$

$$h' = v\exp(\mathrm{i}\omega\tau) \tag{518}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\rho_B^{\mathrm{Thermal}}\right) \tag{519}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\rho_B^{\mathrm{Thermal}}\right) \tag{520}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{521}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{522}$$

$$= \Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{523}$$

$$= \Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\Pi_{\mathbf{k}}\left(\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{524}$$

$$= \Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{525}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{526}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{527}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{528}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\Pi_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{529}$$

$$= \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\Pi_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{530}$$

$$= \Pi_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{531}$$

$$= \Pi_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega\tau+\pi)}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{532}$$

$$= \Pi_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{533}$$

$$= \Pi_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{534}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\Pi_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{535}$$

$$= \text{Tr}_B \left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \tag{536}$$

$$= \prod_{\mathbf{k}} \text{Tr}_B \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i(\omega_{\mathbf{k}}\tau + \pi)} \right) D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \tag{537}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i\sin(\omega\tau + \pi) + \cos(\omega\tau + \pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \exp\left(-\left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{538}$$

$$= \left\langle |B_{1+}\widetilde{B_{0-}}(\tau) B_{0+}\widetilde{B_{1-}}(0) \right\rangle_B \tag{539}$$

$$\left\langle B_{0+}\widetilde{B_{1-}}(\tau)\widetilde{B_{jz}}(0) \right\rangle_B = \text{Tr}_B \left(\prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B^{\text{Thermal}} \right) \tag{540}$$

$$\langle D(h) b \rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | D(h) b | \alpha \rangle \tag{541}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | D(-\alpha) D(h) b D(\alpha) | \alpha \rangle \tag{542}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b D(\alpha) | 0 \rangle \tag{543}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \tag{544}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \tag{545}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) (b + \alpha) | 0 \rangle \tag{546}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha | 0 \rangle \tag{547}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha | 0 \rangle \tag{548}$$

$$= \frac{1}{\pi N} \int \alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|h|^2}{2} \right) d^2\alpha \tag{549}$$

$$= hN \langle D(h) \rangle_B \tag{550}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle \alpha \left| D(h) b^\dagger \right| \alpha \right\rangle \tag{551}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{552}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{553}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{554}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \left\langle 0 \left| D(-\alpha) D(h) D(\alpha) \left(b^\dagger + \alpha^* \right) \right| 0 \right\rangle \tag{555}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \left\langle 0 \left| D(h) \left(b^\dagger + \alpha^* \right) \right| 0 \right\rangle \tag{556}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \tag{557}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) | 1 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \tag{558}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \langle -h | 1 \rangle + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \tag{559}$$

$$\langle -h | = \exp\left(-\frac{|-h^*|^2}{2} \right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n | \tag{560}$$

$$\langle -h | 1 \rangle = \exp\left(-\frac{|-h^*|^2}{2} \right) (-h^*) \tag{561}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|-h^*|^2}{2} \right) (-h^*) + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha^* \exp\left(-\frac{|-h^*|^2}{2} \right) \tag{562}$$

$$= -h^* \langle D(h) \rangle_B (N + 1) \tag{563}$$

$$\langle b D(h) \rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | b D(h) | \alpha \rangle \tag{564}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^*\alpha) \alpha \exp\left(-\frac{|h|^2}{2} \right) \tag{565}$$

$$= h \langle D(h) \rangle_B (N+1) \tag{566}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| b^\dagger D(h) \right| \alpha \right\rangle \tag{567}$$

$$= \frac{1}{\pi N} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h + \frac{1}{\pi N}\int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\frac{|h|^2}{2}\right) \tag{568}$$

$$= -h^* \langle D(h) \rangle_B N \tag{569}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau) \right\rangle_B = \left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right) \right\rangle_B \tag{570}$$

$$= \prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{571}$$

$$= \prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{572}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right) \prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{573}$$

$$= B_{10} \tag{574}$$

The correlation functions can be found readily as:

$$\widetilde{B_{1+}B_{0-}}(\tau) = \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right) \tag{575}$$

$$\widetilde{B_{0+}B_{1-}}(\tau) = \prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{i\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right) \tag{576}$$

$$\widetilde{B_x}(0) = \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2} \tag{577}$$

$$\widetilde{B_y}(0) = \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{01}}{2i} \tag{578}$$

$$B_{10} = \left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega}{2}\right)\right)\right) \tag{579}$$

$$B_{iz} = \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}}-v_{i\mathbf{k}})^* b_{\mathbf{k}}\right) \tag{580}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}}-v_{i\mathbf{k}})^*b_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}\right) \right\rangle_B \tag{581}$$

$$= \sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + \sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}})e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1) \tag{582}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0) \right\rangle_B = \left\langle \frac{B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{01}}{2}\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2} \right\rangle_B \tag{583}$$

$$= \frac{1}{4}\left\langle (B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{01})(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}) \right\rangle_B \tag{584}$$

$$= \frac{1}{4}\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)B_{10} - B_{1+}B_{0-}(\tau)B_{01} + B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} \tag{585}$$

$$-B_{0+}B_{1-}(\tau)B_{10} - B_{0+}B_{1-}(\tau)B_{01} - B_{10}B_{1+}B_{0-} - B_{10}B_{0+}B_{1-} + B_{10}B_{10} + B_{10}B_{01} - B_{01}B_{1+}B_{0-} - B_{01}B_{0+}B_{1-} + B_{01}B_{10} + B_{01}B_{01}\rangle \tag{586}$$

$$= \frac{1}{4}\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)B_{10} - B_{1+}B_{0-}(\tau)B_{01} + B_{0+}B_{1-}(\tau)B_{1+}B_{0-} \tag{587}$$

$$+B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{10} - B_{0+}B_{1-}(\tau)B_{01}\rangle \tag{588}$$

$$\left\langle B\widetilde{_{0+}B_1}_-(\tau)B\widetilde{_{0+}B_1}_-(0)\right\rangle_B = \Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \quad (589)$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \quad (590)$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (591)$$

$$S = \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (592)$$

$$\left\langle B\widetilde{_{0+}B_1}_-(\tau)B\widetilde{_{0+}B_1}_-(0)\right\rangle_B = U\exp(-\phi(\tau))S \quad (593)$$

$$\left\langle B\widetilde{_{1+}B_0}_-(\tau)B\widetilde{_{1+}B_0}_-(0)\right\rangle_B = U^*\exp(-\phi(\tau))S \quad (594)$$

$$\left\langle B\widetilde{_{1+}B_0}_-(\tau)B\widetilde{_{0+}B_1}_-(0)\right\rangle_B = \exp(\phi(\tau))S \quad (595)$$

$$\left\langle B\widetilde{_{0+}B_1}_-(\tau)B\widetilde{_{1+}B_0}_-(0)\right\rangle_B = \left\langle \widetilde{B_{1+}B_0}_-(\tau)\,\widetilde{B_{0+}B_1}_-(0)\right\rangle_B \quad (596)$$

$$\left\langle \widetilde{B_{1+}B_0}_-(\tau)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (597)$$

$$= U^{*1/2}S^{1/2} \quad (598)$$

$$\left\langle \widetilde{B_x}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{01}+B_{0+}B_{1-}(\tau)B_{1+}B_{0-} \quad (599)$$

$$+B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{01}\rangle \quad (600)$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\left\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{01}+B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{0+}B_{1-}\right. \quad (601)$$

$$\left. -B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{01}\right\rangle \quad (602)$$

$$= \frac{1}{4}\left(U^*\exp(-\phi(\tau))S+\exp(\phi(\tau))S-B_{10}^2-|B_{10}|^2+\exp(\phi(\tau))S+U\exp(-\phi(\tau))S-B_{10}^{*2}-|B_{10}|^2\right) \quad (603)$$

$$= \frac{1}{4}\left(2U^{\Re}\exp(-\phi(\tau))S+2\exp(\phi(\tau))S-2\left(B_{10}^2\right)^{\Re}-2|B_{10}|^2\right) \quad (604)$$

$$= \frac{1}{4}\left(2U^{\Re}\exp(-\phi(\tau))S+2\exp(\phi(\tau))S-2\left(U^*\right)^{\Re}S-2S\right) \quad (605)$$

$$= \frac{S}{2}\left(U^{\Re}\exp(-\phi(\tau))+\exp(\phi(\tau))-\left(U^*\right)^{\Re}-1\right) \quad (606)$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle \frac{B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{01}}{2\mathrm{i}}\,\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{01}}{2\mathrm{i}}\right\rangle_B \quad (607)$$

$$= -\frac{1}{4}\left\langle (B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{01})(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{01})\right\rangle_B \quad (608)$$

$$= -\frac{1}{4}\langle B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{01}-B_{1+}B_{0-}(\tau)B_{0+}B_{1-}+B_{1+}B_{0-}(\tau)B_{1+}B_{0-} \quad (609)$$

$$-B_{1+}B_{0-}(\tau)B_{10}+B_{1+}B_{0-}(\tau)B_{01}+B_{10}B_{0+}B_{1-}-B_{10}B_{1+}B_{0-}+B_{10}B_{10}-B_{10}B_{01}-B_{01}B_{0+}B_{1-}+B_{01}B_{1+}B_{0-}-B_{01}B_{10}+B_{01}B_{01}\rangle \quad (610)$$

$$= -\frac{1}{4}\langle B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{01} \quad (611)$$

$$-B_{1+}B_{0-}(\tau)B_{0+}B_{1-}+B_{1+}B_{0-}(\tau)B_{1+}B_{0-}-B_{1+}B_{0-}(\tau)B_{10}+B_{1+}B_{0-}(\tau)B_{01}\rangle \quad (612)$$

$$= -\frac{1}{4}\left\langle B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{01}B_{10}-B_{01}B_{01}-B_{1+}B_{0-}(\tau)B_{0+}B_{1-}+B_{1+}B_{0-}(\tau)B_{1+}B_{0-}-B_{10}B_{10}+B_{10}B_{01}\right\rangle \quad (613)$$

$$= -\frac{1}{4}\left(U\exp(-\phi(\tau))S-\exp(\phi(\tau))S-\exp(\phi(\tau))S+U^*\exp(-\phi(\tau))S+2S-2\left(U^*\right)^{\Re}S\right) \quad (614)$$

$$= -\frac{S}{4}\left(2U^{\Re}\exp(-\phi(\tau))-2\exp(\phi(\tau))+2-2U^{\Re}\right) \quad (615)$$

$$= \frac{S}{2}\left(\exp(\phi(\tau))-U^{\Re}\exp(-\phi(\tau))-1+U^{\Re}\right) \quad (616)$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle \frac{B_{1+}B_{0-}(\tau)+B_{0+}B_{1-}(\tau)-B_{10}-B_{01}}{2}\,\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{01}}{2\mathrm{i}}\right\rangle_B \quad (617)$$

$$= \frac{1}{4\mathrm{i}}\langle(B_{1+}B_{0-}(\tau)+B_{0+}B_{1-}(\tau)-B_{10}-B_{01})(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{01})\rangle_B \quad (618)$$

$$= \frac{1}{4\mathrm{i}}\langle B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{01}+B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-} \quad (619)$$

$$+B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{01}-B_{10}B_{0+}B_{1-}+B_{10}B_{1+}B_{0-}-B_{10}B_{10}+B_{10}B_{01}-B_{01}B_{0+}B_{1-}+B_{01}B_{1+}B_{0-}-B_{01}B_{10}+B_{01}B_{01}\rangle \quad (620)$$

$$= \tfrac{1}{4\mathrm{i}} \langle B_{1+}B_{0-}(\tau)B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)B_{10} - B_{1+}B_{0-}(\tau)B_{01} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{10} \tag{621}$$

$$- B_{0+}B_{1-}(\tau)B_{01} \rangle \tag{622}$$

$$= \tfrac{1}{4\mathrm{i}} \langle B_{1+}B_{0-}(\tau)B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} + B_{10}B_{10} - B_{10}B_{01} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{01}B_{10} - B_{01}B_{01} \rangle \tag{623}$$

$$= \tfrac{1}{4\mathrm{i}} \langle B_{1+}B_{0-}(\tau)B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} + B_{10}B_{10} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{1+}B_{0-} - B_{01}B_{01} \rangle \tag{624}$$

$$= \frac{1}{4\mathrm{i}} \left(\exp\left(\phi\left(\tau\right)\right) S - U^{*}\exp\left(-\phi\left(\tau\right)\right) S + U\exp\left(-\phi\left(\tau\right)\right) S - \exp\left(\phi\left(\tau\right)\right) S + U^{*}S - US \right) \tag{625}$$

$$= \frac{1}{4\mathrm{i}} \left(-U^{*}\exp\left(-\phi\left(\tau\right)\right) S + U\exp\left(-\phi\left(\tau\right)\right) S + U^{*}S - US \right) \tag{626}$$

$$= \frac{S}{4\mathrm{i}} \left(-U^{*}\exp\left(-\phi\left(\tau\right)\right) + U\exp\left(-\phi\left(\tau\right)\right) + U^{*} - U \right) \tag{627}$$

$$= \frac{S\left(U - U^{*}\right)}{4\mathrm{i}} \left(\exp\left(-\phi\left(\tau\right)\right) - 1 \right) \tag{628}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}} \left(\exp\left(-\phi\left(\tau\right)\right) - 1 \right) \tag{629}$$

$$= \frac{U^{\Im}S}{2} \left(\exp\left(-\phi\left(\tau\right)\right) - 1 \right) \tag{630}$$

$$\langle \widetilde{B_{y}}(\tau)\widetilde{B_{x}}(0) \rangle_{B} = \left\langle \frac{B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{01}}{2\mathrm{i}} \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2} \right\rangle_{B} \tag{631}$$

$$= \frac{1}{4\mathrm{i}} \langle \left(B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{01} \right)\left(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01} \right) \rangle_{B} \tag{632}$$

$$= \tfrac{1}{4\mathrm{i}} \langle B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{10} - B_{0+}B_{1-}(\tau)B_{01} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)B_{0+}B_{1-} \tag{633}$$

$$+ B_{1+}B_{0-}(\tau)B_{10} + B_{1+}B_{0-}(\tau)B_{01} + B_{10}B_{1+}B_{0-} + B_{10}B_{0+}B_{1-} - B_{10}B_{10} - B_{10}B_{01} - B_{01}B_{1+}B_{0-} - B_{01}B_{0+}B_{1-} + B_{01}B_{10} + B_{01}B_{01} \rangle \tag{634}$$

$$= \tfrac{1}{4\mathrm{i}} \langle B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{10} - B_{0+}B_{1-}(\tau)B_{01} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)B_{0+}B_{1-} \tag{635}$$

$$+ B_{1+}B_{0-}(\tau)B_{10} + B_{1+}B_{0-}(\tau)B_{01} \rangle \tag{636}$$

$$= \tfrac{1}{4\mathrm{i}} \langle B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{01}B_{10} - B_{01}B_{01} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)B_{0+}B_{1-} + B_{10}B_{10} + B_{10}B_{01} \rangle \tag{637}$$

$$= \tfrac{1}{4\mathrm{i}} \langle B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{01}B_{01} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)B_{0+}B_{1-} + B_{10}B_{10} \rangle \tag{638}$$

$$= \frac{1}{4\mathrm{i}} \left(U\exp\left(-\phi\left(\tau\right)\right) S - U^{*}\exp\left(-\phi\left(\tau\right)\right) S + B_{10}^{2} - B_{10}^{*2} \right) \tag{639}$$

$$= \frac{1}{4\mathrm{i}} \left(U\exp\left(-\phi\left(\tau\right)\right) S - U^{*}\exp\left(-\phi\left(\tau\right)\right) S + U^{*}S - US \right) \tag{640}$$

$$= \frac{S\left(U - U^{*}\right)}{4\mathrm{i}} \left(\exp\left(-\phi\left(\tau\right)\right) - 1 \right) \tag{641}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}} \left(\exp\left(-\phi\left(\tau\right)\right) - 1 \right) \tag{642}$$

$$= -\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*} \left(N_{\mathbf{k}'} + 1 \right) B_{10} \tag{643}$$

$$\langle B_{1+}B_{0-}(\tau)\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*}b_{\mathbf{k}'} \rangle_{B} = \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*} \Pi_{\mathbf{k}} \exp\left(\tfrac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}} \right) \right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} \left\langle \Pi_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right) \right\rangle \tag{644}$$

$$= \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} \tag{645}$$

$$\langle B_{0+}B_{1-}(\tau)\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger} \rangle_{B} = -\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*} \left(N_{\mathbf{k}'} + 1 \right) B_{01} \tag{646}$$

$$\langle B_{0+}B_{1-}(\tau)\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*}b_{\mathbf{k}'} \rangle_{B} = \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*} \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \tag{647}$$

$$\langle \widetilde{B_{x}}(\tau)\widetilde{B_{\mathrm{i}z}}(0) \rangle_{B} = \tfrac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*}\left(N_{\mathbf{k}'} + 1 \right) B_{10} - \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*}\left(N_{\mathbf{k}'} + 1 \right) B_{01} \tag{648}$$

$$+ \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \tag{649}$$

$$= \tfrac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*}\left(N_{\mathbf{k}'} + 1 \right) B_{10} - \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*}\left(N_{\mathbf{k}'} + 1 \right) B_{01} \tag{650}$$

$$+ \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \tag{651}$$

$$= \tfrac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1 \right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*} B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^{*} B_{01} \right) \tag{652}$$

$$+ \left(g_{\mathrm{i}\mathbf{k}'} - v_{\mathrm{i}\mathbf{k}'}\right)^{*} N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) B_{01} \right) \right) \tag{653}$$

$$= \tfrac{1}{2} \sum_{\mathbf{k}'} \Big(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)\Big(\big(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\big)^* B_{10} - \big(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\big)^* B_{01}\Big) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'}\Big(\big(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\big) B_{10} - \big(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\big) B_{01}\Big)\Big) \tag{654}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \Big(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\Big)^* (B_{10} - B_{01}) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'}\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\Big)(B_{10} - B_{01})\Big) \tag{655}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} 2i B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\Big) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\Big)^* \right) \tag{656}$$

$$= i \sum_{\mathbf{k}'} B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\Big) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau}\Big)^* \right) \tag{657}$$

$$= i \sum_{\mathbf{k}'} B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big) e^{i\omega_{\mathbf{k}'}\tau} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* e^{-i\omega_{\mathbf{k}'}\tau} \right) \tag{658}$$

$$\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\rangle_B = \left\langle \sum_{\mathbf{k}'} \Big((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}\Big)\Big(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2}\Big)\right\rangle_B \tag{659}$$

$$= \sum_{\mathbf{k}'} \left\langle \Big((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}\Big)\Big(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2}\Big)\right\rangle_B \tag{660}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} \left\langle \Big((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}\Big)(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01})\right\rangle_B \tag{661}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} \left\langle \Big((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}\Big)(B_{1+}B_{0-} + B_{0+}B_{1-})\right\rangle_B \tag{662}$$

$$= \frac{1}{2}\sum_{\mathbf{k}} \left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_{0+}B_{1-} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-} \right. \tag{663}$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_{0+}B_{1-}\right\rangle \tag{664}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-}\right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-}\right\rangle_B \tag{665}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\Big(D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\Big)\right\rangle_B \tag{666}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau}\Big(D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}'}^* v_{0\mathbf{k}'} - v_{1\mathbf{k}'} v_{0\mathbf{k}'}^*}{\omega_{\mathbf{k}'}^2}\Big)\Big)\Big)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\Big)\right\rangle_B \tag{667}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\right\rangle_B \tag{668}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\right\rangle_B \tag{669}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\right\rangle_B e^{i\omega_{\mathbf{k}'}\tau} \tag{670}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right\rangle_B \Big(-\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*\left\langle D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\right\rangle_B N_{\mathbf{k}'}\Big) e^{i\omega_{\mathbf{k}'}\tau} \tag{671}$$

$$= -\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\left\langle \prod_{\mathbf{k}} D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\right\rangle_B N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \tag{672}$$

$$= -\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{10} N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \tag{673}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_{0+}B_{1-}\right\rangle_B = -\Big(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{01} N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \tag{674}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-}\right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'} B_{1+}B_{0-}\right\rangle_B \tag{675}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\Big(D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\Big)\right\rangle_B \tag{676}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\left\langle b_{\mathbf{k}'} D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\Big)\right\rangle_B \tag{677}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1)\left\langle D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\Big)\right\rangle_B \tag{678}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\Big)\Big)\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1)\left\langle D\Big(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)\Big)\right\rangle_B \tag{679}$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-i\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'}+1) B_{10} \tag{680}$$

$$\left\langle \left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^* b_{\mathbf{k'}} e^{-i\omega_{\mathbf{k'}}\tau} B_{0+}B_{1-}\right\rangle_B = \left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^* e^{-i\omega_{\mathbf{k'}}\tau}\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\left(N_{\mathbf{k'}}+1\right)B_{01} \tag{681}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B = \frac{1}{2}\sum_{\mathbf{k'}}\left(-\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)B_{10}N_{\mathbf{k'}}e^{i\omega_{\mathbf{k'}}\tau}-\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)B_{01}N_{\mathbf{k'}}e^{i\omega_{\mathbf{k'}}\tau}\right. \tag{682}$$

$$\left.+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*e^{-i\omega_{\mathbf{k'}}\tau}\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\left(N_{\mathbf{k'}}+1\right)B_{10}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*e^{-i\omega_{\mathbf{k'}}\tau}\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\left(N_{\mathbf{k'}}+1\right)B_{01}\right) \tag{683}$$

$$=\frac{1}{2}\sum_{\mathbf{k'}}\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)N_{\mathbf{k'}}e^{i\omega_{\mathbf{k'}}\tau}\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*\left(B_{01}-B_{10}\right)+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{-i\omega_{\mathbf{k'}}\tau}\left(N_{\mathbf{k'}}+1\right)\left(B_{10}-B_{01}\right)\right) \tag{684}$$

$$=\frac{1}{2}\sum_{\mathbf{k'}}\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)N_{\mathbf{k'}}e^{i\omega_{\mathbf{k'}}\tau}\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*\left(B_{01}-B_{10}\right)-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{-i\omega_{\mathbf{k'}}\tau}\left(N_{\mathbf{k'}}+1\right)\left(B_{01}-B_{10}\right)\right) \tag{685}$$

$$=i\sum_{\mathbf{k'}}B_{10}\Im\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)N_{\mathbf{k'}}e^{i\omega_{\mathbf{k'}}\tau}\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{-i\omega_{\mathbf{k'}}\tau}\left(N_{\mathbf{k'}}+1\right)\right) \tag{686}$$

$$\left\langle \widetilde{B_y}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_B = \left\langle\left(\frac{B_{0+}B_{1-}\left(\tau\right)-B_{1+}B_{0-}\left(\tau\right)+B_{10}-B_{01}}{2i}\right)\sum_{\mathbf{k'}}\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}\right)\right\rangle_B \tag{687}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left\langle\left(B_{0+}B_{1-}\left(\tau\right)-B_{1+}B_{0-}\left(\tau\right)+B_{10}-B_{01}\right)\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}\right)\right\rangle_B \tag{688}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left\langle\left(B_{0+}B_{1-}\left(\tau\right)-B_{1+}B_{0-}\left(\tau\right)\right)\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}\right)\right\rangle_B \tag{689}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left\langle B_{0+}B_{1-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger-B_{1+}B_{0-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger+B_{0+}B_{1-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}-B_{1+}B_{0-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}\right\rangle \tag{690}$$

$$\left\langle B_{0+}B_{1-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger\right\rangle_B = -\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)^*\left(N_{\mathbf{k'}}+1\right)B_{01} \tag{691}$$

$$\left\langle B_{0+}B_{1-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}\right\rangle_B = \left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)N_{\mathbf{k'}}B_{01} \tag{692}$$

$$\left\langle B_{1+}B_{0-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger\right\rangle_B = -\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)^*\left(N_{\mathbf{k'}}+1\right)B_{10} \tag{693}$$

$$\left\langle B_{1+}B_{0-}\left(\tau\right)\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}\right\rangle_B = \left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)N_{\mathbf{k'}}B_{10} \tag{694}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B=\frac{1}{2i}\sum_{\mathbf{k'}}\left(-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)^*\left(N_{\mathbf{k'}}+1\right)B_{01}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)^*\left(N_{\mathbf{k'}}+1\right)B_{10}\right. \tag{695}$$

$$\left.+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)N_{\mathbf{k'}}B_{01}-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)N_{\mathbf{k'}}B_{10}\right) \tag{696}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left(-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)^*\left(N_{\mathbf{k'}}+1\right)B_{01}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)^*\left(N_{\mathbf{k'}}+1\right)B_{10}\right. \tag{697}$$

$$\left.+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)N_{\mathbf{k'}}B_{01}-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}e^{i\omega_{\mathbf{k'}}\tau}\right)N_{\mathbf{k'}}B_{10}\right) \tag{698}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(N_{\mathbf{k'}}+1\right)e^{-i\omega_{\mathbf{k'}}\tau}\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*\left(B_{10}+B_{01}\right)+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*N_{\mathbf{k'}}e^{i\omega_{\mathbf{k'}}\tau}\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)\left(-B_{10}-B_{01}\right)\right) \tag{699}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left(N_{\mathbf{k'}}+1\right)e^{-i\omega_{\mathbf{k'}}\tau}\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*\left(B_{10}+B_{01}\right)-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*N_{\mathbf{k'}}e^{i\omega_{\mathbf{k'}}\tau}\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)\left(B_{10}+B_{01}\right)\right) \tag{700}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B=\left\langle\sum_{\mathbf{k'}}\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger e^{i\omega_{\mathbf{k'}}\tau}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}e^{-i\omega_{\mathbf{k'}}\tau}\right)\left(\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{01}}{2i}\right)\right\rangle_B \tag{701}$$

$$=\frac{1}{2i}\left\langle\sum_{\mathbf{k'}}\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger e^{i\omega_{\mathbf{k'}}\tau}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}e^{-i\omega_{\mathbf{k'}}\tau}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{01}\right)\right\rangle_B \tag{702}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left\langle\left(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger e^{i\omega_{\mathbf{k'}}\tau}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}e^{-i\omega_{\mathbf{k'}}\tau}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}\right)\right\rangle_B \tag{703}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left\langle\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger e^{i\omega_{\mathbf{k'}}\tau}B_{0+}B_{1-}-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)b_{\mathbf{k'}}^\dagger e^{i\omega_{\mathbf{k'}}\tau}B_{1+}B_{0-}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}e^{-i\omega_{\mathbf{k'}}\tau}B_{0+}B_{1-}-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*b_{\mathbf{k'}}e^{-i\omega_{\mathbf{k'}}\tau}B_{1+}B_{0-}\right\rangle \tag{704}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left(e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left\langle b_{\mathbf{k'}}^\dagger B_{0+}B_{1-}\right\rangle-e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left\langle b_{\mathbf{k'}}^\dagger B_{1+}B_{0-}\right\rangle+e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left\langle b_{\mathbf{k'}}B_{0+}B_{1-}\right\rangle-e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left\langle b_{\mathbf{k'}}B_{1+}B_{0-}\right\rangle\right) \tag{705}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left(e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left\langle b_{\mathbf{k'}}^\dagger B_{0+}B_{1-}\right\rangle-e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left\langle b_{\mathbf{k'}}^\dagger B_{1+}B_{0-}\right\rangle+e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left\langle b_{\mathbf{k'}}B_{0+}B_{1-}\right\rangle-e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left\langle b_{\mathbf{k'}}B_{1+}B_{0-}\right\rangle\right) \tag{706}$$

$$=\frac{1}{2i}\sum_{\mathbf{k'}}\left(e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left\langle b_{\mathbf{k'}}^\dagger B_{0+}B_{1-}\right\rangle-e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\left\langle b_{\mathbf{k'}}^\dagger B_{1+}B_{0-}\right\rangle+e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left\langle b_{\mathbf{k'}}B_{0+}B_{1-}\right\rangle-e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\left\langle b_{\mathbf{k'}}B_{1+}B_{0-}\right\rangle\right) \tag{707}$$

$$\left\langle b_{\mathbf{k'}}^\dagger B_{1+}B_{0-}\right\rangle_B = -\left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*B_{10}N_{\mathbf{k'}} \tag{708}$$

$$\left\langle b_{\mathbf{k'}}^\dagger B_{0+}B_{1-}\right\rangle_B = -\left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)^*B_{01}N_{\mathbf{k'}} \tag{709}$$

$$\left\langle b_{\mathbf{k'}}B_{1+}B_{0-}\right\rangle_B = \left(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)\left(N_{\mathbf{k'}}+1\right)B_{10} \tag{710}$$

$$\left\langle b_{\mathbf{k'}}B_{0+}B_{1-}\right\rangle_B = \left(\frac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\right)\left(N_{\mathbf{k'}}+1\right)B_{01} \tag{711}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \tfrac{1}{2i}\sum_{\mathbf{k'}}\Big(e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(-\Big(\tfrac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{01}N_{\mathbf{k'}}\Big)-e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(-\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{10}N_{\mathbf{k'}}\Big) \tag{712}$$

$$+e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\Big(\tfrac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(N_{\mathbf{k'}}+1)B_{01}\Big)-e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(N_{\mathbf{k'}}+1)B_{10}\Big)\Big) \tag{713}$$

$$=\tfrac{1}{2i}\sum_{\mathbf{k'}}\Big(e^{i\omega_{\mathbf{k'}}\tau}\Big(-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\tfrac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{01}N_{\mathbf{k'}}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{10}N_{\mathbf{k'}}\Big) \tag{714}$$

$$+e^{-i\omega_{\mathbf{k'}}\tau}\Big(\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\Big(\tfrac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(N_{\mathbf{k'}}+1)B_{01}\Big)\Big)-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(N_{\mathbf{k'}}+1)B_{10}\Big)\Big) \tag{715}$$

$$=\tfrac{1}{2i}\sum_{\mathbf{k'}}\Big(e^{i\omega_{\mathbf{k'}}\tau}\Big(\left(-g_{i\mathbf{k'}}+v_{i\mathbf{k'}}\right)\Big(\tfrac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{01}N_{\mathbf{k'}}+\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{10}N_{\mathbf{k'}}\Big) \tag{716}$$

$$+e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\Big(\tfrac{v_{0\mathbf{k'}}-v_{1\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(N_{\mathbf{k'}}+1)B_{01}\Big)-\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(N_{\mathbf{k'}}+1)B_{10}\Big)\Big) \tag{717}$$

$$=\tfrac{1}{2i}\sum_{\mathbf{k'}}\Big(e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*(B_{10}+)N_{\mathbf{k'}}-e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\tfrac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(B_{10}+)(N_{\mathbf{k'}}+1)\Big) \tag{718}$$

$$=\frac{1}{i}\sum_{\mathbf{k'}}\Big(e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{10}^{\Re}N_{\mathbf{k'}}-e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)B_{10}^{\Re}(N_{\mathbf{k'}}+1)\Big) \tag{719}$$

$$=i\sum_{\mathbf{k'}}\Big(e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)B_{10}^{\Re}(N_{\mathbf{k'}}+1)-e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{10}^{\Re}N_{\mathbf{k'}}\Big) \tag{720}$$

$$=i\sum_{\mathbf{k'}}\Big(e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)B_{10}^{\Re}(N_{\mathbf{k'}}+1)-e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*B_{10}^{\Re}N_{\mathbf{k'}}\Big) \tag{721}$$

$$=iB_{10}^{\Re}\sum_{\mathbf{k'}}\Big(e^{-i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)^*\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)(N_{\mathbf{k'}}+1)-e^{i\omega_{\mathbf{k'}}\tau}\left(g_{i\mathbf{k'}}-v_{i\mathbf{k'}}\right)\Big(\frac{v_{1\mathbf{k'}}-v_{0\mathbf{k'}}}{\omega_{\mathbf{k'}}}\Big)^*N_{\mathbf{k'}}\Big) \tag{722}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}\Big(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*e^{i\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\Big) \tag{723}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{724}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-i\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{725}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(U^{\Re}\exp(-\phi(\tau))+\exp(\phi(\tau))-U^{\Re}-1\right) \tag{726}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(\exp(\phi(\tau))-U^{\Re}\exp(-\phi(\tau))-1+U^{\Re}\right) \tag{727}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp(-\phi(\tau))-1\right) \tag{728}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp(-\phi(\tau))-1\right) \tag{729}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B = iB_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\Big(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{730}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = iB_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\Big(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\Big(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)^*e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{731}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = iB_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-i\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\Big(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)(N_{\mathbf{k}}+1)-e^{i\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\Big(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)^*N_{\mathbf{k}}\right) \tag{732}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = iB_{10}^{\Re}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\Big(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)(N_{\mathbf{k}}+1)e^{-i\omega_{\mathbf{k}}\tau}\Big(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\Big)^*\right) \tag{733}$$

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}}|g_{i\mathbf{k}}|^2\delta(\omega-\omega_{\mathbf{k}}) \tag{734}$$

$$v_{i\mathbf{k}} = g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}) \tag{735}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^{\infty}\frac{J_i(\omega)}{\omega}d\omega \tag{736}$$

The integral version of the correlation functions are given by:

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}}+(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}})e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{737}$$

$$= \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}}+(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{738}$$

$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}\left(1-F_i(\omega_{\mathbf{k}})\right)g_{j\mathbf{k}}^*\left(1-F_j(\omega_{\mathbf{k}})\right)^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^*\left(1-F_i(\omega_{\mathbf{k}})\right)^* g_{j\mathbf{k}}\left(1-F_j(\omega_{\mathbf{k}})\right)e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{739}$$

$$\approx \int_0^\infty\left(\sqrt{J_i(\omega)J_j^*(\omega)}\left(1-F_i(\omega)\right)\left(1-F_j^*(\omega)\right)e^{i\omega\tau} N(\omega)+\sqrt{J_i^*(\omega)J_j(\omega)}\left(1-F_i^*(\omega)\right)\left(1-F_j(\omega)\right)e^{-i\omega\tau}(N(\omega)+1)\right)d\omega \tag{740}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{741}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right) \tag{742}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{743}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{744}$$

$$= \exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^* g_{1\mathbf{k}}F_0^*(\omega_{\mathbf{k}})F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}g_{1\mathbf{k}}^* F_0(\omega_{\mathbf{k}})F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{745}$$

$$\approx \exp\left(\int_0^\infty\frac{\sqrt{J_0^*(\omega)J_1(\omega)}F_0^*(\omega)F_1(\omega)-\sqrt{J_0(\omega)J_1^*(\omega)}F_0(\omega)F_1^*(\omega)}{\omega^2}d\omega\right) \tag{746}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-i\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{747}$$

$$= \sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\left(-i\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{748}$$

$$\approx \int_0^\infty\left|\frac{\sqrt{J_1(\omega)}F_1(\omega)-\sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\left(-i\sin(\omega\tau)+\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)d\omega \tag{749}$$

$$B_{10} = \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{750}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{751}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})-g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right)\right) \tag{752}$$

$$\approx \exp\left(-\frac{1}{2}\int_0^\infty\left|\frac{\sqrt{J_1(\omega)}F_1(\omega)-\sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)d\omega\right)\exp\left(\int_0^\infty\frac{1}{2}\left(\frac{\sqrt{J_0(\omega)J_1^*(\omega)}F_0(\omega)F_1^*(\omega)-\sqrt{J_0^*(\omega)J_1(\omega)}F_0^*(\omega)F_1(\omega)}{\omega^2}\right)d\omega\right) \tag{753}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(U^\Re\exp(-\phi(\tau))+\exp(\phi(\tau))-U^\Re-1\right) \tag{754}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(\exp(\phi(\tau))-U^\Re\exp(-\phi(\tau))-1+U^\Re\right) \tag{755}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{U^\Im|B_{10}|^2}{2}\left(\exp(-\phi(\tau))-1\right) \tag{756}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{U^\Im|B_{10}|^2}{2}\left(\exp(-\phi(\tau))-1\right) \tag{757}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B = iB_{10}^\Im\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* - (g_{i\mathbf{k}}-v_{i\mathbf{k}})^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{758}$$

$$= iB_{10}^\Im\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^* - (g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{759}$$

$$= iB_{10}^\Im\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^* - g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{760}$$

$$Q(\omega) = \sqrt{J_i(\omega)}(1-F_i(\omega))\left(\frac{\sqrt{J_1(\omega)}F_1(\omega)-\sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right)^* \tag{761}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B \approx iB_{10}^\Im\int_0^\infty\left(Q(\omega)N(\omega)e^{i\omega\tau}-Q^*(\omega)(N(\omega)+1)e^{-i\omega\tau}\right)d\omega \tag{762}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = iB_{10}^\Im\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})^* N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - (g_{i\mathbf{k}}-v_{i\mathbf{k}})\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{763}$$

$$= iB_{10}^\Im\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^*\left(1-F_i^*(\omega_{\mathbf{k}})\right)\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}N_{\mathbf{k}}e^{i\omega_{\mathbf{k}}\tau}-g_{i\mathbf{k}}(1-F_i(\omega))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* e^{-i\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{764}$$

$$\approx iB_{10}^{\Im} \int_0^\infty \left(Q^*(\omega) N(\omega) e^{i\omega\tau} - Q(\omega)(N(\omega)+1) e^{-i\omega\tau} \right) d\omega \tag{765}$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = iB_{10}^{\Re} \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}}+1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \tag{766}$$

$$= iB_{10}^{\Re} \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}}+1) - e^{i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \tag{767}$$

$$\approx iB_{10}^{\Re} \int_0^\infty \left(e^{-i\omega\tau} Q^*(\omega)(N(\omega)+1) - e^{i\omega\tau} Q(\omega) N(\omega) \right) d\omega \tag{768}$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \rangle_B = iB_{10}^{\Re} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}})(N_{\mathbf{k}}+1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \tag{769}$$

$$= iB_{10}^{\Re} \sum_{\mathbf{k}} \left(g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}}))(N_{\mathbf{k}}+1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \tag{770}$$

$$= iB_{10}^{\Re} \int_0^\infty \left(e^{i\omega\tau} Q^*(\omega) N(\omega) - e^{-i\omega\tau} Q(\omega)(N(\omega)+1) \right) d\omega \tag{771}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}\left(\overline{H_{\bar{S}}}\right) \lambda + \text{Det}\left(\overline{H_{\bar{S}}}\right) = 0 \tag{772}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S |\pm\rangle = \lambda_{\pm} |\pm\rangle$. Using this notation is possible to write $H_{\bar{S}} = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$.

The time-dependence of the system operators $\widetilde{A_i}(t)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_i}(\tau) = e^{i\overline{H_{\bar{S}}}\tau} A_i e^{-i\overline{H_{\bar{S}}}\tau} \tag{773}$$

$$= \sum_w e^{-iw\tau} A_i(w) \tag{774}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.

In order to use the equation (774) to descompose the equation (359) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right) \tag{775}$$

$$= \mathbb{I} + \sum_{n=1}^\infty (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) ... H(t_n) \tag{776}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0,t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right) \equiv \exp(-it H_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-it\left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...\right)\right) \tag{777}$$

The terms can be found expanding $\mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_{\bar{S}}}(t') \, dt' \tag{778}$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'') \right] \tag{779}$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'') \right], \overline{H_{\bar{S}}}(t''') \right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'') \right], \overline{H_{\bar{S}}}(t') \right] \right) \tag{780}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}(t) = e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \tag{781}$$

$$= \sum_{w(t)} e^{-iw(t)t} A_i(w(t)) \tag{782}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the eigenvectors are time dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau, t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau, t) = U(t-\tau) U^\dagger(t) A_j(t) U(t) U^\dagger(t-\tau) \tag{783}$$

$$= e^{-i(t-\tau)H_E(t-\tau)} e^{iH_E(t)t} A_j(t) e^{-iH_E(t)t} e^{i(t-\tau)H_E(t-\tau)} \tag{784}$$

$$= e^{-i(t-\tau)H_E(t-\tau)} \sum_{w(t)} e^{-iw(t)t} A_j(w(t)) e^{i(t-\tau)H_E(t-\tau)} \tag{785}$$

$$= \sum_{w(t), w'(t-\tau)} e^{-iw(t)t} e^{iw'(t-\tau)} A_j'(w(t), w'(t-\tau)) \tag{786}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}(t-\tau)$ and $\overline{H_{\bar{S}}}(t)$ respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (774) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $V = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle | \alpha \rangle\langle \beta | \tag{787}$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{i\overline{H_{\bar{S}}}\tau} = e^{i(\lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|)\tau} \tag{788}$$

$$= e^{i\lambda_+ |+\rangle\langle+|\tau} e^{i\lambda_- |-\rangle\langle-|\tau} \tag{789}$$

$$= \left(|-\rangle\langle-| + e^{i\lambda_+ \tau} |+\rangle\langle+| \right) \left(|+\rangle\langle+| + e^{i\lambda_- \tau} |-\rangle\langle-| \right) \tag{790}$$

$$= e^{i\lambda_+ \tau} |+\rangle\langle+| + e^{i\lambda_- \tau} |-\rangle\langle-| \tag{791}$$

Calculating the transformation (774) directly using the previous relationship we find that:

$$\widetilde{A_i}(\tau) = \left(e^{i\lambda_+ \tau} |+\rangle\langle+| + e^{i\lambda_- \tau} |-\rangle\langle-| \right) \left(\sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle | \alpha \rangle\langle \beta | \right) \left(e^{-i\lambda_+ \tau} |+\rangle\langle+| + e^{-i\lambda_- \tau} |-\rangle\langle-| \right) \tag{792}$$

$$= \langle+|A_i|+\rangle |+\rangle\langle+| + e^{i\eta\tau} \langle+|A_i|-\rangle |+\rangle\langle-| + e^{-i\eta\tau} \langle-|A_i|+\rangle |-\rangle\langle+| + \langle-|A_i|-\rangle |-\rangle\langle-| \tag{793}$$

Here $\eta = \lambda_+ - \lambda-$. Comparing the RHS of the equations (774) and the explicit expression for $\widetilde{A_i}(\tau)$ and we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$A_i(0) = \langle +|A_i|+\rangle |+\rangle\langle +| + \langle -|A_i|-\rangle |-\rangle\langle -| \tag{794}$$

$$A_i(w) = \langle +|A_i|-\rangle |+\rangle\langle -| \tag{795}$$

$$A_i(-w) = \langle -|A_i|+\rangle |-\rangle\langle +| \tag{796}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau) = \widetilde{A_i}^\dagger(\tau)$ and $\widetilde{B_i}(\tau) = \widetilde{B_i}^\dagger(\tau)$ we can use the equation (774) to write the master equation in the following neater form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H}_{\bar{S}}(t), \overline{\rho_S}(t)] - \frac{1}{2}\sum_{ij}\sum_{w,w'}\gamma_{ij}(w,w',t)[A_i, A_j(w,w')\overline{\rho_S}(t) - \overline{\rho_S}(t)A_j^\dagger(w,w')] - i\sum_{ij}\sum_w S_{ij}(w,w',t)[A_i, A_j(w,w')\overline{\rho_S}(t) + \overline{\rho_S}(t)A_j^\dagger(w,w')] \tag{797}$$

where $A_j^\dagger(w) = A(-w)$ as expected from the equations (795) and (796). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}(w,w',t) = 2K_{ij}^\Re(w,w',t)$ and $S_{ij}(w,w',t) = K_{ij}^\Im(w,w',t)$, respectively, defined in terms of the response functions

$$K_{ij}(w,w',t) = \int_0^t C_i(t)C_j(t-\tau)\Lambda_{ij}(\tau)e^{iw\tau}e^{-it(w-w')}d\tau \tag{798}$$

$$= K_{ijww'}(t) \tag{799}$$

If we extend the upper limit of integration to ∞ in the equation (798) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{d\overline{\rho}_S}{dt} = -i[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)] - \sum_{ijww'}K_{ijww'}(t)[A_i, A_{jww'}\overline{\rho}_S(t) - \overline{\rho}_S(t)A_{jww'}^\dagger] \tag{800}$$

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{d\overline{\rho}_S}{dt}e^V = \frac{d(e^{-V}\overline{\rho}_S e^V)}{dt} \tag{801}$$

$$= \frac{d\rho_S}{dt} \tag{802}$$

$$= -ie^{-V}[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)]e^V - \sum_{ijww'}K_{ijww'}(t)e^{-V}[A_i, A_{jww'}\overline{\rho}_S(t) - \overline{\rho}_S(t)A_{jww'}^\dagger]e^V \tag{803}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^V = e^{-V}\overline{A\mathbb{I}B}e^V \tag{804}$$

$$= e^{-V}\overline{A}e^V e^{-V}\overline{B}e^V \tag{805}$$

$$= (e^{-V}\overline{A}e^V)(e^{-V}\overline{B}e^V) \tag{806}$$

$$= AB \tag{807}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^V = e^{-V}\overline{(AB - BA)}e^V \tag{808}$$

$$= e^{-V}\overline{AB}e^V - e^{-V}\overline{BA}e^V \tag{809}$$

$$= AB - BA \tag{810}$$

$$= [A,B] \tag{811}$$

So we will obtain that

$$\frac{d\rho_S}{dt} = -ie^{-V}\left[\overline{H}_{\bar{S}}(t),\overline{\rho}_S(t)\right]e^V - \sum_{ijww'}K_{ijww'}(t)\,e^{-V}\left[A_i,A_{jww'}\overline{\rho}_S(t)-\overline{\rho}_S(t)A_{jww'}^\dagger\right]e^V \tag{812}$$

$$= -i\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}K_{ijww'}(t)\left[e^{-V}A_ie^V,e^{-V}A_{jww'}\overline{\rho}_S(t)e^V - e^{-V}\overline{\rho}_S(t)A_{jww'}^\dagger e^V\right] \tag{813}$$

$$= -i\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}K_{ijww'}(t)\left[e^{-V}A_ie^V,e^{-V}A_{jww'}e^Ve^{-V}\overline{\rho}_S(t)e^V - e^{-V}\overline{\rho}_S(t)e^Ve^{-V}A_{jww'}^\dagger e^V\right] \tag{814}$$

$$= -i\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}K_{ijww'}(t)\left[e^{-V}A_ie^V,e^{-V}A_{jww'}e^V\rho_S(t) - \rho_S(t)e^{-V}A_{jww'}^\dagger e^V\right] \tag{815}$$

$$= -i\left[H_{\bar{S}}(t),\rho_S(t)\right] - \left(\sum_{ijww'}K_{ijww'}(t)\left(\left[e^{-V}A_ie^V,e^{-V}A_{jww'}e^V\rho_S(t)\right] - \left[e^{-V}A_ie^V,\rho_S(t)e^{-V}A_{jww'}^\dagger e^V\right]\right)\right) \tag{816}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (797) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (797) reproduces the results of the reference [1], for the latter case we have that $i,j \in \{1,2,3\}$ and $\omega \in (0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_j g_k\left(b_k^\dagger + b_k\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k \tag{817}$$

After performing the transformation (24) on the Hamiltonian (817) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x \tag{818}$$

$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{819}$$

$$H_B = \sum_k \omega_k b_k^\dagger b_k \tag{820}$$

The Hamiltonian (818) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{821}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{822}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z \tag{823}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2}$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$.

$$\lambda_+ = \frac{\epsilon + \eta}{2} \tag{824}$$

$$\lambda_- = \frac{\epsilon - \eta}{2} \tag{825}$$

$$|+\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} \begin{pmatrix} \epsilon + \eta \\ \Omega_r \end{pmatrix} \tag{826}$$

$$|-\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} \begin{pmatrix} -\Omega_r \\ \epsilon + \eta \end{pmatrix} \tag{827}$$

Using this basis we can find the decomposition matrices using the equations (795)-(796) and the fact that $|+\rangle = \cos(\theta)|1\rangle + \sin(\theta)|0\rangle$ and $|-\rangle = -\sin(\theta)|1\rangle + \cos(\theta)|0\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{828}$$

$$= 2\sin(\theta)\cos(\theta) \tag{829}$$

$$= \sin(2\theta) \tag{830}$$

$$\langle -|\sigma_x|-\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{831}$$

$$= -2\sin(\theta)\cos(\theta) \tag{832}$$

$$= -\sin(2\theta) \tag{833}$$

$$\langle -|\sigma_x|+\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{834}$$

$$= \cos^2(\theta) - \sin^2(\theta) \tag{835}$$

$$= \cos(2\theta) \tag{836}$$

$$\langle +|\sigma_y|+\rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{837}$$

$$= -\mathrm{i}\sin(\theta)\cos(\theta) + \mathrm{i}\sin(\theta)\cos(\theta) \tag{838}$$

$$= 0 \tag{839}$$

$$\langle -|\sigma_y|-\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{840}$$

$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{841}$$

$$= 0 \tag{842}$$

$$\langle -|\sigma_y|+\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{843}$$

$$= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) \tag{844}$$

$$= \mathrm{i} \tag{845}$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{846}$$

$$= \cos(\theta)\cos(\theta) \tag{847}$$

$$= \cos^2(\theta) \tag{848}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{849}$$

$$= \sin(\theta)\sin(\theta) \tag{850}$$

$$= \sin^2(\theta) \tag{851}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{852}$$

$$= -\sin(\theta)\cos(\theta) \tag{853}$$

$$= -\sin(\theta)\cos(\theta) \tag{854}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \tag{855}$$
$$A_1(\eta) = \cos(2\theta)|-\rangle\langle+| \tag{856}$$
$$A_2(0) = 0 \tag{857}$$
$$A_2(\eta) = i|-\rangle\langle+| \tag{858}$$
$$A_3(0) = \cos^2(\theta)|+\rangle\langle+| + \sin^2(\theta)|-\rangle\langle-| \tag{859}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)|-\rangle\langle+| \tag{860}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (425) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau) \tag{861}$$

Using the notation of the master equation (797), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(723)-(733) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (425) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B\right) \tag{862}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{863}$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B\right) \tag{864}$$

$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)}\right) \tag{865}$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega\, J(\omega)\,(1 - F(\omega))^2\,G_+(\tau) \tag{866}$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty d\omega\,\frac{J(\omega)}{\omega}\,F(\omega)\,(1 - F(\omega))\,iG_-(\tau) \tag{867}$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau) \tag{868}$$
$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 \tag{869}$$

Finally taking the Hamiltonian (817) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (340) we have that:

$$f_k = \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{870}$$

$$= \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)} \tag{871}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (423) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (797) and using the fact that

$$\widetilde{A_j}(t - \tau, t) = U(\tau) A_j U(-\tau) \tag{872}$$

$$= \sum_w e^{iw\tau} A_i(-w) \tag{873}$$

$$= \sum_w e^{-iw\tau} A_i(w) \tag{874}$$

because the matrices $U(t)$ and $U(t - \tau)$ commute from the fact that $H_S(t)$ and $H_S(t - \tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -i[H_S(t), \overline{\rho_S}(t)] - \frac{1}{2}\sum_{ij}\sum_w \gamma_{ij}(w, t)\left[A_i, A_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)A_j^\dagger(w)\right] \tag{875}$$

$$- \sum_{ij}\sum_w S_{ij}(w, t)\left[A_i, A_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)A_j^\dagger(w)\right] \tag{876}$$

where $A_j^\dagger(w) = A(-w)$, as we can see the equation (876) contains the rates and energy shifts $\gamma_{ij}(w, t) = 2K_{ij}^{\Re}(w, t)$ and $S_{ij}(w, t) = K_{ij}^{\Im}(w, t)$, respectively, defined in terms of the response functions

$$K_{ij}(w, t) = \int_0^t \Lambda'_{ij}(\tau)e^{iw\tau}\mathrm{d}\tau \tag{877}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (110) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_k \omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{878}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (878) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (227) and (246) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t) \tag{879}$$

$$\overline{H_I} = \frac{\Omega(t)}{2}(B_x\sigma_x + B_y\sigma_y) \tag{880}$$

In this case $R_1 = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} g_{\mathbf{k}} \right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}} \left(-\omega_{\mathbf{k}}^{-1} |g_{\mathbf{k}}|^2 \right) = \sum_{\mathbf{k}} \left(-\omega_{\mathbf{k}} |\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2 \right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (862)-(869) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} G_+(\tau)\, \mathrm{d}\omega \tag{881}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i\sin(\omega\tau)$ so (881) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left(\coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i\sin(\omega\tau) \right) \mathrm{d}\omega \tag{882}$$

Writing the interaction Hamiltonian (880) in the similar way to the equation (246) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (227) with $\delta'|1\rangle\langle1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$ is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t) \tag{883}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B \left(\widetilde{B_1}(\tau) \widetilde{B_1}(0) \rho_B \right) \tag{884}$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right) \tag{885}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B \left(\widetilde{B_2}(\tau) \widetilde{B_2}(0) \rho_B \right) \tag{886}$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right) \tag{887}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(423) is:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i} \left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t) \right] - \sum_{i=1}^2 \int_0^t \mathrm{d}\tau \left(C_i(t) C_i(t-\tau) \Lambda_{ii}(\tau) \left[A_i, \widetilde{A_i}(t-\tau, t) \rho_S(t) \right] \right. \tag{888}$$

$$\left. + C_i(t) C_i(t-\tau) \Lambda_{ii}(-\tau) \left[\rho_S(t) \widetilde{A_i}(t-\tau, t), A_i \right] \right) \tag{889}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau, t) = \widetilde{\sigma_i}(t-\tau, t)$, also using the equations (884) and (887) on the equation (889) we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2} [\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4} \int_0^t \mathrm{d}\tau \Omega(t-\tau) \left([\sigma_x, \widetilde{\sigma_x}(t-\tau, t) \rho_S(t)] \Lambda_x(\tau) \right. \tag{890}$$

$$\left. + [\sigma_y, \widetilde{\sigma_y}(t-\tau, t) \rho_S(t)] \Lambda_y(\tau) + [\rho_S(t) \widetilde{\sigma_x}(t-\tau, t), \sigma_x] \Lambda_x(\tau) + [\rho_S(t) \widetilde{\sigma_y}(t-\tau, t), \sigma_y] \Lambda_y(\tau) \right) \tag{891}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau, t) \rho_S(t) \right]^\dagger = \left[\rho_S(t) \widetilde{A_i}(t-\tau, t), A_j \right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (425) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle1| + \frac{\Omega(t)}{2}(|1\rangle\langle0| + |0\rangle\langle1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right) \tag{892}$$

Using the equation (797), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)\right), \rho_S(t)] - \frac{1}{2}\sum_w \gamma_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) - \rho_S(t)A_3^\dagger(w)\right] \tag{893}$$

$$-\sum_w S_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) + \rho_S(t)A_3^\dagger(w)\right]\Big) \tag{894}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(\tau) \tag{895}$$

$$\Lambda_{33}(-\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(-\tau) \tag{896}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (894) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)\right] - \sum_w \left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^*(w,t)\left[\rho_S(t)A_3(w), A_3\right]\right) \tag{897}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle0| + |0\rangle\langle1|)$, this Hamiltonian have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2} \tag{898}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \tag{899}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2} \tag{900}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |0\rangle) \tag{901}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{902}$$

$$= \frac{1}{2} \tag{903}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{904}$$

$$= \frac{1}{2} \tag{905}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{906}$$

$$= -\frac{1}{2} \tag{907}$$

The decomposition matrices are

$$A_3\left(0\right) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{908}$$

$$= \frac{\mathbb{I}}{2} \tag{909}$$

$$A_3\left(\eta\right) = -\frac{1}{2}|-\rangle\langle+| \tag{910}$$

$$= \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right) \tag{911}$$

$$A_3\left(-\eta\right) = -\frac{1}{2}|+\rangle\langle-| \tag{912}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right) \tag{913}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{914}$$

$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] \tag{915}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty\int_0^\infty J\left(\omega\right)G_+\left(\tau\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{916}$$

$$= \int_0^\infty\int_0^\infty J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)e^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{917}$$

$$= \int_0^\infty\int_0^\infty J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)e^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{918}$$

$$= \int_0^\infty\int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)e^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{919}$$

$$= \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)\pi\delta\left(\widetilde{\Omega}-\omega\right)\mathrm{d}\omega \tag{920}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right) \tag{921}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right) G_+\left(\tau\right) e^{-i\widetilde{\Omega}\tau} d\tau d\omega \tag{922}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-i\widetilde{\Omega}\tau} \left(\left(n\left(\omega\right) + 1\right) e^{-i\tau\omega} + n\left(\omega\right) e^{i\tau\omega}\right) d\tau d\omega \tag{923}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-i\widetilde{\Omega}\tau} n\left(\omega\right) e^{i\tau\omega} d\tau d\omega \tag{924}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) n\left(\omega\right) e^{-i\widetilde{\Omega}\tau + i\tau\omega} d\tau d\omega \tag{925}$$

$$= \int_0^\infty J\left(\omega\right) n\left(\omega\right) \pi\delta\left(-\widetilde{\Omega} + \omega\right) d\omega \tag{926}$$

$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right). \tag{927}$$

Here we have used $\int_0^\infty ds\, e^{\pm i\varepsilon s} = \pi\delta\left(\varepsilon\right) \pm i\frac{V.P.}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3\left(0\right)$ because the spectral density $J\left(\omega\right)$ is equal to zero when $\omega = 0$. Replacing in the equation (914) lead us to obtain:

$$\frac{d\rho_S\left(t\right)}{dt} = -i\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right) + 1\right)\left[\sigma_z, \left(\sigma_z + i\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z - i\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{928}$$

$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right) + 1\right)\left[\rho_S\left(t\right)\left(\sigma_z + i\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S\left(t\right)\left(\sigma_z - i\sigma_y\right), \sigma_z\right]\right) \tag{929}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{d\rho_S\left(t\right)}{dt} = -i\frac{\Omega\left(t\right)}{2}\left[\sigma_x, \rho_S\left(t\right)\right] - \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right) + 1\right)\left[\sigma_z, \left(\sigma_z + i\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\Omega\left(t\right)\right)\left[\sigma_z, \left(\sigma_z - i\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{930}$$

$$- \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right) + 1\right)\left[\rho_S\left(t\right)\left(\sigma_z + i\sigma_y\right), \sigma_z\right] + n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z - i\sigma_y\right), \sigma_z\right]\right) \tag{931}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right) = H_S\left(t\right) + H_I + H_B, \tag{932}$$

$$H_S\left(t\right) = \sum_n \varepsilon_n\left(t\right) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right) |n\rangle\langle m|, \tag{933}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right), \tag{934}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \tag{935}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right) \tag{936}$$

At first let's obtain $e^{\pm V}$ under the transformation (936), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)$, so the equation (936) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n|\hat{\varphi}_n} \tag{937}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n \right)^2}{2!} + ... \tag{938}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + ... \tag{939}$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + ... \tag{940}$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + ... \right) \tag{941}$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \tag{942}$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*} b_{u'\mathbf{k}'} \right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm \alpha_{nu\mathbf{k}} \right) = e^{\pm \left(\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} \tag{943}$$

$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} \right) \tag{944}$$

$$= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm \alpha_{nu\mathbf{k}} \right) \right) \tag{945}$$

$$= \prod_{u\mathbf{k}} D\left(\pm \alpha_{nu\mathbf{k}} \right) \tag{946}$$

$$= \prod_u B_{nu\pm} \tag{947}$$

$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D\left(\pm \alpha_{nu\mathbf{k}} \right) \tag{948}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V} A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \tag{949}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (932):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{950}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{951}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{952}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{953}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{954}$$

$$= |0\rangle\langle 0|. \tag{955}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{956}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{957}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{958}$$

$$= |m\rangle\langle n| \prod_u (B_{mu+} B_{nu-}), \ m \neq n, \tag{959}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D(\alpha_{mu\mathbf{k}}) \prod_{\mathbf{k}} D(-\alpha_{nu\mathbf{k}})\right), \tag{960}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} (D(\alpha_{mu\mathbf{k}}) D(-\alpha_{nu\mathbf{k}})), \tag{961}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}})\right)\right). \tag{962}$$

$$\prod_u (B_{mu+} B_{nu-}) = \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}})\right)\right). \tag{963}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{964}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \cdots\right)\left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \cdots\right), \tag{965}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + \cdots, \tag{966}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \cdots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \cdots\right) \prod_u B_{1u-} + \cdots \tag{967}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \cdots\right) \tag{968}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \cdots\right) + \cdots \tag{969}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}^*}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}^*}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \cdots\right) \tag{970}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}^*}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}^*}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \cdots\right) + \cdots \tag{971}$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \cdots \tag{972}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{973}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{974}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right) \tag{975}$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{976}$$

The transformed Hamiltonians of the equations (933) to (935) written in terms of (950) to (974) are:

$$\overline{\overline{H_S(t)}} = \overline{\sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{977}$$

$$= \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \overline{\sum_{n \neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{978}$$

$$= \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t)\,|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) \tag{979}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right) \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \tag{980}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}} \right) + ... \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \tag{981}$$

$$= \Pi_u\, B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}} \right) \Pi_u\, B_{0u-} + \Pi_u\, B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{1u\mathbf{k}}^* b_{u\mathbf{k}} \right) \Pi_u\, B_{1u-} + ... \tag{982}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \Pi_u\, B_{0u+} b_{u\mathbf{k}}^\dagger \Pi_u\, B_{0u-} + g_{0u\mathbf{k}}^* \Pi_u\, B_{0u+} b_{u\mathbf{k}} \Pi_u\, B_{0u-} \right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \Pi_u\, B_{1u+} b_{u\mathbf{k}}^\dagger \Pi_u\, B_{1u-} + g_{1u\mathbf{k}}^* \Pi_u\, B_{1u+} b_{u\mathbf{k}} \Pi_u\, B_{1u-} \right) + ... \tag{983}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \tfrac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{0u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \tfrac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \tfrac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{1u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \tfrac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + ... \tag{984}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{nu\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{985}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{986}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right) \right) \tag{987}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right) \right) \tag{988}$$

$$+ \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{989}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{990}$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}}) b_{u\mathbf{k}}^\dagger + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^* b_{u\mathbf{k}} \right) \tag{991}$$

Using the previous functions we have that (988) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n R_n(t)|n\rangle\langle n| + \sum_n B_{z,n}(t)|n\rangle\langle n| \tag{992}$$

$$\tag{993}$$

Now in order to separate the elements of the hamiltonian (993) let's follow the references of the equations (227) and (246) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \prod_u (B_{mu+}B_{nu-}) \right\rangle_{\overline{H_0}} = \left\langle \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}}) \right) \right) \right\rangle_{\overline{H_0}} \tag{994}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^*\alpha_{nu\mathbf{k}}) \right) \right) \left\langle \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \right\rangle_{\overline{H_0}} \tag{995}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\tfrac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2} \right) \right) \tag{996}$$

$$\equiv B_{nm} \tag{997}$$

$$\left\langle \prod_u (B_{nu+}B_{mu-}) \right\rangle_{\overline{H_0}} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{nu\mathbf{k}}^* v_{mu\mathbf{k}} - v_{nu\mathbf{k}} v_{mu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\tfrac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2} \right) \right) \tag{998}$$

$$= B_{nm}^* \tag{999}$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+}B_{nu-}) - B_{nm} \tag{1000}$$

As we can see:

$$J_{nm}^\dagger = \left(\prod_u (B_{mu+}B_{nu-}) - B_{nm} \right)^\dagger \tag{1001}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{nm}^* \tag{1002}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{mn} \tag{1003}$$

$$= J_{mn} \tag{1004}$$

We can separate the Hamiltonian (993) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)} = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \tag{1005}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \tag{1006}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \tag{1007}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta} \ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_{\bar{S}}(t)} + H_{\bar{B}})} \right) \right) + \left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{S}}(t)} + H_{\bar{B}}} + O\left(\left\langle \overline{H_{\bar{I}}^2} \right\rangle_{\overline{H_{\bar{S}}(t)} + H_B} \right). \tag{1008}$$

Taking the equations (248)-(256) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}} \right) = C\left(R_0, R_1, ..., R_{d-1}, B_{01}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)} \right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v_{nu\mathbf{k}}^{\Re} + iv_{nu\mathbf{k}}^{\Im}$. So our approach will be based on the derivation respect to $v_{nu\mathbf{k}}^{\Re}$ and $v_{nu\mathbf{k}}^{\Im}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\tilde{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \right) |n\rangle\langle n| \tag{1009}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1010}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}} v_{nu\mathbf{k}}^* + g_{nu\mathbf{k}}^* v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1011}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1012}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{nu\mathbf{k}}^\Im \right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^\Re + i v_{nu\mathbf{k}}^\Im (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1013}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1014}$$

$$v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^* = \left(v_{mu\mathbf{k}}^\Re - i v_{mu\mathbf{k}}^\Im \right)\left(v_{nu\mathbf{k}}^\Re + i v_{nu\mathbf{k}}^\Im \right) - \left(v_{mu\mathbf{k}}^\Re + i v_{mu\mathbf{k}}^\Im \right)\left(v_{nu\mathbf{k}}^\Re - i v_{nu\mathbf{k}}^\Im \right) \tag{1015}$$

$$= \left(v_{mu\mathbf{k}}^\Re v_{nu\mathbf{k}}^\Re + i v_{nu\mathbf{k}}^\Im v_{mu\mathbf{k}}^\Re - i v_{mu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Re + v_{mu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Im \right) \tag{1016}$$

$$- \left(v_{mu\mathbf{k}}^\Re v_{nu\mathbf{k}}^\Re - i v_{nu\mathbf{k}}^\Im v_{mu\mathbf{k}}^\Re + i v_{mu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Re + v_{mu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Im \right) \tag{1017}$$

$$= 2i \left(v_{nu\mathbf{k}}^\Im v_{mu\mathbf{k}}^\Re - v_{mu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Re \right) \tag{1018}$$

$$\overline{H_{\tilde{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{nu\mathbf{k}}^\Im \right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^\Re + i v_{nu\mathbf{k}}^\Im (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1019}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{i\left(v_{nu\mathbf{k}}^\Im v_{mu\mathbf{k}}^\Re - v_{mu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Re \right)}{\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1020}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1021}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}} v_{mu\mathbf{k}}^* + v_{nu\mathbf{k}}^* v_{mu\mathbf{k}}) \tag{1022}$$

$$= \left(v_{mu\mathbf{k}}^\Re \right)^2 + \left(v_{mu\mathbf{k}}^\Im \right)^2 + \left(v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{nu\mathbf{k}}^\Im \right)^2 - \left(v_{nu\mathbf{k}}^\Re + i v_{nu\mathbf{k}}^\Im \right)\left(v_{mu\mathbf{k}}^\Re - i v_{mu\mathbf{k}}^\Im \right) \tag{1023}$$

$$- \left(v_{nu\mathbf{k}}^\Re - i v_{nu\mathbf{k}}^\Im \right)\left(v_{mu\mathbf{k}}^\Re + i v_{mu\mathbf{k}}^\Im \right) \tag{1024}$$

$$= \left(v_{mu\mathbf{k}}^\Re \right)^2 + \left(v_{mu\mathbf{k}}^\Im \right)^2 + \left(v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{nu\mathbf{k}}^\Im \right)^2 - 2 \left(v_{nu\mathbf{k}}^\Re v_{mu\mathbf{k}}^\Re + v_{nu\mathbf{k}}^\Im v_{mu\mathbf{k}}^\Im \right) \tag{1025}$$

$$= \left(v_{mu\mathbf{k}}^\Re - v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{mu\mathbf{k}}^\Im - v_{nu\mathbf{k}}^\Im \right)^2 \tag{1026}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{1027}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{nu\mathbf{k}}^\Im \right)^2 - (g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^\Re - i v_{nu\mathbf{k}}^\Im (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \tag{1028}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{nu\mathbf{k}}^\Im \right)^2 - 2 g_{nu\mathbf{k}}^\Re v_{nu\mathbf{k}}^\Re - 2 g_{nu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Im}{\omega_{u\mathbf{k}}} \right) \tag{1029}$$

$$B_{mn} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1030}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{i\left(v_{nu\mathbf{k}}^\Im v_{mu\mathbf{k}}^\Re - v_{mu\mathbf{k}}^\Im v_{nu\mathbf{k}}^\Re \right)}{\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^\Re - v_{nu\mathbf{k}}^\Re \right)^2 + \left(v_{mu\mathbf{k}}^\Im - v_{nu\mathbf{k}}^\Im \right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1031}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Re}} \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1032}$$

$$= \frac{2v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1033}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Re} - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1034}$$

$$\frac{\partial R_{n'}}{\partial v_{nu\mathbf{k}}^{\Im}} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Im}} \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \right) \tag{1035}$$

$$= \frac{2v_{nu\mathbf{k}}^{\Im} - 2g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1036}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Im} - g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1037}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\Pi_{u\mathbf{k}} \exp\left(\frac{i\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \Pi_u \exp\left(-\frac{1}{2} \Sigma_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \right) \tag{1038}$$

$$= \Sigma_{u\mathbf{k}} \ln\exp\left(\frac{i\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) + \Sigma_u \ln\exp\left(-\frac{1}{2} \Sigma_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \tag{1039}$$

$$= \Sigma_{u\mathbf{k}} \left(\frac{i\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2} \right) + \Sigma_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) \right) \tag{1040}$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{-iv_{mu\mathbf{k}}^{\Im} - \left(v_{nu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1041}$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} = \frac{iv_{mu\mathbf{k}}^{\Re} - \left(v_{nu\mathbf{k}}^{\Im} - v_{mu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1042}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \tag{1043}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \tag{1044}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} \tag{1045}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} \tag{1046}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} - \left(v_{nu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Re}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1047}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1048}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Re}} = \left(\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}}\right)^{\dagger} \tag{1049}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)^{\dagger} \tag{1050}$$

$$= B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1051}$$

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} \tag{1052}$$

$$= B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} - \left(v_{nu\mathbf{k}}^{\Im} - v_{mu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1053}$$

$$= B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1054}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Im}} = \left(\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}}\right)^{\dagger} \tag{1055}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1056}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1057}$$

Introducing this derivates in the equation (1032) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v_{nu\mathbf{k}}^{\Re} - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1058}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1059}$$

$$= 0 \tag{1060}$$

We can obtain the variational parameters:

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)$$
(1061)

$$= -\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)$$
(1062)

$$v^{\Re}_{nu\mathbf{k}} = \frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)}$$
(1063)

$$= \frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}$$
(1064)

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}}-g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right.$$
(1065)

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)$$
(1066)

$$= 0$$
(1067)

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\tag{1068}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)\tag{1069}$$

$$v_{nu\mathbf{k}}^{\Im}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)}\tag{1070}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1071}$$

$$v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+iv_{nu\mathbf{k}}^{\Im}\tag{1072}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1073}$$

$$i\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1074}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2ig_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1075}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-iv_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1076}$$

$$-i\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(iv_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1077}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1078}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1079}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)}\tag{1080}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^V\rho(0)e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)(|0\rangle\langle0|\otimes\rho_B)\left(\sum_n|n\rangle\langle n|B_{n+}\right)\tag{1081}$$

$$0=(B_{0+}|0\rangle\langle0|B_{0-})\otimes\rho_B\tag{1082}$$

$$0=\rho(0)\tag{1083}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t) O U(t) \tag{1084}$$

$$U(t) \equiv \mathcal{T} \exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1085}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t) \overline{\rho_S}(t) U(t), \text{ where} \tag{1086}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{1087}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1088}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1089}$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1090}$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1091}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = (B_{m+}B_{n-} - B_m B_n)^{\dagger} \tag{1092}$$

$$= B_{n-}^{\dagger} B_{m+}^{\dagger} - B_n B_m \tag{1093}$$

$$= B_{n+}B_{m-} - B_n B_m \tag{1094}$$

$$= B_{nm} \tag{1095}$$

So we can say that the set of matrices (1088) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1088) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \tag{1096}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}\right) \tag{1097}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1098}$$

$$\left. + \Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1099}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right) \sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1100}$$

$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right) \sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1101}$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n<m} \left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x} B_{nm,x} - \Im\left(V_{nm}(t)\right) \sigma_{nm,x} B_{nm,y} + \Re\left(V_{nm}(t)\right) \sigma_{nm,y} B_{nm,y}\right. \tag{1102}$$

$$\left. + \Im\left(V_{nm}(t)\right) \sigma_{nm,y} B_{nm,x}\right) \tag{1103}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\right\} \tag{1104}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \tag{1105}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \tag{1106}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1107}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1108}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1109}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1110}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1111}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1112}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1113}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1114}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1115}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1116}$$
$$C_{3,nm}(t) = 1 \tag{1117}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1118}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1119}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t)(A_{jp} \otimes B_{jp}(t)) \tag{1120}$$

Here $J = \{1, 2, 3, 4, 5\}$ and P the set defined in (1104).

We write the interaction Hamiltonian transformed under (1084) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)\right) \tag{1121}$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \tag{1122}$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t)(t) e^{-iH_B t} \tag{1123}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] ds \tag{1124}$$

Replacing the equation (1121) in (1124) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B \left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B \right] \right] \mathrm{d}s \tag{1125}$$

$$= -\int_0^t \mathrm{Tr}_B \left[\sum_{j\in J, p\in P} C_{jp}(t) \left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t) \right), \left[\sum_{j'\in J, p'\in P} C_{j'p'}(s) \left(\widetilde{A_{j'p'}}(s) \otimes \widetilde{B_{j'p'}}(s) \right), \widetilde{\rho_S}(t)\rho_B \right] \right] \mathrm{d}s \tag{1126}$$

$$= -\int_0^t \mathrm{Tr}_B \left[\sum_{j\in J, p\in P} C_{jp}(t) \left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t) \right), \sum_{j'\in J, p'\in P} C_{j'p'}(s) \left(\widetilde{A_{j'p'}}(s) \otimes \widetilde{B_{j'p'}}(s) \right) \widetilde{\rho_S}(t)\rho_B \right. \tag{1127}$$

$$\left. -\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}(s) \left(\widetilde{A_{j'p'}}(s) \otimes \widetilde{B_{j'p'}}(s) \right) \right] \mathrm{d}s \tag{1128}$$

$$= -\int_0^t \mathrm{Tr}_B \left(\sum_{j\in J, p\in P} C_{jp}(t) \left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t) \right) \sum_{j'\in J, p'\in P} C_{j'p'}(s) \left(\widetilde{A_{j'p'}}(s) \otimes \widetilde{B_{j'p'}}(s) \right) \widetilde{\rho_S}(t)\rho_B \right. \tag{1129}$$

$$- \sum_{j\in J, p\in P} C_{jp}(t) \left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t) \right) \widetilde{\rho_S}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}(s) \left(\widetilde{A_{j'p'}}(s) \otimes \widetilde{B_{j'p'}}(s) \right) \tag{1130}$$

$$- \sum_{j'\in J, p'\in P} C_{j'p'}(s) \left(\widetilde{A_{j'p'}}(s) \otimes \widetilde{B_{j'p'}}(s) \right) \widetilde{\rho_S}(t)\rho_B \sum_{j\in J, p\in P} C_{jp}(t) \left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t) \right) \tag{1131}$$

$$+ \widetilde{\rho_S}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}(s) \left(\widetilde{A_{j'p'}}(s) \otimes \widetilde{B_{j'p'}}(s) \right) \sum_{j\in J, p\in P} C_{jp}(t) \left(\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t) \right) \right) \mathrm{d}s \tag{1132}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \tag{1133}$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \widetilde{B_{j'p'}}(0) \right\rangle_B \tag{1134}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t) \widetilde{B_{j'p'}}(t) \right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t) \right) \mathrm{Tr}\left(\widetilde{B_{j'p'}}(t) \right)$. The correlation functions relevant of the master equation (1132) are:

$$\text{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1135}$$

$$= \left\langle \widetilde{B_{jp}}(0)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1136}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1137}$$

$$\text{Tr}_B\left(\widetilde{B_{jp}}(t)\,\rho_B\,\widetilde{B_{j'p'}}(s)\right) = \text{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1138}$$

$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1139}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1140}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1141}$$

$$\text{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\rho_B\,\widetilde{B_{jp}}(t)\right) = \text{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) \tag{1142}$$

$$= \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1143}$$

$$= \left\langle \widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1144}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1145}$$

$$\text{Tr}_B\left(\rho_B\,\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right) = \text{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1146}$$

$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1147}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1148}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1149}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1125)and (1132) and using the equations (1135)-(1149) we can re-write:

$$\frac{d\widetilde{\widetilde{\rho_S}}(t)}{dt} = -\int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{jpj'p'}(\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'p'jp}(-\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\right)\right. \tag{1150}$$

$$\left. + C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{j'p'jp}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jp}}(t)\right)\right)\,ds \tag{1151}$$

$$= -\int_0^t \sum_{jj'pp'}\left(C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{jpj'p'}(\tau)\left[\widetilde{A_{jp}}(t),\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \Lambda_{j'p'jp}(-\tau)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s),\widetilde{A_{jp}}(t)\right]\right)\right) \tag{1152}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{d\,\overline{\rho_S}(t)}{dt} = -\int_0^t \sum_{jj'pp'}\left(C_{jp}(t)\,C_{j'p'}(t-\tau)\left(\Lambda_{jpj'p'}(\tau)\left[A_{jp}(t),A_{j'p'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'p'jp}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'p'}(t-\tau,t),A_{jp}(t)\right]\right)\right)\,d\tau - i\left[H_S(t),\overline{\rho_S}(t)\right] \tag{1153}$$

For this case we used that $A_{jp}(t-\tau,t) = U(t)\,U^\dagger(t-\tau)\,A_{jp}(t)\,U(t-\tau)\,U^\dagger(t)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1154}$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1155}$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right), \tag{1156}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{1157}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right) \right) \tag{1158}$$

At first let's obtain e^V under the transformation (1158), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \tag{1159}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n| \hat{\varphi} \right)^2}{2!} + \dots \tag{1160}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \tag{1161}$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \tag{1162}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1163}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \tag{1164}$$

Given that $\left[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger - b_{\mathbf{k}} \right)} \tag{1165}$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{1166}$$

$$= B_\pm \tag{1167}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-$ because this form imposes that $e^{-V} e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{1168}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1154):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right), \tag{1169}$$

$$= |0\rangle\langle 0|, \tag{1170}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right), \tag{1171}$$

$$= |m\rangle\langle m|B_+|m\rangle\langle n|n\rangle\langle n|B_-, \tag{1172}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1173}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right), \tag{1174}$$

$$= |0\rangle\langle m|B_-\ m \neq 0, \tag{1175}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right) \tag{1176}$$

$$= |0\rangle\langle m|B_+\ m \neq 0, \tag{1177}$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+\right)\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right) \tag{1178}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}} B_+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_- \tag{1179}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(B_+ b_{\mathbf{k}}^\dagger B_-\right)\left(B_+ b_{\mathbf{k}} B_-\right) \tag{1180}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1181}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1182}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right) \tag{1183}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1184}$$

The transformed Hamiltonians of the equations (1155) to (1157) written in terms of (1169) to (1184) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{1185}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|} \tag{1186}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \overline{\sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n|} \tag{1187}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,\overline{|0\rangle\langle n|} + V_{n0}(t)\,\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0} V_{mn}(t)\,\overline{|m\rangle\langle n|} \tag{1188}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,B_-\,|0\rangle\langle n| + V_{n0}(t)\,B_+\,|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1189}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n|B_- + V_{n0}(t)\,|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1190}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+\right)\left(\left(\sum_{n=0} \mu_n(t)\,|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right) \tag{1191}$$

$$= \left(\mu_0(t)\,|0\rangle\langle 0| + \sum_{n=1} \mu_n(t)\,|n\rangle\langle n|B_+\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right) \tag{1192}$$

$$= \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B_+\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) B_- \tag{1193}$$

$$= \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1194}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1195}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n|B_- + V_{n0}(t)\,|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1196}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1197}$$

$$+ \sum_{n=0} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) - \sum_{n=1} \mu_n(t)\,|n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1198}$$

$$= \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\,|0\rangle\langle n|B_- + V_{n0}(t)\,|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1199}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)\,g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)\,|0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1200}$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1201}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)\,g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1202}$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)\,g_{\mathbf{k}}\right) \tag{1203}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1204}$$

Using the previous functions we have that (1201) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1205}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{1206}$$

Now in order to separate the elements of the hamiltonian (1206) let's follow the references of the equations (246) and (227) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \tag{1207}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B_- - B) + V_{n0}(t) |n\rangle\langle 0| (B_+ - B)), \tag{1208}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{1209}$$

Here B is given by:

$$B = \langle B_+ \rangle$$
$$= \langle B_- \rangle$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1210}$$
$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \tag{1211}$$
$$B_x = \frac{B_+ + B_- - 2B}{2} \tag{1212}$$
$$B_y = \frac{B_- - B_+}{2i} \tag{1213}$$

Using this set of hermitian operators to write the Hamiltonians (1155)-(1157)

$$\overline{H_S(t)} = \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{m.n\neq 0}V_{mn}(t)\,|m\rangle\langle n| \tag{1214}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)\,|m\rangle\langle n| + V_{nm}(t)\,|n\rangle\langle m|) \tag{1215}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1216}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t))+i\Im(V_{mn}(t)))\,|m\rangle\langle n| + (\Re(V_{mn}(t))-i\Im(V_{mn}(t)))\,|n\rangle\langle m|) + \varepsilon_0(t)\,|0\rangle\langle 0| \tag{1217}$$

$$+ B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1218}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t))+i\Im(V_{mn}(t)))\frac{\sigma_{nm,x}-i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t))-i\Im(V_{mn}(t)))\frac{\sigma_{nm,x}+i\sigma_{nm,y}}{2}\right) \tag{1219}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x}-i\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x}+i\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1220}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1221}$$

$$+ \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1222}$$

$$\overline{H_I(t)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n|(B_- - B) + V_{n0}(t)\,|n\rangle\langle 0|(B_+ - B)) \tag{1223}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t))+i\Im(V_{0n}(t)))(B_- - B)\frac{\sigma_{0n,x}-i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t))-i\Im(V_{0n}(t)))(B_+ - B)\frac{\sigma_{0n,x}+i\sigma_{0n,y}}{2}\right) \tag{1224}$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1225}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}((B_- - B)(\Re(V_{0n}(t))+i\Im(V_{0n}(t))) + (B_+ - B)(\Re(V_{0n}(t))-i\Im(V_{0n}(t))))\right. \tag{1226}$$

$$\left.+\frac{i\sigma_{0n,y}}{2}((B_+ - B)(\Re(V_{0n}(t))-i\Im(V_{0n}(t))) - (B_- - B)(\Re(V_{0n}(t))+i\Im(V_{0n}(t))))\right) \tag{1227}$$

$$+ \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1228}$$

$$= \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}(B_+ + B_- - 2B)\Re(V_{0n}(t)) + i(B_- - B - B_+ + B)\Im(V_{0n}(t))\right. \tag{1229}$$

$$\left.+\frac{i\sigma_{0n,y}}{2}((B_+ - B - B_- + B)\Re(V_{0n}(t)) + i(B - B_- + B - B_+)\Im(V_{0n}(t)))\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1230}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1231}$$

$$+\sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1232}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right) + \langle\overline{H_I}\rangle_{H_S+H_B} + O\left(\langle\overline{H_I^2}\rangle_{H_S+H_B}\right). \tag{1233}$$

Taking the equations (248)-(256) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (256) and using the chain rule we obtain that:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1234}$$

$$= 0 \tag{1235}$$

Let's recall the equations (1202) and (1204), we can write them in terms of the variational parameters

$$B = \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1236}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1237}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1238}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1239}$$

Introducing this derivates in the equation (1234) give us:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1240}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1241}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}} \tag{1242}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}} \tag{1243}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S}(t)}\right)}{\partial B}}. \tag{1244}$$

C.   Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_+\right)(|0\rangle\langle 0| \otimes \rho_B)\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_-\right) \tag{1245}$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \tag{1246}$$

$$0 = \rho(0) \tag{1247}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t) O U(t) \tag{1248}$$

$$U(t) \equiv \mathcal{T}\exp\left(-i\int_0^t dt' \overline{H_S}(t')\right). \tag{1249}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\overline{\rho_S}(t)U(t), \text{ where} \tag{1250}$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\bar\rho(t)) \tag{1251}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I}(t) = B_{z,0}|0\rangle\langle 0| + \sum_{n=1}\left(\Re(V_{0n}(t))B_x\sigma_{0n,x} + \Re(V_{0n}(t))B_y\sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right) \tag{1252}$$

$$+\Im(V_{0n}(t))B_x\sigma_{0n,y} - \Im(V_{0n}(t))B_y\sigma_{0n,x}) \tag{1253}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}}\mu_0(t)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1254}$$

$$B_{z,n} = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \text{ if } n \neq 0 \tag{1255}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1256}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1257}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1258}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1259}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1260}$$
$$B_{1n}(t) = B_x \tag{1261}$$
$$B_{2n}(t) = B_y \tag{1262}$$
$$B_{3n}(t) = B_{z,n} \tag{1263}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1264}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1265}$$
$$C_{10}(t) = 0 \tag{1266}$$
$$C_{20}(t) = 0 \tag{1267}$$
$$C_{40}(t) = 0 \tag{1268}$$
$$C_{50}(t) = 0 \tag{1269}$$
$$C_{30}(t) = 1 \tag{1270}$$
$$C_{1n}(t) = \Re(V_{0n}(t)) \tag{1271}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1272}$$
$$C_{3n}(t) = 1 \tag{1273}$$
$$C_{4n}(t) = \Im(V_{0n}(t)) \tag{1274}$$
$$C_{5n}(t) = -\Im(V_{0n}(t)) \tag{1275}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J} \sum_{n=1} C_{jn}(t)(A_{jn} \otimes B_{jn}(t)) \tag{1276}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1248) as:

$$\widetilde{H_I}(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1277}$$

$$\widetilde{A_i}(t) = U^\dagger(t) A_i U(t) \tag{1278}$$

$$\widetilde{B_i}(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \tag{1279}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] ds \tag{1280}$$

Replacing the equation (1277) in (1280) we can obtain:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}\,(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}\,(t)\,,\left[\widetilde{H_I}\,(s)\,,\widetilde{\overline{\rho_S}}\,(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1281}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1}C_{jn}\,(t)\left(\widetilde{A_{jn}}\,(t)\otimes\widetilde{B_{jn}}\,(t)\right),\left[\sum_{j'\in J}\sum_{n'=1}C_{j'n'}\,(s)\left(\widetilde{A_{j'n'}}\,(s)\otimes\widetilde{B_{j'n'}}\,(s)\right),\widetilde{\overline{\rho_S}}\,(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1282}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1}C_{jn}\,(t)\left(\widetilde{A_{jn}}\,(t)\otimes\widetilde{B_{jn}}\,(t)\right),\sum_{j'\in J}\sum_{n'=1}C_{j'n'}\,(s)\left(\widetilde{A_{j'n'}}\,(s)\otimes\widetilde{B_{j'n'}}\,(s)\right)\widetilde{\overline{\rho_S}}\,(t)\,\rho_B \tag{1283}\right.$$

$$\left.-\widetilde{\overline{\rho_S}}\,(t)\,\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}\,(s)\left(\widetilde{A_{j'n'}}\,(s)\otimes\widetilde{B_{j'n'}}\,(s)\right)\right]\mathrm{d}s \tag{1284}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1}C_{jn}\,(t)\left(\widetilde{A_{jn}}\,(t)\otimes\widetilde{B_{jn}}\,(t)\right)\sum_{j'\in J}\sum_{n'=1}C_{j'n'}\,(s)\left(\widetilde{A_{j'n'}}\,(s)\otimes\widetilde{B_{j'n'}}\,(s)\right)\widetilde{\overline{\rho_S}}\,(t)\,\rho_B \tag{1285}\right.$$

$$-\sum_{j\in J}\sum_{n=1}C_{jn}\,(t)\left(\widetilde{A_{jn}}\,(t)\otimes\widetilde{B_{jn}}\,(t)\right)\widetilde{\overline{\rho_S}}\,(t)\,\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}\,(s)\left(\widetilde{A_{j'n'}}\,(s)\otimes\widetilde{B_{j'n'}}\,(s)\right) \tag{1286}$$

$$-\sum_{j'\in J}\sum_{n'=1}C_{j'n'}\,(s)\left(\widetilde{A_{j'n'}}\,(s)\otimes\widetilde{B_{j'n'}}\,(s)\right)\widetilde{\overline{\rho_S}}\,(t)\,\rho_B\sum_{j\in J}\sum_{n=1}C_{jn}\,(t)\left(\widetilde{A_{jn}}\,(t)\otimes\widetilde{B_{jn}}\,(t)\right) \tag{1287}$$

$$\left.+\widetilde{\overline{\rho_S}}\,(t)\,\rho_B\sum_{j'\in J}\sum_{n'=1}C_{j'n'}\,(s)\left(\widetilde{A_{j'n'}}\,(s)\otimes\widetilde{B_{j'n'}}\,(s)\right)\sum_{j\in J}\sum_{n=1}C_{jn}\,(t)\left(\widetilde{A_{jn}}\,(t)\otimes\widetilde{B_{jn}}\,(t)\right)\right)\mathrm{d}s \tag{1288}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}\,(\tau) = \left\langle\widetilde{B_{jn}}\,(t)\,(t)\,\widetilde{B_{j'n'}}\,(t)\,(s)\right\rangle_B \tag{1289}$$

$$= \left\langle\widetilde{B_{jn}}\,(\tau)\,\widetilde{B_{j'n'}}\,(0)\right\rangle_B \tag{1290}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}\,(t)\,\widetilde{B_{j'n'}}\,(s)\,\rho_B\right) = \left\langle\widetilde{B_{jn}}\,(t)\,\widetilde{B_{j'n'}}\,(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}\,(t)$ and elements related to the system given by $\widetilde{A_{jn}}\,(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}\,(t)\,\widetilde{B_{j'n'}}\,(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}\,(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}\,(t)\right)$. The correlation functions relevant of the master equation (1288) are:

$$\mathrm{Tr}_B \left(\widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \, \rho_B \right) = \left\langle \widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \right\rangle_B \tag{1291}$$

$$= \left\langle \widetilde{B_{jn}} (0) \, \widetilde{B_{j'n'}} (0) \right\rangle_B \tag{1292}$$

$$= \Lambda_{jnj'n'} (\tau) \tag{1293}$$

$$\mathrm{Tr}_B \left(\widetilde{B_{jn}} (t) \, \rho_B \, \widetilde{B_{j'n'}} (s) \right) = \mathrm{Tr}_B \left(\widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \, \rho_B \right) \tag{1294}$$

$$= \left\langle \widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \right\rangle_B \tag{1295}$$

$$= \left\langle \widetilde{B_{j'n'}} (-\tau) \, \widetilde{B_{jn}} (0) \right\rangle_B \tag{1296}$$

$$= \Lambda_{j'n'jn} (-\tau) \tag{1297}$$

$$\mathrm{Tr}_B \left(\widetilde{B_{j'n'}} (s) \, \rho_B \, \widetilde{B_{jn}} (t) \right) = \mathrm{Tr}_B \left(\widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \, \rho_B \right) \tag{1298}$$

$$= \left\langle \widetilde{B_{jn}} (t) \, \widetilde{B_{j'n'}} (s) \right\rangle_B \tag{1299}$$

$$= \left\langle \widetilde{B_{jn}} (\tau) \, \widetilde{B_{j'n'}} (0) \right\rangle_B \tag{1300}$$

$$= \Lambda_{jnj'n'} (\tau) \tag{1301}$$

$$\mathrm{Tr}_B \left(\rho_B \, \widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \right) = \mathrm{Tr}_B \left(\widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \, \rho_B \right) \tag{1302}$$

$$= \left\langle \widetilde{B_{j'n'}} (s) \, \widetilde{B_{jn}} (t) \right\rangle_B \tag{1303}$$

$$= \left\langle \widetilde{B_{j'n'}} (-\tau) \, \widetilde{B_{jn}} (0) \right\rangle_B \tag{1304}$$

$$= \Lambda_{j'n'jn} (-\tau) \tag{1305}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1281)and (1288) and using the equations (1291)-(1305) we can re-write:

$$\frac{\mathrm{d}\widetilde{\rho_S} (t)}{\mathrm{d}t} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn} (t) \, C_{j'n'} (s) \left(\Lambda_{jnj'n'} (\tau) \, \widetilde{A_{jn}} (t) \, \widetilde{A_{j'n'}} (s) \, \widetilde{\rho_S} (t) - \Lambda_{j'n'jn} (-\tau) \, \widetilde{A_{jn}} (t) \, \widetilde{\rho_S} (t) \, \widetilde{A_{j'n'}} (s) \right) \right.$$

$$\tag{1306}$$

$$\left. + C_{jn} (t) \, C_{j'n'} (s) \left(\Lambda_{j'n'jn} (-\tau) \, \widetilde{\rho_S} (t) \, \widetilde{A_{j'n'}} (s) \, \widetilde{A_{jn}} (t) - \Lambda_{jnj'n'} (\tau) \, \widetilde{A_{j'n'}} (s) \, \widetilde{\rho_S} (t) \, \widetilde{A_{jn}} (t) \right) \right) \mathrm{d}s \tag{1307}$$

$$= - \int_0^t \sum_{j,j',n,n'} \left(C_{jn} (t) \, C_{j'n'} (s) \left(\Lambda_{jnj'n'} (\tau) \left[\widetilde{A_{jn}} (t) , \widetilde{A_{j'n'}} (s) \, \widetilde{\rho_S} (t) \right] + \Lambda_{j'n'jn} (-\tau) \left[\widetilde{\rho_S} (t) \, \widetilde{A_{j'n'}} (s) , \widetilde{A_{jn}} (t) \right] \right) \right)$$

$$\tag{1308}$$

$$\frac{\mathrm{d}\overline{\rho_S} (t)}{\mathrm{d}t} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn} (t) \, C_{j'n'} (t-\tau) \left(\Lambda_{jnj'n'} (\tau) \left[A_{jn} (t) , A_{j'n'} (t-\tau,t) \overline{\rho_S} (t) \right] + \Lambda_{j'n'jn} (-\tau) \left[\overline{\rho_S} (t) A_{j'n'} (t-\tau,t) , A_{jn} (t) \right] \right) \right) \mathrm{d}\tau - i \left[H_S (t) , \overline{\rho_S} (t) \right]$$

$$\tag{1309}$$

For this case we used that $A_{jn} (t - \tau, t) = U (t) U^\dagger (t - \tau) A_{jn} (t) U (t - \tau) U^\dagger (t)$. This is a non-Markovian equation and if we take $n = 2$ (two sites), $\mu_0 (t) = 0$, $\mu_1 (t) = 1$ then we can reproduce a similar expression to (423) as expected.

VIII. BIBLIOGRAPHY

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