Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

Nike Dattani*
Harvard-Smithsonian Center for Astrophysics

Camilo Chaparro Sogamoso†
National University of Colombia

(Dated: 12th November 2018)

I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving the free energy of the system using the variational parameters optimization we consider the generalization in [5] of the Bogoliubov inequality. We consider the partition functions of $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ respect to $\overline{H_0}\left(t\right)$ as:

$$Z\left(t\right)=\left\langle \mathrm{e}^{-\beta\overline{H}\left(t\right)}\right\rangle_{\overline{H_0}\left(t\right)}, \tag{1}$$

$$\overline{H}\left(t\right)=\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right), \tag{2}$$

$$\overline{H_0}\left(t\right)=\overline{H_{\bar{S}}\left(t\right)}+\overline{H_{\bar{B}}}, \tag{3}$$

$$\begin{aligned}Z_0\left(t\right)&=\mathrm{e}^{-\beta\left\langle\overline{H}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}} &(4)\\ &=\mathrm{e}^{-\beta\left\langle\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}} &(5)\\ &=\mathrm{e}^{-\beta\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}-\beta\left\langle\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}} &(6)\\ &=\mathrm{e}^{0}\mathrm{e}^{-\beta\left\langle\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}} &(7)\\ &=\mathrm{e}^{-\beta\left\langle\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}} &(8)\end{aligned}$$

Here $\overline{H_0}\left(t\right)=\overline{H_{\bar{S}}\left(t\right)}+\overline{H_{\bar{B}}}$, also we used $\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}=0$. Taking the Quantum Bogoliubov inequality from [5]:

$$Z\left(t\right)\geq Z_0\left(t\right)\mathrm{e}^{-\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}}\left(1+F_M\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right), \tag{9}$$

$$F_N\left(\overrightarrow{u}\left(t\right)\right)=\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. \tag{10}$$

where

$$\overline{H_{\bar{I}D}}\left(t\right)=\sum_n\left\langle n\left|\overline{H_{\bar{I}}}\left(t\right)\right|n\right\rangle|n\rangle\langle n|\quad\text{(with } |n\rangle \text{ is an eigenstate of } \overline{H_0}\left(t\right)\text{)}, \tag{11}$$

$$\overline{H_0}\left(t\right)|n\rangle=E_{0,n}\left(t\right)|n\rangle, \tag{12}$$

$$Z_0\left(t\right)=\sum_n\mathrm{e}^{-\beta E_{0,n}\left(t\right)}, \tag{13}$$

$$\begin{aligned}u_k\left(t\right)&=\left\langle\left(\overline{H_{\bar{I}D}}\left(t\right)-\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}\right)^k\right\rangle_{\overline{H_0}\left(t\right)} &(14)\\ &=\frac{1}{Z_0\left(t\right)}\sum_n\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|\overline{H_{\bar{I}}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}\right)^k, &(15)\end{aligned}$$

$$v_k\left(t\right)=\frac{1}{Z_0\left(t\right)}\sum_n\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)-\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}\right)^k\right|n\right\rangle. \tag{16}$$

By construction $\left\langle \overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0\left(t\right)}} = 0$, so we arrive to:

$$Z\left(t\right) \geq Z_0\left(t\right)\left(1 + F_M\left(\overrightarrow{u}\left(t\right)\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right)\right)\right), \tag{17}$$

$$u_k\left(t\right) = \left\langle \left(\overline{H_{\overline{I}}}_D\left(t\right)\right)^k \right\rangle_{\overline{H_0\left(t\right)}}, \tag{18}$$

$$v_k\left(t\right) = \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_{\overline{I}}}\left(t\right)\right)^k \right| n \right\rangle. \tag{19}$$

As we can see the expression (17) was written in terms of the expected value of an operator, we want to do the same for (19) in order to write that expressions in a short form, following this we obtained:

$$\begin{aligned}
\left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right)\left|n\right\rangle &= \overline{H_0}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle & (20)\\
&= E_{0,n}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle & (21)\\
&= 0, & (22)\\
\left\langle n\right|\left(\overline{H_0}\left(t\right) - E_{0,n}\right) &= \left\langle n\right|\overline{H_0}\left(t\right) - \left\langle n\right| E_{0,n}\left(t\right) & (23)\\
&= \left\langle n\right| E_{0,n}\left(t\right) - \left\langle n\right| E_{0,n}\left(t\right) & (24)\\
&= 0. & (25)
\end{aligned}$$

At first we calculated $v_1\left(t\right)$ like:

$$\begin{aligned}
v_1\left(t\right) &= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_{\overline{I}}}\left(t\right)\right| n \right\rangle & (26)\\
&= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right| n \right\rangle + \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \overline{H_{\overline{I}}}\left(t\right)\right| n \right\rangle & (27)\\
&= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| E_{0,n}\left(t\right) - E_{0,n}\left(t\right)\right| n \right\rangle + \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \overline{H_{\overline{I}}}\left(t\right)\right| n \right\rangle & (28)\\
&= 0 + \left\langle \overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0\left(t\right)}} & (29)\\
&= 0. & (30)
\end{aligned}$$

For $k \geq 2$ and $k \in N$ we calculated:

$$\begin{aligned}
v_k(t) &= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^k \right| n \right\rangle & (31)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle & (32)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle & (33)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle & (34)\\
&= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \overline{H_{\overline{I}}}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_{\overline{I}}}\left(t\right)\right)^{k-2} \overline{H_{\overline{I}}}\left(t\right) \right| n \right\rangle. & (35)
\end{aligned}$$

We will obtain the explicit form of $v_2\left(t\right)$ and $v_3\left(t\right)$ using (35):

$$\begin{aligned}
v_2\left(t\right) &= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \overline{H_{\overline{I}}}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_{\overline{I}}}\left(t\right)\right)^{2-2} \overline{H_{\overline{I}}}\left(t\right) \right| n \right\rangle & (36)\\
&= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)} \left\langle n \left| \overline{H_{\overline{I}}}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_{\overline{I}}}\left(t\right)\right)^{0} \overline{H_{\overline{I}}}\left(t\right) \right| n \right\rangle & (37)
\end{aligned}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \mathbb{I} \overline{H_I}\left(t\right) \right| n \right\rangle \tag{38}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}^2\left(t\right) \right| n \right\rangle \tag{39}$$

$$= \left\langle \overline{H_I}\left(t\right)^2 \right\rangle_{\overline{H_0}(t)}, \tag{40}$$

$$v_3\left(t\right) = \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right)^{3-2} \overline{H_I}\left(t\right) \right| n \right\rangle \tag{41}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right)^{1} \overline{H_I}\left(t\right) \right| n \right\rangle \tag{42}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right) \overline{H_I}\left(t\right) \right| n \right\rangle, \tag{43}$$

$$\overline{H_0}\left(t\right) \left|n\right\rangle = E_{0,n}\left(t\right) \left|n\right\rangle, \tag{44}$$

$$\left\langle n\right| \overline{H_0}\left(t\right) = \left\langle n\right| E_{0,n}\left(t\right), \tag{45}$$

$$v_3\left(t\right) = \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right) \overline{H_I}\left(t\right) \right| n \right\rangle \tag{46}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right)\ \overline{H_0}\left(t\right)\ \overline{H_I}\left(t\right) - \overline{H_I}\left(t\right) E_{0,n}\left(t\right) \overline{H_I}\left(t\right) + \overline{H_I}\left(t\right)\ \overline{H_I}\left(t\right)\ \overline{H_I}\left(t\right) \right| n \right\rangle \tag{47}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right)\ \overline{H_0}\left(t\right)\ \overline{H_I}\left(t\right) + \overline{H_I}^3\left(t\right) - \overline{H_I}\left(t\right) E_{0,n}\left(t\right) \overline{H_I}\left(t\right) \right| n \right\rangle \tag{48}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right)\ \overline{H_0}\left(t\right)\ \overline{H_I}\left(t\right) + \overline{H_I}^3\left(t\right) - \overline{H_I}\left(t\right)\ \overline{H_I}\left(t\right) E_{0,n}\left(t\right) \right| n \right\rangle \tag{49}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right)\ \overline{H_0}\left(t\right)\ \overline{H_I}\left(t\right) + \overline{H_I}^3\left(t\right) - \overline{H_I}\left(t\right)\ \overline{H_I}\left(t\right)\ \overline{H_0}\left(t\right) \right| n \right\rangle \tag{50}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}^3\left(t\right) + \overline{H_I}\left(t\right) \left(\overline{H_0}\left(t\right)\ \overline{H_I}\left(t\right) - \overline{H_I}\left(t\right)\ \overline{H_0}\left(t\right)\right) \right| n \right\rangle \tag{51}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}^3\left(t\right) + \overline{H_I}\left(t\right) \left[\overline{H_0}\left(t\right), \overline{H_I}\left(t\right)\right] \right| n \right\rangle \tag{52}$$

$$= \left\langle \overline{H_I}^3\left(t\right) + \overline{H_I}\left(t\right) \left[\overline{H_0}\left(t\right), \overline{H_I}\left(t\right)\right] \right\rangle_{\overline{H_0}(t)}. \tag{53}$$

In general we have:

$$v_k\left(t\right) = \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right)^{k-2} \overline{H_I}\left(t\right) \right| n \right\rangle \tag{54}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\overline{H_0}\left(t\right) + \overline{H_I}\left(t\right) - E_{0,n}\left(t\right)\right)^{k-2} \overline{H_I}\left(t\right) \right| n \right\rangle \tag{55}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\overline{H}\left(t\right) - E_{0,n}\left(t\right)\right)^{k-2} \overline{H_I}\left(t\right) \right| n \right\rangle \tag{56}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}\left(t\right) \left(\sum_j^{k-2} \left(-1\right)^j \binom{k-2}{j} \overline{H}^{k-2-j}\left(t\right) E_{0,n}^j\left(t\right)\right) \overline{H_I}\left(t\right) \right| n \right\rangle \tag{57}$$

$$= \frac{1}{Z_0\left(t\right)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} \left(-1\right)^j \binom{k-2}{j} \left\langle n \left| \overline{H_I}\left(t\right) \overline{H}^{k-2-j}\left(t\right) \overline{H_I}\left(t\right) E_{0,n}^j\left(t\right) \right| n \right\rangle \tag{58}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{59}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{60}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t)\ \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{61}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{62}$$

The formula (62) is well defined taking as example $k = 2, 3$.

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{2-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \tag{63}$$

$$= (-1)^0 \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\ \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \tag{64}$$

$$= \left\langle \overline{H_{\overline{I}}}^2(t) \right\rangle_{\overline{H_0}(t)}, \tag{65}$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{3-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{66}$$

$$= \left\langle \sum_{j=0}^{1} (-1)^j \binom{1}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{1-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{67}$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^1 \overline{H_{\overline{I}}}(t)\, \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\, \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \tag{68}$$

$$= \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right) \overline{H_{\overline{I}}}(t)\ \mathbb{I} - \overline{H_{\overline{I}}}(t)\, \mathbb{I} \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{69}$$

$$= \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right) \overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\ \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{70}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t)\ \overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\ \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{71}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t)\ \overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t)\right) \right\rangle_{\overline{H_0}(t)} \tag{72}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t) \left[\overline{H_0}(t), \overline{H_{\overline{I}}}(t)\right] \right\rangle_{\overline{H_0}(t)}. \tag{73}$$

So we summarize:

$$\overline{H_{\overline{I}D}}(t) = \sum_n \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n|, \tag{74}$$

$$u_k(t) = \left\langle \left(\overline{H_{\overline{I}D}}(t)\right)^k \right\rangle_{\overline{H_0}(t)}, \tag{75}$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{76}$$

Then we obtained finally:

$$Z(t) \geq Z_0(t) \left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right), \tag{77}$$

The free energy is defined as:

$$E_{\text{free}}\left(t\right)=-\frac{1}{\beta}\ln\left(Z\left(t\right)\right). \tag{78}$$

It is well known that the function $f\left(x\right)=\ln\left(x\right)$ is monotonic and increasing so we can transform (77):

$$\begin{aligned} E_{\text{free},1}\left(t\right) &= -\frac{1}{\beta}\ln\left(Z_0\left(t\right)\right), && (79)\\ E_{\text{free}}\left(t\right) &\leq E_{\text{free},1}\left(t\right)-\frac{1}{\beta}\ln\left(1+F_M\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right) && (80)\\ &\equiv E_{\text{free,MN}}\left(t\right). && (81)\end{aligned}$$

here $E_{\text{free,MN}}\left(t\right)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of MN order. In our approach we will set $N=M$, so our quantum Bogoliubov inequality of N order is:

$$\begin{aligned} E_{\text{free}}\left(t\right) &\leq E_{\text{free},1}\left(t\right)-\frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right) && (82)\\ &= E_{\text{free,NN}}\left(t\right). && (83)\end{aligned}$$

A weaker form of the inequality (83) making $\overrightarrow{u}\left(t\right)=0$ is:

$$\begin{aligned} E_{\text{free}}\left(t\right) &\leq E_{\text{free},1}\left(t\right)-\frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{v}\left(t\right)\right)\right) && (84)\\ &\equiv E_{\text{free,N}}\left(t\right). && (85)\end{aligned}$$

The algebraic equation associated with $\alpha_{\text{opt}}\left(t\right)$ such that $E_{\text{free,N}}\left(t\right)$ is closer to $E_{\text{free}}\left(t\right)$ is given by the following expression:

$$\begin{aligned} G\left(\alpha_{\text{opt}}\left(t\right)\right) &= \sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\frac{\alpha_{\text{opt}}^{2N-1-k}}{\left(2N-1-k\right)!} && (86)\\ &= 0. && (87)\end{aligned}$$

The elements presented are the required to find variational parameters through the determination of the SCE (self consistent equations) of the system in particular to the order expected.

II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2\left(t\right), v_3\left(t\right), v_4\left(t\right), v_5\left(t\right)$ using the (76), we have already $v_2\left(t\right), v_3\left(t\right)$ and the form of $v_4\left(t\right)$ and $v_5\left(t\right)$ is given by:

$$\begin{aligned} v_4\left(t\right) &= \sum_{j=0}^{4-2}\left(-1\right)^j\binom{4-2}{j}\left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^{4-2-j}\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^{j}\left(t\right)\right\rangle_{\overline{H_0}(t)} && (88)\\ &= \sum_{j=0}^{2}\left(-1\right)^j\binom{2}{j}\left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^{2-j}\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^{j}\left(t\right)\right\rangle_{\overline{H_0}(t)} && (89)\\ &= \left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}(t)+\overline{H_0}(t)\right)^2\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^{0}(t)\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}(t)+\overline{H_0}(t)\right)^1\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^{1}(t)\right\rangle_{\overline{H_0}(t)}+\langle\overline{H_{\overline{I}}}\left(t\right)\big(\overline{H_{\overline{I}}}\left(t\right) && (90)\\ &\quad+\overline{H_0}\left(t\right)\big)^0\,\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^{2}\left(t\right)\Big\rangle_{\overline{H_0}(t)} && (91)\\ &= \left\langle\overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}(t)\right)^2\overline{H_{\overline{I}}}\left(t\right)\mathbb{I}\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^1\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^{1}(t)\right\rangle_{\overline{H_0}(t)}+\left\langle\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}^{2}(t)\right\rangle_{\overline{H_0}(t)} && (92)\\ &= \left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^2\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_{\overline{I}}}\left(t\right)\,\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)}+\left\langle\overline{H_{\overline{I}}}^{2}\left(t\right)\overline{H_0}^{2}\left(t\right)\right\rangle_{\overline{H_0}(t)} && (93)\end{aligned}$$

$$
\begin{aligned}
&=\left\langle \overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)-2\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\right\rangle_{\overline{H_0}(t)} && (94)\\
&=\Big\langle \overline{H_I}(t)\left(\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^2(t)\right)\overline{H_I}(t)-2\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t) && (95)\\
&\quad+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\Big\rangle_{\overline{H_0}(t)} && (96)\\
&=\Big\langle \overline{H_I}^4(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)-2\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t) && (97)\\
&\quad+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\Big\rangle_{\overline{H_0}(t)} && (98)\\
&=\Big\langle \overline{H_I}^4(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)-2\overline{H_I}^3(t)\,\overline{H_0}(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t) && (99)\\
&\quad-2\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\Big\rangle_{\overline{H_0}(t)} && (100)\\
&=\Big\langle \overline{H_I}^4(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)-\overline{H_I}^3(t)\,\overline{H_0}(t) && (101)\\
&\quad+\overline{H_I}^2(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\Big\rangle_{\overline{H_0}(t)} && (102)\\
&=\Big\langle \overline{H_I}^4(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t) && (103)\\
&\quad-\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\Big\rangle_{\overline{H_0}(t)} && (104)\\
&=\Big\langle \overline{H_I}^4(t)+\overline{H_I}(t)\Big(\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}(t)\right]+\left[\overline{H_0}(t),\overline{H_I}^2(t)\right]+\left[\overline{H_0}(t),\overline{H_0}(t)\overline{H_I}(t)\right]+\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\right]\Big)\Big\rangle_{\overline{H_0}(t)}, && (105)\\
v_5(t)&=\sum_{j=0}^{5-2}(-1)^j\binom{5-2}{j}\left\langle \overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^{5-2-j}\overline{H_I}(t)\ \overline{H_0}(t)^j\right\rangle_{\overline{H_0}(t)} && (106)\\
&=\sum_{j=0}^{3}(-1)^j\binom{3}{j}\left\langle \overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^{3-j}\overline{H_I}(t)\ \overline{H_0}(t)^j\right\rangle_{\overline{H_0}(t)} && (107)\\
&=\Big\langle \overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)\overline{H_0}^0(t)-3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\overline{H_0}(t)-\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^0\overline{H_I}(t)\overline{H_0}^3(t) && (108)\\
&\quad+3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^2(t)\Big\rangle_{\overline{H_0}(t)} && (109)\\
&=\Big\langle \overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)-3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^2(t) && (110)\\
&\quad-\overline{H_I}(t)\,\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} && (111)\\
&=\Big\langle \overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)-3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^2(t) && (112)\\
&\quad-\overline{H_I}(t)\,\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} && (113)\\
&=\Big\langle \overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)-3\overline{H_I}(t)\left(\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^2(t)\right)\overline{H_I}(t)\,\overline{H_0}(t) && (114)\\
&\quad+3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} && (115)\\
&=\Big\langle \overline{H_I}(t)\Big(\overline{H_I}^3(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t) && (116)\\
&\quad+\overline{H_I}(t)\,\overline{H_0}^2(t)+\overline{H_0}^3(t)\Big)\overline{H_I}(t)-3\overline{H_I}(t)\left(\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^2(t)\right)\overline{H_I}(t)\,\overline{H_0}(t) && (117)\\
&\quad+3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} && (118)\\
&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}^3(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^3(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t) && (119)\\
&\quad+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}^2(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}^4(t)\,\overline{H_0}(t)-3\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t) && (120)\\
&\quad-3\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)-3\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}^3(t)\,\overline{H_0}^2(t)+3\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}^2(t) && (121)\\
&\quad-\overline{H_I}^2(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} && (122)\\
&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_0}(t)\,\overline{H_I}^3(t)+\overline{H_0}^2(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t) && (123)
\end{aligned}
$$

$$+\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)+\overline{H_0}^3(t)\overline{H_I}(t)-3\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)-3\overline{H_I}^3(t)\overline{H_0}(t)-3\overline{H_0}^2(t)\overline{H_I}(t)\overline{H_0}(t) \quad (124)$$

$$-\overline{H_I}(t)\overline{H_0}^3(t)+3\overline{H_I}^2(t)\overline{H_0}^2(t)+3\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}^2(t)-3\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)\Big)\Big\rangle_{\overline{H_0}(t)} \quad (125)$$

$$=\Big\langle\overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}^3(t) \quad (126)$$

$$-\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}(t)-\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t) \quad (127)$$

$$+\overline{H_0}^3(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}^3(t)+\overline{H_I}^2(t)\overline{H_0}^2(t)-\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)+2\overline{H_I}^2(t)\overline{H_0}^2(t)-2\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t) \quad (128)$$

$$+3\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}^2(t)-3\overline{H_0}^2(t)\overline{H_I}(t)\overline{H_0}(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_0}^2(t)\overline{H_I}^2(t)\Big)\Big\rangle_{\overline{H_0}(t)} \quad (129)$$

$$=\Big\langle\overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\overline{H_0}(t),\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t),\overline{H_I}^3(t)\Big]+\Big[\overline{H_0}(t),\overline{H_0}(t)\overline{H_I}^2(t)\Big] \quad (130)$$

$$+\left[\overline{H_0}(t),\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\right]+\Big[\overline{H_0}^3(t),\overline{H_I}(t)\Big]+\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\overline{H_I}(t)\right]+3\overline{H_0}(t)\left[\overline{H_I}(t),\overline{H_0}(t)\right]\overline{H_0}(t) \quad (131)$$

$$+2\overline{H_I}(t)\left[\overline{H_I}(t),\overline{H_0}(t)\right]\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\overline{H_0}(t),\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}. \quad (132)$$

Summarizing we have that:

$$v_2(t)=\Big\langle\overline{H_I}^2(t)\Big\rangle_{\overline{H_0}(t)}, \quad (133)$$

$$v_3(t)=\Big\langle\overline{H_I}^3(t)+\overline{H_I}(t)\left[\overline{H_0}(t),\overline{H_I}(t)\right]\Big\rangle_{\overline{H_0}(t)}, \quad (134)$$

$$v_4(t)=\Big\langle\overline{H_I}^4(t)+\overline{H_I}(t)\Big(\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}(t)\right]+\Big[\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\left[\overline{H_0}(t),\overline{H_0}(t)\overline{H_I}(t)\right]+\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\right]\Big)\Big\rangle_{\overline{H_0}(t)}, \quad (135)$$

$$v_5(t)=\Big\langle\overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\overline{H_0}(t),\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t),\overline{H_I}^3(t)\Big]+\Big[\overline{H_0}(t),\overline{H_0}(t)\overline{H_I}^2(t)\Big] \quad (136)$$

$$+\left[\overline{H_0}(t),\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\right]+\Big[\overline{H_0}^3(t),\overline{H_I}(t)\Big]+\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\overline{H_I}(t)\right]+3\overline{H_0}(t)\left[\overline{H_I}(t),\overline{H_0}(t)\right]\overline{H_0}(t) \quad (137)$$

$$+2\overline{H_I}(t)\left[\overline{H_I}(t),\overline{H_0}(t)\right]\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\overline{H_0}(t),\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}. \quad (138)$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. We recall that the expected value can be calculated as:

$$\langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)}=\frac{\mathrm{Tr}\left(\overline{H_I}(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}. \quad (139)$$

Recall the hamiltonian of $\overline{H_I}(t)$ then the explicit form of $\overline{H_I}^2(t)$ is:

$$\overline{H_I}^2(t)=\sum_i B_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i B_{iz}(t)|i\rangle\langle i|(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)\sum_i B_{iz}(t)|i\rangle\langle i|(\sigma_x B_y(t)-\sigma_y B_x(t)) \quad (140)$$

$$+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)|i\rangle\langle i|+\left(V_{10}^{\Re}(t)\right)^2(\sigma_x B_x(t)+\sigma_y B_y(t))^2+V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x B_x(t) \quad (141)$$

$$+\sigma_y B_y(t))(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x B_y(t) \quad (142)$$

$$-\sigma_y B_x(t))(\sigma_x B_x(t)+\sigma_y B_y(t))+\left(V_{10}^{\Im}(t)\right)^2(\sigma_x B_y(t)-\sigma_y B_x(t))^2 \quad (143)$$

$$=\sum_i B_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x \quad (144)$$

$$-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2(\sigma_x^2B_x^2(t)+\sigma_x\sigma_yB_x(t)B_y(t) \tag{145}$$

$$+\sigma_y\sigma_xB_y(t)B_x(t)+\sigma_y^2B_y^2(t))+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2(\sigma_x^2B_y^2(t)+\sigma_y^2B_x^2(t) \tag{146}$$

$$-\sigma_x\sigma_yB_y(t)B_x(t)-\sigma_y\sigma_xB_x(t)B_y(t))+V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x^2B_y(t)B_x(t)+\sigma_x\sigma_yB_y^2(t)-\sigma_y\sigma_xB_x^2(t)-\sigma_y^2B_x(t)B_y(t) \tag{147}$$

$$+\sigma_x^2B_x(t)B_y(t)-\sigma_x\sigma_yB_x^2(t)+\sigma_y\sigma_xB_y^2(t)-\sigma_y^2B_y(t)B_x(t)), \tag{148}$$

$$\sigma_x\sigma_y=-\mathrm{i}\sigma_z, \tag{149}$$

$$\overline{H_{\bar{I}}}^2(t)=\sum_i B_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x \tag{150}$$

$$-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2(B_x^2(t)-\mathrm{i}\sigma_zB_x(t)B_y(t) \tag{151}$$

$$+\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t))+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2(B_y^2(t)+B_x^2(t) \tag{152}$$

$$+\mathrm{i}\sigma_zB_y(t)B_x(t)-\mathrm{i}\sigma_zB_x(t)B_y(t)). \tag{153}$$

In order to obtain the expected values of $\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_0(t)}}$ respect to the part related to the bath we need to calculate the following expected values $\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x^2(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_y^2(t)\rangle_{\overline{H_{\bar{B}}}}$:

$$\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\left(\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{154}$$

$$=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)^2+\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\sum_{\mathbf{k}'}((g_{i\mathbf{k}'}\right. \tag{155}$$

$$\left.-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^\dagger+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*b_{\mathbf{k}'}\right)\rangle_{\overline{H_{\bar{B}}}} \tag{156}$$

$$=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{157}$$

$$\times\left\langle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^\dagger+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*b_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{158}$$

$$=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{159}$$

$$=\sum_{\mathbf{k}}\left\langle(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^2\left(b_{\mathbf{k}}^\dagger\right)^2+|g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)|^2\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right)+((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*)^2b_{\mathbf{k}}^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{160}$$

$$=\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^2\left\langle\left(b_{\mathbf{k}}^\dagger\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}|g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)|^2\left\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left\langle((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*)^2b_{\mathbf{k}}^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{161}$$

$$\left\langle\left(b_{\mathbf{k}}^\dagger\right)^2\right\rangle_{\overline{H_{\bar{B}}}}=\frac{\mathrm{Tr}\left(\left(b_{\mathbf{k}}^\dagger\right)^2\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \tag{162}$$

$$=\frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger\right)^2|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \tag{163}$$

$$=\frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\sqrt{(j_{\mathbf{k}}+2)(j_{\mathbf{k}}+1)}|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \tag{164}$$

$$= 0, \tag{165}$$

$$\langle b_{\mathbf{k}}^2 \rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} |j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{166}$$

$$=0, \tag{167}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{168}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{169}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{170}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \tag{171}$$

$$= \frac{1+\mathrm{e}^{-\beta\omega_{\mathbf{k}}}}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \tag{172}$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{173}$$

$$\langle B_{iz}^2(t) \rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{174}$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right) \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{175}$$

$$= \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right) \left(\mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + \mathrm{e}^{\chi_{01}(t)} \right.\right. \tag{176}$$

$$\left.\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{177}$$

$$\left\langle b^{\dagger} D(h) \right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle \alpha | b^{\dagger} D(h) | \alpha \rangle \mathrm{d}^2\alpha \tag{178}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) b^{\dagger} D(\alpha) D(-\alpha) D(h) D(\alpha) | 0 \rangle \mathrm{d}^2\alpha \tag{179}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) b^{\dagger} D(\alpha) D(h) \mathrm{e}^{h\alpha^* - h^*\alpha} | 0 \rangle \mathrm{d}^2\alpha \tag{180}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | \left(b^{\dagger} + \alpha^*\right) D(h) \mathrm{e}^{h\alpha^* - h^*\alpha} | 0 \rangle \mathrm{d}^2\alpha \tag{181}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \langle 0 | \left(b^{\dagger} + \alpha^*\right) | h \rangle \mathrm{d}^2\alpha \tag{182}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \langle 0 | \left(b^{\dagger} + \alpha^*\right) | h \rangle \mathrm{d}^2\alpha, \tag{183}$$

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{184}$$

$$\left\langle b^{\dagger} D(h) \right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \left(\langle 0 | b^{\dagger} | h \rangle + \alpha^* \langle 0 | h \rangle\right) \mathrm{d}^2\alpha \tag{185}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \left(\langle 0 | b^{\dagger} \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle + \alpha^* \langle 0 | h \rangle\right) \mathrm{d}^2\alpha \tag{186}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \left(\langle 0 | \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1} |n+1\rangle + \alpha^* \langle 0 | h \rangle\right) \mathrm{d}^2\alpha \tag{187}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 | h \rangle \mathrm{d}^2\alpha \tag{188}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{189}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(x-\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \tag{190}$$

$$= -h^*N\left(\langle D\left(h\right)\rangle_{\overline{H_{\bar{B}}}}\right)^2 \tag{191}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right.\right. \tag{192}$$

$$\left.\left.+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{193}$$

$$= \frac{1}{2}\left\langle\mathrm{e}^{\chi_{10}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right. \tag{194}$$

$$\left.+\mathrm{e}^{\chi_{01}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{195}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left\langle b_{\mathbf{k}}^\dagger\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\right. \tag{196}$$

$$\left.\times\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{197}$$

$$+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^\dagger\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\right. \tag{198}$$

$$\left.\times\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{199}$$

$$= \frac{B_{10}\left(t\right)}{2}\left(-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{200}$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{201}$$

$$+\frac{B_{01}\left(t\right)}{2}\left(-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{202}$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{203}$$

$$= \frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{204}$$

$$\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{205}$$

$$\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{206}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{207}$$

$$= \frac{B_{10}(t)}{2\mathrm{i}} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} - \sum_{\mathbf{k}} (g_{i\mathbf{k}} \right. \tag{208}$$

$$\left. - v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \mathrm{e}^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{209}$$

$$+ \frac{B_{01}(t)}{2\mathrm{i}} \left(- \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \tag{210}$$

$$\left. + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \mathrm{e}^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{211}$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2\mathrm{i}} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \tag{212}$$

$$\left. - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \mathrm{e}^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right), \tag{213}$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} = \langle B_x(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} \tag{214}$$

$$= \left\langle \frac{B_1^{+}(t) B_0^{-}(t) + B_0^{+}(t) B_1^{-}(t) - B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{215}$$

$$= \frac{1}{2} \left\langle \left(B_1^{+}(t) B_0^{-}(t) + B_0^{+}(t) B_1^{-}(t) \right) \left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{216}$$

$$= \frac{1}{2} \left\langle \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}'} D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{217}$$

$$+ \frac{1}{2} \left\langle \mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}'} D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}}, \tag{218}$$

$$\langle D(h) b \rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle \alpha | D(h) b | \alpha \rangle \mathrm{d}^2\alpha \tag{219}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \mathrm{d}^2\alpha \tag{220}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) \mathrm{e}^{h\alpha^* - h^*\alpha} (b + \alpha) | 0 \rangle \mathrm{d}^2\alpha \tag{221}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha | 0 \rangle \mathrm{d}^2\alpha \tag{222}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha \langle 0 | D(h) | 0 \rangle \mathrm{d}^2\alpha \tag{223}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha \langle 0 | h \rangle \mathrm{d}^2\alpha \tag{224}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N} \int \alpha \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \mathrm{d}^2\alpha \tag{225}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2+y^2}{N}} \mathrm{e}^{h(x - \mathrm{i}y) - h^*(x + \mathrm{i}y)} (x + \mathrm{i}y) \mathrm{d}x\mathrm{d}y \tag{226}$$

$$= Nh\mathrm{e}^{-|h|^2\coth\left(\frac{\beta\omega}{2}\right)} \tag{227}$$

$$= Nh\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^2, \tag{228}$$

$$\left\langle D\left(h\right)b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha|D\left(h\right)b^\dagger|\alpha\right\rangle\mathrm{d}^2\alpha \tag{229}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)D\left(-\alpha\right)b^\dagger D\left(\alpha\right)|0\right\rangle\mathrm{d}^2\alpha \tag{230}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0|D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\left(b^\dagger+\alpha^*\right)|0\right\rangle\mathrm{d}^2\alpha \tag{231}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0|D\left(h\right)b^\dagger|0\right\rangle\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0|D\left(h\right)|0\right\rangle\mathrm{d}^2\alpha \tag{232}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h|1\right\rangle\mathrm{d}^2\alpha + \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha, \tag{233}$$

$$\left\langle\alpha\right| = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\left(\alpha^*\right)^n}{\sqrt{n!}}\left\langle n\right|, \tag{234}$$

$$\left\langle D\left(h\right)b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\left(-h^*\right)\mathrm{d}^2\alpha + \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha \tag{235}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left(-h^*+\alpha^*\right)\mathrm{d}^2\alpha \tag{236}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(-h^*+x-\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \tag{237}$$

$$= -\left(N+1\right)h^*\mathrm{e}^{-|h|^2\coth\left(\frac{\beta\omega}{2}\right)}, \tag{238}$$

$$= -\left(N+1\right)h^*\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^2, \tag{239}$$

$$\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{240}$$

$$\left\langle B_x\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{241}$$

$$+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{242}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right.\right.\right.\right. \tag{243}$$

$$\left.\left.\left.\left.-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\right.\right. \tag{244}$$

$$\left.\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{245}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right.\right.\right.\right. \tag{246}$$

$$\left.\left.\left.\left.-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}\right. \tag{247}$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{248}$$

$$= \frac{B_{10}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*\right.\right. \tag{249}$$

$$+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right)+\frac{B_{01}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}\right.\right.\right. \tag{250}$$

$$\left.-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{251}$$

$$=\frac{B_{10}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+\left(g_{i\mathbf{k}}\right.\right.\right. \tag{252}$$

$$\left.-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right)+\frac{B_{01}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\right.\right. \tag{253}$$

$$\times\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{254}$$

$$=\frac{B_{01}(t)-B_{10}(t)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))(N_{\mathbf{k}}+1)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right. \tag{255}$$

$$\left.\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{256}$$

$$\left\langle D\left(h\right)b\right\rangle_{\overline{H_{\bar{B}}}}=Nh\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{257}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}=-\left(N+1\right)h^{*}\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{258}$$

$$\left\langle B_{y}(t)B_{iz}(t)\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle B_{y}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{259}$$

$$=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{260}$$

$$=\frac{1}{2\mathrm{i}}\left\langle\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{261}$$

$$=\frac{1}{2\mathrm{i}}\left\langle\left(B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{262}$$

$$=\frac{B_{10}(t)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))(N_{\mathbf{k}}+1)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right.\right. \tag{263}$$

$$\left.\left.\times N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right)+\frac{B_{01}\left(t\right)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\right.\right. \tag{264}$$

$$\left.\left.\times\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{265}$$

$$=\frac{B_{10}\left(t\right)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))(N_{\mathbf{k}}+1)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right.\right. \tag{266}$$

$$\left.\left.\times N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right)+\frac{B_{01}\left(t\right)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\right.\right. \tag{267}$$

$$\left.\left.\times\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{268}$$

$$=\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right. \tag{269}$$

$$\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right) \tag{270}$$

$$\left\langle B_{x}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_{x}\left(t\right)\right)+\left\langle B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2} \tag{271}$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)-B_{10}(t)-B_{01}(t)}{2}\right) \tag{272}$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)-B_{10}(t)-B_{01}(t)\right) \tag{273}$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right) \tag{274}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{275}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2+B_1^+(t)B_0^-(t)B_0^+(t)B_1^-(t)+B_0^+(t)B_1^-(t)B_1^+(t)B_0^-(t)+\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right. \tag{276}$$

$$\left.-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{277}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right), \tag{278}$$

$$(D(h))^2 = D(h)\,D(h) \tag{279}$$

$$= D(h+h)\,\mathrm{e}^{\frac{1}{2}\left(\frac{h^*h-hh^*}{\omega^2}\right)} \tag{280}$$

$$= D(2h)\,, \tag{281}$$

$$\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{282}$$

$$= \left\langle\prod_{\mathbf{k}}D\left(2\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{283}$$

$$= \prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\,\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{284}$$

$$\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{285}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{286}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{287}$$

$$= \frac{1}{4}\left(\mathrm{e}^{2\chi_{10}(t)}\prod_{\mathbf{k}}\mathrm{e}^{-2\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}+2+\mathrm{e}^{2\chi_{01}(t)}\prod_{\mathbf{k}}\mathrm{e}^{-2\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{288}$$

$$\left.-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{289}$$

$$= \frac{1}{4}\left(B_{10}^2(t)\left|B_{10}^2(t)\right|+2+B_{01}^2(t)\left|B_{01}^2(t)\right|-\left(B_{10}^2(t)+2B_{10}(t)\,B_{01}(t)+B_{01}^2(t)\right)\right) \tag{290}$$

$$\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_y(t)\right)+\left\langle B_y(t)\right\rangle_{\overline{H_{\bar{B}}}}^2 \tag{291}$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right) \tag{292}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)+B_{10}(t)-B_{01}(t)\right) \tag{293}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right) \tag{294}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right)^2-\left(B_{01}(t)-B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{295}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2-2\mathbb{I}+\left(B_1^+(t)\,B_0^-(t)\right)^2-\left(B_{01}(t)-B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{296}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-2-\left(B_{01}(t)-B_{10}(t)\right)^2\right), \tag{297}$$

$$\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} = \prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\,\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{298}$$

$$
\begin{aligned}
&= \prod_{\mathbf{k}} \mathrm{e}^{\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}} \mathrm{e}^{-\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} \mathrm{e}^{-\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (299)\\
&= B_{ij}^{2}(t)\, B_{ij}(t)\, B_{ji}(t) && (300)\\
&= B_{ij}^{2}(t)\left|B_{ij}(t)\right|^{2}, && (301)\\
\left\langle B_{y}^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} &= -\frac{1}{4}\left(B_{01}^{2}(t)\left|B_{10}(t)\right|^{2}-2+B_{10}^{2}(t)\left|B_{10}(t)\right|^{2}-\left(B_{01}(t)-B_{10}(t)\right)^{2}\right), && (302)\\
\left\langle B_{x}(t)\, B_{y}(t)\right\rangle_{\overline{H_{\bar{B}}}} &= \left\langle \frac{B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\,\frac{B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} && (303)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)\right)\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (304)\\
&= \frac{1}{4\mathrm{i}}\left\langle \mathbb{I}-\left(B_{1}^{+}(t)B_{0}^{-}(t)\right)^{2}+B_{10}^{2}(t)-B_{10}(t)B_{01}(t)+\left(B_{0}^{+}(t)B_{1}^{-}(t)\right)^{2}-\mathbb{I}+B_{10}(t)B_{01}(t)-B_{01}^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} && (305)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_{0}^{+}(t)\,B_{1}^{-}(t)\right)^{2}-\left(B_{1}^{+}(t)\,B_{0}^{-}(t)\right)^{2}-\left(B_{01}^{2}(t)-B_{10}^{2}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (306)\\
&= \frac{1}{4\mathrm{i}}\left(B_{01}^{2}(t)\left|B_{10}(t)\right|^{2}-B_{10}^{2}(t)\left|B_{10}(t)\right|^{2}-\left(B_{01}^{2}(t)-B_{10}^{2}(t)\right)\right), && (307)\\
\left\langle B_{y}(t)B_{x}(t)\right\rangle_{\overline{H_{\bar{B}}}} &= \left\langle \frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\,\frac{B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} && (308)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)\right)\left(B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (309)\\
&= \frac{1}{4\mathrm{i}}\left\langle \mathbb{I}+\left(B_{0}^{+}(t)B_{1}^{-}(t)\right)^{2}-B_{10}(t)B_{01}(t)-B_{01}^{2}(t)-\left(B_{1}^{+}(t)B_{0}^{-}(t)\right)^{2}-\mathbb{I}+B_{10}^{2}(t)+B_{10}(t)B_{01}(t)\right\rangle_{\overline{H_{\bar{B}}}} && (310)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_{0}^{+}(t)\,B_{1}^{-}(t)\right)^{2}-B_{01}^{2}(t)-\left(B_{1}^{+}(t)\,B_{0}^{-}(t)\right)^{2}+B_{10}^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} && (311)\\
&= \frac{1}{4\mathrm{i}}\left(B_{01}^{2}(t)\left|B_{10}(t)\right|^{2}-B_{01}^{2}(t)-\left(B_{10}^{2}(t)\left|B_{10}(t)\right|^{2}-B_{10}^{2}(t)\right)\right). && (312)
\end{aligned}
$$

The density matrix associated to $\rho_{\overline{S}}=\frac{\mathrm{e}^{-\beta\overline{H_0}(t)}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}$ follows the form:

$$
\rho_{\overline{S},00}=\frac{1}{2}+\frac{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}} \tag{313}
$$

$$
\rho_{\overline{S},01}=-\frac{B_{10}^{*}(t)\,V_{10}^{*}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}} \tag{314}
$$

$$
\rho_{\overline{S},10}=-\frac{B_{10}(t)\,V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}} \tag{315}
$$

$$
\rho_{\overline{S},11}=\frac{1}{2}-\frac{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}} \tag{316}
$$

The expected values respect to the system $\overline{S}$ of relevance for calculating $\left\langle\overline{H_{\bar{I}}}^{2}(t)\right\rangle_{H_{\overline{S}}}$ are $\langle|i\rangle\langle i|\rangle_{H_{\overline{S}}}$, $\langle|i\rangle\langle i|\sigma_x\rangle_{H_{\overline{S}}}$, $\langle|i\rangle\langle i|\sigma_y\rangle_{H_{\overline{S}}}$, $\langle\sigma_x|i\rangle\langle i|\rangle_{H_{\overline{S}}}$, $\langle\sigma_y|i\rangle\langle i|\rangle_{H_{\overline{S}}}$ and $\langle\sigma_z\rangle_{H_{\overline{S}}}$, we took account that $\sigma_x\sigma_y=-\mathrm{i}\sigma_z$ and $\sigma_y\sigma_x=\mathrm{i}\sigma_z$. The values needed for our calculation are:

$$\left\langle |0\rangle\langle 0|\right\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{1}{2} - \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{317}$$

$$\left\langle |1\rangle\langle 1|\right\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{1}{2} + \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{318}$$

$$\left\langle |0\rangle\langle 0|\sigma_x\right\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{319}$$

$$\left\langle |1\rangle\langle 1|\sigma_x\right\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}^*(t) V_{10}^*(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{320}$$

$$\left\langle |0\rangle\langle 0|\sigma_y\right\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\mathrm{i}B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{321}$$

$$\left\langle |1\rangle\langle 1|\sigma_y\right\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^*(t) V_{10}^*(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2}}, \tag{322}$$

$$\left\langle \sigma_x|0\rangle\langle 0|\right\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}^*(t) V_{10}^*(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{323}$$

$$\left\langle \sigma_x|1\rangle\langle 1|\right\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{324}$$

$$\left\langle \sigma_y|0\rangle\langle 0|\right\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^*(t) V_{10}^*(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{325}$$

$$\left\langle \sigma_y|1\rangle\langle 1|\right\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\mathrm{i}B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{326}$$

$$\left\langle \sigma_z\right\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}. \tag{327}$$

Summarizing the expected values of the bath we have:

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\sum_{\mathbf{k}}\left|g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right),\tag{328}$$

$$\left\langle B_{iz}\left(t\right)B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right.\right.\tag{329}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\right)\right),\tag{330}$$

$$\left\langle B_{iz}\left(t\right)B_{y}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right.\right.\tag{331}$$

$$\left.\left.-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\right)\right),\tag{332}$$

$$\left\langle B_{x}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{01}\left(t\right)-B_{10}\left(t\right)}{2}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right.\tag{333}$$

$$\left.\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right),\tag{334}$$

$$\left\langle B_{y}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right.\tag{335}$$

$$\left.\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right),\tag{336}$$

$$\left\langle B_{x}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4}\left(B_{10}^{2}\left(t\right)\left|B_{10}^{2}\left(t\right)\right|+2+B_{01}^{2}\left(t\right)\left|B_{01}^{2}\left(t\right)\right|-\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right)^{2}\right),\tag{337}$$

$$\left\langle B_{y}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=-\frac{1}{4}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-2+B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}\left(t\right)-B_{10}\left(t\right)\right)^{2}\right),\tag{338}$$

$$\left\langle B_{x}\left(t\right)B_{y}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4\mathrm{i}}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}^{2}\left(t\right)-B_{10}^{2}\left(t\right)\right)\right),\tag{339}$$

$$\left\langle B_{y}\left(t\right)B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4\mathrm{i}}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}^{2}\left(t\right)-B_{10}^{2}\left(t\right)\right)\right).\tag{340}$$

Our next step is to find $v_3\left(t\right)$, the commutator $\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]$ is a central point for our calculations and it is equal to:

$$\left[\overline{H_0}(t)\,,\overline{H_{\bar{I}}}(t)\right]=\Big[\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)\right.\tag{341}$$

$$\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_xB_x(t)+\sigma_yB_y(t)\right)+V_{10}^{\Im}(t)\left(\sigma_xB_y(t)-\sigma_yB_x(t)\right)\Big]\tag{342}$$

$$=\left[\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right)-\sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right)\right.\tag{343}$$

$$\left.+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_{i}B_{iz}\left(t\right)|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right)+\sigma_yB_y\left(t\right)\right)+V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y\left(t\right)-\sigma_yB_x\left(t\right)\right)\right]\tag{344}$$

$$=\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right)+\sigma_yB_y\left(t\right)\right)+\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y\left(t\right)-\sigma_yB_x\left(t\right)\right)\tag{345}$$

$$+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_xB_x(t)+\sigma_yB_y(t)\right)\tag{346}$$

$$+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(\sigma_xB_y(t)-\sigma_yB_x(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i|\tag{347}$$

$$-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(\sigma_xB_y(t)\tag{348}$$

$$-\sigma_y B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t)\,|i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \tag{349}$$

$$- \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \tag{350}$$

$$- \sum_i B_{iz}(t)\,|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right)|i\rangle\langle i| - V_{10}^{\Re}(t)\left(\sigma_x B_x(t)\right. \tag{351}$$

$$\left.+\sigma_y B_y(t)\right)\sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))\,\sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)(\sigma_x B_x(t) \tag{352}$$

$$+\sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right)|i\rangle\langle i| - V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right)\sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t)\right. \tag{353}$$

$$\left.-B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right)\sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \tag{354}$$

$$= \sum_i \left(\varepsilon_i(t) + R_i(t)\right)|i\rangle\langle i| V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + \sum_i \left(\varepsilon_i(t) + R_i(t)\right)|i\rangle\langle i| V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \tag{355}$$

$$+ \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t)\,|i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t)\left(B_x(t) - \mathrm{i}\sigma_z B_y(t)\right) \tag{356}$$

$$+ \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Im}(t)\left(B_y(t) + \mathrm{i}\sigma_z B_x(t)\right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)\,|i\rangle\langle i| \tag{357}$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Re}(t)\left(\mathrm{i}\sigma_z B_x(t) + B_y(t)\right) - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_z B_y(t) - B_x(t)\right) \tag{358}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t)\,|i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \tag{359}$$

$$- \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \tag{360}$$

$$- \sum_i B_{iz}(t)\,|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right)|i\rangle\langle i| - V_{10}^{\Re}(t)\left(B_x(t) + \mathrm{i}\sigma_z B_y(t)\right)\left(B_{10}^{\Re}(t) V_{10}^{\Re}(t)\right. \tag{361}$$

$$\left.-B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t)\left(-\mathrm{i}\sigma_z B_x(t) + B_y(t)\right)\left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{362}$$

$$- V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right)|i\rangle\langle i| - V_{10}^{\Im}(t)\left(B_y(t) - \mathrm{i}\sigma_z B_x(t)\right)\left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{363}$$

$$+ V_{10}^{\Im}(t)\left(-\mathrm{i}\sigma_z B_y(t) - B_x(t)\right)\left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{364}$$

III. BIBLIOGRAPHY

[1] McCutcheon D P S, Dattani N S, Gauger E M, Lovett B W and Nazir A 2011 Phys. Rev. B 84 081305.
[2] Dara P S McCutcheon and Ahsan Nazir 2010 New J. Phys. 12 113042.
[3] Supplement: Theoretical model of phonon induced dephasing. A.J. Ramsay ey al 2009.
[4] Felix A Pollock et al 2013 New J. Phys. 15 075018.
[5] A. Decoster, J. Phys. A 37, 9051 2004.

* n.dattani@cfa.harvard.edu
† edcchaparroso@unal.edu.co