A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0| + V_{01}(t)\,|0\rangle\langle 1|, \tag{2}$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij} \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}} \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \tag{8}$$

$$V = |0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6)will give us the following result:

$$
\begin{aligned}
e^{\pm V} &= e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)} && (10)\\
&= \mathbb{I}\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)+\frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1))^2}{2!}+... && (11)\\
&= |0\rangle\langle 0|+|1\rangle\langle 1|\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)+\frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!}+\frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!}+... && (12)\\
&= |0\rangle\langle 0|\left(1\pm\hat{\varphi}_0+\frac{\hat{\varphi}_0^2}{2!}\pm...\right)+|1\rangle\langle 1|\left(1\pm\hat{\varphi}_1+\frac{\hat{\varphi}_1^2}{2!}\pm...\right) && (13)\\
&= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0}+|1\rangle\langle 1|e^{\pm\hat{\varphi}_1} && (14)\\
&= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)}+|1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_{0\pm}+|1\rangle\langle 1|B_{1\pm}, && (16)\\
B_{i\pm} &\equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
e^{t(X+Y)}= e^{tX}\ e^{tY}\ e^{-\frac{t^2}{2}[X,Y]}\ e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\ e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\dots \tag{18}
$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger}-\frac{v_{j\mathbf{k}'}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right]=0$ for all $\mathbf{k}^{'},\mathbf{k}$ and i,j we can show making $t=1$ in (18) the following result:

$$
\begin{aligned}
e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\dots && (19)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}0}\dots && (20)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} && (21)
\end{aligned}
$$

By induction of this result we can write expresion of $B_{i\pm}$ as a product of exponentials, which we will call "displacement" operators $D\left(\pm v_{i\mathbf{k}}\right)$:

$$
\begin{aligned}
B_{i\pm} &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), && (22)\\
D\left(\pm v_{i\mathbf{k}}\right) &\equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (23)
\end{aligned}
$$

this will help us to write operators O in the variational frame :

$$
\overline{O}\equiv \mathrm{e}^{V}O\mathrm{e}^{-V}. \tag{24}
$$

We use the following identities:

$$\overline{|0\rangle\langle 0|} = \mathrm{e}^{V}|0\rangle\langle 0|\mathrm{e}^{-V} \tag{25}$$
$$= \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{26}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{27}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{28}$$
$$= |0\rangle\langle 0|B_{0+}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{29}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}B_{1-} \tag{30}$$
$$= |0\rangle\langle 0|, \tag{31}$$
$$\overline{|1\rangle\langle 1|} = \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{32}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 1|B_{0+} + |1\rangle\langle 1|1\rangle\langle 1|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{33}$$
$$= |1\rangle\langle 1|B_{1+}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{34}$$
$$= |1\rangle\langle 1|0\rangle\langle 0|B_{1+}B_{0-} + B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} \tag{35}$$
$$= B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} \tag{36}$$
$$= |1\rangle\langle 1|B_{1+}B_{1-} \tag{37}$$
$$= |1\rangle\langle 1|, \tag{38}$$
$$\overline{|0\rangle\langle 1|} = \mathrm{e}^{V}|0\rangle\langle 1|\mathrm{e}^{-V} \tag{39}$$
$$= \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{40}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_{0+} + |1\rangle\langle 1|B_{1+}|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{41}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_{0+} + |1\rangle\langle 1|0\rangle\langle 1|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{42}$$
$$= |0\rangle\langle 1|B_{0+}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{43}$$
$$= |0\rangle\langle 1|0\rangle\langle 0|B_{0+}B_{0-} + |0\rangle\langle 1|1\rangle\langle 1|B_{0+}B_{1-} \tag{44}$$
$$= |0\rangle\langle 1|B_{0+}B_{1-}, \tag{45}$$
$$\overline{|1\rangle\langle 0|} = \mathrm{e}^{V}|1\rangle\langle 0|\mathrm{e}^{-V} \tag{46}$$
$$= \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{47}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{48}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_{0+} + |1\rangle\langle 1|1\rangle\langle 0|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{49}$$
$$= |1\rangle\langle 0|B_{1+}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{50}$$
$$= |1\rangle\langle 0|B_{1+}|0\rangle\langle 0|B_{0-} + |1\rangle\langle 0|B_{1+}|1\rangle\langle 1|B_{1-} \tag{51}$$
$$= |1\rangle\langle 0|B_{1+}B_{0-} + |1\rangle\langle 0|1\rangle\langle 1|B_{1+}B_{1-} \tag{52}$$
$$= |1\rangle\langle 0|B_{1+}B_{0-}, \tag{53}$$
$$\overline{b_{\mathbf{k}}} = \mathrm{e}^{V}b_{\mathbf{k}}\mathrm{e}^{-V} \tag{54}$$
$$= \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{55}$$
$$= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-}|0\rangle\langle 0| + |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}|1\rangle\langle 1|B_{1-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-}|1\rangle\langle 1| \tag{56}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}B_{1-} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-} \tag{57}$$
$$= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{58}$$
$$= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{59}$$
$$= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{60}$$
$$\overline{b_{\mathbf{k}}}^{\dagger} = \mathrm{e}^{V}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V} \tag{61}$$
$$= \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{62}$$
$$= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-}|0\rangle\langle 0| + |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_{1-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-}|1\rangle\langle 1| \tag{63}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}^{\dagger}B_{1-} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}^{\dagger}B_{0-} + |1\rangle\langle 1|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-} \tag{64}$$
$$= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) \tag{65}$$
$$= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{66}$$

We have used the following:

$$B_{i+}b_{\mathbf{k}}B_{i-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{67}$$

$$B_{i+}b_{\mathbf{k}}^{\dagger}B_{i-} = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{68}$$

We therefore have the following relationships:

$$\begin{aligned}
\overline{\varepsilon_0\left(t\right)|0\rangle\langle 0|} &= \varepsilon_0\left(t\right)|0\rangle\langle 0|, \qquad &(69)\\
\overline{\varepsilon_1\left(t\right)|1\rangle\langle 1|} &= \varepsilon_1\left(t\right)|1\rangle\langle 1|, &(70)\\
\overline{V_{10}\left(t\right)|1\rangle\langle 0|} &= V_{10}\left(t\right)|1\rangle\langle 0|B_{1+}B_{0-}, &(71)\\
\overline{V_{01}\left(t\right)|0\rangle\langle 1|} &= V_{01}\left(t\right)|0\rangle\langle 1|B_{0+}B_{1-}, &(72)\\
\overline{g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}} &= g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\right)+g_{i\mathbf{k}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) &(73)\\
&= g_{i\mathbf{k}}\left(\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left(\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) &(74)\\
&= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}-g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|-g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|-g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| &(75)\\
&= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}-\left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0|-\left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, &(76)\\
\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} &= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) &(77)\\
&= |0\rangle\langle 0|B_{0+}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)|0\rangle\langle 0|B_{0-} &(78)\\
&= |0\rangle\langle 0|B_{0+}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_{0-} &(79)\\
&= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), &(80)\\
\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)} &= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) &(81)\\
&= |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)|1\rangle\langle 1|B_{1-} &(82)\\
&= |1\rangle\langle 1|B_{1+}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_{1-} &(83)\\
&= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), &(84)\\
\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} &= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) &(85)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{0-}+|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{1-}\right) &(86)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) &(87)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) &(88)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) &(89)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) &(90)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) &(91)
\end{aligned}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{92}$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{93}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{94}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \tag{95}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right). \tag{96}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} \tag{97}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0| B_{1+} B_{0-} + V_{01}(t)\,|0\rangle\langle 1| B_{0+} B_{1-}, \tag{98}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{99}$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{100}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{101}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{102}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \tag{103}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right) \tag{104}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right). \tag{105}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{j+} B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{106}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}} \tag{107}$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{108}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \, b_{\mathbf{k}} \right). \tag{109}$$

We assume that the bath is at equilibrium with inverse temperature $\beta = 1/k_{\mathrm{B}}T$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{110}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{111}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{112}$$

So the product of the matrix representation of $b^\dagger$ and b is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{113}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right|, \tag{114}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\!\langle j|, |i\rangle\!\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|, \tag{115}$$

$$\exp\left(-\beta\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|. \tag{116}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\begin{aligned}
\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right)\right) &= \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (117)\\
&= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) && (118)\\
&= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} && (119)\\
&= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} && (120)\\
&\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). && (121)\\
\mathrm{Tr}\left(\exp\left(-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right)\right) &= \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (122)\\
&= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (123)\\
&= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). && (124)
\end{aligned}$$

So the density matrix of the bath is:

$$\begin{aligned}
\rho_B &= \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} && (125)\\
&= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} && (126)\\
&= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. && (127)
\end{aligned}$$

Now, given that creation and annihilation satisfy:

$$\begin{aligned}
b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle &= \sqrt{j_{\mathbf{k}}}\, |j_{\mathbf{k}}-1\rangle, && (128)\\
b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle &= \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. && (129)
\end{aligned}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (128)-(129):

$$\begin{aligned}
\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Tr}\left(\rho_B B_{iz}\right) = \mathrm{Tr}\left(B_{iz}\rho_B\right) && (130)\\
&= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}}\right)\right)\rho_B\right) && (131)\\
&= \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}}\rho_B\right) && (132)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \rho_B\right) + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^{*} \mathrm{Tr}\left(b_{\mathbf{k}} \rho_B\right) && (133)\\
&= \textstyle\sum_{\mathbf{k}} \mathrm{Tr}\Big(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta \omega_{\mathbf{k}}\right)}\Big) + \textstyle\sum_{\mathbf{k}} \mathrm{Tr}\Big(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta \omega_{\mathbf{k}}\right)}\Big) && (134)\\
&= \textstyle\sum_{\mathbf{k}} (\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}}) \mathrm{Tr}\Big(b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta \omega_{\mathbf{k}}\right)}\Big) + \textstyle\sum_{\mathbf{k}} (\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}})^{*} \mathrm{Tr}\Big(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta \omega_{\mathbf{k}}\right)}\Big), && (135)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}\Big(b_{\mathbf{k}}^{\dagger} \textstyle\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\Big) &= \mathrm{Tr}\Big(\big(\textstyle\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}})\big) b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\Big) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}) && (136)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right)\right) \sqrt{j_{\mathbf{k}}+1}\, |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) && (137)\\
&= 0, && (138)\\
\mathrm{Tr}\Big(b_{\mathbf{k}} \textstyle\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\Big) &= \mathrm{Tr}\Big(\big(\textstyle\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}})\big) b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\Big) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}) && (139)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}\right)\right) \sqrt{j_{\mathbf{k}}}\, |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) && (140)\\
&= 0. && (141)
\end{aligned}
$$

we therefore find that:

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0 \tag{142}$$

Another important expected value is $B = \langle B_{\pm}\rangle_{\overline{H_{\bar{B}}}}$, where $B_{\pm} = e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)}$ is given by:

$$
\begin{aligned}
\langle B_{\pm}\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B_{\pm}\right) = \mathrm{Tr}\left(B_{\pm} \rho_B\right) && (143)\\
&= \mathrm{Tr}\left(e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)} \rho_B\right) && (144)\\
&= \prod_{\mathbf{k}} \mathrm{Tr}\left(D\left(\pm \alpha_{\mathbf{k}}\right) \rho_B\right) && (145)\\
&= \prod_{\mathbf{k}} \langle D\left(\pm \alpha_{\mathbf{k}}\right)\rangle . && (146)
\end{aligned}
$$

Given that we can write a density operator as:

$$\rho = \int P\left(\alpha\right) |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha \tag{147}$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right) \mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P\left(\alpha\right)$ is given by:

$$
\begin{aligned}
\langle A\rangle &= \mathrm{Tr}\left(A\rho\right) && (148)\\
&= \int P\left(\alpha\right) \langle\alpha\left|A\right|\alpha\rangle \mathrm{d}^2\alpha && (149)
\end{aligned}
$$

We are typically interested in thermal state density operators, for which it can be shown that $P\left(\alpha\right) = \frac{1}{\pi N} \exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | D(h) | \alpha \rangle \mathrm{d}^2\alpha \tag{150}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle \mathrm{d}^2\alpha \tag{151}$$

$$D(h) D(\alpha) = D(h+\alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{152}$$

$$D(-\alpha)(D(h) D(\alpha)) = D(-\alpha) D(h+\alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{153}$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{154}$$

$$
\begin{aligned}
&= D\left(\alpha\right) e^{\frac{1}{2}\left(-\alpha h^{*}-|\alpha|^{2}+\alpha^{*}h+|\alpha|^{2}\right)} e^{\frac{1}{2}\left(h\alpha^{*}-h^{*}\alpha\right)} && (155)\\
&= D\left(\alpha\right) e^{\left(h\alpha^{*}-h^{*}\alpha\right)}, && (156)\\
\langle D\left(h\right)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^{2}}{N}\right)\langle 0|D\left(h\right)\exp\left(h\alpha^{*}-h^{*}\alpha\right)|0\rangle \mathrm{d}^{2}\alpha && (157)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^{2}}{N}\right)\exp\left(h\alpha^{*}-h^{*}\alpha\right)\langle 0|D\left(h\right)|0\rangle \mathrm{d}^{2}\alpha && (158)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^{2}}{N}\right)\exp\left(h\alpha^{*}-h^{*}\alpha\right)\langle 0|h\rangle \mathrm{d}^{2}\alpha && (159)\\
|\alpha\rangle &= \exp\left(-\frac{|\alpha|^{2}}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle, && (160)\\
\langle D\left(h\right)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^{2}}{N}\right)\exp\left(h\alpha^{*}-h^{*}\alpha\right)\langle 0|\exp\left(-\frac{|h|^{2}}{2}\right)\sum_{n=0}^{\infty}\frac{h^{n}}{\sqrt{n!}}|n\rangle \mathrm{d}^{2}\alpha && (161)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^{2}}{N}\right)\exp\left(h\alpha^{*}-h^{*}\alpha\right)\exp\left(-\frac{|h|^{2}}{2}\right)\mathrm{d}^{2}\alpha && (162)\\
&= \frac{\exp\left(-\frac{|h|^{2}}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^{2}}{N}+h\alpha^{*}-h^{*}\alpha\right)\mathrm{d}^{2}\alpha, && (163)\\
\alpha &= x+\mathrm{i}y, && (164)\\
\langle D\left(h\right)\rangle &= \frac{\exp\left(-\frac{|h|^{2}}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^{2}+y^{2}}{N}+h\left(x-\mathrm{i}y\right)-h^{*}\left(x+\mathrm{i}y\right)\right)\mathrm{d}x\mathrm{d}y && (165)\\
&= \frac{\exp\left(-\frac{|h|^{2}}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^{2}}{N}+hx-h^{*}x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^{2}}{N}-\mathrm{i}hy-\mathrm{i}h^{*}y\right)\mathrm{d}y, && (166)\\
-\tfrac{x^{2}}{N}+hx-h^{*}x &= -\tfrac{1}{N}\left(x^{2}-Nhx+Nh^{*}x\right) && (167)\\
&= -\frac{1}{N}\left(x+\frac{\left(Nh^{*}-Nh\right)}{2}\right)^{2}+\frac{N\left(h^{*}-h\right)^{2}}{4}, && (168)\\
-\tfrac{y^{2}}{N}-\mathrm{i}hy-\mathrm{i}h^{*}y &= -\tfrac{1}{N}\left(y^{2}+\mathrm{i}Nhy+\mathrm{i}Nh^{*}y\right) && (169)\\
&= -\frac{1}{N}\left(y^{2}+\frac{\mathrm{i}N\left(h+h^{*}\right)}{2}\right)-\frac{N\left(h+h^{*}\right)^{2}}{4}, && (170)\\
\langle D(h)\rangle &= \frac{\exp\left(-\frac{|h|^{2}}{2}+\frac{N\left(h^{*}-h\right)^{2}}{4}-\frac{N\left(h+h^{*}\right)^{2}}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{N}\left(x+\frac{\left(Nh^{*}-Nh\right)}{2}\right)^{2}-\frac{1}{N}\left(y^{2}+\frac{\mathrm{i}N\left(h+h^{*}\right)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, && (171)\\
\sqrt{2\pi}\sigma &= \int_{-\infty}^{\infty}\exp\left(-\frac{\left(x-\mu\right)^{2}}{2\sigma^{2}}\right)\mathrm{d}x, && (172)\\
\langle D(h)\rangle &= \frac{\exp\left(-\frac{|h|^{2}}{2}+\frac{N\left(h^{*}-h\right)^{2}}{4}-\frac{N\left(h+h^{*}\right)^{2}}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(x+\frac{\left(Nh^{*}-Nh\right)}{2}\right)^{2}}{2\left(\sqrt{\frac{N}{2}}\right)^{2}}\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{\left(y^{2}+\frac{\mathrm{i}N\left(h+h^{*}\right)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^{2}}\right)\mathrm{d}y && (173)\\
&= \frac{\exp\left(-\frac{|h|^{2}}{2}+\frac{N(h^{*}-h)^{2}}{4}-\frac{N(h+h^{*})^{2}}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^{2} && (174)\\
&= \exp\left(-\frac{|h|^{2}}{2}+\frac{N\left(h^{*}-h\right)^{2}}{4}-\frac{N\left(h+h^{*}\right)^{2}}{4}\right) && (175)\\
&= \exp\left(-\frac{|h|^{2}}{2}+\frac{N\left(h^{*2}-2hh^{*}+h^{2}\right)-N\left(h^{2}+2hh^{*}+h^{*2}\right)}{4}\right) && (176)
\end{aligned}
$$

$$= \exp\left(-|h|^2\left(N+\frac{1}{2}\right)\right) \tag{177}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)\right) \tag{178}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)\right) \tag{179}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{180}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (145) using (180) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{181}$$

We will now force $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_+\sigma_+$and $B_-\sigma_-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H(t)} = \textstyle\sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{j+}B_{j'-} + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{j\mathbf{k}}|j\rangle\langle j|\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}b_{\mathbf{k}} + \frac{\left|v_{j\mathbf{k}}\right|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^{*}\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{182}$$

$$= \textstyle\sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'} + \sum_j |j\rangle\langle j|B_{jz} + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-} - B_{jj'}\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{183}$$

$$\equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{184}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_{1+}B_{0-}\rangle = B_{10} \tag{185}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle \tag{186}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle \tag{187}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)}\right)\right\rangle \tag{188}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)} \tag{189}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)} \tag{190}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)}. \tag{191}$$

From the definition $B_{01} = \langle B_{0+}B_{1-}\rangle$using the displacement operator we have:

$$
\begin{aligned}
\langle B_{0+}B_{1-}\rangle &= B_{01} && (192)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle && (193)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle && (194)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v^*_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v^*_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)}\right)\right\rangle && (195)\\
&= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle e^{\frac{1}{2}\left(\frac{v^*_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v^*_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)} && (196)\\
&= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v^*_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v^*_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)} && (197)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v^*_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v^*_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)} && (198)
\end{aligned}
$$

This can be checked in the following way:

$$
\begin{aligned}
\langle B_{0+}B_{1-}\rangle &= B_{01} && (199)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v^*_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v^*_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)} && (200)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v^*_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*} && (201)\\
&= \langle B_{1+}B_{0-}\rangle^* && (202)\\
&= B^*_{10}. && (203)
\end{aligned}
$$

The parts of the Hamiltonian splitted are:

$$
\begin{aligned}
\overline{H_{\bar{S}}(t)} &\equiv \left(\varepsilon_0\left(t\right) + R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right) + R_1\right)|1\rangle\langle 1| + V_{10}\left(t\right)B_{10}\sigma_+ + V_{01}\left(t\right)B_{01}\sigma_-, && (204)\\
\overline{H_{\bar{I}}} &\equiv V_{10}\left(t\right)\left(B_{1+}B_{0-} - B_{10}\right)\sigma_+ + V_{01}\left(t\right)\left(B_{0+}B_{1-} - B_{01}\right)\sigma_- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, && (205)\\
\overline{H_{\bar{B}}} &\equiv \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}} && (206)\\
&= H_B. && (207)
\end{aligned}
$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (205) we can verify the condition $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$
\begin{aligned}
\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} &= \left\langle\sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})b^\dagger_{\mathbf{k}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^*b_{\mathbf{k}}\right)|n\rangle\langle n|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-}-B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} && (208)\\
&= \left\langle\sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})b^\dagger_{\mathbf{k}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^*b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-}-B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} && (209)\\
&= \sum_{n\mathbf{k}}\left(\left\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}})b^\dagger_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}+\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}})^*b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}(t)B_{j+}B_{j'-}\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle V_{jj'}(t)B_{jj'}\right\rangle_{\overline{H_{\bar{B}}}}\right) && (210)\\
&= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})\left\langle b^\dagger_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(\left\langle B_{j+}B_{j'-}\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle B_{jj'}\right\rangle_{\overline{H_{\bar{B}}}}\right) && (211)\\
&= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})\left\langle b^\dagger_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(B_{jj'}-B_{jj'}\right). && (212)\\
&= 0. && (213)
\end{aligned}
$$

We used (142) and (191) to evaluate the expected values.
Let's consider the following Hermitian combinations:

$$
\begin{aligned}
B_x &= B_x^\dagger \qquad (214)\\
&= \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2}, \qquad (215)\\
B_y &= B_y^\dagger \qquad (216)\\
&= \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{01}}{2\mathrm{i}}, \qquad (217)\\
B_{iz} &= B_{iz}^\dagger \qquad (218)\\
&= \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* b_{\mathbf{k}} \right). \qquad (219)
\end{aligned}
$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$
\begin{aligned}
\overline{H_{\bar{S}}(t)} =& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t)\, B_{10}\sigma_+ + V_{01}(t)\, B_{01}\sigma_- \qquad (220)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t)\, B_{10}\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\, B_{01}\frac{\sigma_x - \mathrm{i}\sigma_y}{2} \qquad (221)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t)\left(\Re(B_{10}(t)) + \mathrm{i}\Im(B_{10}(t))\right)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\left(\Re(B_{10}(t)) - \mathrm{i}\Im(B_{10}(t))\right)\frac{\sigma_x - \mathrm{i}\sigma_y}{2} \qquad (222)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))\left(V_{10}(t)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) + \mathrm{i}\Im(B_{10}(t))\left(V_{10}(t)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} - V_{01}(t)\frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) \qquad (223)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) + V_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t) - V_{01}(t)}{2}\right) + \mathrm{i}\Im(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) - V_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t) + V_{01}(t)}{2}\right) \qquad (224)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) + V_{10}^*(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t) - V_{10}^*(t)}{2}\right) + \mathrm{i}\Im(B_{10}(t))\left(\sigma_x\frac{V_{10}(t) - V_{10}^*(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t) + V_{10}^*(t)}{2}\right) \qquad (225)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \Re(B_{10}(t))\left(\sigma_x\Re(V_{10}(t)) - \sigma_y\Im(V_{10}(t))\right) + \mathrm{i}\Im(B_{10}(t))\left(\mathrm{i}\sigma_x\Im(V_{10}(t)) + \mathrm{i}\sigma_y\Re(V_{10}(t))\right) \qquad (226)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \left(\sigma_x\Re(B_{10}(t))\Re(V_{10}(t)) - \sigma_y\Re(B_{10}(t))\Im(V_{10}(t))\right) - \left(\sigma_x\Im(B_{10}(t))\Im(V_{10}(t)) + \sigma_y\Im(B_{10}(t))\Re(V_{10}(t))\right) \qquad (227)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \sigma_x\left(\Re(B_{10}(t))\Re(V_{10}(t)) - \Im(B_{10}(t))\Im(V_{10}(t))\right) - \sigma_y\left(\Re(B_{10}(t))\Im(V_{10}(t)) + \Im(B_{10}(t))\Re(V_{10}(t))\right) \qquad (228)\\
=& (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)\, V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\, V_{10}^{\Re}(t)\right). \qquad (229)
\end{aligned}
$$

$$
\begin{aligned}
\overline{H_{\bar{I}}} =& V_{10}(t)\left(\sigma_+ B_{1+}B_{0-} - \sigma_+ B_{10}\right) + V_{01}(t)\left(\sigma_- B_{0+}B_{1-} - \sigma_- B_{01}\right) + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} \qquad (230)\\
=& |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} + \left(\Re(V_{10}(t)) + \mathrm{i}\Im(V_{10}(t))\right)\left(\sigma_+ B_{1+}B_{0-} - \sigma_+ B_{10}\right) + \left(\Re(V_{10}(t)) - \mathrm{i}\Im(V_{10}(t))\right)\left(\sigma_- B_{0+}B_{1-} - \sigma_- B_{01}\right) \qquad (231)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_+ B_{1+}B_{0-} - \sigma_+ B_{10} + \sigma_- B_{0+}B_{1-} - \sigma_- B_{01}\right) + \mathrm{i}\Im(V_{10}(t))\left(\sigma_+ B_{1+}B_{0-} - \sigma_+ B_{10} - \sigma_- B_{0+}B_{1-} + \sigma_- B_{01}\right) \qquad (232)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\tfrac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{1+}B_{0-} - \tfrac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10} + \tfrac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{0+}B_{1-} - \tfrac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\right) \qquad (233)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\tfrac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{1+}B_{0-} - \tfrac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10} + \tfrac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{0+}B_{1-} - \tfrac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\right) + \mathrm{i}\Im(V_{10}(t))\left(\tfrac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{1+}B_{0-} - \tfrac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10} - \tfrac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{0+}B_{1-} + \tfrac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01} \right. \qquad (234)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_x\tfrac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2} + \mathrm{i}\sigma_y\tfrac{B_{1+}B_{0-} - B_{0+}B_{1-} - B_{10} + B_{01}}{2}\right) + \mathrm{i}\Im(V_{10}(t))\left(\sigma_x\tfrac{B_{1+}B_{0-} - B_{0+}B_{1-} - B_{10} + B_{01}}{2} + \mathrm{i}\sigma_y\tfrac{B_{1+}B_{0-} + B_{0+}B_{1-} - \cdots}{2} \right. \qquad (235)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_x B_x + \sigma_y B_y\right) + \Im(V_{10}(t))\left(\mathrm{i}\sigma_x\tfrac{B_{1+}B_{0-} - B_{0+}B_{1-} - B_{10} + B_{01}}{2} - \sigma_y\tfrac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2}\right) \qquad (236)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_x B_x + \sigma_y B_y\right) + \Im(V_{10}(t))\left(\mathrm{i}^2\sigma_x\tfrac{B_{1+}B_{0-} - B_{0+}B_{1-} - B_{10} + B_{01}}{2\mathrm{i}} - \sigma_y\tfrac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2}\right) \qquad (237)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_x B_x + \sigma_y B_y\right) + \Im(V_{10}(t))\left(\mathrm{i}^2\sigma_x\tfrac{B_{1+}B_{0-} - B_{0+}B_{1-} - B_{10} + B_{01}}{2\mathrm{i}} - \sigma_y\tfrac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{01}}{2}\right) \qquad (238)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_x B_x + \sigma_y B_y\right) + \Im(V_{10}(t))\left(\mathrm{i}^2\sigma_x(-B_y) - \sigma_y B_x\right) \qquad (239)\\
=& \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))\left(\sigma_x B_x + \sigma_y B_y\right) + \Im(V_{10}(t))\left(\sigma_x B_y - \sigma_y B_x\right). \qquad (240)
\end{aligned}
$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^{2}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{241}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = 0. \tag{242}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\left(t\right), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{243}$$

Then using the fact that $\overline{H_{\bar{S}}}\left(t\right)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{244}$$

So Eq. (242) becomes:

$$\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} &= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}} && (245)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (246)\\
&= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right) + \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}} && (247)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (248)\\
&= 0 \quad \text{(by Eq. (242))}. && (249)
\end{aligned}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\begin{aligned}
\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} &= \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} && (250)\\
&= 0. && (251)
\end{aligned}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} = 0. \tag{252}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}\left(t\right) = -\beta\left(\left(\varepsilon_0\left(t\right)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1| + V_{10}\left(t\right)B_{10}\sigma_+ + V_{01}\left(t\right)B_{01}\sigma_-\right). \tag{253}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}\left(t\right)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right) + \text{Det}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right) = 0. \tag{254}$$

Let's define:

$$\varepsilon\left(t\right) \equiv \text{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), \tag{255}$$

$$\eta \equiv \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}. \tag{256}$$

The solutions of the equation (254) are:

$$\lambda = \beta\frac{-\text{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right) \pm \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}}{2} \tag{257}$$

$$= \beta\frac{-\varepsilon\left(t\right) \pm \eta\left(t\right)}{2} \tag{258}$$

$$= -\beta\frac{\varepsilon\left(t\right) \mp \eta\left(t\right)}{2}. \tag{259}$$

The value of $\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right) + \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) \tag{260}$$

$$= 2\text{exp}\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{261}$$

Given that $v_{i\mathbf{k}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A\left(\varepsilon\left(t\right), \eta\left(t\right)\right)$ to calculate $\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial\Re(v_{i\mathbf{k}})}$ can lead to:

$$\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)} = \frac{\partial\left(2\text{exp}\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)} \tag{262}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) + 2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) \tag{263}$$

$$= -\beta\text{exp}\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{264}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) = 0. \tag{265}$$

The derivates included in the expression given are related to:

$$\left\langle B_{0+}B_{1-}\right\rangle=\left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{266}$$

$$=\left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)}\right)^{*}\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{267}$$

$$=\left\langle B_{1+}B_{0-}\right\rangle^{*}, \tag{268}$$

$$R_{i}=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\right|^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{269}$$

$$=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\right|^{2}}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{270}$$

$$\left\langle B_{0+}B_{1-}\right\rangle=\left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{271}$$

$$=\left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2\omega_{\mathbf{k}}^{2}}\left(v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{272}$$

$$v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}=(\Re(v_{0\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{1\mathbf{k}}))-(\Re(v_{0\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{1\mathbf{k}})) \tag{273}$$

$$=(\Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\mathrm{i}\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}))-(\Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})-\mathrm{i}\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})) \tag{274}$$

$$=2\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})), \tag{275}$$

$$\left|v_{1\mathbf{k}}-v_{0\mathbf{k}}\right|^{2}=\left(v_{1\mathbf{k}}-v_{0\mathbf{k}}\right)\left(v_{1\mathbf{k}}-v_{0\mathbf{k}}\right)^{*} \tag{276}$$

$$=\left|v_{1\mathbf{k}}\right|^{2}+\left|v_{0\mathbf{k}}\right|^{2}-\left(v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}+v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}\right) \tag{277}$$

$$=(\Re(v_{1\mathbf{k}}))^{2}+(\Im(v_{1\mathbf{k}}))^{2}+(\Re(v_{0\mathbf{k}}))^{2}+(\Im(v_{0\mathbf{k}}))^{2}-((\Re(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}}))+(\Re(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}}))) \tag{278}$$

$$=\left(\Re\left(v_{1\mathbf{k}}\right)\right)^{2}+\left(\Im\left(v_{1\mathbf{k}}\right)\right)^{2}+\left(\Re\left(v_{0\mathbf{k}}\right)\right)^{2}+\left(\Im\left(v_{0\mathbf{k}}\right)\right)^{2}-2\left(\Re\left(v_{1\mathbf{k}}\right)\Re\left(v_{0\mathbf{k}}\right)+\Im\left(v_{1\mathbf{k}}\right)\Im\left(v_{0\mathbf{k}}\right)\right) \tag{279}$$

$$=\left(\Re\left(v_{1\mathbf{k}}\right)-\Re\left(v_{0\mathbf{k}}\right)\right)^{2}+\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)^{2}. \tag{280}$$

Rewriting in terms of real and imaginary parts.

$$R_{i}=\sum_{\mathbf{k}}\left(\frac{\Re\left(v_{i\mathbf{k}}\right)^{2}+\Im\left(v_{i\mathbf{k}}\right)^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{\Re\left(v_{i\mathbf{k}}\right)-\mathrm{i}\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{\Re\left(v_{i\mathbf{k}}\right)+\mathrm{i}\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\right)\right) \tag{281}$$

$$=\sum_{\mathbf{k}}\left(\frac{\Re\left(v_{i\mathbf{k}}\right)^{2}+\Im\left(v_{i\mathbf{k}}\right)^{2}}{\omega_{\mathbf{k}}}-\Re\left(v_{i\mathbf{k}}\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}\Im\left(v_{i\mathbf{k}}\right)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{282}$$

$$\left\langle B_{0+}B_{1-}\right\rangle=\left(\prod_{\mathbf{k}}\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{2\omega_{\mathbf{k}}^{2}}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{283}$$

$$=\left(\prod_{\mathbf{k}}\exp\left(\frac{2\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{2\omega_{\mathbf{k}}^{2}}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^{2}+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{284}$$

$$=\left(\prod_{\mathbf{k}}\exp\left(\frac{\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{\omega_{\mathbf{k}}^{2}}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^{2}+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{285}$$

Calculating the derivates respect to $\Re\left(\alpha_{i\mathbf{k}}\right)$ and $\Im\left(\alpha_{i\mathbf{k}}\right)$ we have:

$$\begin{aligned}
\frac{\partial\varepsilon(t)}{\partial\Re(v_{i\mathbf{k}})} &= \frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial\Re(v_{i\mathbf{k}})} &(286)\\
&= \frac{\partial\left(\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\Re(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\Im(v_{i\mathbf{k}})\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial\Re(v_{i\mathbf{k}})} &(287)\\
&= \frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, &(288)\\
\frac{\partial|B_{10}|^2}{\partial\Re(v_{i\mathbf{k}})} &= \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial\Re(v_{i\mathbf{k}})} &(289)\\
&= -\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega^2_{\mathbf{k}}}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial\Re(v_{i\mathbf{k}})}\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) &(290)\\
&= -\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega^2_{\mathbf{k}}}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial\Re(v_{i\mathbf{k}})}|B_{10}|^2, &(291)\\
\frac{\partial\eta(t)}{\partial\Re(v_{i\mathbf{k}})} &= \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial\Re(v_{i\mathbf{k}})} &(292)\\
&= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial\Re(v_{i\mathbf{k}})}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial\Re(v_{i\mathbf{k}})}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} &(293)\\
&= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial\Re(v_{i\mathbf{k}})}}{\eta(t)} &(294)\\
&= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega^2_{\mathbf{k}}}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial\Re(v_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} &(295)\\
&= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2(\Re(v_{i\mathbf{k}})-\Re(v_{i'\mathbf{k}}))}{\omega^2_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} &(296)\\
&= \frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) &(297)\\
&+\frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{\Re(v_{i'\mathbf{k}})}{\omega^2_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) &(298)
\end{aligned}$$

From the equation (265) and replacing the derivates obtained we have:

$$\begin{aligned}
\tanh\left(\frac{\beta\eta(t)}{2}\right) &= \frac{\frac{\partial\varepsilon(t)}{\partial\Re(v_{i\mathbf{k}})}}{\frac{\partial\eta(t)}{\partial\Re(v_{i\mathbf{k}})}} &(299)\\
&= \frac{\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{2\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}}{\frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}+2\frac{\Re(v_{i'\mathbf{k}})}{\omega^2_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}} &(300)
\end{aligned}$$

Rearrannging this equation will lead to:

$$\begin{aligned}
\tanh\left(\frac{\beta\eta(t)}{2}\right) &= \frac{(2\Re(v_{i\mathbf{k}})-g_{i\mathbf{k}}-g^*_{i\mathbf{k}})\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}+g^*_{i\mathbf{k}})(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} &(301)\\
&= \frac{(2\Re(v_{i\mathbf{k}})-2\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2B^2_{10}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Re(g_{i\mathbf{k}})(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} &(302)\\
&= \frac{(2\Re(v_{i\mathbf{k}})-2\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} &(303)\\
&= \frac{(\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} &(304)
\end{aligned}$$

Separating (303) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}}))\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=\Re(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)-\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{305}$$

$$\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}})=\Re(v_{i\mathbf{k}})\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \tag{306}$$

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{307}$$

$$\Re(v_{i\mathbf{k}})=\frac{\Re(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\Re(g_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{308}$$

$$\Re(v_{i\mathbf{k}})=\frac{\Re(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\Re(g_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{309}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon(t)}{\partial\Im\left(v_{i\mathbf{k}}\right)} = \frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial\Im\left(v_{i\mathbf{k}}\right)} \tag{310}$$

$$= \frac{\partial\left(\left(\frac{\Re\left(v_{i\mathbf{k}}\right)^2+\Im\left(v_{i\mathbf{k}}\right)^2}{\omega_{\mathbf{k}}}-\Re\left(v_{i\mathbf{k}}\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}\Im\left(v_{i\mathbf{k}}\right)\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)} \tag{311}$$

$$= 2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{312}$$

$$\frac{\partial|B_{10}|^2}{\partial\Im\left(v_{i\mathbf{k}}\right)} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(\Re\left(v_{1\mathbf{k}}\right)-\Re\left(v_{0\mathbf{k}}\right)\right)^2+\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)} \tag{313}$$

$$= -\frac{2\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(\Re\left(v_{1\mathbf{k}}\right)-\Re\left(v_{0\mathbf{k}}\right)\right)^2+\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{314}$$

$$= -\frac{2\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}|B_{10}|^2 \tag{315}$$

$$\frac{\partial\eta(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(H_{\bar{S}}(t)\right)\right)^2-4\mathrm{Det}\left(H_{\bar{S}}(t)\right)}}{\partial\Re\left(v_{i\mathbf{k}}\right)} \tag{316}$$

$$= \frac{2\mathrm{Tr}\left(H_{\bar{S}}(t)\right)\frac{\partial\mathrm{Tr}\left(H_{\bar{S}}(t)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}-4\frac{\partial\mathrm{Det}\left(H_{\bar{S}}(t)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}}{2\sqrt{\left(\mathrm{Tr}\left(H_{\bar{S}}(t)\right)\right)^2-4\mathrm{Det}\left(H_{\bar{S}}(t)\right)}} \tag{317}$$

$$= \frac{\varepsilon(t)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}}{\eta(t)} \tag{318}$$

$$= \frac{\varepsilon(t)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{319}$$

$$= \frac{\varepsilon(t)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(\Im\left(v_{i\mathbf{k}}\right)-\Im\left(v_{i'\mathbf{k}}\right)\right)}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{320}$$

$$= \frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\beta\omega_{\mathbf{k}}/2\right)}{\eta(t)}\right) \tag{321}$$

$$+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{322}$$

From the equation (265) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial\Im(v_{i\mathbf{k}})}}{\frac{\partial\eta(t)}{\partial\Im(v_{i\mathbf{k}})}} \tag{323}$$

$$=\frac{2\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\Im(g^*_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{\Im(g^*_{i\mathbf{k}})}{\omega_{\mathbf{k}}}+2\frac{\Im(v_{i'\mathbf{k}})}{\omega^2_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)} \tag{324}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{(2\Im(v_{i\mathbf{k}})-\mathrm{i}(g^*_{i\mathbf{k}}-g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}(g^*_{i\mathbf{k}}-g_{i\mathbf{k}})(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{325}$$

$$=\frac{2(\Im(v_{i\mathbf{k}})-\Im(g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2B^2_{10}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Im(g_{i\mathbf{k}})(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{326}$$

$$=\frac{2(\Im(v_{i\mathbf{k}})-\Im(g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Im(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$=\frac{(\Im(v_{i\mathbf{k}})-\Im(g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\Im(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

Separating (**??**) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(\Im(v_{i\mathbf{k}})-\Im(g_{i\mathbf{k}}))\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=\Im(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\Im(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{329}$$

$$\Im(v_{i\mathbf{k}})-\Im(g_{i\mathbf{k}})=\Im(v_{i\mathbf{k}})\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \tag{330}$$

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\Im(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{331}$$

$$\Im(v_{i\mathbf{k}})=\frac{\Im(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{332}$$

$$\Im(v_{i\mathbf{k}})=\frac{\Im(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{333}$$

The variational parameters are:

$$v_{i\mathbf{k}} = \Re\left(v_{i\mathbf{k}}\right) + \mathrm{i}\Im\left(v_{i\mathbf{k}}\right) \tag{334}$$

$$= \frac{\Re\left(g_{i\mathbf{k}}\right)\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{335}$$

$$+\mathrm{i}\frac{\Im\left(g_{i\mathbf{k}}\right)\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{336}$$

$$= \frac{g_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i-\varepsilon\left(t\right)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T\left(0\right) = \rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}$, the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T\left(0\right) e^{-V} \tag{338}$$

$$= \left(|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}\right)\left(\rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}\right)\left(|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}\right) \tag{339}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } |0\rangle\langle 0|0\rangle B_{0+}\langle 0|\rho_B^{\mathrm{Thermal}}|0\rangle\langle 0|B_{0-} \tag{340}$$

$$= |0\rangle B_{0+}\langle 0|\rho_B^{\mathrm{Thermal}}|0\rangle\langle 0|B_{0-} \tag{341}$$

$$= |0\rangle\langle 0| \otimes B_{0+}\rho_B^{\mathrm{Thermal}}B_{0-} \tag{342}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\rho_B^{\mathrm{Thermal}}|1\rangle\langle 1|B_{1-} \tag{343}$$

$$= |1\rangle\langle 1|B_{1+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{344}$$

$$= |1\rangle\langle 1| \otimes B_{1+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{345}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1| : |0\rangle\langle 0|B_{0+}|0\rangle\langle 1|\rho_B^{\mathrm{Thermal}}|1\rangle\langle 1|B_{1-} \tag{346}$$

$$= |0\rangle\langle 1|B_{0+}\rho_B^{\mathrm{Thermal}}|1\rangle\langle 1|B_{1-} \tag{347}$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_{0+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{348}$$

$$= |0\rangle\langle 1| \otimes B_{0+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{349}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0| : |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\rho_B^{\mathrm{Thermal}}|0\rangle\langle 0|B_{0-} \tag{350}$$

$$= |1\rangle\langle 0| \otimes B_{1+}\rho_B^{\mathrm{Thermal}}B_{0-} \tag{351}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}\left(t\right) \equiv U^\dagger(t)O\left(t\right)U(t) \tag{352}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right). \tag{353}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_{\bar{S}}}}(t) = U^\dagger(t)\overline{\rho_{\bar{S}}}(t)U(t), \text{ where} \tag{354}$$

$$\overline{\rho_{\bar{S}}}(t) = \mathrm{Tr}_B\left(\bar{\rho_T}(t)\right) \tag{355}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ \Re(V_{10}(t)) & \Re(V_{10}(t)) & 1 & \Im(V_{10}(t)) & -\Im(V_{10}(t)) & 1 \end{pmatrix} \tag{356}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (**??**):

$$\begin{aligned} \overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}|i\rangle\langle i| + \Re(V_{10}(t))(\sigma_x B_x + \sigma_y B_y) + \Im(V_{10}(t))(\sigma_x B_y - \sigma_y B_x) && (357) \\ &= B_{0z}|0\rangle\langle 0| + B_{1z}|1\rangle\langle 1| + \Re(V_{10}(t))\sigma_x B_x + \Re(V_{10}(t))\sigma_y B_y + \Im(V_{10}(t))\sigma_x B_y - \Im(V_{10}(t))\sigma_y B_x && (358) \\ &= \sum_i C_i(t)(A_i \otimes B_i(t)) && (359) \end{aligned}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\rho_T}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{H}_I(t), \widetilde{\rho_T}(t)]. \tag{360}$$

This equation has the formal solution

$$\widetilde{\rho_T}(t) = \rho(0) \text{ -i} \int_0^t [\widetilde{H}_I(s), \widetilde{\rho_T}(s)]\mathrm{d}s. \tag{361}$$

Replacing the equation (361) in the equation (360)give us:

$$\frac{\mathrm{d}\widetilde{\rho_T}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{H}_I(t), \rho_T(0)]\text{-} \int_0^t [\widetilde{H}_I(t), [\widetilde{H}_I(s), \widetilde{\rho_T}(s)]]\mathrm{d}s. \tag{362}$$

This equation allow us to iterate and write in terms of a series expansion with $\rho_T(0)$ the solution as:

$$\widetilde{\rho_T}(t) = \rho_T(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \cdots [\widetilde{H}_I(t_n), \rho_T(0)]] \cdot\cdot] \tag{363}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho}_{\overline{S}}(t) = \rho_{\overline{S}}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \cdots [\widetilde{H}_I(t_n), \rho_{\overline{S}}(0)\rho_B^{\mathrm{Thermal}}]] \cdot\cdot] \tag{364}$$

here we have assumed that $\rho_T(0) = \rho_{\overline{S}}(0) \otimes \rho_B^{\mathrm{Thermal}}$. Consider the following notation:

$$\begin{aligned} \widetilde{\rho}_{\overline{S}}(t) &= (1 + W_1(t) + W_2(t) + ...)\rho_S(0) && (365) \\ &= W(t)\rho_{\overline{S}}(0) && (366) \end{aligned}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \cdots [\widetilde{H}_I(t_n), (\cdot)\rho_B^{\mathrm{Thermal}}]] \cdot\cdot] \tag{367}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\rho_{\overline{S}}}(t)}{\mathrm{d}t} = \left(\dot{W}_1\left(t\right) + \dot{W}_2\left(t\right) + ...\right)\rho_{\overline{S}}\left(0\right) \tag{368}$$

$$= \left(\dot{W}_1\left(t\right) + \dot{W}_2\left(t\right) + ...\right)W\left(t\right)^{-1}W\left(t\right)\rho_{\overline{S}}\left(0\right) \tag{369}$$

$$= \left(\dot{W}_1\left(t\right) + \dot{W}_2\left(t\right) + ...\right)W\left(t\right)^{-1}\widetilde{\rho_{\overline{S}}}(t) \tag{370}$$

where we assumed that $W\left(t\right)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{H}_I(t)\rho_B(0)] = 0$ so $W_1\left(t\right) = 0$. Thus, to second order and taking $W\left(t\right) \approx \mathbb{I}$ then the equation (368) becomes:

$$\frac{\mathrm{d}\rho_{\overline{S}}\left(\mathrm{t}\right)}{\mathrm{d}t} = -\mathrm{i}\left[H_{\overline{S}}, \rho_{\overline{S}}\left(t\right)\right] - \int_0^t \mathrm{d}\tau\left[H_I, \left[\widetilde{H}_I(-\tau), \rho_{\overline{S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right] \tag{371}$$

Replacing $t_1 \to t-\tau$

$$W_n\left(t\right) = \left(-\mathrm{i}\right)^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_n\mathrm{Tr}_B[\widetilde{H}_I\left(t_1\right), [\widetilde{H}_I\left(t_2\right), \cdots[\widetilde{H}_I\left(t_n\right), \left(\cdot\right)\rho_B^{\mathrm{Thermal}}]]\cdots] \tag{372}$$

Taking as reference state $\rho_B^{\mathrm{Thermal}}$ and truncating at second order in $\overline{H_{\overline{I}}}(t)$, we obtain our master equation in the interaction picture in the transformed frame:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_{\overline{S}}}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\overline{I}}}}\left(t\right), \left[\widetilde{\overline{H_{\overline{I}}}}(s), \widetilde{\overline{\rho_{\overline{S}}}}(t)\rho_B^{\mathrm{Thermal}}\right]\right]\mathrm{d}s \tag{373}$$

From the interaction picture applied on $\overline{H_{\overline{I}}}\left(t\right)$ we find:

$$\widetilde{\overline{H_{\overline{I}}}}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}\overline{H_{\overline{I}}}\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt}U\left(t\right) \tag{374}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\overline{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\overline{I}}}}\left(t\right) = \sum_i C_i\left(t\right)\left(\widetilde{A_i}(t)\otimes\widetilde{B_i}\left(t\right)\right) \tag{375}$$

$$\widetilde{A_i}(t) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}A_i\mathrm{e}^{-\mathrm{i}H_Bt}U\left(t\right) \tag{376}$$

$$= U^{\dagger}\left(t\right)A_iU\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}\mathrm{e}^{-\mathrm{i}H_Bt} \tag{377}$$

$$= U^{\dagger}\left(t\right)A_iU\left(t\right)\mathbb{I} \tag{378}$$

$$= U^{\dagger}\left(t\right)A_iU\left(t\right) \tag{379}$$

$$\widetilde{B_i}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt}U\left(t\right) \tag{380}$$

$$= U^{\dagger}\left(t\right)U\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{381}$$

$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{382}$$

$$= \mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{383}$$

Here we have used the fact that $\left[\overline{H_{\overline{S}}}\left(t\right), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right), \mathrm{e}^{\mathrm{i}H_Bt}\right] = 0$.

Using the expression (375) to replace it in the equation (373)

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}}\right]\right]\mathrm{d}s \tag{384}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \left[\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}}\right]\right]\mathrm{d}s \tag{385}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}} - \widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\right]\mathrm{d}s \tag{386}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}}\right. \tag{387}$$

$$-\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right) \tag{388}$$

$$-\sum_{i\in J} C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right) \tag{389}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B^{\text{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\right)\mathrm{d}s \tag{390}$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \left\langle\widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B \tag{391}$$

$$= \left\langle\widetilde{B_j}(\tau)\,\widetilde{B_i}(0)\right\rangle_B \tag{392}$$

The correlation functions relevant that appear in the equation (384) are:

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B^{\text{Thermal}}\right) = \left\langle\widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B \tag{393}$$

$$= \left\langle\widetilde{B_j}(\tau)\,\widetilde{B_i}(0)\right\rangle_B \tag{394}$$

$$= \Lambda_{ji}(\tau) \tag{395}$$

$$\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\rho_B^{\text{Thermal}}\widetilde{B_i}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B^{\text{Thermal}}\right) \tag{396}$$

$$= \left\langle\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right\rangle_B \tag{397}$$

$$= \left\langle\widetilde{B_i}(-\tau)\,\widetilde{B_j}(0)\right\rangle_B \tag{398}$$

$$= \Lambda_{ij}(-\tau) \tag{399}$$

$$\mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\rho_B^{\text{Thermal}}\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B^{\text{Thermal}}\right) \tag{400}$$

$$= \left\langle\widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B \tag{401}$$

$$= \left\langle\widetilde{B_j}(\tau)\,\widetilde{B_i}(0)\right\rangle_B \tag{402}$$

$$= \Lambda_{ji}(\tau) \tag{403}$$

$$\mathrm{Tr}_B\left(\rho_B^{\text{Thermal}}\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B^{\text{Thermal}}\right) \tag{404}$$

$$= \left\langle\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right\rangle_B \tag{405}$$

$$= \left\langle\widetilde{B_i}(-\tau)\,\widetilde{B_j}(0)\right\rangle_B \tag{406}$$

$$= \Lambda_{ij}(-\tau) \tag{407}$$

The cyclic property of the trace was use widely in the development of equations (393) and (407). Replacing in (384)

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)\widetilde{A_i}(t)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)-\Lambda_{ji}(-\tau)\widetilde{A_i}(t)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\big)+C_i(t)C_j(s)\big(\Lambda_{ji}(-\tau)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\widetilde{A_i}(t)-\Lambda_{ij}(\tau)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t)\big)\Big)\mathrm{d}s$$
$$= -\int_0^t \sum_{ij}\left(C_i\left(t\right)C_j\left(s\right)\left(\Lambda_{ij}\left(\tau\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right]+\Lambda_{ji}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}s$$

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{ij}\Big(C_i\left(t\right)C_j\left(s\right)\Big(\Lambda_{ij}\left(\tau\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)-\Lambda_{ji}\left(-\tau\right)\widetilde{A_i}\left(t\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right)\Big) \tag{408}$$
$$+C_i\left(t\right)C_j\left(s\right)\Big(\Lambda_{ji}\left(-\tau\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\Lambda_{ij}\left(\tau\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}\left(t\right)\Big)\Big)\mathrm{d}s \tag{409}$$
$$= -\int_0^t \sum_{ij}\left(C_i\left(t\right)C_j\left(s\right)\left(\Lambda_{ij}\left(\tau\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right]+\Lambda_{ji}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}s \tag{410}$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}\left(\tau\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)-\Lambda_{ij}\left(\tau\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}\left(t\right)=\Lambda_{ij}\left(\tau\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right] \tag{411}$$
$$\Lambda_{ji}\left(-\tau\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\Lambda_{ji}\left(-\tau\right)\widetilde{A_i}\left(t\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right)=\Lambda_{ji}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j},\widetilde{A_i}\left(t\right)\right] \tag{412}$$

Returning to the interaction picture we have:

$$U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right)=U\left(t\right)\widetilde{A_i}\left(t\right)U^{\dagger}\left(t\right)U\left(t\right)\widetilde{A_j}\left(s\right)U^{\dagger}\left(t\right)U\left(t\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right) \tag{413}$$
$$=\left(U\left(t\right)\widetilde{A_i}\left(t\right)U^{\dagger}\left(t\right)\right)\left(U\left(t\right)\widetilde{A_j}\left(s\right)U^{\dagger}\left(t\right)\right)\left(U\left(t\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right)\right) \tag{414}$$
$$=A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}(t) \tag{415}$$

This procedure applying to the relevant commutators give us:

$$U\left(t\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right]U^{\dagger}\left(t\right)=\left(U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right)-U\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}\left(t\right)U^{\dagger}\left(t\right)\right) \tag{416}$$
$$=A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}(t)-\widetilde{A_j}\left(s,t\right)\overline{\rho_S}(t)A_i \tag{417}$$
$$=\left[A_i,\widetilde{A_j}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{418}$$

Introducing this transformed commutators in the equation (410) allow us to obtain the master equation of the system

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}(t)\right]-\sum_{ij}\int_0^t \mathrm{d}\tau\left(C_i\left(t\right)C_j\left(t-\tau\right)\Lambda_{ij}\left(\tau\right)\left[A_i,\widetilde{A_j}\left(t-\tau,t\right)\overline{\rho_S}(t)\right]\right. \tag{419}$$
$$\left.+C_j\left(t\right)C_i\left(t-\tau\right)\Lambda_{ji}\left(-\tau\right)\left[\overline{\rho_S}(t)\widetilde{A_j}\left(t-\tau,t\right),A_i\right]\right) \tag{420}$$

where $i,j\in\{1,2,3,4,5.6\}$.
Here $\widetilde{A_j}\left(s,t\right)=U\left(t\right)U^{\dagger}\left(s\right)A_jU\left(s\right)U^{\dagger}\left(t\right)$ where $U\left(t\right)$ is given by (353). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. The environmental correlation functions are given by:

$$\Lambda_{ij}\left(\tau\right)=\mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B^{\mathrm{Thermal}}\right) \tag{421}$$
$$=\mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B^{\mathrm{Thermal}}\right) \tag{422}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau)\,\widetilde{B_{jz}}(0)\,\rho_B^{\mathrm{Thermal}}\right) \tag{423}$$

$$= \int \mathrm{d}^2\alpha P(\alpha)\left\langle \alpha\left|\widetilde{B_{jz}}(\tau)\,\widetilde{B_{jz}}(0)\right|\alpha\right\rangle \tag{424}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle \alpha\left|\widetilde{B_{jz}}(\tau)\,\widetilde{B_{jz}}(0)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{425}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle \alpha\left|\widetilde{B_{jz}}(\tau)\,\widetilde{B_{jz}}(0)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{426}$$

$$\widetilde{B_{jz}}(\tau) = \sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{427}$$

$$\widetilde{B_{jz}}(0) = \sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger} + \left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right) \tag{428}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau)\,\widetilde{B_{jz}}(0)\,\rho_B\right) \tag{429}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right)\rho_B\right) \tag{430}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right)\rho_B\right) \tag{431}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}b_{\mathbf{k}}\right)\rho_B\right) \tag{432}$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}} = p_{j\mathbf{k}} \tag{433}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}'}b_{\mathbf{k}'}^{\dagger}+p_{j\mathbf{k}'}^{*}b_{\mathbf{k}'}\right)\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}\right)\rho_B\right) \tag{434}$$

$$= 0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}\right)\rho_B\right) \tag{435}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^{*2} b_{\mathbf{k}} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B\right) \tag{436}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^{*2} b_{\mathbf{k}} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{437}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{438}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\rho_B\right) + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\rho_B\right)\right) \tag{439}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\middle|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{440}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\middle|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{441}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}})\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{442}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{443}$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}})\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{444}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{445}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}})\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{446}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}})\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{447}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|\left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*\right)\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{448}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*\right)\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{449}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{450}$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{451}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^*\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{452}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^*\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{453}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{454}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{455}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{456}$$

$$1 = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{457}$$

$$b_{\mathbf{k}}^\dagger b_{\mathbf{k}}|0\rangle = 0 \tag{458}$$

$$b_{\mathbf{k}} b_{\mathbf{k}}^\dagger|0\rangle = |0\rangle \tag{459}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{460}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\middle|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{461}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{462}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{463}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\frac{1}{\pi N}\int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{464}$$

$$\frac{1}{\pi N}\int\left|\alpha_{\mathbf{k}}\right|^2\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int_0^{2\pi}\int_0^{\infty} r^2\exp\left(-\frac{r^2}{N}\right)r\mathrm{d}r\mathrm{d}\theta \tag{465}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N\right) + \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2 e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{466}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{467}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{468}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1} + \cos\left(\omega_{\mathbf{k}}\tau\right) - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{469}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(2+e^{\beta\omega_{\mathbf{k}}}-1\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{470}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(1+e^{\beta\omega_{\mathbf{k}}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{471}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}+e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}-e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{472}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right) - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{473}$$

$$= \sum_{\mathbf{k}}\left|g_{j\mathbf{k}}-v_{j\mathbf{k}}\right|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right) - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{474}$$

$$\left\langle\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{j'z}}\left(0\right)\right\rangle_B = \int\mathrm{d}^2\alpha_{\mathbf{k}}P\left(\alpha_{\mathbf{k}}\right)\left\langle\alpha_{\mathbf{k}}\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{j'z}}\left(0\right)\right|\alpha_{\mathbf{k}}\right\rangle \tag{475}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{j'z}}\left(0\right)\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{476}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{477}$$

$$\times\sum_{\mathbf{k}'}\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{478}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{479}$$

$$\times\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{480}$$

$$+\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{481}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{482}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{483}$$

$$+\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{484}$$

$$= \sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{485}$$

$$+\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{486}$$

$$\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{487}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{488}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^*\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{489}$$

$$\begin{aligned}
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)|\alpha_{\mathbf{k}}|^2\,\mathrm{d}^2\alpha_{\mathbf{k}} && (490)\\
&= N && (491)\\
\tfrac{1}{\pi N}\textstyle\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle \alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^\dagger D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (492)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^\dagger+\alpha_{\mathbf{k}}^*\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (493)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger+\alpha_{\mathbf{k}}b_{\mathbf{k}}^\dagger+b_{\mathbf{k}}\alpha_{\mathbf{k}}^*+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (494)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (495)\\
&= \tfrac{1}{\pi N}\textstyle\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}+\tfrac{1}{\pi N}\textstyle\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (496)\\
&= N+1 && (497)\\
\left\langle \widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{j'z}}\left(0\right)\right\rangle_B &= \textstyle\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N+\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N+1) && (498)\\
&= \sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} && (499)\\
&+N\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) && (500)\\
D\left(h'\right)D\left(h\right) &= \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)D\left(h'+h\right) && (501)\\
\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B &= \mathrm{Tr}_B\left(\exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)D\left(h'+h\right)\rho_B^{\mathrm{Thermal}}\right) && (502)\\
&= \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)\mathrm{Tr}_B\left(D\left(h'+h\right)\rho_B^{\mathrm{Thermal}}\right) && (503)\\
&= \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)\frac{1}{\pi N}\int \mathrm{d}^2\alpha P\left(\alpha\right)\left\langle \alpha\left|D\left(h'+h\right)\right|\alpha\right\rangle && (504)\\
&= \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) && (505)\\
h' &= h\exp\left(\mathrm{i}\omega\tau\right) && (506)\\
\left\langle D\left(h\exp\left(\mathrm{i}\omega\tau\right)\right)D\left(h\right)\right\rangle_B &= \exp\left(\frac{1}{2}\left(hh^*\exp\left(\mathrm{i}\omega\tau\right)-h^*h\exp\left(-\mathrm{i}\omega\tau\right)\right)\right)\exp\left(-\frac{|h+h\exp\left(\mathrm{i}\omega\tau\right)|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) && (507)\\
\frac{1}{2}\left(hh^*\exp\left(\mathrm{i}\omega\tau\right)-h^*h\exp\left(-\mathrm{i}\omega\tau\right)\right) &= \frac{1}{2}|h|^2\left(\exp\left(\mathrm{i}\omega\tau\right)-\exp\left(-\mathrm{i}\omega\tau\right)\right) && (508)\\
&= \frac{1}{2}|h|^2\left(\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)-\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right) && (509)\\
&= \frac{1}{2}|h|^2\left(2\mathrm{i}\sin\left(\omega\tau\right)\right) && (510)\\
&= \mathrm{i}\,|h|^2\sin\left(\omega\tau\right) && (511)\\
-\frac{|h+h\exp\left(\mathrm{i}\omega\tau\right)|^2}{2} &= -|h|^2\frac{|1+\exp\left(\mathrm{i}\omega\tau\right)|^2}{2} && (512)\\
&= -|h|^2\frac{|1+\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)|^2}{2} && (513)\\
&= -|h|^2\frac{\left(1+\cos\left(\omega\tau\right)\right)^2+\sin^2\left(\omega\tau\right)}{2} && (514)
\end{aligned}$$

$$= -\left|h\right|^2 \frac{\left(1+2\cos\left(\omega\tau\right)+\cos^2\left(\omega\tau\right)\right)+\sin^2\left(\omega\tau\right)}{2} \tag{515}$$

$$= -\left|h\right|^2 \frac{2+2\cos\left(\omega\tau\right)}{2} \tag{516}$$

$$= -\left|h\right|^2 \left(1+\cos\left(\omega\tau\right)\right) \tag{517}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\right\rangle_B = \exp\left(\mathrm{i}|h|^2\sin(\omega\tau)\right)\exp\left(-|h|^2(1+\cos(\omega\tau))\coth\left(\tfrac{\beta\omega}{2}\right)\right) \tag{518}$$

$$= \exp\left(\mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right) - \left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{519}$$

$$= \exp\left(-\left|h\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-\left|h\right|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{520}$$

$$= \left\langle D\left(h\right)\right\rangle_B \exp\left(-\phi\left(\tau\right)\right) \tag{521}$$

$$\exp\left(-\phi\left(\tau\right)\right) = \exp\left(-\left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)\right) \tag{522}$$

$$\phi\left(\tau\right) = \left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{523}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)\exp\left(-\frac{\left|h+h'\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{524}$$

$$h' = v\exp\left(i\omega\tau\right) \tag{525}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}\left(\tau\right)\widetilde{B_{1+}B_{0-}}\left(0\right)\rho_B^{\mathrm{Thermal}}\right) \tag{526}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}\left(\tau\right)\widetilde{B_{1+}B_{0-}}\left(0\right)\rho_B^{\mathrm{Thermal}}\right) \tag{527}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{528}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{529}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{530}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\prod_{\mathbf{k}}\left(\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{531}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{532}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{533}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{534}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{535}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{536}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{537}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{538}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega\tau+\pi)}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) \tag{539}$$

$$= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{540}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{541}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{542}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{543}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}_B\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\left(\omega_{\mathbf{k}}\tau+\pi\right)}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{544}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{545}$$

$$= \left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B \tag{546}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{547}$$

$$\langle D(h)b\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|D(h)b|\alpha\rangle \tag{548}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\left|D(-\alpha)D(h)bD(\alpha)\right|\alpha\rangle \tag{549}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)bD(\alpha)\right|0\rangle \tag{550}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)D(\alpha)D(-\alpha)bD(\alpha)\right|0\rangle \tag{551}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)D(\alpha)(b+\alpha)\right|0\rangle \tag{552}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\left|D(h)(b+\alpha)\right|0\rangle \tag{553}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|D(h)b|0\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|D(h)\alpha|0\rangle \tag{554}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\left|D(h)\alpha\right|0\rangle \tag{555}$$

$$= \frac{1}{\pi N}\int \alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{556}$$

$$= hN\langle D(h)\rangle_B \tag{557}$$

$$\left\langle D(h)b^{\dagger}\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle\alpha\left|D(h)b^{\dagger}\right|\alpha\right\rangle \tag{558}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)b^{\dagger}D(\alpha)\right|0\right\rangle \tag{559}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)b^{\dagger}D(\alpha)\right|0\right\rangle \tag{560}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)D(-\alpha)b^{\dagger}D(\alpha)\right|0\right\rangle \tag{561}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)\left(b^{\dagger}+\alpha^*\right)\right|0\right\rangle \tag{562}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\left\langle 0\left|D(h)\left(b^{\dagger}+\alpha^*\right)\right|0\right\rangle \tag{563}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle 0 \left| D(h) b^\dagger \right| 0 \right\rangle \tag{564}$$

$$+ \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle 0 \left| D(h) \alpha^* \right| 0 \right\rangle \tag{565}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle 0 \left| D(h) \right| 1 \right\rangle \tag{566}$$

$$+ \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha^* \left\langle 0 \left| D(h) \right| 0 \right\rangle \tag{567}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle -h \right| \left. 1 \right\rangle \tag{568}$$

$$+ \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha^* \left\langle 0 \left| D(h) \right| 0 \right\rangle \tag{569}$$

$$\langle -h | = \exp\left(-\frac{|-h^*|^2}{2}\right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n | \tag{570}$$

$$\langle -h | \, 1 \rangle = \exp\left(-\frac{|-h^*|^2}{2}\right) \left(-h^*\right) \tag{571}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|-h^*|^2}{2}\right) \left(-h^*\right) \tag{572}$$

$$+ \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha^* \exp\left(-\frac{|-h^*|^2}{2}\right) \tag{573}$$

$$= -h^* \left\langle D(h) \right\rangle_B (N+1) \tag{574}$$

$$\left\langle bD(h) \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| bD(h) \right| \alpha \right\rangle \tag{575}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h \tag{576}$$

$$+ \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha \exp\left(-\frac{|h|^2}{2}\right) \tag{577}$$

$$= h \left\langle D(h) \right\rangle_B (N+1) \tag{578}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| b^\dagger D(h) \right| \alpha \right\rangle \tag{579}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h \tag{580}$$

$$+ \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha \exp\left(-\frac{|h|^2}{2}\right) \tag{581}$$

$$= -h^* \langle D(h) \rangle_B N \tag{582}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \right\rangle_B \tag{583}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right\rangle_B \tag{584}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right\rangle_B \tag{585}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{586}$$

$$= B_{10} \tag{587}$$

The correlation functions can be found readily as:

$$\widetilde{B_{1+}B_{0-}}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \tag{588}$$

$$\widetilde{B_{0+}B_{1-}}(\tau)=\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\tag{589}$$

$$\widetilde{B_{x}}(0)=\frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{10}^{*}}{2}\tag{590}$$

$$\widetilde{B_{y}}(0)=\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}}{2\mathrm{i}}\tag{591}$$

$$B_{10}=\left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\tag{592}$$

$$B_{iz}=\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}\right)\tag{593}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_{B}=\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j\mathbf{k}}-v_{jk}\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{B}\tag{594}$$

$$=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\tag{595}$$

$$\left\langle\widetilde{B_{x}}(\tau)\widetilde{B_{x}}(0)\right\rangle_{B}=\left\langle\frac{B_{1+}B_{0-}(\tau)+B_{0+}B_{1-}(\tau)-B_{10}-B_{10}^{*}}{2}\frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{10}^{*}}{2}\right\rangle_{B}\tag{596}$$

$$=\frac{1}{4}\left\langle\left(B_{1+}B_{0-}(\tau)+B_{0+}B_{1-}(\tau)-B_{10}-B_{10}^{*}\right)\left(B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{10}^{*}\right)\right\rangle_{B}\tag{597}$$

$$=\frac{1}{4}\left\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{10}^{*}\right.\tag{598}$$

$$+B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*}\tag{599}$$

$$\left.-B_{10}B_{1+}B_{0-}-B_{10}B_{0+}B_{1-}+B_{10}B_{10}+B_{10}B_{10}^{*}-B_{10}^{*}B_{1+}B_{0-}-B_{10}^{*}B_{0+}B_{1-}+B_{10}^{*}B_{10}+B_{10}^{*}B_{10}^{*}\right\rangle\tag{600}$$

$$=\frac{1}{4}\left\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{10}^{*}\right.\tag{601}$$

$$\left.+B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*}\right\rangle\tag{602}$$

$$\left\langle\widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_{B}=\Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\tag{603}$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\tag{604}$$

$$\phi(\tau)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\tag{605}$$

$$S=\prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\tag{606}$$

$$\left\langle\widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_{B}=U\exp\left(-\phi(\tau)\right)S\tag{607}$$

$$\left\langle\widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_{B}=U^{*}\exp\left(-\phi(\tau)\right)S\tag{608}$$

$$\left\langle\widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_{B}=\exp\left(\phi(\tau)\right)S\tag{609}$$

$$\left\langle\widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_{B}=\left\langle\widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_{B}\tag{610}$$

$$\left\langle\widetilde{B_{1+}B_{0-}}(\tau)\right\rangle_{B}=\prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\tag{611}$$

$$=U^{*1/2}S^{1/2}\tag{612}$$

$$\left\langle\widetilde{B_{x}}(\tau)\widetilde{B_{x}}(0)\right\rangle_{B}=\frac{1}{4}\left\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{10}^{*}\right.\tag{613}$$

$$\left.+B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*}\right\rangle\tag{614}$$

$$=\frac{1}{4}\left(U^{*}\exp\left(-\phi(\tau)\right)S+\exp\left(\phi(\tau)\right)S-B_{10}^{2}-\left|B_{10}\right|^{2}+\exp\left(\phi(\tau)\right)S+U\exp\left(-\phi(\tau)\right)S-B_{10}^{*2}-\left|B_{10}\right|^{2}\right)\tag{615}$$

$$=\frac{1}{4}\left(2\Re(U)\exp\left(-\phi(\tau)\right)S+2\exp\left(\phi(\tau)\right)S-2\Re\left(B_{10}^{2}\right)-2\left|B_{10}\right|^{2}\right)\tag{616}$$

$$=\frac{1}{4}\left(2\Re(U)\exp\left(-\phi(\tau)\right)S+2\exp\left(\phi(\tau)\right)S-2\Re\left(U^{*}\right)S-2S\right)\tag{617}$$

$$=\frac{S}{2}\left(\Re(U)\exp\left(-\phi(\tau)\right)+\exp\left(\phi(\tau)\right)-\Re\left(U^{*}\right)-1\right)\tag{618}$$

$$\left\langle\widetilde{B_{y}}(\tau)\widetilde{B_{y}}(0)\right\rangle_{B}=\left\langle\frac{B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{10}^{*}}{2\mathrm{i}}\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}}{2\mathrm{i}}\right\rangle_{B}\tag{619}$$

$$=-\frac{1}{4}\left\langle\left(B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{10}^{*}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}\right)\right\rangle_{B}\tag{620}$$

$$=-\frac{1}{4}\left\langle B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*}\right.\tag{621}$$

$$-B_{1+}B_{0-}(\tau)B_{0+}B_{1-}+B_{1+}B_{0-}(\tau)B_{1+}B_{0-}-B_{1+}B_{0-}(\tau)B_{10}+B_{1+}B_{0-}(\tau)B_{10}^{*}\tag{622}$$

$$\left.+B_{10}B_{0+}B_{1-}-B_{10}B_{1+}B_{0-}+B_{10}B_{10}-B_{10}B_{10}^{*}-B_{10}^{*}B_{0+}B_{1-}+B_{10}^{*}B_{1+}B_{0-}-B_{10}^{*}B_{10}+B_{10}^{*}B_{10}^{*}\right\rangle\tag{623}$$

$$= -\frac{1}{4}\left\langle B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^{*}\right. \tag{624}$$

$$\left. -B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{10} + B_{1+}B_{0-}(\tau)\,B_{10}^{*}\right\rangle \tag{625}$$

$$= -\frac{1}{4}\left\langle B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{10}^{*}B_{10} - B_{10}^{*}B_{10}^{*}\right. \tag{626}$$

$$\left. -B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{10}B_{10} + B_{10}B_{10}^{*}\right\rangle \tag{627}$$

$$= -\frac{1}{4}\left(U\exp\left(-\phi(\tau)\right)S - \exp\left(\phi(\tau)\right)S - \exp\left(\phi(\tau)\right)S + U^{*}\exp\left(-\phi(\tau)\right)S\right. \tag{628}$$

$$\left. +2S - 2\Re\left(U^{*}\right)S\right) \tag{629}$$

$$= -\frac{S}{4}\left(2\Re(U)\exp\left(-\phi(\tau)\right) - 2\exp\left(\phi(\tau)\right) + 2 - 2\Re(U)\right) \tag{630}$$

$$= \frac{S}{2}\left(\exp\left(\phi(\tau)\right) - \Re(U)\exp\left(-\phi(\tau)\right) - 1 + \Re(U)\right) \tag{631}$$

$$\left\langle \widetilde{B_x}(\tau)\,\widetilde{B_y}(0)\right\rangle_B = \left\langle \frac{B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^{*}}{2}\,\frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^{*}}{2\mathrm{i}}\right\rangle_B \tag{632}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^{*}\right)\left(B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^{*}\right)\right\rangle_B \tag{633}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)\,B_{10} - B_{1+}B_{0-}(\tau)\,B_{10}^{*}\right. \tag{634}$$

$$+ B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^{*} \tag{635}$$

$$\left. -B_{10}B_{0+}B_{1-} + B_{10}B_{1+}B_{0-} - B_{10}B_{10} + B_{10}B_{10}^{*} - B_{10}^{*}B_{0+}B_{1-} + B_{10}^{*}B_{1+}B_{0-} - B_{10}^{*}B_{10} + B_{10}^{*}B_{10}^{*}\right\rangle \tag{636}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)\,B_{10} - B_{1+}B_{0-}(\tau)\,B_{10}^{*}\right. \tag{637}$$

$$\left. +B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^{*}\right\rangle \tag{638}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{10}B_{10} - B_{10}B_{10}^{*}\right. \tag{639}$$

$$\left. +B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{10}^{*}B_{10} - B_{10}^{*}B_{10}^{*}\right\rangle \tag{640}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{10}B_{10} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} - B_{10}^{*}B_{10}^{*}\right\rangle \tag{641}$$

$$= \frac{1}{4\mathrm{i}}\left(\exp\left(\phi(\tau)\right)S - U^{*}\exp\left(-\phi(\tau)\right)S + U\exp\left(-\phi(\tau)\right)S - \exp\left(\phi(\tau)\right)S + U^{*}S - US\right) \tag{642}$$

$$= \frac{1}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi(\tau)\right)S + U\exp\left(-\phi(\tau)\right)S + U^{*}S - US\right) \tag{643}$$

$$= \frac{S}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi(\tau)\right) + U\exp\left(-\phi(\tau)\right) + U^{*} - U\right) \tag{644}$$

$$= \frac{S\left(U - U^{*}\right)}{4\mathrm{i}}\left(\exp\left(-\phi(\tau)\right) - 1\right) \tag{645}$$

$$= \frac{2\mathrm{i}\Im(U)\,S}{4\mathrm{i}}\left(\exp\left(-\phi(\tau)\right) - 1\right) \tag{646}$$

$$= \frac{\Im(U)\,S}{2}\left(\exp\left(-\phi(\tau)\right) - 1\right) \tag{647}$$

$$\left\langle \widetilde{B_y}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \left\langle \tfrac{B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^{*}}{2\mathrm{i}}\,\tfrac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^{*}}{2}\right\rangle_B \tag{648}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^{*}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^{*}\right)\right\rangle_B \tag{649}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^{*}\right. \tag{650}$$

$$- B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{10} + B_{1+}B_{0-}(\tau)\,B_{10}^{*} \tag{651}$$

$$\left. +B_{10}B_{1+}B_{0-} + B_{10}B_{0+}B_{1-} - B_{10}B_{10} - B_{10}B_{10}^{*} - B_{10}^{*}B_{1+}B_{0-} - B_{10}^{*}B_{0+}B_{1-} + B_{10}^{*}B_{10} + B_{10}^{*}B_{10}^{*}\right\rangle \tag{652}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^{*}\right. \tag{653}$$

$$\left. -B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)\,B_{10} + B_{1+}B_{0-}(\tau)\,B_{10}^{*}\right\rangle \tag{654}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{10}^{*}B_{10} - B_{10}^{*}B_{10}^{*}\right. \tag{655}$$

$$\left. -B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{10}B_{10} + B_{10}B_{10}^{*}\right\rangle \tag{656}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{10}^{*}B_{10}^{*} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{10}B_{10}\right\rangle \tag{657}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi(\tau)\right)S - U^{*}\exp\left(-\phi(\tau)\right)S + B_{10}^{2} - B_{10}^{*2}\right) \tag{658}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi(\tau)\right)S - U^{*}\exp\left(-\phi(\tau)\right)S + U^{*}S - US\right) \tag{659}$$

$$= \frac{S\left(U - U^{*}\right)}{4\mathrm{i}}\left(\exp\left(-\phi(\tau)\right) - 1\right) \tag{660}$$

$$= \frac{2\mathrm{i}\Im(U)\,S}{4\mathrm{i}}\left(\exp\left(-\phi(\tau)\right) - 1\right) \tag{661}$$

$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} \tag{662}$$

$$\left\langle {}^{B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}} \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle \tag{663}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} \tag{664}$$

$$\left\langle {}^{B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger} \right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}^* \tag{665}$$

$$\left\langle {}^{B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}} \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}^* \tag{666}$$

$$\left\langle \widetilde{B_x}(\tau)\, \widetilde{B_{iz}}(0) \right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}^*\right. \tag{667}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}^*\right) \tag{668}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}^*\right. \tag{669}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}^*\right) \tag{670}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10}^*\right)\right. \tag{671}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10}^*\right)\right) \tag{672}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10}^*\right)\right. \tag{673}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10}^*\right)\right) \tag{674}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(B_{10} - B_{10}^*\right)\right. \tag{675}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{10} - B_{10}^*\right)\right) \tag{676}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} 2\mathrm{i}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{677}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} \Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{678}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} \Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) \tag{679}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\, \widetilde{B_x}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2}\right)\right\rangle_B \tag{680}$$

$$= \sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2}\right)\right\rangle_B \tag{681}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*\right)\right\rangle_B \tag{682}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-}\right)\right\rangle_B \tag{683}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+}B_{1-}\right. \tag{684}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+}B_{1-}\right\rangle \tag{685}$$

$$\left\langle {}^{(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-}} \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-}\right\rangle_B \tag{686}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\right\rangle_B \tag{687}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left\langle b^{\dagger}_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}'} v_{0\mathbf{k}'} - v_{1\mathbf{k}'} v^*_{0\mathbf{k}'}}{\omega^2_{\mathbf{k}'}}\right)\right)\right)\right\rangle_B \tag{688}$$

$$\times \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right)\right)\right\rangle_B \tag{689}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B \left\langle b^{\dagger}_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{690}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B \left\langle b^{\dagger}_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{691}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B \left\langle b^{\dagger}_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{692}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B \left(-\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B N_{\mathbf{k}'}\right) e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{693}$$

$$= -\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{694}$$

$$= -\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{695}$$

$$\left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b^{\dagger}_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+} B_{1-}\right\rangle_B = -\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B^*_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{696}$$

$$\left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+} B_{0-}\right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} B_{1+} B_{0-}\right\rangle_B \tag{697}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right)\right)\right\rangle_B \tag{698}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \left\langle b_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{699}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{700}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}} v_{0\mathbf{k}} - v_{1\mathbf{k}} v^*_{0\mathbf{k}}}{\omega^2_{\mathbf{k}}}\right)\right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{701}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B_{10} \tag{702}$$

$$\left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+} B_{1-}\right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B^*_{10} \tag{703}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right) \widetilde{B_x}\left(0\right)\right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'} \left(-\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B^*_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right. \tag{704}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B^*_{10}\right) \tag{705}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(B^*_{10} - B_{10}\right) + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right) \left(B_{10} - B^*_{10}\right)\right) \tag{706}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(B^*_{10} - B_{10}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right) \left(B^*_{10} - B_{10}\right)\right) \tag{707}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} \Im\left(B_{10}\right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right)\right) \tag{708}$$

$$\left\langle \widetilde{B_y}\left(\tau\right) \widetilde{B_{iz}}\left(0\right)\right\rangle_B = \left\langle \left(\frac{B_{0+}B_{1-}\left(\tau\right) - B_{1+}B_{0-}\left(\tau\right) + B_{10} - B^*_{10}}{2\mathrm{i}}\right) \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b^{\dagger}_{\mathbf{k}'} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{709}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \left\langle \left(B_{0+}B_{1-}\left(\tau\right) - B_{1+}B_{0-}\left(\tau\right) + B_{10} - B^*_{10}\right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b^{\dagger}_{\mathbf{k}'} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{710}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \left\langle \left(B_{0+}B_{1-}\left(\tau\right) - B_{1+}B_{0-}\left(\tau\right)\right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b^{\dagger}_{\mathbf{k}'} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{711}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'} \left\langle B_{0+}B_{1-}\left(\tau\right) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b^{\dagger}_{\mathbf{k}'} - B_{1+}B_{0-}\left(\tau\right) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b^{\dagger}_{\mathbf{k}'}\right. \tag{712}$$

$$\left| + B_{0+}B_{1-}\left(\tau\right) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} - B_{1+}B_{0-}\left(\tau\right) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right\rangle \tag{713}$$

$$\left\langle B_{0+}B_{1-}\left(\tau\right) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b^{\dagger}_{\mathbf{k}'}\right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B^*_{10} \tag{714}$$

$$\left\langle B_{0+}B_{1-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10}^* \tag{715}$$

$$\left\langle B_{1+}B_{0-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger\right\rangle_B = -\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right) \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'}+1\right)B_{10} \tag{716}$$

$$\left\langle B_{1+}B_{0-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10} \tag{717}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'}+1\right)B_{10}^*+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{718}$$

$$+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10}^* - \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10}\Big) \tag{719}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'}+1\right)B_{10}^*+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{720}$$

$$+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10}^* - \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10}\Big) \tag{721}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\left(B_{10}+B_{10}^*\right)\right. \tag{722}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(-B_{10}-B_{10}^*\right)\right) \tag{723}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\left(B_{10}+B_{10}^*\right)\right. = \tag{724}$$

$$\left.-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(B_{10}+B_{10}^*\right)\right) \tag{725}$$

$$= \frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re\left(B_{10}\right)\right. = \tag{726}$$

$$\left.-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\right) \tag{727}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\Re\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\right) \tag{728}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_y}\left(0\right)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^*}{2\mathrm{i}}\right)\right\rangle_B \tag{729}$$

$$= \frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^*\right)\right\rangle_B \tag{730}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}\right)\right\rangle_B \tag{731}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-} \tag{732}$$

$$\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}\Big\rangle \tag{733}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle\right. \tag{734}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_{1+}B_{0-}\right\rangle\right) \tag{735}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle\right. \tag{736}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_{1+}B_{0-}\right\rangle\right) \tag{737}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle\right. \tag{738}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_{1+}B_{0-}\right\rangle\right) \tag{739}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_{1+}B_{0-}\right\rangle_B = -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}N_{\mathbf{k}'} \tag{740}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_{0+}B_{1-}\right\rangle_B = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}^*N_{\mathbf{k}'} \tag{741}$$

$$\left\langle b_{\mathbf{k}'}B_{1+}B_{0-}\right\rangle_B = \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{742}$$

$$\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle_B = \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^* \tag{743}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}^*N_{\mathbf{k}'}\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}N_{\mathbf{k}'}\right)\right. \tag{744}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^*\right)-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\right) \tag{745}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}^*N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}N_{\mathbf{k}'}\right)\right. \tag{746}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^*\right)\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\right) \tag{747}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\left(-g_{i\mathbf{k}'}+v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}^*N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}N_{\mathbf{k}'}\right)\right. \tag{748}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^*\right)\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\right)\right) \tag{749}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\left(B_{10}+B_{10}^*\right)N_{\mathbf{k}'}\right. \tag{750}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(B_{10}+B_{10}^*\right)\left(N_{\mathbf{k}'}+1\right)\right) \tag{751}$$

$$= \frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re\left(B_{10}\right)N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)\right) \tag{752}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re\left(B_{10}\right)N_{\mathbf{k}'}\right) \tag{753}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re\left(B_{10}\right)N_{\mathbf{k}'}\right) \tag{754}$$

$$= \mathrm{i}\Re\left(B_{10}\right)\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*N_{\mathbf{k}'}\right) \tag{755}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{756}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{757}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{758}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{\left|B_{10}\right|^2}{2}\left(\Re(U)\exp\left(-\phi(\tau)\right)+\exp\left(\phi(\tau)\right)-\Re(U)-1\right) \tag{759}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{\left|B_{10}\right|^2}{2}\left(\exp\left(\phi(\tau)\right)-\Re(U)\exp\left(-\phi(\tau)\right)-1+\Re(U)\right) \tag{760}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{\Im(U)\left|B_{10}\right|^2}{2}\left(\exp\left(-\phi(\tau)\right)-1\right) \tag{761}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{\Im(U)\left|B_{10}\right|^2}{2}\left(\exp\left(-\phi(\tau)\right)-1\right) \tag{762}$$

$$\left\langle \widetilde{B}_{iz}(\tau)\widetilde{B_x}(0)\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{763}$$

$$\left\langle \widetilde{B}_x(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{764}$$

$$\left\langle \widetilde{B}_{iz}(\tau)\widetilde{B_y}(0)\right\rangle_B = \mathrm{i}\Re\left(B_{10}\right)\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{765}$$

$$\left\langle \widetilde{B}_y(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}}\Re\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{766}$$

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B^{\mathrm{Thermal}}\right) \tag{767}$$

$$= \frac{B(\tau)\,B(0)}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{768}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B^{\mathrm{Thermal}}\right) \tag{769}$$

$$= \frac{B(\tau)\,B(0)}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) \tag{770}$$

$$\Lambda_{33}(\tau) = \int_0^{\infty} \mathrm{d}\omega J(\omega)\,(1 - F(\omega))^2\,G_+(\tau) \tag{771}$$

$$\Lambda_{32}(\tau) = B(\tau)\int_0^{\infty} \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)\,(1 - F(\omega))\,\mathrm{i}G_-(\tau) \tag{772}$$

$$\Lambda_{23}(\tau) = -B(0)\int_0^{\infty} \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega,\tau)\,(1 - F(\omega,\tau))\,\mathrm{i}G_-(\tau) \tag{773}$$

$$\Lambda_{12}(\tau) = \Lambda_{21}(\tau) = \Lambda_{13}(\tau) = \Lambda_{31}(\tau) = 0 \tag{774}$$

With the phonon propagator given by:

$$\phi(\tau) = \int_0^{\infty} \mathrm{d}\omega \frac{J(\omega)}{\omega^2} F(\omega)^2\,G_+(\tau) \tag{775}$$

defined in terms of $G_{\pm}(\tau) = (n(\omega) + 1)\,\mathrm{e}^{-\mathrm{i}\tau\omega} \pm n(\omega)\,e^{-\mathrm{i}\tau\omega}$ with $n(\omega) = \left(\mathrm{e}^{\beta\omega} - 1\right)^{-1}$ the occupation number.
The eigenvalues of the Hamiltonian $\overline{H_S}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \mathrm{Tr}\left(\overline{H_S}\right)\lambda + \mathrm{Det}\left(\overline{H_S}\right) = 0 \tag{776}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S\left|\pm\right\rangle = \lambda_{\pm}\left|\pm\right\rangle$. Using this notation is possible to write $H_S = \lambda_+\left|+\right\rangle\left\langle+\right| + \lambda_-\left|-\right\rangle\left\langle-\right|$.
The time-dependence of the system operators $\widetilde{A_i}(t)$ may be made explicit using the Fourier decomposition:

$$\widetilde{A_i}(\tau) = e^{\mathrm{i}\overline{H_S}\tau} A_i e^{-\mathrm{i}\overline{H_S}\tau} \tag{777}$$

$$= \sum_w e^{-\mathrm{i}w\tau} A_i(w) \tag{778}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.
In order to use the equation (778) to descompose the equation (353) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}(t')\right)$ like:

$$U(t) \equiv \mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}(t')\right) \tag{779}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty}(-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1)\,H(t_2)\,...H(t_n) \tag{780}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}(t')\right) \equiv \exp(-\mathrm{i}tH_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U\left(t\right)=\exp\left(-\mathrm{i}t\left(H_E^{(0)}\left(t\right)+H_E^{(1)}\left(t\right)+H_E^{(2)}\left(t\right)+...\right)\right) \tag{781}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}\left(t'\right)\right)$ and $U\left(t\right)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}\left(t\right)=\frac{1}{t}\int_0^t \overline{H_S}\left(t'\right)\mathrm{d}t' \tag{782}$$

$$H_E^{(1)}\left(t\right)=-\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\left[\overline{H_S}\left(t'\right),\overline{H_S}\left(t''\right)\right] \tag{783}$$

$$H_E^{(2)}\left(t\right)=\frac{1}{6t}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\int_0^{t''}\mathrm{d}t'''\left(\left[\left[\overline{H_S}\left(t'\right),\overline{H_S}\left(t''\right)\right],\overline{H_S}\left(t'''\right)\right]+\left[\left[\overline{H_S}\left(t'''\right),\overline{H_S}\left(t''\right)\right],\overline{H_S}\left(t'\right)\right]\right) \tag{784}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}\left(t\right)=e^{\mathrm{i}H_E(t)t}A_i(t)e^{-\mathrm{i}H_E(t)t} \tag{785}$$

$$=\sum_{w(t)}e^{-\mathrm{i}w(t)t}A_i\left(w\left(t\right)\right) \tag{786}$$

$w\left(t\right)$ belongs to the set of differences of eigenvalues that depends of the time. As we can see the eigenvectors are time dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}\left(t-\tau,t\right)$ using the Magnus expansion generates:

$$\widetilde{A_j}\left(t-\tau,t\right)=U\left(t-\tau\right)U^\dagger\left(t\right)A_j\left(t\right)U\left(t\right)U^\dagger\left(t-\tau\right) \tag{787}$$

$$=e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}e^{\mathrm{i}H_E(t)t}A_j(t)e^{-\mathrm{i}H_E(t)t}e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{788}$$

$$=e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}\sum_{w(t)}e^{-\mathrm{i}w(t)t}A_j\left(w\left(t\right)\right)e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{789}$$

$$=\sum_{w(t),w'(t-\tau)}e^{-\mathrm{i}w(t)t}e^{\mathrm{i}w'(t-\tau)}A_j'\left(w\left(t\right),w'\left(t-\tau\right)\right) \tag{790}$$

where $w'\left(t-\tau\right)$ and $w\left(t\right)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $H_S\left(t-\tau\right)$ and $H_S\left(t\right)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (778) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V}=\left\{\left|+\right\rangle,\left|-\right\rangle\right\}$ in the following way:

$$A_i=\sum_{\alpha,\beta\in\mathrm{V}}\left\langle\alpha\right|A_i\left|\beta\right\rangle\left|\alpha\right\rangle\left\langle\beta\right| \tag{791}$$

Given that $\left[\left|+\right\rangle\left\langle+\right|,\left|-\right\rangle\left\langle-\right|\right]=0$, then using the Zassenhaus formula we obtain:

$$e^{\mathrm{i}\overline{H_S}\tau}=e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{792}$$

$$=e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} \tag{793}$$

$$=\left(\left|-\right\rangle\left\langle-\right|+e^{\mathrm{i}\lambda_+\tau}\left|+\right\rangle\left\langle+\right|\right)\left(\left|+\right\rangle\left\langle+\right|+e^{\mathrm{i}\lambda_-\tau}\left|-\right\rangle\left\langle-\right|\right) \tag{794}$$

$$=e^{\mathrm{i}\lambda_+\tau}\left|+\right\rangle\left\langle+\right|+e^{\mathrm{i}\lambda_-\tau}\left|-\right\rangle\left\langle-\right| \tag{795}$$

Calculating the transformation (778) directly using the previous relationship we find that:

$$\widetilde{A_i}(\tau) = \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in\mathrm{V}}\langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{796}$$

$$= \langle+|A_i|+\rangle|+\rangle\langle+| + e^{\mathrm{i}\eta\tau}\langle+|A_i|-\rangle|+\rangle\langle-| + e^{-\mathrm{i}\eta\tau}\langle-|A_i|+\rangle|-\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-| \tag{797}$$

Here $\eta = \lambda_+ - \lambda-$. Comparing the RHS of the equations (778) and the explicit expression for $\widetilde{A_i}(\tau)$ and we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$A_i(0) = \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-| \tag{798}$$

$$A_i(w) = \langle+|A_i|-\rangle|+\rangle\langle-| \tag{799}$$

$$A_i(-w) = \langle-|A_i|+\rangle|-\rangle\langle+| \tag{800}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau) = \widetilde{A_i}^\dagger(\tau)$ and $\widetilde{B_i}(\tau) = \widetilde{B_i}^\dagger(\tau)$ we can use the equation (778) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho}_S}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\overline{S}}(t), \overline{\rho}_S(t)\right] - \frac{1}{2}\sum_{ij}\sum_{w,w'}\gamma_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho}_S(t) - \overline{\rho}_S(t)A_j^\dagger(w,w')\right] - \mathrm{i}\sum_{ij}\sum_{w}S_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho}_S(t) + \overline{\rho}_S(t)A_j^\dagger(w,w')\right] \tag{801}$$

where $A_j^\dagger(w) = A(-w)$ as expected from the equations (799) and (800). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}(w,w',t) = 2\Re(K_{ij}(w,w',t))$ and $S_{ij}(w,w',t) = \Im(K_{ij}(w,w',t))$, respectively, defined in terms of the response functions

$$K_{ij}(w,w',t) = \int_0^t C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{802}$$

$$= K_{ijww'}(t) \tag{803}$$

If we extend the upper limit of integration to ∞ in the equation (802) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.
We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{\mathrm{d}\overline{\rho}_{\overline{S}}}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\overline{S}}(t), \overline{\rho}_{\overline{S}}(t)\right] - \sum_{ijww'}K_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho}_{\overline{S}}(t) - \overline{\rho}_{\overline{S}}(t)A_{jww'}^\dagger\right] \tag{804}$$

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_{\overline{S}}}{\mathrm{d}t}e^V = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_{\overline{S}}e^V\right)}{\mathrm{d}t} \tag{805}$$

$$= \frac{\mathrm{d}\rho_{\overline{S}}}{\mathrm{d}t} \tag{806}$$

$$= -\mathrm{i}e^{-V}\left[\overline{H}_{\overline{S}}(t), \overline{\rho}_{\overline{S}}(t)\right]e^V - \sum_{ijww'}K_{ijww'}(t)\,e^{-V}\left[A_i, A_{jww'}\overline{\rho}_{\overline{S}}(t) - \overline{\rho}_{\overline{S}}(t)A_{jww'}^\dagger\right]e^V \tag{807}$$

We can proof the following property for the inverse transformation of a commutator:

$$
\begin{aligned}
e^{-V}[A,B]e^{V} &= e^{-V}(AB-BA)e^{V} && (808)\\
&= e^{-V}ABe^{V}-e^{-V}BAe^{V} && (809)\\
&= e^{-V}A\mathbb{I}Be^{V}-e^{-V}B\mathbb{I}Ae^{V} && (810)\\
&= e^{-V}Ae^{V}e^{-V}Be^{V}-e^{-V}Be^{V}e^{-V}Ae^{V} && (811)\\
&= \left(e^{-V}Ae^{V}\right)\left(e^{-V}Be^{V}\right)-\left(e^{-V}Be^{V}\right)\left(e^{-V}Ae^{V}\right) && (812)\\
&= \left[e^{-V}Ae^{V},e^{-V}Be^{V}\right] && (813)
\end{aligned}
$$

For a product we have the following property:

$$
\begin{aligned}
e^{-V}ABe^{V} &= e^{-V}A\mathbb{I}Be^{V} && (814)\\
&= e^{-V}Ae^{V}e^{-V}Be^{V} && (815)\\
&= \left(e^{-V}Ae^{V}\right)\left(e^{-V}Be^{V}\right) && (816)
\end{aligned}
$$

So we will obtain that

$$
\begin{aligned}
\frac{\mathrm{d}\rho_{\overline{S}}}{\mathrm{d}t} &= -\,\mathrm{i}e^{-V}\left[\overline{H}_{\overline{S}}(t),\overline{\rho}_{\overline{S}}(t)\right]e^{V}-\sum_{ijww'}K_{ijww'}(t)\,e^{-V}\left[A_i,A_{jww'}\overline{\rho}_{\overline{S}}(t)-\overline{\rho}_{\overline{S}}(t)\,A^{\dagger}_{jww'}\right]e^{V} && (817)\\
&= -\,\mathrm{i}\left[e^{-V}\overline{H}_{\overline{S}}(t)\,e^{V},e^{-V}\overline{\rho}_{\overline{S}}(t)\,e^{V}\right]-\sum_{ijww'}K_{ijww'}(t)\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}\overline{\rho}_{\overline{S}}(t)\,e^{V}-e^{-V}\overline{\rho}_{\overline{S}}(t)\,A^{\dagger}_{jww'}e^{V}\right] && (818)\\
&= -\,\mathrm{i}\left[H_{\overline{S}}(t),\rho_{\overline{S}}(t)\right]-\sum_{ijww'}K_{ijww'}(t)\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}e^{V}e^{-V}\overline{\rho}_{\overline{S}}(t)\,e^{V}-e^{-V}\overline{\rho}_{\overline{S}}(t)\,e^{V}e^{-V}A^{\dagger}_{jww'}e^{V}\right] && (819)\\
&= -\,\mathrm{i}\left[H_{\overline{S}}(t),\rho_{\overline{S}}(t)\right]-\sum_{ijww'}K_{ijww'}(t)\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}e^{V}\rho_{\overline{S}}(t)-\rho_{\overline{S}}(t)\,e^{-V}A^{\dagger}_{jww'}e^{V}\right] && (820)\\
&= -\,\mathrm{i}\left[H_{\overline{S}}(t),\rho_{\overline{S}}(t)\right]-\left(\sum_{ijww'}K_{ijww'}(t)\left(\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}e^{V}\rho_{\overline{S}}(t)\right]-\left[e^{-V}A_ie^{V},\rho_{\overline{S}}(t)\,e^{-V}A^{\dagger}_{jww'}e^{V}\right]\right)\right) && (821)
\end{aligned}
$$

In the lab frame we will obtain that:

$$
\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}\left[H_{\overline{S}}(t),\rho_{\overline{S}}(t)\right]-\left(\sum_{ijww'}K_{ijww'}(t)\left(\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}e^{V}\rho(t)\right]-\left[e^{-V}A_ie^{V},\rho(t)\,e^{-V}A^{\dagger}_{jww'}e^{V}\right]\right)\right) \quad (822)
$$

V. LIMIT CASES

In order to show the plausibility of the master equation (801) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (801) reproduces the results of the reference [1], for the latter case we have that $i,j\in\{1,2,3\}$ and $\omega\in(0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$
H = \left(\delta+\sum_j g_k\left(b_k^{\dagger}+b_k\right)\right)|1\rangle\langle 1|+\frac{\Omega}{2}\sigma_x+\sum_k\omega_k b_k^{\dagger}b_k \quad (823)
$$

After performing the transformation (24) on the Hamiltonian (823) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)\,|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x \tag{824}$$

$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{825}$$

$$H_B = \sum_k \omega_k b_k^\dagger b_k \tag{826}$$

The Hamiltonian (824) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{827}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{828}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z \tag{829}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2}$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_S}$.

$$\lambda_+ = \frac{\epsilon + \eta}{2} \tag{830}$$

$$\lambda_- = \frac{\epsilon - \eta}{2} \tag{831}$$

$$|+\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}\begin{pmatrix}\epsilon + \eta \\ \Omega_r\end{pmatrix} \tag{832}$$

$$|-\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}\begin{pmatrix}-\Omega_r \\ \epsilon + \eta\end{pmatrix} \tag{833}$$

Using this basis we can find the decomposition matrices using the equations (799)-(800) and the fact that $|+\rangle = \cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle$ and $|-\rangle = -\sin(\theta)\,|1\rangle + \cos(\theta)\,|0\rangle$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\begin{aligned}
\langle + | \, \sigma_x \, | + \rangle &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} && (834) \\
&= 2 \sin(\theta) \cos(\theta) && (835) \\
&= \sin(2\theta) && (836) \\
\langle - | \, \sigma_x \, | - \rangle &= \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} && (837) \\
&= -2 \sin(\theta) \cos(\theta) && (838) \\
&= -\sin(2\theta) && (839) \\
\langle - | \, \sigma_x \, | + \rangle &= \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} && (840) \\
&= \cos^2(\theta) - \sin^2(\theta) && (841) \\
&= \cos(2\theta) && (842) \\
\langle + | \, \sigma_y \, | + \rangle &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} && (843) \\
&= \mathrm{i} \sin(\theta) \cos(\theta) - \mathrm{i} \sin(\theta) \cos(\theta) && (844) \\
&= 0 && (845) \\
\langle - | \, \sigma_y \, | - \rangle &= \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} && (846) \\
&= -\mathrm{i} \sin(\theta) \cos(\theta) + \mathrm{i} \sin(\theta) \cos(\theta) && (847) \\
&= 0 && (848) \\
\langle - | \, \sigma_y \, | + \rangle &= \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} && (849) \\
&= \mathrm{i} \cos^2(\theta) + \mathrm{i} \sin^2(\theta) && (850) \\
&= \mathrm{i} && (851) \\
\langle + | \, \frac{1 + \sigma_z}{2} \, | + \rangle &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} && (852) \\
&= \cos(\theta) \cos(\theta) && (853) \\
&= \cos^2(\theta) && (854) \\
\langle - | \, \frac{1 + \sigma_z}{2} \, | - \rangle &= \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} && (855) \\
&= \sin(\theta) \sin(\theta) && (856) \\
&= \sin^2(\theta) && (857) \\
\langle - | \, \frac{1 + \sigma_z}{2} \, | + \rangle &= \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} && (858) \\
&= -\sin(\theta) \cos(\theta) && (859) \\
&= -\sin(\theta) \cos(\theta) && (860)
\end{aligned}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$\begin{aligned}
A_1(0) &= \sin(2\theta) \left(|+\rangle \langle +| - |-\rangle \langle -| \right) && (861) \\
A_1(\eta) &= \cos(2\theta) \, |-\rangle \langle +| && (862) \\
A_2(0) &= 0 && (863) \\
A_2(\eta) &= \mathrm{i} \, |-\rangle \langle +| && (864) \\
A_3(0) &= \cos^2(\theta) \, |+\rangle \langle +| + \sin^2(\theta) \, |-\rangle \langle -| && (865) \\
A_3(\eta) &= -\sin(\theta) \cos(\theta) \, |-\rangle \langle +| && (866)
\end{aligned}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (422) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau) \tag{867}$$

Using the notation of the master equation (801), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(767)-(774) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (422) are equal to:

$$\begin{aligned}
\Lambda'_{11}(\tau) &= \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B\right) && (868)\\
&= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) && (869)\\
\Lambda'_{22}(\tau) &= \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B\right) && (870)\\
&= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) && (871)\\
\Lambda'_{33}(\tau) &= \int_0^\infty \mathrm{d}\omega J(\omega)\,(1-F(\omega))^2\, G_+(\tau) && (872)\\
\Lambda'_{32}(\tau) &= \frac{\Omega_r}{2}\int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)\,(1-F(\omega))\,\mathrm{i}G_-(\tau) && (873)\\
\Lambda'_{32}(\tau) &= -\Lambda'_{23}(\tau) && (874)\\
\Lambda'_{12}(\tau) &= \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 && (875)
\end{aligned}$$

Finally taking the Hamiltonian (823) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (334) we have that:

$$\begin{aligned}
f_k &= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} && (876)\\
&= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)} && (877)
\end{aligned}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (420) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)\,U^\dagger(t-\tau) = U(\tau)$. From the equation (801) and using the fact that

$$\begin{aligned}
\widetilde{A_j}(t-\tau, t) &= U(\tau)\, A_j U(-\tau) && (878)\\
&= \sum_w e^{iw\tau} A_i(-w) && (879)\\
&= \sum_w e^{-iw\tau} A_i(w) && (880)
\end{aligned}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_{w}\gamma_{ij}(w,t)\left[A_i, A_j(w)\,\overline{\rho}_S(t) - \overline{\rho}_S(t)\,A_j^{\dagger}(w)\right] \tag{881}$$

$$-\sum_{ij}\sum_{w}S_{ij}(w,t)\left[A_i, A_j(w)\,\overline{\rho}_S(t) + \overline{\rho}_S(t)\,A_j^{\dagger}(w)\right] \tag{882}$$

where $A_j^{\dagger}(w) = A(-w)$, as we can see the equation (882) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2\Re(K_{ij}(w,t))$ and $S_{ij}(w,t) = \Im(K_{ij}(w,t))$, respectively, defined in terms of the response functions

$$K_{ij}(w,t) = \int_0^t \Lambda'_{ij}(\tau)\, e^{\mathrm{i}w\tau}\mathrm{d}\tau \tag{883}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (109) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_{k}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{884}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (884) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (223) and (**??**) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)\,|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t) \tag{885}$$

$$\overline{H_{\mathrm{I}}} = \frac{\Omega(t)}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{886}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (767)-(774) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^{\infty}\frac{J(\omega)}{\omega^2}G_{+}(\tau)\,\mathrm{d}\omega \tag{887}$$

Writing $G_{+}(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (887) can be written as:

$$\phi(\tau) = \int_0^{\infty}\frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega \tag{888}$$

Writing the interaction Hamiltonian (886) in the similar way to the equation (**??**) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (223) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t) \tag{889}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{890}$$
$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{891}$$
$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{892}$$
$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) \tag{893}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(420) is:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{894}$$
$$\left.+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right) \tag{895}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (890) and (893) on the equation (895) we obtain that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{896}$$
$$\left.+\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right) \tag{897}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (422) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right) \tag{898}$$

Using the equation (801), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)\right), \rho_S(t)\right] - \frac{1}{2}\sum_w \gamma_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) - \rho_S(t)A_3^\dagger(w)\right] \tag{899}$$
$$\left.-\sum_w S_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) + \rho_S(t)A_3^\dagger(w)\right]\right) \tag{900}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(\tau) \tag{901}$$

$$\Lambda_{33}(-\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(-\tau) \tag{902}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (900) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)\right)] - \sum_w \left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^*(w,t)\left[\rho_S(t) A_3(w), A_3\right]\right) \tag{903}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2} \tag{904}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right) \tag{905}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2} \tag{906}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(-|1\rangle + |0\rangle\right) \tag{907}$$

The elements of the decomposition matrices are:

$$\langle +| \frac{1+\sigma_z}{2} |+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{908}$$

$$= \frac{1}{2} \tag{909}$$

$$\langle -| \frac{1+\sigma_z}{2} |-\rangle = \frac{1}{2}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} \tag{910}$$

$$= \frac{1}{2} \tag{911}$$

$$\langle -| \frac{1+\sigma_z}{2} |+\rangle = \frac{1}{2}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{912}$$

$$= -\frac{1}{2} \tag{913}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \tag{914}$$

$$= \frac{\mathbb{I}}{2} \tag{915}$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle +| \tag{916}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right) \tag{917}$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle -| \tag{918}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right) \tag{919}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{920}$$

$$- K_{33}^{*}\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^{*}\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] \tag{921}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right) G_+\left(\tau\right) e^{\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{922}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{923}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{924}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{925}$$

$$= \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)\pi\delta\left(\widetilde{\Omega} - \omega\right)\mathrm{d}\omega \tag{926}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right) \tag{927}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J\left(\omega\right) G_+\left(\tau\right) e^{-\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{928}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{929}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) e^{-\mathrm{i}\widetilde{\Omega}\tau} n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{930}$$

$$= \int_0^\infty \int_0^\infty J\left(\omega\right) n\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau + \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{931}$$

$$= \int_0^\infty J\left(\omega\right) n\left(\omega\right)\pi\delta\left(-\widetilde{\Omega} + \omega\right)\mathrm{d}\omega \tag{932}$$

$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right) \tag{933}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s} = \pi\delta\left(\varepsilon\right) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3\left(0\right)$ because the spectral density $J\left(\omega\right)$ is equal to zero when $\omega = 0$. Replacing in the equation (920) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{934}$$

$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S\left(t\right)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right) \tag{935}$$

This is the same result than the equation ($S17$), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega\left(\mathrm{t}\right)}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\Omega\left(t\right)\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{936}$$

$$- \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right) \tag{937}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{938}$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{939}$$

$$H_I = \left(\sum_{n=0} \mu_n(t)\,|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{940}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{941}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t)\,|n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{942}$$

At first let's obtain e^V under the transformation (942), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$\begin{aligned}
e^V &= e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} && (943)\\
&= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + \ldots && (944)\\
&= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + \ldots && (945)\\
&= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n|\left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \ldots\right) && (946)\\
&= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} && (947)\\
&= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ && (948)
\end{aligned}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$\begin{aligned}
e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} &= \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} && (949)\\
&= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (950)\\
&= B_{\pm} && (951)
\end{aligned}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{952}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (938):

$$\begin{aligned}
\overline{|0\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), && (953)\\
&= |0\rangle\langle 0|, && (954)\\
\overline{|m\rangle\langle n|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), && (955)\\
&= |m\rangle\langle m| B_+ |m\rangle\langle n| n\rangle\langle n| B_-, && (956)\\
&= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, && (957)\\
\overline{|0\rangle\langle m|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), && (958)\\
&= |0\rangle\langle m| B_-\ m \neq 0, && (959)\\
\overline{|m\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) && (960)\\
&= |0\rangle\langle m| B_+\ m \neq 0, && (961)\\
\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) && (962)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B_+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_- && (963)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_+ b_{\mathbf{k}}^\dagger B_- \right) \left(B_+ b_{\mathbf{k}} B_- \right) && (964)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) && (965)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) && (966)\\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) && (967)\\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) && (968)
\end{aligned}$$

The transformed Hamiltonians of the equations (939) to (941) written in terms of (953) to (968) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{969}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{970}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \overline{\sum_{n=1} (V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|)} + \overline{\sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n|} \tag{971}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\overline{|0\rangle\langle n|} + V_{n0}(t)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0} V_{mn}(t)\overline{|m\rangle\langle n|} \tag{972}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} (V_{0n}(t)B_-|0\rangle\langle n| + V_{n0}(t)B_+|n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{973}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} (V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{974}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)\left(\left(\sum_{n=0}\mu_n(t)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{975}$$

$$= \left(\mu_0(t)|0\rangle\langle 0| + \sum_{n=1}\mu_n(t)|n\rangle\langle n|B_+\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{976}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}B_+\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)B_- \tag{977}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{978}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{979}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} (V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{980}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{981}$$

$$+\sum_{n=0}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} 2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{982}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} (V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{983}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{984}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{985}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{986}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\right) \tag{987}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{988}$$

Using the previous functions we have that (985) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B_- + V_{n0}\left(t\right)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0} V_{mn}\left(t\right)|m\rangle\langle n| \tag{989}$$

$$+ \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n|n\rangle\langle n| + \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{990}$$

Now in order to separate the elements of the hamiltonian (990) let's follow the references of the equations (**??**) and (223) to separate the hamiltonian like:

$$\overline{H_S\left(t\right)} = \sum_{n=0} \varepsilon_n\left(t\right)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}\left(t\right)|m\rangle\langle n| + \sum_{n=1} R_n|n\rangle\langle n| \tag{991}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|\left(B_- - B\right) + V_{n0}\left(t\right)|n\rangle\langle 0|\left(B_+ - B\right)\right), \tag{992}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{993}$$

Here B is given by (**??**) The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{994}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{995}$$

$$B_x = \frac{B_+ + B_- - 2B}{2} \tag{996}$$

$$B_y = \frac{B_- - B_+}{2\mathrm{i}} \tag{997}$$

Using this set of hermitian operators to write the Hamiltonians (939)-(941)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{998}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \tag{999}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1000}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \tag{1001}$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1002}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1003}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x}-\mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x}+\mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1004}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1005}$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1006}$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(V_{0n}(t)|0\rangle\langle n|(B_- - B) + V_{n0}(t)|n\rangle\langle 0|(B_+ - B)) \tag{1007}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))(B_- - B)\frac{\sigma_{0n,x}-\mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))(B_+ - B)\frac{\sigma_{0n,x}+\mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1008}$$

$$+ \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1009}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}((B_- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + (B_+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))))\right. \tag{1010}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}((B_+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - (B_- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))))\right) \tag{1011}$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1012}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}(B_+ + B_- - 2B)\Re(V_{0n}(t)) + \mathrm{i}(B_- - B - B_+ + B)\Im(V_{0n}(t))\right. \tag{1013}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}((B_+ - B - B_- + B)\Re(V_{0n}(t)) + \mathrm{i}(B - B_- + B - B_+)\Im(V_{0n}(t)))\right) + \sum_{n=1} B_{z,n}|n\rangle\langle n| \tag{1014}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1015}$$

$$+ \sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1016}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_S+H_B}\right)}\right)\right) + \left\langle \overline{H_I}\right\rangle_{\overline{H_S+H_B}} + O\left(\left\langle \overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1017}$$

Taking the equations (242)-(250) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (250) and using the chain rule we obtain that:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1018}$$

$$= 0 \tag{1019}$$

Let's recall the equations (986) and (988), we can write them in terms of the variational parameters

$$B = \exp\left(-\left(1/2\right)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1020}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1021}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1022}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1023}$$

Introducing this derivates in the equation (1018) give us:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1024}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1025}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1026}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1027}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1028}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_{B}$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho\left(0\right)e^{-V}=\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B_{+}\right)\left(|0\rangle\langle 0|\otimes\rho_{B}\right)\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B_{-}\right) \tag{1029}$$

$$0=|0\rangle\langle 0|\otimes\rho_{B} \tag{1030}$$

$$0=\rho\left(0\right) \tag{1031}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}\equiv U^{\dagger}(t)OU(t) \tag{1032}$$

$$U(t)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{S}}\left(t'\right)\right). \tag{1033}$$

Therefore:

$$\widetilde{\overline{\rho_{S}}}(t)=U^{\dagger}(t)\overline{\rho_{S}}(t)U(t),\text{ where} \tag{1034}$$

$$\overline{\rho_{S}}(t)=\mathrm{Tr}_{B}\left(\bar{\rho}(t)\right) \tag{1035}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_{I}\left(t\right)}=B_{z,0}|0\rangle\langle 0|+\sum_{n=1}\left(\Re\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,x}+\Re\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,y}+B_{z,n}|n\rangle\langle n|\right. \tag{1036}$$

$$\left.+\Im\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,y}-\Im\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,x}\right) \tag{1037}$$

where

$$B_{z,0}=\sum_{\mathbf{k}}g_{\mathbf{k}}\mu_{0}\left(t\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1038}$$

$$B_{z,n}=\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_{n}\left(t\right)-\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\ \text{if} n\neq 0 \tag{1039}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1040}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1041}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1042}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1043}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1044}$$
$$B_{1n}(t) = B_x \tag{1045}$$
$$B_{2n}(t) = B_y \tag{1046}$$
$$B_{3n}(t) = B_{z,n} \tag{1047}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1048}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1049}$$
$$C_{10}(t) = 0 \tag{1050}$$
$$C_{20}(t) = 0 \tag{1051}$$
$$C_{40}(t) = 0 \tag{1052}$$
$$C_{50}(t) = 0 \tag{1053}$$
$$C_{30}(t) = 1 \tag{1054}$$
$$C_{1n}(t) = \Re\left(V_{0n}(t)\right) \tag{1055}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1056}$$
$$C_{3n}(t) = 1 \tag{1057}$$
$$C_{4n}(t) = \Im\left(V_{0n}(t)\right) \tag{1058}$$
$$C_{5n}(t) = -\Im\left(V_{0n}(t)\right) \tag{1059}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn} \otimes B_{jn}(t)\right) \tag{1060}$$

Here $J = \{1,2,3,4,5\}$.

We write the interaction Hamiltonian transformed under (1032) as:

$$\widetilde{H_I}(t) = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1061}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\, A_i U(t) \tag{1062}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{1063}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1064}$$

Replacing the equation (1061)in (1064)we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1065}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1066}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1067}$$

$$\left.-\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1068}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1069}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1070}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1071}$$

$$\left.+\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1072}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1073}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1074}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1072) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\right\rangle_B \tag{1075}$$
$$= \left\langle\widetilde{B_{jn}}\left(0\right)\widetilde{B_{j'n'}}\left(0\right)\right\rangle_B \tag{1076}$$
$$= \Lambda_{jnj'n'}\left(\tau\right) \tag{1077}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}\left(t\right)\rho_B\widetilde{B_{j'n'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\rho_B\right) \tag{1078}$$
$$= \left\langle\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\right\rangle_B \tag{1079}$$
$$= \left\langle\widetilde{B_{j'n'}}\left(-\tau\right)\widetilde{B_{jn}}\left(0\right)\right\rangle_B \tag{1080}$$
$$= \Lambda_{j'n'jn}\left(-\tau\right) \tag{1081}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}\left(s\right)\rho_B\widetilde{B_{jn}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\rho_B\right) \tag{1082}$$
$$= \left\langle\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\right\rangle_B \tag{1083}$$
$$= \left\langle\widetilde{B_{jn}}\left(\tau\right)\widetilde{B_{j'n'}}\left(0\right)\right\rangle_B \tag{1084}$$
$$= \Lambda_{jnj'n'}\left(\tau\right) \tag{1085}$$
$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\rho_B\right) \tag{1086}$$
$$= \left\langle\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\right\rangle_B \tag{1087}$$
$$= \left\langle\widetilde{B_{j'n'}}\left(-\tau\right)\widetilde{B_{jn}}\left(0\right)\right\rangle_B \tag{1088}$$
$$= \Lambda_{j'n'jn}\left(-\tau\right) \tag{1089}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1065)and (1072) and using the equations (1075)-(1089) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}\left(t\right)C_{j'n'}\left(s\right)\left(\Lambda_{jnj'n'}\left(\tau\right)\widetilde{A_{jn}}\left(t\right)\widetilde{A_{j'n'}}\left(s\right)\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'n'jn}\left(-\tau\right)\widetilde{A_{jn}}\left(t\right)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_{j'n'}}\left(s\right)\right)\right. \tag{1090}$$
$$\left.+C_{jn}\left(t\right)C_{j'n'}\left(s\right)\left(\Lambda_{j'n'jn}\left(-\tau\right)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_{j'n'}}\left(s\right)\widetilde{A_{jn}}\left(t\right) - \Lambda_{jnj'n'}\left(\tau\right)\widetilde{A_{j'n'}}\left(s\right)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_{jn}}\left(t\right)\right)\right)\mathrm{d}s \tag{1091}$$
$$= -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}\left(t\right)C_{j'n'}\left(s\right)\left(\Lambda_{jnj'n'}\left(\tau\right)\left[\widetilde{A_{jn}}\left(t\right),\widetilde{A_{j'n'}}\left(s\right)\widetilde{\widetilde{\rho_S}}(t)\right] + \Lambda_{j'n'jn}\left(-\tau\right)\left[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_{j'n'}}\left(s\right),\widetilde{A_{jn}}\left(t\right)\right]\right)\right) \tag{1092}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}\left(t\right)C_{j'n'}\left(t-\tau\right)\left(\Lambda_{jnj'n'}\left(\tau\right)\left[A_{jn}\left(t\right),A_{j'n'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}\left(-\tau\right)\left[\overline{\rho_S}(t)A_{j'n'}\left(t-\tau,t\right),A_{jn}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}(t)\right] \tag{1093}$$

For this case we used that $A_{jn}\left(t-\tau,t\right) = U\left(t\right)U^\dagger\left(t-\tau\right)A_{jn}\left(t\right)U\left(t-\tau\right)U^\dagger\left(t\right)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0\left(t\right)=0$, $\mu_1\left(t\right)=1$ then we can reproduce a similar expression to (420) as expected.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1094}$$
$$H_S(t) = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|, \tag{1095}$$
$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} + g^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}\right), \tag{1096}$$
$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^{\dagger}_{u\mathbf{k}} b_{u\mathbf{k}}. \tag{1097}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}\right) \tag{1098}$$

At first let's obtain $e^{\pm V}$ under the transformation (1098), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}\right)$, so the equation (1098) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$\begin{aligned}
e^{\pm V} &= e^{\pm\sum_n |n\rangle\langle n|\hat{\varphi}_n} && (1099)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!} + \ldots && (1100)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}^2_n}{2!} + \ldots && (1101)\\
&= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}^2_n}{2!} + \ldots && (1102)\\
&= \sum_n |n\rangle\langle n|\left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}^2_n}{2!} + \ldots\right) && (1103)\\
&= \sum_n |n\rangle\langle n| e^{\pm\hat{\varphi}_n} && (1104)
\end{aligned}$$

Given that $\left[f_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b^{\dagger}_{u'\mathbf{k}'} - f^{*}_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}^{'}$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D(\pm\alpha_{nu\mathbf{k}}) = e^{\pm\left(\alpha_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} - \alpha^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$\begin{aligned}
e^{\pm\sum_{u\mathbf{k}} \omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}\right)} &= \prod_u e^{\pm\sum_{\mathbf{k}} \omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}\right)} && (1105)\\
&= \prod_u\left(\prod_{\mathbf{k}} e^{\pm\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}} b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}} b_{u\mathbf{k}}\right)}\right) && (1106)\\
&= \prod_u\left(\prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}})\right) && (1107)\\
&= \prod_{u\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}) && (1108)\\
&= \prod_u B_{nu\pm} && (1109)\\
B_{nu\pm} &\equiv \prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}) && (1110)
\end{aligned}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1111}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1094):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1112}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1113}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1114}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1115}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1116}$$

$$= |0\rangle\langle 0|. \tag{1117}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1118}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1119}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1120}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, \tag{1121}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1122}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1123}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1124}$$

$$\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1125}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1126}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + ...\right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right) \left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + ...\right), \tag{1127}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + ..., \tag{1128}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + ...\right) \prod_u B_{0u-} \tag{1129}$$

$$+ |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + ...\right) \prod_u B_{1u-} + ... \tag{1130}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + ...\right) \tag{1131}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + ...\right) + ... \tag{1132}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + ...\right) \tag{1133}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + ...\right) + ... \tag{1134}$$

The transformed Hamiltonians of the equations (1095) to (1097) written in terms of (1112) to (1137) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1140}$$

$$= \overline{\sum_n \varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1141}$$

$$= \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) \tag{1142}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1143}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{0u\mathbf{k}}b_{u\mathbf{k}}\right) + \ldots\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1144}$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{0u\mathbf{k}}b_{u\mathbf{k}}\right) \prod_u B_{0u-} \tag{1145}$$

$$+ \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{1u\mathbf{k}}b_{u\mathbf{k}}\right) \prod_u B_{1u-} + \ldots \tag{1146}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \prod_u B_{0u+} b^\dagger_{u\mathbf{k}} \prod_u B_{0u-} + g^*_{0u\mathbf{k}} \prod_u B_{0u+} b_{u\mathbf{k}} \prod_u B_{0u-}\right) \tag{1147}$$

$$+ \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \prod_u B_{1u+} b^\dagger_{u\mathbf{k}} \prod_u B_{1u-} + g^*_{1u\mathbf{k}} \prod_u B_{1u+} b_{u\mathbf{k}} \prod_u B_{1u-}\right) + \ldots \tag{1148}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}\left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{0u\mathbf{k}}\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1149}$$

$$+ \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}}\left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{1u\mathbf{k}}\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1150}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}\left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{nu\mathbf{k}}\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1151}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}}b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1152}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)\right) \tag{1153}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}\right. \tag{1154}$$

$$\left. - \left(v_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)\right) + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}}b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1155}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1156}$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}})\, b^\dagger_{u\mathbf{k}} + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^*\, b_{u\mathbf{k}}\right) \tag{1157}$$

Using the previous functions we have that (1154) can be re-written in the following way:

$$\overline{H} = \sum_{n} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \prod_{u} (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}} \tag{1158}$$

$$+ \sum_{n} R_n(t) |n\rangle\langle n| + \sum_{n} B_{z,n}(t) |n\rangle\langle n| \tag{1159}$$

Now in order to separate the elements of the hamiltonian (1159) let's follow the references of the equations (223) and (**??**) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \prod_{u} (B_{mu+}B_{nu-}) \right\rangle_{\overline{H_0}} = \left\langle \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*} + \alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1160}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*} + \alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right) \left\langle \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \right\rangle_{\overline{H_0}} \tag{1161}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}} - v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right) \prod_{u} \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}} \coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1162}$$

$$\equiv B_{nm} \tag{1163}$$

$$\left\langle \prod_{u} (B_{nu+}B_{mu-}) \right\rangle_{\overline{H_0}} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{nu\mathbf{k}}^{*}v_{mu\mathbf{k}} - v_{nu\mathbf{k}}v_{mu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right) \prod_{u} \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}} \coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1164}$$

$$= B_{nm}^{*} \tag{1165}$$

Following the reference [4] we define:

$$J_{nm} = \prod_{u} (B_{mu+}B_{nu-}) - B_{nm} \tag{1166}$$

As we can see:

$$J_{nm}^{\dagger} = \left(\prod_{u} (B_{mu+}B_{nu-}) - B_{nm}\right)^{\dagger} \tag{1167}$$

$$= \prod_{u} (B_{nu+}B_{mu-}) - B_{nm}^{*} \tag{1168}$$

$$= \prod_{u} (B_{nu+}B_{mu-}) - B_{mn} \tag{1169}$$

$$= J_{mn} \tag{1170}$$

We can separate the Hamiltonian (1159) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)} = \sum_{n} (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \tag{1171}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_{n} B_{z,n}(t) |n\rangle\langle n|, \tag{1172}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}} \tag{1173}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_B}\right)}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_B}} + O\left(\left\langle\overline{H_{\bar{I}}^2}\right\rangle_{\overline{H_{\bar{S}}(t)+H_B}}\right). \tag{1174}$$

Taking the equations (242)-(250) and given that Tr $\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right) = C\left(R_0, R_1, R_2, ..., R_{d-1}, B_{01}, B_{02}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = \Re\left(v_{nu\mathbf{k}}\right) + \mathrm{i}\Im\left(v_{nu\mathbf{k}}\right)$. So our approach will be based on the derivation respect to $\Re\left(v_{nu\mathbf{k}}\right)$ and $\Im\left(v_{nu\mathbf{k}}\right)$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1175}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \tag{1176}$$

$$\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1177}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}}v_{nu\mathbf{k}}^* + g_{nu\mathbf{k}}^* v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1178}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \tag{1179}$$

$$\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1180}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*)\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n \tag{1181}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \tag{1182}$$

$$\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1183}$$

$$v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^* = (\Re(v_{mu\mathbf{k}}) - \mathrm{i}\Im(v_{mu\mathbf{k}}))(\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})) - (\Re(v_{mu\mathbf{k}}) + \mathrm{i}\Im(v_{mu\mathbf{k}}))(\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}})) \tag{1184}$$

$$= (\Re(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) - \mathrm{i}\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) + \Im(v_{mu\mathbf{k}})\Im(v_{nu\mathbf{k}})) \tag{1185}$$

$$- (\Re(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) + \mathrm{i}\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}) + \Im(v_{mu\mathbf{k}})\Im(v_{nu\mathbf{k}})) \tag{1186}$$

$$= 2\mathrm{i}(\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) - \Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}})) \tag{1187}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*)\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n \tag{1188}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}(\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) - \Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}))}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1189}$$

$$\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1190}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1191}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}}v_{mu\mathbf{k}}^* + v_{nu\mathbf{k}}^* v_{mu\mathbf{k}}) \tag{1192}$$

$$= (\Re(v_{mu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}))^2 + (\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2 \tag{1193}$$

$$- ((\Re(v_{nu\mathbf{k}}) + \mathrm{i}\Im(v_{nu\mathbf{k}}))(\Re(v_{mu\mathbf{k}}) - \mathrm{i}\Im(v_{mu\mathbf{k}})) + (\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}}))(\Re(v_{mu\mathbf{k}}) + \mathrm{i}\Im(v_{mu\mathbf{k}}))) \tag{1194}$$

$$= (\Re(v_{mu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}))^2 + (\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2$$

$$- 2(\Re(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}}) + \Im(v_{nu\mathbf{k}})\Im(v_{mu\mathbf{k}})) \tag{1195}$$

$$= (\Re(v_{mu\mathbf{k}}) - \Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}) - \Im(v_{nu\mathbf{k}}))^2 \tag{1196}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1197}$$

$$= \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2 - (g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*)\Re(v_{nu\mathbf{k}}) - \mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right) \tag{1198}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \frac{\partial}{\partial \Re\left(v_{nu\mathbf{k}}\right)} \sum_{u\mathbf{k}} \left(\frac{\left(\Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{nu\mathbf{k}}\right)\right)^2 - 2\Re\left(g_{nu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} \right) \tag{1203}$$

$$= \frac{2\Re\left(v_{nu\mathbf{k}}\right) - 2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1204}$$

$$= 2\frac{\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1205}$$

$$\frac{\partial R_{n'}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \frac{\partial}{\partial \Im\left(v_{nu\mathbf{k}}\right)} \sum_{u\mathbf{k}} \left(\frac{\left(\Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{nu\mathbf{k}}\right)\right)^2 - 2\Re\left(g_{nu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} \right) \tag{1206}$$

$$= \frac{2\Im\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1207}$$

$$= 2\frac{\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} \tag{1208}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2}\right)\right)\right. \tag{1209}$$

$$\left.\prod_{u} \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) \tag{1210}$$

$$= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1211}$$

$$+ \sum_{u} \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1212}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1213}$$

$$+ \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1214}$$

$$\frac{\partial \ln B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) - \left(\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1215}$$

$$\frac{\partial \ln B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) - \left(\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \tag{1216}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}}\frac{\partial B_{mn}}{\partial a} \tag{1217}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn}\frac{\partial \ln B_{mn}}{\partial a} \tag{1218}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial\left(B_{nm}\right)^\dagger}{\partial a} \tag{1219}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = B_{mn}\frac{\partial \ln B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} \tag{1220}$$

$$= B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) - \left(\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1221}$$

$$= B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1222}$$

$$\frac{\partial B_{nm}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \left(\frac{\partial B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)}\right)^{\dagger} \tag{1223}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)^{\dagger} \tag{1224}$$

$$= B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1225}$$

$$\frac{\partial B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = B_{mn}\frac{\partial \ln B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} \tag{1226}$$

$$= B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) - \left(\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1227}$$

$$= B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1228}$$

$$\frac{\partial B_{nm}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \left(\frac{\partial B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)}\right)^{\dagger} \tag{1229}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1230}$$

$$= B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1231}$$

Introducing this derivates in the equation (1203) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1232}$$

$$+ \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1233}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1234}$$

$$= 0 \tag{1235}$$

We can obtain the variational parameters:

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Re\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\Re\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\Re\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1236}$$

$$=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1237}$$

$$\Re\left(v_{nu\mathbf{k}}\right)=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1238}$$

$$=\frac{2\Re\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1239}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial\Im\left(v_{nu\mathbf{k}}\right)}=\frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1240}$$

$$+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)-\left(\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1241}$$

$$\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)-\left(\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1242}$$

$$=0 \tag{1243}$$

Rearranging we obtain

$$- 2\frac{\partial A_{\rm B}}{\partial R_n}\frac{\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\frac{\Im\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\frac{\Im\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1244}$$

$$= -2\frac{\partial A_{\rm B}}{\partial R_n}\frac{\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1245}$$

$$\Im\left(v_{nu\mathbf{k}}\right)=\frac{2\frac{\partial A_{\rm B}}{\partial R_n}\frac{\Im(g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re(v_{mu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\rm B}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1246}$$

$$=\frac{2\Im\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1247}$$

$$v_{nu\mathbf{k}}=\Re\left(v_{nu\mathbf{k}}\right)+\mathrm{i}\Im\left(v_{nu\mathbf{k}}\right) \tag{1248}$$

$$=\frac{2\Re\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1249}$$

$$\mathrm{i}\frac{2\Im\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1250}$$

$$=\frac{2\Re\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}+2\mathrm{i}\Im\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1251}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1252}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1253}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1254}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1255}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\rm B}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\rm B}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\rm B}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1256}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^{V}\rho\left(0\right)e^{-V}=\left(\sum_{n}|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_{n}|n\rangle\langle n|B_{n+}\right) \tag{1257}$$

$$0=\left(B_{0+}|0\rangle\langle 0|B_{0-}\right)\otimes\rho_B \tag{1258}$$

$$0=\rho\left(0\right) \tag{1259}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t) O U(t) \tag{1260}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}\left(t'\right)\right). \tag{1261}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\overline{\rho_S}(t)U(t), \text{ where} \tag{1262}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1263}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1264}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1265}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1266}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1267}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1268}$$
$$= B_{n-}^{\dagger}B_{m+}^{\dagger} - B_n B_m \tag{1269}$$
$$= B_{n+}B_{m-} - B_n B_m \tag{1270}$$
$$= B_{nm} \tag{1271}$$

So we can say that the set of matrices (1264) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1264) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} + \sum_n B_{z,n}\left(t\right)|n\rangle\langle n|, \tag{1272}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} + V_{mn}\left(t\right)|m\rangle\langle n|B_{mn}\right) \tag{1273}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}\left(t\right)\right)B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}\Im\left(V_{nm}\left(t\right)\right)B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1274}$$
$$\left.+\Re\left(V_{nm}\left(t\right)\right)B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}\Im\left(V_{nm}\left(t\right)\right)B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1275}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1276}$$
$$\left.+\mathrm{i}\Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1277}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1278}$$
$$\left.+\Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}B_{nm,x}\right) \tag{1279}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \land (n = m \lor n < m)\right\} \tag{1280}$$

Now consider the following set of operators,

$$A_{1,nm}\left(t\right) = \sigma_{nm,x}\left(1-\delta_{mn}\right) \tag{1281}$$
$$A_{2,nm}\left(t\right) = \sigma_{nm,y}\left(1-\delta_{mn}\right) \tag{1282}$$
$$A_{3,nm}\left(t\right) = \delta_{mn}|n\rangle\langle m| \tag{1283}$$
$$A_{4,nm}\left(t\right) = A_{2,mn}\left(t\right) \tag{1284}$$
$$A_{5,nm}\left(t\right) = A_{1,nm}\left(t\right) \tag{1285}$$
$$B_{1,nm}\left(t\right) = B_{nm,x} \tag{1286}$$
$$B_{2,nm}\left(t\right) = B_{nm,y} \tag{1287}$$
$$B_{3,nm}\left(t\right) = B_{z,n}\left(t\right) \tag{1288}$$
$$B_{4,nm}\left(t\right) = B_{1,nm}\left(t\right) \tag{1289}$$
$$B_{5,nm}\left(t\right) = B_{2,nm}\left(t\right) \tag{1290}$$
$$C_{1,nm}\left(t\right) = \Re\left(V_{nm}\left(t\right)\right) \tag{1291}$$
$$C_{2,nm}\left(t\right) = C_{1,nm}\left(t\right) \tag{1292}$$
$$C_{3,nm}\left(t\right) = 1 \tag{1293}$$
$$C_{4,nm}\left(t\right) = \Im\left(V_{nm}\left(t\right)\right) \tag{1294}$$
$$C_{5,nm}\left(t\right) = -\Im\left(V_{nm}\left(t\right)\right) \tag{1295}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}\left(t\right)$ as:

$$\overline{H_I} = \sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(A_{jp}\otimes B_{jp}\left(t\right)\right) \tag{1296}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1280).

We write the interaction Hamiltonian transformed under (1260) as:

$$\widetilde{H_I}\left(t\right) = \sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1297}$$
$$\widetilde{A_{jp}}(t) = U^{\dagger}\left(t\right)A_{jp}U\left(t\right) \tag{1298}$$
$$\widetilde{B_{jp}}\left(t\right) = \mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\left(t\right)(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_Bt} \tag{1299}$$

Taking as reference state ρ_B and truncating at second order in $H_I\left(t\right)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right),\left[\widetilde{H_I}\left(s\right),\widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1300}$$

Replacing the equation (1297) in (1300) we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right), \left[\widetilde{H_I}\left(s\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1301}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right), \left[\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1302}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right), \sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1303}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\right]\mathrm{d}s \tag{1304}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1305}$$

$$-\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right) \tag{1306}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1307}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1308}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}\left(\tau\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1309}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1310}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}\left(t\right)$ and elements related to the system given by $\widetilde{A_{jp}}\left(t\right)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(t\right)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}\left(t\right)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}\left(t\right)\right)$. The correlation functions relevant of the master equation (1308) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1311}$$

$$= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1312}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1313}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1314}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1315}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1316}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1317}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) \tag{1318}$$

$$= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1319}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1320}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1321}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1322}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1323}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1324}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1325}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1301)and (1308) and using the equations (1311)-(1325) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}(t) - \Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'p'}}\left(s\right)\right)\right. \tag{1326}$$

$$\left.+C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right) - \Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1327}$$

$$= -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[\widetilde{A_{jp}}\left(t\right),\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'p'}}\left(s\right),\widetilde{A_{jp}}\left(t\right)\right]\right)\right) \tag{1328}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right),A_{j'p'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}(t)A_{j'p'}\left(t-\tau,t\right),A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}(t)\right] \tag{1329}$$

For this case we used that $A_{jp}\left(t-\tau,t\right) = U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation.

VIII. BIBLIOGRAPHY

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