A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$
$$H_S(t) = \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| + V_{01}(t)|0\rangle\langle 1|, \tag{2}$$
$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij} \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}} \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$
$$V = |0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6)will give us the following result:

$$\begin{aligned}
e^{\pm V} &= e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)} && (10)\\
&= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{\left(\pm\left(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1\right)\right)^2}{2!} + ... && (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + ... && (12)\\
&= |0\rangle\langle 0|\left(1 \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm ...\right) + |1\rangle\langle 1|\left(1 \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm ...\right) && (13)\\
&= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} && (14)\\
&= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} - \alpha_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} - \alpha_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_0^{\pm} + |1\rangle\langle 1|B_1^{\pm}, && (16)\\
B_i^{\pm} &\equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX}\ e^{tY}\ e^{-\frac{t^2}{2}[X,Y]}\ e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\ e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\dots \tag{18}$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i, j we can show making $t = 1$ in (18) the following result:

$$\begin{aligned}
e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\dots && (19)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}0}\dots && (20)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} && (21)
\end{aligned}$$

By induction of this result we can write expresion of $B_i^{\pm}$ as a product of exponentials, which we will call "displacement" operators $D\left(\pm v_{i\mathbf{k}}\right)$:

$$\begin{aligned}
B_i^{\pm} &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), && (22)\\
D\left(\pm v_{i\mathbf{k}}\right) &\equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (23)\\
B_i^{\pm} &= e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} && (24)
\end{aligned}$$

this will help us to write operators O in the variational frame:

$$\overline{O} \equiv \mathrm{e}^{V} O \mathrm{e}^{-V}. \tag{25}$$

We use the following identities:

$$\overline{|0\rangle\langle 0|} = \mathrm{e}^{V}|0\rangle\langle 0|\mathrm{e}^{-V} \tag{26}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{27}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{28}$$
$$= |0\rangle\langle 0|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{29}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+B_1^- \tag{30}$$
$$= |0\rangle\langle 0|, \tag{31}$$
$$\overline{|1\rangle\langle 1|} = \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{32}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+ + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{33}$$
$$= |1\rangle\langle 1|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{34}$$
$$= |1\rangle\langle 1|0\rangle\langle 0|B_1^+B_0^- + B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- \tag{35}$$
$$= B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- \tag{36}$$
$$= |1\rangle\langle 1|, \tag{37}$$
$$\overline{|0\rangle\langle 1|} = \mathrm{e}^{V}|0\rangle\langle 1|\mathrm{e}^{-V} \tag{38}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{39}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|B_1^+|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{40}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{41}$$
$$= |0\rangle\langle 1|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{42}$$
$$= |0\rangle\langle 1|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 1|1\rangle\langle 1|B_0^+B_1^- \tag{43}$$
$$= |0\rangle\langle 1|B_0^+B_1^-, \tag{44}$$
$$\overline{|1\rangle\langle 0|} = \mathrm{e}^{V}|1\rangle\langle 0|\mathrm{e}^{-V} \tag{45}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{46}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{47}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{48}$$
$$= |1\rangle\langle 0|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{49}$$
$$= |1\rangle\langle 0|B_1^+|0\rangle\langle 0|B_0^- + |1\rangle\langle 0|B_1^+|1\rangle\langle 1|B_1^- \tag{50}$$
$$= |1\rangle\langle 0|B_1^+B_0^- + |1\rangle\langle 0|1\rangle\langle 1|B_1^+B_1^- \tag{51}$$
$$= |1\rangle\langle 0|B_1^+B_0^-, \tag{52}$$
$$\overline{b_{\mathbf{k}}} = \mathrm{e}^{V}b_{\mathbf{k}}\mathrm{e}^{-V} \tag{53}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{54}$$
$$= |0\rangle\langle 0|B_0^+b_{\mathbf{k}}B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+b_{\mathbf{k}}|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}B_1^-|1\rangle\langle 1| \tag{55}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+b_{\mathbf{k}}B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+b_{\mathbf{k}}B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+b_{\mathbf{k}}B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}B_1^- \tag{56}$$
$$= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{57}$$
$$= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{58}$$
$$= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{59}$$
$$\overline{b_{\mathbf{k}}}^{\dagger} = \mathrm{e}^{V}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V} \tag{60}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{61}$$
$$= |0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}B_1^-|1\rangle\langle 1| \tag{62}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+b_{\mathbf{k}}^{\dagger}B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+b_{\mathbf{k}}^{\dagger}B_0^- + |1\rangle\langle 1|1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}B_1^- \tag{63}$$
$$= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) \tag{64}$$
$$= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{65}$$

We have used the following:

$$B_i^+ b_{\mathbf{k}} B_i^- = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{66}$$

$$B_i^+ b_{\mathbf{k}}^\dagger B_i^- = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \tag{67}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{68}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{69}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|} = V_{10}(t)\,|1\rangle\langle 0|B_1^+B_0^-, \tag{70}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|} = V_{01}(t)\,|0\rangle\langle 1|B_0^+B_1^-, \tag{71}$$

$$\overline{g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}}} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\right) + g_{\mathbf{ik}}^*\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{72}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^*\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{73}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{74}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{75}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{76}$$

$$= |0\rangle\langle 0|B_0^+|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^- \tag{77}$$

$$= |0\rangle\langle 0|B_0^+\left(g_{0\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}\right)B_0^- \tag{78}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^*\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{79}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{80}$$

$$= |1\rangle\langle 1|B_1^+|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1k}^* b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^- \tag{81}$$

$$= |1\rangle\langle 1|B_1^+\left(g_{1\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}\right)B_1^- \tag{82}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^*\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{83}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_0^- + |1\rangle\langle 1|B_1^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_1^-\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right. \tag{86}$$

$$\left.+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^\dagger b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{90}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right)\right) \tag{91}$$

$$
\begin{aligned}
&= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) && (92) \\
&= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) && (93) \\
&= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) && (94) \\
&= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) && (95) \\
&= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right). && (96)
\end{aligned}
$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$
\begin{aligned}
\overline{H_S(t)} &= \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} && (97) \\
&= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0| B_1^+ B_0^- + V_{01}(t)\,|0\rangle\langle 1| B_0^+ B_1^-, && (98) \\
\overline{H_I} &= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} && (99) \\
&= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} && (100) \\
&= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) && (101) \\
&= \textstyle\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), && (102) \\
\overline{H_B} &= \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} && (103) \\
&= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right) && (104) \\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \right) \right). && (105)
\end{aligned}
$$

Finally merging these expressions gives the transformed Hamiltonian:

$$
\overline{H(t)} = \textstyle\sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (106)
$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$
\overline{H(t)} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}} \quad (107)
$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{108}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \, b_{\mathbf{k}} \right). \tag{109}$$

$$\chi_{ij} \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \tag{110}$$

χ_{ij} is an imaginary number so $e^{\chi_{ij}}$ is the phase associated to B_{ij}. With the following definitions that we will proof and use from now:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_{ij} \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \, b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{\chi_{ij}} \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2\mathrm{i}} & \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix} \tag{111}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot) \tag{112}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot) \tag{113}$$

We assume that the bath is at equilibrium with inverse temperature $\beta = 1/k_{\mathrm{B}}T$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{114}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{115}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{116}$$

So the product of the matrix representation of $b^\dagger$ and b is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{117}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right|, \tag{118}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \tag{119}$$

$$\exp\left(-\beta\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{120}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{121}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{122}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{123}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{124}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{125}$$

$$\mathrm{Tr}\left(\exp\left(-\beta\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{126}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{127}$$

$$= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{128}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{129}$$

$$= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{130}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{131}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{132}$$
$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. \tag{133}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (132)-(133):

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Tr}\left(\rho_B B_{iz}\right) = \mathrm{Tr}\left(B_{iz}\rho_B\right) \tag{134}$$
$$= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{135}$$
$$= \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}}\rho_B\right) \tag{136}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) \tag{137}$$
$$= \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})^{*} b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) \tag{138}$$
$$= \sum_{\mathbf{k}} (\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}})\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} (\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}})^{*}\mathrm{Tr}\left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right), \tag{139}$$
$$\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}\text{)} \tag{140}$$
$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{141}$$
$$= 0, \tag{142}$$
$$\mathrm{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} \tag{143}$$
$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) \tag{144}$$
$$= 0. \tag{145}$$

we therefore find that:

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0 \tag{146}$$

Another important expected value is $B = \langle B^{\pm}\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm} = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\left\langle B^{\pm}\right\rangle_{H_B} = \mathrm{Tr}\left(\rho_B B_{\pm}\right) = \mathrm{Tr}\left(B_{\pm}\rho_B\right) \tag{147}$$
$$= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) \tag{148}$$
$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\right)\rho_B\right) \tag{149}$$
$$= \prod_{\mathbf{k}} \left\langle D\left(\pm\alpha_{\mathbf{k}}\right)\right\rangle. \tag{150}$$

Given that we can write a density operator as:

$$\rho = \int P\left(\alpha\right)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha \tag{151}$$

where $P(\alpha)$ satisfies $\int P(\alpha)\,\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P(\alpha)$ is given by:

$$\langle A\rangle = \mathrm{Tr}\,(A\rho) \tag{152}$$
$$= \int P(\alpha)\,\langle\alpha\,|A|\,\alpha\rangle \mathrm{d}^2\alpha \tag{153}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (153) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle\alpha|D(h)\,|\alpha\rangle \mathrm{d}^2\alpha \tag{154}$$
$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(-\alpha)\,D(h)\,D(\alpha)\,|0\rangle \mathrm{d}^2\alpha \tag{155}$$
$$D(h)\,D(\alpha) = D(h+\alpha)\,e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{156}$$
$$D(-\alpha)\,(D(h)\,D(\alpha)) = D(-\alpha)\,D(h+\alpha)\,e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{157}$$
$$= D(h)\,e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))}e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{158}$$
$$= D(\alpha)\,e^{\frac{1}{2}\left(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2\right)}e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{159}$$
$$= D(\alpha)\,e^{(h\alpha^* - h^*\alpha)}, \tag{160}$$
$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(h)\exp(h\alpha^* - h^*\alpha)\,|0\rangle \mathrm{d}^2\alpha \tag{161}$$
$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^* - h^*\alpha)\,\langle 0|D(h)\,|0\rangle \mathrm{d}^2\alpha \tag{162}$$
$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^* - h^*\alpha)\,\langle 0|h\rangle \mathrm{d}^2\alpha \tag{163}$$
$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{164}$$
$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^* - h^*\alpha)\,\langle 0|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha \tag{165}$$
$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^* - h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{166}$$
$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha\right)\mathrm{d}^2\alpha, \tag{167}$$
$$\alpha = x + \mathrm{i}y, \tag{168}$$
$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{N} + h(x - \mathrm{i}y) - h^*(x + \mathrm{i}y)\right)\mathrm{d}x\mathrm{d}y \tag{169}$$
$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{N} + hx - h^*x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^*y\right)\mathrm{d}y, \tag{170}$$
$$-\tfrac{x^2}{N} + hx - h^*x = -\tfrac{1}{N}\left(x^2 - Nhx + Nh^*x\right) \tag{171}$$

$$= -\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 + \frac{N\left(h^* - h\right)^2}{4}, \tag{172}$$

$$-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^* y = -\frac{1}{N}\left(y^2 + \mathrm{i}Nhy + \mathrm{i}Nh^* y\right) \tag{173}$$

$$= -\frac{1}{N}\left(y^2 + \frac{\mathrm{i}N\left(h + h^*\right)}{2}\right) - \frac{N\left(h + h^*\right)^2}{4}, \tag{174}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^* - h\right)^2}{4} - \frac{N\left(h + h^*\right)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 - \frac{1}{N}\left(y^2 + \frac{\mathrm{i}N\left(h + h^*\right)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, \tag{175}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x - \mu\right)^2}{2\sigma^2}\right)\mathrm{d}x, \tag{176}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^* - h\right)^2}{4} - \frac{N\left(h + h^*\right)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x + \frac{\left(Nh^* - Nh\right)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}x \int_{-\infty}^{\infty} \exp\left(-\frac{\left(y^2 + \frac{\mathrm{i}N\left(h + h^*\right)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}y \tag{177}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^* - h\right)^2}{4} - \frac{N\left(h + h^*\right)^2}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{178}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^* - h\right)^2}{4} - \frac{N\left(h + h^*\right)^2}{4}\right) \tag{179}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^{*2} - 2hh^* + h^2\right) - N\left(h^2 + 2hh^* + h^{*2}\right)}{4}\right) \tag{180}$$

$$= \exp\left(-|h|^2\left(N + \frac{1}{2}\right)\right) \tag{181}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2}\right)\right) \tag{182}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1}\right)\right) \tag{183}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{184}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (149) using (184) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{185}$$

We will now force $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_+\sigma^+$and $B_-\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H\left(t\right)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j|\left((g_{j\mathbf{k}} - v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{\left|v_{j\mathbf{k}}\right|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{186}$$

$$= \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'} + \sum_j |j\rangle\langle j|B_{jz} + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+ B_{j'}^- - B_{jj'}\right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{187}$$

$$\equiv \overline{H_{\bar{S}}\left(t\right)} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{188}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\begin{aligned}
\langle B_1^+ B_0^- \rangle &= B_{10} && (189)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle && (190)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle && (191)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right) \right\rangle && (192)\\
&= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (193)\\
&= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (194)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}. && (195)
\end{aligned}$$

From the definition $B_{01} = \langle B_0^+ B_1^- \rangle$ using the displacement operator we have:

$$\begin{aligned}
\langle B_0^+ B_1^- \rangle &= B_{01} && (196)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle && (197)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle && (198)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right) \right\rangle && (199)\\
&= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (200)\\
&= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (201)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (202)
\end{aligned}$$

We can check:

$$\begin{aligned}
\langle B_0^+ B_1^- \rangle &= B_{01} && (203)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (204)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)^*} && (205)\\
&= \langle B_1^+ B_0^- \rangle^* && (206)\\
&= B_{10}^*. && (207)
\end{aligned}$$

The parts of the Hamiltonian splitted are:

$$\overline{H_{\bar{S}}\left(t\right)} \equiv \left(\varepsilon_0\left(t\right)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1| + V_{10}\left(t\right)B_{10}\sigma^+ + V_{01}\left(t\right)B_{01}\sigma^-, \tag{208}$$

$$\overline{H_{\bar{I}}} \equiv V_{10}\left(t\right)\left(B_1^+B_0^- - B_{10}\right)\sigma^+ + V_{01}\left(t\right)\left(B_0^+B_1^- - B_{01}\right)\sigma^- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, \tag{209}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{210}$$

$$= H_B. \tag{211}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (209) we can verify the condition $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}}=0$ in the following way:

$$\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = \left\langle\sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|n\rangle\langle n| + \sum_{j\neq j'}V_{jj'}\left(t\right)|j\rangle\langle j'|\left(B_j^+B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{212}$$

$$= \left\langle\sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle\sum_{j\neq j'}V_{jj'}\left(t\right)|j\rangle\langle j'|\left(B_j^+B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{213}$$

$$= \sum_{n\mathbf{k}}\left(\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}})b_{\mathbf{k}}^{\dagger}\rangle_{\overline{H_{\bar{B}}}}+\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}})^*b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}(t)B_j^+B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle V_{jj'}(t)B_{jj'}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{214}$$

$$= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})\langle b_{\mathbf{k}}^{\dagger}\rangle_{\overline{H_{\bar{B}}}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(\left\langle B_j^+B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle B_{jj'}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{215}$$

$$= \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)\left\langle b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}\left(t\right)\left(B_{jj'}-B_{jj'}\right) \tag{216}$$

$$= 0. \tag{217}$$

We used (146) and (195) to evaluate the expected values.

Let's consider the following Hermitian combinations:

$$B_x = B_x^{\dagger} \tag{218}$$

$$= \frac{B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}}{2}, \tag{219}$$

$$B_y = B_y^{\dagger} \tag{220}$$

$$= \frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}, \tag{221}$$

$$B_{iz} = B_{iz}^{\dagger} \tag{222}$$

$$= \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*b_{\mathbf{k}}\right). \tag{223}$$

Writing the equations (208) and (209) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}\left(t\right)=\left(\varepsilon_{0}\left(t\right)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}\left(t\right)+R_{1}\right)|1\rangle\langle1|+V_{10}\left(t\right)B_{10}\sigma^{+}+V_{01}\left(t\right)B_{01}\sigma^{-} \tag{224}$$

$$=\left(\varepsilon_{0}\left(t\right)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}\left(t\right)+R_{1}\right)|1\rangle\langle1|+V_{10}\left(t\right)B_{10}\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}+V_{01}\left(t\right)B_{01}\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2} \tag{225}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+V_{10}(t)\left(B_{10}^{\Re}(t)+\mathrm{i}B_{10}^{\Im}(t)\right)\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}+V_{01}(t)\left(B_{10}^{\Re}(t)-\mathrm{i}B_{10}^{\Im}(t)\right)\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2} \tag{226}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+B_{10}^{\Re}(t)\left(V_{10}(t)\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}+V_{01}(t)\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(V_{10}(t)\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}-V_{01}(t)\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}\right) \tag{227}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+B_{10}^{\Re}(t)\left(\sigma_{x}\frac{V_{10}(t)+V_{01}(t)}{2}+\mathrm{i}\sigma_{y}\frac{V_{10}(t)-V_{01}(t)}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\sigma_{x}\frac{V_{10}(t)-V_{01}(t)}{2}+\mathrm{i}\sigma_{y}\frac{V_{10}(t)+V_{01}(t)}{2}\right) \tag{228}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+B_{10}^{\Re}(t)\left(\sigma_{x}\frac{V_{10}(t)+V_{10}^{*}(t)}{2}+\mathrm{i}\sigma_{y}\frac{V_{10}(t)-V_{10}^{*}(t)}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\sigma_{x}\frac{V_{10}(t)-V_{10}^{*}(t)}{2}+\mathrm{i}\sigma_{y}\frac{V_{10}(t)+V_{10}^{*}(t)}{2}\right) \tag{229}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+B_{10}^{\Re}\left(t\right)\left(\sigma_{x}V_{10}^{\Re}\left(t\right)-\sigma_{y}V_{10}^{\Im}\left(t\right)\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\mathrm{i}\sigma_{x}V_{10}^{\Im}(t)+\mathrm{i}\sigma_{y}V_{10}^{\Re}(t)\right) \tag{230}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+\left(\sigma_{x}B_{10}^{\Re}(t)V_{10}^{\Re}(t)-\sigma_{y}B_{10}^{\Re}(t)V_{10}^{\Im}(t)\right)-\left(\sigma_{x}B_{10}^{\Im}(t)V_{10}^{\Im}(t)+\sigma_{y}B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{231}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+\sigma_{x}\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_{y}\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{232}$$

$$=\left(\varepsilon_{0}(t)+R_{0}\right)|0\rangle\langle0|+\left(\varepsilon_{1}(t)+R_{1}\right)|1\rangle\langle1|+\sigma_{x}\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_{y}\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right). \tag{233}$$

$$\overline{H_{\bar{I}}}=V_{10}\left(t\right)\left(\sigma^{+}B_{1}^{+}B_{0}^{-}-\sigma^{+}B_{10}\right)+V_{01}\left(t\right)\left(\sigma^{-}B_{0}^{+}B_{1}^{-}-\sigma^{-}B_{01}\right)+|0\rangle\langle0|B_{0z}+|1\rangle\langle1|B_{1z} \tag{234}$$

$$=|0\rangle\langle0|B_{0z}+|1\rangle\langle1|B_{1z}+\left(V_{10}^{\Re}\left(t\right)+\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{+}B_{1}^{+}B_{0}^{-}-\sigma^{+}B_{10}\right)+\left(V_{10}^{\Re}\left(t\right)-\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{-}B_{0}^{+}B_{1}^{-}-\sigma^{-}B_{01}\right) \tag{235}$$

$$=\sum\nolimits_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma^{+}B_{1}^{+}B_{0}^{-}-\sigma^{+}B_{10}+\sigma^{-}B_{0}^{+}B_{1}^{-}-\sigma^{-}B_{01}\right)+\mathrm{i}V_{10}^{\Im}(t)\left(\sigma^{+}B_{1}^{+}B_{0}^{-}-\sigma^{+}B_{10}-\sigma^{-}B_{0}^{+}B_{1}^{-}+\sigma^{-}B_{01}\right) \tag{236}$$

$$=\sum_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}B_{1}^{+}B_{0}^{-}-\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}B_{10}+\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}B_{0}^{+}B_{1}^{-}-\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}B_{01}\right) \tag{237}$$

$$=\sum\nolimits_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}B_{1}^{+}B_{0}^{-}-\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}B_{10}+\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}B_{0}^{+}B_{1}^{-}-\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}B_{01}\right)+\mathrm{i}V_{10}^{\Im}(t)\left(\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}B_{1}^{+}B_{0}^{-}-\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}B_{10}-\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}B_{0}^{+}B_{1}^{-}\right. \tag{238}$$

$$\left.+\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}B_{01}\right) \tag{239}$$

$$=\sum\nolimits_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_{x}\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}+\mathrm{i}\sigma_{y}\frac{B_{1}^{+}B_{0}^{-}-B_{0}^{+}B_{1}^{-}-B_{10}+B_{01}}{2}\right)+\mathrm{i}V_{10}^{\Im}(t)\left(\sigma_{x}\frac{B_{1}^{+}B_{0}^{-}-B_{0}^{+}B_{1}^{-}-B_{10}+B_{01}}{2}\right. \tag{240}$$

$$\left.+\mathrm{i}\sigma_{y}\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right) \tag{241}$$

$$=\sum_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_{x}B_{x}+\sigma_{y}B_{y}\right)+V_{10}^{\Im}\left(t\right)\left(\mathrm{i}\sigma_{x}\frac{B_{1}^{+}B_{0}^{-}-B_{0}^{+}B_{1}^{-}-B_{10}+B_{01}}{2}-\sigma_{y}\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right) \tag{242}$$

$$=\sum_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_{x}B_{x}+\sigma_{y}B_{y}\right)+V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^{2}\sigma_{x}\frac{B_{1}^{+}B_{0}^{-}-B_{0}^{+}B_{1}^{-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_{y}\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right) \tag{243}$$

$$=\sum_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_{x}B_{x}+\sigma_{y}B_{y}\right)+V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^{2}\sigma_{x}\frac{B_{1}^{+}B_{0}^{-}-B_{0}^{+}B_{1}^{-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_{y}\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right) \tag{244}$$

$$=\sum_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_{x}B_{x}+\sigma_{y}B_{y}\right)+V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^{2}\sigma_{x}\left(-B_{y}\right)-\sigma_{y}B_{x}\right) \tag{245}$$

$$=\sum_{i}B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_{x}B_{x}+\sigma_{y}B_{y}\right)+V_{10}^{\Im}\left(t\right)\left(\sigma_{x}B_{y}-\sigma_{y}B_{x}\right). \tag{246}$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A\leq A_{\mathrm{B}}\equiv-\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)+\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}+O\left(\left\langle\overline{H_{\bar{I}}}^{2}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{247}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}=0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}}=0. \tag{248}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\left(t\right), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{249}$$

Then using the fact that $\overline{H_{\bar{S}}}\left(t\right)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{250}$$

So Eq. (248) becomes:

$$\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} &= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}} && (251)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (252)\\
&= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}} && (253)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (254)\\
&= 0 \quad (\text{by Eq. (248)}). && (255)
\end{aligned}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\begin{aligned}
\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} &= \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} && (256)\\
&= 0. && (257)
\end{aligned}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} = 0. \tag{258}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}\left(t\right) = -\beta\left(\left(\varepsilon_0\left(t\right)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1| + V_{10}\left(t\right)B_{10}\sigma^{+} + V_{01}\left(t\right)B_{01}\sigma^{-}\right). \tag{259}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}\left(t\right)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right) + \mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right) = 0. \tag{260}$$

Let's define:

$$\begin{aligned}
\varepsilon\left(t\right) &\equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), && (261)\\
\eta &\equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}. && (262)
\end{aligned}$$

The solutions of the equation (260) are:

$$\lambda = \beta \frac{-\text{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right) \pm \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2} - 4\text{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}}{2} \tag{263}$$

$$= \beta \frac{-\varepsilon\left(t\right) \pm \eta\left(t\right)}{2} \tag{264}$$

$$= -\beta \frac{\varepsilon\left(t\right) \mp \eta\left(t\right)}{2}. \tag{265}$$

The value of $\text{Tr}\left(e^{-\beta \overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr}\left(e^{-\beta \overline{H_{\bar{S}}}(t)}\right) = \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right) + \exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) \tag{266}$$

$$= 2\text{exp}\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{267}$$

Given that $v_{i\mathbf{k}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr}\left(e^{-\beta \overline{H_{\bar{S}}}(t)}\right) = A\left(\varepsilon\left(t\right), \eta\left(t\right)\right)$ to calculate $\frac{\partial \text{Tr}\left(e^{-\beta \overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}$ can lead to:

$$\frac{\partial \text{Tr}\left(e^{-\beta \overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(2\text{exp}\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{268}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) + 2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) \tag{269}$$

$$= -\beta\text{exp}\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{270}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right) - \frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) = 0. \tag{271}$$

The derivates included in the expression given are related to:

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \tag{272}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right)^* \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \tag{273}$$

$$= \langle B_1^+ B_0^- \rangle^*, \tag{274}$$

$$R_i = \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{275}$$

$$= \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{276}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \tag{277}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2\omega_{\mathbf{k}}^2} \left(v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* \right) \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right), \tag{278}$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = \left(v_{0\mathbf{k}}^{\Re} - \mathrm{i} v_{0\mathbf{k}}^{\Im}\right)\left(v_{1\mathbf{k}}^{\Re} + \mathrm{i} v_{1\mathbf{k}}^{\Im}\right) - \left(v_{0\mathbf{k}}^{\Re} + \mathrm{i} v_{0\mathbf{k}}^{\Im}\right)\left(v_{1\mathbf{k}}^{\Re} - \mathrm{i} v_{1\mathbf{k}}^{\Im}\right) \tag{279}$$

$$= \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} + \mathrm{i} v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - \mathrm{i} v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}\right) - \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} - \mathrm{i} v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} + \mathrm{i} v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}\right) \tag{280}$$

$$= 2\mathrm{i} \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} \right), \tag{281}$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}} - v_{0\mathbf{k}})(v_{1\mathbf{k}} - v_{0\mathbf{k}})^* \tag{282}$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - (v_{1\mathbf{k}} v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}}) \tag{283}$$

$$= \left(v_{1\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im}\right)^2 + \left(v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{0\mathbf{k}}^{\Im}\right)^2 - \left(\left(v_{1\mathbf{k}}^{\Re} + \mathrm{i} v_{1\mathbf{k}}^{\Im}\right)\left(v_{0\mathbf{k}}^{\Re} - \mathrm{i} v_{0\mathbf{k}}^{\Im}\right) + \left(v_{1\mathbf{k}}^{\Re} - \mathrm{i} v_{1\mathbf{k}}^{\Im}\right)\left(v_{0\mathbf{k}}^{\Re} + \mathrm{i} v_{0\mathbf{k}}^{\Im}\right)\right) \tag{284}$$

$$= \left(v_{1\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im}\right)^2 + \left(v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{0\mathbf{k}}^{\Im}\right)^2 - 2\left(v_{1\mathbf{k}}^{\Re} v_{0\mathbf{k}}^{\Re} + v_{1\mathbf{k}}^{\Im} v_{0\mathbf{k}}^{\Im}\right) \tag{285}$$

$$= \left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2. \tag{286}$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re} - \mathrm{i} v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re} + \mathrm{i} v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \right) \right) \tag{287}$$

$$= \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - \mathrm{i} v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{288}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} \exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \tag{289}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re}\right)}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \tag{290}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right), \tag{291}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{292}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - \mathrm{i}v_{i\mathbf{k}}^{\Im}\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{293}$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{294}$$

$$\frac{\partial\left|B_{10}\right|^2}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{295}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{296}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}\left|B_{10}\right|^2, \tag{297}$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{298}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} - 4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{299}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0) - |V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta(t)} \tag{300}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{301}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{i\mathbf{k}}^{\Re} - v_{i'\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{302}$$

$$= \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) - \frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + \frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right. \tag{303}$$

$$\left. + 4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{304}$$

From the equation (271) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \tag{305}$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) - \frac{2}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + 2\frac{\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right)\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} + 2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}} \tag{306}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v^{\Re}_{i\mathbf{k}}-g_{i\mathbf{k}}-g^{*}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2B^2_{10}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{308}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{309}$$

$$=\frac{\left(v^{\Re}_{i\mathbf{k}}-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{310}$$

Separating (309) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Re}_{i\mathbf{k}}-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Re}_{i\mathbf{k}}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{311}$$

$$v^{\Re}_{i\mathbf{k}}-g^{\Re}_{i\mathbf{k}}=v^{\Re}_{i\mathbf{k}}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right) \tag{312}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{313}$$

$$v^{\Re}_{i\mathbf{k}}=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}}{g^{\Re}_{i\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{314}$$

$$v^{\Re}_{i\mathbf{k}}=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}}{g^{\Re}_{i\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{315}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon(t)}{\partial v^{\Im}_{i\mathbf{k}}} = \frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial v^{\Im}_{i\mathbf{k}}} \tag{316}$$

$$= \frac{\partial\left(\left(\frac{\left(v^{\Re}_{i\mathbf{k}}\right)^2+\left(v^{\Im}_{i\mathbf{k}}\right)^2}{\omega_{\mathbf{k}}}-v^{\Re}_{i\mathbf{k}}\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}v^{\Im}_{i\mathbf{k}}\frac{g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}} \tag{317}$$

$$= 2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{318}$$

$$\frac{\partial|B_{10}|^2}{\partial v^{\Im}_{i\mathbf{k}}} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)^2+\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}} \tag{319}$$

$$= -\frac{2\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)}{\partial v^{\Im}_{i\mathbf{k}}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)^2+\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{320}$$

$$= -\frac{2\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)}{\partial v^{\Im}_{i\mathbf{k}}}|B_{10}|^2 \tag{321}$$

$$\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v^{\Re}_{i\mathbf{k}}} \tag{322}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v^{\Im}_{i\mathbf{k}}}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v^{\Im}_{i\mathbf{k}}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{323}$$

$$= \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v^{\Im}_{i\mathbf{k}}}}{\eta(t)} \tag{324}$$

$$= \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)}{\partial v^{\Im}_{i\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{325}$$

$$= \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Im}_{i\mathbf{k}}-v^{\Im}_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{326}$$

$$= \frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2(\varepsilon(t)-\varepsilon_i(t)-R_i)\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{327}$$

$$\left.+4\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{328}$$

From the equation (271) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v^{\Im}_{i\mathbf{k}}}}{\frac{\partial\eta(t)}{\partial v^{\Im}_{i\mathbf{k}}}} \tag{329}$$

$$= \frac{2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\left(g^*_{i\mathbf{k}}\right)^{\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\frac{\left(g^*_{i\mathbf{k}}\right)^{\Im}}{\omega_{\mathbf{k}}}+2\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)} \tag{330}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v^{\Im}_{i\mathbf{k}}-\mathrm{i}\left(g^*_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g^*_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{331}$$

$$= \frac{2\left(v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{332}$$

$$= \frac{2\left(v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{333}$$

$$= \frac{\left(v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{334}$$

Separating (334) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{335}$$

$$v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}=v^{\Im}_{i\mathbf{k}}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \tag{336}$$

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{337}$$

$$v^{\Im}_{i\mathbf{k}}=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{338}$$

$$v^{\Im}_{i\mathbf{k}}=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{339}$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}})=v^{\Re}_{i\mathbf{k}}(\omega_{\mathbf{k}})+\mathrm{i}v^{\Im}_{i\mathbf{k}}(\omega_{\mathbf{k}}) \tag{340}$$

$$=\frac{g^{\Re}_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{341}$$

$$+\mathrm{i}\frac{g^{\Im}_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{342}$$

$$=\frac{g_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{343}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0)=\rho_S(0)\otimes\rho_B^{\mathrm{Thermal}}$, the transformed density operator is equal to:

$$\begin{aligned}
\overline{\rho_T(0)} &\equiv e^V \rho_T\left(0\right) e^{-V} && (344)\\
&= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)\left(\rho_S(0)\otimes\rho_B^{\text{Thermal}}\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (345)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } & |0\rangle\langle 0|0\rangle B_0^+\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_0^- && (346)\\
&= |0\rangle B_0^+\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_0^- && (347)\\
&= |0\rangle\langle 0|\otimes B_0^+\rho_B^{\text{Thermal}}B_0^- && (348)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } & |1\rangle\langle 1|B_1^+|1\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_1^- && (349)\\
&= |1\rangle\langle 1|B_1^+\rho_B^{\text{Thermal}}B_1^- && (350)\\
&= |1\rangle\langle 1|\otimes B_1^+\rho_B^{\text{Thermal}}B_1^- && (351)\\
\text{for } \rho_S(0) = |0\rangle\langle 1| : & |0\rangle\langle 0|B_0^+|0\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_1^- && (352)\\
&= |0\rangle\langle 1|B_0^+\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_1^- && (353)\\
&= |0\rangle\langle 1|1\rangle\langle 1|B_0^+\rho_B^{\text{Thermal}}B_1^- && (354)\\
&= |0\rangle\langle 1|\otimes B_0^+\rho_B^{\text{Thermal}}B_1^- && (355)\\
\text{for } \rho_S(0) = |1\rangle\langle 0| : & |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_0^- && (356)\\
&= |1\rangle\langle 0|\otimes B_1^+\rho_B^{\text{Thermal}}B_0^- && (357)
\end{aligned}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}\left(t\right) \equiv U^{\dagger}\left(t\right) O\left(t\right) U\left(t\right) \tag{358}$$

$$U\left(t\right) \equiv \mathcal{T}\text{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right). \tag{359}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}\left(t\right) = U^{\dagger}\left(t\right)\overline{\rho_S}\left(t\right)U\left(t\right), \text{ where} \tag{360}$$

$$\overline{\rho_S}\left(t\right) = \text{Tr}_B\left(\overline{\rho}_T\left(t\right)\right) \tag{361}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda\left(t\right)$ such that $\Lambda_{1i}\left(t\right) = A_i$, $\Lambda_{2i}\left(t\right) = B_i$ and $\Lambda_{3i}\left(t\right) = C_i\left(t\right)$ written as:

$$\begin{pmatrix} A\left(t\right)\\ B\left(t\right)\\ C\left(t\right)\end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2}\\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z}\\ V_{10}^{\Re}\left(t\right) & V_{10}^{\Re}\left(t\right) & 1 & V_{10}^{\Im}\left(t\right) & -V_{10}^{\Im}\left(t\right) & 1\end{pmatrix} \tag{362}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}\left(t\right)$ as pointed in the equation (246):

$$\begin{aligned}
\overline{H_{\bar{I}}}\left(t\right) &= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y - \sigma_y B_x\right) && (363)\\
&= B_{0z}|0\rangle\langle 0| + B_{1z}|1\rangle\langle 1| + V_{10}^{\Re}\left(t\right)\sigma_x B_x + V_{10}^{\Re}\left(t\right)\sigma_y B_y + V_{10}^{\Im}\left(t\right)\sigma_x B_y - V_{10}^{\Im}\left(t\right)\sigma_y B_x && (364)\\
&= \sum_i C_i\left(t\right)\left(A_i\otimes B_i\left(t\right)\right) && (365)
\end{aligned}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\widetilde{H_{\bar{I}}}}(t)\,,\widetilde{\widetilde{\rho_T}}(t)]. \tag{366}$$

This equation has the formal solution

$$\widetilde{\widetilde{\rho_T}}(t) = \overline{\rho_T}(0)\,\text{-i}\int_0^t [\widetilde{\widetilde{H_{\bar{I}}}}(s)\,,\widetilde{\widetilde{\rho_T}}(s)]\mathrm{d}s. \tag{367}$$

Replacing the equation (367) in the equation (366) gives us:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\widetilde{H_{\bar{I}}}}(t)\,,\overline{\rho_T}(0)]\text{-}\int_0^t [\widetilde{\widetilde{H_{\bar{I}}}}(t)\,,[\widetilde{\widetilde{H_{\bar{I}}}}(s)\,,\widetilde{\widetilde{\rho_T}}(s)]]\mathrm{d}s. \tag{368}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\widetilde{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 ... \int_0^{t_{n-1}}\mathrm{d}t_n[\widetilde{\widetilde{H_{\bar{I}}}}(t_1)\,,[\widetilde{\widetilde{H_{\bar{I}}}}(t_2)\,,\cdots[\widetilde{\widetilde{H_{\bar{I}}}}(t_n)\,,\overline{\rho_T}(0)]]\cdots] \tag{369}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\widetilde{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 ... \int_0^{t_{n-1}}\mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\widetilde{H_{\bar{I}}}}(t_1)\,,[\widetilde{\widetilde{H_{\bar{I}}}}(t_2)\,,\cdots[\widetilde{\widetilde{H_{\bar{I}}}}(t_n)\,,\overline{\rho_S}(0)\,\rho_B^{\mathrm{Thermal}}]]\cdots] \tag{370}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0)\otimes\rho_B^{\mathrm{Thermal}}$. Consider the following notation:

$$\begin{aligned}\widetilde{\widetilde{\rho_S}}(t) &= (1 + W_1(t) + W_2(t) + ...)\,\overline{\rho_S}(0) && (371)\\ &= W(t)\,\overline{\rho_S}(0) && (372)\end{aligned}$$

in this case

$$W_n(t) = (-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 ... \int_0^{t_{n-1}}\mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\widetilde{H_{\bar{I}}}}(t_1)\,,[\widetilde{\widetilde{H_{\bar{I}}}}(t_2)\,,\cdots[\widetilde{\widetilde{H_{\bar{I}}}}(t_n)\,,(\cdot)\,\rho_B^{\mathrm{Thermal}}]]\cdots] \tag{373}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} &= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) && (374)\\ &= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}\,W(t)\,\overline{\rho_S}(0) && (375)\\ &= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}\,\widetilde{\widetilde{\rho_S}}(t) && (376)\end{aligned}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\widetilde{H_{\bar{I}}}}(t)\,\rho_B^{\mathrm{Thermal}}] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t)\approx\mathbb{I}$ then the equation (374) becomes:

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} &= \dot{W}_2(t)\,\widetilde{\widetilde{\rho_S}}(t) && (377)\\ &= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B\left[\widetilde{\widetilde{H_{\bar{I}}}}(t)\,,\left[\widetilde{\widetilde{H_{\bar{I}}}}(t_1)\,,\widetilde{\widetilde{\rho_S}}(t)\,\rho_B^{\mathrm{Thermal}}\right]\right] && (378)\end{aligned}$$

Replacing $t_1 \to t-\tau$ and moving back into the Schrödinger picture gives:

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}\left(t\right), \left[\widetilde{\overline{H_{\bar{I}}}}\left(-\tau\right), \overline{\rho_S}\left(t\right)\rho_B^{\mathrm{Thermal}}\right]\right] \tag{379}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}\left(t\right)$ we find:

$$\widetilde{\widetilde{\overline{H_{\bar{I}}}}}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) \tag{380}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\begin{aligned}
\widetilde{\widetilde{\overline{H_{\bar{I}}}}}\left(t\right) &= \sum_i C_i\left(t\right)\left(\widetilde{A_i}\left(t\right) \otimes \widetilde{B_i}\left(t\right)\right) &(381)\\
\widetilde{A_i}\left(t\right) &= U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}A_i \mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) &(382)\\
&= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} &(383)\\
&= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathbb{I} &(384)\\
&= U^{\dagger}\left(t\right)A_i U\left(t\right) &(385)\\
\widetilde{B_i}\left(t\right) &= U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) &(386)\\
&= U^{\dagger}\left(t\right)U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} &(387)\\
&= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} &(388)\\
&= \mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} &(389)
\end{aligned}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}\left(t\right), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.

Using the expression (381) to replace it in the equation (378)

$$\begin{aligned}
\tfrac{\mathrm{d}\widetilde{\widetilde{\overline{\rho_S}}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{\widetilde{\overline{H_{\bar{I}}}}}\left(t\right), \left[\widetilde{\widetilde{\overline{H_{\bar{I}}}}}\left(s\right), \widetilde{\widetilde{\overline{\rho_S}}}\left(t\right)\rho_B^{\mathrm{Thermal}}\right]\right]\mathrm{d}s &(390)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j\left(t\right)\left(\widetilde{A_j}\left(t\right)\otimes\widetilde{B_j}\left(t\right)\right), \left[\sum_i C_i\left(s\right)\left(\widetilde{A_i}\left(s\right)\otimes\widetilde{B_i}\left(s\right)\right), \widetilde{\widetilde{\overline{\rho_S}}}\left(t\right)\rho_B^{\mathrm{Thermal}}\right]\right]\mathrm{d}s &(391)\\
&= -\textstyle\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\overline{\rho_S}}}(t)\rho_B^{\mathrm{Thermal}} - \widetilde{\widetilde{\overline{\rho_S}}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\right]\mathrm{d}s &(392)\\
&= -\textstyle\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\overline{\rho_S}}}(t)\rho_B^{\mathrm{Thermal}} - \sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\widetilde{\overline{\rho_S}}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right) &(393)\\
&\quad -\textstyle\sum_{i\in J} C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\overline{\rho_S}}}(t)\rho_B^{\mathrm{Thermal}}\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right) + \widetilde{\widetilde{\overline{\rho_S}}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\Big)\mathrm{d}s &(394)
\end{aligned}$$

In order to calculate the correlation functions we define:

$$\begin{aligned}
\Lambda_{ji}\left(\tau\right) &= \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\right\rangle_B &(395)\\
&= \left\langle \widetilde{B_j}\left(\tau\right)\widetilde{B_i}\left(0\right)\right\rangle_B &(396)
\end{aligned}$$

The correlation functions relevant that appear in the equation (394) are:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\rho_B^{\mathrm{Thermal}}\right) &= \left\langle\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\right\rangle_B && (397)\\
&= \left\langle\widetilde{B_j}\left(\tau\right)\widetilde{B_i}\left(0\right)\right\rangle_B && (398)\\
&= \Lambda_{ji}\left(\tau\right) && (399)\\
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B^{\mathrm{Thermal}}\widetilde{B_i}\left(s\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\rho_B^{\mathrm{Thermal}}\right) && (400)\\
&= \left\langle\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right\rangle_B && (401)\\
&= \left\langle\widetilde{B_i}\left(-\tau\right)\widetilde{B_j}\left(0\right)\right\rangle_B && (402)\\
&= \Lambda_{ij}\left(-\tau\right) && (403)\\
\mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\rho_B^{\mathrm{Thermal}}\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\rho_B^{\mathrm{Thermal}}\right) && (404)\\
&= \left\langle\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\right\rangle_B && (405)\\
&= \left\langle\widetilde{B_j}\left(\tau\right)\widetilde{B_i}\left(0\right)\right\rangle_B && (406)\\
&= \Lambda_{ji}\left(\tau\right) && (407)\\
\mathrm{Tr}_B\left(\rho_B^{\mathrm{Thermal}}\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\rho_B^{\mathrm{Thermal}}\right) && (408)\\
&= \left\langle\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right\rangle_B && (409)\\
&= \left\langle\widetilde{B_i}\left(-\tau\right)\widetilde{B_j}\left(0\right)\right\rangle_B && (410)\\
&= \Lambda_{ij}\left(-\tau\right) && (411)
\end{aligned}$$

The cyclic property of the trace was use widely in the development of equations (397) and (411). Replacing in (394)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)\widetilde{A_i}(t)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)-\Lambda_{ji}(-\tau)\widetilde{A_i}(t)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\big)+C_i(t)C_j(s)\big(\Lambda_{ji}(-\tau)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\widetilde{A_i}(t)-\Lambda_{ij}(\tau)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t)\big)\Big)\mathrm{d}s && (412)\\
&= -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\Big(\Lambda_{ij}(\tau)\Big[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\Big]+\Lambda_{ji}(-\tau)\Big[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)\Big]\Big)\Big)\mathrm{d}s && (413)
\end{aligned}$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}\left(\tau\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)-\Lambda_{ij}\left(\tau\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)=\Lambda_{ij}\left(\tau\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_s}}\left(t\right)\right] \quad (414)$$

$$\Lambda_{ji}\left(-\tau\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\Lambda_{ji}\left(-\tau\right)\widetilde{A_i}\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)=\Lambda_{ji}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j},\widetilde{A_i}\left(t\right)\right] \quad (415)$$

Returning to the Schroedinger picture we have:

$$\begin{aligned}
U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right) &= U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{A_j}\left(s\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right) && (416)\\
&= \left(U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{A_j}\left(s\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right)\right) && (417)\\
&= A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right) && (418)
\end{aligned}$$

This procedure applying to the relevant commutators give us:

$$\begin{aligned}
U\left(t\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\right]U^\dagger\left(t\right) &= \left(U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right)-U\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right) && (419)\\
&= A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right)-\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right)A_i && (420)\\
&= \left[A_i,\widetilde{A_j}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right] && (421)
\end{aligned}$$

Introducing this transformed commutators in the equation (413) allow us to obtain the master equation of the system

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t}=-\mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ij}\int_0^t\mathrm{d}\tau\left(C_i\left(t\right)C_j\left(t-\tau\right)\Lambda_{ij}\left(\tau\right)\left[A_i,\widetilde{A_j}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right]\right. \tag{422}$$

$$\left.+C_j\left(t\right)C_i\left(t-\tau\right)\Lambda_{ji}\left(-\tau\right)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}\left(t-\tau,t\right),A_i\right]\right) \tag{423}$$

where $i,j\in\{1,2,3,4,5.6\}$.
Here $\widetilde{A_j}\left(s,t\right)=U\left(t\right)U^{\dagger}\left(s\right)A_jU\left(s\right)U^{\dagger}\left(t\right)$ where $U\left(t\right)$ is given by (359). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I\left(t\right)$. The environmental correlation functions are given by:

$$\Lambda_{ij}\left(\tau\right)=\mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B^{\mathrm{Thermal}}\right) \tag{424}$$

$$=\mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B^{\mathrm{Thermal}}\right) \tag{425}$$

Calculating the correlation functions allow us to obtain:

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B=\mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B^{\mathrm{Thermal}}\right) \tag{426}$$

$$=\int\mathrm{d}^2\alpha P\left(\alpha\right)\left\langle\alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle \tag{427}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha\right|^2}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle\mathrm{d}^2\alpha \tag{428}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha\right|^2}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle\mathrm{d}^2\alpha \tag{429}$$

$$\widetilde{B_{jz}}\left(\tau\right)=\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{430}$$

$$\widetilde{B_{jz}}\left(0\right)=\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right) \tag{431}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B=\mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B\right) \tag{432}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\rho_B\right) \tag{433}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\rho_B\right) \tag{434}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}\right)\rho_B\right) \tag{435}$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}}=p_{j\mathbf{k}} \tag{436}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}'}b_{\mathbf{k}'}^{\dagger}+p_{j\mathbf{k}'}^*b_{\mathbf{k}'}\right)\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^*b_{\mathbf{k}}\right)\rho_B\right) \tag{437}$$

$$=0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^*b_{\mathbf{k}}\right)\rho_B\right) \tag{438}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+|p_{j\mathbf{k}}|^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+|p_{j\mathbf{k}}|^2b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B\right) \tag{439}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}p_{j\mathbf{k}}^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{440}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{441}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right)+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right)\right) \tag{442}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{443}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{444}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{445}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{446}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{447}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{448}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| \left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{449}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{450}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right)\right. \tag{451}$$

$$\left. + \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right)\right) \tag{452}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{453}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{454}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{455}$$

$$1 = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \tag{456}$$

$$b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} |0\rangle = 0 \tag{457}$$

$$b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} |0\rangle = |0\rangle \tag{458}$$

$$\left\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \right\rangle_B = \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{459}$$

$$+ \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{460}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{461}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}}\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{462}$$

$$\frac{1}{\pi N} \int_0^{2\pi} \int_0^{\infty} r^2 \exp\left(-\frac{r^2}{N}\right) r \mathrm{d}r \mathrm{d}\theta = \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \mathrm{d}^2\alpha_{\mathbf{k}} \tag{463}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(2\cos\left(\omega_{\mathbf{k}}\tau\right) N\right) + \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{464}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(2\cos\left(\omega_{\mathbf{k}}\tau\right) N + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{465}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{466}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} + \cos\left(\omega_{\mathbf{k}}\tau\right) - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{467}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\frac{\left(2 + e^{\beta\omega_{\mathbf{k}}} - 1\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{468}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\frac{\left(1 + e^{\beta\omega_{\mathbf{k}}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}} - 1} - \mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{469}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos(\omega_{\mathbf{k}}\tau)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}} - \mathrm{i}\sin(\omega_{\mathbf{k}}\tau)\right) \tag{470}$$

$$= \sum_{\mathbf{k}} \left|p_{j\mathbf{k}}\right|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau) - \mathrm{i}\sin(\omega_{\mathbf{k}}\tau)\right) \tag{471}$$

$$= \sum_{\mathbf{k}} \left|g_{j\mathbf{k}} - v_{j\mathbf{k}}\right|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau) - \mathrm{i}\sin(\omega_{\mathbf{k}}\tau)\right) \tag{472}$$

$$\left\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \right\rangle_B = \int \mathrm{d}^2\alpha_{\mathbf{k}} P(\alpha_{\mathbf{k}}) \left\langle \alpha_{\mathbf{k}} \middle| \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \middle| \alpha_{\mathbf{k}} \right\rangle \tag{473}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{474}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b^\dagger_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left(\left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right) b^\dagger_{\mathbf{k}'} + \left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{475}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b^\dagger_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right) b^\dagger_{\mathbf{k}'} + \left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{476}$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b^\dagger_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b^\dagger_{\mathbf{k}} + \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* b_{\mathbf{k}} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{477}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b^\dagger_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b^\dagger_{\mathbf{k}} + \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* b_{\mathbf{k}} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{478}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* b^\dagger_{\mathbf{k}} b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b_{\mathbf{k}} b^\dagger_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{479}$$

$$= \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} + \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{480}$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b^\dagger_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{481}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D\left(-\alpha_{\mathbf{k}}\right) b^\dagger_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right) D\left(-\alpha_{\mathbf{k}}\right) b_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{482}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| \left(b^\dagger_{\mathbf{k}} + \alpha^*_{\mathbf{k}}\right) \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{483}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}|^2 \, \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{484}$$

$$= N \tag{485}$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b^\dagger_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{486}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) \left(b^\dagger_{\mathbf{k}} + \alpha^*_{\mathbf{k}}\right) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{487}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \alpha_{\mathbf{k}} b^\dagger_{\mathbf{k}} + b_{\mathbf{k}} \alpha^*_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{488}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{489}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{490}$$

$$= N + 1 \tag{491}$$

$$\left\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \right\rangle_B = \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N+1) \tag{492}$$

$$= \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} + N \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right) \tag{493}$$

$$D\left(h'\right) D\left(h\right) = \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) D\left(h' + h\right) \tag{494}$$

$$\left\langle D\left(h'\right) D\left(h\right) \right\rangle_B = \mathrm{Tr}_B \left(\exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) D\left(h' + h\right) \rho_B^{\mathrm{Thermal}} \right) \tag{495}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \mathrm{Tr}_B \left(D\left(h' + h\right) \rho_B^{\mathrm{Thermal}} \right) \tag{496}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \frac{1}{\pi N} \int \mathrm{d}^2\alpha P\left(\alpha\right) \left\langle \alpha \middle| D\left(h' + h\right) \middle| \alpha \right\rangle \tag{497}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{498}$$

$$h' = h\exp\left(\mathrm{i}\omega\tau\right) \tag{499}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau)) D(h) \right\rangle_B = \exp\left(\frac{1}{2}\left(hh^*\exp(\mathrm{i}\omega\tau) - h^*h\exp(-\mathrm{i}\omega\tau)\right)\right) \exp\left(-\frac{|h + h\exp(\mathrm{i}\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{500}$$

$$\frac{1}{2}|h|^2 \left(\exp\left(\mathrm{i}\omega\tau\right) - \exp\left(-\mathrm{i}\omega\tau\right)\right) = \frac{1}{2}\left(hh^*\exp\left(\mathrm{i}\omega\tau\right) - h^*h\exp\left(-\mathrm{i}\omega\tau\right)\right) \tag{501}$$

$$= \frac{1}{2}|h|^2 \left(\cos\left(\omega\tau\right) + \mathrm{i}\sin\left(\omega\tau\right) - \cos\left(\omega\tau\right) + \mathrm{i}\sin\left(\omega\tau\right)\right) \tag{502}$$

$$= \frac{1}{2}|h|^2 \left(2\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{503}$$

$$= \mathrm{i}\,|h|^2 \sin\left(\omega\tau\right) \tag{504}$$

$$-\frac{|h + h\exp\left(\mathrm{i}\omega\tau\right)|^2}{2} = -|h|^2 \frac{|1 + \exp\left(\mathrm{i}\omega\tau\right)|^2}{2} \tag{505}$$

$$= -\left|h\right|^{2}\frac{\left|1+\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right|^{2}}{2} \tag{506}$$

$$= -\left|h\right|^{2}\frac{\left(1+\cos\left(\omega\tau\right)\right)^{2}+\sin^{2}\left(\omega\tau\right)}{2} \tag{507}$$

$$= -\left|h\right|^{2}\frac{\left(1+2\cos\left(\omega\tau\right)+\cos^{2}\left(\omega\tau\right)\right)+\sin^{2}\left(\omega\tau\right)}{2} \tag{508}$$

$$= -\left|h\right|^{2}\frac{2+2\cos\left(\omega\tau\right)}{2} \tag{509}$$

$$= -\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right) \tag{510}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\right\rangle_{B} = \exp\left(\mathrm{i}|h|^{2}\sin(\omega\tau)\right)\exp\left(-|h|^{2}(1+\cos(\omega\tau))\coth\left(\tfrac{\beta\omega}{2}\right)\right) \tag{511}$$

$$= \exp\left(\mathrm{i}\left|h\right|^{2}\sin\left(\omega\tau\right)-\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{512}$$

$$= \exp\left(-\left|h\right|^{2}\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-\left|h\right|^{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{513}$$

$$= \left\langle D\left(h\right)\right\rangle_{B}\exp\left(-\phi\left(\tau\right)\right) \tag{514}$$

$$\exp\left(-\phi\left(\tau\right)\right) = \exp\left(-\left|h\right|^{2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)\right) \tag{515}$$

$$\phi\left(\tau\right) = \left|h\right|^{2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{516}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_{B} = \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\exp\left(-\frac{\left|h+h'\right|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{517}$$

$$h' = v\exp\left(i\omega\tau\right) \tag{518}$$

$$\left\langle \widetilde{B_{1}^{+}B_{0}^{-}}(\tau)\widetilde{B_{1}^{+}B_{0}^{-}}(0)\right\rangle_{B} = \mathrm{Tr}_{B}\left(\widetilde{B_{1}^{+}B_{0}^{-}}\left(\tau\right)\widetilde{B_{1}^{+}B_{0}^{-}}\left(0\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{519}$$

$$= \mathrm{Tr}_{B}\left(\widetilde{B_{1}^{+}B_{0}^{-}}\left(\tau\right)\widetilde{B_{1}^{+}B_{0}^{-}}\left(0\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{520}$$

$$= \mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{521}$$

$$= \mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{522}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{523}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\prod_{\mathbf{k}}\left(\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{524}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{525}$$

$$\left\langle \widetilde{B_{0}^{+}B_{1}^{-}}(\tau)\widetilde{B_{0}^{+}B_{1}^{-}}(0)\right\rangle_{B} = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{526}$$

$$\left\langle \widetilde{B_{1}^{+}B_{0}^{-}}(\tau)\widetilde{B_{0}^{+}B_{1}^{-}}(0)\right\rangle_{B} = \mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{527}$$

$$= \mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{528}$$

$$= \mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{529}$$

$$= \mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{530}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_{B}\left(\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{531}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_{B}\left(\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega\tau+\pi)}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{532}$$

$$= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{533}$$

$$= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{534}$$

$$\left\langle \widetilde{B_{0}^{+}B_{1}^{-}}(\tau)\widetilde{B_{1}^{+}B_{0}^{-}}(0)\right\rangle_{B} = \mathrm{Tr}_{B}\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)}\right)\rho_{B}^{\mathrm{Thermal}}\right) \tag{535}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{536}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}_B\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{537}$$

$$=\prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{538}$$

$$= \left\langle |\widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B \tag{539}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B =\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{540}$$

$$\langle D(h)b\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|D(h)b|\alpha\rangle \tag{541}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\left|D(-\alpha)D(h)bD(\alpha)\right|\alpha\rangle \tag{542}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)bD(\alpha)\right|0\rangle \tag{543}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)D(\alpha)D(-\alpha)bD(\alpha)\right|0\rangle \tag{544}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)D(\alpha)(b+\alpha)\right|0\rangle \tag{545}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0\left|D(h)(b+\alpha)\right|0\rangle \tag{546}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)b|0\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \tag{547}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0\left|D(h)\alpha\right|0\rangle \tag{548}$$

$$= \frac{1}{\pi N}\int \alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{549}$$

$$= hN\langle D(h)\rangle_B \tag{550}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle\alpha\left|D(h)b^\dagger\right|\alpha\right\rangle \tag{551}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)b^\dagger D(\alpha)\right|0\right\rangle \tag{552}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)b^\dagger D(\alpha)\right|0\right\rangle \tag{553}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)D(-\alpha)b^\dagger D(\alpha)\right|0\right\rangle \tag{554}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{555}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\left\langle 0\left|D(h)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{556}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\left\langle 0\left|D(h)b^\dagger\right|0\right\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha^*|0\rangle \tag{557}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|1\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle \tag{558}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle -h|1\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle \tag{559}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right)\sum_n\frac{(-h^*)^n}{\sqrt{n!}}\langle n| \tag{560}$$

$$\langle -h|1\rangle = \exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*) \tag{561}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*)+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\exp\left(-\frac{|-h^*|^2}{2}\right) \tag{562}$$

$$= -h^*\langle D(h)\rangle_B(N+1) \tag{563}$$

$$\langle bD(h)\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\left|bD(h)\right|\alpha\rangle \tag{564}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)h+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\frac{|h|^2}{2}\right) \tag{565}$$

$$= h \langle D(h) \rangle_B (N+1) \tag{566}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| b^\dagger D(h) \right| \alpha \right\rangle \tag{567}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha \exp\left(-\tfrac{|h|^2}{2}\right) \tag{568}$$

$$= -h^* \langle D(h) \rangle_B N \tag{569}$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\tfrac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle_B \tag{570}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\tfrac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{571}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\tfrac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{572}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(\tfrac{1}{2} \left(\tfrac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp\left(-\tfrac{1}{2} \left| \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \tag{573}$$

$$= B_{10} \tag{574}$$

The correlation functions can be found readily as:

$$\widetilde{B_1^+ B_0^-}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \tag{575}$$

$$\widetilde{B_0^+ B_1^-}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \tag{576}$$

$$\widetilde{B_x}(0) = \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \tag{577}$$

$$\widetilde{B_y}(0) = \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2\mathrm{i}} \tag{578}$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega}{2} \right) \right) \right) \tag{579}$$

$$B_{iz} = \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* b_{\mathbf{k}} \right) \tag{580}$$

$$\left\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}} - v_{jk}\right)^* b_{\mathbf{k}} \right) \right\rangle_B \tag{581}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left(N_{\mathbf{k}} + 1\right) \tag{582}$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \right\rangle_B = \left\langle \frac{B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01}}{2} \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right\rangle_B \tag{583}$$

$$= \frac{1}{4} \left\langle \left(B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01} \right) \left(B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01} \right) \right\rangle_B \tag{584}$$

$$= \frac{1}{4} \Big\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- \tag{585}$$

$$- B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_{10} B_1^+ B_0^- - B_{10} B_0^+ B_1^- + B_{10} B_{10} + B_{10} B_{01} - B_{01} B_1^+ B_0^- - B_{01} B_0^+ B_1^- + B_{01} B_{10} + B_{01} B_{01} \Big\rangle \tag{586}$$

$$= \frac{1}{4} \Big\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \tag{587}$$

$$+ B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \Big\rangle \tag{588}$$

$$
\begin{aligned}
\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B &= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (589)\\
U &= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) && (590)\\
\phi(\tau) &= \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) && (591)\\
S &= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) && (592)\\
\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B &= U\exp(-\phi(\tau))\,S && (593)\\
\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B &= U^*\exp(-\phi(\tau))S && (594)\\
\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B &= \exp(\phi(\tau))\,S && (595)\\
\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B &= \left\langle \widetilde{B_1^+B_0^-}(\tau)\,\widetilde{B_0^+B_1^-}(0)\right\rangle_B && (596)\\
\left\langle \widetilde{B_1^+B_0^-}(\tau)\right\rangle_B &= \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) && (597)\\
&= U^{*1/2}S^{1/2} && (598)\\
\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B &= \frac{1}{4}\Big\langle B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_1^+B_0^- && (599)\\
&\quad + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01}\Big\rangle && (600)\\
\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B &= \frac{1}{4}\Big\langle B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- && (601)\\
&\quad - B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01}\Big\rangle && (602)\\
&= \frac{1}{4}\left(U^*\exp(-\phi(\tau))\,S + \exp(\phi(\tau))\,S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau))\,S + U\exp(-\phi(\tau))\,S - B_{10}^{*2} - |B_{10}|^2\right) && (603)\\
&= \frac{1}{4}\left(2U^{\Re}\exp(-\phi(\tau))\,S + 2\exp(\phi(\tau))\,S - 2\left(B_{10}^2\right)^{\Re} - 2|B_{10}|^2\right) && (604)\\
&= \frac{1}{4}\left(2U^{\Re}\exp(-\phi(\tau))\,S + 2\exp(\phi(\tau))\,S - 2\left(U^*\right)^{\Re}S - 2S\right) && (605)\\
&= \frac{S}{2}\left(U^{\Re}\exp(-\phi(\tau)) + \exp(\phi(\tau)) - \left(U^*\right)^{\Re} - 1\right) && (606)\\
\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B &= \left\langle \frac{B_0^+B_1^-(\tau) - B_1^+B_0^-(\tau) + B_{10} - B_{01}}{2\mathrm{i}}\,\frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B && (607)\\
&= -\frac{1}{4}\left\langle\left(B_0^+B_1^-(\tau) - B_1^+B_0^-(\tau) + B_{10} - B_{01}\right)\left(B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}\right)\right\rangle_B && (608)\\
&= -\frac{1}{4}\Big\langle B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_1^+B_0^-(\tau)B_0^+B_1^- + B_1^+B_0^-(\tau)B_1^+B_0^- && (609)\\
&\quad - B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01} + B_{10}B_0^+B_1^- - B_{10}B_1^+B_0^- + B_{10}B_{10} - B_{10}B_{01} - B_{01}B_0^+B_1^- + B_{01}B_1^+B_0^- - B_{01}B_{10} + B_{01}B_{01}\Big\rangle && (610)\\
&= -\frac{1}{4}\Big\langle B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} && (611)\\
&\quad - B_1^+B_0^-(\tau)B_0^+B_1^- + B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01}\Big\rangle && (612)\\
&= -\frac{1}{4}\Big\langle B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_{01}B_{10} - B_{01}B_{01} - B_1^+B_0^-(\tau)B_0^+B_1^- + B_1^+B_0^-(\tau)B_1^+B_0^- - B_{10}B_{10} + B_{10}B_{01}\Big\rangle && (613)\\
&= -\frac{1}{4}\left(U\exp(-\phi(\tau))\,S - \exp(\phi(\tau))\,S - \exp(\phi(\tau))\,S + U^*\exp(-\phi(\tau))\,S + 2S - 2\left(U^*\right)^{\Re}S\right) && (614)\\
&= -\frac{S}{4}\left(2U^{\Re}\exp(-\phi(\tau)) - 2\exp(\phi(\tau)) + 2 - 2U^{\Re}\right) && (615)\\
&= \frac{S}{2}\left(\exp(\phi(\tau)) - U^{\Re}\exp(-\phi(\tau)) - 1 + U^{\Re}\right) && (616)\\
\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B &= \left\langle \frac{B_1^+B_0^-(\tau) + B_0^+B_1^-(\tau) - B_{10} - B_{01}}{2}\,\frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B && (617)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_1^+B_0^-(\tau) + B_0^+B_1^-(\tau) - B_{10} - B_{01}\right)\left(B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}\right)\right\rangle_B && (618)\\
&= \frac{1}{4\mathrm{i}}\Big\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- && (619)\\
&\quad + B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_{10}B_0^+B_1^- + B_{10}B_1^+B_0^- - B_{10}B_{10} + B_{10}B_{01} - B_{01}B_0^+B_1^- + B_{01}B_1^+B_0^- - B_{01}B_{10} + B_{01}B_{01}\Big\rangle && (620)
\end{aligned}
$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10} \tag{621}$$

$$- B_0^+B_1^-(\tau)B_{01}\Big\rangle \tag{622}$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_{10}B_{10} - B_{10}B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_{01}B_{10} - B_{01}B_{01}\Big\rangle \tag{623}$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_{10}B_{10} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- - B_{01}B_{01}\Big\rangle \tag{624}$$

$$= \frac{1}{4\mathrm{i}}\left(\exp\left(\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^*S - US\right) \tag{625}$$

$$= \frac{1}{4\mathrm{i}}\left(-U^*\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S + U^*S - US\right) \tag{626}$$

$$= \frac{S}{4\mathrm{i}}\left(-U^*\exp\left(-\phi\left(\tau\right)\right) + U\exp\left(-\phi\left(\tau\right)\right) + U^* - U\right) \tag{627}$$

$$= \frac{S\left(U - U^*\right)}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{628}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{629}$$

$$= \frac{U^{\Im}S}{2}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{630}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B = \left\langle \tfrac{B_0^+B_1^-(\tau) - B_1^+B_0^-(\tau) + B_{10} - B_{01}}{2\mathrm{i}}\ \tfrac{B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}}{2}\right\rangle_B \tag{631}$$

$$= \frac{1}{4\mathrm{i}}\left\langle \left(B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}\right)\left(B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}\right)\right\rangle_B \tag{632}$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_0^+B_1^- \tag{633}$$

$$+ B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01} + B_{10}B_1^+B_0^- + B_{10}B_0^+B_1^- - B_{10}B_{10} - B_{10}B_{01} - B_{01}B_1^+B_0^- - B_{01}B_0^+B_1^- + B_{01}B_{10} + B_{01}B_{01}\Big\rangle \tag{634}$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_0^+B_1^- \tag{635}$$

$$+ B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01}\Big\rangle \tag{636}$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- - B_{01}B_{10} - B_{01}B_{01} - B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_0^+B_1^- + B_{10}B_{10} + B_{10}B_{01}\Big\rangle \tag{637}$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- - B_{01}B_{01} - B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_0^+B_1^- + B_{10}B_{10}\Big\rangle \tag{638}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + B_{10}^2 - B_{10}^{*2}\right) \tag{639}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi\left(\tau\right)\right)S - U^*\exp\left(-\phi\left(\tau\right)\right)S + U^*S - US\right) \tag{640}$$

$$= \frac{S\left(U - U^*\right)}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{641}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{642}$$

$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{10} \tag{643}$$

$$\left\langle B_1^+B_0^-(\tau)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\prod_{\mathbf{k}}\exp\left(\tfrac{1}{2}\left(\tfrac{v_{1\mathbf{k}}^*v_{0\mathbf{k}} - v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}\left\langle\prod_{\mathbf{k}}\left(D\left(\tfrac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle \tag{644}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} \tag{645}$$

$$\left\langle B_0^+B_1^-(\tau)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger\right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{01} \tag{646}$$

$$\left\langle B_0^+B_1^-(\tau)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01} \tag{647}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \tfrac{1}{2}\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{01} \tag{648}$$

$$+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\Big) \tag{649}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)B_{01} \tag{650}$$

$$+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\Big) \tag{651}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\Big(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\Big(\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*B_{10} + \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*B_{01}\Big) \tag{652}$$

$$+\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*N_{\mathbf{k}'}\Big(\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{10} + \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{01}\Big)\Big) \tag{653}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Bigg(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{01}\right)+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{10}-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)B_{01}\right)\Bigg) \tag{654}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Bigg(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(B_{10}-B_{01}\right)+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{10}-B_{01}\right)\Bigg) \tag{655}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}2\mathrm{i}B_{10}^{\Im}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{656}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{657}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) \tag{658}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{x}}(0)\right\rangle_{B}=\left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_{B} \tag{659}$$

$$=\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_{B} \tag{660}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}\right)\right\rangle_{B} \tag{661}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}\right)\right\rangle_{B} \tag{662}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Big\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-} \tag{663}$$

$$+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\Big\rangle \tag{664}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B}=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B} \tag{665}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{666}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^{*}v_{0\mathbf{k}'}-v_{1\mathbf{k}'}v_{0\mathbf{k}'}^{*}}{\omega_{\mathbf{k}'}^{2}}\right)\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{667}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{668}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{669}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{670}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}N_{\mathbf{k}'}\right)e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{671}$$

$$=-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{672}$$

$$=-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{673}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\right\rangle_{B}=-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{01}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{674}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B}=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\right\rangle_{B} \tag{675}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{676}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle b_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{677}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{678}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{679}$$

$$=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{680}$$

$$\left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- \right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B_{01} \tag{681}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\, \widetilde{B_x}(0) \right\rangle_B = \frac{1}{2} \sum_{\mathbf{k}'} \Bigg(- \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) B_{01} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{682}$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'} + 1) B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} (N_{\mathbf{k}'} + 1) B_{01} \Bigg) \tag{683}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{01} - B_{10}) + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right) (B_{10} - B_{01}) \right) \tag{684}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{01} - B_{10}) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right) (B_{01} - B_{10}) \right) \tag{685}$$

$$= \mathrm{i} \sum_{\mathbf{k}'} B_{10}^{\Im} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right) \right) \tag{686}$$

$$\left\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \right\rangle_B = \left\langle \left(\frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2\mathrm{i}} \right) \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{687}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01} \right) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{688}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) \right) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{689}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle B_0^+ B_1^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger - B_1^+ B_0^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + B_0^+ B_1^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} - B_1^+ B_0^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right\rangle \tag{690}$$

$$\left\langle B_0^+ B_1^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger \right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{01} \tag{691}$$

$$\left\langle B_0^+ B_1^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \tag{692}$$

$$\left\langle B_1^+ B_0^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger \right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} \tag{693}$$

$$\left\langle B_1^+ B_0^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} \tag{694}$$

$$\left\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \right\rangle_B = \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{01} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} \right. \tag{695}$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} \right) \tag{696}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{01} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} \right. \tag{697}$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} \right) \tag{698}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(N_{\mathbf{k}'} + 1\right) e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + B_{01}) + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (-B_{10} - B_{01}) \right) \tag{699}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(N_{\mathbf{k}'} + 1\right) e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + B_{01}) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (B_{10} + B_{01}) \right) \tag{700}$$

$$\left\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) \left(\frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2\mathrm{i}} \right) \right\rangle_B \tag{701}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) \left(B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01} \right) \right\rangle_B \tag{702}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) \left(B_0^+ B_1^- - B_1^+ B_0^- \right) \right\rangle_B \tag{703}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- \right\rangle \tag{704}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left\langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- \right\rangle - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left\langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- \right\rangle + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left\langle b_{\mathbf{k}'} B_0^+ B_1^- \right\rangle - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left\langle b_{\mathbf{k}'} B_1^+ B_0^- \right\rangle \right) \tag{705}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left\langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- \right\rangle - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left\langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- \right\rangle + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left\langle b_{\mathbf{k}'} B_0^+ B_1^- \right\rangle - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left\langle b_{\mathbf{k}'} B_1^+ B_0^- \right\rangle \right) \tag{706}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left\langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- \right\rangle - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left\langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- \right\rangle + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left\langle b_{\mathbf{k}'} B_0^+ B_1^- \right\rangle - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left\langle b_{\mathbf{k}'} B_1^+ B_0^- \right\rangle \right) \tag{707}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- \right\rangle_B = -\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \tag{708}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- \right\rangle_B = -\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} \tag{709}$$

$$\left\langle b_{\mathbf{k}'} B_1^+ B_0^- \right\rangle_B = \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \left(N_{\mathbf{k}'} + 1\right) B_{10} \tag{710}$$

$$\left\langle b_{\mathbf{k}'} B_0^+ B_1^- \right\rangle_B = \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \left(N_{\mathbf{k}'} + 1\right) B_{01} \tag{711}$$

$$\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(-\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* B_{01}N_{\mathbf{k}'}\Big)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(-\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* B_{10}N_{\mathbf{k}'}\Big) \tag{712}$$

$$+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{01}\Big)-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{10}\Big)\Big) \tag{713}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* B_{01}N_{\mathbf{k}'}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* B_{10}N_{\mathbf{k}'}\Big) \tag{714}$$

$$+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{01}\Big)\Big)-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{10}\Big)\Big) \tag{715}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big((-g_{i\mathbf{k}'}+v_{i\mathbf{k}'})\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* B_{01}N_{\mathbf{k}'}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^* B_{10}N_{\mathbf{k}'}\Big) \tag{716}$$

$$+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{01}\Big)\Big)-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{10}\Big)\Big) \tag{717}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*(B_{10}+)N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(B_{10}+)(N_{\mathbf{k}'}+1)\Big) \tag{718}$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}^{\Re}N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(N_{\mathbf{k}'}+1)\right) \tag{719}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}^{\Re}N_{\mathbf{k}'}\right) \tag{720}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}^{\Re}N_{\mathbf{k}'}\right) \tag{721}$$

$$=\mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* N_{\mathbf{k}'}\right) \tag{722}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}(\tau)\,\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}})e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{723}$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{724}$$

$$\phi(\tau)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{725}$$

$$\left\langle \widetilde{B_x}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(U^{\Re}\exp(-\phi(\tau))+\exp(\phi(\tau))-U^{\Re}-1\right) \tag{726}$$

$$\left\langle \widetilde{B_y}(\tau)\,\widetilde{B_y}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(\exp(\phi(\tau))-U^{\Re}\exp(-\phi(\tau))-1+U^{\Re}\right) \tag{727}$$

$$\left\langle \widetilde{B_x}(\tau)\,\widetilde{B_y}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp(-\phi(\tau))-1\right) \tag{728}$$

$$\left\langle \widetilde{B_y}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp(-\phi(\tau))-1\right) \tag{729}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*-(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{730}$$

$$\left\langle \widetilde{B_x}(\tau)\,\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}})\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{731}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\,\widetilde{B_y}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}})\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^* N_{\mathbf{k}}\right) \tag{732}$$

$$\left\langle \widetilde{B_y}(\tau)\,\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}})(N_{\mathbf{k}}+1)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{733}$$

The spectral density is defined in the usual way:

$$J_i(\omega)\equiv\sum_{\mathbf{k}}|g_{i\mathbf{k}}|^2\,\delta(\omega-\omega_{\mathbf{k}}) \tag{734}$$

$$v_{i\mathbf{k}}=g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}) \tag{735}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i=\int_0^{\infty}\frac{J_i(\omega)}{\omega}\mathrm{d}\omega \tag{736}$$

The integral version of the correlation functions are given by:

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B=\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}})e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{737}$$

$$=\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{738}$$

$$=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}\left(1-F_i\left(\omega_{\mathbf{k}}\right)\right)g_{j\mathbf{k}}^*\left(1-F_j\left(\omega_{\mathbf{k}}\right)\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+g_{i\mathbf{k}}^*\left(1-F_i\left(\omega_{\mathbf{k}}\right)\right)^*g_{j\mathbf{k}}\left(1-F_j\left(\omega_{\mathbf{k}}\right)\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{739}$$

$$\approx\int_0^\infty\left(\sqrt{J_i(\omega)J_j^*(\omega)}(1-F_i(\omega))\left(1-F_j^*(\omega)\right)e^{\mathrm{i}\omega\tau}N(\omega)+\sqrt{J_i^*(\omega)J_j(\omega)}\left(1-F_i^*(\omega)\right)\left(1-F_j(\omega)\right)e^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{740}$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{741}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right) \tag{742}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^*F_0^*\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}^*F_1^*\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\right) \tag{743}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^*F_0^*\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}^*F_1^*\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\right) \tag{744}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^*g_{1\mathbf{k}}F_0^*\left(\omega_{\mathbf{k}}\right)F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}g_{1\mathbf{k}}^*F_0\left(\omega_{\mathbf{k}}\right)F_1^*\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\right) \tag{745}$$

$$\approx\exp\left(\int_0^\infty\frac{\sqrt{J_0^*\left(\omega\right)J_1\left(\omega\right)}F_0^*\left(\omega\right)F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)J_1^*\left(\omega\right)}F_0\left(\omega\right)F_1^*\left(\omega\right)}{\omega^2}\mathrm{d}\omega\right) \tag{746}$$

$$\phi\left(\tau\right)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{747}$$

$$=\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{748}$$

$$\approx\int_0^\infty\left|\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega\right)}{\omega}\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega \tag{749}$$

$$B_{10}=\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{750}$$

$$=\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{751}$$

$$=\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^*F_1^*(\omega_{\mathbf{k}})-g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})g_{0\mathbf{k}}^*F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right)\right) \tag{752}$$

$$\approx\exp\left(-\frac{1}{2}\int_0^\infty\left|\frac{\sqrt{J_1(\omega)}F_1(\omega)-\sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega\right)\exp\left(\int_0^\infty\frac{1}{2}\left(\frac{\sqrt{J_0(\omega)J_1^*(\omega)}F_0(\omega)F_1^*(\omega)-\sqrt{J_0^*(\omega)J_1(\omega)}F_0^*(\omega)F_1(\omega)}{\omega^2}\right)\mathrm{d}\omega\right) \tag{753}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B=\frac{|B_{10}|^2}{2}\left(U^{\Re}\exp(-\phi(\tau))+\exp(\phi(\tau))-U^{\Re}-1\right) \tag{754}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B=\frac{|B_{10}|^2}{2}\left(\exp(\phi(\tau))-U^{\Re}\exp(-\phi(\tau))-1+U^{\Re}\right) \tag{755}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B=\frac{U^{\Im}|B_{10}|^2}{2}(\exp(-\phi(\tau))-1) \tag{756}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B=\frac{U^{\Im}|B_{10}|^2}{2}(\exp(-\phi(\tau))-1) \tag{757}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{758}$$

$$=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{759}$$

$$=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{760}$$

$$Q\left(\omega\right)=\sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega\right)\right)\left(\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega\right)}{\omega}\right)^* \tag{761}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B\approx\mathrm{i}B_{10}^{\Im}\int_0^\infty\left(Q\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{762}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{763}$$

$$=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^*\left(1-F_i^*(\omega_{\mathbf{k}})\right)\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}-g_{i\mathbf{k}}(1-F_i(\omega))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{764}$$

$$\approx \mathrm{i}B_{10}^{\Im}\int_0^{\infty}\left(Q^*\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{765}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{766}$$

$$= \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}g_{i\mathbf{k}}^*\left(1-F_i^*(\omega_{\mathbf{k}})\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{767}$$

$$\approx \mathrm{i}B_{10}^{\Re}\int_0^{\infty}\left(e^{-\mathrm{i}\omega\tau}Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right)-e^{\mathrm{i}\omega\tau}Q\left(\omega\right)N\left(\omega\right)\right)\mathrm{d}\omega \tag{768}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{769}$$

$$= \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^*\left(1-F_i^*(\omega_{\mathbf{k}})\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))(N_{\mathbf{k}}+1)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{770}$$

$$= \mathrm{i}B_{10}^{\Re}\int_0^{\infty}\left(e^{\mathrm{i}\omega\tau}Q^*(\omega)N(\omega)-e^{-\mathrm{i}\omega\tau}Q(\omega)(N(\omega)+1)\right)\mathrm{d}\omega \tag{771}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda+\mathrm{Det}\left(\overline{H_{\bar{S}}}\right)=0 \tag{772}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm}=\frac{\xi\pm\eta}{2}$ and they satisfy $H_S\left|\pm\right\rangle=\lambda_{\pm}\left|\pm\right\rangle$. Using this notation is possible to write $H_{\bar{S}}=\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_i}\left(\tau\right)=e^{\mathrm{i}\overline{H_{\bar{S}}}\tau}A_ie^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} \tag{773}$$

$$=\sum_{w}e^{-\mathrm{i}w\tau}A_i\left(w\right) \tag{774}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w\in\{0,\pm\eta\}$.
In order to use the equation (774) to descompose the equation (359) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U\left(t\right)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_0^t\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ like:

$$U\left(t\right)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_0^t\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right) \tag{775}$$

$$=\mathbb{I}+\sum_{n=1}^{\infty}\left(-\mathrm{i}\right)^n\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_nH\left(t_1\right)H\left(t_2\right)...H\left(t_n\right) \tag{776}$$

Here $0<t_1<t_2<...<t_{n-1}<t_n=t$ is a partition of the set $[0,t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E\left(t\right)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)\equiv\exp\left(-\mathrm{i}tH_E\left(t\right)\right)$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E\left(t\right)=H_E^{(0)}\left(t\right)+H_E^{(1)}\left(t\right)+H_E^{(2)}\left(t\right)+...$ so we can write:

$$U\left(t\right)=\exp\left(-\mathrm{i}t\left(H_E^{(0)}\left(t\right)+H_E^{(1)}\left(t\right)+H_E^{(2)}\left(t\right)+...\right)\right) \tag{777}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ and $U\left(t\right)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t' \tag{778}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right] \tag{779}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right) \tag{780}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}(t) = e^{\mathrm{i}H_E(t)t} A_i(t)\, e^{-\mathrm{i}H_E(t)t} \tag{781}$$

$$= \sum_{w(t)} e^{-\mathrm{i}w(t)t} A_i(w(t)) \tag{782}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the eigenvectors are time dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau, t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau, t) = U(t-\tau)\, U^\dagger(t)\, A_j(t)\, U(t)\, U^\dagger(t-\tau) \tag{783}$$

$$= e^{-\mathrm{i}(t-\tau)H_E(t-\tau)} e^{\mathrm{i}H_E(t)t} A_j(t)\, e^{-\mathrm{i}H_E(t)t} e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{784}$$

$$= e^{-\mathrm{i}(t-\tau)H_E(t-\tau)} \sum_{w(t)} e^{-\mathrm{i}w(t)t} A_j(w(t))\, e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{785}$$

$$= \sum_{w(t), w'(t-\tau)} e^{-\mathrm{i}w(t)t} e^{\mathrm{i}w'(t-\tau)} A'_j(w(t), w'(t-\tau)) \tag{786}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}(t-\tau)$ and $\overline{H_{\bar{S}}}(t)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (774) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V} = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta\in \mathrm{V}} \langle\alpha|A_i|\beta\rangle |\alpha\rangle\langle\beta| \tag{787}$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{\mathrm{i}\overline{H_{\bar{S}}}\tau} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{788}$$

$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau} e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} \tag{789}$$

$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{790}$$

$$= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-| \tag{791}$$

Calculating the transformation (774) directly using the previous relationship we find that:

$$\widetilde{A_i}(\tau) = \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in \mathrm{V}} \langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{792}$$

$$= \langle+|A_i|+\rangle|+\rangle\langle+| + e^{\mathrm{i}\eta\tau}\langle+|A_i|-\rangle|+\rangle\langle-| + e^{-\mathrm{i}\eta\tau}\langle-|A_i|+\rangle|-\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-| \tag{793}$$

Here $\eta = \lambda_+ - \lambda-$. Comparing the RHS of the equations (774) and the explicit expression for $\widetilde{A_i}(\tau)$ and we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$A_i(0) = \langle +|A_i|+\rangle |+\rangle\langle +| + \langle -|A_i|-\rangle |-\rangle\langle -| \tag{794}$$
$$A_i(w) = \langle +|A_i|-\rangle |+\rangle\langle -| \tag{795}$$
$$A_i(-w) = \langle -|A_i|+\rangle |-\rangle\langle +| \tag{796}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau) = \widetilde{A_i}^{\dagger}(\tau)$ and $\widetilde{B_i}(\tau) = \widetilde{B_i}^{\dagger}(\tau)$ we can use the equation (774) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\bar{S}}(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_{w,w'}\gamma_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho_S}(t) - \overline{\rho_S}(t)A_j^{\dagger}(w,w')\right] - \mathrm{i}\sum_{ij}\sum_{w}S_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho_S}(t) + \overline{\rho_S}(t)A_j^{\dagger}(w,w')\right] \tag{797}$$

where $A_j^{\dagger}(w) = A(-w)$ as expected from the equations (795) and (796). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}(w,w',t) = 2K_{ij}^{\Re}(w,w',t)$ and $S_{ij}(w,w',t) = K_{ij}^{\Im}(w,w',t)$, respectively, defined in terms of the response functions

$$K_{ij}(w,w',t) = \int_0^t C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}w\tau} e^{-\mathrm{i}t(w-w')}\mathrm{d}\tau \tag{798}$$
$$= K_{ijww'}(t) \tag{799}$$

If we extend the upper limit of integration to ∞ in the equation (798) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.
We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{\mathrm{d}\overline{\rho}_S}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)\right] - \sum_{ijww'} K_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho}_S(t) - \overline{\rho}_S(t)A_{jww'}^{\dagger}\right] \tag{800}$$

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_S}{\mathrm{d}t}e^{V} = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^{V}\right)}{\mathrm{d}t} \tag{801}$$
$$= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} \tag{802}$$
$$= -\mathrm{i}e^{-V}\left[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)\right]e^{V} - \sum_{ijww'} K_{ijww'}(t)\, e^{-V}\left[A_i, A_{jww'}\overline{\rho}_S(t) - \overline{\rho}_S(t)A_{jww'}^{\dagger}\right]e^{V} \tag{803}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V} = e^{-V}\overline{A\mathbb{I}B}e^{V} \tag{804}$$
$$= e^{-V}\overline{A}e^{V}e^{-V}\bar{B}e^{V} \tag{805}$$
$$= \left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\bar{B}e^{V}\right) \tag{806}$$
$$= AB \tag{807}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V} = e^{-V}\overline{(AB - BA)}e^{V} \tag{808}$$
$$= e^{-V}\overline{AB}e^{V} - e^{-V}\overline{BA}e^{V} \tag{809}$$
$$= AB - BA \tag{810}$$
$$= [A,B] \tag{811}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\,\mathrm{i}e^{-V}\left[\overline{H}_{\bar{S}}\left(t\right),\overline{\rho}_S\left(t\right)\right]e^{V} - \sum_{ijww'} K_{ijww'}\left(t\right)e^{-V}\left[A_i, A_{jww'}\overline{\rho}_S\left(t\right) - \overline{\rho}_S\left(t\right)A^{\dagger}_{jww'}\right]e^{V} \tag{812}$$

$$= -\,\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'} K_{ijww'}\left(t\right)\left[e^{-V}A_i e^{V}, e^{-V}A_{jww'}\overline{\rho}_S\left(t\right)e^{V} - e^{-V}\overline{\rho}_S\left(t\right)A^{\dagger}_{jww'}e^{V}\right] \tag{813}$$

$$= -\,\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'} K_{ijww'}\left(t\right)\left[e^{-V}A_i e^{V}, e^{-V}A_{jww'}e^{V}e^{-V}\overline{\rho}_S\left(t\right)e^{V} - e^{-V}\overline{\rho}_S\left(t\right)e^{V}e^{-V}A^{\dagger}_{jww'}e^{V}\right] \tag{814}$$

$$= -\,\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'} K_{ijww'}\left(t\right)\left[e^{-V}A_i e^{V}, e^{-V}A_{jww'}e^{V}\rho_S\left(t\right) - \rho_S\left(t\right)e^{-V}A^{\dagger}_{jww'}e^{V}\right] \tag{815}$$

$$= -\,\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \left(\sum_{ijww'} K_{ijww'}\left(t\right)\left(\left[e^{-V}A_i e^{V}, e^{-V}A_{jww'}e^{V}\rho_S\left(t\right)\right] - \left[e^{-V}A_i e^{V}, \rho_S\left(t\right)e^{-V}A^{\dagger}_{jww'}e^{V}\right]\right)\right) \tag{816}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (797) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (797) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1, 2, 3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} \tag{817}$$

After performing the transformation (25) on the Hamiltonian (817) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = \left(\delta + R\right)|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x \tag{818}$$

$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{819}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} \tag{820}$$

The Hamiltonian (818) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$\left(\delta + R\right)|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{821}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{822}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z \tag{823}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix

$\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$
\begin{aligned}
0 &= \lambda^2 - R\lambda + \frac{R^2-\epsilon^2-\Omega_r^2}{4} && (824)\\
\lambda_\pm &= \frac{R \pm \sqrt{\left(-R\right)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} && (825)\\
&= \frac{R \pm \sqrt{R^2 - \left(R^2-\epsilon^2-\Omega_r^2\right)}}{2} && (826)\\
&= \frac{R \pm \sqrt{\epsilon^2+\Omega_r^2}}{2} && (827)\\
\eta &= \sqrt{\epsilon^2+\Omega_r^2} && (828)\\
\lambda_\pm &= \frac{R\pm\eta}{2} && (829)
\end{aligned}
$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$
\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix} \tag{830}
$$

so the eigenvector $\left|+\right\rangle = a\left|0\right\rangle + b\left|1\right\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $\left|+\right\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\left|0\right\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\left|1\right\rangle$ with $\sin\left(\theta\right) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos\left(\theta\right) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $\left|+\right\rangle = \sin\left(\theta\right)\left|0\right\rangle + \cos\left(\theta\right)\left|1\right\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$
\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix} \tag{831}
$$

so the eigenvector $\left|+\right\rangle = a\left|0\right\rangle + b\left|1\right\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $\left|-\right\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\left|0\right\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\left|1\right\rangle$. The vector is written in reduced way like $\left|-\right\rangle = \cos\left(\theta\right)\left|0\right\rangle - \sin\left(\theta\right)\left|1\right\rangle$. Summarizing these results we can write:

$$
\begin{aligned}
\lambda_+ &= \frac{\epsilon+\eta}{2} && (832)\\
\lambda_- &= \frac{\epsilon-\eta}{2} && (833)\\
\left|+\right\rangle &= \sin\left(\theta\right)\left|0\right\rangle + \cos\left(\theta\right)\left|1\right\rangle && (834)\\
\left|-\right\rangle &= \cos\left(\theta\right)\left|0\right\rangle - \sin\left(\theta\right)\left|1\right\rangle && (835)\\
\sin\left(\theta\right) &= \frac{\Omega_r}{\sqrt{\left(\epsilon+\eta\right)^2+\Omega_r^2}} && (836)\\
\cos\left(\theta\right) &= \frac{\epsilon+\eta}{\sqrt{\left(\epsilon+\eta\right)^2+\Omega_r^2}} && (837)
\end{aligned}
$$

This result is plausible because in the paper [1] we have that:

$$
\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right) \tag{838}
$$

We can obtain the value of $\tan\left(\theta\right)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1} \tag{839}$$

So the value of $\tan(\theta)$ is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}+1} \tag{840}$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{841}$$

$$= \frac{\Omega_r}{\epsilon+\eta} \tag{842}$$

This proves our assertion.
Using this basis we can find the decomposition matrices using the equations (795)-(796) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ with$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\langle+|\sigma_x|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{843}$$

$$= 2\sin(\theta)\cos(\theta) \tag{844}$$

$$= \sin(2\theta) \tag{845}$$

$$\langle-|\sigma_x|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{846}$$

$$= -2\sin(\theta)\cos(\theta) \tag{847}$$

$$= -\sin(2\theta) \tag{848}$$

$$\langle-|\sigma_x|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{849}$$

$$= \cos^2(\theta) - \sin^2(\theta) \tag{850}$$

$$= \cos(2\theta) \tag{851}$$

$$\langle+|\sigma_y|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{852}$$

$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{853}$$

$$= 0 \tag{854}$$

$$\langle-|\sigma_y|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{855}$$

$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{856}$$

$$= 0 \tag{857}$$

$$\langle-|\sigma_y|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{858}$$

$$= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) \tag{859}$$

$$= \mathrm{i} \tag{860}$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{861}$$
$$= \cos(\theta)\cos(\theta) \tag{862}$$
$$= \cos^2(\theta) \tag{863}$$
$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{864}$$
$$= \sin(\theta)\sin(\theta) \tag{865}$$
$$= \sin^2(\theta) \tag{866}$$
$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{867}$$
$$= -\sin(\theta)\cos(\theta) \tag{868}$$
$$= -\sin(\theta)\cos(\theta) \tag{869}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \tag{870}$$
$$A_1(\eta) = \cos(2\theta)\,|-\rangle\langle+| \tag{871}$$
$$A_2(0) = 0 \tag{872}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+| \tag{873}$$
$$A_3(0) = \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-| \tag{874}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)\,|-\rangle\langle+| \tag{875}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (425) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau) \tag{876}$$

Using the notation of the master equation (797), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(723)-(733) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (425) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B\right) \tag{877}$$
$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{878}$$
$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B\right) \tag{879}$$
$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) \tag{880}$$
$$\Lambda'_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)\left(1-F(\omega)\right)^2 G_+(\tau) \tag{881}$$
$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)\left(1-F(\omega)\right)\mathrm{i}G_-(\tau) \tag{882}$$
$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau) \tag{883}$$
$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 \tag{884}$$

Finally taking the Hamiltonian (817) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that Det $\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, Tr $\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (340) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{885}$$

$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)} \tag{886}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (423) under a time-independent approach providing similar results.
Given that the Hamiltonian of this system is time-independent, then $U(t)U^{\dagger}(t-\tau) = U(\tau)$. From the equation (797) and using the fact that

$$\widetilde{A_j}(t-\tau,t) = U(\tau)A_jU(-\tau) \tag{887}$$

$$= \sum_w e^{\mathrm{i}w\tau}A_i(-w) \tag{888}$$

$$= \sum_w e^{-\mathrm{i}w\tau}A_i(w) \tag{889}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_w \gamma_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)A_j^{\dagger}(w)\right] \tag{890}$$

$$-\sum_{ij}\sum_w S_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)A_j^{\dagger}(w)\right] \tag{891}$$

where $A_j^{\dagger}(w) = A(-w)$, as we can see the equation (891) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2K_{ij}^{\Re}(w,t)$ and $S_{ij}(w,t) = K_{ij}^{\Im}(w,t)$, respectively, defined in terms of the response functions

$$K_{ij}(w,t) = \int_0^t \Lambda'_{ij}(\tau)e^{\mathrm{i}w\tau}\mathrm{d}\tau \tag{892}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (109) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{893}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (893) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that

$v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (227) and (246) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1) |1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t) \tag{894}$$

$$\overline{H_{\mathrm{I}}} = \frac{\Omega(t)}{2}(B_x\sigma_x + B_y\sigma_y) \tag{895}$$

In this case $R_1 = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}} \left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}} \left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (877)-(884) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^{\infty} \frac{J(\omega)}{\omega^2} G_+(\tau)\,\mathrm{d}\omega \tag{896}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (896) can be written as:

$$\phi(\tau) = \int_0^{\infty} \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega \tag{897}$$

Writing the interaction Hamiltonian (895) in the similar way to the equation (246) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (227) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t) \tag{898}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{899}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{900}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{901}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) \tag{902}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(423) is:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{903}$$

$$\left. + C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right) \tag{904}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (899) and (902) on the equation (904) we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{905}$$

$$\left. + \left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right) \tag{906}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^{\dagger} = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (425) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*} b_{\mathbf{k}}\right) \tag{907}$$

Using the equation (797), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)), \rho_S(t)\right] - \frac{1}{2}\sum_{w} \gamma_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) - \rho_S(t) A_3^{\dagger}(w)\right] \tag{908}$$

$$- \sum_{w} S_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) + \rho_S(t) A_3^{\dagger}(w)\right]\Bigg) \tag{909}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^{\infty} \mathrm{d}\omega J(\omega) G_{+}(\tau) \tag{910}$$

$$\Lambda_{33}(-\tau) = \int_0^{\infty} \mathrm{d}\omega J(\omega) G_{+}(-\tau) \tag{911}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (909) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t))\right] - \sum_{w}\left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^{*}(w,t)\left[\rho_S(t) A_3(w), A_3\right]\right) \tag{912}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_{+} = \frac{\widetilde{\Omega}}{2} \tag{913}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \tag{914}$$

$$\lambda_{-} = -\frac{\widetilde{\Omega}}{2} \tag{915}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{916}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{917}$$

$$= \frac{1}{2} \tag{918}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{919}$$

$$= \frac{1}{2} \tag{920}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{921}$$

$$= -\frac{1}{2} \tag{922}$$

The decomposition matrices are

$$A_3\left(0\right) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{923}$$

$$= \frac{\mathbb{I}}{2} \tag{924}$$

$$A_3\left(\eta\right) = -\frac{1}{2}|-\rangle\langle+| \tag{925}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right) \tag{926}$$

$$A_3\left(-\eta\right) = -\frac{1}{2}|+\rangle\langle-| \tag{927}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right) \tag{928}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{929}$$

$$- K_{33}^{*}\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^{*}\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] \tag{930}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty\int_0^\infty J\left(\omega\right)G_+\left(\tau\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{931}$$

$$= \int_0^\infty\int_0^\infty J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{932}$$

$$= \int_0^\infty\int_0^\infty J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{933}$$

$$= \int_0^\infty\int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{934}$$

$$= \int_0^\infty J\left(\omega\right)\left(n\left(\omega\right)+1\right)\pi\delta\left(\widetilde{\Omega}-\omega\right)\mathrm{d}\omega \tag{935}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right) \tag{936}$$

$$K_{33}\left(-\widetilde{\Omega}\right)=\int_0^\infty\int_0^\infty J(\omega)\,G_+(\tau)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{937}$$

$$=\int_0^\infty\int_0^\infty J(\omega)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega}+n(\omega)\,e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{938}$$

$$=\int_0^\infty\int_0^\infty J(\omega)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}n(\omega)\,\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{939}$$

$$=\int_0^\infty\int_0^\infty J(\omega)\,n(\omega)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau+\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{940}$$

$$=\int_0^\infty J(\omega)\,n(\omega)\,\pi\delta\left(-\widetilde{\Omega}+\omega\right)\mathrm{d}\omega \tag{941}$$

$$=\pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right) \tag{942}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s}=\pi\delta(\varepsilon)\pm\mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega=0$. Replacing in the equation (929) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}=-\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x,\rho_S(t))\right]-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\,\rho_S(t)\right]+n\left(\widetilde{\Omega}\right)\left[\sigma_z,(\sigma_z-\mathrm{i}\sigma_y)\,\rho_S(t)\right]\right) \tag{943}$$

$$-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S(t)\,(\sigma_z+\mathrm{i}\sigma_y)\,,\sigma_z\right]+n\left(\widetilde{\Omega}\right)\left[\rho_S(t)\,(\sigma_z-\mathrm{i}\sigma_y)\,,\sigma_z\right]\right) \tag{944}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}=-\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x,\rho_S(t)\right]-\frac{\pi}{8}J(\Omega(t))\left((n(\Omega(t))+1)[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\rho_S(t)]+n(\Omega(t))[\sigma_z,(\sigma_z-\mathrm{i}\sigma_y)\,\rho_S(t)]\right) \tag{945}$$

$$-\frac{\pi}{8}J(\Omega(t))\left((n(\Omega(t))+1)\left[\rho_S(t)\,(\sigma_z+\mathrm{i}\sigma_y)\,,\sigma_z\right]+n(\Omega(t))\left[\rho_S(t)\,(\sigma_z-\mathrm{i}\sigma_y)\,,\sigma_z\right]\right) \tag{946}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t)=H_S(t)+H_I+H_B, \tag{947}$$

$$H_S(t)=\sum_n\varepsilon_n(t)\,|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|, \tag{948}$$

$$H_I=\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger+g_{nu\mathbf{k}}^*b_{u\mathbf{k}}\right), \tag{949}$$

$$H_B=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \tag{950}$$

A. Variational Transformation

We consider the following operator:

$$V=\sum_{nu\mathbf{k}}|n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{951}$$

At first let's obtain $e^{\pm V}$ under the transformation (951), consider $\hat{\varphi}_n=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)$, so the equation (951) can be written as $V=\sum_n|n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$\begin{aligned}
e^{\pm V} &= e^{\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n} && (952)\\
&= \mathbb{I}\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\left(\sum_n|n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!}+\ldots && (953)\\
&= \mathbb{I}\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\sum_n|n\rangle\langle n|\hat{\varphi}_n^2}{2!}+\ldots && (954)\\
&= \sum_n|n\rangle\langle n|\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\sum_n|n\rangle\langle n|\hat{\varphi}_n^2}{2!}+\ldots && (955)\\
&= \sum_n|n\rangle\langle n|\left(\mathbb{I}\pm\hat{\varphi}_n+\frac{\hat{\varphi}_n^2}{2!}+\ldots\right) && (956)\\
&= \sum_n|n\rangle\langle n|e^{\pm\hat{\varphi}_n} && (957)
\end{aligned}$$

Given that $\left[f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}},f_{nu'\mathbf{k}'}b_{u'\mathbf{k}'}^{\dagger}-f_{nu'\mathbf{k}'}^{*}b_{u'\mathbf{k}'}\right]=0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u,u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right)=e^{\pm\left(\alpha_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-\alpha_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}}=\frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$\begin{aligned}
e^{\pm\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} &= \prod_u e^{\pm\sum_{\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} && (958)\\
&= \prod_u\left(\prod_{\mathbf{k}}e^{\pm\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}\right) && (959)\\
&= \prod_u\left(\prod_{\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) && (960)\\
&= \prod_{u\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right) && (961)\\
&= \prod_u B_{nu\pm} && (962)\\
B_{nu\pm} &\equiv \prod_{\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right) && (963)
\end{aligned}$$

As we can see $e^{-V}=\sum_n|n\rangle\langle n|\prod_u B_{nu-}$ and $e^{V}=\sum_n|n\rangle\langle n|\prod_u B_{nu+}$ this implies that $e^{-V}e^{V}=\mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V}=\left(\sum_n|n\rangle\langle n|\prod_u B_{nu+}\right)A\left(\sum_n|n\rangle\langle n|\prod_u B_{nu-}\right) \tag{964}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (947):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{965}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{966}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{967}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{968}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{969}$$

$$= |0\rangle\langle 0|. \tag{970}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{971}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{972}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{973}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, \tag{974}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{975}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{976}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{977}$$

$$\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{978}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{979}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right) \left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), \tag{980}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, \tag{981}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots \tag{982}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) \tag{983}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots \tag{984}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) \tag{985}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots \tag{986}$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{987}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{988}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{989}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) \tag{990}$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{991}$$

The transformed Hamiltonians of the equations (948) to (950) written in terms of (965) to (989) are:

$$\overline{H_S\left(t\right)} = \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} \tag{992}$$

$$= \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} \tag{993}$$

$$= \sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m| \prod_u \left(B_{mu+}B_{nu-}\right) \tag{994}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{995}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right) + ...\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{996}$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \prod_u B_{0u-} + \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \prod_u B_{1u-} + \ldots \tag{997}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}\prod_u B_{0u+}b_{u\mathbf{k}}^{\dagger} \prod_u B_{0u-} + g_{0u\mathbf{k}}^{*}\prod_u B_{0u+}b_{u\mathbf{k}}\prod_u B_{0u-}\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}}\prod_u B_{1u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{1u-} + g_{1u\mathbf{k}}^{*}\prod_u B_{1u+}b_{u\mathbf{k}}\prod_u B_{1u-}\right) + \ldots \tag{998}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger} - \frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right) + g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger} - \frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right) + g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{999}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger} - \frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right) + g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1000}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1001}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1002}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1003}$$

$$+ \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1004}$$

Let's define the following functions:

$$R_n\left(t\right) = \sum_{u\mathbf{k}} \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1005}$$

$$B_{z,n}\left(t\right) = \sum_{u\mathbf{k}} \left(\left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right)b_{u\mathbf{k}}^{\dagger} + \left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right)^{*}b_{u\mathbf{k}}\right) \tag{1006}$$

Using the previous functions we have that (1003) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} + \sum_n R_n(t)|n\rangle\langle n| + \sum_n B_{z,n}(t)|n\rangle\langle n| \tag{1007}$$

$$\tag{1008}$$

Now in order to separate the elements of the hamiltonian (1008) let's follow the references of the equations (227) and (246) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle\prod_{u}\left(B_{mu+}B_{nu-}\right)\right\rangle_{\overline{H_0}}=\left\langle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*}+\alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1009}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*}+\alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right)\left\langle\prod_{u\mathbf{k}}D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1010}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1011}$$

$$\equiv B_{nm} \tag{1012}$$

$$\left\langle\prod_{u}\left(B_{nu+}B_{mu-}\right)\right\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{nu\mathbf{k}}^{*}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v_{mu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1013}$$

$$=B_{nm}^{*} \tag{1014}$$

Following the reference [4] we define:

$$J_{nm}=\prod_{u}\left(B_{mu+}B_{nu-}\right)-B_{nm} \tag{1015}$$

As we can see:

$$J_{nm}^{\dagger}=\left(\prod_{u}\left(B_{mu+}B_{nu-}\right)-B_{nm}\right)^{\dagger} \tag{1016}$$

$$=\prod_{u}\left(B_{nu+}B_{mu-}\right)-B_{nm}^{*} \tag{1017}$$

$$=\prod_{u}\left(B_{nu+}B_{mu-}\right)-B_{mn} \tag{1018}$$

$$=J_{mn} \tag{1019}$$

We can separate the Hamiltonian (1008) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}\left(t\right)}=\sum_{n}\left(\varepsilon_{n}\left(t\right)+R_{n}\right)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} \tag{1020}$$

$$\overline{H_{\bar{I}}}=\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|J_{nm}+\sum_{n}B_{z,n}\left(t\right)|n\rangle\langle n|, \tag{1021}$$

$$\overline{H_{\bar{B}}}=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} \tag{1022}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A\leq A_{\mathrm{B}}\equiv-\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right)+\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}+O\left(\left\langle\overline{H_{\bar{I}}^{2}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_{B}}}\right). \tag{1023}$$

Taking the equations (248)-(256) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right)=C\left(R_{0},R_{1},...,R_{d-1},B_{01},...,B_{0(d-1)},...,B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}$. So our approach will be based on the derivation respect to $v_{nu\mathbf{k}}^{\Re}$ and $v_{nu\mathbf{k}}^{\Im}$. The Hamiltonian $\overline{H_{\bar{S}}\left(t\right)}$ can be written like:

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1024}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1025}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\frac{g_{nu\mathbf{k}}v_{nu\mathbf{k}}^*+g_{nu\mathbf{k}}^*v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1026}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1027}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^*\right)v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(g_{nu\mathbf{k}}^*-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1028}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1029}$$

$$v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*=\left(v_{mu\mathbf{k}}^{\Re}-\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right)\left(v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right)-\left(v_{mu\mathbf{k}}^{\Re}+\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right)\left(v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right) \tag{1030}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-\mathrm{i}v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Im}\right) \tag{1031}$$

$$-\left(v_{mu\mathbf{k}}^{\Re}v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}+\mathrm{i}v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Im}\right) \tag{1032}$$

$$=2\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}\right) \tag{1033}$$

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^*\right)v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(g_{nu\mathbf{k}}^*-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1034}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1035}$$

$$\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2=\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)^* \tag{1036}$$

$$=\left|v_{mu\mathbf{k}}\right|^2+\left|v_{nu\mathbf{k}}\right|^2-\left(v_{nu\mathbf{k}}v_{mu\mathbf{k}}^*+v_{nu\mathbf{k}}^*v_{mu\mathbf{k}}\right) \tag{1037}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}\right)^2+\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-\left(v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right)\left(v_{mu\mathbf{k}}^{\Re}-\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right) \tag{1038}$$

$$-\left(v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right)\left(v_{mu\mathbf{k}}^{\Re}+\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right) \tag{1039}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}\right)^2+\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-2\left(v_{nu\mathbf{k}}^{\Re}v_{mu\mathbf{k}}^{\Re}+v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Im}\right) \tag{1040}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}-v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}-v_{nu\mathbf{k}}^{\Im}\right)^2 \tag{1041}$$

$$R_n\left(t\right)=\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1042}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^*\right)v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(g_{nu\mathbf{k}}^*-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1043}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-2g_{nu\mathbf{k}}^{\Re}v_{nu\mathbf{k}}^{\Re}-2g_{nu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}\right) \tag{1044}$$

$$B_{mn}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1045}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{mu\mathbf{k}}^{\Re}-v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}-v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1046}$$

Then we can obtain using the chain rule that:

$$\begin{aligned}
\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1047)\\
&= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1048)\\
&= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1049)\\
\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1050)\\
&= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1051)\\
&= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1052)
\end{aligned}$$

Given that:

$$\begin{aligned}
\ln B_{mn} &= \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) && (1053)\\
&= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1054)\\
&= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1055)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1056)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1057)\\
\frac{\partial \ln B_{mn}}{\partial a} &= \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} && (1058)\\
\frac{\partial B_{mn}}{\partial a} &= B_{mn} \frac{\partial \ln B_{mn}}{\partial a} && (1059)\\
\frac{\partial B_{mn}}{\partial a} &= \frac{\partial \left(B_{nm}\right)^\dagger}{\partial a} && (1060)
\end{aligned}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1061}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1062}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1063}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1064}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1065}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1066}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1067}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1068}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1069}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1070}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1071}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1072}$$

Introducing this derivates in the equation (1047) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1073}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1074}$$

$$= 0 \tag{1075}$$

We can obtain the variational parameters:

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1076}$$

$$=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1077}$$

$$v^{\Re}_{nu\mathbf{k}}=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1078}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1079}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}}=\frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}}-g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1080}$$

$$\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1081}$$

$$=0 \tag{1082}$$

Rearranging we obtain

$$
\begin{aligned}
&-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) && (1083)\\
&=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) && (1084)\\
v^{\Im}_{nu\mathbf{k}}&=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} && (1085)\\
&=\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1086)\\
v_{nu\mathbf{k}}&=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}} && (1087)\\
&=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1088)\\
&\mathrm{i}\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1089)\\
&=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1090)\\
&-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1091)\\
&-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1092)\\
&=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1093)\\
&-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1094)\\
&=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1095)
\end{aligned}
$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$
\begin{aligned}
e^{V}\rho(0)\,e^{-V}&=\left(\sum_n|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_n|n\rangle\langle n|B_{n+}\right) && (1096)\\
0&=\left(B^{+}_{0}|0\rangle\langle 0|B^{-}_{0}\right)\otimes\rho_B && (1097)\\
0&=\rho(0) && (1098)
\end{aligned}
$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1099}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1100}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\,,\ \text{where} \tag{1101}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1102}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1103}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1104}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1105}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1106}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1107}$$
$$= B_{n-}^{\dagger}B_{m+}^{\dagger} - B_n B_m \tag{1108}$$
$$= B_{n+}B_{m-} - B_n B_m \tag{1109}$$
$$= B_{nm} \tag{1110}$$

So we can say that the set of matrices (1103) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1103) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t)\,|n\rangle\langle n|, \tag{1111}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)\,|n\rangle\langle m| B_{nm} + V_{mn}(t)\,|m\rangle\langle n| B_{mn}\right) \tag{1112}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t)\, B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1113}$$
$$+\Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t)\, B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\Bigg) \tag{1114}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1115}$$
$$+\mathrm{i}\Im\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\Bigg) \tag{1116}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1117}$$
$$+\Im\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,x}\Big) \tag{1118}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1119}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1-\delta_{mn}) \tag{1120}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1-\delta_{mn}) \tag{1121}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1122}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1123}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1124}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1125}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1126}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1127}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1128}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1129}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1130}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1131}$$
$$C_{3,nm}(t) = 1 \tag{1132}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1133}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1134}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J, p\in P} C_{jp}(t)\left(A_{jp}\otimes B_{jp}(t)\right) \tag{1135}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1119).

We write the interaction Hamiltonian transformed under (1099) as:

$$\widetilde{H_I}(t) = \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1136}$$
$$\widetilde{A_{jp}}(t) = U^{\dagger}(t)\,A_{jp}U(t) \tag{1137}$$
$$\widetilde{B_{jp}}(t) = \mathrm{e}^{\mathrm{i}H_Bt}B_{jp}(t)\,(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_Bt} \tag{1138}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1139}$$

Replacing the equation (1136) in (1139) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B \left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] \mathrm{d}s \tag{1140}$$

$$= -\int_0^t \mathrm{Tr}_B \left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right), \widetilde{\rho_S}(t)\rho_B\right]\right] \mathrm{d}s \tag{1141}$$

$$= -\int_0^t \mathrm{Tr}_B \left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1142}$$

$$\left.-\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right] \mathrm{d}s \tag{1143}$$

$$= -\int_0^t \mathrm{Tr}_B \left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1144}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1145}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\rho_S}(t)\rho_B \sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1146}$$

$$\left.+\widetilde{\rho_S}(t)\rho_B \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right) \mathrm{d}s \tag{1147}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1148}$$

$$= \left\langle \widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1149}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1147) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1150}$$

$$= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1151}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1152}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1153}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1154}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1155}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1156}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) \tag{1157}$$

$$= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1158}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1159}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1160}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1161}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1162}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1163}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1164}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1140)and (1147) and using the equations (1150)-(1164) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}\left(t\right)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right) - \Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\right)\right. \tag{1165}$$

$$\left.+C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right) - \Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1166}$$

$$= -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[\widetilde{A_{jp}}\left(t\right),\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right),\widetilde{A_{jp}}\left(t\right)\right]\right)\right) \tag{1167}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right),A_{j'p'}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}\left(t\right)A_{j'p'}\left(t-\tau,t\right),A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}\left(t\right)\right] \tag{1168}$$

For this case we used that $A_{jp}\left(t-\tau,t\right) = U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1169}$$
$$H_S(t) = \sum_{n=0} \varepsilon_n(t)\, |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\, |n\rangle\langle m|, \tag{1170}$$
$$H_I = \left(\sum_{n=0} \mu_n(t)\, |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1171}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1172}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t)\, |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1173}$$

At first let's obtain e^V under the transformation (1173), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{1174}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + ... \tag{1175}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + ... \tag{1176}$$
$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + ...\right) \tag{1177}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1178}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \tag{1179}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1180}$$
$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1181}$$
$$= B_{\pm} \tag{1182}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-\right) \tag{1183}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1169):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right), \tag{1184}$$

$$= |0\rangle\langle 0|, \tag{1185}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right), \tag{1186}$$

$$= |m\rangle\langle m|B_+|m\rangle\langle n|n\rangle\langle n|B_-, \tag{1187}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1188}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right), \tag{1189}$$

$$= |0\rangle\langle m|B_-\ m \neq 0, \tag{1190}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{1191}$$

$$= |0\rangle\langle m|B_+\ m \neq 0, \tag{1192}$$

$$\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{1193}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B_+b_{\mathbf{k}}^\dagger b_{\mathbf{k}}B_- \tag{1194}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_+b_{\mathbf{k}}^\dagger B_-\right)\left(B_+b_{\mathbf{k}}B_-\right) \tag{1195}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1196}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1197}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right) \tag{1198}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1199}$$

The transformed Hamiltonians of the equations (1170) to (1172) written in terms of (1184) to (1199) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1200}$$

$$= \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1201}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \overline{\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n|} \tag{1202}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)\overline{|0\rangle\langle n|} + V_{n0}(t)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0}V_{mn}(t)\overline{|m\rangle\langle n|} \tag{1203}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)B_-|0\rangle\langle n| + V_{n0}(t)B_+|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1204}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1205}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)\left(\left(\sum_{n=0}\mu_n(t)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{1206}$$

$$= \left(\mu_0(t)|0\rangle\langle 0| + \sum_{n=1}\mu_n(t)|n\rangle\langle n|B_+\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{1207}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}B_+\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)B_- \tag{1208}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1209}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1210}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1211}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1212}$$

$$+\sum_{n=0}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1213}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1214}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1215}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1216}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1217}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\right) \tag{1218}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1219}$$

Using the previous functions we have that (1216) can be re-written in the following way:

$$\overline{H} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1220}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}R_n|n\rangle\langle n| + \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1221}$$

Now in order to separate the elements of the hamiltonian (1221) let's follow the references of the equations (246) and (227) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| + \sum_{n=1}R_n|n\rangle\langle n| \tag{1222}$$

$$\overline{H_I} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|(B_- - B) + V_{n0}(t)|n\rangle\langle 0|(B_+ - B)\right), \tag{1223}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{1224}$$

Here B is given by:

$$\begin{aligned} B &= \langle B_+\rangle \\ &= \langle B_-\rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1225}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1226}$$

$$B_x = \frac{B_+ + B_- - 2B}{2} \tag{1227}$$

$$B_y = \frac{B_- - B_+}{2\mathrm{i}} \tag{1228}$$

Using this set of hermitian operators to write the Hamiltonians (1170)-(1172)

$$\overline{H_S\left(t\right)} = \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1229}$$

$$= \varepsilon_0\left(t\right)|0\rangle\langle 0| + B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{0<m<n}\left(V_{mn}\left(t\right)|m\rangle\langle n| + V_{nm}\left(t\right)|n\rangle\langle m|\right) \tag{1230}$$

$$+\sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1231}$$

$$= \sum_{0<m<n}\left(\left(\Re\left(V_{mn}\left(t\right)\right)+\mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)|m\rangle\langle n| + \left(\Re\left(V_{mn}\left(t\right)\right)-\mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)|n\rangle\langle m|\right) + \varepsilon_0\left(t\right)|0\rangle\langle 0| \tag{1232}$$

$$+B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1233}$$

$$= \sum_{0<m<n}\left(\left(\Re\left(V_{nm}\left(t\right)\right)+\mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2} + \left(\Re\left(V_{nm}\left(t\right)\right)-\mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1234}$$

$$+B\sum_{n=1}\left(V_{0n}\left(t\right)\frac{\sigma_{0n,x}-\mathrm{i}\sigma_{0n,y}}{2} + V_{n0}\left(t\right)\frac{\sigma_{0n,x}+\mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1235}$$

$$= \sum_{0<m<n}\left(\Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,x} + \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}\right) + B\sum_{n=1}\left(\Re\left(V_{0n}\left(t\right)\right)\sigma_{0n,x} + \Im\left(V_{mn}\left(t\right)\right)\sigma_{0n,y}\right) \tag{1236}$$

$$+\varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1237}$$

$$\overline{H_I\left(t\right)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|\left(B_- - B\right) + V_{n0}\left(t\right)|n\rangle\langle 0|\left(B_+ - B\right)\right) \tag{1238}$$

$$= \sum_{n=1}\left(\left(\Re\left(V_{0n}\left(t\right)\right)+\mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\left(B_- - B\right)\frac{\sigma_{0n,x}-\mathrm{i}\sigma_{0n,y}}{2} + \left(\Re\left(V_{0n}\left(t\right)\right)-\mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\left(B_+ - B\right)\frac{\sigma_{0n,x}+\mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1239}$$

$$+\sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1240}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B_- - B\right)\left(\Re\left(V_{0n}\left(t\right)\right)+\mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right) + \left(B_+ - B\right)\left(\Re\left(V_{0n}\left(t\right)\right)-\mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\right)\right. \tag{1241}$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B_+ - B\right)\left(\Re\left(V_{0n}\left(t\right)\right)-\mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right) - \left(B_- - B\right)\left(\Re\left(V_{0n}\left(t\right)\right)+\mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\right)\right) \tag{1242}$$

$$+\mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1243}$$

$$= \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B_+ + B_- - 2B\right)\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\left(B_- - B - B_+ + B\right)\Im\left(V_{0n}\left(t\right)\right)\right. \tag{1244}$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B_+ - B - B_- + B\right)\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\left(B - B_- + B - B_+\right)\Im\left(V_{0n}\left(t\right)\right)\right)\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1245}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\sigma_{0n,x}\left(B_x\Re\left(V_{0n}\left(t\right)\right) - B_y\Im\left(V_{0n}\left(t\right)\right)\right)\right. \tag{1246}$$

$$\left.+\sigma_{0n,y}\left(B_y\Re\left(V_{0n}\left(t\right)\right) + B_x\Im\left(V_{0n}\left(t\right)\right)\right)\right) \tag{1247}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right) + \langle\overline{H_I}\rangle_{\overline{H_S+H_B}} + O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1248}$$

Taking the equations (248)-(256) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (256) and using the chain rule we obtain that:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1249}$$

$$= 0 \tag{1250}$$

Let's recall the equations (1217) and (1219), we can write them in terms of the variational parameters

$$B = \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1251}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1252}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1253}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1254}$$

Introducing this derivates in the equation (1249) give us:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1255}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1256}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1257}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1258}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1259}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+\right) (|0\rangle\langle 0| \otimes \rho_B) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-\right) \tag{1260}$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \tag{1261}$$

$$0 = \rho(0) \tag{1262}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t) O U(t) \tag{1263}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1264}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \tag{1265}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{1266}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle 0| + \sum_{n=1} \left(\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right. \tag{1267}$$

$$\left.+\Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x}\right) \tag{1268}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1269}$$

$$B_{z,n} = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \text{ if} n \neq 0 \tag{1270}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1271}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1272}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1273}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1274}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1275}$$
$$B_{1n}(t) = B_x \tag{1276}$$
$$B_{2n}(t) = B_y \tag{1277}$$
$$B_{3n}(t) = B_{z,n} \tag{1278}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1279}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1280}$$
$$C_{10}(t) = 0 \tag{1281}$$
$$C_{20}(t) = 0 \tag{1282}$$
$$C_{40}(t) = 0 \tag{1283}$$
$$C_{50}(t) = 0 \tag{1284}$$
$$C_{30}(t) = 1 \tag{1285}$$
$$C_{1n}(t) = \Re(V_{0n}(t)) \tag{1286}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1287}$$
$$C_{3n}(t) = 1 \tag{1288}$$
$$C_{4n}(t) = \Im(V_{0n}(t)) \tag{1289}$$
$$C_{5n}(t) = -\Im(V_{0n}(t)) \tag{1290}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn} \otimes B_{jn}(t)\right) \tag{1291}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1263) as:

$$\widetilde{H_I}(t) = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1292}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\, A_i U(t) \tag{1293}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{1294}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t))$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1295}$$

Replacing the equation (1292)in (1295)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1296}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1297}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1298}$$

$$\left. -\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1299}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1300}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1301}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1302}$$

$$\left. +\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1303}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle\widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1304}$$

$$= \left\langle\widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1305}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1303) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right)=\left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1306}$$
$$=\left\langle\widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1307}$$
$$=\Lambda_{jnj'n'}(\tau) \tag{1308}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right)=\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1309}$$
$$=\left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1310}$$
$$=\left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1311}$$
$$=\Lambda_{j'n'jn}(-\tau) \tag{1312}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right)=\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1313}$$
$$=\left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1314}$$
$$=\left\langle\widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1315}$$
$$=\Lambda_{jnj'n'}(\tau) \tag{1316}$$
$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right)=\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1317}$$
$$=\left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1318}$$
$$=\left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1319}$$
$$=\Lambda_{j'n'jn}(-\tau) \tag{1320}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1296)and (1303) and using the equations (1306)-(1320) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t}=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)-\Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1321}$$
$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t)-\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1322}$$
$$=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t),\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\right]+\Lambda_{j'n'jn}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s),\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1323}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t}=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t),A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right]+\Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t),A_{jn}(t)\right]\right)\right)\mathrm{d}\tau-\mathrm{i}\left[H_S(t),\overline{\rho_S}(t)\right] \tag{1324}$$

For this case we used that $A_{jn}(t-\tau,t)=U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (423) as expected.

VIII. BIBLIOGRAPHY

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