

Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality. Let's define the partition functions of $\bar{H}(t)$ and $\bar{H}_0(t)$ given by $Z(t)$ and $Z_0(t)$ respectively as:

$$Z(t) \equiv \text{Tr} \left(e^{-\beta \bar{H}(t)} \right), \quad (1)$$

$$Z_0(t) \equiv \text{Tr} \left(e^{-\beta \bar{H}_0(t)} \right). \quad (2)$$

where the transformed hamiltonians $\bar{H}(t)$ and $\bar{H}_0(t)$ are defined as:

$$\bar{H}(t) \equiv \bar{H}_I(t) + \bar{H}_0(t), \quad (3)$$

$$\bar{H}_0(t) \equiv \bar{H}_S(t) + \bar{H}_B. \quad (4)$$

For any operator $A(t)$ we define the expected value respect to $\bar{H}_0(t)$ as:

$$\langle A(t) \rangle_{\bar{H}_0(t)} \equiv \frac{\text{Tr} \left(A(t) e^{-\beta \bar{H}_0(t)} \right)}{\text{Tr} \left(e^{-\beta \bar{H}_0(t)} \right)}. \quad (5)$$

The terms $\bar{H}_S(t)$, \bar{H}_B and $\bar{H}_I(t)$ are related to the variational transformation performed in [1,2], this transformation allowed us to construct $\bar{H}_I(t)$ such that $\langle \bar{H}_I(t) \rangle_{\bar{H}_0(t)} = 0$. The diagonalization of $\bar{H}_0(t)$ in terms of it's eigenstates and eigenvalues such that $\bar{H}_0(t) |n\rangle = E_{0,n}(t) |n\rangle$, being $|n\rangle$ an eigenstate of $\bar{H}_0(t)$ with eigenvalue $E_{0,n}(t)$ is $\bar{H}_0(t) = \sum_n E_{0,n}(t) |n\rangle \langle n|$, with $\langle n|n'\rangle = \delta_{nn'}$, so a simple form of $e^{-\beta \bar{H}_0(t)}$ can be found as follows:

$$e^{r(X+Y)} = e^{rX} e^{rY} e^{-\frac{r^2}{2}[X,Y]} e^{\frac{r^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} \dots \text{ (Zassenhaus formula),} \quad (6)$$

$$e^{X+Y} = e^X e^Y e^{-\frac{r^2}{2}0} e^{\frac{r^3}{6}(2[Y,0] + [X,0])} \dots \text{ (setting } r = 1 \text{ and } [X,Y] = 0 \text{ in (6))} \quad (7)$$

$$= e^X e^Y \mathbb{I} \quad (8)$$

$$= e^X e^Y, \quad (9)$$

$$e^{-\beta \bar{H}_0(t)} = e^{-\sum_n \beta E_{0,n}(t) |n\rangle \langle n|} \text{ (by the diagonalization of } \bar{H}_0(t)) \quad (10)$$

$$e^{-\sum_n \beta E_{0,n}(t) |n\rangle\langle n|} = \prod_n e^{-\beta E_{0,n}(t) |n\rangle\langle n|} \text{ (by (9) and } [|n\rangle\langle n|, |n'\rangle\langle n'|] = 0) \quad (11)$$

$$= \prod_n \sum_{j=0}^{\infty} \frac{(-\beta E_{0,n}(t) |n\rangle\langle n|)^j}{j!} \text{ (by the exponential formula)} \quad (12)$$

$$= \prod_n \left(\mathbb{I} + \sum_{j=1}^{\infty} \frac{(-\beta E_{0,n}(t))^j |n\rangle\langle n|}{j!} \right) \text{ (using } (aA)^j = a^j A^j \text{ and } (|n\rangle\langle n|)^2 = |n\rangle\langle n|) \quad (13)$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{(-\beta E_{0,n}(t))^j |n\rangle\langle n|}{j!} \right) \quad (14)$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left(\sum_{j=0}^{\infty} \frac{(-\beta E_{0,n}(t))^j}{j!} \right) \right) \quad (15)$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right) \text{ (by the exponential formula)} \quad (16)$$

$$= \prod_n \left(\mathbb{I} + \left(e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right). \quad (17)$$

We will prove by induction a neat form for (17), we will show that:

$$\prod_{j=1}^n \left(\mathbb{I} + \left(e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \mathbb{I} + \sum_{j=1}^n \left(e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \quad (18)$$

For $n = 1$ the formula is trivial, in the case $n = 2$ we obtain that:

$$\prod_{j=1}^2 \left(\mathbb{I} + \left(e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \left(\mathbb{I} + \left(e^{-\beta E_{0,1}(t)} - 1 \right) |1\rangle\langle 1| \right) \left(\mathbb{I} + \left(e^{-\beta E_{0,2}(t)} - 1 \right) |2\rangle\langle 2| \right) \quad (19)$$

$$= \mathbb{I} + \left(e^{-\beta E_{0,1}(t)} - 1 \right) |1\rangle\langle 1| + \left(e^{-\beta E_{0,2}(t)} - 1 \right) |2\rangle\langle 2| \text{ (by } \langle i|j \rangle = \delta_{ij}) \quad (20)$$

$$= \mathbb{I} + \sum_{j=1}^2 \left(e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \quad (21)$$

It is our case base, our induction step is (18), in the case $n + 1$ we will have:

$$\prod_{j=1}^{n+1} \left(\mathbb{I} + \left(e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \left(\prod_{j=1}^n \left(\mathbb{I} + \left(e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) \right) \left(\mathbb{I} + \left(e^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| \right) \quad (22)$$

$$= \left(\mathbb{I} + \sum_n \left(e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right) \left(\mathbb{I} + \left(e^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| \right) \text{ (by induction step)} \quad (23)$$

$$= \mathbb{I} + \left(e^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| + \sum_n \left(e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \text{ (by } \langle i|j \rangle = \delta_{ij}) \quad (24)$$

$$= \mathbb{I} + \sum_{j=1}^{n+1} \left(e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \quad (25)$$

By mathematical induction we proved that (18) is true for all $n \in \mathbb{N}$. Given that the resolution of the identity is $\mathbb{I} = \sum_n |n\rangle\langle n|$ so we find that:

$$e^{-\beta \overline{H}_0(t)} = \prod_n \left(\mathbb{I} + \left(e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right) \quad (26)$$

$$= \mathbb{I} + \sum_n \left(e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \quad (27)$$

$$= \mathbb{I} + \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| - \sum_n |n\rangle\langle n| \quad (28)$$

$$= \mathbb{I} + \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| - \mathbb{I} \text{ (by the resolution of the identity)} \quad (29)$$

$$= \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n|. \quad (30)$$

The partition function $Z_0(t)$ is equal to:

$$Z_0(t) = \text{Tr} \left(\sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right) \quad (31)$$

$$= \sum_n e^{-\beta E_{0,n}(t)} \text{Tr}(|n\rangle\langle n|) \quad (32)$$

$$= \sum_n e^{-\beta E_{0,n}(t)}. \quad (33)$$

The explicit form of the average value $\langle A(t) \rangle_{\overline{H}_0(t)}$ can be found from the partition function $Z_0(t)$:

$$\langle A(t) \rangle_{\overline{H}_0(t)} = \frac{\text{Tr} \left(A(t) e^{-\beta \overline{H}_0(t)} \right)}{Z_0(t)} \quad (34)$$

$$= \frac{\text{Tr} \left(\sum_n A(t) e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right)}{\text{Tr} \left(\sum_n e^{-\beta \overline{H}_0(t)} \right)} \quad (35)$$

$$= \frac{\text{Tr} \left(\sum_n e^{-\beta E_{0,n}(t)} A(t) |n\rangle\langle n| \right)}{\text{Tr} \left(\sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right)} \quad (36)$$

$$= \frac{\text{Tr} \left(\sum_n e^{-\beta E_{0,n}(t)} A(t) |n\rangle\langle n| \right)}{\sum_n e^{-\beta E_{0,n}(t)}} \quad (37)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \text{Tr} \left(A(t) |n\rangle\langle n| \right)}{\sum_n e^{-\beta E_{0,n}(t)}}. \quad (38)$$

At first we show a double sequence of inequalities of order M, N which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$Z(t) \geq Z_0(t) e^{-\langle \overline{H}_T(t) \rangle_{\overline{H}_0(t)}} (1 + F_M(\vec{u}(t); \alpha) + F_N(\vec{v}(t) - \vec{u}(t); \alpha)). \quad (39)$$

where the function $F_N(\vec{v}(t); \alpha)$ is defined as:

$$F_N(\vec{w}(t); \alpha) \equiv e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{w_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!}. \quad (40)$$

In this case α is a parameter that can be optimized, $\beta \equiv \frac{1}{k_B T}$, $\vec{w}(t)$ is a vector such that $\vec{w}(t) = (w_1, w_2, \dots)$ and $\vec{u}(t)$ and $\vec{v}(t)$ are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian $\overline{H}_{TD}(t)$ respect to the basis of eigenstates of $\overline{H}_0(t)$ as:

$$\overline{H_{ID}}(t) \equiv \sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n|. \quad (41)$$

We will prove an important property related to $\overline{H_{ID}}(t)$, which is a Hamiltonian written as a linear combination of a set of orthonormal operators. Let's consider a vector space R with two operations $+$ and \cdot , if there exist $a, b \in R$ such that $a \cdot b = 0$ and $b \cdot a = 0$ then for any $k \in \mathbb{N}$ we have $(a + b)^k = a^k + b^k$ where $a^k = a^{k-1} \cdot a$ is a recursive definition of the power of an element written in terms of \cdot . At first we prove that this result yields for any $k \in \mathbb{N}$ by induction, the case $k = 1$ is trivial so we will focus on the case $k = 2$, we have that:

$$(a + b)^2 = (a + b) \cdot (a + b) \quad (42)$$

$$= a \cdot a + a \cdot b + b \cdot a + b \cdot b \quad (43)$$

$$= a^2 + a \cdot b + b \cdot a + b^2 \quad (44)$$

$$= a^2 + 0 + 0 + b^2 \text{ (because } a \cdot b = b \cdot a = 0) \quad (45)$$

$$= a^2 + b^2. \quad (46)$$

This is the base case. By induction step we will consider that $(a + b)^k = a^k + b^k$ with $k \geq 2$, now for $k + 1$ we will have that:

$$(a + b)^{k+1} = (a + b)^k \cdot (a + b) \quad (47)$$

$$= (a^k + b^k) \cdot (a + b) \text{ (by induction step)} \quad (48)$$

$$= a^k \cdot a + a^k \cdot b + b^k \cdot a + b^k \cdot b \quad (49)$$

$$= a^{k+1} + a^{k-1} \cdot a \cdot b + b^{k-1} \cdot b \cdot a + b^{k+1} \text{ (by recursive definition of } a^k) \quad (50)$$

$$= a^{k+1} + a^{k-1} \cdot (a \cdot b) + b^{k-1} \cdot (b \cdot a) + b^{k+1} \text{ (by associativity on } R) \quad (51)$$

$$= a^{k+1} + a^{k-1} \cdot (0) + b^{k-1} \cdot (0) + b^{k+1} \text{ (because } a \cdot b = b \cdot a = 0) \quad (52)$$

$$= a^{k+1} + b^{k+1}. \quad (53)$$

By the principle of mathematical induction we can conclude that the proposition is true for all $k \in \mathbb{N}$. Now we will extend the result, let $a_1, \dots, a_n \in R$ such that $a_i \cdot a_j = 0$ for all $i \neq j$ then $(a_1 + \dots + a_n)^k = a_1^k + \dots + a_n^k$. The case $n = 1$ is trivial as well so we will focus on $n = 2$, this case was proved in the precedent lines so it will be our base case. By induction step we will consider that $(a_1 + \dots + a_n)^k = a_1^k + \dots + a_n^k$ with $n \geq 2$, now for $n + 1$ we will have that:

$$a_{n+1} \cdot (a_1 + \dots + a_n) = a_{n+1} \cdot a_1 + \dots + a_{n+1} \cdot a_n \quad (54)$$

$$= 0 \text{ (because } a_i \cdot a_j = 0 \text{ for all } i \neq j), \quad (55)$$

$$(a_1 + \dots + a_n + a_{n+1})^k = ((a_1 + \dots + a_n) + a_{n+1})^k \quad (56)$$

$$= (a_1 + \dots + a_n)^k + a_{n+1}^k \text{ (by (47) and (55))} \quad (57)$$

$$= a_1^k + \dots + a_n^k + a_{n+1}^k \text{ (by inductive step).} \quad (58)$$

So we can conclude by mathematical induction that the proposition is true for all $n \in \mathbb{N}$. We can prove the following property for $(\overline{H_{ID}}(t))^k$:

$$\langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \langle n' | \overline{H_I}(t) | n' \rangle |n'\rangle\langle n'| = \langle n | \overline{H_I}(t) | n \rangle \langle n' | \overline{H_I}(t) | n' \rangle |n\rangle\langle n| n' \langle n'| \quad (59)$$

$$= \langle n | \overline{H_I}(t) | n \rangle \langle n' | \overline{H_I}(t) | n' \rangle |n\rangle\langle n'| \delta_{nn'}, \quad (60)$$

$$(\overline{H_{ID}}(t))^k = \left(\sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \right)^k \quad (\text{by (41)}) \quad (61)$$

$$= \sum_n \left(\langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \right)^k \quad (\text{by (58) and (60)}), \quad (62)$$

$$(aA)^k = a^k A^k \quad (\text{by the property of the power of a matrix}), \quad (63)$$

$$(|n\rangle\langle n|)^k = |n\rangle\langle n| \quad (\text{because } |n\rangle\langle n| \text{ is a projector and } k \in \mathbb{N}^*), \quad (64)$$

$$(\overline{H_{ID}}(t))^k = \sum_n \left(\langle n | \overline{H_I}(t) | n \rangle \right)^k |n\rangle\langle n| \quad (\text{by (63) and (64)}). \quad (65)$$

The vectors $\vec{u}(t)$ and $\vec{v}(t)$ are defined as $\vec{u}(t) \equiv (u_1, u_2, \dots)$ and $\vec{v}(t) \equiv (v_1, v_2, \dots)$. We can define the elements of $\vec{u}(t)$ and $\vec{v}(t)$ in terms of the matrix $\overline{H_{ID}}(t)$:

$$u_k(t) \equiv \left\langle \left(\overline{H_{ID}}(t) - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k \right\rangle_{\overline{H_0}(t)} \quad (66)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \text{Tr} \left(\left(\sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n\rangle\langle n| \right)}{Z_0(t)} \quad (\text{by (38)}), \quad (67)$$

$$\left(\sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k = \sum_{j=0}^k (-1)^j \binom{k}{j} \left(\sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \right)^j \left(\langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^{k-j} \quad (\text{by binomial theorem}) \quad (68)$$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} \left(\sum_n \langle n | \overline{H_I}(t) | n \rangle^j |n\rangle\langle n| \right) \left(\langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^{k-j} \quad (\text{by (65)}) \quad (69)$$

$$= \sum_n \left(\sum_{j=0}^k (-1)^j \binom{k}{j} \langle n | \overline{H_I}(t) | n \rangle^j \left(\langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^{k-j} \right) |n\rangle\langle n| \quad (70)$$

$$= \sum_n \left(\langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n\rangle\langle n|, \quad (71)$$

$$= \sum_n \left(\langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n\rangle\langle n|, \quad (72)$$

$$u_k(t) = \frac{\sum_n e^{-\beta E_{0,n}(t)} \text{Tr} \left(\sum_{n'} \left(\langle n' | \overline{H_I}(t) | n' \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n'\rangle\langle n'| n \rangle \right)}{Z_0(t)} \quad (73)$$

$$= \frac{\sum_{nn'} e^{-\beta E_{0,n}(t)} \text{Tr} \left(\left(\langle n' | \overline{H_I}(t) | n' \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n'\rangle\langle n'| \delta_{nn'} \right)}{Z_0(t)} \quad (74)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left(\langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k \text{Tr}(|n\rangle\langle n|)}{Z_0(t)} \quad (75)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left(\langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k 1}{Z_0(t)} \quad (76)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left(\langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k}{Z_0(t)}, \quad (77)$$

$$v_k(t) \equiv \frac{\sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t) - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k \right| n \right\rangle}{Z_0(t)}. \quad (78)$$

By construction $\langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} = 0$, so we summarize the double inequality that generalizes the Bogoliubov inequality and it's coefficients as:

$$Z(t) \geq Z_0(t) (1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))), \quad (79)$$

$$Z(t) = \text{Tr} \left(e^{-\beta \overline{H}(t)} \right), \quad (80)$$

$$Z_0(t) = \sum_n e^{-\beta E_{0,n}(t)}, \quad (81)$$

$$F_N(\vec{u}(t)) = e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!}, \quad (82)$$

$$u_k(t) = \frac{\sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) | n \rangle^k}{Z_0(t)}, \quad (83)$$

$$v_k(t) = \frac{\sum_n e^{-\beta E_{0,n}(t)} \langle n | (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^k | n \rangle}{Z_0(t)}. \quad (84)$$

As we can see the expression (83) was written in shorter terms, we want to do the same for (84) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for $v_k(t)$:

$$(\overline{H_0}(t) - E_{0,n}(t)) |n\rangle = \overline{H_0}(t) |n\rangle - E_{0,n}(t) |n\rangle \quad (85)$$

$$= E_{0,n}(t) |n\rangle - E_{0,n}(t) |n\rangle \quad (86)$$

$$= 0, \quad (87)$$

$$\langle n | (\overline{H_0}(t) - E_{0,n}(t)) = \langle n | \overline{H_0}(t) - \langle n | E_{0,n}(t) \quad (88)$$

$$= \langle n | E_{0,n}(t) - \langle n | E_{0,n}(t) \quad (89)$$

$$= 0. \quad (90)$$

At first we calculated $v_1(t)$ using the definition (84):

$$v_1(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t) | n \rangle \quad (91)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_0}(t) - E_{0,n}(t) | n \rangle + \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) | n \rangle \quad (92)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} (\langle n | \overline{H_0}(t) | n \rangle - \langle n | E_{0,n}(t) | n \rangle) + \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \quad (93)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} (\langle n | E_{0,n}(t) | n \rangle - \langle n | E_{0,n}(t) | n \rangle) + \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \quad (94)$$

$$= 0 + \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \quad (\text{by construction } \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} = 0) \quad (95)$$

$$= 0. \quad (96)$$

For $k \geq 2$ and $k \in \mathbb{N}$ we calculated:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^k \right| n \right\rangle \quad (97)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (98)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (99)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (E_{0,n}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (E_{0,n}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (100)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle. \quad (101)$$

In general we can write a formula for $v_k(t)$ that implies an expected value of a dependent expression of $\overline{H_I}(t)$ and $\overline{H_0}(t)$:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle \quad (102)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) + \overline{H_I}(t) - E_{0,n}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle \quad (103)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H}(t) - E_{0,n}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle \quad (104)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) \left(\sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j}(t) E_{0,n}^j(t) \right) \overline{H_I}(t) \right| n \right\rangle \quad (105)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) E_{0,n}^j(t) \right| n \right\rangle \quad (106)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right| n \right\rangle \quad (107)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right| n \right\rangle \quad (108)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (109)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{k-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \quad (110)$$

The formula (110) is well defined taking as example $k = 2, 3$.

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{2-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (111)$$

$$= (-1)^0 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^0 \overline{H_I}(t) \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \quad (112)$$

$$= \left\langle \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)}, \quad (113)$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{3-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (114)$$

$$= \left\langle \sum_{j=0}^1 (-1)^j \binom{1}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{1-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (115)$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^1 \overline{H_I}(t) \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^0 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \quad (116)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \mathbb{I} - \overline{H_I}(t) \mathbb{I} \overline{H_I}(t) \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \quad (117)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \quad (118)$$

$$= \langle \overline{H_I}(t)^3 + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \quad (119)$$

$$= \langle \overline{H_I}(t)^3 + \overline{H_I}(t) (\overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t)) \rangle_{\overline{H_0}(t)} \quad (120)$$

$$= \langle \overline{H_I}(t)^3 + \overline{H_I}(t) [\overline{H_0}(t), \overline{H_I}(t)] \rangle_{\overline{H_0}(t)} \quad (\text{because } [\overline{H_0}(t), \overline{H_I}(t)] = \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t)). \quad (121)$$

So we summarize:

$$\overline{H_{ID}}(t) = \sum_n \langle n | \overline{H_I}(t) | n \rangle | n \rangle \langle n |, \quad (122)$$

$$u_k(t) = \left\langle (\overline{H_{ID}}(t))^k \right\rangle_{\overline{H_0}(t)}, \quad (123)$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{k-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \quad (124)$$

The free energy is defined as:

$$E_{\text{free}}(t) \equiv -\frac{1}{\beta} \ln(Z(t)). \quad (125)$$

We define the free energy at first order as:

$$E_{\text{free},1}(t) \equiv -\frac{1}{\beta} \ln(Z_0(t)). \quad (126)$$

It is well-known that the function $f(x) = -\ln(x)$ is a decreasing function so we can transform (39):

$$E_{\text{free}}(t) \leq -\frac{1}{\beta} \ln(Z_0(t) (1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t)))) \quad (127)$$

$$= -\frac{1}{\beta} \ln(Z_0(t)) - \frac{1}{\beta} \ln(1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (128)$$

$$= E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (129)$$

$$\equiv E_{\text{free,MN}}(t). \quad (130)$$

here $E_{\text{free,MN}}(t)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of M, N order. In our approach we will set $N = M$, so our quantum Bogoliubov inequality of N, N order is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (131)$$

$$= E_{\text{free,NN}}(t). \quad (132)$$

A weaker form of the inequality (132) is obtained making $\vec{u}(t) = 0$ as suggest [3]:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{v}(t))) \quad (133)$$

$$\equiv E_{\text{free},N}(t). \quad (134)$$

The algebraic equation associated with $\alpha_{\text{opt}}(t)$ such that $E_{\text{free},N}(t)$ is closer to $E_{\text{free}}(t)$ follows from the fact that in the optimal parameter $\frac{\partial E_{\text{free},N}(t)}{\partial \alpha}|_{\alpha_{\text{opt}}(t)} = 0$, calculating this derivate we have:

$$\frac{\partial E_{\text{free},N}(t)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left(E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{v}(t))) \right) \quad (135)$$

$$= -\frac{1}{\beta} \frac{\frac{\partial}{\partial \alpha} (F_N(\vec{v}(t)))}{1 + F_N(\vec{v}(t))} \quad (136)$$

$$= 0. \quad (137)$$

The precedent equation is equivalent to make the numerator equal to 0:

$$\frac{\partial F_N(\vec{v}(t))}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left(e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} \right) \quad (138)$$

$$= -e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} + e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\partial}{\partial \alpha} \frac{\alpha^i}{i!} \quad (\text{by product rule}) \quad (139)$$

$$= -e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} + e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=1}^{2N-1-k} \frac{\alpha^{i-1}}{(i-1)!} \quad (140)$$

$$= e^{-\alpha} \left(\sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=1}^{2N-1-k} \frac{\alpha^{i-1}}{(i-1)!} - \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} \right) \quad (141)$$

$$= e^{-\alpha} \left(\sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{j=0}^{2N-2-k} \frac{\alpha^j}{j!} - \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} \right) \quad (\text{setting } j = i - 1) \quad (142)$$

$$= e^{-\alpha} \left(- \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \frac{\alpha^{2N-1-k}}{(2N-1-k)!} \right) \quad (\text{performing the difference}) \quad (143)$$

$$= 0. \quad (144)$$

Then the optimal value $\alpha_{\text{opt}}(t)$ will satisfy the following equation:

$$G(\alpha_{\text{opt}}(t)) \equiv \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \quad (145)$$

$$= 0. \quad (146)$$

The elements presented are the required to find variational parameters of the system using the inequality (134) and the self consistent equation (SCE) (145) to a particular order required.

II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2(t)$, $v_3(t)$, $v_4(t)$, $v_5(t)$ using the (124), we have already $v_2(t)$, $v_3(t)$ and the form of $v_4(t)$ and $v_5(t)$ is given by:

$$v_4(t) = \sum_{j=0}^{4-2} (-1)^j \binom{4-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{4-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (147)$$

$$= \sum_{j=0}^2 (-1)^j \binom{2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (148)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^1 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}(t) \overline{H_I}(t) \right\rangle_{\overline{H_0}(t)} \quad (149)$$

$$\left. + \overline{H}_0(t) \right)^0 \overline{H}_I(t) \overline{H}_0^{-2}(t) \Big\rangle_{\overline{H}_0(t)} \quad (150)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \mathbb{I} \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^1 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (151)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (152)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) - 2\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (153)$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^2(t) \right) \overline{H_I}(t) - 2\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}^2(t) \right\rangle \quad (154)$$

$$\times \overline{H_0}^2(t) \Big\rangle_{\overline{H_0}(t)} \quad (155)$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - 2\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) \right\rangle \quad (156)$$

$$+\overline{H_I}^2(t)\overline{H_0}^2(t)\Big\rangle_{\overline{H_0}(t)} \quad (157)$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - 2\overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) - 2\overline{H_I}(t) \right\rangle \quad (158)$$

$$\times \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \quad (159)$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - \overline{H_I}^3(t) \overline{H_0}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}^2(t) \rangle \quad (160)$$

$$\times \overline{H_0^2}(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \Big\rangle_{\overline{H_0}(t)} \quad (161)$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t) - \overline{H_I}^3(t)\overline{H_0}(t) + \overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t) - \overline{H_I}^3(t)\overline{H_0}(t) + \overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}(t) - \overline{H_I}(t) \rangle \quad (162)$$

$$\times \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \Big\rangle_{\overline{H_0}(t)} \quad (163)$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \left(((\overline{H_I}(t) \overline{H_0}(t)) \overline{H_I}(t) - \overline{H_I}(t) (\overline{H_I}(t) \overline{H_0}(t))) \right) + \left(\overline{H_0}(t) \overline{H_I}^2(t) - \overline{H_I}^2(t) \overline{H_0}(t) \right) + (\overline{H_0}(t) (\overline{H_0}(t) \right. \quad (164)$$

$$\times \overline{H_I}(t)) - (\overline{H_0}(t) \overline{H_I}(t)) \overline{H_0}(t)) + ((\overline{H_I}(t) \overline{H_0}(t)) \overline{H_0}(t) - \overline{H_0}(t) (\overline{H_I}(t) \overline{H_0}(t))) \rangle_{\overline{H_0}(t)} \quad (165)$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \left([\overline{H_I}(t) \overline{H_0}(t), \overline{H_I}(t)] + [\overline{H_0}(t), \overline{H_I}^2(t)] + [\overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t)] + [\overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t)] \right) \right\rangle_{\overline{H_0}(t)}, \quad (166)$$

$$v_5(t) = \sum_{j=0}^{5-2} (-1)^j \binom{5-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{5-2-j} \overline{H_I}(t) \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \quad (167)$$

$$= \sum_{j=0}^3 (-1)^j \binom{3}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{3-j} \overline{H_I}(t) \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \quad (168)$$

$$= \langle \overline{H}_T(t) (\overline{H}_T(t) + \overline{H}_0(t))^3 \overline{H}_T(t) \overline{H}_0^0(t) - 3 \overline{H}_T(t) (\overline{H}_T(t) + \overline{H}_0(t))^2 \overline{H}_T(t) \overline{H}_0(t) - \overline{H}_T(t) (\overline{H}_T(t) + \overline{H}_0(t))^0 \overline{H}_T(t) \overline{H}_0^3(t) \rangle \quad (169)$$

$$+3\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t) \right) \overline{H_I}(t) \overline{H_0}^2(t) \Big\rangle_{\overline{H_0}(t)} \quad (170)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^3 \overline{H_I}(t) - 3\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}(t) \rangle \quad (171)$$

$$\times \overline{H_I}(t) \overline{H_0}^3(t) \Big\rangle_{\overline{H_0}(t)} \quad (172)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^3 \overline{H_I}(t) - 3\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}^2(t) \quad (173)$$

$$-\overline{H_I}(t) \overline{H_I}(t) \overline{H_0^3}(t) \rangle_{\overline{H_0}(t)} \quad (174)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^3 \overline{H_I}(t) - 3\overline{H_I}(t) (\overline{H_I}^2(t) + \overline{H_I}(t)\overline{H_0}(t) + \overline{H_0}(t)\overline{H_I}(t) + \overline{H_0}^2(t)) \overline{H_I}(t)\overline{H_0}(t) + 3\overline{H_I}(t) \quad (175)$$

$$\times \left(\overline{H_I}(t) + \overline{H_0}(t) \right) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}^3(t) \Big\rangle_{\overline{H_0}(t)} \quad (176)$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}^3(t) + \overline{H_I}^2(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_0}^2(t) \overline{H_I}(t) + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}(t) \right. \right. \quad (177)$$

$$\left. \times \overline{H_0}^2(t) + \overline{H_0}^3(t) \right) \overline{H_I}(t) - 3\overline{H_I}(t) \left(\overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^2(t) \right) \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) \left(\overline{H_I}(t) \right. \quad (178)$$

$$\left. + \overline{H_0}(t) \right) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}^3(t) \Big\rangle_{\overline{H_0}(t)} \quad (179)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}^3(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^3(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \right. \quad (180)$$

$$\left. \times \overline{H_0}^2(t) \overline{H_I}^2(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}^3(t) \overline{H_I}(t) - 3\overline{H_I}^4(t) \overline{H_0}(t) - 3\overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) - 3\overline{H_I}(t) \right. \quad (181)$$

$$\left. \times \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) - 3\overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}^3(t) \overline{H_0}^2(t) + 3\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}^2(t) \overline{H_0}^3(t) \right\rangle_{\overline{H_0}(t)} \quad (182)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left(\overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_0}(t) \overline{H_I}^3(t) + \overline{H_0}^2(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) \right. \right. \quad (183)$$

$$\left. + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^3(t) \overline{H_I}(t) - 3\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) - 3\overline{H_I}^3(t) \overline{H_0}(t) - 3\overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}(t) \right. \quad (184)$$

$$\left. \times \overline{H_0}^3(t) + 3\overline{H_I}^2(t) \overline{H_0}^2(t) + 3\overline{H_0}(t) \overline{H_I}(t) \overline{H_0}^2(t) - 3\overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \quad (185)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left(\overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}^3(t) - \overline{H_0}(t) \overline{H_I}^2(t) \right. \right. \quad (186)$$

$$\left. \times \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) + \overline{H_0}^3(t) \overline{H_I}(t) \right. \quad (187)$$

$$\left. - \overline{H_I}(t) \overline{H_0}^3(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) - \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) + 2\overline{H_I}^2(t) \overline{H_0}^2(t) - 2\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_0}(t) \overline{H_I}(t) \right. \quad (188)$$

$$\left. \times \overline{H_0}^2(t) - 3\overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_0}^2(t) \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (189)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left(\left(\overline{H_I}^2(t) \overline{H_0}(t) \right) \overline{H_I}(t) - \overline{H_I}(t) \left(\overline{H_I}^2(t) \overline{H_0}(t) \right) \right) + \left(\overline{H_I}(t) \overline{H_0}(t) \right) \overline{H_I}^2(t) - \overline{H_I}^2(t) \left(\overline{H_I}(t) \overline{H_0}(t) \right) \right. \quad (190)$$

$$\left. + \left(\overline{H_0}(t) \overline{H_I}^3(t) - \overline{H_I}^3(t) \overline{H_0}(t) \right) + \left(\overline{H_0}(t) \left(\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right) - \left(\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right) \overline{H_0}(t) \right) + \left(\overline{H_0}(t) \left(\overline{H_0}(t) \overline{H_I}^2(t) \right) \right. \quad (191)$$

$$\left. - \left(\overline{H_0}(t) \overline{H_I}^2(t) \right) \overline{H_0}(t) \right) + \left(\overline{H_0}^3(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}^3(t) \right) + \left(\left(\overline{H_I}(t) \overline{H_0}(t) \right) \left(\overline{H_0}(t) \overline{H_I}(t) \right) - \left(\overline{H_0}(t) \overline{H_I}(t) \right) \left(\overline{H_I}(t) \overline{H_0}(t) \right) \right) \quad (192)$$

$$\left. + 3\overline{H_0}(t) \left(\overline{H_I}(t) \overline{H_0}(t) - \overline{H_0}(t) \overline{H_I}(t) \right) \overline{H_0}(t) + 2\overline{H_I}(t) \left(\overline{H_I}(t) \overline{H_0}(t) - \overline{H_0}(t) \overline{H_I}(t) \right) \overline{H_0}(t) + \left(\left(\overline{H_I}^2(t) \overline{H_0}(t) \right) \left(\overline{H_0}(t) \right) \right. \quad (193)$$

$$\left. - \left(\overline{H_0}(t) \right) \left(\overline{H_I}^2(t) \overline{H_0}(t) \right) \right\rangle_{\overline{H_0}(t)} \quad (194)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left(\left[\overline{H_I}^2(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[\overline{H_0}(t), \overline{H_I}^3(t) \right] + \left[\overline{H_0}(t), \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right] \right. \quad (195)$$

$$\left. + \left[\overline{H_0}(t), \overline{H_0}(t) \overline{H_I}^2(t) \right] + \left[\overline{H_0}^3(t), \overline{H_I}(t) \right] + \left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + 3\overline{H_0}(t) \left[\overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + 2\overline{H_I}(t) \right. \quad (196)$$

$$\left. \times \left[\overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + \left[\overline{H_I}^2(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right\rangle_{\overline{H_0}(t)}. \quad (197)$$

Summarizing we have that:

$$v_2(t) = \left\langle \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)}, \quad (198)$$

$$v_3(t) = \left\langle \overline{H_I}^3(t) + \overline{H_I}(t) \left[\overline{H_0}(t), \overline{H_I}(t) \right] \right\rangle_{\overline{H_0}(t)}, \quad (199)$$

$$v_4(t) = \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \left(\left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[\overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[\overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + \left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right) \right\rangle_{\overline{H_0}(t)}, \quad (200)$$

$$v_5(t) = \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left(\left[\overline{H_I}^2(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[\overline{H_0}(t), \overline{H_I}^3(t) \right] + \left[\overline{H_0}(t), \overline{H_0}(t) \overline{H_I}^2(t) \right] \right. \quad (201)$$

$$\left. + \left[\overline{H_0}(t), \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right] + \left[\overline{H_0}^3(t), \overline{H_I}(t) \right] + \left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + 3\overline{H_0}(t) \left[\overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) \right. \quad (202)$$

$$\left. + 2\overline{H_I}(t) \left[\overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + \left[\overline{H_I}^2(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right\rangle_{\overline{H_0}(t)}. \quad (203)$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_S}(t) \equiv (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right), \quad (204)$$

$$\overline{H_I}(t) \equiv \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)), \quad (205)$$

$$\overline{H_B} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (206)$$

$$= H_B. \quad (207)$$

In this case $\varepsilon_j(t)$, $R_j(t)$ for $j \in \{0, 1\}$, $B_{10}^{\Re}(t)$, $B_{10}^{\Im}(t)$, $V_{10}^{\Re}(t)$ and $V_{10}^{\Im}(t)$ are scalars and the other operators are:

$$\sigma_x \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \quad (208)$$

$$\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (209)$$

$$\sigma_y \equiv -i|1\rangle\langle 0| + i|0\rangle\langle 1| \quad (210)$$

$$\equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (211)$$

$$\sigma_z \equiv |1\rangle\langle 1| - |0\rangle\langle 0| \quad (212)$$

$$\equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (213)$$

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} e^{\chi_{ij}(t)} \end{pmatrix}, \quad (214)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right), \quad (215)$$

$$D(\pm v_{\mathbf{k}}(t)) \equiv e^{\pm \left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}. \quad (216)$$

As we can see they verify the relationship $\sigma_x \sigma_y = i \sigma_z$. The explicit form of $\overline{H_I}^2(t)$ is:

$$\overline{H_I}^2(t) = \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i B_{iz}(t) |i\rangle\langle i| (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \sum_i B_{iz}(t) |i\rangle\langle i| (\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Re}(t) \quad (217)$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i B_{iz}(t) |i\rangle\langle i| + \left(V_{10}^{\Re}(t) \right)^2 (\sigma_x B_x(t) + \sigma_y B_y(t))^2 + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (218)$$

$$\times (\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) (\sigma_x B_x(t) \quad (219)$$

$$+ \sigma_y B_y(t)) + \left(V_{10}^{\Im}(t) \right)^2 (\sigma_x B_y(t) - \sigma_y B_x(t))^2 \quad (220)$$

$$= \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}(t) \quad (221)$$

$$\times B_x(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t)) + \left(V_{10}^{\Re}(t) \right)^2 (\sigma_x^2 B_x^2(t) + \sigma_x \sigma_y B_x(t) B_y(t) + \sigma_y \quad (222)$$

$$\times \sigma_x B_y(t) B_x(t) + \sigma_y^2 B_y^2(t)) + V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) + \left(V_{10}^{\Im}(t) \right)^2 (\sigma_x^2 B_y^2(t) + \sigma_y^2 B_x^2(t) \quad (223)$$

$$- \sigma_x \sigma_y B_y(t) B_x(t) - \sigma_y \sigma_x B_x(t) B_y(t)) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x^2 B_y(t) B_x(t) + \sigma_x \sigma_y B_y^2(t) - \sigma_y \sigma_x B_x^2(t) - \sigma_y^2 B_x(t) B_y(t) \quad (224)$$

$$+ \sigma_x^2 B_x(t) B_y(t) - \sigma_x \sigma_y B_x^2(t) + \sigma_y \sigma_x B_y^2(t) - \sigma_y^2 B_y(t) B_x(t)), \quad (225)$$

$$\sigma_x \sigma_y = i\sigma_z \text{ (by Pauli matrices properties),} \quad (226)$$

$$\overline{H_T}^2(t) = \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}(t) \quad (227)$$

$$\times B_x(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_x^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z \quad (228)$$

$$\times B_y(t) B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_y^2(t) + B_x^2(t) - i\sigma_z \quad (229)$$

$$\times B_y(t) B_x(t) + i\sigma_z B_x(t) B_y(t)). \quad (230)$$

To introduce the direct calculation of the expected values recall that the hamiltonian $\overline{H_0}(t)$ is a direct sum of the hamiltonians of two Hilbert spaces given by $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$, so we can write in general the hamiltonian $\overline{H_0}(t)$ as:

$$\overline{H_0}(t) = \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} + \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}. \quad (231)$$

where $\mathbb{I}_{\bar{B}}$ and $\mathbb{I}_{\bar{S}}$ are the identity of the systems \bar{B} and \bar{S} respectively.

We can show that:

$$[\overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}}, \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}] = \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} \cdot \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} \cdot \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} \quad (232)$$

$$= \overline{H_{\bar{S}}}(t) \mathbb{I}_{\bar{S}} \otimes \mathbb{I}_{\bar{B}} \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} \mathbb{I}_{\bar{B}} \quad (233)$$

$$= \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} - \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} \text{ (by definition of identity operator)} \quad (234)$$

$$= 0. \quad (235)$$

Let's introduce the following partition functions $Z_{\bar{S}}(t)$ and $Z_{\bar{B}}$ related to the systems \bar{S} and \bar{B} respectively.:

$$Z_{\bar{S}}(t) \equiv \text{Tr} \left(e^{-\beta \overline{H_{\bar{S}}}(t)} \right), \quad (236)$$

$$Z_{\bar{B}} \equiv \text{Tr} \left(e^{-\beta \overline{H_{\bar{B}}}} \right) \quad (237)$$

Using (9), (232) and $\text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B)$ we can infer that the partition function $Z_0(t)$ can be factorized as:

$$Z_0(t) = \text{Tr} \left(e^{-\beta \overline{H_0}(t)} \right). \quad (238)$$

$$= \text{Tr} \left(e^{-\beta (\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}})} \right) \text{ (by (4))}, \quad (239)$$

$$= \text{Tr} \left(e^{-\beta \overline{H_{\bar{S}}}(t)} e^{-\beta \overline{H_{\bar{B}}}} \right) \text{ (by (9))} \quad (240)$$

$$= \text{Tr} \left(e^{-\beta \overline{H_{\bar{S}}}(t)} \otimes e^{-\beta \overline{H_{\bar{B}}}} \right) \text{ (because } \bar{S} \text{ and } \bar{B} \text{ are disjoint Hilbert spaces)} \quad (241)$$

$$= \text{Tr} \left(e^{-\beta \overline{H_{\bar{S}}}(t)} \right) \text{Tr} \left(e^{-\beta \overline{H_{\bar{B}}}} \right) \text{ (by } \text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B)), \quad (242)$$

$$= Z_{\bar{S}}(t) Z_{\bar{B}} \text{ (by (236) and (237))}. \quad (243)$$

For an operator $J(t)$ that can be factorized as $J(t) = S(t) \otimes B(t)$ with $S(t) \in \text{gen}(\overline{H_{\bar{S}}}(t))$ and $B(t) \in \text{gen}(\overline{H_{\bar{B}}})$, being $\text{gen}(A)$ the vectorial space generated by the eigenvectors of the operator A , we calculate it's expected value respect to $\overline{H_0}(t)$ using a simple way as follows:

$$\langle J(t) \rangle_{\overline{H_0(t)}} = \frac{\text{Tr} \left(J(t) e^{-\beta \overline{H_0(t)}} \right)}{\text{Tr} \left(e^{-\beta \overline{H_0(t)}} \right)} \text{ (by (5))} \quad (244)$$

$$= \frac{\text{Tr} \left((S(t) \otimes B(t)) \left(e^{-\beta \overline{H_S(t)}} \otimes e^{-\beta \overline{H_B}} \right) \right)}{\text{Tr} \left(e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right)} \text{ (by } J(t) = S(t) \otimes B(t) \text{ and } e^{-\beta \overline{H_0(t)}} = e^{-\beta \overline{H_S(t)}} \otimes e^{-\beta \overline{H_B}} \text{)} \quad (245)$$

$$= \frac{\text{Tr} \left(\left(S(t) e^{-\beta \overline{H_S(t)}} \right) \otimes \left(B(t) e^{-\beta \overline{H_B}} \right) \right)}{\text{Tr} \left(e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right)} \text{ (rearranging and factorizing)} \quad (246)$$

$$= \frac{\text{Tr} \left(S(t) e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left(B(t) e^{-\beta \overline{H_B}} \right)}{\text{Tr} \left(e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right)} \text{ (by } \text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B) \text{)} \quad (247)$$

$$= \frac{\text{Tr} \left(S(t) e^{-\beta \overline{H_S(t)}} \right)}{\text{Tr} \left(e^{-\beta \overline{H_S(t)}} \right)} \frac{\text{Tr} \left(B(t) e^{-\beta \overline{H_B}} \right)}{\text{Tr} \left(e^{-\beta \overline{H_B}} \right)} \quad (248)$$

$$= \langle S(t) \rangle_{\overline{H_S(t)}} \langle B(t) \rangle_{\overline{H_B}} \text{ (by (5))}. \quad (249)$$

The factorization of $\langle \overline{H_I}^2(t) \rangle_{\overline{H_0(t)}}$ in terms of expected values of elements from $\text{gen}(\overline{H_S}(t))$ and $\text{gen}(\overline{H_B})$ is:

$$\langle \overline{H_I}^2(t) \rangle_{\overline{H_0(t)}} = \sum_i \langle |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_{iz}^2(t) \rangle_{\overline{H_B}} + V_{10}^{\Re}(t) \sum_i \left(\langle |i\rangle\langle i| \sigma_x \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} + \langle |i\rangle\langle i| \sigma_y \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} \right) \quad (250)$$

$$+ V_{10}^{\Im}(t) \sum_i \left(\langle |i\rangle\langle i| \sigma_x \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} - \langle |i\rangle\langle i| \sigma_y \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} \right) + V_{10}^{\Re}(t) \sum_i \left(\langle \sigma_x |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} \right) \quad (251)$$

$$\times \langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} + \langle \sigma_y |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} + \left(V_{10}^{\Re}(t) \right)^2 \left(\langle B_x^2(t) \rangle_{\overline{H_B}} + i \langle \sigma_z \rangle_{\overline{H_S(t)}} \langle B_x(t) B_y(t) \rangle_{\overline{H_B}} \right) \quad (252)$$

$$- i \langle \sigma_z \rangle_{\overline{H_S(t)}} \langle B_y(t) B_x(t) \rangle_{\overline{H_B}} + \langle B_y^2(t) \rangle_{\overline{H_B}} + V_{10}^{\Im}(t) \sum_i \left(\langle \sigma_x |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} - \langle \sigma_y |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} \right) \quad (253)$$

$$\times \langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} + \left(V_{10}^{\Im}(t) \right)^2 \left(\langle B_y^2(t) \rangle_{\overline{H_B}} + \langle B_x^2(t) \rangle_{\overline{H_B}} - i \langle \sigma_z \rangle_{\overline{H_S(t)}} \langle B_y(t) B_x(t) \rangle_{\overline{H_B}} + i \langle \sigma_z \rangle_{\overline{H_S(t)}} \langle B_x(t) B_y(t) \rangle_{\overline{H_B}} \right) \quad (254)$$

$$\times \langle B_x(t) B_y(t) \rangle_{\overline{H_B}}. \quad (255)$$

In order to obtain the expected values of $\langle \overline{H_I}^2(t) \rangle_{\overline{H_0(t)}}$ respect to the part related to the bath we need to calculate the following expected values that appear in the equation (230) and can be obtained using the factorization of (250). The expected values relevant for calculations are $\langle B_{iz}^2(t) \rangle_{\overline{H_B}}$, $\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}}$, $\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}}$, $\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}}$, $\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}}$, $\langle B_x^2(t) \rangle_{\overline{H_B}}$, $\langle B_x(t) B_y(t) \rangle_{\overline{H_B}}$, $\langle B_y(t) B_x(t) \rangle_{\overline{H_B}}$, $\langle B_y^2(t) \rangle_{\overline{H_B}}$. Recalling the form of the hamiltonian $\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$ we can extend the result (243), also we introduce the notation:

$$A_1 \otimes \cdots \otimes A_n \equiv \bigotimes_k A_k, \quad (256)$$

$$Z_{\mathbf{k}} \equiv \text{Tr} \left(e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right) \quad (257)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}})^{-1} \quad (258)$$

$$= f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}}). \quad (259)$$

with the creation $b_{\mathbf{k}}$ and annihilation $b_{\mathbf{k}}^{\dagger}$ operators defined in terms of their actions as:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle, \quad (260)$$

$$b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle. \quad (261)$$

being $|j_{\mathbf{k}}\rangle$ an eigenstate of $H_{\mathbf{k}} \equiv \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$. With this notation we can write the partition function as:

$$Z_{\bar{B}} = \text{Tr} \left(e^{-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right), \quad (262)$$

$$e^{-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}, \quad (263)$$

$$Z_{\bar{B}} = \text{Tr} \left(\bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right) \text{ (by (263))} \quad (264)$$

$$= \prod_{\mathbf{k}} \text{Tr} \left(e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right) \text{ (by } \text{Tr} (A \otimes B) = \text{Tr} (A) \text{Tr} (B) \text{)} \quad (265)$$

$$= \prod_{\mathbf{k}} Z_{\mathbf{k}} \text{ (by (263))}. \quad (266)$$

For a function $f(t)$ which can be factorized as:

$$f(t) \equiv \prod_{\mathbf{k}} f_{\mathbf{k}}(t). \quad (267)$$

with $f_{\mathbf{k}}(t) \in \text{gen}(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}})$, it's expected value is given by:

$$\langle f(t) \rangle_{\overline{H_B}} = \frac{\text{Tr} \left(f(t) e^{-\beta \overline{H_B}} \right)}{\text{Tr} \left(e^{-\beta \overline{H_B}} \right)} \quad (268)$$

$$= \frac{\text{Tr} \left(\prod_{\mathbf{k}} f_{\mathbf{k}}(t) \bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\text{Tr} \left(\bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \text{ (by (263) and (267))} \quad (269)$$

$$= \frac{\text{Tr} \left(\bigotimes_{\mathbf{k}} f_{\mathbf{k}}(t) e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\text{Tr} \left(\bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \quad (270)$$

$$= \frac{\prod_{\mathbf{k}} \text{Tr} \left(f_{\mathbf{k}}(t) e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\prod_{\mathbf{k}} \text{Tr} \left(e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \quad (271)$$

$$= \prod_{\mathbf{k}} \frac{\text{Tr} \left(f_{\mathbf{k}}(t) e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\text{Tr} \left(e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \quad (272)$$

$$= \prod_{\mathbf{k}} \langle f_{\mathbf{k}}(t) \rangle_{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}. \quad (273)$$

It means that for an operator that can be factorized in terms of functions generated by $\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$ for each \mathbf{k} we only require to calculate the expected value respect to the Hilbert space where the operator belongs. This process lead us to the following explicit forms of the expected values relevant for our calculations:

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \left\langle \left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right)^2 \right\rangle_{\overline{H_B}} \quad (274)$$

$$= \sum_{\mathbf{k}} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 + \sum_{\mathbf{k} \neq \mathbf{k}'} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_B}} \quad (275)$$

$$- v_{i\mathbf{k}'}(t)) b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} \rangle_{\overline{H_B}} \quad (276)$$

$$= \sum_{\mathbf{k}} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (277)$$

$$\times \left\langle \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_B}} \text{ (by (273))}, \quad (278)$$

$$\langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} = \frac{\text{Tr} \left(b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (279)$$

$$= \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (280)$$

$$= \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 1)} |j_{\mathbf{k}} + 1\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (281)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 1)} \text{Tr}(|j_{\mathbf{k}} + 1\rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (282)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 1)} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (283)$$

$$= 0, \quad (284)$$

$$\langle b_{\mathbf{k}} \rangle_{\overline{H_B}} = \frac{\text{Tr} \left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (285)$$

$$= \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (286)$$

$$= \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} |j_{\mathbf{k}} - 1\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (287)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \text{Tr}(|j_{\mathbf{k}} - 1\rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (288)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (289)$$

$$= 0, \quad (290)$$

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) \quad (291)$$

$$\times \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \langle b_{\mathbf{k}'}^\dagger \rangle_{\overline{H_B}} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \langle b_{\mathbf{k}'} \rangle_{\overline{H_B}} \right) \quad (292)$$

$$= \sum_{\mathbf{k}} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k} \neq \mathbf{k}'} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \cdot 0 + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \cdot 0) ((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \cdot 0 \quad (293)$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \cdot 0) \text{ (by (279) and (285))} \quad (294)$$

$$= \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} \quad (295)$$

$$= \sum_{\mathbf{k}} \left\langle (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 (b_{\mathbf{k}}^\dagger)^2 + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*)^2 b_{\mathbf{k}}^2 \right\rangle_{\overline{H_B}} \quad (296)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 \left\langle (b_{\mathbf{k}}^\dagger)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*)^2 \langle b_{\mathbf{k}}^2 \rangle_{\overline{H_B}}, \quad (297)$$

$$\left\langle (b_{\mathbf{k}}^\dagger)^2 \right\rangle_{\overline{H_B}} = \frac{\text{Tr} \left((b_{\mathbf{k}}^\dagger)^2 \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (299)$$

$$= \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger} \right)^2 |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (300)$$

$$= \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 2)(j_{\mathbf{k}} + 1)} |j_{\mathbf{k}} + 2 \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (301)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 2)(j_{\mathbf{k}} + 1)} \text{Tr}(|j_{\mathbf{k}} + 2 \rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (302)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 2)(j_{\mathbf{k}} + 1)} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (303)$$

$$= 0, \quad (304)$$

$$\langle b_{\mathbf{k}}^2 \rangle_{\overline{H_B}} = \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^2 |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (305)$$

$$= \frac{\text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}(j_{\mathbf{k}} - 1)} |j_{\mathbf{k}} - 2 \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (306)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}(j_{\mathbf{k}} - 1)} \text{Tr}(|j_{\mathbf{k}} - 2 \rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (307)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}(j_{\mathbf{k}} - 1)} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (308)$$

$$= 0, \quad (309)$$

$$\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \rangle_{\overline{H_B}} = (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left(\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (310)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (311)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (312)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1 \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (313)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}} + 1} \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (314)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} j_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (j_{\mathbf{k}} + 1) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (315)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (316)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) \text{Tr}(|j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|) \quad (317)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) \quad (318)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \sum_{j_{\mathbf{k}}=0}^{\infty} (e^{-\beta \omega_{\mathbf{k}}})^{j_{\mathbf{k}}} (2j_{\mathbf{k}} + 1), \quad (319)$$

$$\sum_{j_{\mathbf{k}}=0}^{\infty} x^{j_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) = \frac{1+x}{(1-x)^2}, \quad (320)$$

$$\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \rangle_{\overline{H_B}} = \left(1 - e^{-\beta\omega_{\mathbf{k}}}\right) \frac{e^{-\beta\omega_{\mathbf{k}}} + 1}{(1 - e^{-\beta\omega_{\mathbf{k}}})^2} \text{ (setting } x = e^{-\beta\omega_{\mathbf{k}}} \text{ in (320) and by (310))}, \quad (321)$$

$$= \frac{1 + e^{-\beta\omega_{\mathbf{k}}}}{1 - e^{-\beta\omega_{\mathbf{k}}}} \quad (322)$$

$$= \frac{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} e^{\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} e^{\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}} \quad (323)$$

$$= \frac{\cosh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\sinh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (324)$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (325)$$

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \text{ (by (299), (305) and (325))}, \quad (326)$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_B}} \quad (327)$$

$$= \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left(e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + e^{\chi_{01}(t)} \right. \right. \quad (328)$$

$$\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \right) \right\rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (329)$$

$$= \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left(e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + e^{\chi_{01}(t)} \right. \right. \quad (330)$$

$$\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \right) \right\rangle_{\overline{H_B}} \text{ (by (279) and (285))}, \quad (331)$$

$$\langle F(h) \rangle_{\overline{H_B}} \equiv \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | F(h) | \alpha \rangle d^2 \alpha \text{ (using the coherent representation with } N = (e^{\beta\omega} - 1)^{-1} \text{)}, \quad (332)$$

$$D(\alpha_{\mathbf{k}}) \equiv e^{\left(\frac{\alpha_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{\alpha_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)}, \quad (333)$$

$$|\alpha\rangle \equiv D(\alpha) |0\rangle, \quad (334)$$

$$\langle \alpha | \equiv \langle 0 | D(-\alpha), \quad (335)$$

$$D(-\alpha) D(h) D(\alpha) \equiv D(h) e^{h\alpha^* - h^* \alpha}, \quad (336)$$

$$D(0) \equiv \mathbb{I}, \quad (337)$$

$$D(-\alpha) D(0) D(\alpha) = D(0) e^{0 \cdot \alpha^* - 0^* \cdot \alpha} \quad (338)$$

$$= D(0) \quad (339)$$

$$= \mathbb{I}, \quad (340)$$

$$D(-\alpha) b^{\dagger} D(\alpha) = b^{\dagger} + \alpha^*, \quad (341)$$

$$D(-\alpha) b D(\alpha) = b + \alpha, \quad (342)$$

$$\langle D(h) \rangle_{\overline{H_B}} = e^{-\frac{|h|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}, \quad (343)$$

$$\langle b^{\dagger} D(h) \rangle_{\overline{H_B}} = -h^* \langle D(h) \rangle_{\overline{H_B}} N \quad (344)$$

$$\langle b D(h) \rangle_B = h \langle D(h) \rangle_B (N+1), \quad (345)$$

$$\langle D(h) b \rangle_B = h N \langle D(h) \rangle_B, \quad (346)$$

$$\langle D(h) b^{\dagger} \rangle_B = -h^* \langle D(h) \rangle_B (N+1) \quad (347)$$

$$\langle B_{iz}(t)B_x(t) \rangle_{\overline{H_B}} = \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left(e^{\chi_{10}(t)} \prod_{\mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right) \right. \quad (348)$$

$$\left. + e^{\chi_{01}(t)} \prod_{\mathbf{k}'} \left(D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \quad (349)$$

$$= \frac{1}{2} \left\langle e^{\chi_{10}(t)} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \prod_{\mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) + e^{\chi_{01}(t)} \right. \quad (350)$$

$$\left. \times \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \prod_{\mathbf{k}'} \left(D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \quad (351)$$

$$= \frac{e^{\chi_{10}(t)}}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \right. \quad (352)$$

$$\times \prod_{\mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \rangle_{\overline{H_B}} + \frac{e^{\chi_{01}(t)}}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right. \right. \quad (353)$$

$$\left. - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} \left(D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \right), \quad (354)$$

$$\langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = e^{-\frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (355)$$

$$N_{\mathbf{k}} = (e^{\beta\omega_{\mathbf{k}}} - 1)^{-1}, \quad (356)$$

$$\langle b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (357)$$

$$\langle b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (358)$$

$$\prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (359)$$

$$\left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} = \left\langle b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) \right\rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (360)$$

$$= -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (361)$$

$$= -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (362)$$

$$= -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (363)$$

$$\left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} = \langle b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (364)$$

$$= \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (365)$$

$$= \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (366)$$

$$= \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (367)$$

$$\langle B_{iz}(t)B_x(t) \rangle_{\overline{H_B}} = \frac{e^{\chi_{10}(t)}}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \right. \quad (368)$$

$$\langle B_{iz}(t)B_y(t) \rangle_{\overline{H_B}} = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right\rangle_{\overline{H_B}} \quad (388)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}}}{2i} + \frac{(B_{10}(t) - B_{01}(t))}{2i} \quad (389)$$

$$\times \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad (390)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}}}{2i} + \frac{(B_{10}(t) - B_{01}(t))}{2i} \cdot 0 \quad (391)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}}}{2i} \quad (392)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left(e^{\chi_{01}(t)} \Pi_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - e^{\chi_{10}(t)} \Pi_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle}{2i} \overline{H_B} \quad (393)$$

$$= \frac{1}{2i} \left\langle \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} e^{\chi_{01}(t)} \prod_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} e^{\chi_{10}(t)} \prod_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \quad (394)$$

$$+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}e^{\chi_{01}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}e^{\chi_{10}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\Bigg\rangle_{\overline{H_B}} \quad (395)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{01}(t)} \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_B}} - e^{\chi_{10}(t)} \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_B}} \right) \quad (396)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* e^{i\chi_{01}(t)} \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\overline{B}}}} - \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\overline{B}}}} \quad (397)$$

$$\times e^{\chi_{10}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \Big) \quad (398)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{01}(t)} \left(- \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) - e^{\chi_{10}(t)} \right) \quad (399)$$

$$\times \left(- \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* e^{\chi_{01}(t)} \quad (400)$$

$$\times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \quad (401)$$

$$\times e^{\chi_{10}(t)} e^{-\sum_{\mathbf{k}} \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (402)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} B_{01}(t) \right) - \left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} B_{10}(t) \right) \right) \quad (403)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \quad (404)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \quad (405)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}B_{01}}(t) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}B_{10}}(t) \right) \quad (406)$$

$$-(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \quad (407)$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \quad (408)$$

$$\langle B_x(t)B_{iz}(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (409)$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \quad (410)$$

$$\times \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (411)$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \cdot 0 \quad (412)$$

$$= \frac{1}{2} \left\langle \left(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) \right) \left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_B}} \quad (413)$$

$$= \frac{1}{2} \sum_{\mathbf{k}} \left\langle \left(e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) + e^{\chi_{01}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \right. \right. \quad (414)$$

$$\left. + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (415)$$

$$\langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}} \rangle_{\overline{H_B}} = \alpha_{\mathbf{k}} N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (416)$$

$$\langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} = -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (417)$$

$$\left\langle \left(\prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} = \langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (418)$$

$$= -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (419)$$

$$= -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (420)$$

$$= -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (421)$$

$$\left\langle \left(\prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} = \langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}} \rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (422)$$

$$= \alpha_{\mathbf{k}} N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (423)$$

$$= \alpha_{\mathbf{k}} N_{\mathbf{k}} \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (424)$$

$$= \alpha_{\mathbf{k}} N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (425)$$

$$\langle B_x(t)B_{iz}(t) \rangle_{\overline{H_B}} = \frac{1}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{10}(t)} \left\langle \left(\prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + e^{\chi_{01}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \right. \quad (426)$$

$$\left. \times \left\langle \left(\prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \left(\prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right. \quad (427)$$

$$\left. \times e^{\chi_{10}(t)} + e^{\chi_{01}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \left(\prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right) \quad (428)$$

$$= \frac{1}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{10}(t)} \left(- \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \right. \quad (429)$$

$$\left. + e^{\chi_{01}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(- \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + e^{\chi_{10}(t)} \right) \quad (430)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) + e^{\chi_{01}(t)} \quad (431)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (432)$$

$$(433)$$

$$(434)$$

$$(435)$$

$$(436)$$

$$(437)$$

$$\times \left\langle \prod_{\mathbf{k}'} \left(D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_{\overline{B}}}} \quad (445)$$

$$-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\bigg)\bigg\rangle_{\overline{H_B}} + \frac{e^{\chi_{01}(t)}}{2} \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \prod_{\mathbf{k}'} \left(D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \right. \quad (447)$$

$$= \frac{e^{\chi_{10}(t)}}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle \prod_{\mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) t_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H\bar{5}}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \prod_{\mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H\bar{5}}} \right) \quad (449)$$

$$-v_{i\mathbf{k}}(t))^* \left\langle \prod_{\mathbf{k}'} \left(D \left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \quad (451)$$

$$-v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \Big) + \frac{B_{01}(t)}{2} \left(\sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) (-g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \right) \quad (453)$$

$$= \frac{B_{10}(t)}{2} \left(\sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left(-(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(N_{\mathbf{k}} + 1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* + g_{i\mathbf{k}} \right. \right. \quad (455)$$

$$-v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \Big) + \frac{B_{01}(t)}{2} \left(\sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \right. \quad (456)$$

$$= \frac{B_{01}(t) - B_{10}(t)}{2} \left(\sum_{\mathbf{k}} e^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(N_{\mathbf{k}} + 1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right. \right. \quad (458)$$

$$- (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (459)$$

$$(460)$$

$$(461)$$

$$(462)$$

$$\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (463)$$

$$= \frac{1}{2i} \left\langle \left(B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t) \right) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (464)$$

$$= \frac{1}{2i} \left\langle \left(B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) \right) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (465)$$

$$= \frac{B_{10}(t)}{2i} \left(\sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(N_{\mathbf{k}} + 1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \right. \quad (466)$$

$$\left. \times N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right) + \frac{B_{01}(t)}{2i} \left(\sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) (- (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(N_{\mathbf{k}} + 1) \right. \quad (467)$$

$$\left. \times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right) \quad (468)$$

$$= \frac{B_{10}(t)}{2i} \left(\sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(N_{\mathbf{k}} + 1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \right. \quad (469)$$

$$\left. \times N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right) + \frac{B_{01}(t)}{2i} \left(\sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(N_{\mathbf{k}} + 1) \right. \quad (470)$$

$$\left. \times \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right) \quad (471)$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))(N_{\mathbf{k}} + 1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right. \quad (472)$$

$$\left. - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right), \quad (473)$$

$$\text{Var}_{\overline{H_B}}(A) \equiv \langle A^2 \rangle_{\overline{H_B}} - \langle A \rangle_{\overline{H_B}}^2, \quad (474)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \text{Var}_{\overline{H_B}}(B_x(t)) + \langle B_x(t) \rangle_{\overline{H_B}}^2 \quad (475)$$

$$= \text{Var}_{\overline{H_B}} \left(\frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (476)$$

$$= \frac{1}{4} \text{Var}_{\overline{H_B}} (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)) \quad (477)$$

$$= \frac{1}{4} \text{Var}_{\overline{H_B}} (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \quad (478)$$

$$= \frac{1}{4} \left(\left\langle (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (479)$$

$$= \frac{1}{4} \left(\left\langle (B_1^+(t) B_0^-(t))^2 + B_1^+(t) B_0^-(t) B_0^+(t) B_1^-(t) + B_0^+(t) B_1^-(t) B_1^+(t) B_0^-(t) + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} \right. \quad (480)$$

$$\left. - (B_{10}(t) + B_{01}(t))^2 \right) \quad (481)$$

$$= \frac{1}{4} \left(\left\langle (B_1^+(t) B_0^-(t))^2 + 2\mathbb{I} + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right), \quad (482)$$

$$(D(h))^2 = D(h) D(h) \quad (483)$$

$$= D(h+h) e^{\frac{1}{2} \left(\frac{h^* h - h h^*}{\omega^2} \right)} \quad (484)$$

$$= D(2h), \quad (485)$$

$$\langle (B_i^+(t) B_j^-(t))^2 \rangle_{\overline{H_B}} = \left\langle \left(\prod_{\mathbf{k}} D \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right)^2 \right\rangle_{\overline{H_B}} \quad (486)$$

$$= \left\langle \prod_{\mathbf{k}} D \left(2 \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} \right\rangle_{\overline{H_B}} \quad (487)$$

$$= \prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} e^{-2 \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (488)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \frac{1}{4} \left(\left\langle (B_1^+(t) B_0^-(t))^2 + 2\mathbb{I} + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (489)$$

$$= \frac{1}{4} \left(\left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} + 2 + \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (490)$$

$$= \frac{1}{4} \left(\left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} + 2 + \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (491)$$

$$= \frac{1}{4} \left(e^{2\chi_{10}(t)} \prod_{\mathbf{k}} e^{-2 \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} + 2 + e^{2\chi_{01}(t)} \prod_{\mathbf{k}} e^{-2 \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) \quad (492)$$

$$- (B_{10}(t) + B_{01}(t))^2 \quad (493)$$

$$= \frac{1}{4} (B_{10}^2(t) |B_{10}^2(t)| + 2 + B_{01}^2(t) |B_{01}^2(t)| - (B_{10}^2(t) + 2B_{10}(t) B_{01}(t) + B_{01}^2(t))) \quad (494)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = \text{Var}_{\overline{H_B}}(B_y(t)) + \langle B_y(t) \rangle_{\overline{H_B}}^2 \quad (495)$$

$$= \text{Var}_{\overline{H_B}} \left(\frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right) \quad (496)$$

$$= -\frac{1}{4} \text{Var}_{\overline{H_B}} (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)) \quad (497)$$

$$= -\frac{1}{4} \text{Var}_{\overline{H_B}} (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \quad (498)$$

$$= -\frac{1}{4} \left(\left\langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t))^2 - (B_{01}(t) - B_{10}(t))^2 \right\rangle_{\overline{H_B}} \right) \quad (499)$$

$$= -\frac{1}{4} \left(\left\langle (B_0^+(t) B_1^-(t))^2 - 2\mathbb{I} + (B_1^+(t) B_0^-(t))^2 - (B_{01}(t) - B_{10}(t))^2 \right\rangle_{\overline{H_B}} \right) \quad (500)$$

$$= -\frac{1}{4} \left(\left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} + \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} - 2 - (B_{01}(t) - B_{10}(t))^2 \right), \quad (501)$$

$$\langle (B_i^+(t) B_j^-(t))^2 \rangle_{\overline{H_B}} = \prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} e^{-2 \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (502)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}} e^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right)^2 \left(\prod_{\mathbf{k}} e^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right)^2 \quad (503)$$

$$= B_{ij}^2(t) |B_{ij}(t)|^2, \quad (504)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = -\frac{1}{4} (B_{01}^2(t) |B_{10}(t)|^2 - 2 + B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}(t) - B_{10}(t))^2), \quad (505)$$

$$\langle B_x(t) B_y(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right\rangle_{\overline{H_B}} \quad (506)$$

$$= \frac{1}{4i} \langle (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)) \rangle_{\overline{H_B}} \quad (507)$$

$$= \frac{1}{4i} \langle \mathbb{I} - (B_1^+(t) B_0^-(t))^2 + B_{10}^2(t) - B_{10}(t) B_{01}(t) + (B_0^+(t) B_1^-(t))^2 - \mathbb{I} + B_{10}(t) B_{01}(t) - B_{01}^2(t) \rangle_{\overline{H_B}} \quad (508)$$

$$= \frac{1}{4i} \left\langle (B_0^+(t) B_1^-(t))^2 - (B_1^+(t) B_0^-(t))^2 - (B_{01}^2(t) - B_{10}^2(t)) \right\rangle_{\overline{H_B}} \quad (509)$$

$$= \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{01}(t)|^2 - (B_{01}^2(t) - B_{10}^2(t))), \quad (510)$$

$$\langle B_y(t) B_x(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_B}} \quad (511)$$

$$= \frac{1}{4i} \langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)) (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)) \rangle_{\overline{H_B}} \quad (512)$$

$$= \frac{1}{4i} \langle \mathbb{I} + (B_0^+(t) B_1^-(t))^2 - B_{10}(t) B_{01}(t) - B_{01}^2(t) - (B_1^+(t) B_0^-(t))^2 - \mathbb{I} + B_{10}^2(t) + B_{10}(t) B_{01}(t) \rangle_{\overline{H_B}} \quad (513)$$

$$= \frac{1}{4i} \langle (B_0^+(t) B_1^-(t))^2 - B_{01}^2(t) - (B_1^+(t) B_0^-(t))^2 + B_{10}^2(t) \rangle_{\overline{H_B}} \quad (514)$$

$$= \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{01}^2(t) - (B_{10}^2(t) |B_{01}(t)|^2 - B_{10}^2(t))). \quad (515)$$

The density matrix associated to $\rho_{\overline{S}} = \frac{e^{-\beta \overline{H_S}(t)}}{\text{Tr}(e^{-\beta \overline{H_S}(t)})}$ follows the form:

$$\rho_{\overline{S},00} = \frac{1}{2} - \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{2 \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (516)$$

$$\rho_{\overline{S},01} = - \frac{B_{10}^*(t) V_{10}^*(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (517)$$

$$\rho_{\overline{S},10} = - \frac{B_{10}(t) V_{10}(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (518)$$

$$\rho_{\overline{S},11} = \frac{1}{2} + \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{2 \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}}. \quad (519)$$

The expected values respect to the system \overline{S} of relevance for calculating $\langle \overline{H_I}^2(t) \rangle_{\overline{H_S}(t)}$ are $\langle |i\rangle\langle i| \rangle_{\overline{H_S}(t)}$, $\langle |i\rangle\langle i| \sigma_x \rangle_{\overline{H_S}(t)}$, $\langle |i\rangle\langle i| \sigma_y \rangle_{\overline{H_S}(t)}$, $\langle \sigma_x |i\rangle\langle i| \rangle_{\overline{H_S}(t)}$, $\langle \sigma_y |i\rangle\langle i| \rangle_{\overline{H_S}(t)}$ and $\langle \sigma_z \rangle_{\overline{H_S}(t)}$, we took account that $\sigma_x \sigma_y = i \sigma_z$ and $\sigma_y \sigma_x = -i \sigma_z$. The values needed for our calculation are:

$$\langle |0\rangle\langle 0| \rangle_{\overline{H_S}(t)} = \frac{1}{2} - \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{2 \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (520)$$

$$\langle |1\rangle\langle 1| \rangle_{\overline{H_S}(t)} = \frac{1}{2} + \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{2 \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (521)$$

$$\langle |0\rangle\langle 0| \sigma_x \rangle_{\overline{H_S}(t)} = - \frac{B_{10}(t) V_{10}(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (522)$$

$$\langle |1\rangle\langle 1| \sigma_x \rangle_{\overline{H_S}(t)} = - \frac{B_{10}^*(t) V_{10}^*(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (523)$$

$$\langle |0\rangle\langle 0| \sigma_y \rangle_{\overline{H_S}(t)} = - \frac{i B_{10}(t) V_{10}(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (524)$$

$$\langle |1\rangle\langle 1| \sigma_y \rangle_{\overline{H_S}(t)} = \frac{i B_{10}^*(t) V_{10}^*(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (525)$$

$$\langle \sigma_x | 0\rangle\langle 0| \rangle_{\overline{H_S}(t)} = - \frac{B_{10}^*(t) V_{10}^*(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (526)$$

$$\langle \sigma_x | 1\rangle\langle 1| \rangle_{\overline{H_S}(t)} = - \frac{B_{10}(t) V_{10}(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (527)$$

$$\langle \sigma_y | 0\rangle\langle 0| \rangle_{\overline{H_S}(t)} = \frac{i B_{10}^*(t) V_{10}^*(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (528)$$

$$\langle \sigma_y | 1\rangle\langle 1| \rangle_{\overline{H_S}(t)} = - \frac{i B_{10}(t) V_{10}(t) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (529)$$

$$\langle \sigma_z \rangle_{\overline{H_S}(t)} = \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}. \quad (530)$$

Summarizing the expected values obtained in (250) respect to the bath we have:

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (531)$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left(e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left(-(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right. \right. \right. \quad (532)$$

$$\left. \left. \left. + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \right) \right), \quad (533)$$

$$\left. \left. \left. + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \right) \right), \quad (534)$$

$$\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} \left(e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right. \right. \right. \quad (535)$$

$$\left. \left. \left. - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \right) \right), \quad (536)$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} = \frac{B_{01}(t) - B_{10}(t)}{2} \sum_{\mathbf{k}} \left(e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (N_{\mathbf{k}} + 1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right. \right. \right. \quad (537)$$

$$\left. \left. \left. - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right) \right), \quad (538)$$

$$\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} \left(e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (N_{\mathbf{k}} + 1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right. \right. \right. \quad (539)$$

$$\left. \left. \left. - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right) \right), \quad (540)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \frac{1}{4} \left(B_{10}^2(t) |B_{10}^2(t)| + 2 + B_{01}^2(t) |B_{01}^2(t)| - (B_{10}(t) + B_{01}(t))^2 \right), \quad (541)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = -\frac{1}{4} \left(B_{01}^2(t) |B_{10}(t)|^2 - 2 + B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}(t) - B_{10}(t))^2 \right), \quad (542)$$

$$\langle B_x(t) B_y(t) \rangle_{\overline{H_B}} = \frac{1}{4i} \left(B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}^2(t) - B_{10}^2(t)) \right), \quad (543)$$

$$\langle B_y(t) B_x(t) \rangle_{\overline{H_B}} = \frac{1}{4i} \left(B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}^2(t) - B_{10}^2(t)) \right). \quad (544)$$

Our next step is to find $v_3(t)$, the commutator $[\overline{H_0}(t), \overline{H_I}(t)]$ is a central point for our calculations and it is equal to:

$$[\overline{H_0(t)}, \overline{H_I(t)}] = \left[(\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) \right] \quad (545)$$

$$+B_{10}^{\mathfrak{S}}(t)V_{10}^{\mathfrak{R}}(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \Big] \quad (546)$$

$$= \left[\sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \right] \quad (547)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_i B_{iz}(t) |i\rangle \langle i| + V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\mathfrak{I}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \Big] \quad (548)$$

$$= \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\mathcal{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\mathcal{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \quad (549)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (550)$$

$$+ \sigma_x \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{S}}(t) \right) V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sigma_y \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{S}}(t) + B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{R}}(t) \right) \sum_i B_{iz}(t) |i\rangle \langle i| \quad (551)$$

$$-\sigma_y \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{S}}(t) + B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{R}}(t) \right) V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) - \sigma_y \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{S}}(t) + B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{R}}(t) \right) V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) \quad (552)$$

$$-\sigma_y B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle \langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (553)$$

$$-\sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{I}}(t) V_{10}^{\mathfrak{I}}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{I}}(t) + B_{10}^{\mathfrak{I}}(t) V_{10}^{\mathfrak{R}}(t) \right) - \sum_i B_{iz}(t) |i\rangle\langle i| \quad (554)$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{-\mathbf{k}} - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (555)$$

$$+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \quad (556)$$

$$\times \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{S}}(t) \right) + V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \quad (557)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \quad (558)$$

$$= \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\mathfrak{Z}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{Z}}(t) V_{10}^{\mathfrak{Z}}(t) \right) \quad (559)$$

$$\times \sum_i B_{iz}(t) |i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (B_x(t) + i\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) \quad (560)$$

$$\times (B_y(t) - i\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) (-i\sigma_z B_x(t) + B_y(t)) \quad (561)$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (-i\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (562)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) \quad (563)$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)) - \sum_i B_{iz}(t)|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Re}(t)(B_x(t) - i\sigma_z B_y(t)) \quad (564)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Re}(t) (i\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (565)$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\mathfrak{S}}(t) (B_y(t) + i\sigma_z B_x(t)) \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{S}}(t) \right) \quad (566)$$

$$+ V_{10}^{\mathfrak{S}}(t) (\mathrm{i} \sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{S}}(t) + B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{R}}(t) \right) - V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \quad (567)$$

The term $\overline{H_I}(t) [\overline{H_0}(t), \overline{H_I}(t)]$ is given by:

$$\overline{H_I}(t) [\overline{H_0}(t), \overline{H_I}(t)] = \left(\sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \right) \left(\sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Re}(t) \right) \quad (568)$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (569)$$

$$\times \sum_i B_{iz}(t) |i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (B_x(t) + i\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (570)$$

$$\times V_{10}^{\Im}(t) (B_y(t) - i\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (571)$$

$$\times V_{10}^{\Re}(t) (-i\sigma_z B_x(t) + B_y(t)) - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (-i\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) \quad (572)$$

$$\times |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \quad (573)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (574)$$

$$- V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t) (B_x(t) - i\sigma_z B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (575)$$

$$+ V_{10}^{\Re}(t) (i\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (576)$$

$$- V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (B_y(t) + i\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (577)$$

$$+ V_{10}^{\Im}(t) (i\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (578)$$

$$= \sum_i B_{iz}(t) |i\rangle \langle i| \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i B_{iz}(t) |i\rangle \langle i| \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\mathfrak{I}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (579)$$

$$+ \sum_i B_{iz}(t) |i\rangle \langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}(t) |i\rangle \langle i| + \sum_i B_{iz}(t) |i\rangle \langle i| \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (B_x(t) \quad (580)$$

$$+i\sigma_z B_y(t) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) (B_y(t) - i\sigma_z B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right. \quad (581)$$

$$+B_{10}^{\mathfrak{S}}(t)V_{10}^{\mathfrak{R}}(t))\sum_i B_{iz}(t)|i\rangle\langle i| - \sum_i B_{iz}(t)|i\rangle\langle i|\left(B_{10}^{\mathfrak{R}}(t)V_{10}^{\mathfrak{S}}(t) + B_{10}^{\mathfrak{S}}(t)V_{10}^{\mathfrak{R}}(t)\right)V_{10}^{\mathfrak{R}}(t)(-i\sigma_z B_x(t) + B_y(t)) - \sum_i B_{iz}(t)|i\rangle\langle i| \quad (582)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (-i\sigma_z B_y(t) - B_x(t)) + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (583)$$

$$\times V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \quad (584)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - \sum_i B_{iz}(t) |i\rangle\langle i| \sum_i B_{iz}(t) |i\rangle\langle i| \quad (585)$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{R}}(t) (B_x(t) - i\sigma_z B_y(t)) \quad (586)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Re}(t) (i\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Re}(t) \quad (587)$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{Z}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{Z}}(t) \quad (588)$$

$$\times (B_y(t) + i\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Im}(t) (i\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (589)$$

$$-\sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (590)$$

$$+\sigma_y B_y(t)) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_x \quad (591)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}(t) |i\rangle \langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (B_x(t) + i\sigma_z B_y(t)) \quad (592)$$

$$+ V_{10}^{\mathcal{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_x \left(B_{10}^{\mathcal{R}}(t) V_{10}^{\mathcal{R}}(t) - B_{10}^{\mathcal{I}}(t) V_{10}^{\mathcal{I}}(t) \right) V_{10}^{\mathcal{I}}(t) (B_y(t) - i\sigma_z B_x(t)) - V_{10}^{\mathcal{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_y \quad (593)$$

$$X \left(D_{10}^{\text{reg}}(t) \vee_{10}(t) + D_{10}^{\text{reg}}(t) \vee_{10}(t) \right) \sum_i D_{iz}(t) |t| \wedge |t| (-\vee_{10}(t) (x X(t) + y Y(t))) \left(D_{10}^{\text{reg}}(t) \vee_{10}(t) + D_{10}^{\text{reg}}(t) \vee_{10}(t) \right) \vee_{10}(t) (-10z X(t) + D_y(t)) \quad (594)$$

$$= \sum_{\mathbf{k}} \mathbf{B}_{\mathbf{k}}^{\dagger}(\ell) \left[\mathbf{V}_{10}(\ell) + \mathbf{V}_{10}^{\mathcal{R}}(\ell) \right] + \mathbf{B}_{\mathbf{k}}(\ell) + \mathbf{B}_{\mathbf{k}}(\ell) \sum_{\mathbf{k}'} \mathbf{J}_{\mathbf{k}\mathbf{k}'}^{\dagger} \mathbf{V}_{10}^{\mathcal{R}}(\ell) + \mathbf{B}_{\mathbf{k}}(\ell) + \mathbf{B}_{\mathbf{k}}(\ell) + \mathbf{V}_{10}^{\mathcal{R}}(\ell) + \mathbf{B}_{\mathbf{k}}(\ell) + \mathbf{B}_{\mathbf{k}}(\ell) \sum_{\mathbf{k}'} \mathbf{J}_{\mathbf{k}\mathbf{k}'}^{\dagger} \quad (50c)$$

$$+ \sum_k \omega_k \mathbf{k} \omega_k \left(\frac{1}{10} (v) (\partial_x D_x(v) + \partial_y D_y(v)) + \frac{1}{10} (v) (\partial_x D_x(v) + \partial_y D_y(v)) \right) \sum_k \omega_k \mathbf{k} \omega_k (v) \\ + V^{\text{ex}}(v) (- P_+(v) - P_-(v)) + U^{\text{ex}}(v) (- P_+(v) - P_-(v)) \sum_k P_+(v) + \frac{1}{2} (v) (U^{\text{ex}}(v) U^{\text{ex}}(v) - P^{\text{ex}}(v) V^{\text{ex}}(v)) + U^{\text{ex}}(v) (- P_+(v) - P_-(v)) \quad (507)$$

$$+ \frac{1}{\sqrt{10}} \left(\langle \phi_x | L_y | \phi \rangle - \langle \phi_y | L_x | \phi \rangle \right) + \frac{1}{\sqrt{10}} \left(\langle \phi_x | L_x | \phi \rangle + \langle \phi_y | L_y | \phi \rangle \right) \sum_i L_{2i}(\phi) |\phi\rangle \langle \phi| \left(L_{10}^+(\phi) \sqrt{10}(\phi) - L_{10}^-(\phi) \sqrt{10}(\phi) \right) + \frac{1}{\sqrt{10}} \left(\langle \phi_x | L_x | \phi \rangle - \langle \phi_y | L_y | \phi \rangle \right) \sum_i L_{2i}(\phi) |\phi\rangle \langle \phi| \left(L_{10}^+(\phi) \sqrt{10}(\phi) + L_{10}^-(\phi) \sqrt{10}(\phi) \right) \quad (59)$$

$$+ \partial_y D_y(v) \sum_i D_{iz}(v) |\lambda| \lambda_i \partial_y (D_{10}(v) v_{10}(v) + D_{10}(v) v_{10}(v)) - v_{10}(v) (\partial_x D_x(v) + \partial_y D_y(v)) \sum_i D_{iz}(v) |\lambda| \lambda_i \sum_{\mathbf{k}} \omega_{\mathbf{k}} \mathbf{k} \cdot \mathbf{k} - v_{10}(v) \quad (59b)$$

$$+ \frac{1}{10}(\psi)(\psi_X L_X(\psi) + \psi_Y L_Y(\psi)) + \frac{1}{10}(\psi)(L_X(\psi) + L_Y L_Y(\psi)) \quad (5.5)$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i \omega_k \mathbf{k}_k^\dagger \mathbf{k}_k - V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i t) \quad (601)$$

$$+R_i(t)|i\rangle\langle i| - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))V_{10}^{\Im}(t)(B_y(t) + \mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (602)$$

$$\times V_{10}^{\mathfrak{Z}}(t)(i\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{Z}}(t) + B_{10}^{\mathfrak{Z}}(t) V_{10}^{\mathfrak{R}}(t) \right) - V_{10}^{\mathfrak{R}}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\mathfrak{Z}}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (603)$$

$$+ V_{10}^{\mathfrak{S}}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\mathfrak{S}}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| \quad (604)$$

$$\times V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{S}}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (605)$$

$$-\sigma_y B_x(t)) \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{I}}(t) V_{10}^{\mathfrak{I}}(t) \right) V_{10}^{\mathfrak{R}}(t) (B_x(t) + i\sigma_z B_y(t)) + V_{10}^{\mathfrak{I}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{I}}(t) V_{10}^{\mathfrak{I}}(t) \right) \quad (606)$$

$$\times V_{10}^{\mathfrak{S}}(t) (B_y(t) - \mathrm{i}\sigma_z B_x(t)) - V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \left(B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{S}}(t) + B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{R}}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| - V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \quad (607)$$

$$-\sigma_y B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) (-i\sigma_z B_x(t) + B_y(t)) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (608)$$

$$\times V_{10}^{\Im}(t) (-i\sigma_z B_y(t) - B_x(t)) + V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (609)$$

$$\times V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (610)$$

$$\times \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (611)$$

$$- V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) \quad (612)$$

$$\times |i\rangle\langle i| - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (B_x(t) - i\sigma_z B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) \quad (613)$$

$$\times (i\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (614)$$

$$- V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (B_y(t) \quad (615)$$

$$+ i\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (i\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (616)$$

$$- V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (617)$$

$$= V_{10}^{\Re}(t) \sum_i (\varepsilon_i(t) + R_i(t)) (|i\rangle\langle i| \sigma_x B_{iz}(t) B_x(t) + |i\rangle\langle i| \sigma_y B_{iz}(t) B_y(t)) + V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t)) (|i\rangle\langle i| \sigma_x B_{iz}(t) B_y(t) - |i\rangle\langle i| \sigma_y \quad (618)$$

$$\times B_{iz}(t) B_x(t)) + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_{i \neq i'} B_{iz}(t) B_{i'z}(t) |i\rangle\langle i'| + \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) - B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) \sum_i (|i\rangle\langle i| B_{iz}(t) \quad (619)$$

$$\times B_x(t) + |i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t)) + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) \sum_i (|i\rangle\langle i| \sigma_x B_{iz}(t) B_y(t) - |i\rangle\langle i| \sigma_y B_{iz}(t) B_x(t)) - \left(B_{10}^{\Re}(t) \quad (620)$$

$$\times V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_{i \neq i'} B_{iz}(t) B_{i'z}(t) |i\rangle\langle i| \sigma_y |i'\rangle\langle i'| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) \sum_i (-|i\rangle\langle i| \sigma_z B_{iz}(t) B_x(t) + B_{iz}(t) \quad (621)$$

$$\times B_y(t) |i\rangle\langle i|) + \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) \sum_i (|i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t) + |i\rangle\langle i| B_{iz}(t) B_x(t)) + \sum_{i, \mathbf{k}} |i\rangle\langle i| B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) \quad (622)$$

$$+ V_{10}^{\Re}(t) \sum_{i, \mathbf{k}} \left(|i\rangle\langle i| \sigma_x B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + |i\rangle\langle i| \sigma_y B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) \right) + V_{10}^{\Im}(t) \sum_{i, \mathbf{k}} \left(|i\rangle\langle i| \sigma_x B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - |i\rangle\langle i| \sigma_y B_{iz}(t) \quad (623)$$

$$\times \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) \right) - \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}^2(t) |i\rangle\langle i| \sigma_x + \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i B_{iz}^2(t) |i\rangle\langle i| \sigma_y - \sum_{i, \mathbf{k}} |i\rangle\langle i| \quad (624)$$

$$\times B_{iz}^2(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t) \sum_{i, i'} (\varepsilon_{i'}(t) + R_{i'}(t)) (|i\rangle\langle i| \sigma_x |i'\rangle\langle i'| B_{iz}(t) B_x(t) + |i\rangle\langle i| \sigma_y |i'\rangle\langle i'| B_{iz}(t) B_y(t)) - V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) \quad (625)$$

$$\times V_{10}^{\Im}(t) \right) \sum_i (|i\rangle\langle i| B_{iz}(t) B_x(t) - |i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t)) + V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i (|i\rangle\langle i| \sigma_z B_{iz}(t) B_x(t) + |i\rangle\langle i| \quad (626)$$

$$\times B_{iz}(t) B_y(t)) - V_{10}^{\Im}(t) \sum_{i, \mathbf{k}} \left(|i\rangle\langle i| \sigma_x B_{iz}(t) B_x(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |i\rangle\langle i| \sigma_y B_{iz}(t) B_y(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) - V_{10}^{\Re}(t) \sum_{i \neq i'} (\varepsilon_{i'}(t) + R_{i'}(t)) (|i\rangle\langle i| \sigma_x |i'\rangle\langle i'| \quad (627)$$

$$\times B_{iz}(t) B_y(t) - |i\rangle\langle i| \sigma_y |i'\rangle\langle i'| B_{iz}(t) B_x(t)) - V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i (|i\rangle\langle i| B_{iz}(t) B_y(t) + |i\rangle\langle i| \sigma_z B_x(t)) + V_{10}^{\Re}(t) \quad (628)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i (|i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t) - |i\rangle\langle i| B_{iz}(t) B_x(t)) - V_{10}^{\Im}(t) \sum_{i, \mathbf{k}} \left(|i\rangle\langle i| \sigma_x B_{iz}(t) B_y(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - |i\rangle\langle i| \sigma_y \quad (629)$$

$$\times B_{iz}(t) B_x(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) + \left(V_{10}^{\Re}(t) \right)^2 \sum_i (\varepsilon_i(t) + R_i(t)) (\sigma_x |i\rangle\langle i| \sigma_x B_x^2(t) + \sigma_x |i\rangle\langle i| \sigma_y B_x(t) B_y(t) + \sigma_y |i\rangle\langle i| \sigma_x B_y(t) B_x(t) + \sigma_y |i\rangle\langle i| \quad (630)$$

$$\times \sigma_y B_y^2(t)) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t)) (\sigma_x |i\rangle\langle i| \sigma_x B_x(t) B_y(t) - \sigma_x |i\rangle\langle i| \sigma_y B_x^2(t) + \sigma_y |i\rangle\langle i| \sigma_x B_y^2(t) - \sigma_y |i\rangle\langle i| \sigma_y B_y(t) B_x(t)) \quad (631)$$

$$+ V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i (|i\rangle\langle i| B_x(t) B_{iz}(t) - i\sigma_z |i\rangle\langle i| B_y(t) B_{iz}(t)) + \left(V_{10}^{\Re}(t) \right)^2 \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (632)$$

$$\times (\sigma_x B_x^2(t) + \sigma_y B_x(t) B_y(t) + \sigma_y B_y(t) B_x(t) - \sigma_x B_y^2(t)) + V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) (B_x(t) B_y(t) - i\sigma_z B_y^2(t) \quad (633)$$

$$- i\sigma_z B_x^2(t) - B_y(t) B_x(t)) - V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i (i\sigma_z |i\rangle\langle i| B_x(t) B_{iz}(t) + |i\rangle\langle i| B_y(t) B_{iz}(t)) - V_{10}^{\Re}(t) \quad (634)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) (-\sigma_y B_x^2(t) + \sigma_x B_y(t) B_x(t) + \sigma_x B_x(t) B_y(t) + \sigma_y B_y^2(t)) - V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) \quad (635)$$

$$\times V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (-\sigma_y B_x(t) B_y(t) + \sigma_x B_y^2(t) - \sigma_x B_x^2(t) - \sigma_y B_y(t) B_x(t)) + V_{10}^{\Re}(t) \sum_{i, \mathbf{k}} (\sigma_x |i\rangle\langle i| B_x(t) + \sigma_y |i\rangle\langle i| B_y(t)) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) \quad (636)$$

$$\begin{aligned}
& + \left(V_{10}^{\Re}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) - i\sigma_z B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) + B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) \right) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \quad (637) \\
& \times \left(B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - i\sigma_z B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) - B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) \right) - V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (638) \\
& \times \sum_i \left(\sigma_x |i\rangle\langle i| \sigma_x B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| \sigma_x B_y(t) B_{iz}(t) \right) + V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i \left(\sigma_x |i\rangle\langle i| \sigma_y B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| \right. \quad (639) \\
& \times \sigma_y B_y(t) B_{iz}(t) \left. - V_{10}^{\Re}(t) \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) - \left(V_{10}^{\Re}(t) \right)^2 \sum_i \left(\varepsilon_i(t) + R_i(t) \right) \left(|i\rangle\langle i| B_x^2(t) \right. \right. \quad (640) \\
& - i\sigma_z |i\rangle\langle i| B_y(t) B_x(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_y(t) + |i\rangle\langle i| B_y^2(t) \left. - \left(V_{10}^{\Re}(t) \right)^2 \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left(\sigma_x B_x^2(t) + \sigma_y B_y(t) B_x(t) \right. \right. \quad (641) \\
& - \sigma_y B_x(t) B_y(t) + \sigma_x B_y^2(t) \left. \right) + \left(V_{10}^{\Re}(t) \right)^2 \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left(\sigma_y B_x^2(t) - \sigma_x B_y(t) B_x(t) + \sigma_x B_x(t) B_y(t) + \sigma_y B_y^2(t) \right) \quad (642) \\
& - \left(V_{10}^{\Re}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_x^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) - V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_i \left(\varepsilon_i(t) + R_i(t) \right) \quad (643) \\
& \times \left(|i\rangle\langle i| B_x(t) B_y(t) - i\sigma_z |i\rangle\langle i| B_y^2(t) - i\sigma_z |i\rangle\langle i| B_x^2(t) - |i\rangle\langle i| B_y(t) B_x(t) \right) - V_{10}^{\Re}(t) V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left(\sigma_x B_x(t) \right. \quad (644) \\
& \times B_y(t) + \sigma_y B_y^2(t) + \sigma_y B_x^2(t) - \sigma_x B_y(t) B_x(t) \left. \right) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left(\sigma_y B_x(t) B_y(t) - \sigma_x B_y^2(t) - \sigma_x B_x^2(t) \right) \quad (645) \\
& - \sigma_y B_y(t) B_x(t) - V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_x^2(t) - B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) + V_{10}^{\Im}(t) V_{10}^{\Re}(t) \sum_i \left(\varepsilon_i(t) \right. \quad (646) \\
& + R_i(t) \left. \right) \left(\sigma_x |i\rangle\langle i| \sigma_x B_y(t) B_x(t) - \sigma_y |i\rangle\langle i| \sigma_x B_x^2(t) + \sigma_x |i\rangle\langle i| \sigma_y B_y^2(t) - \sigma_y |i\rangle\langle i| \sigma_y B_x(t) B_y(t) \right) + \left(V_{10}^{\Im}(t) \right)^2 \sum_i \left(\varepsilon_i(t) + R_i(t) \right) \left(\sigma_x |i\rangle\langle i| \right. \quad (647) \\
& \times \sigma_x B_y^2(t) - \sigma_y |i\rangle\langle i| \sigma_x B_x(t) B_y(t) - \sigma_x |i\rangle\langle i| \sigma_y B_y(t) B_x(t) + \sigma_y |i\rangle\langle i| \sigma_y B_x^2(t) \left. \right) + V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i \left(|i\rangle\langle i| B_y(t) \right. \quad (648) \\
& \times B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_{iz}(t) \left. \right) + V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) \left(\sigma_x B_y(t) B_x(t) - \sigma_y B_x^2(t) + \sigma_y B_y^2(t) + \sigma_x B_x(t) B_y(t) \right) \quad (649) \\
& + \left(V_{10}^{\Im}(t) \right)^2 \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left(B_y^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z B_y(t) B_x(t) + B_x^2(t) \right) - V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (650) \\
& \times \sum_i \left(i\sigma_z |i\rangle\langle i| B_y(t) B_{iz}(t) - |i\rangle\langle i| B_x(t) B_{iz}(t) \right) - V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) \left(-\sigma_y B_y(t) B_x(t) - \sigma_x B_x^2(t) + \sigma_x B_y^2(t) \right. \quad (651) \\
& - \sigma_y B_x(t) B_y(t) \left. \right) - \left(V_{10}^{\Im}(t) \right)^2 \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left(-\sigma_y B_y^2(t) - \sigma_x B_x(t) B_y(t) - \sigma_x B_y(t) B_x(t) + \sigma_y B_x^2(t) \right) + V_{10}^{\Im}(t) \quad (652) \\
& \times \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x |i\rangle\langle i| B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) \right) + V_{10}^{\Im}(t) V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + i\sigma_z \right. \quad (653) \\
& \times B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) \left. \right) + \left(V_{10}^{\Im}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) + i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - i\sigma_z B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + B_x(t) \right. \quad (654) \\
& \times b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) \left. \right) - V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i \left(\sigma_x |i\rangle\langle i| \sigma_x B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| \sigma_x B_x(t) B_{iz}(t) \right) + V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right. \quad (655) \\
& + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \left. \right) \sum_i \left(\sigma_x |i\rangle\langle i| \sigma_y B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| \sigma_y B_x(t) B_{iz}(t) \right) - V_{10}^{\Im}(t) \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) \quad (656) \\
& - V_{10}^{\Im}(t) V_{10}^{\Re}(t) \sum_i \left(\varepsilon_i(t) + R_i(t) \right) \left(|i\rangle\langle i| B_y(t) B_x(t) + i\sigma_z |i\rangle\langle i| B_x^2(t) + i\sigma_z |i\rangle\langle i| B_y^2(t) - |i\rangle\langle i| B_x(t) B_y(t) \right) - V_{10}^{\Im}(t) V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) \right. \quad (657) \\
& - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \left. \right) \left(\sigma_x B_y(t) B_x(t) - \sigma_y B_x^2(t) - \sigma_y B_y^2(t) - \sigma_x B_x(t) B_y(t) \right) + V_{10}^{\Im}(t) V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left(\sigma_y B_y(t) B_x(t) \right. \quad (658) \\
& + \sigma_x B_x^2(t) + \sigma_x B_y^2(t) - \sigma_y B_x(t) B_y(t) \left. \right) - \sum_{\mathbf{k}} V_{10}^{\Im}(t) V_{10}^{\Re}(t) \omega_{\mathbf{k}} \left(B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_x^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) \quad (659) \\
& - \left(V_{10}^{\Im}(t) \right)^2 \sum_i \left(\varepsilon_i(t) + R_i(t) \right) \left(|i\rangle\langle i| B_y^2(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_y(t) - i\sigma_z |i\rangle\langle i| B_y(t) B_x(t) + |i\rangle\langle i| B_x^2(t) \right) - \left(V_{10}^{\Im}(t) \right)^2 \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) \right. \quad (660) \\
& - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \left. \right) \left(\sigma_x B_y^2(t) - \sigma_y B_x(t) B_y(t) + \sigma_y B_y(t) B_x(t) + \sigma_x B_x^2(t) \right) + \left(V_{10}^{\Im}(t) \right)^2 \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left(\sigma_y B_y^2(t) \right. \quad (661) \\
& + \sigma_x B_x(t) B_y(t) - \sigma_x B_y(t) B_x(t) + \sigma_y B_x^2(t) \left. \right) - \left(V_{10}^{\Im}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + B_x^2(t) \right. \quad (662) \\
& \times b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \left. \right), \quad (663)
\end{aligned}$$

Now let's obtain the form of $\overline{H_T}^3(t)$:

$$\overline{H_T}^3(t) = \left(\sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \right) \left(\sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x \right. \quad (664)$$

$$+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) \quad (665)$$

$$+\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+i\sigma_zB_x(t)B_y(t)-i\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2 \quad (666)$$

$$\times\left(B_y^2(t)+B_x^2(t)-i\sigma_zB_y(t)B_x(t)+i\sigma_zB_x(t)B_y(t)\right) \quad (667)$$

$$=\sum_iB_{iz}(t)|i\rangle\langle i|\sum_iB_{iz}^2(t)|i\rangle\langle i|+\sum_iB_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+\sum_iB_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t) \quad (668)$$

$$\times\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+\sum_iB_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\sum_iB_{iz}(t)|i\rangle\langle i| \quad (669)$$

$$\times\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+i\sigma_zB_x(t)B_y(t)-i\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+\sum_iB_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) \quad (670)$$

$$+\sum_iB_{iz}(t)|i\rangle\langle i|\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-i\sigma_zB_y(t)B_x(t)+i\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\sum_iB_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t) \quad (671)$$

$$\times(\sigma_xB_x(t)+\sigma_yB_y(t))V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t) \quad (672)$$

$$\times|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t) \quad (673)$$

$$+\sigma_yB_y(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+i\sigma_zB_x(t)B_y(t)-i\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) \quad (674)$$

$$-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-i\sigma_zB_y(t)B_x(t)+i\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t) \quad (675)$$

$$-\sigma_yB_x(t))\sum_iB_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_y \quad (676)$$

$$\times B_x(t))V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i| \quad (677)$$

$$\times B_y(t)B_{iz}(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+i\sigma_zB_x(t)B_y(t)-i\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t)) \quad (678)$$

$$\times V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-i\sigma_zB_y(t)B_x(t) \quad (679)$$

$$+i\sigma_zB_x(t)B_y(t)) \quad (680)$$

$$=\sum_iB_{iz}^3(t)|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i(B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_y) \quad (681)$$

$$+V_{10}^{\Re}(t)\sum_{i\neq i'}(|i'\rangle\langle i'|)\sigma_x|i\rangle\langle i|B_{i'z}(t)B_x(t)B_{iz}(t)+|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2\sum_i(|i\rangle\langle i|B_{iz}(t)B_x^2(t)+i|i\rangle\langle i|\sigma_zB_{iz}(t) \quad (682)$$

$$\times B_x(t)B_y(t)-i|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+|i\rangle\langle i|B_{iz}(t)B_y^2(t))+V_{10}^{\Im}(t)\sum_{i\neq i'}(|i'\rangle\langle i'|)\sigma_x|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)-|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t) \quad (683)$$

$$B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2\sum_i(|i\rangle\langle i|B_{iz}(t)B_y^2(t)+|i\rangle\langle i|B_{iz}(t)B_x^2(t)-i|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+i|i\rangle\langle i|\sigma_zB_{iz}(t)B_x(t)B_y(t))+V_{10}^{\Re}(t) \quad (684)$$

$$\times\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}^2(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}^2(t))+\left(V_{10}^{\Re}(t)\right)^2\sum_i(B_x(t)B_{iz}(t)B_x(t)\sigma_x|i\rangle\langle i|\sigma_x+B_x(t)B_{iz}(t)B_y(t)\sigma_x|i\rangle\langle i|\sigma_y+B_y(t) \quad (685)$$

$$\times B_{iz}(t)B_x(t)\sigma_y|i\rangle\langle i|\sigma_x+B_y(t)B_{iz}(t)B_y(t)\sigma_y|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(B_x(t)B_{iz}(t)B_y(t)\sigma_x|i\rangle\langle i|\sigma_x-B_x(t)B_{iz}(t)B_x(t)\sigma_x|i\rangle\langle i|\sigma_y \quad (686)$$

$$+B_y(t)B_{iz}(t)B_y(t)\sigma_y|i\rangle\langle i|\sigma_x-B_y(t)B_{iz}(t)B_x(t)\sigma_y|i\rangle\langle i|\sigma_y)+\left(V_{10}^{\Re}(t)\right)^2\sum_i(|i\rangle\langle i|B_x^2(t)B_{iz}(t)+i\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-i\sigma_z|i\rangle\langle i| \quad (687)$$

$$\times B_y(t)B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y^2(t)B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^3(\sigma_xB_x^3(t)+\sigma_yB_x^2(t)B_y(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_xB_x(t)B_y^2(t)+\sigma_yB_y(t)B_x^2(t) \quad (688)$$

$$-\sigma_xB_y(t)B_x(t)B_y(t)+\sigma_xB_y^2(t)B_x(t)+\sigma_yB_y^3(t))+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-i\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-i|i\rangle\langle i|\sigma_zB_y^2(t) \quad (689)$$

$$\times B_{iz}(t)+i|i\rangle\langle i|\sigma_zB_y(t)B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)\left(V_{10}^{\Im}(t)\right)^2(\sigma_xB_x(t)B_y^2(t)+\sigma_xB_x^3(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_yB_x^2(t)B_y(t)+\sigma_yB_y^3(t) \quad (690)$$

$$+\sigma_yB_y(t)B_x^2(t)+\sigma_xB_y^2(t)B_x(t)-\sigma_xB_y(t)B_x(t)B_y(t))+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}^2(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}^2(t))+V_{10}^{\Im}(t)V_{10}^{\Im}(t) \quad (691)$$

$$\times(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_x(t)+\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t)-\sigma_y|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t))+\left(V_{10}^{\Im}(t)\right)^2 \quad (692)$$

$$\times(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_x(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x(t)B_{iz}(t)B_y(t)+\sigma_y|i\rangle\langle i|\sigma_yB_x(t)B_{iz}(t)B_x(t))+V_{10}^{\Re}(t) \quad (693)$$

$$\times V_{10}^{\mathfrak{S}}(t) \sum_i (|i\rangle\langle i| B_y(t) B_x(t) B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_y^2(t) B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_x^2(t) B_{iz}(t) - |i\rangle\langle i| B_x(t) B_y(t) B_{iz}(t)) + V_{10}^{\mathfrak{S}}(t) \left(V_{10}^{\mathfrak{R}}(t)\right)^2 \quad (694)$$

$$\times (\sigma_x B_y(t) B_x^2(t) + \sigma_y B_y(t) B_x(t) B_y(t) - \sigma_y B_y^2(t) B_x(t) + \sigma_x B_y^3(t) - \sigma_y B_x^3(t) + \sigma_x B_x^2(t) B_y(t) - \sigma_x B_x(t) B_y(t) B_x(t) - \sigma_y B_x(t) B_y^2(t)) \quad (695)$$

$$+ \left(V_{10}^{\mathfrak{S}}(t)\right)^2 \sum_i (|i\rangle\langle i| B_y^2(t) B_{iz}(t) - i\sigma_z |i\rangle\langle i| B_y(t) B_x(t) B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_y(t) B_{iz}(t) + |i\rangle\langle i| B_x^2(t) B_{iz}(t)) + \left(V_{10}^{\mathfrak{S}}(t)\right)^3 (\sigma_x B_y^3(t) \quad (696)$$

$$+ \sigma_x B_y(t) B_x^2(t) - \sigma_y B_y^2(t) B_x(t) + \sigma_y B_y(t) B_x(t) B_y(t) - \sigma_y B_x(t) B_y^2(t) - \sigma_y B_x^3(t) - \sigma_x B_x(t) B_y(t) B_x(t) + \sigma_x B_x^2(t) B_y(t)) . \quad (697)$$

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