

A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1)$$

$$H_S(t) = \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|, \quad (2)$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (3)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (4)$$

For the states $|0\rangle, |1\rangle$ we have the orthonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \quad (5)$$

I. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right). \quad (6)$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}(t)$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}}(t). \quad (7)$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \quad (8)$$

$$V(t) = |0\rangle\langle 0| \hat{\varphi}_0(t) + |1\rangle\langle 1| \hat{\varphi}_1(t). \quad (9)$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V(t)}$ using the notation (6) will give us the following result:

$$e^{\pm V(t)} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t))} \quad (10)$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)))^2}{2!} + \dots \quad (11)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2(t)}{2!} + \dots \quad (12)$$

$$= |0\rangle\langle 0| \left(\mathbb{I} \pm \hat{\varphi}_0(t) + \frac{\hat{\varphi}_0^2(t)}{2!} \pm \dots \right) + |1\rangle\langle 1| \left(\mathbb{I} \pm \hat{\varphi}_1(t) + \frac{\hat{\varphi}_1^2(t)}{2!} \pm \dots \right) \quad (13)$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1(t)} \quad (14)$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*(t)b_{\mathbf{k}})} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*(t)b_{\mathbf{k}})} \quad (15)$$

$$= |0\rangle\langle 0|B_0^\pm(t) + |1\rangle\langle 1|B_1^\pm(t), \quad (16)$$

$$B_i^\pm(t) \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (17)$$

Let's recall the Zassenhaus formula:

$$e^{r(X+Y)} = e^{rX} e^{rY} e^{-\frac{r^2}{2}[X,Y]} e^{\frac{r^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} e^{-\frac{r^4}{24}([[[X,Y],X],X] + 3[[[X,Y],X],Y] + 3[[[X,Y],Y],Y])} \dots \quad (18)$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}, i, j$ we can show plugging $r = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right)} = e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right]} \dots \quad (19)$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}0} \dots \quad (20)$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} \mathbb{I} \quad (21)$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}}. \quad (22)$$

By induction of this result we can write an expression of $B_i^\pm(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$D(\pm v_{i\mathbf{k}}(t)) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, \quad (23)$$

$$B_i^\pm(t) = \prod_{\mathbf{k}} D\left(\pm \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). \quad (24)$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$\overline{O(t)} \equiv e^{V(t)} O(t) e^{-V(t)}. \quad (25)$$

We will use the following identities:

$$\overline{|0\rangle\langle 0|}(t) = e^{V(t)}|0\rangle\langle 0|e^{-V(t)} \quad (26)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|0\rangle\langle 0|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (27)$$

$$= (|0\rangle\langle 0|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (28)$$

$$= |0\rangle\langle 0|B_0^+(t)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (29)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)B_1^-(t) \quad (30)$$

$$= |0\rangle\langle 0|, \quad (31)$$

$$\overline{|1\rangle\langle 1|}(t) = (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|1\rangle\langle 1|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (32)$$

$$= (|0\rangle\langle 0|1\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (33)$$

$$= |1\rangle\langle 1|B_1^+(t)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (34)$$

$$= |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)B_0^-(t) + B_1^+(t)|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) \quad (35)$$

$$= B_1^+(t)|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) \quad (36)$$

$$= |1\rangle\langle 1|, \quad (37)$$

$$\overline{|0\rangle\langle 1|}(t) = e^{V(t)}|0\rangle\langle 1|e^{-V(t)} \quad (38)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|0\rangle\langle 1|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (39)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)|0\rangle\langle 1|)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (40)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (41)$$

$$= |0\rangle\langle 1|B_0^+(t)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (42)$$

$$= |0\rangle\langle 1|0\rangle\langle 0|B_0^+(t)B_0^-(t) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+(t)B_1^-(t) \quad (43)$$

$$= |0\rangle\langle 1|B_0^+(t)B_1^-(t), \quad (44)$$

$$\overline{|1\rangle\langle 0|}(t) = e^{V(t)}|1\rangle\langle 0|e^{-V(t)} \quad (45)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|1\rangle\langle 0|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (46)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)|1\rangle\langle 0|)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (47)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (48)$$

$$= |1\rangle\langle 0|0\rangle\langle 0|B_1^+(t)B_0^-(t) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+(t)B_1^-(t) \quad (49)$$

$$= |1\rangle\langle 0|B_1^+(t)B_0^-(t), \quad (50)$$

$$\overline{b_{\mathbf{k}}}(t) = e^{V(t)}b_{\mathbf{k}}e^{-V(t)} \quad (51)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))b_{\mathbf{k}}(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (52)$$

$$= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| \quad (53)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t) \quad (54)$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (55)$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (56)$$

$$= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (57)$$

$$\overline{b_{\mathbf{k}}(t)}^\dagger = e^{V(t)}b_{\mathbf{k}}^\dagger e^{-V(t)} \quad (58)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))b_{\mathbf{k}}^\dagger(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (59)$$

$$= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^\dagger B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^\dagger|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^\dagger|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^\dagger B_1^-(t)|1\rangle\langle 1| \quad (60)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^\dagger B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^\dagger B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^\dagger B_0^-(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^\dagger B_1^-(t) \quad (61)$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) \quad (62)$$

$$= b_{\mathbf{k}}^\dagger - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. \quad (63)$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$:

$$B_i^+(t) b_{\mathbf{k}} B_i^-(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \quad (64)$$

$$B_i^+(t) b_{\mathbf{k}}^\dagger B_i^-(t) = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. \quad (65)$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t) |0\rangle\langle 0| (t)} = \varepsilon_0(t) |0\rangle\langle 0|, \quad (66)$$

$$\overline{\varepsilon_1(t) |1\rangle\langle 1| (t)} = \varepsilon_1(t) |1\rangle\langle 1|, \quad (67)$$

$$\overline{V_{10}(t) |1\rangle\langle 0| (t)} = V_{10}(t) |1\rangle\langle 0| B_1^+(t) B_0^-(t), \quad (68)$$

$$\overline{V_{01}(t) |0\rangle\langle 1| (t)} = V_{01}(t) |0\rangle\langle 1| B_0^+(t) B_1^-(t), \quad (69)$$

$$\overline{(g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}})(t)} = g_{i\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) \right) + g_{i\mathbf{k}}^* \left(|0\rangle\langle 0| \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (70)$$

$$= g_{i\mathbf{k}} \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (71)$$

$$= g_{i\mathbf{k}} \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (72)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| \quad (73)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) |0\rangle\langle 0| - \left(g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) |1\rangle\langle 1|, \quad (74)$$

$$\overline{|0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}})(t)} = (|0\rangle\langle 0| B_0^+(t) + |1\rangle\langle 1| B_1^+(t)) |0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) (|0\rangle\langle 0| B_0^-(t) + |1\rangle\langle 1| B_1^-(t)) \quad (75)$$

$$= |0\rangle\langle 0| B_0^+(t) |0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) |0\rangle\langle 0| B_0^-(t) \quad (76)$$

$$= |0\rangle\langle 0| B_0^+(t) (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) B_0^-(t) \quad (77)$$

$$= |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (78)$$

$$\overline{|1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}})(t)} = (|0\rangle\langle 0| B_0^+(t) + |1\rangle\langle 1| B_1^+(t)) |1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) (|0\rangle\langle 0| B_0^-(t) + |1\rangle\langle 1| B_1^-(t)) \quad (79)$$

$$= |1\rangle\langle 1| B_1^+(t) |1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) |1\rangle\langle 1| B_1^-(t) \quad (80)$$

$$= |1\rangle\langle 1| B_1^+(t) (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) B_1^-(t) \quad (81)$$

$$= |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (82)$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}(t)} = \omega_{\mathbf{k}} (|0\rangle\langle 0| B_0^+(t) + |1\rangle\langle 1| B_1^+(t)) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} (|0\rangle\langle 0| B_0^-(t) + |1\rangle\langle 1| B_1^-(t)) \quad (83)$$

$$= \omega_{\mathbf{k}} (|0\rangle\langle 0| \Pi_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} (|0\rangle\langle 0| \Pi_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)) \quad (84)$$

$$= \omega_{\mathbf{k}} (|0\rangle\langle 0| B_0^+(t) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_0^-(t) + |1\rangle\langle 1| B_1^+(t) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_1^-(t)) \quad (85)$$

$$= \omega_{\mathbf{k}} \sum_j |j\rangle\langle j| D\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}} \left(D\left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \right) \quad (86)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \mathbb{I} \right) \quad (87)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (88)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger + \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger + \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) \right) \quad (89)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (92)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \quad (93)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right). \quad (94)$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)} |0\rangle\langle 0| + \overline{\varepsilon_1(t)} |1\rangle\langle 1| + \overline{V_{10}(t)} |1\rangle\langle 0| + \overline{V_{01}(t)} |0\rangle\langle 1| \quad (95)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_1^+(t) B_0^-(t) + V_{01}(t) |0\rangle\langle 1| B_0^+(t) B_1^-(t), \quad (96)$$

$$\overline{H_I} = \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (97)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (98)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (99)$$

$$= \sum_{\mathbf{k}, i} |i\rangle\langle i| \left(g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (100)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} \quad (101)$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right) \quad (102)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right). \quad (103)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right). \quad (104)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_S}(t) + \overline{H_I}(t) + \overline{H_B}. \quad (105)$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (106)$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right), \quad (107)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right). \quad (108)$$

$\chi_{ij}(t)$ is an imaginary number so $e^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix} B_{iz}(t) & B_i^\pm(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\chi_{ij}(t)} \end{pmatrix}, \quad (109)$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \quad (110)$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \quad (111)$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_B T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})}. \quad (112)$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (113)$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (114)$$

So the product of the matrix representation of b^\dagger and b with $-\beta$ is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (115)$$

$$= \sum_{j=0}^{\infty} -j\beta\omega |j\rangle\langle j|, \quad (116)$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j .

$$e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \quad (117)$$

$$e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \quad (118)$$

The value of $\text{Tr} \left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} \right)$ is:

$$\text{Tr} \left(e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} \right) = \text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \quad (119)$$

$$= \sum_{j_{\mathbf{k}}} \left(e^{-\beta\omega_{\mathbf{k}}} \right)^{j_{\mathbf{k}}} \quad (120)$$

$$= \frac{1}{1 - e^{-\beta\omega_{\mathbf{k}}}} \quad (\text{by geometric series}) \quad (121)$$

$$\equiv f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}), \quad (122)$$

$$\text{Tr} \left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} \right) = \text{Tr} \left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \quad (123)$$

$$= \prod_{\mathbf{k}} \text{Tr} \left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \quad (124)$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}). \quad (125)$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})} \quad (126)$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (127)$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}. \quad (128)$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle, \quad (129)$$

$$b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle. \quad (130)$$

Then we can prove that $\langle B_{iz} \rangle_{\overline{H_B}} = 0$ using the following property based on (129)-(130):

$$\langle B_{iz}(t) \rangle_{\overline{H_B}} = \text{Tr} (B_{iz}(t) \rho_B) \quad (131)$$

$$= \text{Tr} \left(\left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \rho_B \right) \quad (132)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \rho_B \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \rho_B \right) \quad (133)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \text{Tr} (b_{\mathbf{k}}^\dagger \rho_B) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \text{Tr} (b_{\mathbf{k}} \rho_B) \quad (134)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) \quad (135)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \text{Tr} \left(b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \text{Tr} \left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right), \quad (136)$$

$$\text{Tr} \left(b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}^\dagger) \quad (137)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \right) \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (138)$$

$$= 0, \quad (139)$$

$$\text{Tr} \left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}) \quad (140)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \right) \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (141)$$

$$= 0. \quad (142)$$

we therefore find that:

$$\langle B_{iz}(t) \rangle_{\overline{H_B}} = 0. \quad (143)$$

Another important expected value is $B(t) = \langle B^\pm(t) \rangle_{\overline{H_B}}$, where $B^\pm(t) = \pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)$ is given by:

$$\langle B^\pm(t) \rangle_{H_B} = \text{Tr} (\rho_B B^\pm(t)) = \text{Tr} (B^\pm(t) \rho_B) \quad (144)$$

$$= \text{Tr} \left(e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \rho_B \right) \quad (145)$$

$$= \prod_{\mathbf{k}} \text{Tr} (D(\pm \alpha_{\mathbf{k}}(t)) \rho_B) \quad (146)$$

$$= \prod_{\mathbf{k}} \langle D(\pm \alpha_{\mathbf{k}}(t)) \rangle. \quad (147)$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha) |\alpha \rangle \langle \alpha| d^2 \alpha. \quad (148)$$

where $P(\alpha)$ satisfies $\int P(\alpha) d^2 \alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P(\alpha)$ is given by:

$$\langle A \rangle = \text{Tr} (A \rho) \quad (149)$$

$$= \int P(\alpha) \langle \alpha | A | \alpha \rangle d^2 \alpha. \quad (150)$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}}$ where $N = (e^{\beta\omega} - 1)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = \frac{1}{k_B T}$.

Using the integral representation (150) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | D(h) | \alpha \rangle d^2 \alpha \quad (151)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle d^2 \alpha, \quad (152)$$

$$D(h) D(\alpha) = D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)}, \quad (153)$$

$$D(-\alpha) (D(h) D(\alpha)) = D(-\alpha) D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (154)$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (155)$$

$$= D(\alpha) e^{\frac{1}{2}(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2)} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (156)$$

$$= D(\alpha) e^{(h\alpha^* - h^*\alpha)}, \quad (157)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) e^{(h\alpha^* - h^*\alpha)} | 0 \rangle d^2 \alpha \quad (158)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} \langle 0 | D(h) | 0 \rangle d^2 \alpha \quad (159)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} \langle 0 | h \rangle d^2 \alpha, \quad (160)$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (161)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} \langle 0 | e^{-\frac{|h|^2}{2}} \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle d^2 \alpha \quad (162)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^*\alpha)} e^{-\frac{|h|^2}{2}} d^2 \alpha \quad (163)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int e^{-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha} d^2 \alpha, \quad (164)$$

$$\alpha = x + iy, \quad (165)$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{N} + h(x-iy) - h^*(x+iy)} dx dy \quad (166)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} e^{-\frac{x^2}{N} + hx - h^*x} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{N} + hy - h^*y} dy, \quad (167)$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N} (x^2 - Nhx + Nh^*x) \quad (168)$$

$$= -\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 + \frac{N(h^* - h)^2}{4}, \quad (169)$$

$$-\frac{y^2}{N} - ihy - ih^*y = -\frac{1}{N} (y^2 + iNhy + iNh^*y) \quad (170)$$

$$= -\frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right) - \frac{N(h + h^*)^2}{4}, \quad (171)$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx, \quad (172)$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 - \frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right)} dx dy, \quad (173)$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}}}{\pi N} \int_{-\infty}^{\infty} e^{-\frac{\left(x + \frac{(Nh^* - Nh)}{2} \right)^2}{2 \left(\sqrt{\frac{N}{2}} \right)^2}} dx \int_{-\infty}^{\infty} e^{-\frac{\left(y^2 + \frac{iN(h + h^*)}{2} \right)}{2 \left(\sqrt{\frac{N}{2}} \right)^2}} dy \quad (174)$$

$$= \frac{e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}}}{\pi N} \left(\sqrt{2\pi} \sqrt{\frac{N}{2}} \right)^2 \quad (175)$$

$$= e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}} \quad (176)$$

$$= e^{-\frac{|h|^2}{2} + \frac{N(h^{*2} - 2hh^* + h^2)}{4} - \frac{N(h^2 + 2hh^* + h^{*2})}{4}} \quad (177)$$

$$= e^{-|h|^2 \left(N + \frac{1}{2} \right)} \quad (178)$$

$$= e^{-|h|^2 \left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2} \right)} \quad (179)$$

$$= e^{-\frac{|h|^2}{2} \left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} \right)} \quad (180)$$

$$= e^{-\frac{|h|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}. \quad (181)$$

In the last line we used $\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (146) using (181) is given by:

$$B = e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \quad (182)$$

We will now force $\langle \overline{H_I}(t) \rangle_{\overline{H_B}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}(t)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+(t) \sigma^+$ and $B_i^-(t) \sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}(t)$ and we subtract their expected value in order to satisfy $\langle \overline{H_I}(t) \rangle_{\overline{H_B}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}(t)$ is:

$$\overline{H}(t) = \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle \langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (183)$$

$$= \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| B_{jj'}^-(t) + \sum_j |j\rangle \langle j| B_{jz}(t) + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_j^+(t) B_{j'}^-(t) - B_{jj'}(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (184)$$

$$\equiv \overline{H_S}(t) + \overline{H_I}(t) + \overline{H_B}. \quad (185)$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_i^+(t) B_j^-(t) \rangle = B_{ij}(t) \quad (186)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (187)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (188)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (189)$$

$$= \prod_{\mathbf{k}} \left\langle D \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \quad (190)$$

$$= \prod_{\mathbf{k}} e^{-\frac{1}{2} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \quad (191)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \quad (192)$$

From the definition $B_{01}(t) = \langle B_0^+(t) B_1^-(t) \rangle$ using the displacement operator we have:

$$\langle B_0^+(t) B_1^-(t) \rangle = B_{01}(t) \quad (193)$$

$$= \left\langle \prod_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D \left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \quad (194)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D \left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \quad (195)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (196)$$

$$= \prod_{\mathbf{k}} \left(\left\langle D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (197)$$

$$= \prod_{\mathbf{k}} \left(e^{-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (198)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \quad (199)$$

We can check:

$$\langle B_0^+(t) B_1^-(t) \rangle = B_{01}(t) \quad (200)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \quad (201)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}^* \quad (202)$$

$$= \langle B_1^+(t) B_0^-(t) \rangle^* \quad (203)$$

$$= B_{10}^*(t). \quad (204)$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma^+ + V_{01}(t) B_{01} \sigma^-, \quad (205)$$

$$\overline{H_{\bar{I}}(t)} \equiv V_{10}(t) (B_1^+(t) B_0^-(t) - B_{10}(t)) \sigma^+ + V_{01}(t) (B_0^+(t) B_1^-(t) - B_{01}(t)) \sigma^- + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t), \quad (206)$$

$$\overline{H_{\bar{B}}(t)} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (207)$$

$$= H_B. \quad (208)$$

Note that $\overline{H_B}$, which is the bath acting on the effective “system” \bar{S} in the variational frame, is just the original bath, H_B , before transforming to the variational frame.

For the Hamiltonian (206) we can verify the condition $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$ in the following way:

$$\langle \overline{H_I} \rangle_{\overline{H_B}} = \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+(t) B_{j'}^-(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_B}} \quad (209)$$

$$= \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_B}} + \left\langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+(t) B_{j'}^-(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_B}} \quad (210)$$

$$= \sum_{n\mathbf{k}} \left(\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + \langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| \left(\langle V_{jj'}(t) B_j^+(t) B_{j'}^-(t) \rangle_{\overline{H_B}} - \langle V_{jj'}(t) B_{jj'}(t) \rangle_{\overline{H_B}} \right) \quad (211)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(\langle B_j^+(t) B_{j'}^-(t) \rangle_{\overline{H_B}} - \langle B_{jj'}(t) \rangle_{\overline{H_B}} \right) \quad (212)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) (B_{jj'}(t) - B_{jj'}(t)) \quad (213)$$

$$= 0. \quad (214)$$

We used (143) and (199) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_x(t) = B_x^\dagger(t) \quad (215)$$

$$= \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}, \quad (216)$$

$$B_y(t) = B_y^\dagger(t) \quad (217)$$

$$= \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i}, \quad (218)$$

$$B_{iz}(t) = B_{iz}^\dagger(t) \quad (219)$$

$$= \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right). \quad (220)$$

Writing the equations (205) and (206) using the previous combinations we obtain that:

$$\overline{H_S}(t) = \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \sigma^+ + V_{01}(t) B_{01}(t) \sigma^- \quad (221)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \quad (222)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) \left(B_{10}^{\Re}(t) + iB_{10}^{\Im}(t) \right) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \left(B_{10}^{\Re}(t) - iB_{10}^{\Im}(t) \right) \frac{\sigma_x - i\sigma_y}{2} \quad (223)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) + iB_{10}^{\Im}(t) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) \quad (224)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + iB_{10}^{\Im}(t) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (225)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + iB_{10}^{\Im}(t) \left(i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t) \right) \quad (226)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + iB_{10}^{\Im}(t) \left(i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t) \right) \quad (227)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) - \left(\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (228)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (229)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right), \quad (230)$$

$$\overline{H_I}(t) = V_{10}(t) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) \right) + V_{01}(t) \left(\sigma^- B_0^+(t) B_1^-(t) - \sigma^- B_{01}(t) \right) + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t) \quad (231)$$

$$= |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t) + \left(V_{10}^{\Re}(t) + i V_{10}^{\Im}(t) \right) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) \right) + \left(V_{10}^{\Re}(t) - i V_{10}^{\Im}(t) \right) \left(\sigma^- B_0^+(t) B_1^-(t) - \sigma^- B_{01}(t) \right) \quad (232)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) + \sigma^- B_0^+(t) B_1^-(t) - \sigma^- B_{01}(t) \right) + i V_{10}^{\Im}(t) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) - \sigma^- B_0^+(t) B_1^-(t) + \sigma^- B_{01}(t) \right) \quad (233)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+(t) B_0^-(t) - \frac{\sigma_x + i\sigma_y}{2} B_{10}(t) + \frac{\sigma_x - i\sigma_y}{2} B_0^+(t) B_1^-(t) - \frac{\sigma_x - i\sigma_y}{2} B_{01}(t) \right) \quad (234)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+(t) B_0^-(t) - \frac{\sigma_x + i\sigma_y}{2} B_{10}(t) + \frac{\sigma_x - i\sigma_y}{2} B_0^+(t) B_1^-(t) - \frac{\sigma_x - i\sigma_y}{2} B_{01}(t) \right) \quad (235)$$

$$+ i V_{10}^{\Im}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+(t) B_0^-(t) - \frac{\sigma_x + i\sigma_y}{2} B_{10}(t) - \frac{\sigma_x - i\sigma_y}{2} B_0^+(t) B_1^-(t) + \frac{\sigma_x - i\sigma_y}{2} B_{01}(t) \right) \quad (236)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\sigma_x \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} + i\sigma_y \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} \right) \quad (237)$$

$$+ i V_{10}^{\Im}(t) \left(\sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} + i\sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (238)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \left(i\sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} - \sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (239)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \left(i^2 \sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2i} - \sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (240)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \left(i^2 \sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2i} - \sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (241)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (i^2 \sigma_x (-B_y(t)) - \sigma_y B_x(t)) \quad (242)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)). \quad (243)$$

II. FREE-ENERGY MINIMIZATION

The true free energy $E_{\text{Free}}(t)$ is bounded by the Bogoliubov inequality:

$$E_{\text{Free}}(t) \leq E_{\text{Free,B}}(t) \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t) + \overline{H_B}} \right) \right) + \langle \overline{H_I}(t) \rangle_{\overline{H_S}(t) + \overline{H_B}} + O \left(\langle \overline{H_I}^2(t) \rangle_{\overline{H_S}(t) + \overline{H_B}} \right). \quad (244)$$

We will optimize the set of variational parameters $\{v_{\mathbf{k}}(t)\}$ in order to minimize $E_{\text{Free,B}}(t)$ (i.e. to make it as close to the true free energy $E_{\text{Free}}(t)$ as possible). Neglecting the higher order terms and using $\langle \overline{H_I}(t) \rangle_{\overline{H_S}(t) + \overline{H_B}} = 0$ we can obtain the following condition to obtain the set $\{v_{\mathbf{k}}(t)\}$:

$$\frac{\partial E_{\text{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = 0. \quad (245)$$

Using this condition and given that $[\overline{H_S}(t), \overline{H_B}] = 0$, we have:

$$e^{-\beta(\overline{H_S}(t) + \overline{H_B})} = e^{-\beta \overline{H_S}(t)} e^{-\beta \overline{H_B}}. \quad (246)$$

Then using the fact that $\overline{H_S}(t)$ and $\overline{H_B}$ relate to different Hilbert spaces, we obtain:

$$\text{Tr} \left(e^{-\beta \overline{H_S}(t)} e^{-\beta \overline{H_B}} \right) = \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right). \quad (247)$$

So Eq. (245) becomes:

$$\frac{\partial E_{\text{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (248)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (249)$$

$$= -\frac{1}{\beta} \frac{\partial \left(\ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right) + \ln \left(\text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (250)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (251)$$

$$= 0 \quad (\text{by Eq. (245)}). \quad (252)$$

But since $\overline{H_B} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{e^{-\beta \overline{H_S}(t)}} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}(t)} \quad (253)$$

$$= 0. \quad (254)$$

This means we need to impose:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}(t)} = 0. \quad (255)$$

First we look at:

$$-\beta \overline{H_S}(t) = -\beta \left((\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + V_{10}(t) B_{10}(t) \sigma^+ + V_{01}(t) B_{01}(t) \sigma^- \right). \quad (256)$$

Then the eigenvalues of $-\beta \overline{H_S}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr} \left(-\beta \overline{H_S}(t) \right) + \text{Det} \left(-\beta \overline{H_S}(t) \right) = 0. \quad (257)$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr} \left(\overline{H_S}(t) \right), \quad (258)$$

$$\eta \equiv \sqrt{\left(\text{Tr} \left(\overline{H_S}(t) \right) \right)^2 - 4 \text{Det} \left(\overline{H_S}(t) \right)}. \quad (259)$$

The solutions of the equation (257) are:

$$\lambda = \beta \frac{-\text{Tr} \left(\overline{H_S}(t) \right) \pm \sqrt{\left(\text{Tr} \left(\overline{H_S}(t) \right) \right)^2 - 4 \text{Det} \left(\overline{H_S}(t) \right)}}{2} \quad (260)$$

$$= \beta \frac{-\varepsilon(t) \pm \eta(t)}{2} \quad (261)$$

$$= -\beta \frac{\varepsilon(t) \mp \eta(t)}{2}. \quad (262)$$

The value of $\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) = e^{-\frac{\varepsilon(t)\beta}{2}} e^{\frac{\eta(t)\beta}{2}} + e^{-\frac{\varepsilon(t)\beta}{2}} e^{-\frac{\eta(t)\beta}{2}} \quad (263)$$

$$= 2e^{-\frac{\varepsilon(t)\beta}{2}} \cosh \left(\frac{\eta(t)\beta}{2} \right). \quad (264)$$

Given that $v_{i\mathbf{k}}(t)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) = A(\varepsilon(t), \eta(t))$ to calculate $\frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial \left(2e^{-\frac{\varepsilon(t)\beta}{2}} \cosh \left(\frac{\eta(t)\beta}{2} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (265)$$

$$= 2 \left(-\frac{\beta}{2} \frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \right) e^{-\frac{\varepsilon(t)\beta}{2}} \cosh \left(\frac{\eta(t)\beta}{2} \right) + 2 \left(\frac{\beta}{2} \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \right) e^{-\frac{\varepsilon(t)\beta}{2}} \sinh \left(\frac{\eta(t)\beta}{2} \right) \quad (266)$$

$$= -\beta e^{-\frac{\varepsilon(t)\beta}{2}} \left(\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \cosh \left(\frac{\eta(t)\beta}{2} \right) - \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \sinh \left(\frac{\eta(t)\beta}{2} \right) \right). \quad (267)$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \cosh \left(\frac{\eta(t)\beta}{2} \right) - \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \sinh \left(\frac{\eta(t)\beta}{2} \right) = 0. \quad (268)$$

The derivates included in the expression given are related to:

$$\langle B_0^+(t) B_1^-(t) \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (269)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right)^* e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (270)$$

$$= \langle B_1^+(t) B_0^-(t) \rangle^*, \quad (271)$$

$$R_i(t) = \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (272)$$

$$= \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right), \quad (273)$$

$$\langle B_0^+(t) B_1^-(t) \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (274)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}} \right) e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}, \quad (275)$$

$$v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t) = (v_{0\mathbf{k}}^{\Re}(t) - i v_{0\mathbf{k}}^{\Im}(t)) (v_{1\mathbf{k}}^{\Re}(t) + i v_{1\mathbf{k}}^{\Im}(t)) - (v_{0\mathbf{k}}^{\Re}(t) + i v_{0\mathbf{k}}^{\Im}(t)) (v_{1\mathbf{k}}^{\Re}(t) - i v_{1\mathbf{k}}^{\Im}(t)) \quad (276)$$

$$= (v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Re}(t) + i v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) - i v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Im}(t)) \quad (277)$$

$$- (v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Re}(t) - i v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) + i v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Im}(t)) \quad (278)$$

$$= 2i (v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t)), \quad (279)$$

$$|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2 = (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))^* \quad (280)$$

$$= |v_{1\mathbf{k}}(t)|^2 + |v_{0\mathbf{k}}(t)|^2 - (v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t) + v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)) \quad (281)$$

$$= (v_{1\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t))^2 + (v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{0\mathbf{k}}^{\Im}(t))^2 - 2(v_{1\mathbf{k}}^{\Re}(t)v_{0\mathbf{k}}^{\Re}(t) + v_{1\mathbf{k}}^{\Im}(t)v_{0\mathbf{k}}^{\Im}(t)) \quad (282)$$

$$= (v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2. \quad (283)$$

Rewriting in terms of real and imaginary parts.

$$R_i(t) = \sum_{\mathbf{k}} \left(\frac{(v_{i\mathbf{k}}^{\Re}(t))^2 + (v_{i\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re}(t) - i v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re}(t) + i v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (284)$$

$$= \sum_{\mathbf{k}} \left(\frac{(v_{i\mathbf{k}}^{\Re}(t))^2 + (v_{i\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im}(t) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (285)$$

$$\langle B_0^+(t) B_1^-(t) \rangle = \left(\prod_{\mathbf{k}} e^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}} \right) \left(e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (286)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{2i(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t))}{2\omega_{\mathbf{k}}^2}} \right) \left(e^{-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (287)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{i(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2}} \right) \left(e^{-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right), \quad (288)$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (289)$$

$$= \frac{\partial \left(\left(\frac{(v_{i\mathbf{k}}^{\Re}(t))^2 + (v_{i\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im}(t) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (290)$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \quad (291)$$

$$\frac{\partial |B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial \left(e^{-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (292)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} e^{-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (293)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} |B_{10}(t)|^2, \quad (294)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial \sqrt{(\text{Tr}(\overline{H_{\bar{S}}(t)}))^2 - 4\text{Det}(\overline{H_{\bar{S}}(t)})}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (295)$$

$$= \frac{2\text{Tr}(\overline{H_{\bar{S}}(t)}) \frac{\partial \text{Tr}(\overline{H_{\bar{S}}(t)})}{\partial v_{i\mathbf{k}}^{\Re}(t)} - 4 \frac{\partial \text{Det}(\overline{H_{\bar{S}}(t)})}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{(\text{Tr}(\overline{H_{\bar{S}}(t)}))^2 - 4\text{Det}(\overline{H_{\bar{S}}(t)})}} \quad (296)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1(t))(\varepsilon_0(t) + R_0(t)) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \quad (297)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} |B_{10}(t) V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (298)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{i\mathbf{k}}^{\Re}(t) - v_{i'\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (299)$$

$$= \frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}(t) V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(-\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \quad (300)$$

$$+ 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^2} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (301)$$

From the equation (268) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}} \quad (302)$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} \left(2 \frac{(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right))}{\eta(t)} \right) + 2 \frac{(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} + 2 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \varepsilon(t)}{\eta(t)}} \quad (303)$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{(2v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}} - g_{i\mathbf{k}}^*) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4|V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} + g_{i\mathbf{k}}^*) (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t))) + 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (304)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re}(t) - 2g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4|V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re} (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t))) + 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (305)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re}(t) - 2g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4|V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (306)$$

$$= \frac{(v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |V_{10}(t)B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (307)$$

Separating (306) such that the terms with $v_{i\mathbf{k}}(t)$ are located at one side of the equation permit us to write:

$$\frac{(v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Re}(t) \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re} (2(\varepsilon_i(t) + R_i(t)) - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (308)$$

$$v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re}(t) = v_{i\mathbf{k}}^{\Re}(t) \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) \quad (309)$$

$$+ 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (310)$$

$$v_{i\mathbf{k}}^{\Re}(t) = \frac{g_{i\mathbf{k}}^{\Re} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}, \quad (311)$$

$$v_{i\mathbf{k}}^{\Re}(t) = \frac{g_{i\mathbf{k}}^{\Re} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}. \quad (312)$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} = \frac{\partial (\varepsilon_1(t) + R_1(t) + \varepsilon_0(t) + R_0(t))}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} \quad (313)$$

$$= \frac{\partial \left(\left(\frac{(v_{i\mathbf{k}}^{\mathfrak{R}}(t))^2 + (v_{i\mathbf{k}}^{\mathfrak{S}}(t))^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\mathfrak{R}}(t) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\mathfrak{S}}(t) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} \quad (314)$$

$$= 2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (315)$$

$$\frac{\partial |B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} = \frac{\partial \left(e^{-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\mathfrak{R}}(t) - v_{0\mathbf{k}}^{\mathfrak{R}}(t))^2 + (v_{1\mathbf{k}}^{\mathfrak{S}}(t) - v_{0\mathbf{k}}^{\mathfrak{S}}(t))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \right)}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} \quad (316)$$

$$= - \frac{2(v_{1\mathbf{k}}^{\mathfrak{S}}(t) - v_{0\mathbf{k}}^{\mathfrak{S}}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\mathfrak{S}}(t) - v_{0\mathbf{k}}^{\mathfrak{S}}(t))}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} e^{-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\mathfrak{R}}(t) - v_{0\mathbf{k}}^{\mathfrak{R}}(t))^2 + (v_{1\mathbf{k}}^{\mathfrak{S}}(t) - v_{0\mathbf{k}}^{\mathfrak{S}}(t))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \quad (317)$$

$$= - \frac{2(v_{1\mathbf{k}}^{\mathfrak{S}}(t) - v_{0\mathbf{k}}^{\mathfrak{S}}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\mathfrak{S}}(t) - v_{0\mathbf{k}}^{\mathfrak{S}}(t))}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} |B_{10}(t)|^2, \quad (318)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\mathfrak{R}}(t)} = \frac{\partial \sqrt{\left(\text{Tr}(\overline{H_{\mathfrak{S}}}(t)) \right)^2 - 4 \text{Det}(\overline{H_{\mathfrak{S}}}(t))}}{\partial v_{i\mathbf{k}}^{\mathfrak{R}}(t)} \quad (319)$$

$$= \frac{2 \text{Tr}(\overline{H_{\mathfrak{S}}}(t)) \frac{\partial \text{Tr}(\overline{H_{\mathfrak{S}}}(t))}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} - 4 \frac{\partial \text{Det}(\overline{H_{\mathfrak{S}}}(t))}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)}}{2 \sqrt{\left(\text{Tr}(\overline{H_{\mathfrak{S}}}(t)) \right)^2 - 4 \text{Det}(\overline{H_{\mathfrak{S}}}(t))}} \quad (320)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1(t))(\varepsilon_0(t) + R_0(t)) - |B_{10}(t)|^2 |V_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)}}{\eta(t)} \quad (321)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1(t))(\varepsilon_0(t) + R_0(t)) - |B_{10}(t)|^2 |V_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)}}{\eta(t)} \quad (322)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \left(2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{i\mathbf{k}}^{\mathfrak{S}}(t) - v_{i\mathbf{k}}^{\mathfrak{R}}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{i\mathbf{k}}^{\mathfrak{S}}(t) - v_{i\mathbf{k}}^{\mathfrak{R}}(t))}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (323)$$

$$\delta_{1i} - \delta_{0i} = \frac{\partial (v_{1\mathbf{k}}^{\mathfrak{S}}(t) - v_{0\mathbf{k}}^{\mathfrak{S}}(t))}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)} \quad (324)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\mathfrak{R}}(t)} = \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \left(2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{i\mathbf{k}}^{\mathfrak{S}}(t) - v_{i\mathbf{k}}^{\mathfrak{R}}(t))}{\omega_{\mathbf{k}}^2} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (325)$$

$$= \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t) \left(4(\varepsilon_i(t) + R_i(t)) - 2\varepsilon(t) - \frac{4|B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}{\eta(t)} + \frac{1}{\eta(t)} \left(2 \frac{g_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} \varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} + 4 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t) |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^2} \right). \quad (326)$$

From the equation (268) and replacing the derivatives obtained we have:

$$\frac{\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)}}{\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\mathfrak{S}}(t)}} = \tanh\left(\frac{\beta \eta(t)}{2}\right) \quad (327)$$

$$= \frac{2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{v_{i\mathbf{k}}^{\mathfrak{S}}(t) \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4|B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)}{\eta(t) \omega_{\mathbf{k}}} \right) + \frac{2}{\eta(t)} \left(\frac{g_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}} + 2 \frac{v_{i\mathbf{k}}^{\mathfrak{S}}(t)}{\omega_{\mathbf{k}}^2} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right) \right)}. \quad (328)$$

Rearranging this equation will lead to:

$$a_i(\omega_{\mathbf{k}}, t) = \frac{\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)|^2|V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \quad (342)$$

$$b_i(\omega_{\mathbf{k}}, t) = \frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{1}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)|^2|V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \quad (343)$$

So the equation (338) written in explicit form is:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) + v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t) b_0(\omega_{\mathbf{k}}, t), \quad (344)$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) + v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t). \quad (345)$$

This system of equations has the following solutions:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) + v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t) b_0(\omega_{\mathbf{k}}, t) \quad (346)$$

$$= g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) + (g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) + v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)) b_0(\omega_{\mathbf{k}}, t) \quad (347)$$

$$= g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) + g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) b_0(\omega_{\mathbf{k}}, t) + v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t) b_0(\omega_{\mathbf{k}}, t) \quad (348)$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t)(1 - b_1(\omega_{\mathbf{k}}, t)b_0(\omega_{\mathbf{k}}, t)) = g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) + g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) b_0(\omega_{\mathbf{k}}, t) \quad (349)$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t) = \frac{g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) + g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) b_0(\omega_{\mathbf{k}}, t)}{1 - b_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)}, \quad (350)$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) + \frac{g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) + g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) b_0(\omega_{\mathbf{k}}, t)}{1 - b_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)} b_1(\omega_{\mathbf{k}}, t) \quad (351)$$

$$= \frac{g_1(\omega_{\mathbf{k}}) a_1(\omega_{\mathbf{k}}, t) + g_0(\omega_{\mathbf{k}}) a_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)}{1 - b_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)}. \quad (352)$$

For a shorter representation let's define:

$$r_i(\omega_{\mathbf{k}}, t) = \frac{a_i(\omega_{\mathbf{k}}, t)}{1 - b_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)}, \quad (353)$$

$$s_i(\omega_{\mathbf{k}}, t) = \frac{a_{(i+1) \bmod 2}(\omega_{\mathbf{k}}, t) b_{i \bmod 2}(\omega_{\mathbf{k}}, t)}{1 - b_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)}. \quad (354)$$

So the variational parameters are given by:

$$\begin{pmatrix} v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t) \\ v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t) \end{pmatrix} \equiv \begin{pmatrix} r_0(\omega_{\mathbf{k}}, t) & s_0(\omega_{\mathbf{k}}, t) \\ r_1(\omega_{\mathbf{k}}, t) & s_1(\omega_{\mathbf{k}}, t) \end{pmatrix} \begin{pmatrix} g_0(\omega_{\mathbf{k}}) \\ g_1(\omega_{\mathbf{k}}) \end{pmatrix}. \quad (355)$$

Given that $v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t) \equiv g_i(\omega_{\mathbf{k}}) F_i(\omega_{\mathbf{k}}, t)$ then we can write:

$$F_0(\omega_{\mathbf{k}}, t) = r_0(\omega_{\mathbf{k}}, t) + \frac{g_1(\omega_{\mathbf{k}})}{g_0(\omega_{\mathbf{k}})} s_0(\omega_{\mathbf{k}}, t) \quad (356)$$

$$F_1(\omega_{\mathbf{k}}, t) = \frac{g_0(\omega_{\mathbf{k}})}{g_1(\omega_{\mathbf{k}})} r_1(\omega_{\mathbf{k}}, t) + s_1(\omega_{\mathbf{k}}, t) \quad (357)$$

III. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$, where $\rho_B^{\text{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^{V(0)} \rho_T(0) e^{-V(0)} \quad (358)$$

$$= (|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)) (\rho_S(0) \otimes \rho_B) (|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)), \quad (359)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: \quad |0\rangle\langle 0|B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \quad (360)$$

$$= |0\rangle\langle 0|B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \quad (361)$$

$$= |0\rangle\langle 0| \otimes B_0^+(0) \rho_B B_0^-(0), \quad (362)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: \quad |1\rangle\langle 1|B_1^+(0) |1\rangle\langle 1|\rho_B |1\rangle\langle 1|B_1^-(0) \quad (363)$$

$$= |1\rangle\langle 1|B_1^+(0) \rho_B B_1^-(0) \quad (364)$$

$$= |1\rangle\langle 1| \otimes B_1^+(0) \rho_B B_1^-(0), \quad (365)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1|: \quad |0\rangle\langle 1|B_0^+(0) |0\rangle\langle 1|\rho_B |1\rangle\langle 1|B_1^-(0) \quad (366)$$

$$= |0\rangle\langle 1|B_0^+(0) \rho_B |1\rangle\langle 1|B_1^-(0) \quad (367)$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0) \rho_B B_1^-(0) \quad (368)$$

$$= |0\rangle\langle 1| \otimes B_0^+(0) \rho_B B_1^-(0), \quad (369)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0|: \quad |1\rangle\langle 0|B_1^+(0) |1\rangle\langle 0|\rho_B |0\rangle\langle 0|B_0^-(0) \quad (370)$$

$$= |1\rangle\langle 0| \otimes B_1^+(0) \rho_B B_0^-(0). \quad (371)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O}(t) \equiv U^\dagger(t) O(t) U(t), \quad (372)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (373)$$

Here \mathcal{T} denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (374)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\overline{\rho_T}(t)). \quad (375)$$

In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \quad (376)$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_I}(t)$ as pointed in the equation (243):

$$\overline{H_I}(t) = \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (377)$$

$$= B_{0z}(t) |0\rangle\langle 0| + B_{1z}(t) |1\rangle\langle 1| + V_{10}^{\Re}(t) \sigma_x B_x(t) + V_{10}^{\Re}(t) \sigma_y B_y(t) + V_{10}^{\Im}(t) \sigma_x B_y(t) - V_{10}^{\Im}(t) \sigma_y B_x(t) \quad (378)$$

$$= \sum_i C_i(t) (A_i \otimes B_i(t)). \quad (379)$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \widetilde{\rho_T}(t)]. \quad (380)$$

This equation has the formal solution

$$\widetilde{\rho_T}(t) = \widetilde{\rho_T}(0) - i \int_0^t [\widetilde{H_I}(t'), \widetilde{\rho_T}(t')] dt'. \quad (381)$$

Replacing the equation (381) in the equation (380) gives us:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \widetilde{\rho_T}(0)] - \int_0^t [\widetilde{H_I}(t), [\widetilde{H_I}(t'), \widetilde{\rho_T}(t')]] dt'. \quad (382)$$

This equation allow us to iterate and write in terms of a series expansion with $\widetilde{\rho_T}(0)$ the solution as:

$$\widetilde{\rho_T}(t) = \widetilde{\rho_T}(0) + \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \widetilde{\rho_T}(0)]] \dots]. \quad (383)$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho_S}(t) = \widetilde{\rho_S}(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \widetilde{\rho_S}(0) \rho_B]] \dots]. \quad (384)$$

here we have assumed that $\widetilde{\rho_T}(0) = \widetilde{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\rho_S}(t) = (1 + W_1(t) + W_2(t) + \dots) \widetilde{\rho_S}(0) \quad (385)$$

$$= W(t) \widetilde{\rho_S}(0). \quad (386)$$

in this case

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), (\cdot) \rho_B]] \dots]. \quad (387)$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = (\dot{W}_1(t) + \dot{W}_2(t) + \dots) \widetilde{\rho_S}(0) \quad (388)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} W(t) \widetilde{\rho_S}(0) \quad (389)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} \widetilde{\rho_S}(t). \quad (390)$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\text{Tr}_B [\widetilde{H_I}(t) \rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (388) becomes:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = \dot{W}_2(t) \widetilde{\rho_S}(t) \quad (391)$$

$$= - \int_0^t dt_1 \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(t_1), \widetilde{\rho_S}(t) \rho_B]]. \quad (392)$$

Replacing $t_1 \rightarrow t - \tau$

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \int_0^t d\tau \text{Tr}_B [\overline{H_I}(t), [\widetilde{\overline{H_I}}(-\tau), \overline{\rho_S}(t) \rho_B]] . \quad (393)$$

From the interaction picture applied on $\overline{H_I}(t)$ we find:

$$\widetilde{\overline{H_I}}(t) = U^\dagger(t) e^{iH_B t} \overline{H_I}(t) e^{-iH_B t} U(t) . \quad (394)$$

we use the time-ordering operator \mathcal{T} because in general $\overline{H_S}(t)$ doesn't commute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_I}}(t) = \sum_i C_i(t) (\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)) , \quad (395)$$

$$\widetilde{A_i}(t) = U^\dagger(t) e^{iH_B t} A_i e^{-iH_B t} U(t) \quad (396)$$

$$= U^\dagger(t) A_i U(t) e^{iH_B t} e^{-iH_B t} \quad (397)$$

$$= U^\dagger(t) A_i U(t) \mathbb{I} \quad (398)$$

$$= U^\dagger(t) A_i U(t) , \quad (399)$$

$$\widetilde{B_i}(t) = U^\dagger(t) e^{iH_B t} B_i(t) e^{-iH_B t} U(t) \quad (400)$$

$$= U^\dagger(t) U(t) e^{iH_B t} B_i(t) e^{-iH_B t} \quad (401)$$

$$= \mathbb{I} e^{iH_B t} B_i(t) e^{-iH_B t} \quad (402)$$

$$= e^{iH_B t} B_i(t) e^{-iH_B t} . \quad (403)$$

Here we have used the fact that $[\overline{H_S}(t), H_B] = 0$ because these operators belong to different Hilbert spaces, so $[U(t), e^{iH_B t}] = 0$.

Using the expression (395) to replace it in the equation (392)

$$\frac{d\widetilde{\overline{\rho_S}}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{\overline{H_I}}(t), [\widetilde{\overline{H_I}}(t'), \widetilde{\overline{\rho_S}}(t) \rho_B]] dt' \quad (404)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \left[\sum_i C_i(t') (\widetilde{A_i}(t') \otimes \widetilde{B_i}(t')) , \widetilde{\overline{\rho_S}}(t) \rho_B \right] \right] dt' \quad (405)$$

$$= - \int_0^t \text{Tr}_B [\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \sum_i C_i(t') (\widetilde{A_i}(t') \otimes \widetilde{B_i}(t')) \widetilde{\overline{\rho_S}}(t) \rho_B - \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(t') (\widetilde{A_i}(t') \otimes \widetilde{B_i}(t'))] dt' \quad (406)$$

$$= - \int_0^t \text{Tr}_B (\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \sum_i C_i(t') (\widetilde{A_i}(t') \otimes \widetilde{B_i}(t')) \widetilde{\overline{\rho_S}}(t) \rho_B - \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(t') (\widetilde{A_i}(t') \otimes \widetilde{B_i}(t')) \quad (407)$$

$$- \sum_i C_i(t') (\widetilde{A_i}(t') \otimes \widetilde{B_i}(t')) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) + \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(t') (\widetilde{A_i}(t') \otimes \widetilde{B_i}(t')) \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t))) dt' . \quad (408)$$

In order to calculate the correlation functions we define:

$$\mathcal{B}_{ij}(t, t') = \text{Tr}_B (\widetilde{B_i}(t) \widetilde{B_j}(t') \rho_B) . \quad (409)$$

An useful property is

$$\mathcal{B}_{ji}^*(t, t') = \text{Tr}_B (\widetilde{B_j}(t) \widetilde{B_i}(t') \rho_B)^\dagger \quad (410)$$

$$= \text{Tr}_B (\rho_B^\dagger \widetilde{B_i}^\dagger(t') \widetilde{B_j}^\dagger(t)) \quad (411)$$

$$= \text{Tr}_B (\rho_B \widetilde{B_i}(t') \widetilde{B_j}(t)) \quad (412)$$

$$= \text{Tr}_B (\widetilde{B_i}(t') \widetilde{B_j}(t) \rho_B) \quad (413)$$

$$= \mathcal{B}_{ij}(t', t) . \quad (414)$$

The correlation functions relevant that appear in the equation (408) are:

$$\text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(t') \rho_B \right) = \left\langle \widetilde{B}_j(t) \widetilde{B}_i(t') \right\rangle_B \quad (415)$$

$$= \mathcal{B}_{ji}(t, t') \quad (416)$$

$$= \mathcal{B}_{ij}^*(t', t) \quad (417)$$

$$\text{Tr}_B \left(\widetilde{B}_j(t) \rho_B \widetilde{B}_i(t') \right) = \text{Tr}_B \left(\widetilde{B}_i(t') \widetilde{B}_j(t) \rho_B \right) \quad (418)$$

$$= \mathcal{B}_{ij}(t', t) \quad (419)$$

$$\text{Tr}_B \left(\widetilde{B}_i(t') \rho_B \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(t') \rho_B \right) \quad (420)$$

$$= \mathcal{B}_{ij}^*(t', t) \quad (421)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B}_i(t') \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_i(t') \widetilde{B}_j(t) \rho_B \right) \quad (422)$$

$$= \mathcal{B}_{ij}(t', t) \quad (423)$$

The cyclic property of the trace was use widely in the development of equations (415) and (423). Replacing in (408)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left(\sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \sum_i C_i(t') (\widetilde{A}_i(t') \otimes \widetilde{B}_i(t')) \widetilde{\rho_S}(t) \rho_B - \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \widetilde{\rho_S}(t) \rho_B \sum_i C_i(t') (\widetilde{A}_i(t') \otimes \widetilde{B}_i(t')) \right) \quad (424)$$

$$- \sum_i C_i(t') (\widetilde{A}_i(t') \otimes \widetilde{B}_i(t')) \widetilde{\rho_S}(t) \rho_B \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) + \widetilde{\rho_S}(t) \rho_B \sum_i C_i(t') (\widetilde{A}_i(t') \otimes \widetilde{B}_i(t')) \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) dt'. \quad (425)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ji} C_j(t) C_i(t') (\widetilde{A}_j(t) \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(t') \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{B}_j(t) \rho_B \widetilde{B}_i(t') \right) \quad (426)$$

$$+ \sum_{ij} C_i(t') C_j(t) (\widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{A}_j(t) \rho_B \widetilde{B}_i(t') \widetilde{B}_j(t) - \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(t') \rho_B \widetilde{B}_j(t)) dt' \quad (427)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ji} C_j(t) C_i(t') (\widetilde{A}_j(t) \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(t') \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{B}_j(t) \rho_B \widetilde{B}_i(t') \right) \quad (428)$$

$$+ \sum_{ij} C_i(t') C_j(t) (\widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{A}_j(t) \rho_B \widetilde{B}_i(t') \widetilde{B}_j(t) - \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(t') \rho_B \widetilde{B}_j(t)) dt' \quad (429)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ij} C_j(t) C_i(t') (\widetilde{A}_j(t) \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(t') \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{B}_j(t) \rho_B \widetilde{B}_i(t') \right) \text{ (by permuting i and j because } ij \in J) \quad (430)$$

$$+ \sum_{ij} C_i(t') C_j(t) (\widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{A}_j(t) \rho_B \widetilde{B}_i(t') \widetilde{B}_j(t) - \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(t') \rho_B \widetilde{B}_j(t)) dt' \quad (431)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ij} C_j(t) C_i(t') (\widetilde{A}_j(t) \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(t') \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{B}_j(t) \rho_B \widetilde{B}_i(t') \right) \quad (432)$$

$$+ \widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{A}_j(t) \rho_B \widetilde{B}_i(t') \widetilde{B}_j(t) - \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(t') \rho_B \widetilde{B}_j(t)) dt' \quad (433)$$

$$= - \int_0^t \left(\sum_{ij} C_j(t) C_i(t') (\widetilde{A}_j(t) \widetilde{A}_i(t') \widetilde{\rho_S}(t) \mathcal{B}_{ji}(t, t') - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(t') \mathcal{B}_{ij}(t', t) \right) \quad (434)$$

$$+ \widetilde{\rho_S}(t) \widetilde{A}_i(t') \widetilde{A}_j(t) \mathcal{B}_{ij}(t, t) - \widetilde{A}_i(t') \widetilde{\rho_S}(t) \widetilde{A}_j(t) \mathcal{B}_{ji}(t, t)) dt' \quad (435)$$

$$= - \int_0^t \left(\sum_{ij} C_j(t) C_i(t') \left(\mathcal{B}_{ji}(t, t') [\widetilde{A}_j(t), \widetilde{A}_i(t') \widetilde{\rho_S}(t)] + \mathcal{B}_{ij}(t', t) [\widetilde{\rho_S}(t) \widetilde{A}_i(t'), \widetilde{A}_j(t)] \right) \right) dt' \quad (436)$$

$$= - \int_0^t \left(\sum_{ij} C_i(t) C_j(t') \left(\mathcal{B}_{ij}(t, t') [\widetilde{A}_i(t), \widetilde{A}_j(t') \widetilde{\rho_S}(t)] + \mathcal{B}_{ji}(t', t) [\widetilde{\rho_S}(t) \widetilde{A}_j(t'), \widetilde{A}_i(t)] \right) \right) dt' \text{ (exchanging i and j)} \quad (437)$$

$$= - \int_0^t \left(\sum_{ij} C_i(t) C_j(t') \left(\mathcal{B}_{ij}(t, t') [\widetilde{A}_i(t), \widetilde{A}_j(t') \widetilde{\rho_S}(t)] + \mathcal{B}_{ij}^*(t, t') [\widetilde{\rho_S}(t) \widetilde{A}_j(t'), \widetilde{A}_i(t)] \right) \right) dt' \quad (438)$$

$$= - \int_0^t \left(\sum_{ij} C_i(t) C_j(t') \left(\mathcal{B}_{ij}(t, t') [\widetilde{A}_i(t), \widetilde{A}_j(t') \widetilde{\rho_S}(t)] - \mathcal{B}_{ij}^*(t, t') [\widetilde{A}_i(t), \widetilde{\rho_S}(t) \widetilde{A}_j(t')] \right) \right) dt' \quad (439)$$

We could identify the following commutators in the equation deduced:

$$\mathcal{B}_{ij}(t, t') \widetilde{A}_i(t) \widetilde{A}_j(t') \widetilde{\rho_S}(t) - \mathcal{B}_{ij}(t, t') \widetilde{A}_j(t') \widetilde{\rho_S}(t) \widetilde{A}_i(t) = \mathcal{B}_{ij}(t, t') [\widetilde{A}_i(t), \widetilde{A}_j(t') \widetilde{\rho_S}(t)], \quad (440)$$

$$\mathcal{B}_{ij}^*(t, t') \widetilde{\rho_S}(t) \widetilde{A}_j(t') \widetilde{A}_i(t) - \mathcal{B}_{ij}^*(t, t') \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(t') = \mathcal{B}_{ij}^*(t, t') [\widetilde{\rho_S}(t) \widetilde{A}_j(t'), \widetilde{A}_i(t)]. \quad (441)$$

Returning to the Schroedinger picture we have:

$$U(t) \widetilde{A}_i(t) \widetilde{A}_j(t') \widetilde{\rho_S}(t) U^\dagger(t) = U(t) \widetilde{A}_i(t) U^\dagger(t) U(t) \widetilde{A}_j(t') U^\dagger(t) U(t) \widetilde{\rho_S}(t) U^\dagger(t), \quad (442)$$

$$= \left(U(t) \widetilde{A}_i(t) U^\dagger(t) \right) \left(U(t) \widetilde{A}_j(t') U^\dagger(t) \right) \left(U(t) \widetilde{\rho_S}(t) U^\dagger(t) \right), \quad (443)$$

$$= A_i(t) \widetilde{A}_j(t', t) \overline{\rho_S}(t). \quad (444)$$

This procedure applying to the relevant commutators give us:

$$U(t) \left[\widetilde{A}_i(t), \widetilde{A}_j(t') \widetilde{\rho_S}(t) \right] U^\dagger(t) = \left(U(t) \widetilde{A}_i(t) \widetilde{A}_j(t') \widetilde{\rho_S}(t) U^\dagger(t) - U(t) \widetilde{A}_j(t') \widetilde{\rho_S}(t) \widetilde{A}_i(t) U^\dagger(t) \right) \quad (445)$$

$$= A_i(t) \widetilde{A}_j(t', t) \overline{\rho_S}(t) - \widetilde{A}_j(t', t) \overline{\rho_S}(t) A_i \quad (446)$$

$$= \left[A_i(t), \widetilde{A}_j(t', t) \overline{\rho_S}(t) \right]. \quad (447)$$

Introducing this transformed commutators in the equation (439) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathcal{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t ds C_i(t) C_j(t') \left(\mathcal{B}_{ij}(t, t') \left[A_i(t), \widetilde{A}_j(t', t) \overline{\rho_S}(t) \right] + \mathcal{B}_{ij}^*(t, t') \left[\overline{\rho_S}(t) \widetilde{A}_j(t', t), A_i \right] \right), \quad (448)$$

$$t' = t - \tau \text{ (Change of variables in the integration process),} \quad (449)$$

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau C_i(t) C_j(t') \left(\mathcal{B}_{ij}(t, t') \left[A_i(t), \widetilde{A}_j(t', t) \overline{\rho_S}(t) \right] + \mathcal{B}_{ij}^*(t, t') \left[\overline{\rho_S}(t) \widetilde{A}_j(t', t), A_i(t) \right] \right). \quad (450)$$

where $i, j \in \{1, 2, 3, 4, 5, 6\}$ and $t' = t - \tau$.

Here $\widetilde{A}_j(t - \tau, t) = U(t) U^\dagger(t - \tau) A_j(t) U(t - \tau) U^\dagger(t)$ where $U(t)$ is given by (373). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. In order to write in a simplified way we define the following notation:

$$\mathcal{B}_{ij}(t, t') = \text{Tr}_B \left(\widetilde{B}_i(t) \widetilde{B}_j(t') \rho_B \right) \quad (451)$$

$$= \text{Tr}_B \left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') e^{-iH_B t'} \rho_B \right) \quad (452)$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad (453)$$

$$e^{-iH_B t'} e^{-\beta H_B} = \sum_{m=0}^{\infty} \frac{(-iH_B t')^m}{m!} \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \quad (454)$$

$$= \sum_{m,n} \frac{(-iH_B t')^m}{m!} \frac{(-\beta H_B)^n}{n!} \quad (455)$$

$$= \sum_{m,n} \frac{(-it')^m}{m!} \frac{(-\beta)^n}{n!} H_B^m H_B^n \quad (456)$$

$$= \sum_{m,n} \frac{(-it')^m}{m!} \frac{(-\beta)^n}{n!} H_B^n H_B^m \text{ (because the powers of a matrix commute)} \quad (457)$$

$$= \sum_{m,n} \frac{(-\beta)^n}{n!} H_B^n \frac{(-it')^m}{m!} H_B^m \quad (458)$$

$$= \sum_{m,n} \frac{(-\beta H_B)^n}{n!} \frac{(-it' H_B)^m}{m!} \quad (459)$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_B t')^m}{m!} \quad (460)$$

$$= e^{-\beta H_B} e^{-iH_B t'} \quad (461)$$

$$0 = e^{-iH_B t'} e^{-\beta H_B} - e^{-\beta H_B} e^{-iH_B t'} \text{ (then } e^{-iH_B t'} \text{ and } \rho_B \text{ commute)} \quad (462)$$

$$\mathcal{B}_{ij}(t, t') = \text{Tr}_B \left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \rho_B e^{-iH_B t'} \right) \text{ (by permuting } e^{-iH_B t'} \text{ and } \rho_B) \quad (463)$$

$$= \text{Tr}_B \left(\left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \right) \rho_B e^{-iH_B t'} \right) \text{ (by associative property)} \quad (464)$$

$$= \text{Tr}_B \left(e^{-iH_B t'} \left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \right) \rho_B \right) \text{ (by cyclic property of the trace)} \quad (465)$$

$$= \text{Tr}_B \left(\left(e^{-iH_B t'} e^{iH_B t} \right) B_i(t) \left(e^{-iH_B t} e^{iH_B t'} \right) B_j(t') \rho_B \right) \text{ (by associative property)} \quad (466)$$

$$[iH_B t, -iH_B t'] = iH_B t (-iH_B t') - (-iH_B t') iH_B t \quad (467)$$

$$= tt' H_B^2 - t't' H_B^2 \quad (468)$$

$$= 0 \text{ (so } iH_B t \text{ and } -iH_B t' \text{ commute)} \quad (469)$$

$$e^{-iH_B t'} e^{iH_B t} = e^{iH_B t - iH_B t'} \text{ (by the Zassenhaus formula because } iH_B t \text{ and } -iH_B t' \text{ commute)} \quad (470)$$

$$= e^{iH_B(t-t')} \quad (471)$$

$$= e^{iH_B \tau} \quad (472)$$

$$e^{iH_B t'} e^{-iH_B t} = e^{-iH_B t + iH_B t'} \text{ (by the Zassenhaus formula because } -iH_B t \text{ and } iH_B t' \text{ commute)} \quad (473)$$

$$= e^{iH_B(-t+t')} \quad (474)$$

$$= e^{-iH_B \tau} \quad (475)$$

$$\mathcal{B}_{ij}(t, t') = \text{Tr}_B \left(e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(t') \rho_B \right) \quad (476)$$

$$B_i(t, \tau) \equiv e^{iH_B \tau} B_i(t) e^{-iH_B \tau} \quad (477)$$

$$\mathcal{B}_{ij}(t, t') = \text{Tr}_B \left(e^{iH_B(t-t')} B_i(t) e^{-iH_B(t-t')} B_j(t') \rho_B \right) \quad (478)$$

$$t' = t - \tau \quad (479)$$

$$\mathcal{B}_{ij}(t, t') = \text{Tr}_B \left(e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(t') \rho_B \right) \quad (480)$$

$$= \text{Tr}_B \left(B_i(t, \tau) B_j(t', 0) \rho_B \right) \quad (481)$$

For the following results $i, j \in \{3, 6\}$, calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{j\text{mod}2z}(t)} \widetilde{B_{j\text{mod}2z}(t')} \right\rangle_B = \text{Tr}_B \left(B_{j\text{mod}2z}(t, \tau) B_{j\text{mod}2z}(t', 0) \rho_B \right) \quad (482)$$

$$= \int d^2\alpha P(\alpha) \langle \alpha | B_{j\text{mod}2z}(t, \tau) B_{j\text{mod}2z}(t', 0) | \alpha \rangle \quad (483)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | B_{j\text{mod}2z}(t, \tau) B_{j\text{mod}2z}(t', 0) | \alpha \rangle d^2\alpha, \quad (484)$$

$$q_{j\mathbf{k}}(t) = g_{j\text{mod}2\mathbf{k}} - v_{j\text{mod}2\mathbf{k}}(t) \quad (485)$$

$$B_{j\text{mod}2z}(t, \tau) = \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right), \quad (486)$$

$$B_{j\text{mod}2z}(t', 0) = \sum_{\mathbf{k}'} \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right), \quad (487)$$

$$\left\langle \widetilde{B_{j\text{mod}2z}(t)} \widetilde{B_{j\text{mod}2z}(t')} \right\rangle_B = \text{Tr}_B \left(B_{j\text{mod}2z}(t, \tau) B_{j\text{mod}2z}(t', 0) \rho_B \right) \quad (488)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \quad (489)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} (q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) \sum_{\mathbf{k}'} (q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'}) \rho_B \right) \quad (490)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \quad (491)$$

$$+ \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{i\mathbf{k}}(t') b_{\mathbf{k}}^{\dagger} + q_{j\mathbf{k}}^*(t') b_{\mathbf{k}} \right) \rho_B \right), \quad (492)$$

$$\left\langle \widetilde{B_{j\text{mod}2z}(t)} \widetilde{B_{j\text{mod}2z}(t')} \right\rangle_B = \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(q_{j\mathbf{k}}(t) \mathbf{b}_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) \mathbf{b}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}'}(t') \mathbf{b}_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') \mathbf{b}_{\mathbf{k}'} \right) \rho_B \right) \quad (493)$$

$$+ \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}}(t') b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*(t') b_{\mathbf{k}} \right) \rho_B \right) \quad (494)$$

$$0 = \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(q_{\mathbf{j}\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} e^{i\omega_{\mathbf{k}}\tau} + q_{\mathbf{j}\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{\mathbf{j}\mathbf{k}'}(t') b_{\mathbf{k}'}^{\dagger} + q_{\mathbf{j}\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \quad (495)$$

$$\left\langle \widetilde{B_{j\text{mod}2z}(t)} \widetilde{B_{j\text{mod}2z}(t')} \right\rangle_B = 0 + \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}}(t') b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*(t') b_{\mathbf{k}} \right) \rho_B \right) \quad (496)$$

$$= \text{Tr}_B \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}(t') \left(b_{\mathbf{k}}^\dagger \right)^2 e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}^*(t') b_{\mathbf{k}}^2 e^{-i\omega_{\mathbf{k}}\tau} \right) \rho_B \quad (497)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) \quad (498)$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} \text{Tr}_B \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rho_B \right) + \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \text{Tr}_B \left(b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rho_B \right) \quad (499)$$

$$= \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle \right) d^2 \alpha_{\mathbf{k}} \quad (500)$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left\langle 0 \left| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle d^2\alpha_{\mathbf{k}} \right) \quad (501)$$

$$+ \sum_{\mathbf{k}} q_{\mathbf{k}}^*(t) q_{\mathbf{k}}(t') \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) | 0 \rangle d^2\alpha_{\mathbf{k}} \right) \quad (502)$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2\alpha_{\mathbf{k}} \right) \quad (503)$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left\langle 0 \left| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle d^2\alpha_{\mathbf{k}} \right) \quad (504)$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) | 0 \rangle d^2\alpha_{\mathbf{k}} \right) \quad (505)$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') \left(e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) | 0 \rangle d^2\alpha_{\mathbf{k}} \right), \quad (506)$$

$$= \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} \frac{\tau}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \right) \quad (507)$$

$$+q_{j\mathbf{k}}^*(t)q_{j\mathbf{k}}(t')e^{-i\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\langle 0|b_{\mathbf{k}}b_{\mathbf{k}}^\dagger+b_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^*+|\alpha_{\mathbf{k}}|^2|0\rangle d^2\alpha_{\mathbf{k}}\rangle \quad (508)$$

$$= \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} \langle 0 | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 | 0 \rangle + q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} \langle 0 | b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* | 0 \rangle \right) \quad (509)$$

$$+ q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 | 0 \rangle + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \langle 0 | b_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* | 0 \rangle \rangle d^2 \alpha_{\mathbf{k}} \quad (510)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \sum_{\mathbf{k}} \left(\langle 0 | q_{\mathbf{k}}(t) q_{\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} (b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2) + q_{\mathbf{k}}^*(t) q_{\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} (b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2) | 0 \rangle \right) d^2 \alpha_{\mathbf{k}} \quad (511)$$

$$= \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} \left(\langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle + \langle 0 | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | 0 \rangle \right) d^2 \alpha_{\mathbf{k}} \quad (512)$$

$$+ \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \left(\langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle + \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle \right) d^2\alpha_{\mathbf{k}}, \quad (513)$$

$$1 = \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} d^2 \alpha_{\mathbf{k}}, \quad (514)$$

$$b_{\mathbf{k}}^\dagger b_{\mathbf{k}} |0\rangle = 0, \quad (515)$$

$$b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} |0\rangle = |0\rangle, \quad (516)$$

$$\langle \widetilde{B_{j\text{mod}2z}}(t) \widetilde{B_{j\text{mod}2z}}(t') \rangle_B = \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left((q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau}) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle \right. \quad (517)$$

$$\left. + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') \left(e^{-i\omega_{\mathbf{k}}\tau} \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle \right) \right) d^2 \alpha_{\mathbf{k}} \quad (518)$$

$$= \frac{1}{\pi N} \int \sum_{\mathbf{k}} e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left((q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau}) |\alpha_{\mathbf{k}}|^2 + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \right) d^2 \alpha_{\mathbf{k}}, \quad (519)$$

$$\int_0^{2\pi} \int_0^{+\infty} r^2 e^{-\frac{r^2}{N}} r dr d\theta = \int |\alpha_{\mathbf{k}}|^2 e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} d^2 \alpha_{\mathbf{k}} \quad (520)$$

$$= \pi N^2 \quad (521)$$

$$\langle \widetilde{B_{j\text{mod}2z}}(t) \widetilde{B_{j\text{mod}2z}}(t') \rangle_B = \sum_{\mathbf{k}} \left((q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau}) N + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \right) \quad (522)$$

$$\langle \widetilde{B_{j\text{mod}2z}}(t) \widetilde{B_{j'\text{mod}2z}}(t') \rangle_B = \text{Tr}_B (B_{j\text{mod}2z}(t, \tau) B_{j'\text{mod}2z}(t', 0) \rho_B) \quad (523)$$

$$= \int d^2 \alpha P(\alpha) \langle \alpha | B_{j\text{mod}2z}(t, \tau) B_{j'\text{mod}2z}(t', 0) | \alpha \rangle \quad (524)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | B_{j\text{mod}2z}(t, \tau) B_{j'\text{mod}2z}(t', 0) | \alpha \rangle d^2 \alpha \quad (525)$$

$$= \langle \alpha | \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} (q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) \sum_{\mathbf{k}'} (q_{j'\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j'\mathbf{k}'}^*(t') b_{\mathbf{k}'}) | \alpha \rangle d^2 \alpha_{\mathbf{k}} \quad (526)$$

$$= \langle \alpha | \sum_{\mathbf{k} \neq \mathbf{k}'} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} (q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) (q_{j'\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j'\mathbf{k}'}^*(t') b_{\mathbf{k}'}) | \alpha \rangle d^2 \alpha_{\mathbf{k}} \quad (527)$$

$$+ \langle \alpha | \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} (q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) (q_{j'\mathbf{k}'}(t') b_{\mathbf{k}}^\dagger + q_{j'\mathbf{k}}^*(t') b_{\mathbf{k}}) | \alpha \rangle d^2 \alpha_{\mathbf{k}} \quad (528)$$

$$= \langle \alpha | \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} (q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) (q_{j'\mathbf{k}}(t') b_{\mathbf{k}}^\dagger + q_{j'\mathbf{k}}^*(t') b_{\mathbf{k}}) | \alpha \rangle d^2 \alpha_{\mathbf{k}} \quad (529)$$

$$= \left\langle \alpha \left| \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q_{j\mathbf{k}}(t) q_{j'\mathbf{k}}^*(t') b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \right| \alpha \right\rangle d^2 \alpha_{\mathbf{k}} \quad (530)$$

$$+ \left\langle \alpha \left| \sum_{\mathbf{k}} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q_{j\mathbf{k}}^*(t) q_{j'\mathbf{k}}(t') b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \right| \alpha \right\rangle d^2 \alpha_{\mathbf{k}} \quad (531)$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j'\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (532)$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j'\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}}, \quad (533)$$

$$\langle b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rangle_B = \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (534)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (535)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (536)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (537)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (538)$$

$$= N + 1, \quad (539)$$

$$\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rangle_B = \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \langle 0 | (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (540)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} |\alpha_{\mathbf{k}}|^2 d^2 \alpha_{\mathbf{k}} \quad (541)$$

$$= N, \quad (542)$$

$$\left\langle \widetilde{B_{j \bmod 2z}}(t) \widetilde{B_{j' \bmod 2z}}(t') \right\rangle_B = \sum_{\mathbf{k}} (q_{j\mathbf{k}}(t) q_{j'\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau} N + q_{j\mathbf{k}}^*(t) q_{j'\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} (N+1)) \quad (543)$$

$$= \sum_{\mathbf{k}} 2N (q_{j\mathbf{k}}(t) q_{j'\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}}\tau})^{\Re} + \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j'\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}}\tau} \quad (544)$$

$$D(h') D(h) = e^{\frac{1}{2}(h'h^* - h'^*h)} D(h' + h), \quad (545)$$

$$\langle D(h') D(h) \rangle_B = \text{Tr}_B \left(e^{\frac{1}{2}(h'h^* - h'^*h)} D(h' + h) \rho_B \right) \quad (546)$$

$$= e^{\frac{1}{2}(h'h^* - h'^*h)} \text{Tr}_B (D(h' + h) \rho_B) \quad (547)$$

$$= e^{\frac{1}{2}(h'h^* - h'^*h)} \frac{1}{\pi N} \int d^2\alpha P(\alpha) \langle \alpha | D(h' + h) | \alpha \rangle \quad (548)$$

$$= e^{\frac{1}{2}(h'h^* - h'^*h)} e^{-\frac{|h+h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}, \quad (549)$$

$$h' = h e^{i\omega\tau}, \quad (550)$$

$$\langle D(h e^{i\omega\tau}) D(h) \rangle_B = e^{\frac{1}{2}(h h^* e^{i\omega\tau} - h^* h e^{-i\omega\tau})} e^{-\frac{|h+h e^{i\omega\tau}|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}, \quad (551)$$

$$\frac{1}{2} |h|^2 (e^{i\omega\tau} - e^{-i\omega\tau}) = \frac{1}{2} (h h^* e^{i\omega\tau} - h^* h e^{-i\omega\tau}) \quad (552)$$

$$= \frac{1}{2} |h|^2 (\cos(\omega\tau) + i \sin(\omega\tau) - \cos(\omega\tau) + i \sin(\omega\tau)) \quad (553)$$

$$= \frac{1}{2} |h|^2 (2i \sin(\omega\tau)) \quad (554)$$

$$= i |h|^2 \sin(\omega\tau), \quad (555)$$

$$-\frac{|h + h e^{i\omega\tau}|^2}{2} = -|h|^2 \frac{|1 + e^{i\omega\tau}|^2}{2} \quad (556)$$

$$= -|h|^2 \frac{(1 + 2 \cos(\omega\tau) + \cos^2(\omega\tau)) + \sin^2(\omega\tau)}{2} \quad (557)$$

$$= -|h|^2 \frac{2 + 2 \cos(\omega\tau)}{2} \quad (558)$$

$$= -|h|^2 (1 + \cos(\omega\tau)), \quad (559)$$

$$\langle D(h e^{i\omega\tau}) D(h) \rangle_B = e^{i|h|^2 \sin(\omega\tau)} e^{-|h|^2 (1 + \cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)} \quad (560)$$

$$= e^{i|h|^2 \sin(\omega\tau) - |h|^2 (1 + \cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)} \quad (561)$$

$$= e^{-|h|^2 (-i \sin(\omega\tau) + \cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right))} e^{-|h|^2 \coth\left(\frac{\beta\omega}{2}\right)} \quad (562)$$

$$= \langle D(h) \rangle_B e^{-\phi(\tau)}, \quad (563)$$

$$e^{-\phi(\tau)} = e^{-|h|^2 (\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau))}, \quad (564)$$

$$\phi(\tau) = |h|^2 \left(\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau) \right), \quad (565)$$

$$\langle D(h') D(h) \rangle_B = e^{\frac{1}{2}(h'h^* - h'^*h)} e^{-\frac{|h+h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}, \quad (566)$$

$$h' = v e^{i\omega\tau}, \quad (567)$$

$$m_{\mathbf{k}}(t) = \frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \quad (568)$$

$$\Gamma_{\mathbf{k}}(t) = \frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \quad (569)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(t) \widetilde{B_1^+ B_0^-}(t') \right\rangle_B = \langle B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(t', 0) \rangle_B \quad (570)$$

$$= \langle B_{10}(t, \tau) B_{10}(t', 0) \rangle_B \quad (571)$$

$$= \text{Tr}_B (B_{10}(t, \tau) B_{10}(t', 0) \rho_B) \quad (572)$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \text{Tr}_B \left(\prod_{\mathbf{k}} (D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau})) \prod_{\mathbf{k}} (D(m_{\mathbf{k}}(t'))) \rho_B \right) \quad (573)$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \text{Tr}_B \left(\prod_{\mathbf{k}} (D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) D(m_{\mathbf{k}}(t'))) \rho_B \right) \quad (574)$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} e^{\frac{i}{2} (m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} m_{\mathbf{k}}^*(t') - (m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau})^* m_{\mathbf{k}}(t')) - \frac{|m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} + m_{\mathbf{k}}(t')|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (575)$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} e^{i(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} m_{\mathbf{k}}^*(t'))^{\Im} - \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (576)$$

$$= e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} e^{i(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} m_{\mathbf{k}}^*(t'))^{\Im} - \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (577)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(t) \widetilde{B_0^+ B_1^-}(t') \right\rangle_B = e^{\chi_{10}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} \left(e^{i(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau} m_{\mathbf{k}}^*(t'))^{\Im} - \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (578)$$

$$\langle D(h) b \rangle_B = \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle \alpha | D(h) b | \alpha \rangle \quad (579)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle \alpha | D(-\alpha) D(h) b D(\alpha) | \alpha \rangle \quad (580)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) b D(\alpha) | 0 \rangle \quad (581)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \quad (582)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \quad (583)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) (b + \alpha) | 0 \rangle \quad (584)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \alpha | 0 \rangle \quad (585)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \alpha | 0 \rangle \quad (586)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} e^{-\frac{|h|^2}{2}} d^2\alpha \quad (587)$$

$$= hN \langle D(h) \rangle_B, \quad (588)$$

$$\langle D(h) b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle \alpha | D(h) b^\dagger | \alpha \rangle \quad (589)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (590)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (591)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) | 0 \rangle \quad (592)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} \langle 0 | D(-\alpha) D(h) D(\alpha) (b^\dagger + \alpha^*) | 0 \rangle \quad (593)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) (b^\dagger + \alpha^*) | 0 \rangle \quad (594)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) \alpha^* | 0 \rangle \quad (595)$$

$$= \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \langle 0 | D(h) | 1 \rangle + \frac{1}{\pi N} \int d^2\alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 | D(h) | 0 \rangle \quad (596)$$

$$= \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} \langle -h|1 \rangle + \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} \alpha^* \langle 0|D(h)|0 \rangle, \quad (597)$$

$$\langle -h| = e^{-\frac{|-h^*|^2}{2}} \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n| \quad (598)$$

$$\langle -h|1 \rangle = e^{-\frac{|-h^*|^2}{2}} \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n|1 \rangle \quad (599)$$

$$\langle -h|1 \rangle = e^{-\frac{|-h^*|^2}{2}} (-h^*), \quad (600)$$

$$\langle D(h)b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} e^{-\frac{|-h^*|^2}{2}} (-h^*) + \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} \alpha^* e^{-\frac{|-h^*|^2}{2}} \quad (601)$$

$$\langle D(h)b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} e^{-\frac{|-h^*|^2}{2}} (-h^*) + \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} \alpha^* e^{-\frac{|-h^*|^2}{2}} \quad (602)$$

$$= -h^* \langle D(h) \rangle_B (N+1), \quad (603)$$

$$\langle bD(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} \langle \alpha | bD(h) | \alpha \rangle \quad (604)$$

$$= \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} e^{-\frac{|h|^2}{2}} h + \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} e^{-\frac{|h|^2}{2}} \quad (605)$$

$$= h \langle D(h) \rangle_B (N+1), \quad (606)$$

$$\langle b^\dagger D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} \langle \alpha | b^\dagger D(h) | \alpha \rangle \quad (607)$$

$$= \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} e^{-\frac{|h|^2}{2}} h + \frac{1}{\pi N} \int d^2 \alpha e^{-\frac{|\alpha|^2}{2}} e^{h\alpha^* - h^* \alpha} e^{-\frac{|h|^2}{2}} \quad (608)$$

$$= -h^* \langle D(h) \rangle_B N. \quad (609)$$

The correlation functions can be found readily as:

$$B_1^+ B_0^- (t, \tau) = \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} e^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right), \quad (610)$$

$$B_0^+ B_1^- (t, \tau) = \prod_{\mathbf{k}} \left(D(-m_{\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} e^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right), \quad (611)$$

$$B_{10}(t) = e^{\chi_{10}(t)} \left(e^{-\frac{1}{2} \sum_{\mathbf{k}} |m_{\mathbf{k}}(t)|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \right), \quad (612)$$

$$B_x(t, \tau) = \frac{B_1^+ B_0^- (t, \tau) + B_0^+ B_1^- (t, \tau) - B_{10}(t) - B_{01}(t)}{2}, \quad (613)$$

$$B_y(t, \tau) = \frac{B_0^+ B_1^- (t, \tau) - B_1^+ B_0^- (t, \tau) + B_{10}(t) - B_{01}(t)}{2i}, \quad (614)$$

$$B_{i \bmod 2z}(t, \tau) = \sum_{\mathbf{k}} \left(q_{i\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} \tau} + q_{i\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} \tau} \right), \quad (615)$$

$$\left\langle \widetilde{B_{i \bmod 2z}}(t) \widetilde{B_{j \bmod 2z}}(t') \right\rangle_B = \langle B_{i \bmod 2z}(t, \tau) B_{j \bmod 2z}(t', 0) \rangle_B \quad (616)$$

$$= \left\langle \sum_{\mathbf{k}} \left(q_{i\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} \tau} + q_{i\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} \tau} \right) \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t') b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*(t') b_{\mathbf{k}} \right) \right\rangle_B \quad (617)$$

$$= \sum_{\mathbf{k}} q_{i\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}} \tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} q_{i\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') e^{-i\omega_{\mathbf{k}} \tau} (N_{\mathbf{k}} + 1), \quad (618)$$

$$\left\langle \widetilde{B_x}(t) \widetilde{B_x}(t') \right\rangle_B = \langle B_x(t, \tau) B_x(t', 0) \rangle_B \quad (619)$$

$$= \left\langle \left(\frac{B_1^+ B_0^- (t, \tau) + B_0^+ B_1^- (t, \tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left(\frac{B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t', 0) - B_{10}(t') - B_{01}(t')}{2} \right) \right\rangle_B \quad (620)$$

$$= \frac{1}{4} \left\langle \left(B_1^+ B_0^- (t, \tau) + B_0^+ B_1^- (t, \tau) - B_{10}(t) - B_{01}(t) \right) \left(B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t', 0) - B_{10}(t') - B_{01}(t') \right) \right\rangle_B \quad (621)$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) + B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) - B_1^+ B_0^- (t, \tau) B_{10}(t') - B_1^+ B_0^- (t, \tau) B_{01}(t') \right. \quad (622)$$

$$\left. + B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) - B_0^+ B_1^- (t, \tau) B_{10}(t') - B_0^+ B_1^- (t, \tau) B_{01}(t') \right. \quad (623)$$

$$\left. - B_{10}(t) B_1^+ B_0^- (t', 0) - B_{10}(t) B_0^+ B_1^- (t', 0) + B_{10}(t) B_{10}(t') + B_{10}(t) B_{01}(t') - B_{01}(t) B_1^+ B_0^- (t', 0) \right. \quad (624)$$

$$\left. - B_{01}(t) B_0^+ B_1^- (t', 0) + B_{01}(t) B_{10}(t') + B_{01}(t) B_{01}(t') \right\rangle \quad (625)$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) + B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) + B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) \right. \quad (626)$$

$$\left. + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) \right\rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(t') + B_{10}(t'))}{4}, \quad (627)$$

$$U_{10}(t, t') = \prod_{\mathbf{k}} e^{i(m_{\mathbf{k}}(t) m_{\mathbf{k}}^*(t') e^{i\omega_{\mathbf{k}} \tau})^3}, \quad (628)$$

$$\langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) \rangle_B = \left\langle \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}} \tau}) e^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \prod_{\mathbf{k}} \left(D(-m_{\mathbf{k}}(t')) e^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \quad (629)$$

$$= e^{\chi_{10}(t) + \chi_{01}(t')} \left\langle \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}} \tau}) \right) \prod_{\mathbf{k}} \left(D(-m_{\mathbf{k}}(t')) \right) \right\rangle_B \quad (630)$$

$$= e^{\chi_{10}(t) + \chi_{01}(t')} \prod_{\mathbf{k}} \left\langle \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}} \tau}) D(-m_{\mathbf{k}}(t')) \right) \right\rangle_B \quad (631)$$

$$= e^{\chi_{10}(t) + \chi_{01}(t')} U_{10}^*(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (632)$$

$$\langle B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) \rangle_B = \left\langle \prod_{\mathbf{k}} \left(D(-m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}} \tau}) e^{-\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t')) e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \quad (633)$$

$$= \prod_{\mathbf{k}} e^{-\frac{\Gamma_{\mathbf{k}}(t)}{2}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle D(-m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}} \tau}) D(m_{\mathbf{k}}(t')) \right\rangle_B \quad (634)$$

$$= e^{\chi_{01}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} \left\langle D(-m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}} \tau}) D(m_{\mathbf{k}}(t')) \right\rangle_B \quad (635)$$

$$= e^{\chi_{01}(t) + \chi_{10}(t')} U_{10}^*(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (636)$$

$$\langle B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) \rangle_B = e^{\chi_{10}(t) + \chi_{10}(t')} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (637)$$

$$\langle B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) \rangle_B = e^{\chi_{01}(t) + \chi_{01}(t')} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (638)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_x(t') \rangle_B = \frac{1}{4} \left\langle B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) + B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) + B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) \right. \quad (639)$$

$$\left. + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) \right\rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(t') + B_{10}(t'))}{4}, \quad (640)$$

$$= \frac{1}{4} \left(2U_{10}(t, t') \left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (641)$$

$$\left. + 2U_{10}^*(t, t') \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (642)$$

$$- \left(e^{\chi_{10}(t)} e^{-\frac{1}{2} \sum_{\mathbf{k}} |m_{\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \left(e^{\chi_{01}(t')} e^{-\frac{1}{2} \sum_{\mathbf{k}} |m_{\mathbf{k}}(t')|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \quad (643)$$

$$= \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}} \tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (644)$$

$$+ \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (645)$$

$$- \left(e^{\chi_{10}(t)} e^{-\frac{1}{2} \sum_{\mathbf{k}} |m_{\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \left(e^{\chi_{01}(t')} e^{-\frac{1}{2} \sum_{\mathbf{k}} (|m_{\mathbf{k}}(t')|^2) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \quad (646)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_y(t') \rangle_B = \langle B_y(t, \tau) B_y(t', 0) \rangle_B \quad (647)$$

$$= \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \right. \quad (648)$$

$$\left. + \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (649)$$

$$= \left\langle \left(\frac{B_0^+ B_1^-(t, \tau) - B_1^+ B_0^-(t, \tau) + B_{10}(t) - B_{01}(t)}{2i} \right) \left(\frac{B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t', 0) + B_{10}(t') - B_{01}(t')}{2i} \right) \right\rangle_B \quad (650)$$

$$= -\frac{1}{4} \left\langle \left(B_0^+ B_1^-(t, \tau) - B_1^+ B_0^-(t, \tau) + B_{10}(t) - B_{01}(t) \right) \left(B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t', 0) + B_{10}(t') - B_{01}(t') \right) \right\rangle_B \quad (651)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(t', 0) - B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(t', 0) + B_0^+ B_1^-(t, \tau) B_{10}(t') - B_0^+ B_1^-(t, \tau) B_{01}(t') - B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(t', 0) \right. \quad (652)$$

$$\left. + B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(t', 0) - B_1^+ B_0^-(t, \tau) B_{10}(t') + B_1^+ B_0^-(t, \tau) B_{01}(t') + B_{10}(t) B_0^+ B_1^-(t', 0) - B_{10}(t) B_1^+ B_0^-(t', 0) \right. \quad (653)$$

$$\left. + B_{10}(t) B_{10}(t') - B_{10}(t) B_{01}(t') - B_{01}(t) B_0^+ B_1^-(t', 0) + B_{01}(t) B_1^+ B_0^-(t', 0) - B_{01}(t) B_{10}(t') + B_{01}(t) B_{01}(t') \right\rangle \quad (654)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(t', 0) - B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(t', 0) - B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(t', 0) + B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(t', 0) \right. \quad (655)$$

$$\left. + (B_{01}(t))^{\Im} (B_{10}(t'))^{\Im} \right\rangle \quad (656)$$

$$= -\frac{1}{4} \left(2 \left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (657)$$

$$\left. - 2 \left(e^{\chi_{01}(t) + \chi_{10}(t')} \right)^{\Re} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (658)$$

$$+ \left(e^{\chi_{01}(t)} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Im} \left(e^{\chi_{10}(t')} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Im} \quad (659)$$

$$= -\frac{1}{2} \left(\left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Re} U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (660)$$

$$\left. - \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (661)$$

$$+ \left(e^{\chi_{01}(t)} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Im} \left(e^{\chi_{10}(t')} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Im} \quad (662)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_y(t') \rangle_B = \langle B_x(t, \tau) B_y(t', 0) \rangle_B \quad (663)$$

$$= \left\langle \left(\frac{B_1^+ B_0^-(t, \tau) + B_0^+ B_1^-(t, \tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left(\frac{B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t', 0) + B_{10}(t') - B_{01}(t')}{2i} \right) \right\rangle_B \quad (664)$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(t', 0) + B_1^+ B_0^-(t, \tau) B_{10}(t') - B_1^+ B_0^-(t, \tau) B_{01}(t') + B_{01}(t) B_1^+ B_0^-(t', 0) \right. \quad (665)$$

$$\left. + B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(t', 0) - B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(t', 0) + B_0^+ B_1^-(t, \tau) B_{10}(t') - B_0^+ B_1^-(t, \tau) B_{01}(t') + B_{01}(t) B_{01}(t') \right. \quad (666)$$

$$\left. - B_{10}(t) B_0^+ B_1^-(t', 0) + B_{10}(t) B_1^+ B_0^-(t', 0) - B_{10}(t) B_{10}(t') + B_{10}(t) B_{01}(t') - B_{01}(t) B_0^+ B_1^-(t', 0) - B_{01}(t) B_{10}(t') \right\rangle_B \quad (667)$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(t', 0) + B_1^+ B_0^-(t, \tau) B_{10}(t') - B_1^+ B_0^-(t, \tau) B_{01}(t') \right. \quad (668)$$

$$\left. + B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(t', 0) - B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(t', 0) + B_0^+ B_1^-(t, \tau) B_{10}(t') - B_0^+ B_1^-(t, \tau) B_{01}(t') \right. \quad (669)$$

$$\left. - B_{10}(t) B_0^+ B_1^-(t', 0) + B_{10}(t) B_1^+ B_0^-(t', 0) - B_{10}(t) B_{10}(t') + B_{10}(t) B_{01}(t') - B_{01}(t) B_0^+ B_1^-(t', 0) \right. \quad (670)$$

$$\left. + B_{01}(t) B_1^+ B_0^-(t', 0) - B_{01}(t) B_{10}(t') + B_{01}(t) B_{01}(t') \right\rangle_B \quad (671)$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) - B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) \right\rangle \quad (672)$$

$$+ \frac{1}{4i} (B_{10} (t) + B_{01} (t)) (B_{10} (t') - B_{01} (t')) \quad (673)$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) \right. \quad (674)$$

$$\left. - B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) \right\rangle + \frac{1}{4i} (B_{10} (t) + B_{01} (t)) (B_{10} (t') - B_{01} (t')) \quad (675)$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) \right. \quad (676)$$

$$\left. - B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) \right\rangle + (B_{10} (t))^{\Re} (B_{10} (t'))^{\Im} \quad (677)$$

$$= \frac{1}{4i} \left(\left(e^{\chi_{10}(t) + \chi_{01}(t')} - e^{\chi_{01}(t) + \chi_{10}(t')} \right) U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (678)$$

$$\left. + \left(e^{\chi_{01}(t) + \chi_{01}(t')} - e^{\chi_{10}(t) + \chi_{10}(t')} \right) U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (679)$$

$$+ (B_{10} (t))^{\Re} (B_{10} (t'))^{\Im} \quad (680)$$

$$= \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (681)$$

$$\left. + \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + (B_{10}(t))^{\Re} (B_{10}(t'))^{\Im} \quad (682)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_x(t') \rangle_B = \left\langle \left(\frac{B_0^+ B_1^- (t, \tau) - B_1^+ B_0^- (t, \tau) + B_{10} (t) - B_{01} (t)}{2i} \right) \left(\frac{B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t', 0) - B_{10} (t') - B_{01} (t')}{2} \right) \right\rangle_B \quad (683)$$

$$= \frac{1}{4i} \left\langle \left(B_0^+ B_1^- (t, \tau) - B_1^+ B_0^- (t, \tau) + B_{10} (t) - B_{01} (t) \right) \left(B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t', 0) - B_{10} (t') - B_{01} (t') \right) \right\rangle_B \quad (684)$$

$$= \frac{1}{4i} \left\langle B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) - B_0^+ B_1^- (t, \tau) B_{10} (t') - B_0^+ B_1^- (t, \tau) B_{01} (t') + B_{01} (t) B_{01} (t') \right. \quad (685)$$

$$\left. - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) - B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) + B_1^+ B_0^- (t, \tau) B_{10} (t') + B_1^+ B_0^- (t, \tau) B_{01} (t') \right. \quad (686)$$

$$\left. + B_{10} (t) B_0^+ B_1^- (t', 0) - B_{10} (t) B_{10} (t') - B_{10} (t) B_{01} (t') - B_{01} (t) B_1^+ B_0^- (t', 0) - B_{01} (t) B_0^+ B_1^- (t', 0) \right. \quad (687)$$

$$\left. + B_{01} (t) B_{10} (t') + B_{10} (t) B_1^+ B_0^- (t', 0) \right\rangle \quad (688)$$

$$= \frac{1}{4i} \left\langle B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (t', 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (t', 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (t', 0) - B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (t', 0) \right\rangle \quad (689)$$

$$+ (B_{10} (t))^{\Im} (B_{10} (t'))^{\Re} \quad (690)$$

$$= \frac{1}{4i} \left(e^{\chi_{01}(t) + \chi_{10}(t')} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (691)$$

$$\left. + e^{\chi_{01}(t) + \chi_{01}(t')} U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (692)$$

$$- e^{\chi_{10}(t) + \chi_{10}(t')} U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (693)$$

$$- e^{\chi_{10}(t) + \chi_{01}(t')} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \left. \right) + (B_{10} (t))^{\Im} (B_{10} (t'))^{\Re} \quad (694)$$

$$= \frac{1}{4i} \left(2i \left(e^{\chi_{01}(t) + \chi_{10}(t')} \right)^{\Im} U_{10}^* (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (695)$$

$$\left. + 2i \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10} (t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + (B_{10}(t))^{\Im} (B_{10}(t'))^{\Re} \quad (696)$$

$$= (B_{10}(t))^{\Im} (B_{10}(t'))^{\Re} + \frac{1}{2} \left(\left(e^{\chi_{01}(t) + \chi_{10}(t')} \right)^{\Im} U_{10}^*(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (697)$$

$$\left. + \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (698)$$

$$\langle b^\dagger D(h) \rangle_B = -h^* \langle D(h) \rangle_B N \quad (699)$$

$$\langle b D(h) \rangle_B = h \langle D(h) \rangle_B (N+1) \quad (700)$$

$$\langle D(h) b^\dagger \rangle_B = -h^* \langle D(h) \rangle_B (N+1) \quad (701)$$

$$\langle D(h) b \rangle_B = h \langle D(h) \rangle_B N \quad (702)$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) = q_{j\mathbf{k}}(t) \quad (703)$$

$$\langle B_1^+ B_0^-(t, \tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \rangle_B = \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \left\langle \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) \right) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \right\rangle_B \quad (704)$$

$$= e^{\chi_{10}(t)} \left\langle D(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau}) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) \right) \right\rangle_B \quad (705)$$

$$= e^{\chi_{10}(t)} q_{i\mathbf{k}'}(t') \left\langle D(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau}) b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) \right) \right\rangle_B \quad (706)$$

$$= e^{\chi_{10}(t)} q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) \left\langle \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) \right) \right\rangle_B \quad (707)$$

$$= q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) e^{\chi_{10}(t)} \left\langle \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) \right) \right\rangle_B \quad (708)$$

$$= -q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) \quad (709)$$

$$\langle B_1^+ B_0^-(t, \tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \rangle_B = q_{i\mathbf{k}'}^*(t') \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} \left\langle \prod_{\mathbf{k}} D(m_{\mathbf{k}}(t) e^{i\omega_{\mathbf{k}}\tau}) \right\rangle \quad (710)$$

$$= q_{i\mathbf{k}'}^*(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}(t), \quad (711)$$

$$\langle B_0^+ B_1^-(t, \tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \rangle_B = -q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01}(t), \quad (712)$$

$$\langle B_0^+ B_1^-(t, \tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \rangle_B = q_{i\mathbf{k}'}^*(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}(t), \quad (713)$$

$$q_{i\mathbf{k}'}(0) = g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \quad (714)$$

$$\langle B_x(t, \tau) B_{\text{imod}2z}(t', 0) \rangle_B = \left\langle \left(\frac{B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{10}^*}{2} \right) \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(0) b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(0) b_{\mathbf{k}'} \right) \right\rangle_B \quad (715)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle (B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{10}^*) \left(q_{i\mathbf{k}'}(0) b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(0) b_{\mathbf{k}'} \right) \right\rangle_B \quad (716)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle (B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau)) \left(q_{i\mathbf{k}'}(0) b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(0) b_{\mathbf{k}'} \right) \right\rangle_B \quad (717)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle B_{1+} B_{0-}(\tau) q_{i\mathbf{k}'}(0) b_{\mathbf{k}'}^\dagger + B_{0+} B_{1-}(\tau) q_{i\mathbf{k}'}(0) b_{\mathbf{k}'}^\dagger + B_{1+} B_{0-}(\tau) q_{i\mathbf{k}'}^*(0) b_{\mathbf{k}'} \right. \quad (718)$$

$$\left. + B_{0+} B_{1-}(\tau) q_{i\mathbf{k}'}^*(0) b_{\mathbf{k}'} \right\rangle_B \quad (719)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) + q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01}(t) \right. \quad (720)$$

$$\left. + q_{i\mathbf{k}'}^*(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}(t) + q_{i\mathbf{k}'}^*(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}(t) \right) \quad (721)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) + q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01}(t) \right. \quad (722)$$

$$\left. + q_{i\mathbf{k}'}^*(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}(t) + q_{i\mathbf{k}'}^*(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}(t) \right) \quad (723)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \left(\left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{10}(t) + \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{01}(t) \right) \right. \quad (724)$$

$$\left. + q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} B_{10}(t) - m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} B_{01}(t) \right) \right) \quad (725)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \left(\left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{10}(t) - \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{01}(t) \right) \right. \quad (726)$$

$$\left. + q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} B_{10}(t) - m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} B_{01}(t) \right) \right) \quad (727)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (B_{10}(t) - B_{01}(t)) + q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} (B_{10}(t) \right. \quad (728)$$

$$-B_{01}(t))) \quad (729)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} 2iB_{10}^{\mathbb{S}}(t) \left(q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) - q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* \right) \quad (730)$$

$$= i \sum_{\mathbf{k}'} B_{10}^{\mathbb{S}}(t) \left(q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} - q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* \right) \quad (731)$$

$$= i \sum_{\mathbf{k}'} B_{10}^{\mathbb{S}}(t) \left(q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} - q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) (m_{\mathbf{k}'}(t))^* e^{-i\omega_{\mathbf{k}'}\tau} \right), \quad (732)$$

$$= i \sum_{\mathbf{k}'} B_{10}^{\mathbb{S}}(t) \left(q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} - q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) (m_{\mathbf{k}'}(t))^* e^{-i\omega_{\mathbf{k}'}\tau} \right) \quad (733)$$

$$\langle B_{\text{imod}2z}(t, \tau) B_x(t', 0) \rangle_B = \left\langle \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(\frac{B_1^+ B_0^-(t', 0) + B_0^+ B_1^-(t', 0) - B_{10}(t') - B_{01}(t')}{2} \right) \right\rangle_B \quad (734)$$

$$= \sum_{\mathbf{k}'} \left\langle \left(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(\frac{B_1^+ B_0^-(t', 0) + B_0^+ B_1^-(t', 0) - B_{10}(t') - B_{01}(t')}{2} \right) \right\rangle_B \quad (735)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle \left(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(B_1^+ B_0^-(t', 0) + B_0^+ B_1^-(t', 0) - B_{10}(t') - B_{01}(t') \right) \right\rangle_B \quad (736)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle \left(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(B_1^+ B_0^-(t', 0) + B_0^+ B_1^-(t', 0) \right) \right\rangle_B \quad (737)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-(t', 0) + q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^-(t', 0) + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-(t', 0) \right. \quad (738)$$

$$\left. + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^-(t', 0) \right\rangle, \quad (739)$$

$$\langle q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-(t', 0) \rangle_B = q_{i\mathbf{k}'}(t) \langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-(t', 0) \rangle_B \quad (740)$$

$$= q_{i\mathbf{k}'}(t) \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t')) e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \quad (741)$$

$$= q_{i\mathbf{k}'}(t) \left\langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} \left(D(m_{\mathbf{k}'}(t')) e^{\frac{\Gamma_{\mathbf{k}'}(t')}{2}} \right) \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D(m_{\mathbf{k}}(t')) e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \quad (742)$$

$$= q_{i\mathbf{k}'}(t) \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D(m_{\mathbf{k}}(t')) \right\rangle_B \langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} D(m_{\mathbf{k}'}(t')) \rangle_B \quad (743)$$

$$= q_{i\mathbf{k}'}(t) \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D(m_{\mathbf{k}}(t')) \right\rangle_B \langle b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} D(m_{\mathbf{k}'}(t')) \rangle_B \quad (744)$$

$$= q_{i\mathbf{k}'}(t) \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D(m_{\mathbf{k}}(t')) \right\rangle_B \langle b_{\mathbf{k}'}^\dagger D(m_{\mathbf{k}'}(t')) \rangle_B e^{i\omega_{\mathbf{k}'}\tau} \quad (745)$$

$$= q_{i\mathbf{k}'}(t) \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} D(m_{\mathbf{k}}(t')) \right\rangle_B (-m_{\mathbf{k}'}^*(t') \langle D(m_{\mathbf{k}'}(t')) \rangle_B N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \quad (746)$$

$$= -m_{\mathbf{k}'}^*(t') q_{i\mathbf{k}'}(t) \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle \prod_{\mathbf{k}} D(m_{\mathbf{k}}(t')) \right\rangle_B N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \quad (747)$$

$$= -m_{\mathbf{k}'}^*(t') q_{i\mathbf{k}'}(t) B_{10}(t') N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau}, \quad (748)$$

$$\langle q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^-(t', 0) \rangle_B = m_{\mathbf{k}'}^*(t') q_{i\mathbf{k}'}(t) B_{01}(t') N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau}, \quad (749)$$

$$\langle q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-(t', 0) \rangle_B = q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} \langle b_{\mathbf{k}'} B_1^+ B_0^-(t', 0) \rangle_B \quad (750)$$

$$= q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} \prod_{\mathbf{k}} \left(D(m_{\mathbf{k}}(t')) e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \quad (751)$$

$$= q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \langle b_{\mathbf{k}'} D(m_{\mathbf{k}'}(t')) \rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} (D(m_{\mathbf{k}}(t')))) \right\rangle_B \quad (752)$$

$$= q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \langle D(m_{\mathbf{k}'}(t')) \rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} (D(m_{\mathbf{k}}(t')))) \right\rangle_B \quad (753)$$

$$= q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} e^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \langle D(m_{\mathbf{k}'}(t')) \rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} (D(m_{\mathbf{k}}(t')))) \right\rangle_B \quad (754)$$

$$= q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{10}(t'), \quad (755)$$

$$\left\langle (q_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_{0+B_1}^-(t', 0) \right\rangle_B = q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} (-m_{\mathbf{k}'}(t')) (N_{\mathbf{k}'} + 1) B_{01}(t'), \quad (756)$$

$$\langle B_{\text{imod}2z}(t, \tau) B_x(t', 0) \rangle_B = \frac{1}{2} \sum_{\mathbf{k}'} \left(-m_{\mathbf{k}'}^*(t') q_{i\mathbf{k}'}(t) B_{10}(t') N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} - (-m_{\mathbf{k}'}^*(t')) q_{i\mathbf{k}'}(t) B_{01}(t') N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \right. \quad (757)$$

$$\left. + q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{10}(t') + q_{i\mathbf{k}'}^*(t) e^{-i\omega_{\mathbf{k}'}\tau} (-m_{\mathbf{k}'}(t')) (N_{\mathbf{k}'} + 1) B_{01}(t') \right) \quad (758)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t) N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t') (B_{01}(t') - B_{10}(t')) + q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') e^{-i\omega_{\mathbf{k}'}\tau} \right. \quad (759)$$

$$\left. \times (N_{\mathbf{k}'} + 1) (B_{10}(t') - B_{01}(t')) \right) \quad (760)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t) N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t') (B_{01}(t') - B_{10}(t')) - q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') e^{-i\omega_{\mathbf{k}'}\tau} \right. \quad (761)$$

$$\left. \times (N_{\mathbf{k}'} + 1) (B_{01}(t') - B_{10}(t')) \right) \quad (762)$$

$$= i \sum_{\mathbf{k}'} B_{01}^{\text{S}}(t') \left(q_{i\mathbf{k}'}(t) N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t') - q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') e^{-i\omega_{\mathbf{k}'}\tau} (N_{\mathbf{k}'} + 1) \right), \quad (763)$$

$$\langle B_y(t, \tau) B_{\text{imod}2z}(t', 0) \rangle_B = \left\langle \left(\frac{B_0^+ B_1^-(t, \tau) - B_1^+ B_0^-(t, \tau) + B_{10}(t) - B_{01}(t)}{2i} \right) \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \right\rangle_B \quad (764)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left\langle (B_0^+ B_1^-(t, \tau) - B_1^+ B_0^-(t, \tau) + B_{10}(t) - B_{01}(t)) \left(q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \right\rangle_B \quad (765)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left\langle (B_0^+ B_1^-(t, \tau) - B_1^+ B_0^-(t, \tau)) \left(q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \right\rangle_B \quad (766)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left\langle B_0^+ B_1^-(t, \tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger - B_1^+ B_0^-(t, \tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + B_0^+ B_1^-(t, \tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right. \quad (767)$$

$$\left. - B_1^+ B_0^-(t, \tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right\rangle, \quad (768)$$

$$\left\langle B_0^+ B_1^-(t, \tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \right\rangle_B = -q_{i\mathbf{k}'}(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01}(t), \quad (769)$$

$$\left\langle B_0^+ B_1^-(t, \tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right\rangle_B = q_{i\mathbf{k}'}^*(t') \left(-m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}(t), \quad (770)$$

$$\left\langle B_1^+ B_0^-(t, \tau) q_{i\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger \right\rangle_B = -q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}(t), \quad (771)$$

$$\left\langle B_1^+ B_0^-(t, \tau) q_{i\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right\rangle_B = q_{i\mathbf{k}'}^*(t') m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} N_{\mathbf{k}'} B_{10}(t), \quad (772)$$

$$\langle B_y(t, \tau) B_{\text{imod}2z}(t', 0) \rangle_B = \frac{1}{2i} \sum_{\mathbf{k}'} (B_{01}(t) + B_{10}(t)) \left(q_{i\mathbf{k}'}(t') \left(m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) - q_{i\mathbf{k}'}^*(t') m_{\mathbf{k}'}(t) e^{i\omega_{\mathbf{k}'}\tau} N_{\mathbf{k}'} \right) \quad (773)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}^*(t') (N_{\mathbf{k}'} + 1) e^{-i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t) - q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t) \right) (B_{10}(t) + B_{01}(t)), \quad (774)$$

$$= \frac{2}{2i} \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) e^{-i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t) (B_{10}(t))^{\Re} - q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t) (B_{10}(t))^{\Re} \right), \quad (775)$$

$$= \frac{(B_{10}(t))^{\Re}}{i} \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) e^{-i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}^*(t) - q_{i\mathbf{k}'}^*(t') N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t) \right), \quad (776)$$

$$\langle B_{\text{imod}2z}(t, \tau) B_y(t', 0) \rangle_B = \left\langle \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(\frac{B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t', 0) + B_{10}(t') - B_{01}(t')}{2i} \right) \right\rangle_B \quad (777)$$

$$= \frac{1}{2i} \left\langle \sum_{\mathbf{k}'} \left(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) (B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t', 0) + B_{10}(t') \right. \quad (778)$$

$$\left. - B_{01}(t') \right) \rangle_B \quad (779)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left\langle \left(q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) (B_0^+ B_1^-(t', 0) - B_1^+ B_0^-(t', 0)) \right\rangle_B \quad (780)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left\langle q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^-(t', 0) - q_{i\mathbf{k}'}(t) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-(t', 0) \right. \quad (781)$$

$$+q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- (t', 0) - q_{i\mathbf{k}'}^*(t) b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- (t', 0) \rangle \quad (782)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left\langle e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) \langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- (t', 0) \rangle - e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) \langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- (t', 0) \rangle \right. \quad (783)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) \langle b_{\mathbf{k}'} B_0^+ B_1^- (t', 0) \rangle - e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) \langle b_{\mathbf{k}'} B_1^+ B_0^- (t', 0) \rangle \right\rangle \quad (784)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) \langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- (t', 0) \rangle - e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) \langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- (t', 0) \rangle + e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) \langle b_{\mathbf{k}'} B_0^+ B_1^- (t', 0) \rangle \quad (785)$$

$$- e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) \langle b_{\mathbf{k}'} B_1^+ B_0^- (t', 0) \rangle \quad (786)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) \langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- (t', 0) \rangle - e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) \langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- (t', 0) \rangle + e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) \langle b_{\mathbf{k}'} B_0^+ B_1^- (t', 0) \rangle \right. \quad (787)$$

$$\left. - e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) \langle b_{\mathbf{k}'} B_1^+ B_0^- (t', 0) \rangle \right) \quad (788)$$

$$\langle b_{\mathbf{k}'}^\dagger B_1^+ B_0^- (t', 0) \rangle_B = -m_{\mathbf{k}'}^*(t') B_{10}(t') N_{\mathbf{k}'}, \quad (789)$$

$$\langle b_{\mathbf{k}'}^\dagger B_0^+ B_1^- (t', 0) \rangle_B = m_{\mathbf{k}'}^*(t') B_{01}(t') N_{\mathbf{k}'}, \quad (790)$$

$$\langle b_{\mathbf{k}'} B_1^+ B_0^- (t', 0) \rangle_B = m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{10}(t'), \quad (791)$$

$$\langle b_{\mathbf{k}'} B_0^+ B_1^- (t', 0) \rangle_B = -m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{01}(t'), \quad (792)$$

$$\langle B_{i\text{mod}2z}(t, \tau) B_y(t', 0) \rangle_B = \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) (-(-m_{\mathbf{k}'}^*(t')) B_{01}(t') N_{\mathbf{k}'}) - e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) (-m_{\mathbf{k}'}^*(t') B_{10}(t') N_{\mathbf{k}'}) \right. \quad (793)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) (-m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{01}(t')) - e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{10}(t') \right) \quad (794)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (-q_{i\mathbf{k}'}(t) (-m_{\mathbf{k}'}^*(t')) B_{01}(t') N_{\mathbf{k}'} + q_{i\mathbf{k}'}(t) m_{\mathbf{k}'}^*(t') B_{10}(t') N_{\mathbf{k}'}) \right. \quad (795)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} (q_{i\mathbf{k}'}^*(t) (-m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{01}(t') - q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) B_{10}(t')) \right) \quad (796)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} (B_{10}(t') + B_{01}(t')) \left(e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) m_{\mathbf{k}'}^*(t') N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) \right) \quad (797)$$

$$= \frac{1}{i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) m_{\mathbf{k}'}^*(t') B_{10}^{\Re}(t') N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') B_{10}^{\Re}(t') (N_{\mathbf{k}'} + 1) \right) \quad (798)$$

$$= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') B_{10}^{\Re}(t') (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) m_{\mathbf{k}'}^*(t') B_{10}^{\Re}(t') N_{\mathbf{k}'} \right) \quad (799)$$

$$= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') B_{10}^{\Re}(t') (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) m_{\mathbf{k}'}^*(t') B_{10}^{\Re}(t') N_{\mathbf{k}'} \right) \quad (800)$$

$$= i B_{10}^{\Re}(t') \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}^*(t) m_{\mathbf{k}'}(t') (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} q_{i\mathbf{k}'}(t) m_{\mathbf{k}'}^*(t') N_{\mathbf{k}'} \right). \quad (801)$$

The correlation functions are equal to:

$$\langle \widetilde{B_{i\text{mod}2z}}(t) \widetilde{B_{j\text{mod}2z}}(t') \rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t'))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t')) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \quad (802)$$

$$\langle \widetilde{B_x}(t) \widetilde{B_x}(t') \rangle_B = \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (803)$$

$$\left. + \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}^*(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (804)$$

$$- \left(e^{\chi_{10}(t)} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \left(e^{\chi_{01}(t')} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \quad (805)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_y(t') \rangle_B = -\frac{1}{2} \left(\left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (806)$$

$$\left. - \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}^*(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \quad (807)$$

$$+ \left(e^{\chi_{01}(t)} \left(e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \right)^{\Im} \left(e^{\chi_{10}(t')} \left(e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \right)^{\Im} \quad (808)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_y(t') \rangle_B = \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}^*(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (809)$$

$$+ \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \Bigg) + (B_{10}(t))^{\Re} (B_{10}(t'))^{\Im} \quad (810)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_x(t') \rangle_B = \frac{1}{2} \left(\left(e^{\chi_{01}(t) + \chi_{10}(t')} \right)^{\Im} U_{10}^*(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \quad (811)$$

$$+ \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}(t, t') \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \Bigg) + (B_{10}(t))^{\Im} (B_{10}(t'))^{\Re} \quad (812)$$

$$\langle \widetilde{B}_{i\text{mod}2z}(t) \widetilde{B}_x(t') \rangle_B = i \sum_{\mathbf{k}} B_{01}^{\Im}(t') \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (813)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_{i\text{mod}2z}(t') \rangle_B = i B_{10}^{\Im}(t) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t'))^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t')) \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (814)$$

$$\langle \widetilde{B}_{i\text{mod}2z}(t) \widetilde{B}_y(t') \rangle_B = i B_{10}^{\Re}(t') \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right), \quad (815)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_{i\text{mod}2z}(t') \rangle_B = i B_{10}^{\Re}(t) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t'))^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t')) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right). \quad (816)$$

Let's consider the following expression related to the sum of coupling constants for a bath over all the frequencies:

$$L_i(\omega) \equiv \sum_{\mathbf{k}} g_{i\mathbf{k}} \sqrt{\delta(\omega - \omega_{\mathbf{k}})}. \quad (817)$$

Under this definition we have the following expression for a function $f(\omega) \in L^2$:

$$\int_0^\infty f(\omega) L_i(\omega) L_j^*(\omega) d\omega \approx \int_0^\infty f(\omega) \sum_{\mathbf{k}} g_i(\omega_{\mathbf{k}}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} \sum_{\mathbf{k}'} g_j(\omega_{\mathbf{k}'}) \sqrt{\delta(\omega - \omega_{\mathbf{k}'})} d\omega \quad (818)$$

$$= \int_0^\infty f(\omega) \sum_{\mathbf{k}, \mathbf{k}'} g_i(\omega_{\mathbf{k}}) g_j(\omega_{\mathbf{k}'}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} \sqrt{\delta(\omega - \omega_{\mathbf{k}'})} d\omega, \quad (819)$$

$$\int_0^\infty f(\omega) \sum_{\mathbf{k}} g_i(\omega_{\mathbf{k}}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} d\omega = \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} d\omega. \quad (820)$$

Now we will approach to the function $\sqrt{\delta(\omega - \omega_{\mathbf{k}})}$ using the normal distribution, so:

$$\delta(\omega - \omega_{\mathbf{k}}) = \lim_{\sigma \rightarrow 0^+} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\omega - \omega_{\mathbf{k}})^2}{2\sigma^2}} \quad (821)$$

$$\sqrt{\delta(\omega - \omega_{\mathbf{k}})} = \lim_{\sigma \rightarrow 0^+} \sqrt{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\omega - \omega_{\mathbf{k}})^2}{2\sigma^2}}} \quad (822)$$

$$= \lim_{\sigma \rightarrow 0^+} \sqrt{\sqrt{2\pi}\sigma} \frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2\sigma}} e^{-\frac{(\omega - \omega_{\mathbf{k}})^2}{4\sigma^2}} \quad (823)$$

$$= \lim_{\sigma \rightarrow 0^+} \sqrt{\sqrt{2\pi}\sigma} \frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2\sigma}} e^{-\frac{(\omega - \omega_{\mathbf{k}})^2}{2(\sqrt{2}\sigma)^2}} \quad (824)$$

$$= \lim_{\sigma \rightarrow 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma} N(x; \omega_{\mathbf{k}}, \sqrt{2}\sigma). \quad (825)$$

So we can obtain that:

$$\sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} d\omega = \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} d\omega \quad (826)$$

$$= \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) \left(\lim_{\sigma \rightarrow 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma} N(x; \omega_{\mathbf{k}}, \sqrt{2}\sigma) \right) d\omega \quad (827)$$

$$= \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) \left(\lim_{\sigma \rightarrow 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma} N(x; \omega_{\mathbf{k}}, \sqrt{2}\sigma) \right) d\omega \quad (828)$$

$$= \lim_{\sigma \rightarrow 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma} \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) N(x; \omega_{\mathbf{k}}, \sqrt{2}\sigma) d\omega \quad (829)$$

$$= \sum_{\mathbf{k}} \left(\lim_{\sigma \rightarrow 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma} \right) \left(\lim_{\sigma \rightarrow 0^+} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) N(x; \omega_{\mathbf{k}}, \sqrt{2}\sigma) d\omega \right) \quad (830)$$

$$= \sum_{\mathbf{k}} \left(\lim_{\sigma \rightarrow 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma} \right) f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) \text{ (with } f(\omega), g_i(\omega) \in L^2 \text{)} \quad (831)$$

$$= \lim_{\sigma \rightarrow 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma} \sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) \text{ (with } f(\omega), g_i(\omega) \in L^2 \text{)} \quad (832)$$

$$= 0 \text{ (because the sum } \sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) \text{ is finite).} \quad (833)$$

Then we can proof the following:

$$\int_0^\infty f(\omega) L_i(\omega) L_j^*(\omega) d\omega \approx \int_0^\infty f(\omega) \sum_{\mathbf{k}} g_i(\omega_{\mathbf{k}}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} \sum_{\mathbf{k}'} g_j^*(\omega_{\mathbf{k}'}) \sqrt{\delta(\omega - \omega_{\mathbf{k}'})} d\omega \quad (834)$$

$$= \int_0^\infty f(\omega) \sum_{\mathbf{k}, \mathbf{k}'} g_i(\omega_{\mathbf{k}}) g_j^*(\omega_{\mathbf{k}'}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} \sqrt{\delta(\omega - \omega_{\mathbf{k}'})} d\omega \quad (835)$$

$$= \sum_{\mathbf{k} \neq \mathbf{k}'} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) g_j^*(\omega_{\mathbf{k}'}) \sqrt{\delta(\omega - \omega_{\mathbf{k}})} \sqrt{\delta(\omega - \omega_{\mathbf{k}'})} d\omega + \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) g_j^*(\omega_{\mathbf{k}}) \delta(\omega - \omega_{\mathbf{k}}) d\omega \quad (836)$$

$$= 0 + \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) g_j^*(\omega_{\mathbf{k}}) \delta(\omega - \omega_{\mathbf{k}}) d\omega \quad (837)$$

$$= \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) g_j^*(\omega_{\mathbf{k}}) \delta(\omega - \omega_{\mathbf{k}}) d\omega \quad (838)$$

$$= \sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) g_j^*(\omega_{\mathbf{k}}) \quad (839)$$

if $i = j$ we recover the spectral density defined in the usual way when we integrate for a function $f(\omega)$ that belongs to the set L^2 :

$$\sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) = \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) \delta(\omega - \omega_{\mathbf{k}}) d\omega \quad (840)$$

$$= \int_0^\infty f(\omega) J_{ii}(\omega) d\omega \quad (841)$$

$$= \int_0^\infty f(\omega) |L_i(\omega)|^2 d\omega. \quad (842)$$

where

$$J_{ii}(\omega) = \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \quad (843)$$

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t). \quad (844)$$

In this case $g_i(\omega)$ and $v_i(\omega, t)$ are the continuous version of $g_i(\omega_{\mathbf{k}})$ and $v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t)$ respectively.

The integral version of the correlation functions can be obtained as follows:

$$\langle \widetilde{B_{iz}(t) B_{j \bmod 2z}(t')} \rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t'))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t')) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \quad (845)$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}}, t)) g_{j\mathbf{k}}^* (1 - F_j(\omega_{\mathbf{k}}, t'))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}, t))^* g_{j\mathbf{k}} (1 - F_j(\omega_{\mathbf{k}}, t')) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (846)$$

$$\approx \int_0^\infty \left(L_i(\omega) L_j^*(\omega) (1 - F_i(\omega, t)) (1 - F_j^*(\omega, t')) e^{i\omega\tau} N(\omega) + L_i^*(\omega) L_j(\omega) (1 - F_i^*(\omega, t)) (1 - F_j(\omega, t')) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega, \quad (847)$$

$$\chi_{10}(t) = \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \quad (848)$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \quad (849)$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g_{1\mathbf{k}}^* g_{0\mathbf{k}} F_1^*(\omega_{\mathbf{k}}, t) F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} g_{0\mathbf{k}}^* F_1(\omega_{\mathbf{k}}, t) F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \quad (850)$$

$$\approx \int_0^\infty \frac{L_0(\omega) L_1^*(\omega) F_1^*(\omega, t) F_0(\omega, t) - L_1(\omega) L_0^*(\omega) F_1(\omega, t) F_0^*(\omega, t)}{2\omega^2} d\omega, \quad (851)$$

$$U_{10}(t, t') = \prod_{\mathbf{k}} e^{i \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* e^{i\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)} \quad (852)$$

$$= e^{i \sum_{\mathbf{k}} \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* e^{i\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)} \quad (853)$$

$$= e^{i \left(\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* e^{i\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)} \quad (854)$$

$$= e^{i \left(\sum_{\mathbf{k}} \frac{(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t))(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t'))^* e^{i\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)} \quad (855)$$

$$\approx e^{i \left(\int_0^\infty \frac{(L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t))(L_1(\omega) F_1(\omega, t') - L_0(\omega) F_0(\omega, t'))^* e^{i\omega\tau}}{\omega^2} d\omega \right)} \quad (856)$$

$$B_{10}(t) = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \left(e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right), \quad (857)$$

$$= e^{\chi_{10}(t)} e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (858)$$

$$\approx e^{\chi_{10}(t)} e^{-\frac{1}{2} \int_0^\infty \left| \frac{L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t)}{\omega} \right|^2 \coth\left(\frac{\beta\omega}{2}\right) d\omega} \quad (859)$$

$$\xi^+(t, t') = \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (860)$$

$$= e^{-\sum_{\mathbf{k}} \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (861)$$

$$= e^{-\sum_{\mathbf{k}} \frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}}, t))e^{i\omega_{\mathbf{k}}\tau} + g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}}, t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (862)$$

$$\approx e^{-\int_0^\infty \frac{|(L_1(\omega)F_1(\omega, t) - L_0(\omega)F_0(\omega, t))e^{i\omega\tau} + L_1(\omega)F_1(\omega, t') - L_0(\omega)F_0(\omega, t')|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \quad (863)$$

$$\xi^-(t, t') = \prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (864)$$

$$= e^{-\sum_{\mathbf{k}} \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (865)$$

$$= e^{-\sum_{\mathbf{k}} \frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}}, t))e^{i\omega_{\mathbf{k}}\tau} - (g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}}, t'))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (866)$$

$$\approx e^{-\int_0^\infty \frac{|(L_1(\omega)F_1(\omega, t) - L_0(\omega)F_0(\omega, t))e^{i\omega\tau} - (L_1(\omega)F_1(\omega, t') - L_0(\omega)F_0(\omega, t'))|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \quad (867)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_x(t') \rangle_B = \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} U_{10}(t, t') \xi^+(t, t') + \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}^*(t, t') \xi^-(t, t') \right) - (B_{10}(t))^{\Re} (B_{01}(t'))^{\Re} \quad (868)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_y(t') \rangle_B = -\frac{1}{2} \left(\left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}(t, t') \xi^+(t, t') - \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} U_{10}^*(t, t') \xi^-(t, t') \right) + (B_{01}(t))^{\Im} (B_{10}(t'))^{\Im} \quad (869)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_y(t') \rangle_B = \frac{1}{2} \left(\left(e^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}^*(t, t') \xi^-(t, t') + \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}(t, t') \xi^+(t, t') \right) + (B_{10}(t))^{\Re} (B_{10}(t'))^{\Im} \quad (870)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_x(t') \rangle_B = \frac{1}{2} \left(\left(e^{\chi_{01}(t) + \chi_{10}(t')} \right)^{\Im} U_{10}^*(t, t') \xi^-(t, t') + \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right)^{\Im} U_{10}(t, t') \xi^+(t, t') \right) + (B_{10}(t))^{\Im} (B_{10}(t'))^{\Re} \quad (871)$$

$$\langle \widetilde{B}_{iz}(t) \widetilde{B}_x(t') \rangle_B = iB_{01}^{\Im}(t') \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (872)$$

$$= iB_{01}^{\Im}(t') \sum_{\mathbf{k}} \left(-g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}, t))^* \frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}}, t')}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right. \quad (873)$$

$$\left. + g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}}, t)) \left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}}, t')}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \right), \quad (874)$$

$$Q(\omega, t) = \frac{L_1(\omega)F_1(\omega, t) - L_0(\omega)F_0(\omega, t)}{\omega} \quad (875)$$

$$\langle \widetilde{B}_{iz}(t) \widetilde{B}_x(t') \rangle_B \approx iB_{01}^{\Im}(t') \int_0^\infty \left(L_i(\omega) (1 - F_i(\omega, t)) Q^*(\omega, t') N(\omega) e^{i\omega\tau} - L_i^*(\omega) (1 - F_i^*(\omega, t)) Q(\omega, t') e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega, \quad (876)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_{iz}(t') \rangle_B = iB_{10}^{\Im}(t) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t'))^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (877)$$

$$\approx iB_{01}^{\Im}(t) \int_0^\infty \left(L_i^*(\omega) (1 - F_i^*(\omega, t')) Q(\omega, t) N(\omega) e^{i\omega\tau} - L_i(\omega) (1 - F_i(\omega, t')) Q^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega \quad (878)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_y(t') \rangle_B = iB_{10}^{\Re}(t') \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right), \quad (879)$$

$$\approx iB_{10}^{\Re}(t') \int_0^\infty \left(L_i^*(\omega) (1 - F_i^*(\omega, t')) Q(\omega, t') (N(\omega) + 1) e^{-i\omega\tau} - L_i(\omega) (1 - F_i(\omega, t')) Q^*(\omega, t') e^{i\omega\tau} N(\omega) \right) d\omega \quad (880)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_{iz}(t') \rangle_B = iB_{10}^{\Re}(t) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t'))^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t')) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right) \quad (881)$$

$$\approx iB_{10}^{\Re}(t) \int_0^\infty \left(L_i^*(\omega) (1 - F_i^*(\omega, t')) Q(\omega, t) N(\omega) e^{i\omega\tau} - L_i(\omega) (1 - F_i(\omega, t')) Q^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega. \quad (882)$$

The integral version of $F_0(\omega, t)$ and $F_1(\omega, t)$ are:

$$a_i(\omega_{\mathbf{k}}, t) = \frac{\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)|^2|V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}, \quad (883)$$

$$b_i(\omega_{\mathbf{k}}, t) = \frac{2 \frac{\tanh(\frac{\beta\eta(t)}{2})}{\eta(t)} \frac{1}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh(\frac{\beta\eta(t)}{2})}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}, \quad (884)$$

$$r_i(\omega_{\mathbf{k}}, t) = \frac{a_i(\omega_{\mathbf{k}}, t)}{1 - b_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)}, \quad (885)$$

$$s_i(\omega_{\mathbf{k}}, t) = \frac{a_{(i+1) \bmod 2}(\omega_{\mathbf{k}}, t) b_{i \bmod 2}(\omega_{\mathbf{k}}, t)}{1 - b_0(\omega_{\mathbf{k}}, t) b_1(\omega_{\mathbf{k}}, t)}. \quad (886)$$

$$F_0(\omega, t) = r_0(\omega_{\mathbf{k}}, t) + \frac{g_1(\omega_{\mathbf{k}})}{g_0(\omega_{\mathbf{k}})} s_0(\omega_{\mathbf{k}}, t) \quad (887)$$

$$\approx r_0(\omega, t) + \frac{g_1(\omega)}{g_0(\omega)} s_0(\omega, t) \quad (888)$$

$$= r_0(\omega, t) + \frac{L_1(\omega)}{L_0(\omega)} s_0(\omega, t), \quad (889)$$

$$F_1(\omega, t) = \frac{g_0(\omega_{\mathbf{k}})}{g_1(\omega_{\mathbf{k}})} r_1(\omega_{\mathbf{k}}, t) + s_1(\omega_{\mathbf{k}}, t) \quad (890)$$

$$\approx \frac{g_0(\omega)}{g_1(\omega)} r_1(\omega, t) + s_1(\omega, t) \quad (891)$$

$$= \frac{L_0(\omega)}{L_1(\omega)} r_1(\omega, t) + s_1(\omega, t). \quad (892)$$

The expressions showed are well defined because the relevant products present in the correlations functions are of the form:

$$\int_0^\infty f(\omega) L_j(\omega) F_j(\omega, t) L_i^*(\omega) F_i^*(\omega, t) d\omega = \int_0^\infty f(\omega) L_j(\omega) \left(r_j(\omega, t) + \frac{L_i(\omega)}{L_j(\omega)} s_j(\omega, t) \right) L_i^*(\omega) \left(r_i^*(\omega, t) + \frac{L_j^*(\omega)}{L_i^*(\omega)} s_i^*(\omega, t) \right) d\omega \quad (893)$$

$$= \int_0^\infty f(\omega) (L_j(\omega) r_j(\omega, t) + L_i(\omega) s_j(\omega, t)) (L_i^*(\omega) r_i^*(\omega, t) + L_j^*(\omega) s_i^*(\omega, t)) d\omega \quad (894)$$

$$= \int_0^\infty f(\omega) (L_j(\omega) L_i^*(\omega) r_j(\omega, t) r_i^*(\omega, t) + |L_j(\omega)|^2 r_j(\omega, t) s_i^*(\omega, t) \quad (895)$$

$$+ |L_i(\omega)|^2 s_j(\omega, t) r_i^*(\omega, t) + L_i(\omega) L_j^*(\omega) s_j(\omega, t) s_i^*(\omega, t)) d\omega \quad (896)$$

here $f(\omega) \in L^2$. As we could proof these integral are convergent.

So the integral version of the correlation functions $\mathcal{B}_{ij}(t, t')$ is can be written in a neater form as:

$$\mathcal{B}(t, t') = \begin{pmatrix} \mathcal{B}_{11}(t, t') & \mathcal{B}_{12}(t, t') & \mathcal{B}_{13}(t, t') & \mathcal{B}_{12}(t, t') & \mathcal{B}_{11}(t, t') & \mathcal{B}_{16}(t, t') \\ \mathcal{B}_{21}(t, t') & \mathcal{B}_{22}(t, t') & \mathcal{B}_{23}(t, t') & \mathcal{B}_{22}(t, t') & \mathcal{B}_{21}(t, t') & \mathcal{B}_{26}(t, t') \\ \mathcal{B}_{31}(t, t') & \mathcal{B}_{32}(t, t') & \mathcal{B}_{33}(t, t') & \mathcal{B}_{32}(t, t') & \mathcal{B}_{31}(t, t') & \mathcal{B}_{36}(t, t') \\ \mathcal{B}_{21}(t, t') & \mathcal{B}_{22}(t, t') & \mathcal{B}_{23}(t, t') & \mathcal{B}_{22}(t, t') & \mathcal{B}_{21}(t, t') & \mathcal{B}_{26}(t, t') \\ \mathcal{B}_{11}(t, t') & \mathcal{B}_{12}(t, t') & \mathcal{B}_{13}(t, t') & \mathcal{B}_{12}(t, t') & \mathcal{B}_{11}(t, t') & \mathcal{B}_{16}(t, t') \\ \mathcal{B}_{61}(t, t') & \mathcal{B}_{62}(t, t') & \mathcal{B}_{63}(t, t') & \mathcal{B}_{62}(t, t') & \mathcal{B}_{61}(t, t') & \mathcal{B}_{66}(t, t') \end{pmatrix}, \quad (897)$$

$$\mathcal{B}_{11}(t, t') = \frac{1}{2} \left(\Re \left(e^{X_{10}(t) + X_{10}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') + \Re \left(e^{X_{10}(t) + X_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) - B_{10}^{\Re}(t) B_{01}^{\Re}(t'), \quad (898)$$

$$\mathcal{B}_{22}(t, t') = -\frac{1}{2} \left(\Re \left(e^{X_{01}(t) + X_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') - \Re \left(e^{X_{10}(t) + X_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) + B_{01}^{\Im}(t) B_{10}^{\Im}(t'), \quad (899)$$

$$\mathcal{B}_{12}(t, t') = \frac{1}{2} \left(\Im \left(e^{X_{10}(t) + X_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{X_{01}(t) + X_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Re}(t) B_{10}^{\Im}(t'), \quad (900)$$

$$\mathcal{B}_{21}(t, t') = \frac{1}{2} \left(\Im \left(e^{X_{01}(t) + X_{10}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{X_{01}(t) + X_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Im}(t) B_{10}^{\Re}(t'), \quad (901)$$

$$\mathcal{B}_{ij}(t, t') = \int_0^\infty (P_i(\omega, t) P_j^*(\omega, t') e^{i\omega\tau} N(\omega) + P_i^*(\omega, t) P_j(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i, j \in \{3, 6\}, \quad (902)$$

$$\mathcal{B}_{i1}(t, t') = iB_{01}^{\mathfrak{S}}(t') \int_0^\infty (P_i(\omega, t) Q_{10}^*(\omega, t') N(\omega) e^{i\omega\tau} - P_i^*(\omega, t) Q_{10}(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\}, \quad (903)$$

$$\mathcal{B}_{1i}(t, t') = iB_{01}^{\mathfrak{S}}(t) \int_0^\infty (P_i^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\}, \quad (904)$$

$$\mathcal{B}_{i2}(t, t') = iB_{10}^{\mathfrak{R}}(t') \int_0^\infty (P_i^*(\omega, t') Q_{10}(\omega, t') (N(\omega) + 1) e^{-i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t') e^{i\omega\tau} N(\omega)) d\omega, i \in \{3, 6\}, \quad (905)$$

$$\mathcal{B}_{2i}(t, t') = iB_{10}^{\mathfrak{R}}(t) \int_0^\infty (P_i^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\}, \quad (906)$$

$$\zeta_{ij}(t, t') = e^{i\Im \left(\int_0^\infty \frac{(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t))(L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t'))^* e^{i\omega\tau}}{\omega^2} d\omega \right)}, \quad (907)$$

$$\xi_{ij}^\pm(t, t') = e^{-\int_0^\infty \frac{|(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t)) e^{i\omega\tau} \pm (L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t'))|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega}, \quad (908)$$

$$P_i(\omega, t) = L_{i \bmod 2}(\omega) (1 - F_{i \bmod 2}(\omega, t)), \quad (909)$$

$$Q_{ij}(\omega, t) = \frac{L_i(\omega) F_1(\omega, t) - L_j(\omega) F_j(\omega, t)}{\omega}, \quad (910)$$

$$a_i(\omega, t) = \frac{\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)}, \quad (911)$$

$$b_i(\omega, t) = \frac{2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{1}{\omega} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)}, \quad (912)$$

$$r_i(\omega, t) = \frac{a_i(\omega, t)}{1 - b_0(\omega, t) b_1(\omega, t)}, \quad (913)$$

$$s_i(\omega, t) = \frac{a_{(i+1) \bmod 2}(\omega, t) b_{i \bmod 2}(\omega, t)}{1 - b_0(\omega, t) b_1(\omega, t)}, \quad (914)$$

$$F_0(\omega, t) = r_0(\omega, t) + \frac{L_1(\omega)}{L_0(\omega)} s_0(\omega, t), \quad (915)$$

$$F_1(\omega, t) = \frac{L_0(\omega)}{L_1(\omega)} r_1(\omega, t) + s_1(\omega, t). \quad (916)$$

The time-dependence of the system operators $\widetilde{A}_i(t)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\mathfrak{S}}}$ we will obtain:

$$\widetilde{A}_i(\tau) = e^{i\overline{H_{\mathfrak{S}}}\tau} A_i(t) e^{-i\overline{H_{\mathfrak{S}}}\tau} \quad (917)$$

$$= \sum_w e^{-i\omega\tau} A_i(\omega). \quad (918)$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.

In order to use the equation (918) to descompose the equation (373) we need to consider the time ordering operator \mathcal{T} , it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_{\mathfrak{S}}}(t')\right)$ like:

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right) \quad (919)$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) \dots H(t_n). \quad (920)$$

Here $0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_{\text{eff}}(t)$ such that $\mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right) \equiv \exp(-it H_{\text{eff}}(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_{\text{eff}}(t) = H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + \dots$ so we can write:

$$U(t) = \exp \left(-it \left(H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + \dots \right) \right). \quad (921)$$

The terms can be found expanding $\mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_S}(t') dt', \quad (922)$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' [\overline{H_S}(t'), \overline{H_S}(t'')], \quad (923)$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' ([\overline{H_S}(t'), \overline{H_S}(t'')], \overline{H_S}(t''')) + [[\overline{H_S}(t'''), \overline{H_S}(t'')], \overline{H_S}(t')]. \quad (924)$$

We can summarize that:

$$\tilde{O}(t) \equiv U^\dagger(t) O(t) U(t), \quad (925)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_T}(t') \right) \quad (926)$$

$$= \exp(-i \overline{H_{T,\text{eff}}}(t)), \text{ where} \quad (927)$$

$$H_X^{\text{eff}}(t) \equiv \frac{1}{t} \int_0^t H_X(t') dt' - \frac{i}{2t} \int_0^t \int_0^{t'} [H_X(t'), H_X(t'')] dt' dt'' + \frac{1}{6t} \int_0^t \int_0^{t'} \int_0^{t''} ([\overline{H_S}(t'), \overline{H_S}(t'')], \overline{H_S}(t''')) + [[\overline{H_S}(t'''), \overline{H_S}(t'')], \overline{H_S}(t')] dt' dt'' dt''' + \dots \quad (928)$$

In order to show the explicit form of the matrices present in the RHS of the equation (918) for a general 2×2 matrix in a given time let's write the matrix $A_i(t)$ in the base $W(t) = \{ |\overline{H_{S,\text{eff},1}}(t)\rangle, |\overline{H_{S,\text{eff},0}}(t)\rangle \}$, formed by the time-dependent eigenvectors of $\overline{H_{S,\text{eff}}}(t)$ in the following way:

$$A_i(t) = \sum_{j,j'} \langle \overline{H_{S,\text{eff},j}}(t-\tau) | A_i(t) | \overline{H_{S,\text{eff},j'}}(t-\tau) \rangle | \overline{H_{S,\text{eff},j}}(t-\tau) \rangle \langle \overline{H_{S,\text{eff},j'}}(t-\tau) |. \quad (929)$$

Let's obtain $U^\dagger(t') A_i(t) U(t')$ in explicit form:

$$U^\dagger(t') A_i(t) U(t') = U^\dagger(t') \left(\sum_{j,j'} \langle \overline{H_{S,\text{eff},j}}(t-\tau) | A_i(t) | \overline{H_{S,\text{eff},j'}}(t-\tau) \rangle | \overline{H_{S,\text{eff},j}}(t-\tau) \rangle \langle \overline{H_{S,\text{eff},j'}}(t-\tau) | \right) U(t') \quad (930)$$

$$= \sum_{j,j'} \langle \overline{H_{S,\text{eff},j}}(t-\tau) | A_i(t) | \overline{H_{S,\text{eff},j'}}(t-\tau) \rangle U^\dagger(t') | \overline{H_{S,\text{eff},j}}(t-\tau) \rangle \langle \overline{H_{S,\text{eff},j'}}(t-\tau) | U(t') \quad (931)$$

$$= \sum_{j,j'} \langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) | A_i(t) | \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \rangle e^{i(t-\tau)\lambda_j(t-\tau)} | \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \rangle \langle \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) | e^{-i(t-\tau)\lambda_{j'}(t-\tau)} \quad (932)$$

$$= \sum_{j,j'} \langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) | A_i(t) | \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \rangle e^{i(t-\tau)(\lambda_j(t-\tau)-\lambda_{j'}(t-\tau))} | \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \rangle \langle \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) |, \quad (933)$$

$$M_{jj'}(t-\tau) = \langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) | A_i(t) | \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \rangle | \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \rangle \langle \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) |, \quad (934)$$

$$U^\dagger(t') A_i(t) U(t') = M_{00}(t-\tau) + M_{01}(t-\tau) e^{i(t-\tau)(\lambda_0(t-\tau)-\lambda_1(t-\tau))} + M_{10}(t-\tau) e^{i(t-\tau)(\lambda_1(t-\tau)-\lambda_0(t-\tau))} + M_{11}(t-\tau), \quad (935)$$

$$w(t-\tau) = \lambda_1(t-\tau) - \lambda_0(t-\tau), \quad (936)$$

$$U^\dagger(t') A_i(t) U(t') = M_{00}(t-\tau) + M_{01}(t-\tau) e^{-i(t-\tau)w(t-\tau)} + M_{10} e^{i(t-\tau)w(t-\tau)} + M_{11} \quad (937)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} A_i(w(t-\tau)) \quad (938)$$

$$= A_i(0) + A_i(w(t-\tau)) e^{-i(t-\tau)w(t-\tau)} + A_i(-w(t-\tau)) e^{i(t-\tau)w(t-\tau)}. \quad (939)$$

By direct comparison we obtain that:

$$A_i(w(t-\tau)) = \langle \overline{H_{\bar{S},\text{eff},0}}(t-\tau) | A_i(t) | \overline{H_{\bar{S},\text{eff},1}}(t-\tau) \rangle | \overline{H_{\bar{S},\text{eff},0}}(t-\tau) \rangle \langle \overline{H_{\bar{S},\text{eff},1}}(t-\tau) |, \quad (940)$$

$$A_i(-w(t-\tau)) = \langle \overline{H_{\bar{S},\text{eff},1}}(t-\tau) | A_i(t) | \overline{H_{\bar{S},\text{eff},0}}(t-\tau) \rangle | \overline{H_{\bar{S},\text{eff},1}}(t-\tau) \rangle \langle \overline{H_{\bar{S},\text{eff},0}}(t-\tau) |, \quad (941)$$

$$A_i(0) = \sum_j \langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) | A_i(t) | \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \rangle | \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \rangle \langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) |. \quad (942)$$

These matrix have the following property $A_i(w(t-\tau)) = A_i^\dagger(-w(t-\tau))$. Now in order to perform the double Fourier decomposition let's recall:

$$\widetilde{A}_i(t, t') \equiv U(t) U^\dagger(t') A_i(t) U(t') U^\dagger(t). \quad (943)$$

In this case the decomposition can be written as:

$$\widetilde{A}_i(t, t-\tau) \equiv U(t) U^\dagger(t-\tau) A_i(t) U(t-\tau) U^\dagger(t) \quad (944)$$

$$= U(t) \left(\sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} A_i(w(t-\tau)) \right) U^\dagger(t). \quad (945)$$

Now writting $A_i(w(t-\tau))$ in terms of the eigenstates of $\overline{H_{\bar{S},\text{eff}}}(t)$ we find:

$$A_i(w(t-\tau)) = \sum_{j,j'} \langle \overline{H_{\bar{S},\text{eff},j}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},j'}}(t) \rangle | \overline{H_{\bar{S},\text{eff},j}}(t) \rangle \langle \overline{H_{\bar{S},\text{eff},j'}}(t) |. \quad (946)$$

Then the time evolution is given by:

$$\widetilde{A}_i(t, t-\tau) = U(t) \left(\sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} A_i(w(t-\tau)) \right) U^\dagger(t) \quad (947)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} U(t) A_i(w(t-\tau)) U^\dagger(t) \quad (948)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} U(t) \left(\sum_{j,j'} \langle \overline{H_{\bar{S},\text{eff},j}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},j'}}(t) \rangle | \overline{H_{\bar{S},\text{eff},j}}(t) \rangle \langle \overline{H_{\bar{S},\text{eff},j'}}(t) | \right) U^\dagger(t) \quad (949)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} \sum_{j,j'} \langle \overline{H_{\tilde{S},\text{eff},j}}(t) | A_i(w(t-\tau)) | \overline{H_{\tilde{S},\text{eff},j'}}(t) \rangle U(t) | \overline{H_{\tilde{S},\text{eff},j}}(t) \rangle \langle \overline{H_{\tilde{S},\text{eff},j'}}(t) | U^\dagger(t) \quad (950)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} \sum_{j,j'} \langle \overline{H_{\tilde{S},\text{eff},j}}(t) | A_i(w(t-\tau)) | \overline{H_{\tilde{S},\text{eff},j'}}(t) \rangle e^{-it\lambda_{j'}(t)} |\overline{H_{\tilde{S},\text{eff},j}}(t)\rangle \langle \overline{H_{\tilde{S},\text{eff},j'}}(t)| e^{it\lambda_{j'}(t)}, \quad (951)$$

$$w'(t) = \lambda_1(t) - \lambda_0(t), \quad (952)$$

$$\widetilde{A}_i(t, t-\tau) = \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} \left(\langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle \right) \left| \overline{H_{\bar{S}, \text{eff}, 0}}(t) \right\rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | \quad (953)$$

$$+ \langle \overline{H_{\bar{S},\text{eff},1}}(t) | A_i(w(t - \tau)) | \overline{H_{\bar{S},\text{eff},1}}(t) \rangle | \overline{H_{\bar{S},\text{eff},1}}(t) \rangle \langle \overline{H_{\bar{S},\text{eff},1}}(t) | \quad (954)$$

$$+ e^{-itw'(t)} \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(w(t - \tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | \quad (955)$$

$$+e^{itw'(t)} \left\langle \overline{H_{\bar{S},\text{eff},0}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\text{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\text{eff},1}}(t) \right\rangle \right) \quad (956)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} \sum_{w'(t)} e^{itw'(t)} A_{iww'}(t-\tau, t) \quad (957)$$

$$= \sum_{w(t-\tau), w'(t)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_{iww'}(t-\tau, t) \quad (958)$$

$$= \sum_{w(t-\tau), w'(t)} e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} A_{iww'}(t-\tau, t). \quad (959)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} \left(\langle \overline{H_{S,\text{eff},0}}(t) | A_i(w(t-\tau)) | \overline{H_{S,\text{eff},0}}(t) \rangle \right) | \overline{H_{S,\text{eff},0}}(t) \rangle \langle \overline{H_{S,\text{eff},0}}(t) | \quad (960)$$

$$+ \langle \overline{H_{\bar{S},\text{eff},1}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},1}}(t) \rangle | \overline{H_{\bar{S},\text{eff},1}}(t) \rangle \langle \overline{H_{\bar{S},\text{eff},1}}(t) | \quad (961)$$

$$+ e^{-itw'(t)} \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(w(t - \tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) |$$

$$+e^{itw'(t)} \left\langle \overline{H_{\bar{S},\text{eff},0}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\text{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\text{eff},1}}(t) \right\rangle \right) \quad (963)$$

$$= \langle \overline{H_{\bar{S},\text{eff},0}(t)} | A_i(0) | \overline{H_{\bar{S},\text{eff},0}(t)} \rangle \langle \overline{H_{\bar{S},\text{eff},0}(t)} | \overline{H_{\bar{S},\text{eff},0}(t)} \rangle + \langle \overline{H_{\bar{S},\text{eff},1}(t)} | A_i(0) | \overline{H_{\bar{S},\text{eff},1}(t)} \rangle \langle \overline{H_{\bar{S},\text{eff},1}(t)} | \overline{H_{\bar{S},\text{eff},1}(t)} \rangle \quad (964)$$

$$+ e^{-itw'(t)} \langle \overline{H_{\bar{S},\text{eff},1}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},0}}(t) \rangle | \overline{H_{\bar{S},\text{eff},1}}(t) \rangle \langle \overline{H_{\bar{S},\text{eff},0}}(t) |$$

$$+ e^{itw'(t)} \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_i(w(t - \tau)) | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | \quad (966)$$

$$+ e^{-i(t-\tau)w(t-\tau)} (\langle \overline{H_{\bar{S},\text{eff},0}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},0}}(t) \rangle | \overline{H_{\bar{S},\text{eff},0}}(t) \rangle \langle \overline{H_{\bar{S},\text{eff},0}}(t) |) \quad (967)$$

$$+ \langle \overline{H_{\bar{S},\text{eff},1}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},1}}(t) \rangle \left| \overline{H_{\bar{S},\text{eff},1}}(t) \right\rangle \left\langle \overline{H_{\bar{S},\text{eff},1}}(t) \right| \quad (968)$$

$$+ e^{-itw'(t)} \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(w(t - \tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | \quad (969)$$

$$+e^{itw'(t)} \left\langle \overline{H_{\bar{S},\text{eff},0}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S},\text{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\text{eff},0}}(t) \right\rangle \left\langle \overline{H_{\bar{S},\text{eff},1}}(t) \right| \right) \quad (970)$$

$$+ e^{i(t-\tau)w(t-\tau)} \left(\langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_i(-w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle \overline{H_{\bar{S}, \text{eff}, 0}}(t) \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | \right) \quad (971)$$

$$+ \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(-w(t - \tau)) | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | \quad (972)$$

$$+ e^{-itw'(t)} \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(-w(t - \tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | \quad (973)$$

$$+e^{itw'(t)} \left\langle \overline{H_{\bar{S},\text{eff},0}}(t) | A_i(-w(t-\tau)) | \overline{H_{\bar{S},\text{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\text{eff},0}}(t) \right\rangle \left\langle \overline{H_{\bar{S},\text{eff},1}}(t) \right| \right). \quad (974)$$

Directly we can find that the decomposition matrices are:

$$A_{i0w'}(t - \tau, t) = \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_i(0) | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) |, \quad (975)$$

$$A_{iww'}(t - \tau, t) = \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_i(w(t - \tau)) | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle, \quad (976)$$

$$A_{iw(-w')}(t-\tau, t) = \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t-\tau) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t-\tau) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t-\tau) |, \quad (977)$$

$$A_{iw0}(t-\tau, t) = \sum_j \langle \overline{H_{\bar{S}, \text{eff}, j}}(t-\tau) | A_i(w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, j}}(t-\tau) \rangle | \overline{H_{\bar{S}, \text{eff}, j}}(t-\tau) \rangle \langle \overline{H_{\bar{S}, \text{eff}, j}}(t-\tau) |, \quad (978)$$

$$A_{i00}(t-\tau, t) = \sum_j \langle \overline{H_{\bar{S}, \text{eff}, j}}(t) | A_i(0) | \overline{H_{\bar{S}, \text{eff}, j}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, j}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, j}}(t) |, \quad (979)$$

$$A_{i0(-w')}(t-\tau, t) = \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(0) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) |, \quad (980)$$

$$A_{i(-w)0}(t-\tau, t) = \sum_j \langle \overline{H_{\bar{S}, \text{eff}, j}}(t) | A_i(-w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, j}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, j}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, j}}(t) |, \quad (981)$$

$$A_{i(-w)w'}(t-\tau, t) = \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_i(-w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) |, \quad (982)$$

$$A_{i(-w)(-w')}(t-\tau, t) = \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_i(-w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t-\tau) |. \quad (983)$$

Let's prove that $A_{jww'}(t-\tau, t) = A_{j(-w)(-w')}(t-\tau, t)$:

$$\left(\langle \overline{H_{\bar{S}, \text{eff}, j}}(t) | A_i(-w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, j'}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, j}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, j'}}(t-\tau) | \right)^\dagger = \langle \overline{H_{\bar{S}, \text{eff}, j}}(t) | A_i(-w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, j'}}(t) \rangle^* | \overline{H_{\bar{S}, \text{eff}, j'}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, j}}(t-\tau) | \quad (984)$$

$$= \langle \overline{H_{\bar{S}, \text{eff}, j'}}(t) | A_i^\dagger(-w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, j}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, j'}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, j}}(t-\tau) | \quad (985)$$

$$= \langle \overline{H_{\bar{S}, \text{eff}, j'}}(t) | A_i(w(t-\tau)) | \overline{H_{\bar{S}, \text{eff}, j}}(t) \rangle | \overline{H_{\bar{S}, \text{eff}, j'}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, j}}(t-\tau) |. \quad (986)$$

It can be seen that the index $-w$ and $-w'$ change to the functions w and w' .

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A}_i(\tau) = \widetilde{A}_i^\dagger(\tau)$ and $\widetilde{B}_i(\tau) = \widetilde{B}_i^\dagger(\tau)$ we can use the equation (918) to write the master equation in the following neater form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau C_i(t) C_j(t-\tau) \left(\mathcal{B}_{ij}(\tau) \left[A_i, \widetilde{A}_j(t-\tau, t) \overline{\rho_S}(t) \right] + \mathcal{B}_{ji}(-\tau) \left[\overline{\rho_S}(t) \widetilde{A}_j(t-\tau, t), A_i \right] \right) \quad (987)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \left(\mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} A_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \right. \quad (988)$$

$$\left. - \mathcal{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} A_{jww'}(t-\tau, t) \right] \right). \quad (989)$$

Now let's consider the following trace recalling that $\text{Tr}(A)^* = \text{Tr}(A^\dagger)$ so:

$$\text{Tr}_B \left(\widetilde{B}_j(-\tau) \widetilde{B}_i(0) \rho_B \right) = \text{Tr}_B \left(e^{-i\tau H_B(\tau)} B_j e^{i\tau H_B(\tau)} B_i \rho_B \right) \quad (990)$$

$$= \text{Tr}_B \left(B_j e^{i\tau H_B(\tau)} B_i \rho_B e^{-i\tau H_B(\tau)} \right) \text{ (by cyclic permutivity of trace)} \quad (991)$$

$$= \text{Tr}_B \left(B_j e^{i\tau H_B(\tau)} B_i e^{-i\tau H_B(\tau)} \rho_B \right) \text{ (by commutativity of } e^{-i\tau H_B(\tau)} \text{ and } \rho_B) \quad (992)$$

$$= \text{Tr}_B \left(B_j \widetilde{B}_i(\tau) \rho_B \right) \text{ (by definition of time evolution)} \quad (993)$$

$$= \text{Tr}_B \left(B_j \widetilde{B}_i(\tau) \rho_B \right) \quad (994)$$

$$= \text{Tr}_B \left(\rho_B B_j \widetilde{B}_i(\tau) \right) \quad (995)$$

$$= \text{Tr}_B \left(\left(\widetilde{B}_i(\tau) B_j \rho_B \right)^\dagger \right) \text{ (by definition of adjoint)} \quad (996)$$

$$= \text{Tr}_B \left(\widetilde{B}_i(\tau) B_j \rho_B \right)^* \quad (997)$$

$$= \mathcal{B}_{ij}^*(\tau) \quad (998)$$

Given that $A_{jww'}(t-\tau, t) = A_{j(-w)(-w')}(t-\tau, t)$ and $w(t-\tau), w'(t)$ belong to the set of differences of eigenvalues of $\overline{H_S^{\text{eff}}}(t-\tau)$ and $\overline{H_S^{\text{eff}}}(t)$ denoted by J_t and $J_{t-\tau}$ respectively that depends of the time we can take an application where $w(t-\tau) \rightarrow -w(t-\tau)$ and $w'(t) \rightarrow -w'(t)$ such that the sum:

$$\sum_{ww'} \int_0^t d\tau e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} A_{jww'}(t-\tau, t) = \sum_{ww'} e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} A_{j(-w)(-w')}(t-\tau, t) \quad (999)$$

$$= \sum_{ww'} e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} A_{jww'}^\dagger(t-\tau, t). \quad (1000)$$

is invariant because if $(w(t-\tau), w'(t)) \in J_{t-\tau} \times J_t$ then $(-w(t-\tau), -w'(t)) \in J_{t-\tau} \times J_t$ where J_t denotes the set of differences of eigenvalues at time t . So the master equation can be written as:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \left(\mathcal{B}_{ij}(\tau) \left[A_i(t), e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} A_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \right. \quad (1001)$$

$$\left. + \mathcal{B}_{ij}^*(\tau) \left[\overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} A_{jww'}^\dagger(t-\tau, t), A_i(t) \right] \right) \quad (1002)$$

With the definition:

$$L_{ijww'}(t, t') \equiv \int_0^t C_i(t) C_j(t') \mathcal{B}_{ij}(\tau) e^{i\tau w(t')} e^{-it(w(t')-w'(t))} A_{jww'}(t, t') d\tau. \quad (1003)$$

We can show that:

$$L_{ijww'}^\dagger(t, t') = \int_0^t \left(C_i(t) C_j(t') \mathcal{B}_{ij}(t, t') e^{i\tau w(t')} e^{-it(w(t')-w'(t))} A_{jww'}(t, t') d\tau \right)^\dagger \quad (1004)$$

$$= \int_0^t C_i^*(t) C_j^*(t') \mathcal{B}_{ij}^*(t, t') e^{-i\tau w^*(t')} e^{it(w(t')-w'(t))^*} A_{jww'}^\dagger(t, t') d\tau \quad (1005)$$

$$= \int_0^t C_i(t) C_j(t') \mathcal{B}_{ij}^*(\tau) e^{-i\tau w(t')} e^{it(w(t')-w'(t))} A_{jww'}^\dagger(t, t') d\tau. \quad (1006)$$

So we can write the master equation as:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \left(\mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} A_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \right. \quad (1007)$$

$$\left. - \mathcal{B}_{ij}^*(\tau) \left[A_i, \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} A_{jww'}^\dagger(t-\tau, t) \right] \right) \quad (1008)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left([A_i, L_{ijww'}(t) \overline{\rho_S}(t)] + [\overline{\rho_S}(t) L_{ijww'}^\dagger(t), A_i] \right). \quad (1009)$$

If we extend the upper limit of integration to ∞ in the equation (1006) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V} \overline{AB} e^V = e^{-V} \overline{A} \overline{B} e^V \quad (1010)$$

$$= e^{-V} \overline{A} e^V e^{-V} \overline{B} e^V \quad (1011)$$

$$= (e^{-V} \overline{A} e^V) (e^{-V} \overline{B} e^V) \quad (1012)$$

$$= \overline{AB}. \quad (1013)$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}[A, B]e^V = e^{-V}(AB - BA)e^V \quad (1014)$$

$$= e^{-V}ABe^V - e^{-V}BAe^V \quad (1015)$$

$$= AB - BA \quad (1016)$$

$$= [A, B]. \quad (1017)$$

So we will obtain that

$$\frac{d\rho}{dt} = -ie^{-V} [\overline{H_S}(t), \overline{\rho_S}(t)] e^V - e^{-V} \sum_{ijww'} \left([A_i, L_{ijww'}(t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) L_{ijww'}^\dagger(t)] \right) e^V \quad (1018)$$

$$= -ie^{-V} [\overline{H_S}(t), \overline{\rho_S}(t)] e^V - \sum_{ijww'} \left(e^{-V} [A_i, L_{ijww'}(t) \overline{\rho_S}(t)] e^V - e^{-V} [A_i, \overline{\rho_S}(t) L_{ijww'}^\dagger(t)] e^V \right) \quad (1019)$$

$$= -i[H(t), \rho(t)] - \sum_{ijww'} \left([A_i, e^{-V} L_{ijww'}(t) \overline{\rho_S}(t) e^V] - [A_i, e^{-V} \overline{\rho_S}(t) L_{ijww'}^\dagger(t) e^V] \right) \quad (1020)$$

$$= -i[H(t), \rho(t)] - \sum_{ijww'} \left([A_i, e^{-V} L_{ijww'}(t) e^V e^{-V} \overline{\rho_S}(t) e^V] - [e^{-V} A_i e^V, e^{-V} \overline{\rho_S}(t) e^V e^{-V} L_{ijww'}^\dagger(t) e^V] \right) \quad (1021)$$

$$= -i[H(t), \rho(t)] - \sum_{ijww'} ([A_i, L_{ijww'}(t) \rho(t)] - [A_i, \rho(t) L_{ijww'}^\dagger(t)]). \quad (1022)$$

Our master equation in the variationally optimized frame is:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left([A_i, L_{ijww'}(t) \overline{\rho_S}(t)] + [\overline{\rho_S}(t) L_{ijww'}^\dagger(t), A_i] \right), \quad (1023)$$

$$\dot{\rho}(t) = -i[H(t), \rho(t)] - \sum_{ijww'} \left([A_i, L_{ijww'}(t) \rho(t)] + [\rho(t) L_{ijww'}^\dagger(t), A_i] \right). \quad (1024)$$

IV. LIMIT CASES

In order to show the plausibility of the master equation (1023) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-dependent VPQME for 2LS with real-valued system Hamiltonian and real-valued uniform coupling

This hamiltonian has as particular feature that the coupling constants are real, so we know that $g_{\mathbf{k}} = g_{\mathbf{k}}^*$ then:

$$H_T(t) = H_S(t) + H_I + H_B, \quad (1025)$$

$$H_S(t) = \sum_i \varepsilon_i(t) |i\rangle\langle i| + \sum_{i \neq j} V_{ij}(t) |i\rangle\langle j|, \quad (1026)$$

$$H_I = \sum_i |i\rangle\langle i| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}), \quad (1027)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1028)$$

The transformed hamiltonian is:

$$\overline{H_S}(t) \equiv \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)), \quad (1029)$$

We can summarize the principal results of the elements of the variational parameters and the transformed hamiltonians as:

$$\overline{H_{\bar{S}}}(t) \equiv \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| + \sigma_x B_{10}(t) V_{10}(t) - \sigma_y B_{10}(t) V_{10}(t), \quad (1030)$$

$$R_i(t) = \int_0^\infty \frac{J(\omega)}{\omega} (F_i^2(\omega, t) - 2F_i(\omega, t)) d\omega, \quad (1031)$$

$$\chi_{ij}(t) = 0, \quad (1032)$$

$$B_{ij}(t) = e^{-\frac{1}{2} \int_0^\infty \frac{J(\omega)(F_i(\omega, t) - F_j(\omega, t))^2}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega}, \quad (1033)$$

$$F_i(\omega, t) = \frac{\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{F_{i'}(\omega, t)g(\omega)}{\omega} B_{10}^2(t) V_{10}^2(t) \coth(\beta\omega/2)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2B_{10}^2(t)V_{10}^2(t) \coth(\beta\omega/2)}{\omega}\right)}, \quad (1034)$$

$$\eta(t) \equiv \sqrt{(\text{Tr}(\overline{H_{\bar{S}}}(t)))^2 - 4\text{Det}(\overline{H_{\bar{S}}}(t))}, \quad (1035)$$

$$\varepsilon(t) \equiv \text{Tr}(\overline{H_{\bar{S}}}(t)), \quad (1036)$$

$$J(\omega) \equiv \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}). \quad (1037)$$

The Fourier decomposition remains without change:

$$L_{ijww'}(t, t') \equiv \int_0^t C_i(t) C_j(t') \mathcal{B}_{ij}(t, t') e^{i\tau w(t')} e^{-it(w(t') - w'(t))} A_{jww'}(t, t') d\tau, \quad (1038)$$

$$t' = t - \tau, \quad (1039)$$

$$A \equiv \begin{pmatrix} \sigma_x & \sigma_y & \frac{I - \sigma_z}{2} & \sigma_x & \sigma_y & \frac{I + \sigma_z}{2} \end{pmatrix}, \quad (1040)$$

$$C(t) \equiv \begin{pmatrix} V_{10}(t) & V_{10}(t) & 1 & 0 & 0 & 1 \end{pmatrix}, \quad (1041)$$

$$A_{j00}(t, t') = \sum_i \langle \overline{H_{\bar{S}, \text{eff}, i}}(t) | A_{j0}(t') | \overline{H_{\bar{S}, \text{eff}, i}}(t') \rangle | \overline{H_{\bar{S}, \text{eff}, i}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, i}}(t') |, \quad (1042)$$

$$A_{j0w'}(t, t') = \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_{j0}(t') | \overline{H_{\bar{S}, \text{eff}, 1}}(t') \rangle | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t') |, \quad (1043)$$

$$A_{jw0}(t, t') = \sum_i \langle \overline{H_{\bar{S}, \text{eff}, i}}(t) | A_{jw}(t') | \overline{H_{\bar{S}, \text{eff}, i}}(t') \rangle | \overline{H_{\bar{S}, \text{eff}, i}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, i}}(t') |, \quad (1044)$$

$$A_{jww'}(t, t') = \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t) | A_{jw}(t') | \overline{H_{\bar{S}, \text{eff}, 1}}(t') \rangle | \overline{H_{\bar{S}, \text{eff}, 0}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t') |, \quad (1045)$$

$$A_{jw(-w')}(t, t') = \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t) | A_{jw}(t') | \overline{H_{\bar{S}, \text{eff}, 0}}(t - \tau) \rangle | \overline{H_{\bar{S}, \text{eff}, 1}}(t) \rangle \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t') |, \quad (1046)$$

$$A_{j(-w)(-w')}(t, t') = A_{jww'}^\dagger(t, t') \quad (1047)$$

$$A_{j0}(t') = \sum_i \langle \overline{H_{\bar{S}, \text{eff}, i}}(t') | A_j(t) | \overline{H_{\bar{S}, \text{eff}, i}}(t') \rangle | \overline{H_{\bar{S}, \text{eff}, i}}(t') \rangle \langle \overline{H_{\bar{S}, \text{eff}, i}}(t') |, \quad (1048)$$

$$A_{jw}(t') = \langle \overline{H_{\bar{S}, \text{eff}, 0}}(t') | A_j(t) | \overline{H_{\bar{S}, \text{eff}, 1}}(t') \rangle | \overline{H_{\bar{S}, \text{eff}, 0}}(t') \rangle \langle \overline{H_{\bar{S}, \text{eff}, 1}}(t') |, \quad (1049)$$

$$A_{j(-w)}(t') = A_{jw}^\dagger(t'). \quad (1050)$$

The effective hamiltonian is:

$$H_X^{\text{eff}}(t) \equiv \frac{1}{t} \int_0^t H_X(t') dt' - \frac{i}{2t} \int_0^t \int_0^{t'} [H_X(t'), H_X(t'')] dt' dt'' + \frac{1}{6t} \int_0^t \int_0^{t'} \int_0^{t''} \left([[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')], \overline{H_{\bar{S}}}(t''')] + [[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')], \overline{H_{\bar{S}}}(t')] \right) dt' dt'' dt''' + \dots, \quad (1051)$$

The correlation functions are:

$$\mathcal{B}(t, t') \equiv \begin{pmatrix} \mathcal{B}_{11}(t, t') & 0 & 0 & 0 & \mathcal{B}_{11}(t, t') & 0 \\ 0 & \mathcal{B}_{22}(t, t') & \mathcal{B}_{23}(t, t') & \mathcal{B}_{22}(t, t') & 0 & \mathcal{B}_{26}(t, t') \\ 0 & \mathcal{B}_{32}(t, t') & \mathcal{B}_{33}(t, t') & \mathcal{B}_{32}(t, t') & 0 & \mathcal{B}_{36}(t, t') \\ 0 & \mathcal{B}_{22}(t, t') & \mathcal{B}_{23}(t, t') & \mathcal{B}_{22}(t, t') & 0 & \mathcal{B}_{26}(t, t') \\ \mathcal{B}_{11}(t, t') & 0 & 0 & 0 & \mathcal{B}_{11}(t, t') & 0 \\ 0 & \mathcal{B}_{62}(t, t') & \mathcal{B}_{63}(t, t') & \mathcal{B}_{62}(t, t') & 0 & \mathcal{B}_{66}(t, t') \end{pmatrix}, \quad (1052)$$

$$v_{i\mathbf{k}}^*(t) = v_{i\mathbf{k}}(t), \quad (1053)$$

$$\chi_{10}(t) = \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2} \right) \quad (1054)$$

$$= \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}(t)}{2\omega_{\mathbf{k}}^2} \right) \quad (1055)$$

$$= 0, \quad (1056)$$

$$B_{10}(t) = B_{10}^*(t), \quad (1057)$$

$$Q_{ij}(\omega, t) = Q_{ij}^*(\omega, t), \quad (1058)$$

$$\zeta_{ij}(t, t') = e^{i\Im \left(\int_0^\infty \frac{(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t)) (L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t'))^* e^{i\omega\tau}}{\omega^2} d\omega \right)} \quad (1059)$$

$$= e^{i\Im \left(\int_0^\infty \frac{(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t)) (L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t')) e^{i\omega\tau}}{\omega^2} d\omega \right)} \quad (1060)$$

$$= \zeta_{ji}(t, t'), \quad (1061)$$

$$\mathcal{B}_{11}(t, t') = \frac{1}{2} \left(\Re \left(e^{\chi_{10}(t) + \chi_{10}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') + \Re \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) - B_{10}^{\Re}(t) B_{01}^{\Re}(t') \quad (1062)$$

$$= \frac{1}{2} \left(\Re \left(e^{0+0} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') + \Re \left(e^{0+0} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) - B_{10}(t) B_{01}(t') \quad (1063)$$

$$= \frac{1}{2} \left(\zeta_{10}(t, t') \xi_{10}^+(t, t') + \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) - B_{10}(t) B_{01}(t'), \quad (1064)$$

$$\mathcal{B}_{22}(t, t') = -\frac{1}{2} \left(\Re \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') - \Re \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) + B_{01}^{\Im}(t) B_{10}^{\Im}(t') \quad (1065)$$

$$= -\frac{1}{2} \left(\Re \left(e^{0+0} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') - \Re \left(e^{0+0} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) \quad (1066)$$

$$= -\frac{1}{2} \left(\zeta_{10}(t, t') \xi_{10}^+(t, t') - \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right), \quad (1067)$$

$$\mathcal{B}_{12}(t, t') = \frac{1}{2} \left(\Im \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Re}(t) B_{10}^{\Im}(t') \quad (1068)$$

$$= \frac{1}{2} \left(\Im \left(e^{0+0} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{0+0} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Re}(t) B_{10}^{\Im}(t') \quad (1069)$$

$$= \frac{1}{2} \left(0 \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + 0 \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Re}(t) 0 \quad (1070)$$

$$= 0, \quad (1071)$$

$$\mathcal{B}_{21}(t, t') = \frac{1}{2} \left(\Im \left(e^{\chi_{01}(t) + \chi_{10}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Im}(t) B_{10}^{\Re}(t') \quad (1072)$$

$$= \frac{1}{2} \left(\Im \left(e^{0+0} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{0+0} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + 0 B_{10}^{\Re}(t') \quad (1073)$$

$$= 0, \quad (1074)$$

$$\mathcal{B}_{i2}(t, t') = i B_{10}^{\Re}(t') \int_0^\infty \left(P_i^*(\omega, t') Q_{10}(\omega, t') (N(\omega) + 1) e^{-i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t') e^{i\omega\tau} N(\omega) \right) d\omega, i \in \{3, 6\} \quad (1075)$$

$$= i B_{10}(t') \int_0^\infty \left(P_i(\omega, t') Q_{10}(\omega, t') (N(\omega) + 1) e^{-i\omega\tau} - P_i(\omega, t') Q_{10}(\omega, t') e^{i\omega\tau} N(\omega) \right) d\omega, i \in \{3, 6\} \quad (1076)$$

$$= i B_{10}(t') \int_0^\infty P_i(\omega, t') Q_{10}(\omega, t') \left((N(\omega) + 1) e^{-i\omega\tau} - e^{i\omega\tau} N(\omega) \right) d\omega, i \in \{3, 6\}, \quad (1077)$$

$$\mathcal{B}_{2i}(t, t') = iB_{10}^{\Re}(t) \int_0^\infty (P_i^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\}, \quad (1078)$$

$$= iB_{10}(t) \int_0^\infty (P_i(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_i(\omega, t') Q_{10}(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\}, \quad (1079)$$

$$= iB_{10}(t) \int_0^\infty P_i(\omega, t') Q_{10}(\omega, t) (N(\omega) e^{i\omega\tau} - e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\}, \quad (1080)$$

$$P_i(\omega, t) = P_i^*(\omega, t), \quad (1081)$$

$$\mathcal{B}_{ij}(t, t') = \int_0^\infty (P_i(\omega, t) P_j^*(\omega, t') e^{i\omega\tau} N(\omega) + P_i^*(\omega, t) P_j(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i, j \in \{3, 6\}, \quad (1082)$$

$$= \int_0^\infty (P_i(\omega, t) P_j(\omega, t') e^{i\omega\tau} N(\omega) + P_i(\omega, t) P_j(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i, j \in \{3, 6\}, \quad (1083)$$

$$= \int_0^\infty P_i(\omega, t) P_j(\omega, t') e^{i\omega\tau} (N(\omega) + e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i, j \in \{3, 6\}, \quad (1084)$$

$$\mathcal{B}_{i1}(t, t') = iB_{01}^{\Im}(t') \int_0^\infty (P_i(\omega, t) Q_{10}^*(\omega, t') N(\omega) e^{i\omega\tau} - P_i^*(\omega, t) Q_{10}(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1085)$$

$$= i0 \int_0^\infty (P_i(\omega, t) Q_{10}^*(\omega, t') N(\omega) e^{i\omega\tau} - P_i^*(\omega, t) Q_{10}(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1086)$$

$$= 0, i \in \{3, 6\}, \quad (1087)$$

$$\mathcal{B}_{1i}(t, t') = iB_{01}^{\Im}(t) \int_0^\infty (P_i^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1088)$$

$$= i0 \int_0^\infty (P_i^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1089)$$

$$= 0, i \in \{3, 6\}, \quad (1090)$$

$$\zeta_{ij}(t, t') = e^{i\Im \left(\int_0^\infty \frac{(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t))(L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t')) e^{i\omega\tau}}{\omega^2} d\omega \right)}, \quad (1091)$$

$$\xi_{ij}^\pm(t, t') = e^{-\int_0^\infty \frac{|(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t)) e^{i\omega\tau} \pm (L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t'))|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega}, \quad (1092)$$

$$P_i(\omega, t) = L_{i \bmod 2}(\omega) (1 - F_{i \bmod 2}(\omega, t)), \quad (1093)$$

$$Q_{ij}(\omega, t) = \frac{L_i(\omega) F_1(\omega, t) - L_j(\omega) F_j(\omega, t)}{\omega}. \quad (1094)$$

B. Time-independent variational quantum master equation

At first let's show that the master equation (1023) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1, 2, 3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) |1\rangle\langle 1| + \frac{\Omega}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1095)$$

After performing the transformation (25) on the Hamiltonian (1095) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H}_S = (\delta + R) |1\rangle\langle 1| + \frac{\Omega_r}{2} \sigma_x, \quad (1096)$$

$$\overline{H}_I = B_z |1\rangle\langle 1| + \frac{\Omega}{2} (B_x \sigma_x + B_y \sigma_y), \quad (1097)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1098)$$

The Hamiltonian (1096) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$, where $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ (this base for the Pauli matrix is different from the base assumed in PRB2011 which is $\sigma'_z = |1\rangle\langle 1| - |0\rangle\langle 0|$) by a term proportional to the identity given by $-\frac{\delta}{2}\mathbb{I}$ which is independent of the variational parameters, this can be seen in the following way, with $\epsilon = \delta + R$:

$$(\delta + R) |1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = (\delta + R) |1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| - \frac{\delta}{2}|1\rangle\langle 1| \quad (1099)$$

$$= \left(\frac{\delta}{2} + R\right) |1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \quad (1100)$$

$$= \left(\frac{\delta}{2} + R\right) |1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \quad (1101)$$

$$= \frac{R}{2}|1\rangle\langle 1| + \left(\frac{\delta}{2} + \frac{R}{2}\right) |1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \quad (1102)$$

$$= \frac{R}{2}|1\rangle\langle 1| + \frac{R}{2}|0\rangle\langle 0| + \left(\frac{\delta}{2} + \frac{R}{2}\right) |1\rangle\langle 1| - \frac{R}{2}|0\rangle\langle 0| - \frac{\delta}{2}|0\rangle\langle 0| \quad (1103)$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2} (|1\rangle\langle 1| - |0\rangle\langle 0|) \quad (1104)$$

$$= \frac{R}{2}\mathbb{I} - \frac{\delta + R}{2} (|0\rangle\langle 0| - |1\rangle\langle 1|) \quad (1105)$$

$$= \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2} (|0\rangle\langle 0| - |1\rangle\langle 1|) \quad (1106)$$

$$= \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\sigma_z. \quad (1107)$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_S}$. Given that $\overline{H_S} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\text{Tr}(\overline{H_S}) = R$ and $\text{Det}(\overline{H_S}) = \frac{R^2 - \epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \quad (1108)$$

$$\lambda_{\pm} = \frac{R \pm \sqrt{(-R)^2 - 4 \left(\frac{R^2 - \epsilon^2 - \Omega_r^2}{4} \right)}}{2} \quad (1109)$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \quad (1110)$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \quad (1111)$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \quad (1112)$$

$$\lambda_{\pm} = \frac{R \pm \eta}{2}. \quad (1113)$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\left(\frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} \quad \frac{\Omega_r}{2} \right) \begin{pmatrix} \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \\ \frac{\Omega_r}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \quad (1114)$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.

For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} & -\frac{\Omega_r}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \quad (1115)$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$. Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon + \eta}{2}, \quad (1116)$$

$$\lambda_- = \frac{\epsilon - \eta}{2}, \quad (1117)$$

$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \quad (1118)$$

$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, \quad (1119)$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}, \quad (1120)$$

$$\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}. \quad (1121)$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{\Omega_r}{\epsilon} \right). \quad (1122)$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1} \left(\frac{\Omega_r}{\epsilon} \right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x) + 1}. \quad (1123)$$

So the value of $\tan(\theta)$ using (1123) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}} + 1} \quad (1124)$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}}{\frac{\epsilon + \sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}} \quad (1125)$$

$$= \frac{\Omega_r}{\epsilon + \eta}. \quad (1126)$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (940)-(942) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$:

$$\langle +|\sigma_x|+ \rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1127)$$

$$= 2 \sin(\theta) \cos(\theta) \quad (1128)$$

$$= \sin(2\theta), \quad (1129)$$

$$\langle -|\sigma_x|- \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (1130)$$

$$= -2 \sin(\theta) \cos(\theta) \quad (1131)$$

$$= -\sin(2\theta), \quad (1132)$$

$$\langle -|\sigma_x|+ \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1133)$$

$$= \cos^2(\theta) - \sin^2(\theta) \quad (1134)$$

$$= \cos(2\theta), \quad (1135)$$

$$\langle +|\sigma_y|+ \rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1136)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (1137)$$

$$= 0, \quad (1138)$$

$$\langle -|\sigma_y|- \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (1139)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (1140)$$

$$= 0, \quad (1141)$$

$$\langle -|\sigma_y|+ \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1142)$$

$$= i \cos^2(\theta) + i \sin^2(\theta) \quad (1143)$$

$$= i. \quad (1144)$$

$$\langle +|\frac{1+\sigma_z}{2}|+ \rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1145)$$

$$= \cos(\theta) \cos(\theta) \quad (1146)$$

$$= \cos^2(\theta), \quad (1147)$$

$$\langle -|\frac{1+\sigma_z}{2}|- \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (1148)$$

$$= \sin(\theta) \sin(\theta) \quad (1149)$$

$$= \sin^2(\theta), \quad (1150)$$

$$\langle -|\frac{1+\sigma_z}{2}|+ \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1151)$$

$$= -\sin(\theta) \cos(\theta) \quad (1152)$$

$$= -\sin(\theta) \cos(\theta). \quad (1153)$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta) (|+\rangle\langle +| - |-\rangle\langle -|), \quad (1154)$$

$$A_1(\eta) = \cos(2\theta) |-\rangle\langle +|, \quad (1155)$$

$$A_2(0) = 0, \quad (1156)$$

$$A_2(\eta) = i|\chi|, \quad (1157)$$

$$A_3(0) = \cos^2(\theta)|\chi| + \sin^2(\theta)|-\chi|, \quad (1158)$$

$$A_3(\eta) = -\sin(\theta)\cos(\theta)|-\chi|. \quad (1159)$$

Now to prove the fact that the model of the “Time-independent variational quantum master equation” is a special case the master equation (1023) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (1003) is equivalent to:

$$L_{ijww'}(t, t - \tau) \equiv \int_0^t C_i(t) C_j(t - \tau) \mathcal{B}_{ij}(t, t - \tau) e^{i\tau w(t - \tau)} e^{-it(w(t - \tau) - w'(t))} A_{jww'}(t, t - \tau) d\tau, \quad (1160)$$

$$= \int_0^t C_i(t) C_j(t - \tau) \mathcal{B}_{ij}(\tau) e^{i\tau w} e^{-it(w - w')} A_j(w, w') d\tau. \quad (1161)$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda_{ij}(\tau)$ relate with the correlation functions defined in the equation (414) in the following way:

$$\Lambda_{ij}(\tau) = C_i(t) C_j(t - \tau) \mathcal{B}_{ij}(\tau). \quad (1162)$$

So the response matrix can be rewritten as:

$$L_{ijww'}(t, t - \tau) = \left(\int_0^t d\tau \Lambda_{ij}(\tau) e^{i\tau w} e^{-it(w - w')} \right) A_j(w, w'). \quad (1163)$$

Let's define the response function like:

$$K_{ij}(w, w', t) = \int_0^t C_i(t) C_j(t - \tau) \mathcal{B}_{ij}(\tau) e^{i\tau w} e^{-it(w - w')} d\tau \quad (1164)$$

$$= \int_0^t \Lambda_{ij}(\tau) e^{i\tau w} e^{-it(w - w')} d\tau \quad (1165)$$

$$= K_{ijww'}(t). \quad (1166)$$

Then we have the following equivalence:

$$L_{ijww'}(t) = K_{ijww'}(t) A_j(w, w') \quad (1167)$$

$$= K_{ijww'}(t) A_{jww'}. \quad (1168)$$

We can proof that

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left([A_i, L_{ijww'}(t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) L_{ijww'}^\dagger(t)] \right) \quad (1169)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left([A_i, K_{ijww'}(t) A_{jww'} \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) K_{ijww'}^*(t) A_{jww'}^\dagger] \right) \quad (1170)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left(K_{ijww'}(t) [A_i, A_{jww'} \overline{\rho_S}(t)] - K_{ijww'}^*(t) [A_i, \overline{\rho_S}(t) A_{jww'}^\dagger] \right) \quad (1171)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left((K_{ijww'}^{\Re}(t) + iK_{ijww'}^{\Im}(t)) [A_i, A_{jww'} \overline{\rho_S}(t)] - (K_{ijww'}^{\Re}(t) - iK_{ijww'}^{\Im}(t)) [A_i, \overline{\rho_S}(t) A_{jww'}^\dagger] \right) \quad (1172)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} K_{ijww'}^{\Re}(t) [A_i, A_{jww'} \overline{\rho_S}(t) - \overline{\rho_S}(t) A_{jww'}^\dagger] - i \sum_{ijww'} K_{ijww'}^{\Im}(t) [A_i, A_{jww'} \overline{\rho_S}(t) + \overline{\rho_S}(t) A_{jww'}^\dagger] \quad (1173)$$

For the time-independent PRB2011 we have the following correlations obtained from the general model, we take account from the fact that $L_0(\omega) = 0$, $\Im(L_1(\omega)) = 0$, $\Im(F_1(\omega)) = 0$, $F_0(\omega) = 0$ and $\int_0^\infty |L_1(\omega)|^2 f(\omega) d\omega = \int_0^\infty J(\omega) f(\omega) d\omega$ for $f(\omega) \in L^2$. We can drop the time vector (t, t') and instead write the correlation functions as function of τ , we will drop t and t' from the expressions that contain them given the time independence of the hamiltonian:

$$\chi_{ij}(t) = \int_0^\infty \frac{L_1^*(\omega) L_0(\omega) F_1^*(\omega, t) F_0(\omega, t) - L_1(\omega) L_0^*(\omega) F_1(\omega, t) F_0^*(\omega, t)}{2\omega^2} d\omega, \quad (1174)$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2} \left(\Re \left(e^{\chi_{10}(t) + \chi_{10}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') + \Re \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) - B_{10}^{\Re}(t) B_{01}^{\Re}(t'), \quad (1175)$$

$$= \frac{1}{2} (e^{\chi_{10} + \chi_{10}} \zeta_{10} \xi_{10}^+ + e^{\chi_{10} + \chi_{01}} \zeta_{10}^* \xi_{10}^-) - B^2, \quad (1176)$$

$$\chi_{ij} = \int_0^\infty \frac{L_1(\omega) L_0(\omega) F_1(\omega) F_0(\omega) - L_1(\omega) L_0(\omega) F_1(\omega) F_0(\omega)}{2\omega^2} d\omega, \quad (1177)$$

$$= 0 \quad (1178)$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2} (e^0 \zeta_{10} \xi_{10}^+ + e^0 \zeta_{10}^* \xi_{10}^-) - B^2, \quad (1179)$$

$$\zeta_{10} = e^{i\Im \left(\int_0^\infty \frac{(L_1(\omega) F_1(\omega) - L_0(\omega) F_0(\omega))(L_1(\omega) F_1(\omega) - L_0(\omega) F_0(\omega))^* e^{i\omega\tau}}{\omega^2} d\omega \right)} \quad (1180)$$

$$= e^{i\Im \left(\int_0^\infty \frac{(L_1(\omega) F_1(\omega))(L_1(\omega) F_1(\omega))^* e^{i\omega\tau}}{\omega^2} d\omega \right)} \quad (1181)$$

$$= e^{i\Im \left(\int_0^\infty \frac{J(\omega) F^2(\omega) e^{i\omega\tau}}{\omega^2} d\omega \right)} \quad (1182)$$

$$= e^{i\Im \left(\int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} (\cos(\omega\tau) + i \sin(\omega\tau)) d\omega \right)} \quad (1183)$$

$$= e^{i \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \sin(\omega\tau) d\omega} \quad (1184)$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2} \left(e^{i \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \sin(\omega\tau) d\omega} \xi_{10}^+ + e^{-i \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \sin(\omega\tau) d\omega} \xi_{10}^- \right) - B^2, \quad (1185)$$

$$\xi_{10}^\pm = e^{-\int_0^\infty \frac{|(L_1(\omega) F_1(\omega) - L_0(\omega) F_0(\omega)) e^{i\omega\tau} \pm L_1(\omega) F_1(\omega) \mp L_0(\omega) F_0(\omega)|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \quad (1186)$$

$$= e^{-\int_0^\infty \frac{|L_1(\omega) F_1(\omega) e^{i\omega\tau} \pm L_1(\omega) F_1(\omega)|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \quad (1187)$$

$$= e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) |e^{i\omega\tau} \pm 1|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \quad (1188)$$

$$|e^{i\omega\tau} \pm 1|^2 = 2(1 \pm \cos(\omega\tau)) \quad (1189)$$

$$\xi_{10}^\pm = e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (1 \pm \cos(\omega\tau))}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \quad (1190)$$

$$\mathcal{B}_{11}(\tau) = -B^2 + \frac{1}{2} \left(e^{i \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \sin(\omega\tau) d\omega} e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (1 + \cos(\omega\tau))}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \right. \quad (1191)$$

$$\left. + e^{-i \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \sin(\omega\tau) d\omega} e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (1 - \cos(\omega\tau))}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \right) \quad (1192)$$

$$= -B^2 + \frac{e^{-\int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega}}{2} \left(e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau))}{\omega^2} d\omega} \right. \quad (1193)$$

$$\left. + e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (-\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) + i \sin(\omega\tau))}{\omega^2} d\omega} \right) \quad (1194)$$

$$B = e^{-\frac{1}{2} \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega}, \quad (1195)$$

$$G_+(\omega) = e^{i\omega\tau} N(\omega) + e^{-i\omega\tau} (N(\omega) + 1) \quad (1196)$$

$$= (\cos(\omega\tau) + i \sin(\omega\tau)) N(\omega) + (\cos(\omega\tau) - i \sin(\omega\tau)) (N(\omega) + 1) \quad (1197)$$

$$= \cos(\omega\tau) (2N(\omega) + 1) - i \sin(\omega\tau) \quad (1198)$$

$$= \cos(\omega\tau) \left(\frac{2}{e^{\beta\omega} - 1} + 1 \right) - i \sin(\omega\tau) \quad (1199)$$

$$= \cos(\omega\tau) \left(\frac{1 + e^{\beta\omega}}{e^{\beta\omega} - 1} \right) - i \sin(\omega\tau) \quad (1200)$$

$$= \cos(\omega\tau) \left(\frac{e^{-\beta\omega/2} + e^{\beta\omega/2}}{-e^{-\beta\omega/2} + e^{\beta\omega/2}} \right) - i \sin(\omega\tau) \quad (1201)$$

$$= \cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau), \quad (1202)$$

$$\phi(\tau) = \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} G_+(\omega, \tau) d\omega \quad (1203)$$

$$= \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \left(\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau) \right) d\omega, \quad (1204)$$

$$\mathcal{B}_{11}(\tau) = -B^2 + \frac{e^{-\int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega}}{2} \left(e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau))}{\omega^2} d\omega} \right. \quad (1205)$$

$$\left. + e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (-\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) + i \sin(\omega\tau))}{\omega^2} d\omega} \right) \quad (1206)$$

$$= \frac{B^2}{2} (e^{-\phi(\tau)} + e^{\phi(\tau)} - 2) \quad (1207)$$

$$\mathcal{B}_{22}(\tau) = -\frac{1}{2} \left(\Re \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') - \Re \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') \right) + B_{01}^{\Im}(t) B_{10}^{\Im}(t'), \quad (1208)$$

$$= -\frac{1}{2} \left((e^{\chi_{01} + \chi_{01}}) \zeta_{10} \xi_{10}^+ - (e^{\chi_{10} + \chi_{01}}) \zeta_{10}^* \xi_{10}^- \right), \quad (1209)$$

$$= \frac{1}{2} (\zeta_{10}^* \xi_{10}^- - \zeta_{10} \xi_{10}^+), \quad (1210)$$

$$= \frac{1}{2} \left(e^{-i \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \sin(\omega\tau) d\omega} e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (1 - \cos(\omega\tau))}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \right. \quad (1211)$$

$$\left. - e^{i \int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \sin(\omega\tau) d\omega} e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (1 + \cos(\omega\tau))}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \right) \quad (1212)$$

$$= \frac{1}{2} e^{-\int_0^\infty \frac{J(\omega) F^2(\omega)}{\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega} \left(e^{\int_0^\infty \frac{J(\omega) F^2(\omega) (\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau))}{\omega^2} d\omega} \right. \quad (1213)$$

$$\left. - e^{-\int_0^\infty \frac{J(\omega) F^2(\omega) (\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau))}{\omega^2} d\omega} \right) \quad (1214)$$

$$= \frac{B^2}{2} (e^{\phi(\tau)} - e^{-\phi(\tau)}), \quad (1215)$$

$$\mathcal{B}_{12}(\tau) = \frac{1}{2} \left(\Im \left(e^{\chi_{10}(t) + \chi_{01}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Re}(t) B_{10}^{\Im}(t') \quad (1216)$$

$$= 0, \quad (1217)$$

$$\mathcal{B}_{21}(\tau) = \frac{1}{2} \left(\Im \left(e^{\chi_{01}(t) + \chi_{10}(t')} \right) \zeta_{10}^*(t, t') \xi_{10}^-(t, t') + \Im \left(e^{\chi_{01}(t) + \chi_{01}(t')} \right) \zeta_{10}(t, t') \xi_{10}^+(t, t') \right) + B_{10}^{\Im}(t) B_{10}^{\Re}(t'), \quad (1218)$$

$$= 0, \quad (1219)$$

$$\mathcal{B}_{ij}(t, t') = \int_0^\infty (P_i(\omega, t) P_j^*(\omega, t') e^{i\omega\tau} N(\omega) + P_i^*(\omega, t) P_j(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i, j \in \{3, 6\}, \quad (1220)$$

$$\mathcal{B}_{66}(t, t') = \int_0^\infty (P_6(\omega, t) P_6^*(\omega, t') e^{i\omega\tau} N(\omega) + P_6^*(\omega, t) P_6(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1221)$$

$$P_6(\omega, t) = L_{6\text{mod}2}(\omega) (1 - F_{6\text{mod}2}(\omega, t)), \quad (1222)$$

$$= L_0(\omega) (1 - F_0(\omega, t)), \quad (1223)$$

$$= 0, \quad (1224)$$

$$\mathcal{B}_{66}(\tau) = 0, \quad (1225)$$

$$\mathcal{B}_{36}(t, t') = \int_0^\infty (P_3(\omega, t) P_6^*(\omega, t') e^{i\omega\tau} N(\omega) + P_3^*(\omega, t) P_6(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1226)$$

$$= 0, \quad (1227)$$

$$\mathcal{B}_{63}(t, t') = \int_0^\infty (P_6(\omega, t) P_3^*(\omega, t') e^{i\omega\tau} N(\omega) + P_6^*(\omega, t) P_3(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1228)$$

$$= 0, \quad (1229)$$

$$\mathcal{B}_{33}(t, t') = \int_0^\infty (P_3(\omega, t) P_3^*(\omega, t') e^{i\omega\tau} N(\omega) + P_3^*(\omega, t) P_3(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1230)$$

$$= \int_0^\infty (P_3(\omega, t) P_3^*(\omega, t') e^{i\omega\tau} N(\omega) + P_3^*(\omega, t) P_3(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1231)$$

$$P_3(\omega, t) = L_{3\text{mod}2}(\omega) (1 - F_{3\text{mod}2}(\omega, t)), \quad (1232)$$

$$= L_1(\omega) (1 - F_1(\omega, t)), \quad (1233)$$

$$P_3(\omega, t) P_3^*(\omega, t') = L_1(\omega) (1 - F_1(\omega)) L_1^*(\omega) (1 - F_1(\omega)), \quad (1234)$$

$$= |L_1(\omega)|^2 (1 - F_1(\omega))^2 \quad (1235)$$

$$\mathcal{B}_{33}(t, t') = \int_0^\infty |L_1(\omega)|^2 (1 - F_1(\omega))^2 (e^{i\omega\tau} N(\omega) + e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1236)$$

$$= \int_0^\infty J(\omega) (1 - F(\omega))^2 G_+(\omega) d\omega \quad (1237)$$

$$\mathcal{B}_{i1}(t, t') = iB_{01}^{\Im}(t') \int_0^\infty (P_i(\omega, t) Q_{10}^*(\omega, t') N(\omega) e^{i\omega\tau} - P_i^*(\omega, t) Q_{10}(\omega, t') e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\} \quad (1238)$$

$$= 0, \quad (1239)$$

$$\mathcal{B}_{1i}(t, t') = iB_{01}^{\Im}(t) \int_0^\infty (P_i^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_i(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega, i \in \{3, 6\}, \quad (1240)$$

$$= 0, \quad (1241)$$

$$\mathcal{B}_{62}(t, t') = iB_{10}^{\Re}(t') \int_0^\infty (P_6^*(\omega, t') Q_{10}(\omega, t') (N(\omega) + 1) e^{-i\omega\tau} - P_6(\omega, t') Q_{10}^*(\omega, t') e^{i\omega\tau} N(\omega)) d\omega \quad (1242)$$

$$= 0, \quad (1243)$$

$$\mathcal{B}_{26}(t, t') = iB_{10}^{\Re}(t) \int_0^\infty (P_6^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_6(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1244)$$

$$= 0, \quad (1245)$$

$$\mathcal{B}_{32}(t, t') = iB_{10}^{\Re}(t') \int_0^\infty (P_3^*(\omega, t') Q_{10}(\omega, t') (N(\omega) + 1) e^{-i\omega\tau} - P_3(\omega, t') Q_{10}^*(\omega, t') e^{i\omega\tau} N(\omega)) d\omega \quad (1246)$$

$$= iB \int_0^\infty (P_3^*(\omega) Q_{10}(\omega) (N(\omega) + 1) e^{-i\omega\tau} - P_3(\omega) Q_{10}^*(\omega) e^{i\omega\tau} N(\omega)) d\omega, \quad (1247)$$

$$Q_{ij}(\omega, t) = \frac{L_i(\omega) F_j(\omega, t) - L_i(\omega) F_j(\omega, t)}{\omega}, \quad (1248)$$

$$Q_{10}(\omega, t) = \frac{L_1(\omega) F_1(\omega, t)}{\omega}, \quad (1249)$$

$$\mathcal{B}_{32}(t, t') = iB \int_0^\infty \left(L_1^*(\omega) (1 - F_1(\omega, t)) \frac{L_1(\omega) F_1(\omega, t)}{\omega} (N(\omega) + 1) e^{-i\omega\tau} \right. \quad (1250)$$

$$\left. - L_1(\omega) (1 - F_1(\omega, t)) \frac{L_1^*(\omega) F_1(\omega, t)}{\omega} e^{i\omega\tau} N(\omega) \right) d\omega \quad (1251)$$

$$= iB \int_0^\infty |L_1(\omega)|^2 \left((1 - F_1(\omega)) \frac{F_1(\omega)}{\omega} (N(\omega) + 1) e^{-i\omega\tau} - (1 - F_1(\omega)) \frac{F_1(\omega)}{\omega} e^{i\omega\tau} N(\omega) \right) d\omega \quad (1252)$$

$$= iB \int_0^\infty J(\omega) (1 - F(\omega, t)) \frac{F(\omega)}{\omega} ((N(\omega) + 1) e^{-i\omega\tau} - e^{i\omega\tau} N(\omega)) d\omega \quad (1253)$$

$$= iB \int_0^\infty J(\omega) (1 - F(\omega, t)) \frac{F(\omega)}{\omega} G_-(\omega) d\omega \quad (1254)$$

$$\mathcal{B}_{23}(t, t') = iB_{10}^{\Re}(t) \int_0^\infty (P_3^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_3(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1255)$$

$$= iB \int_0^\infty (P_3^*(\omega, t') Q_{10}(\omega, t) N(\omega) e^{i\omega\tau} - P_3(\omega, t') Q_{10}^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1256)$$

$$= iB \int_0^\infty \left(L_1^*(\omega) (1 - F_1(\omega, t)) \frac{L_1(\omega) F_1(\omega, t)}{\omega} N(\omega) e^{i\omega\tau} - L_1(\omega) (1 - F_1(\omega, t)) \frac{L_1^*(\omega) F_1(\omega, t)}{\omega} e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega \quad (1257)$$

$$= iB \int_0^\infty J(\omega) (1 - F_1(\omega, t)) \frac{F_1(\omega, t)}{\omega} (N(\omega) e^{i\omega\tau} - e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1258)$$

$$= -iB \int_0^\infty J(\omega) (1 - F_1(\omega, t)) \frac{F_1(\omega, t)}{\omega} (-N(\omega) e^{i\omega\tau} + e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1259)$$

$$= -\mathcal{B}_{32}(t, t') \quad (1260)$$

$$\zeta_{ij}(t, t') = e^{i\Im \left(\int_0^\infty \frac{(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t))(L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t'))^* e^{i\omega\tau}}{\omega^2} d\omega \right)}, \quad (1261)$$

$$\xi_{ij}^\pm(t, t') = e^{-\int_0^\infty \frac{|(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t)) e^{i\omega\tau} \pm L_i(\omega) F_i(\omega, t') \mp L_j(\omega) F_j(\omega, t')|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) d\omega}, \quad (1262)$$

$$P_i(\omega, t) = L_{i \bmod 2}(\omega) (1 - F_{i \bmod 2}(\omega, t)), \quad (1263)$$

$$Q_{ij}(\omega, t) = \frac{L_i(\omega) F_j(\omega, t) - L_i(\omega) F_j(\omega, t)}{\omega}, \quad (1264)$$

$$\mathcal{B}(\tau) \equiv \begin{pmatrix} \mathcal{B}_{11}(\tau) & 0 & 0 & 0 & \mathcal{B}_{11}(\tau) & 0 \\ 0 & \mathcal{B}_{22}(\tau) & \mathcal{B}_{23}(\tau) & \mathcal{B}_{22}(\tau) & 0 & 0 \\ 0 & -\mathcal{B}_{23}(\tau) & \mathcal{B}_{33}(\tau) & -\mathcal{B}_{23}(\tau) & 0 & 0 \\ 0 & \mathcal{B}_{22}(\tau) & \mathcal{B}_{23}(\tau) & \mathcal{B}_{22}(\tau) & 0 & 0 \\ \mathcal{B}_{11}(\tau) & 0 & 0 & 0 & \mathcal{B}_{11}(\tau) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1265)$$

The correlation functions as we can see in PRB2011 can be obtained using the following definition:

$$\Lambda_{ij}(\tau) \equiv C_i C_j \mathcal{B}_{ij}(\tau) \quad (1266)$$

Also the matrix C(t) can be decomposed as:

$$C_1 = C_2 = \frac{\Omega}{2}, \quad (1267)$$

$$C_3 = C_6 = 1, \quad (1268)$$

$$C_4 = C_4 = 0 \quad (1269)$$

Let's recall that $\Omega_r = B\Omega$. So the correlation functions Λ are:

$$\Lambda_{11}(\tau) = C_1 C_1 \mathcal{B}_{11}(\tau) \quad (1270)$$

$$= \left(\frac{\Omega}{2}\right)^2 \frac{B^2}{2} (e^{\phi(\tau)} + e^{-\phi(\tau)} - 2) \quad (1271)$$

$$= \frac{(\Omega B)^2}{8} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right) \quad (1272)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \quad (1273)$$

$$\Lambda_{22}(\tau) = C_2 C_2 \mathcal{B}_{22}(\tau) \quad (1274)$$

$$= \left(\frac{\Omega}{2} \right)^2 \frac{B^2}{2} \left(e^{\phi(\tau)} - e^{-\phi(\tau)} \right) \quad (1275)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} - e^{-\phi(\tau)} \right), \quad (1276)$$

$$\Lambda_{33}(\tau) = C_3 C_3 \mathcal{B}_{33}(\tau) \quad (1277)$$

$$= (1)^2 \int_0^\infty J(\omega) (1 - F(\omega))^2 G_+(\omega) d\omega \quad (1278)$$

$$= \int_0^\infty J(\omega) (1 - F(\omega))^2 G_+(\omega) d\omega, \quad (1279)$$

$$\Lambda_{23}(\tau) = C_2 C_3 \mathcal{B}_{23}(\tau) \quad (1280)$$

$$= \frac{\Omega}{2} i B \int_0^\infty J(\omega) (1 - F(\omega, t)) \frac{F(\omega)}{\omega} G_-(\omega) d\omega \quad (1281)$$

$$= i \frac{\Omega_r}{2} \int_0^\infty J(\omega) (1 - F(\omega, t)) \frac{F(\omega)}{\omega} G_-(\omega) d\omega, \quad (1282)$$

$$\Lambda_{12}(\tau) = \Lambda_{13}(\tau) = \Lambda_{16}(\tau) \quad (1283)$$

$$= \Lambda_{21}(\tau) = \Lambda_{26}(\tau) \quad (1284)$$

$$= \Lambda_{31}(\tau) = \Lambda_{36}(\tau) = 0. \quad (1285)$$

Now let's define:

$$K_{ijw}(t) = \int_0^t C_i C_j \mathcal{B}_{ij}(\tau) e^{i\omega\tau} d\tau. \quad (1286)$$

So

$$L_{ijw}(t) = A_{jw} K_{ijw}(t). \quad (1287)$$

Now for a time-independent hamiltonian is possible to show that for the decomposition matrix $A_j(w(t)) = A_j(w)$:

$$U^\dagger(t) A_j(t) U(t) = \sum_w e^{-i\omega t} A_j(w). \quad (1288)$$

It means that a decomposition matrix of $\widetilde{A}_j(t)$ associated to the eigenvector under evolution for the same time-independent hamiltonian $U(t) A_j(w) U^\dagger(t)$ generates the same decomposition matrix multiplied by a phase $e^{i\omega t}$. It means that the decomposition matrix $A_{jww'}$ for a time-independent hamiltonian fulfill $A_{jww'} = A_j(w) \delta_{ww'}$ so only if $w = w'$ then the response function is relevant for taking account and it's equal to:

$$K_{ijww}(t) = \int_0^t C_i C_j \mathcal{B}_{ij}(\tau) e^{i\omega\tau} d\tau \quad (1289)$$

$$\equiv K_{ijw}(t). \quad (1290)$$

Finally taking the Hamiltonian (1095) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\text{Det}(\overline{H_S}) = -\frac{\Omega_r^2}{4}$, $\text{Tr}(\overline{H_S}) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (338) we have that:

$$f_k = \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k} \right)} \quad (1291)$$

$$= \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k} \right)}. \quad (1292)$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (450) under a time-independent approach providing similar results.

The master equation for this special case is:

$$L_{ijww'}(t) = \delta_{ww'} A_{jw} K_{ijw}(t), \quad (1293)$$

$$\frac{d\bar{\rho}_S(t)}{dt} = -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} \left([A_i, L_{ijww'}(t) \bar{\rho}_S(t)] + [\bar{\rho}_S(t) L_{ijww'}^\dagger(t), A_i] \right) \quad (1294)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} \left([A_i, L_{ijw}(t) \bar{\rho}_S(t)] + [\bar{\rho}_S(t) L_{ijw}^\dagger(t), A_i] \right) \quad (1295)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} \left([A_i, A_{jw} K_{ijw}(t) \bar{\rho}_S(t)] + [\bar{\rho}_S(t) A_{jw}^* K_{ijw}^\dagger(t), A_i] \right) \quad (1296)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} \left(\left(K_{ijw}^{\Re}(t) + i K_{ijw}^{\Im}(t) \right) [A_i, A_{jw} \bar{\rho}_S(t)] + \left(K_{ijw}^{\Re}(t) - i K_{ijw}^{\Im}(t) \right) [\bar{\rho}_S(t) A_{jw}^\dagger, A_i] \right) \quad (1297)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} \left(K_{ijw}^{\Re}(t) \left([A_i, A_{jw} \bar{\rho}_S(t)] + [\bar{\rho}_S(t) A_{jw}^\dagger, A_i] \right) + i K_{ijw}^{\Im}(t) \left([A_i, A_{jw} \bar{\rho}_S(t)] - [\bar{\rho}_S(t) A_{jw}^\dagger, A_i] \right) \right) \quad (1298)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} \left(K_{ijw}^{\Re}(t) \left([A_i, A_{jw} \bar{\rho}_S(t)] - [A_i, \bar{\rho}_S(t) A_{jw}^\dagger] \right) + i K_{ijw}^{\Im}(t) \left([A_i, A_{jw} \bar{\rho}_S(t)] + [A_i, \bar{\rho}_S(t) A_{jw}^\dagger] \right) \right) \quad (1299)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} \left(K_{ijw}^{\Re}(t) [A_i, A_{jw} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jw}^\dagger] + i K_{ijw}^{\Im}(t) [A_i, A_{jw} \bar{\rho}_S(t) + \bar{\rho}_S(t) A_{jw}^\dagger] \right) \quad (1300)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijw} K_{ijw}^{\Re}(t) [A_i, A_{jw} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jw}^\dagger] - i \sum_{ijw} K_{ijw}^{\Im}(t) [A_i, A_{jw} \bar{\rho}_S(t) + \bar{\rho}_S(t) A_{jw}^\dagger], \quad (1301)$$

$$\gamma_{ij}(w, t) \equiv 2K_{ijw}^{\Re}(t) \quad (1302)$$

$$S_{ij}(w, t) \equiv K_{ijw}^{\Im}(t) \quad (1303)$$

$$A_j(w) \equiv A_{jw} \quad (1304)$$

$$\frac{d\bar{\rho}_S(t)}{dt} = -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \frac{1}{2} \sum_{ijw} \gamma_{ij}(w, t) [A_i, A_{jw} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jw}^\dagger] - i \sum_{ijw} S_{ij}(w, t) [A_i, A_{jw} \bar{\rho}_S(t) + \bar{\rho}_S(t) A_{jw}^\dagger] \quad (1305)$$

$$= -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \frac{1}{2} \sum_{ijw} \gamma_{ij}(w, t) [A_i, A_j(w) \bar{\rho}_S(t) - \bar{\rho}_S(t) A_j^\dagger(w)] - i \sum_{ijw} S_{ij}(w, t) [A_i, A_j(w) \bar{\rho}_S(t) + \bar{\rho}_S(t) A_j^\dagger(w)]. \quad (1306)$$

C. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right) \right) |1\rangle\langle 1| + \frac{\Omega(t)}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1307)$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (1307) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given

that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (1096) and (1097) we obtain the following transformed Hamiltonians:

$$\overline{H}_S = (\delta + R_1) |1\rangle\langle 1| + \frac{B\sigma_x}{2} \Omega(t), \quad (1308)$$

$$\overline{H}_I = \frac{\Omega(t)}{2} (B_x \sigma_x + B_y \sigma_y). \quad (1309)$$

In this case $R_1 = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} g_{\mathbf{k}} \right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}}^{-1} |g_{\mathbf{k}}|^2) = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}} |\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (1270)-(1285) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} G_+(\tau) d\omega. \quad (1310)$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau)$ so (1310) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left(\coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau) \right) d\omega. \quad (1311)$$

Writing the interaction Hamiltonian (1309) in the similar way to the equation (1097) allow us to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (1096) with $\delta' |1\rangle\langle 1| = \frac{\delta'}{2} \sigma_z + \frac{\delta'}{2} \mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then \overline{H}_S is equal to:

$$\overline{H}_S = \frac{\delta'}{2} \sigma_z + \frac{B\sigma_x}{2} \Omega(t). \quad (1312)$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \text{Tr}_B \left(\widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (1313)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \quad (1314)$$

$$\Lambda_{22}(\tau) = \text{Tr}_B \left(\widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (1315)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right). \quad (1316)$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation (450) is:

$$\frac{d\rho_S(t)}{dt} = -i \left[\frac{\delta'}{2} \sigma_z + \frac{\Omega_r(t) \sigma_x}{2}, \rho_S(t) \right] - \sum_{i=1}^2 \int_0^t d\tau \left(C_i(t) C_i(t-\tau) \Lambda_{ii}(\tau) \left[A_i, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right] \right. \quad (1317)$$

$$\left. + C_i(t) C_i(t-\tau) \Lambda_{ii}(-\tau) \left[\rho_S(t) \widetilde{A}_i(t-\tau, t), A_i \right] \right). \quad (1318)$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A}_i(t-\tau, t) = \widetilde{\sigma}_i(t-\tau, t)$, also using the equations (1313) and (1316) on the equation (1318) we obtain that:

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{2} [\delta' \sigma_z + \Omega_r(t) \sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4} \int_0^t d\tau \Omega(t-\tau) ([\sigma_x, \widetilde{\sigma}_x(t-\tau, t) \rho_S(t)] \Lambda_x(\tau) \quad (1319)$$

$$+ [\sigma_y, \widetilde{\sigma}_y(t-\tau, t) \rho_S(t)] \Lambda_y(\tau) + [\rho_S(t) \widetilde{\sigma}_x(t-\tau, t), \sigma_x] \Lambda_x(\tau) + [\rho_S(t) \widetilde{\sigma}_y(t-\tau, t), \sigma_y] \Lambda_y(\tau)). \quad (1320)$$

As we can see $\left[A_j, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right]^\dagger = \left[\rho_S(t) \widetilde{A}_i(t-\tau, t), A_j \right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

D. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation (S17) of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (414) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\hbar = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} (g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^* b_{\mathbf{k}}). \quad (1321)$$

Using the equation (1023), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \frac{1}{2} \sum_w \gamma_{33}(w, t) [A_3, A_3(w) \rho_S(t) - \rho_S(t) A_3^{\dagger}(w)] \quad (1322)$$

$$- \sum_w S_{33}(w, t) [A_3, A_3(w) \rho_S(t) + \rho_S(t) A_3^{\dagger}(w)] \Bigg). \quad (1323)$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^{\infty} d\omega J(\omega) G_+(\tau), \quad (1324)$$

$$\Lambda_{33}(-\tau) = \int_0^{\infty} d\omega J(\omega) G_+(-\tau). \quad (1325)$$

In our case $A_3 = \frac{\mathbb{I} + \sigma_z}{2}$, the equation (1323) can be transformed in

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \sum_w (K_{33}(w, t) [A_3, A_3(w) \rho_S(t)] + K_{33}^*(w, t) [\rho_S(t) A_3(w), A_3]). \quad (1326)$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthermore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\tilde{\Omega}$. To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian using the approximation $\tilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\tilde{\Omega}}{2}, \quad (1327)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle), \quad (1328)$$

$$\lambda_- = -\frac{\tilde{\Omega}}{2}, \quad (1329)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \quad (1330)$$

The elements of the decomposition matrices are:

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1331)$$

$$= \frac{1}{2}, \quad (1332)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1333)$$

$$= \frac{1}{2}, \quad (1334)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1335)$$

$$= -\frac{1}{2}. \quad (1336)$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2} |+\rangle\langle +| + \frac{1}{2} |-\rangle\langle -| \quad (1337)$$

$$= \frac{\mathbb{I}}{2}, \quad (1338)$$

$$A_3(\eta) = -\frac{1}{2} |-\rangle\langle +| \quad (1339)$$

$$= \frac{1}{4} (\sigma_z + i\sigma_y), \quad (1340)$$

$$A_3(-\eta) = -\frac{1}{2} |+\rangle\langle -| \quad (1341)$$

$$= \frac{1}{4} (\sigma_z - i\sigma_y). \quad (1342)$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - K_{33}(\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z + i\sigma_y) \rho_S(t) \right] - K_{33}(-\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z - i\sigma_y) \rho_S(t) \right] \quad (1343)$$

$$- K_{33}^*(\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4} (\sigma_z + i\sigma_y), \frac{\sigma_z}{2} \right] - K_{33}^*(-\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4} (\sigma_z - i\sigma_y), \frac{\sigma_z}{2} \right]. \quad (1344)$$

Calculating the response functions extending the upper limit of τ to ∞ , we obtain:

$$K_{33}(\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{i\tilde{\Omega}\tau} d\tau d\omega \quad (1345)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (1346)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} (n(\omega) + 1) e^{-i\tau\omega} d\tau d\omega \quad (1347)$$

$$= \int_0^\infty \int_0^\infty J(\omega) (n(\omega) + 1) e^{i\tilde{\Omega}\tau - i\tau\omega} d\tau d\omega \quad (1348)$$

$$= \int_0^\infty J(\omega) (n(\omega) + 1) \pi \delta(\tilde{\Omega} - \omega) d\omega \quad (1349)$$

$$= \pi J(\tilde{\Omega}) (n(\tilde{\Omega}) + 1), \quad (1350)$$

$$K_{33}(-\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{-i\tilde{\Omega}\tau} d\tau d\omega \quad (1351)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (1352)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} n(\omega) e^{i\tau\omega} d\tau d\omega \quad (1353)$$

$$= \int_0^\infty \int_0^\infty J(\omega) n(\omega) e^{-i\tilde{\Omega}\tau + i\tau\omega} d\tau d\omega \quad (1354)$$

$$= \int_0^\infty J(\omega) n(\omega) \pi \delta(-\tilde{\Omega} + \omega) d\omega \quad (1355)$$

$$= \pi J(\tilde{\Omega}) n(\tilde{\Omega}). \quad (1356)$$

Here we have used $\int_0^\infty ds e^{\pm i\epsilon s} = \pi \delta(\epsilon) \pm i \frac{\text{V.P.}}{\epsilon}$, where V.P. denotes the Cauchy's principal value. These principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1343) lead us to obtain:

$$\frac{d\rho_S(t)}{dt} = -i \frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\tilde{\Omega}) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (1357)$$

$$- \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\tilde{\Omega}) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right). \quad (1358)$$

This is the same result than the equation (S17), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{d\rho_S(t)}{dt} = -i \frac{\Omega(t)}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\Omega(t)) \left((n(\Omega(t)) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\Omega(t)) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (1359)$$

$$- \frac{\pi}{8} J(\Omega(t)) \left((n(\Omega(t)) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\Omega(t)) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right). \quad (1360)$$

V. FREE ENERGY MINIMIZATION WITH 3TH BOGOLIUBOV INEQUALITY

If we recall the form of the third Bogoliubov inequality using the

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1361)$$

$$H_S(t) = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1362)$$

$$H_I = \sum_{nuk} |n\rangle\langle n| \left(g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk} \right), \quad (1363)$$

$$H_B = \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk}. \quad (1364)$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nuk} |n\rangle\langle n| \omega_{uk}^{-1} \left(f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk} \right) \quad (1365)$$

At first let's obtain $e^{\pm V}$ under the transformation (1365), consider $\hat{\varphi}_n = \sum_{uk} \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})$, so the equation (1365) can be written as $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \quad (1366)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{(\sum_n |n\rangle\langle n| \hat{\varphi}_n)^2}{2!} + \dots \quad (1367)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1368)$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1369)$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots \right) \quad (1370)$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \quad (1371)$$

Given that $[f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk}, f_{nu'k'} b_{u'k'}^\dagger - f_{nu'k'}^* b_{u'k'}] = 0$ for all k', k and u, u' then we can proof using the Zassenhaus formula and defining $D(\pm \alpha_{nuk}) = e^{\pm (\alpha_{nuk} b_{uk}^\dagger - \alpha_{nuk}^* b_{uk})}$ in the same way than (24) with $\alpha_{nuk} = \frac{f_{nuk}}{\omega_{uk}}$:

$$e^{\pm \sum_{uk} \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} = \prod_u e^{\pm \sum_k \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} \quad (1372)$$

$$= \prod_u \left(\prod_k e^{\pm \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} \right) \quad (1373)$$

$$= \prod_u \left(\prod_k D(\pm \alpha_{nuk}) \right) \quad (1374)$$

$$= \prod_{uk} D(\pm \alpha_{nuk}) \quad (1375)$$

$$= \prod_u B_{nu\pm} \quad (1376)$$

$$B_{nu\pm} \equiv \prod_k D(\pm \alpha_{nuk}) \quad (1377)$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1378)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1361):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1379)$$

$$= \prod_u B_{0u+} |0\rangle\langle 0| \prod_u B_{0u-}, \quad (1380)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \quad (1381)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \quad (1382)$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \quad (1383)$$

$$= |0\rangle\langle 0|. \quad (1384)$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1385)$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n| \prod_u B_{nu-}, \quad (1386)$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \quad (1387)$$

$$= |m\rangle\langle n| \prod_u (B_{mu+} B_{nu-}), \quad m \neq n, \quad (1388)$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D(\alpha_{muk}) \prod_{\mathbf{k}} D(-\alpha_{nuk}) \right), \quad (1389)$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} (D(\alpha_{muk}) D(-\alpha_{nuk})), \quad (1390)$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D(\alpha_{muk} - \alpha_{nuk}) \exp \left(\frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (1391)$$

$$\Pi_u(B_{mu+} B_{nu-}) = \prod_{u\mathbf{k}} \left(D(\alpha_{muk} - \alpha_{nuk}) \exp \left(\frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (1392)$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = (\sum_n |n\rangle\langle n| \prod_u B_{nu+}) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} (\sum_n |n\rangle\langle n| \prod_u B_{nu-}), \quad (1393)$$

$$= (|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \dots) (\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}) (|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \dots), \quad (1394)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + \dots, \quad (1395)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} + (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} + (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{1u-} + \dots \quad (1396)$$

$$= |0\rangle\langle 0| (\prod_u B_{0u+} + \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \dots) \quad (1397)$$

$$+ |1\rangle\langle 1| (\prod_u B_{1u+} + \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \dots) + \dots \quad (1398)$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) \quad (1399)$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) + \dots \quad (1400)$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \dots \quad (1401)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\omega_{n\mathbf{k}} \left(b_{n\mathbf{k}}^\dagger - \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} \right) \left(b_{n\mathbf{k}} - \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} \right) \right) \quad (1402)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\omega_{n\mathbf{k}} \left(b_{n\mathbf{k}}^\dagger b_{n\mathbf{k}} - \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} b_{n\mathbf{k}}^\dagger - \frac{v_{n\mathbf{k}}^*}{\omega_{n\mathbf{k}}} b_{n\mathbf{k}} + \left| \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} \right|^2 \right) \right) \quad (1403)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \omega_{n\mathbf{k}} b_{n\mathbf{k}}^\dagger b_{n\mathbf{k}} + \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{n\mathbf{k}}|^2}{\omega_{n\mathbf{k}}} - v_{n\mathbf{k}} b_{n\mathbf{k}}^\dagger - v_{n\mathbf{k}}^* b_{n\mathbf{k}} \right) \quad (1404)$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{n\mathbf{k}}|^2}{\omega_{n\mathbf{k}}} - (v_{n\mathbf{k}} b_{n\mathbf{k}}^\dagger + v_{n\mathbf{k}}^* b_{n\mathbf{k}}) \right) \quad (1405)$$

The transformed Hamiltonians of the equations (1362) to (1364) written in terms of (1379) to (1403) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1406)$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1407)$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) \quad (1408)$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{n \neq m} |n\rangle\langle m| (g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}) \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1409)$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}) + \dots \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1410)$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}) \prod_u B_{0u-} + \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| (g_{1u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{1u\mathbf{k}}^* b_{u\mathbf{k}}) \prod_u B_{1u-} + \dots \quad (1411)$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} \prod_u B_{0u+} b_{u\mathbf{k}}^\dagger \prod_u B_{0u-} + g_{0u\mathbf{k}}^* \prod_u B_{0u+} b_{u\mathbf{k}} \prod_u B_{0u-}) + \sum_{u\mathbf{k}} |1\rangle\langle 1| (g_{1u\mathbf{k}} \prod_u B_{1u+} b_{u\mathbf{k}}^\dagger \prod_u B_{1u-} + g_{1u\mathbf{k}}^* \prod_u B_{1u+} b_{u\mathbf{k}} \prod_u B_{1u-}) + \dots \quad (1412)$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{0u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{1u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \dots \quad (1413)$$

$$= \sum_{n \neq m} |n\rangle\langle m| \left(g_{nu\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{nu\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1414)$$

$$= \sum_{n \neq m} |n\rangle\langle m| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1415)$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{n \neq m} |n\rangle\langle m| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - (v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}) \right) \quad (1416)$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{n \neq m} |n\rangle\langle m| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - (v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}) \right) \quad (1417)$$

$$+ \sum_{n \neq m} |n\rangle\langle m| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1418)$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1419)$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}}) b_{u\mathbf{k}}^\dagger + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^* b_{u\mathbf{k}} \right) \quad (1420)$$

Using the previous functions we have that (1417) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \quad (1421)$$

$$(1422)$$

Now in order to separate the elements of the hamiltonian (1422) let's follow the references of the equations (??) and (??) to separate the hamiltonian, before proceeding to do this we need to consider the term of the form:

$$\langle \Pi_u (B_{mu+} B_{nu-}) \rangle_{\overline{H_0}} = \langle \Pi_{uk} (D(\alpha_{muk} - \alpha_{nuk}) \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk})) \rangle_{\overline{H_0}} \quad (1423)$$

$$= (\Pi_{uk} \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}))) \langle \Pi_{uk} D(\alpha_{muk} - \alpha_{nuk}) \rangle_{\overline{H_0}} \quad (1424)$$

$$= \left(\Pi_{uk} \exp\left(\frac{(v_{muk}^* v_{nuk} - v_{nuk} v_{muk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (1425)$$

$$\equiv B_{nm} \quad (1426)$$

$$\langle \Pi_u (B_{nu+} B_{mu-}) \rangle_{\overline{H_0}} = \left(\Pi_{uk} \exp\left(\frac{(v_{nuk}^* v_{muk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (1427)$$

$$= B_{nm}^* \quad (1428)$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+} B_{nu-}) - B_{nm} \quad (1429)$$

As we can see:

$$J_{nm}^\dagger = \left(\prod_u (B_{mu+} B_{nu-}) - B_{nm} \right)^\dagger \quad (1430)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{nm}^* \quad (1431)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{mn} \quad (1432)$$

$$= J_{mn} \quad (1433)$$

We can separate the Hamiltonian (1422) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}}(t) = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \quad (1434)$$

$$\overline{H_{\bar{I}}}(t) = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1435)$$

$$\overline{H_{\bar{B}}}(t) = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \quad (1436)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}})} \right) \right) + \langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} + O \left(\left\langle \overline{H_{\bar{I}}}^2 \right\rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} \right). \quad (1437)$$

Taking the equations (245)-(253) and given that $\text{Tr} \left(e^{-\beta \overline{H_{\bar{S}}}(t)} \right) = C(R_0, R_1, \dots, R_{d-1}, B_{01}, \dots, B_{0(d-1)}, \dots, B_{(d-2)(d-1)})$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nuk}\}$. Given that the numbers v_{nuk} are complex then we can separate them as $v_{nuk} = v_{nuk}^{\Re} + i v_{nuk}^{\Im}$. So our approach will be based on the derivation respect to v_{nuk}^{\Re} and v_{nuk}^{\Im} . The Hamiltonian $\overline{H_{\bar{S}}}(t)$ can be written like:

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \right) |n\rangle\langle n| \quad (1438)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1439)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \frac{g_{nuk} v_{nuk}^* + g_{nuk}^* v_{nuk}}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1440)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1441)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2}{\omega_{uk}} - \frac{(g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} + i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1442)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1443)$$

$$v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^* = (v_{muk}^{\Re} - i v_{muk}^{\Im})(v_{nuk}^{\Re} + i v_{nuk}^{\Im}) - (v_{muk}^{\Re} + i v_{muk}^{\Im})(v_{nuk}^{\Re} - i v_{nuk}^{\Im}) \quad (1444)$$

$$= (v_{muk}^{\Re} v_{nuk}^{\Re} + i v_{nuk}^{\Im} v_{muk}^{\Re} - i v_{muk}^{\Re} v_{nuk}^{\Im} + v_{muk}^{\Im} v_{nuk}^{\Im}) \quad (1445)$$

$$- (v_{muk}^{\Re} v_{nuk}^{\Re} - i v_{nuk}^{\Im} v_{muk}^{\Re} + i v_{muk}^{\Re} v_{nuk}^{\Im} + v_{muk}^{\Im} v_{nuk}^{\Im}) \quad (1446)$$

$$= 2i (v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re}) \quad (1447)$$

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2}{\omega_{uk}} - \frac{(g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} + i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1448)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1449)$$

$$|v_{muk} - v_{nuk}|^2 = (v_{muk} - v_{nuk})(v_{muk} - v_{nuk})^* \quad (1450)$$

$$= |v_{muk}|^2 + |v_{nuk}|^2 - (v_{nuk} v_{muk}^* + v_{muk} v_{nuk}^*) \quad (1451)$$

$$= (v_{muk}^{\Re})^2 + (v_{muk}^{\Im})^2 + (v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - (v_{nuk}^{\Re} + i v_{nuk}^{\Im})(v_{muk}^{\Re} - i v_{muk}^{\Im}) \quad (1452)$$

$$- (v_{nuk}^{\Re} - i v_{nuk}^{\Im})(v_{muk}^{\Re} + i v_{muk}^{\Im}) \quad (1453)$$

$$= (v_{muk}^{\Re})^2 + (v_{muk}^{\Im})^2 + (v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2(v_{nuk}^{\Re} v_{muk}^{\Re} + v_{nuk}^{\Im} v_{muk}^{\Im}) \quad (1454)$$

$$= (v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2 \quad (1455)$$

$$R_n(t) = \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1456)$$

$$= \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - (g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} - i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \quad (1457)$$

$$= \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1458)$$

$$B_{mn} = \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1459)$$

$$= \left(\Pi_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1460)$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Re}} = \frac{\partial}{\partial v_{nuk}^{\Re}} \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1461)$$

$$= \frac{2v_{nuk}^{\Re} - 2g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1462)$$

$$= 2 \frac{v_{nuk}^{\Re} - g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1463)$$

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Im}} = \frac{\partial}{\partial v_{nuk}^{\Im}} \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1464)$$

$$= \frac{2v_{nuk}^{\Im} - 2g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1465)$$

$$= 2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1466)$$

Given that:

$$\ln B_{mn} = \ln \left(\left(\prod_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \right) \quad (1467)$$

$$= \sum_{uk} \ln \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_u \ln \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1468)$$

$$= \sum_{uk} \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_{uk} \left(-\frac{1}{2} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1469)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Re}} = \frac{-i v_{muk}^{\Im} - (v_{nuk}^{\Re} - v_{muk}^{\Re}) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1470)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Im}} = \frac{i v_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1471)$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \quad (1472)$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \quad (1473)$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial (B_{nm})^{\dagger}}{\partial a} \quad (1474)$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \quad (1475)$$

$$= B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} - (v_{n\mathbf{k}}^{\Re} - v_{m\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1476)$$

$$= B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1477)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Re}} = \left(\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \right)^{\dagger} \quad (1478)$$

$$= \left(B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \quad (1479)$$

$$= B_{nm} \left(\frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1480)$$

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \quad (1481)$$

$$= B_{mn} \left(\frac{iv_{m\mathbf{k}}^{\Re} - (v_{n\mathbf{k}}^{\Im} - v_{m\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1482)$$

$$= B_{mn} \left(\frac{iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1483)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Im}} = \left(\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \right)^{\dagger} \quad (1484)$$

$$= (B_{mn})^{\dagger} \quad (1485)$$

$$= B_{nm} \left(\frac{-iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1486)$$

Introducing this derivatives in the equation (1461) give us:

$$\frac{\partial A_{\mathbf{B}}}{\partial v_{n\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathbf{B}}}{\partial R_n} \left(2 \frac{v_{n\mathbf{k}}^{\Re} - g_{u\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n < m} \left(\frac{\partial A_{\mathbf{B}}}{\partial B_{nm}} B_{nm} \left(\frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \quad (1487)$$

$$+ \frac{\partial A_{\mathbf{B}}}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1488)$$

$$= 0 \quad (1489)$$

We can obtain the variational parameters:

$$-2 \frac{\partial A_B}{\partial R_n} \frac{v_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1490)$$

$$= -\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1491)$$

$$v_{nuk}^{\Re} = \frac{\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right)}{2 \frac{\partial A_B}{\partial R_n} \frac{1}{\omega_{uk}} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right)} \quad (1492)$$

$$= \frac{2g_{nuk}^{\Re} \omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} (iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} (-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) \right)}{2\omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right)} \quad (1493)$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_B}{\partial v_{nuk}^{\Im}} = \frac{\partial A_B}{\partial R_n} \left(2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \right) + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{-iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1494)$$

$$+ \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1495)$$

$$= 0 \quad (1496)$$

Rearranging we obtain

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1513)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1514)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1515)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\bar{\rho}(t)) \quad (1516)$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1517)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1518)$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \quad (1519)$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2i} \quad (1520)$$

We can proof that $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+}B_{n-} - B_m B_n)^\dagger \quad (1521)$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \quad (1522)$$

$$= B_{n+} B_{m-} - B_n B_m \quad (1523)$$

$$= B_{nm} \quad (1524)$$

So we can say that the set of matrices (1517) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1517) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1525)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}) \quad (1526)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) + iV_{nm}^\Im(t) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) \right) \quad (1527)$$

$$+ \Re(V_{nm}(t)) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) - iV_{nm}^\Im(t) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1528)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} + B_{mn}}{2} \right) + \Re(V_{nm}(t)) \sigma_{nm,y} \frac{i(B_{mn} - B_{nm})}{2} \right) \quad (1529)$$

$$+ i\Im(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} - B_{mn}}{2} \right) + \Im(V_{nm}(t)) \sigma_{nm,y} \left(\frac{B_{nm} + B_{mn}}{2} \right) \quad (1530)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (\Re(V_{nm}(t)) \sigma_{nm,x} B_{nm,x} - \Im(V_{nm}(t)) \sigma_{nm,x} B_{nm,y} + \Re(V_{nm}(t)) \sigma_{nm,y} B_{nm,y} \quad (1531)$$

$$+ \Im(V_{nm}(t)) \sigma_{nm,y} B_{nm,x}) \quad (1532)$$

Let's define the set

$$P = \{(n, m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\} \quad (1533)$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \quad (1534)$$

$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \quad (1535)$$

$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \quad (1536)$$

$$A_{4,nm}(t) = A_{2,mn}(t) \quad (1537)$$

$$A_{5,nm}(t) = A_{1,nm}(t) \quad (1538)$$

$$B_{1,nm}(t) = B_{nm,x} \quad (1539)$$

$$B_{2,nm}(t) = B_{nm,y} \quad (1540)$$

$$B_{3,nm}(t) = B_{z,n}(t) \quad (1541)$$

$$B_{4,nm}(t) = B_{1,nm}(t) \quad (1542)$$

$$B_{5,nm}(t) = B_{2,nm}(t) \quad (1543)$$

$$C_{1,nm}(t) = \Re(V_{nm}(t)) \quad (1544)$$

$$C_{2,nm}(t) = C_{1,nm}(t) \quad (1545)$$

$$C_{3,nm}(t) = 1 \quad (1546)$$

$$C_{4,nm}(t) = \Im(V_{nm}(t)) \quad (1547)$$

$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \quad (1548)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t) (A_{jp} \otimes B_{jp}(t)) \quad (1549)$$

Here $J = \{1, 2, 3, 4, 5\}$ and P the set defined in (1533).

We write the interaction Hamiltonian transformed under (1513) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t) (\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)) \quad (1550)$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \quad (1551)$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t) e^{-iH_B t} \quad (1552)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1553)$$

Replacing the equation (1550) in (1553) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1554)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \left[\sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1555)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] \quad (1556)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \Big] ds \quad (1557)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right) \quad (1558)$$

$$- \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \quad (1559)$$

$$- \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \quad (1560)$$

$$+ \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \Big) ds \quad (1561)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B \quad (1562)$$

$$= \left\langle \widetilde{B}_{jp}(\tau) \widetilde{B}_{j'p'}(0) \right\rangle_B \quad (1563)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jp}(t)$ and elements related to the system given by $\widetilde{A}_{jp}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jp}(t) \widetilde{B}_{j'p'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jp}(t) \right) \text{Tr} \left(\widetilde{B}_{j'p'}(t) \right)$. The correlation functions relevant of the master equation (1561) are:

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1564)$$

$$= \left\langle \widetilde{B_{jp}}(0) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1565)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1566)$$

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \rho_B \widetilde{B_{j'p'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1567)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1568)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1569)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1570)$$

$$\text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \rho_B \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) \quad (1571)$$

$$= \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1572)$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1573)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1574)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1575)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1576)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1577)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1578)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1554) and (1561) and using the equations (1564)-(1578) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \widetilde{A_{jp}}(t) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau) \widetilde{A_{jp}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \right) \right. \quad (1579)$$

$$\left. + C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{j'p'jp}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jp}}(t) \right) \right) ds \quad (1580)$$

$$= - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \left[\widetilde{A_{jp}}(t), \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s), \widetilde{A_{jp}}(t) \right] \right) \right) \quad (1581)$$

Rearranging and identifying the commutators allow us to write a more simplified version

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(t-\tau) \left(\Lambda_{jpj'p'}(\tau) \left[A_{jp}(t), A_{j'p'}(t-\tau, t) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) A_{j'p'}(t-\tau, t), A_{jp}(t) \right] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1582)$$

For this case we used that $A_{jp}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jp}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1583)$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1584)$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right), \quad (1585)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1586)$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}) \right) \quad (1587)$$

At first let's obtain e^V under the transformation (1587), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \quad (1588)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{(\sum_{n=1} |n\rangle\langle n| \hat{\varphi})^2}{2!} + \dots \quad (1589)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \quad (1590)$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \quad (1591)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \quad (1592)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \quad (1593)$$

Given that $[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})}$ in the same way than (24):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \quad (1594)$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (1595)$$

$$= B_{\pm} \quad (1596)$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B$. because this form imposes that $e^{-V} e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1597)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1583):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1598)$$

$$= |0\rangle\langle 0|, \quad (1599)$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1600)$$

$$= |m\rangle\langle m| B^+ |m\rangle\langle n| n\rangle\langle n| B^-, \quad (1601)$$

$$= |m\rangle\langle n|, \quad m \neq 0, \quad n \neq 0, \quad (1602)$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1603)$$

$$= |0\rangle\langle m| B^- \quad m \neq 0, \quad (1604)$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1605)$$

$$= |0\rangle\langle m| B^+ \quad m \neq 0, \quad (1606)$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1607)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B^- \quad (1608)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B^+ b_{\mathbf{k}}^\dagger B^- \right) \left(B^+ b_{\mathbf{k}} B^- \right) \quad (1609)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1610)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1611)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \quad (1612)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1613)$$

The transformed Hamiltonians of the equations (1584) to (1586) written in terms of (1598) to (1613) are:

$$\overline{H_S(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1614)$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1615)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1616)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|}) + \sum_{m,n \neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \quad (1617)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) B^- |0\rangle\langle n| + V_{n0}(t) B^+ |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1618)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1619)$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \left(\left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1620)$$

$$= \left(\mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B^+ \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1621)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B^+ (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) B^- \quad (1622)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1623)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1624)$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1625)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1626)$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (1627)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1628)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1629)$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1630)$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1631)$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \right) \quad (1632)$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1633)$$

Using the previous functions we have that (1630) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1634)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1635)$$

Now in order to separate the elements of the hamiltonian (1635) let's follow the references of the equations (??) and (??) to separate the hamiltonian like:

$$\overline{H_S}(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \quad (1636)$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)), \quad (1637)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (1638)$$

Here B is given by:

$$\begin{aligned} B &= \langle B^+ \rangle \\ &= \langle B^- \rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1639)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1640)$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \quad (1641)$$

$$B_y = \frac{B^- - B^+}{2i} \quad (1642)$$

Using this set of hermitian operators to write the Hamiltonians (1584)-(1586)

$$\overline{H_S(t)} = \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1643)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{0 < m < n} (V_{mn}(t) |m\rangle\langle n| + V_{nm}(t) |n\rangle\langle m|) \quad (1644)$$

$$+ \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1645)$$

$$= \sum_{0 < m < n} ((\Re(V_{mn}(t)) + i\Im(V_{mn}(t))) |m\rangle\langle n| + (\Re(V_{mn}(t)) - i\Im(V_{mn}(t))) |n\rangle\langle m|) + \varepsilon_0(t) |0\rangle\langle 0| \quad (1646)$$

$$+ B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1647)$$

$$= \sum_{0 < m < n} \left((\Re(V_{nm}(t)) + i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1648)$$

$$+ B \sum_{n=1} \left(V_{0n}(t) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + V_{n0}(t) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) + \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1649)$$

$$= \sum_{0 < m < n} (\Re(V_{nm}(t)) \sigma_{nm,x} + \Im(V_{nm}(t)) \sigma_{nm,y}) + B \sum_{n=1} (\Re(V_{0n}(t)) \sigma_{0n,x} + \Im(V_{mn}(t)) \sigma_{0n,y}) \quad (1650)$$

$$+ \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1651)$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)) \quad (1652)$$

$$= \sum_{n=1} \left((\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) (B^- - B) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) (B^+ - B) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) \quad (1653)$$

$$+ \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1654)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} ((B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) + (B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))) \right) \quad (1655)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) - (B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))) \quad (1656)$$

$$+ \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1657)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} (B^+ + B^- - 2B) \Re(V_{0n}(t)) + i(B^- - B - B^+ + B) \Im(V_{0n}(t)) \right) \quad (1658)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B - B^- + B) \Re(V_{0n}(t)) + i(B - B^- + B - B^+) \Im(V_{0n}(t))) + \sum_{n=1} B_{z,n} |n\rangle\langle n| \quad (1659)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (\sigma_{0n,x} (B_x \Re(V_{0n}(t)) - B_y \Im(V_{0n}(t))) \quad (1660)$$

$$+ \sigma_{0n,y} (B_y \Re(V_{0n}(t)) + B_x \Im(V_{0n}(t)))) \quad (1661)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H}_S + \overline{H}_B)} \right) \right) + \langle \overline{H}_I \rangle_{\overline{H}_S + \overline{H}_B} + O \left(\langle \overline{H}_I^2 \rangle_{\overline{H}_S + \overline{H}_B} \right). \quad (1662)$$

Taking the equations (245)-(253) and given that $\text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right) = C(R_1, R_2, \dots, R_{d-1}, B)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (253) and using the chain rule we obtain that:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \frac{\partial R_n}{\partial v_{\mathbf{k}}}, \quad (1663)$$

$$= 0 \quad (1664)$$

Let's recall the equations (1631) and (1633), we can write them in terms of the variational parameters

$$B = \exp \left(- (1/2) \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) \right) \quad (1665)$$

$$R_n = \sum_{\mathbf{k}} \omega_{\mathbf{k}}^{-1} (v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}} v_{\mathbf{k}}) \quad (1666)$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \quad (1667)$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1668)$$

Introducing this derivates in the equation (1663) give us:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \right) + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1669)$$

$$= v_{\mathbf{k}} \left(\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t) \quad (1670)$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1671)$$

$$= \frac{2g_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1672)$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}} v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}}. \quad (1673)$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) (|0\rangle\langle 0| \otimes \rho_B) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1674)$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \quad (1675)$$

$$0 = \rho(0) \quad (1676)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1677)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1678)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1679)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}(t)) \quad (1680)$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I}(t) = B_{z,0} |0\rangle\langle 0| + \sum_{n=1} (\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n} |n\rangle\langle n|) \quad (1681)$$

$$+ \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x} \quad (1682)$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1683)$$

$$B_{z,n} = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \text{ if } n \neq 0 \quad (1684)$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \quad (1685)$$

$$A_{2n}(t) = \sigma_{0n,y} \quad (1686)$$

$$A_{3n}(t) = |n\rangle\langle n| \quad (1687)$$

$$A_{4n}(t) = A_{2n}(t) \quad (1688)$$

$$A_{5n}(t) = A_{1n}(t) \quad (1689)$$

$$B_{1n}(t) = B_x \quad (1690)$$

$$B_{2n}(t) = B_y \quad (1691)$$

$$B_{3n}(t) = B_{z,n} \quad (1692)$$

$$B_{4n}(t) = B_{1n}(t) \quad (1693)$$

$$B_{5n}(t) = B_{2n}(t) \quad (1694)$$

$$C_{10}(t) = 0 \quad (1695)$$

$$C_{20}(t) = 0 \quad (1696)$$

$$C_{40}(t) = 0 \quad (1697)$$

$$C_{50}(t) = 0 \quad (1698)$$

$$C_{30}(t) = 1 \quad (1699)$$

$$C_{1n}(t) = \Re(V_{0n}(t)) \quad (1700)$$

$$C_{2n}(t) = C_{1n}(t) \quad (1701)$$

$$C_{3n}(t) = 1 \quad (1702)$$

$$C_{4n}(t) = \Im(V_{0n}(t)) \quad (1703)$$

$$C_{5n}(t) = -\Im(V_{0n}(t)) \quad (1704)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H}_I(t)$ as:

$$\overline{H}_I = \sum_{j \in J} \sum_{n=1} C_{jn}(t) (A_{jn} \otimes B_{jn}(t)) \quad (1705)$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1677) as:

$$\widetilde{H}_I(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1706)$$

$$\widetilde{A}_i(t) = U^\dagger(t) A_i U(t) \quad (1707)$$

$$\widetilde{B}_i(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \quad (1708)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho}_S(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho}_S(t) \rho_B \right] \right] ds \quad (1709)$$

Replacing the equation (1706) in (1709) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1710)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \left[\sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1711)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1712)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \Big] ds \quad (1713)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right) ds \quad (1714)$$

$$- \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \quad (1715)$$

$$- \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1716)$$

$$+ \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \Big) ds \quad (1717)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jn j'n'}(\tau) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B \quad (1718)$$

$$= \left\langle \widetilde{B}_{jn}(\tau) \widetilde{B}_{j'n'}(0) \right\rangle_B \quad (1719)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \rho_B \right) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jn}(t)$ and elements related to the system given by $\widetilde{A}_{jn}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jn}(t) \widetilde{B}_{j'n'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jn}(t) \right) \text{Tr} \left(\widetilde{B}_{j'n'}(t) \right)$. The correlation functions relevant of the master equation (1717) are:

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1720)$$

$$= \left\langle \widetilde{B_{jn}}(0) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1721)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1722)$$

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \rho_B \widetilde{B_{j'n'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1723)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1724)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1725)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1726)$$

$$\text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \rho_B \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) \quad (1727)$$

$$= \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1728)$$

$$= \left\langle \widetilde{B_{jn}}(\tau) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1729)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1730)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1731)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1732)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1733)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1734)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1710) and (1717) and using the equations (1720)-(1734) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jn j'n'}(\tau) \widetilde{A_{jn}}(t) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'n' jn}(-\tau) \widetilde{A_{jn}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \right) \right. \quad (1735)$$

$$\left. + C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{j'n' jn}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \widetilde{A_{jn}}(t) - \Lambda_{jn j'n'}(\tau) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jn}}(t) \right) \right) ds \quad (1736)$$

$$= - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jn j'n'}(\tau) \left[\widetilde{A_{jn}}(t), \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'n' jn}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s), \widetilde{A_{jn}}(t) \right] \right) \right) \quad (1737)$$

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(t-\tau) \left(\Lambda_{jn j'n'}(\tau) [A_{jn}(t), A_{j'n'}(t-\tau, t) \widetilde{\rho_S}(t)] + \Lambda_{j'n' jn}(-\tau) [\widetilde{\rho_S}(t) A_{j'n'}(t-\tau, t), A_{jn}(t)] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1738)$$

For this case we used that $A_{jn}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jn}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation and if we take $n = 2$ (two sites), $\mu_0(t) = 0$, $\mu_1(t) = 1$ then we can reproduce a similar expression to (450) as expected.

VIII. BIBLIOGRAPHY

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