Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong or anything in between

Nike Dattani*
Harvard-Smithsonian Center for Astrophysics

Camilo Chaparro Sogamoso†
National University of Colombia

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I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality on $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ and its partition function given by $Z\left(t\right)$ and $Z_0\left(t\right)$ respectively as:

$$Z\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta \overline{H}(t)}\right), \tag{1}$$

$$Z_0\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta \overline{H}_0(t)}\right). \tag{2}$$

where the transformed hamiltonians $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ are defined as:

$$\overline{H}\left(t\right) \equiv \overline{H_{\bar{I}}}\left(t\right) + \overline{H_0}\left(t\right), \tag{3}$$

$$\overline{H_0}\left(t\right) \equiv \overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}. \tag{4}$$

For any operator $A\left(t\right)$ we define the expected value respect to $\overline{H_0}\left(t\right)$ as:

$$\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)} \equiv \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}. \tag{5}$$

The terms $\overline{H_{\bar{S}}}\left(t\right)$, $\overline{H_{\bar{B}}}$ and $\overline{H_{\bar{I}}}\left(t\right)$ are related to the variational transformation performed in [1, 2], this transformation allowed us to construct $\overline{H_{\bar{I}}}\left(t\right)$ such that $\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)} = 0$. The diagonalization of $\overline{H_0}\left(t\right)$ in terms of it's eigenstates and eigenvalues, such that $\overline{H_0}\left(t\right)|n\rangle = E_{0,n}\left(t\right)|n\rangle$ being $|n\rangle$ an eigenstate of $\overline{H_0}\left(t\right)$ with eigenvalue $E_{0,n}\left(t\right)$ is $\overline{H_0}\left(t\right) = \sum_n E_{0,n}\left(t\right)|n\rangle\!\langle n|$, with $\langle n|n'\rangle = \delta_{nn'}$. A simple form of $\mathrm{e}^{-\beta\overline{H_0}(t)}$ can be found as follows:

$$\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\mathrm{e}^{rY}\mathrm{e}^{-\frac{r^2}{2}[X,Y]}\mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\cdots \text{ (Zassenhaus formula)}, \tag{6}$$

$$\mathrm{e}^{X+Y} = \mathrm{e}^{X}\mathrm{e}^{Y}\mathrm{e}^{-\frac{1^2}{2}0}\mathrm{e}^{\frac{1^3}{6}(2[Y,0]+[X,0])}\cdots \text{ (setting } r=1 \text{ and } [X,Y]=0 \text{ in (6))} \tag{7}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}\mathbb{I} \tag{8}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}, \tag{9}$$

$$\mathrm{e}^{-\beta\overline{H_0}(t)} = \mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\!\langle n|} \text{ (by the diagonalization of } \overline{H_0}\left(t\right)\text{)} \tag{10}$$

$$= \prod_n \mathrm{e}^{-\beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by (9) and } [|n\rangle\langle n|, |n'\rangle\langle n'|] = 0) \tag{11}$$

$$= \prod_n \sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)|n\rangle\langle n|\right)^j}{j!} \text{ (by the exponential formula)} \tag{12}$$

$$= \prod_n \left(\mathbb{I} + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)\right)^j |n\rangle\langle n|}{j!}\right) \text{ (using } (aA)^j = a^j A^j \text{ and } (|n\rangle\langle n|)^2 = |n\rangle\langle n|) \tag{13}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)\right)^j |n\rangle\langle n|}{j!}\right) \tag{14}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left(\sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)\right)^j}{j!}\right)\right) \text{ (introducing } |n\rangle\langle n| \text{ inside the sum)} \tag{15}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \text{ (by the exponential formula)} \tag{16}$$

$$= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right). \tag{17}$$

We will prove by induction a neat form for (17), we will show that:

$$\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \mathbb{I} + \sum_{j=1}^{n} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \tag{18}$$

For $n = 1$ the formula is trivial, in the case $n = 2$ we obtain that:

$$\begin{aligned}
\prod_{j=1}^{2} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) &= \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2|\right) &(19)\\
&= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1| + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2| \text{ (by } \langle i|j\rangle = \delta_{ij}) &(20)\\
&= \mathbb{I} + \sum_{j=1}^{2} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. &(21)
\end{aligned}$$

It was our case base, our induction step is (18). In the case $n + 1$ we will have:

$$\begin{aligned}
\prod_{j=1}^{n+1}\left(\mathbb{I}+\left(\mathrm{e}^{-\beta E_{0,j}(t)}-1\right)|j\rangle\langle j|\right) &= \left(\prod_{j=1}^{n}\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right)\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) &(22)\\
&= \left(\mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) \text{ (by induction step)} &(23)\\
&= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1| + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| \text{ (by } \langle i|j\rangle = \delta_{ij}) &(24)\\
&= \mathbb{I} + \sum_{j=1}^{n+1} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. &(25)
\end{aligned}$$

By mathematical induction we proved that (18) is true for all $n \in \mathbb{N}$. Given that the resolution of the identity is $\mathbb{I} = \sum_n |n\rangle\langle n|$ then we find that:

$$
\begin{aligned}
\mathrm{e}^{-\beta\overline{H_0}(t)} &= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right) &(26)\\
&= \mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| \text{ (by (18))} &(27)\\
&= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \sum_n |n\rangle\langle n| \text{ (separating the terms of the sum)} &(28)\\
&= \cancel{\mathbb{I}} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \cancel{\mathbb{I}} \text{ (by the resolution of the identity } \mathbb{I} = \sum_n |n\rangle\langle n|) &(29)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|. &(30)
\end{aligned}
$$

The partition function $Z_0\left(t\right)$ is equal to:

$$
\begin{aligned}
Z_0\left(t\right) &= \mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \text{ (by (30))} &(31)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(|n\rangle\langle n|\right) \text{ (by trace linearity)} &(32)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \text{ (because } \mathrm{Tr}\left(|n\rangle\langle n|\right) = 1). &(33)
\end{aligned}
$$

The explicit form of the average value $\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)}$ can be found from the partition function $Z_0\left(t\right)$:

$$
\begin{aligned}
\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)} &= \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{Z_0\left(t\right)} \text{ (by (5))} &(34)\\
&= \frac{\mathrm{Tr}\left(\sum_n A\left(t\right)\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \text{ (by (30))} &(35)\\
&= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)} \text{ (by commutativity in scalar product)} &(36)\\
&= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} \text{ (by (33))} &(37)\\
&= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} \text{ (by trace linearity).} &(38)
\end{aligned}
$$

At first we show a double sequence of inequalities of order M, N which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$
Z\left(t\right) \geq Z_0\left(t\right)\mathrm{e}^{-\left\langle \overline{H_I}(t)\right\rangle_{\overline{H_0}(t)}}\left(1 + F_M\left(\overrightarrow{u}\left(t\right);\alpha\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right);\alpha\right)\right). \tag{39}
$$

where the function $F_N\left(\overrightarrow{u}\left(t\right);\alpha\right)$ is defined as:

$$
F_N\left(\overrightarrow{w}\left(t\right);\alpha\right) \equiv \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{w_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. \tag{40}
$$

In this case α is a parameter that can be optimized, $\beta \equiv \frac{1}{\mathrm{k_B}T}$, $\overrightarrow{w}\left(t\right)$ is a vector such that $\overrightarrow{w}\left(t\right) = \left(w_1, w_2, \ldots\right)$ and $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian $\overline{H_I}_D\left(t\right)$ respect to the basis of eigenstates of $\overline{H_0}\left(t\right)$ as:

$$\overline{H_{\overline{I}D}}(t) \equiv \sum_n \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n|. \tag{41}$$

We will prove an important property related to $\overline{H_{\overline{I}D}}(t)$ which is a Hamiltonian written as a linear combination of a set of orthonormal operators. Let's consider a ring R with two operations $+$ and $\cdot$, if there exist $a, b \in R$ such that $a \cdot b = 0$ and $b \cdot a = 0$ then for any $k \in \mathbb{N}$ we have $(a+b)^k = a^k + b^k$ where $a^k = a^{k-1} \cdot a$ is a recursive definition of the power of an element written in terms of $\cdot$. At first we prove that this result yields for any $k \in \mathbb{N}$ by induction, the case $k = 1$ is trivial so we will focus on the case $k = 2$, we have that:

$$\begin{aligned}
(a+b)^2 &= (a+b)\cdot(a+b) \ \text{(by definition of the power of an element)} && (42)\\
&= a\cdot a + a\cdot b + b\cdot a + b\cdot b \ \text{(by distributive multiplication respect addition)} && (43)\\
&= a^2 + a\cdot b + b\cdot a + b^2 \ \text{(by definition of the power of an element)} && (44)\\
&= a^2 + 0 + 0 + b^2 \ \text{(because } a\cdot b = b\cdot a = 0\text{)} && (45)\\
&= a^2 + b^2. && (46)
\end{aligned}$$

It was the base case. By induction step we will consider that $(a+b)^k = a^k + b^k$ with $k \geq 2$, now for $k+1$ we will have that:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)^k \cdot (a+b) \ \text{(by definition of the power of an element)} && (47)\\
&= \left(a^k + b^k\right)\cdot(a+b) \ \text{(by induction step)} && (48)\\
&= a^k\cdot a + a^k\cdot b + b^k\cdot a + b^k\cdot b \ \text{(by distributive multiplication respect addition)} && (49)\\
&= a^{k+1} + a^{k-1}\cdot a\cdot b + b^{k-1}\cdot b\cdot a + b^{k+1} \ \text{(by recursive definition of } a^k\text{)} && (50)\\
&= a^{k+1} + a^{k-1}\cdot(a\cdot b) + b^{k-1}\cdot(b\cdot a) + b^{k+1} \ \text{(by associativity on } R \text{ respect } \cdot\text{)} && (51)\\
&= a^{k+1} + a^{k-1}\cdot(0) + b^{k-1}\cdot(0) + b^{k+1} \ \text{(because } a\cdot b = b\cdot a = 0\text{)} && (52)\\
&= a^{k+1} + b^{k+1}. && (53)
\end{aligned}$$

By the principle of mathematical induction we can conclude that the proposition is true for all $k \in \mathbb{N}$. Now we will extend the result, let $a_1, ..., a_n \in R$ such that $a_i \cdot a_j = 0$ for all $i \neq j$ then $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$. The case $n = 1$ is trivial as well so we will focus on $n = 2$, this case was proved in the precedent lines so it will be our base case. By induction step we will consider that $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$ with $n \geq 2$, now for $n+1$ we will have that:

$$\begin{aligned}
a_{n+1}\cdot(a_1 + ... + a_n) &= a_{n+1}\cdot a_1 + ... + a_{n+1}\cdot a_n \ \text{(by distributive multiplication respect addition)} && (54)\\
&= 0 \ \text{(because } a_i\cdot a_j = 0 \text{ for all } i \neq j\text{),} && (55)\\
(a_1 + ... + a_n + a_{n+1})^k &= ((a_1 + ... + a_n) + a_{n+1})^k \ \text{(by associative property of +)} && (56)\\
&= (a_1 + ... + a_n)^k + a_{n+1}^k \ \text{(by (53) and (55))} && (57)\\
&= a_1^k + ... + a_n^k + a_{n+1}^k \ \text{(by inductive step).} && (58)
\end{aligned}$$

So we can conclude by mathematical induction that the proposition is true for all $n \in \mathbb{N}$. We can prove the following property for $\left(\overline{H_{\overline{I}D}}(t)\right)^k$:

$$\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n|\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle |n'\rangle\langle n'| = \left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle \left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle |n\rangle\langle n|n'\rangle\langle n'| \tag{59}$$

$$= \left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle \left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle |n\rangle\langle n'|\delta_{nn'} \text{ (by } \delta \text{ properties)}, \tag{60}$$

$$\left(\overline{H_{ID}}(t)\right)^{k} = \left(\sum_{n}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n|\right)^{k} \text{ (by (41))} \tag{61}$$

$$= \sum_{n}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n|\right)^{k} \text{ (by (58) and (60))}, \tag{62}$$

$$(aA)^{k} = a^{k}A^{k} \text{ (by the property of the power of a matrix)}, \tag{63}$$

$$(|n\rangle\langle n|)^{k} = |n\rangle\langle n| \text{ (because } |n\rangle\langle n| \text{ is a projector and } k\in\mathbb{N}^{*}), \tag{64}$$

$$\left(\overline{H_{ID}}(t)\right)^{k} = \sum_{n}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle\right)^{k} |n\rangle\langle n| \text{ (by (63) and (64))}. \tag{65}$$

The vectors $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ are defined as $\overrightarrow{u}(t) \equiv (u_1(t), u_2(t), ...)$ and $\overrightarrow{v}(t) \equiv (v_1(t), v_2(t), ...)$. We can define the elements of $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ in terms of the matrix $\overline{H_{ID}}(t)$:

$$u_k(t) \equiv \left\langle \left(\overline{H_{ID}}(t) - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}\right\rangle_{\overline{H_0(t)}} \tag{66}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\sum_n\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n| - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}|n\rangle\langle n|\right)}{Z_0(t)} \text{ (by (38))}, \tag{67}$$

$$\left(\sum_{n}\langle n|\overline{H_{I}}(t)|n\rangle|n\rangle\langle n| - \langle\overline{H_{I}}(t)\rangle_{\overline{H_0(t)}}\right)^{k} = \sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left(\sum_{n}\langle n|\overline{H_{I}}(t)|n\rangle|n\rangle\langle n|\right)^{j}\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k-j} \text{ (by binomial theorem)} \tag{68}$$

$$= \sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left(\sum_{n}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle^{j}|n\rangle\langle n|\right)\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k-j} \text{ (by (65))} \tag{69}$$

$$= \sum_{n}\left(\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle^{j}\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k-j}\right)|n\rangle\langle n| \text{ (by exchange of } \textstyle\sum) \tag{70}$$

$$= \sum_{n}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}|n\rangle\langle n| \text{ (by binomial theorem)}, \tag{71}$$

$$u_k(t) = \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\sum_{n'}\left(\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}|n'\rangle\langle n'|n\rangle\langle n|\right)}{Z_0(t)} \tag{72}$$

$$= \frac{\sum_{nn'} \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}|n'\rangle\langle n|\langle n'|n\rangle\right)}{Z_0(t)} \tag{73}$$

$$= \frac{\sum_{nn'} \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}|n'\rangle\langle n|\delta_{nn'}\right)}{Z_0(t)} \tag{74}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}\mathrm{Tr}\left(|n\rangle\langle n|\right)}{Z_0(t)} \text{ (by } \delta \text{ properties)} \tag{75}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}1}{Z_0(t)} \text{ (by } \mathrm{Tr}\left(|n\rangle\langle n|\right) = 1) \tag{76}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}}{Z_0(t)}, \tag{77}$$

$$v_k(t) \equiv \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{I}}(t) - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k}\right|n\right\rangle}{Z_0(t)}. \tag{78}$$

By construction $\left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)}=0$, so we summarize the double inequality that generalizes the Bogoliubov inequality and it's coefficients as:

$$Z\left(t\right)\geq Z_0\left(t\right)\left(1+F_M\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right), \tag{79}$$

$$Z\left(t\right)=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}\left(t\right)}\right), \tag{80}$$

$$Z_0\left(t\right)=\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}, \tag{81}$$

$$F_N\left(\overrightarrow{u}\left(t\right);\alpha\right)=\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}, \tag{82}$$

$$u_k\left(t\right)=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle^k}{Z_0\left(t\right)}, \tag{83}$$

$$v_k\left(t\right)=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_I}\left(t\right)\right)^k\right|n\right\rangle}{Z_0\left(t\right)}. \tag{84}$$

As we can see the expression (83) was written in shorter terms, we want to do the same for (84) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for $v_k\left(t\right)$:

$$
\begin{aligned}
\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)\right)\left|n\right\rangle &= \overline{H_0}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle \text{ (by distributive property)} && (85)\\
&= E_{0,n}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle \text{ (by } \overline{H_0}\left(t\right)\left|n\right\rangle = E_{0,n}\left(t\right)\left|n\right\rangle\text{)} && (86)\\
&= 0, && (87)\\
\left\langle n\right|\left(\overline{H_0}\left(t\right)-E_{0,n}\right) &= \left\langle n\right|\overline{H_0}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) \text{ (by distributive property)} && (88)\\
&= \left\langle n\right|E_{0,n}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) \text{ (by } \left\langle n\right|\overline{H_0}\left(t\right) = \left\langle n\right|E_{0,n}\left(t\right)\text{)} && (89)\\
&= 0. && (90)
\end{aligned}
$$

At first we calculated $v_1\left(t\right)$ using the definition (84) :

$$
\begin{aligned}
v_1(t) =& \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_0}(t)-E_{0,n}(t)+\overline{H_I}(t)\right|n\right\rangle \text{ (by (84))} && (91)\\
=& \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_0}(t)-E_{0,n}(t)\right|n\right\rangle+\frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_I}(t)\right|n\right\rangle \text{ (by distributive property)} && (92)\\
=& \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_0}(t)\right|n\right\rangle-\left\langle n\left|E_{0,n}(t)\right|n\right\rangle\right)+\left\langle\overline{H_I}(t)\right\rangle_{\overline{H_0}(t)} \text{ (by (5))} && (93)\\
=& \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|E_{0,n}(t)\right|n\right\rangle-\left\langle n\left|E_{0,n}(t)\right|n\right\rangle\right)+\left\langle\overline{H_I}(t)\right\rangle_{\overline{H_0}(t)} \text{ (by } \overline{H_0}\left(t\right)\left|n\right\rangle = E_{0,n}\left(t\right)\left|n\right\rangle\text{)} && (94)\\
=& 0+\left\langle\overline{H_I}(t)\right\rangle_{\overline{H_0}(t)} && (95)\\
=& 0 \text{ (by construction } \left\langle\overline{H_I}(t)\right\rangle_{\overline{H_0}(t)}=0\text{)}. && (96)
\end{aligned}
$$

For $k\geq 2$ and $k\in\mathbb{N}$ we calculated:

$$
\begin{aligned}
v_k(t) &= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^k \right| n \right\rangle & (97)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle & (98)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle & (99)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle & (100)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle . & (101)
\end{aligned}
$$

In general we can write a formula for $v_k(t)$ that implies an expected value of a dependent expression of $\overline{H_{\overline{I}}}(t)$ and $\overline{H_0}(t)$:

$$
\begin{aligned}
v_k(t) &= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle & (102)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) + \overline{H_{\overline{I}}}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle & (103)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \text{ (by (3))} & (104)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j}(t)\, E_{0,n}^j(t) \right) \overline{H_{\overline{I}}}(t) \right| n \right\rangle \text{ (by binomial theorem)} & (105)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, E_{0,n}^j(t) \right| n \right\rangle \text{ (exchanging } \Sigma \text{ and } \langle n | \cdots | n\rangle) & (106)\\
&= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \text{ (by } E_{0,n}(t)\,|n\rangle = \overline{H_0}(t)\,|n\rangle) & (107)\\
&= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle & (108)\\
&= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \text{ (by (5))} & (109)\\
&= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \text{ (rewriting using (3)).} & (110)
\end{aligned}
$$

The formula (110) is well defined taking as example $k = 2, 3$.

$$
\begin{aligned}
v_2(t) &= \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{2-2-j} \overline{H_{\overline{I}}}(t)\; \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} & (111)\\
&= (-1)^0 \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\; \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} & (112)\\
&= \left\langle \overline{H_{\overline{I}}}^2(t) \right\rangle_{\overline{H_0}(t)} . & (113)
\end{aligned}
$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{3-2-j} \overline{H_I}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{114}$$

$$= \left\langle \sum_{j=0}^{1} (-1)^j \binom{1}{j} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{1-j} \overline{H_I}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{115}$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^1 \overline{H_I}(t)\, \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^0 \overline{H_I}(t)\, \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \tag{116}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \mathbb{I} - \overline{H_I}(t)\, \mathbb{I} \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{117}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{118}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t)\, \overline{H_0}(t)\, \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{119}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) \left(\overline{H_0}(t)\, \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_0}(t)\right) \right\rangle_{\overline{H_0}(t)} \tag{120}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) \left[\overline{H_0}(t), \overline{H_I}(t)\right] \right\rangle_{\overline{H_0}(t)} \ \text{(because } \left[\overline{H_0}(t), \overline{H_I}(t)\right] = \overline{H_0}(t)\, \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_0}(t)\text{)}. \tag{121}$$

So we summarize:

$$\overline{H_I}_D(t) = \sum_n \left\langle n \left| \overline{H_I}(t) \right| n \right\rangle |n\rangle\langle n|, \tag{122}$$

$$u_k(t) = \left\langle \left(\overline{H_I}_D(t)\right)^k \right\rangle_{\overline{H_0}(t)}, \tag{123}$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_I}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{124}$$

The free energy $E_{\text{free}}(t)$ and free energy $E_{\text{free},1}(t)$ at first order are respectively:

$$E_{\text{free}}(t) \equiv -\frac{1}{\beta} \ln(Z(t)), \tag{125}$$

$$E_{\text{free},1}(t) \equiv -\frac{1}{\beta} \ln(Z_0(t)). \tag{126}$$

It is well-known that the function $f(x) = -\ln(x)$ is a decreasing function so we can transform (39):

$$E_{\text{free}}(t) = -\frac{1}{\beta} \ln(Z(t)) \ \text{(by (125))} \tag{127}$$

$$\leq -\frac{1}{\beta} \ln\left(Z_0(t)\left(1 + F_M\left(\overrightarrow{u}(t); \alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t); \alpha\right)\right)\right) \tag{128}$$

$$= -\frac{1}{\beta} \ln(Z_0(t)) - \frac{1}{\beta} \ln\left(1 + F_M\left(\overrightarrow{u}(t); \alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t); \alpha\right)\right) \tag{129}$$

$$= E_{\text{free},1}(t) - \frac{1}{\beta} \ln\left(1 + F_M\left(\overrightarrow{u}(t); \alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t); \alpha\right)\right) \ \text{(by (126))} \tag{130}$$

$$\equiv E_{\text{free,MN}}(t). \tag{131}$$

here $E_{\text{free,MN}}(t)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of M, N order. In our approach we will set $N = M$, so the inequality (131) of N, N order is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln\left(1 + F_N\left(\overrightarrow{u}(t); \alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t); \alpha\right)\right) \tag{132}$$

$$= E_{\text{free,NN}}(t). \tag{133}$$

A weaker form of the inequality (133) is obtained making $\overrightarrow{u}(t)=0$ as suggest [3]:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{v}(t);\alpha\right)\right) \tag{134}$$

$$\equiv E_{\text{free,N}}(t). \tag{135}$$

The algebraic equation associated with $\alpha_{\text{opt}}(t)$ such that $E_{\text{free,N}}(t)$ is closer to $E_{\text{free}}(t)$ follows from the fact that in the optimal parameter $\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha}|_{\alpha=\alpha_{\text{opt}(t)}}=0$, calculating this derivative we have:

$$\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{v}(t);\alpha\right)\right)\right) \tag{136}$$

$$= -\frac{1}{\beta}\frac{\frac{\partial}{\partial\alpha}\left(F_N\left(\overrightarrow{v}(t);\alpha\right)\right)}{1+F_N\left(\overrightarrow{v}(t);\alpha\right)} \tag{137}$$

$$= 0. \tag{138}$$

The precedent equation is equivalent to make the numerator equal to 0:

$$\frac{\partial F_N\left(\overrightarrow{v}(t);\alpha\right)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{139}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\partial}{\partial\alpha}\frac{\alpha^i}{i!}\ \text{(by product rule)} \tag{140}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} \tag{141}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{142}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{j=0}^{2N-2-k}\frac{\alpha^j}{j!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right)\ \text{(setting } j=i-1\text{)} \tag{143}$$

$$= \mathrm{e}^{-\alpha}\left(-\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha^{2N-1-k}}{(2N-1-k)!}\right)\ \text{(performing the difference leaving } i=2N-1-k\text{)} \tag{144}$$

$$= 0. \tag{145}$$

Then the optimal value $\alpha_{\text{opt}}(t)$ will satisfy the following equation:

$$G\left(\alpha_{\text{opt}}(t)\right) \equiv \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \tag{146}$$

$$= 0. \tag{147}$$

The elements presented are the required to find variational parameters of the system using the inequality (135) and the self consistent equation (SCE) (146) to a particular order required.

II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2(t), v_3(t), v_4(t), v_5(t)$ using the (124) because the order $N=3$ requires to obtain the elements $v_k(t)$ until $k=2N-1=5$. We already have $v_2(t), v_3(t)$, so we will find $v_4(t)$ and $v_5(t)$:

$$v_4(t) = \sum_{j=0}^{4-2} (-1)^j \binom{4-2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{4-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{148}$$

$$= \sum_{j=0}^{2} (-1)^j \binom{2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{149}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^1 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) \right.\right. \tag{150}$$

$$\left.\left. + \overline{H_0}(t)\right)^0 \overline{H_I}(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{151}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) \mathbb{I} \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^1 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{152}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{153}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) - 2\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{154}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^2(t)\right) \overline{H_I}(t) - 2\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}^2(t) \right. \tag{155}$$

$$\left. \times \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{156}$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - 2\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) \overline{H_0}(t) \right. \tag{157}$$

$$\left. + \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{158}$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - 2\overline{H_I}^3(t) \overline{H_0}(t) - 2\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \right. \tag{159}$$

$$\left. + \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{160}$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - \overline{H_I}^3(t) \overline{H_0}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}^2(t) \right. \tag{161}$$

$$\left. \times \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{162}$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - \overline{H_I}(t) \right. \tag{163}$$

$$\left. \times \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \quad \text{(rewriting (162))} \tag{164}$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \Big(\Big(\left(\overline{H_I}(t) \overline{H_0}(t)\right) \overline{H_I}(t) - \overline{H_I}(t) \left(\overline{H_I}(t) \overline{H_0}(t)\right) \Big) + \left(\overline{H_0}(t) \overline{H_I}^2(t) - \overline{H_I}^2(t) \overline{H_0}(t) \right) + \Big(\overline{H_0}(t) \Big(\overline{H_0}(t) \right. \tag{165}$$

$$\left. \times \overline{H_I}(t) \Big) - \left(\overline{H_0}(t) \overline{H_I}(t)\right) \overline{H_0}(t) \Big) + \Big(\left(\overline{H_I}(t) \overline{H_0}(t)\right) \overline{H_0}(t) - \overline{H_0}(t) \left(\overline{H_I}(t) \overline{H_0}(t)\right) \Big) \Big) \right\rangle_{\overline{H_0}(t)} \tag{166}$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \Big(\left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[\overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[\overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + \left[\overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \right] \Big) \right\rangle_{\overline{H_0}(t)}, \tag{167}$$

$$v_5(t) = \sum_{j=0}^{5-2} (-1)^j \binom{5-2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{5-2-j} \overline{H_I}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{168}$$

$$= \sum_{j=0}^{3} (-1)^j \binom{3}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{3-j} \overline{H_I}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{169}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t) \overline{H_0}^0(t) - 3\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) \overline{H_0}^2(t) \right. \tag{170}$$

$$\left. - \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^0 \overline{H_I}(t) \overline{H_0}^3(t) \right\rangle_{\overline{H_0}(t)} \tag{171}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t) - 3\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) \overline{H_0}^2(t) \right. \tag{172}$$

$$\left. - \overline{H_I}^2(t) \overline{H_0}^3(t) \right\rangle_{\overline{H_0}(t)} \tag{173}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t) - 3\overline{H_I}(t) \left(\overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^2(t)\right) \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) \right. \tag{174}$$

$$\left. \times \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}^2(t) \overline{H_0}^3(t) \right\rangle_{\overline{H_0}(t)} \tag{175}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t) - 3\overline{H_I}(t) \left(\overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^2(t)\right) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}^2(t) \right. \tag{176}$$

$$\left. \times \overline{H_0}^3(t) + 3\overline{H_I}^3(t) \overline{H_0}^2(t) + 3\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{177}$$

$$
\begin{aligned}
&=\Big\langle \overline{H_I}(t)\Big(\overline{H_I}^3(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}(t) \qquad (178)\\
&\quad\times\overline{H_0}^2(t)+\overline{H_0}^3(t)\Big)\,\overline{H_I}(t)-3\overline{H_I}(t)\Big(\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^2(t)\Big)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}(t)\big(\overline{H_I}(t) \qquad (179)\\
&\quad+\overline{H_0}(t)\big)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-\overline{H_I}^2(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} \qquad (180)\\
&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}^3(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^3(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t) \qquad (181)\\
&\quad\times\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}^4(t)\,\overline{H_0}(t)-3\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-3\overline{H_I}(t) \qquad (182)\\
&\quad\times\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)-3\overline{H_I}(t)\overline{H_0}^2(t)\,\overline{H_I}(t)\overline{H_0}(t)+3\overline{H_I}^3(t)\overline{H_0}^2(t)+3\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}^2(t)-\overline{H_I}^2(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} \qquad (183)\\
&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\overline{H_I}^2(t)+\overline{H_0}(t)\overline{H_I}^3(t)+\overline{H_0}^2(t)\overline{H_I}^2(t)+\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t) \qquad (184)\\
&\quad+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}^3(t)\overline{H_0}(t)-3\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-3\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)-3\overline{H_0}^2(t) \qquad (185)\\
&\quad\times\overline{H_I}(t)\overline{H_0}(t)+3\overline{H_I}^2(t)\,\overline{H_0}^2(t)+3\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_0}^3(t)\Big)\Big\rangle_{\overline{H_0}(t)} \qquad (186)\\
&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}^3(t)-\overline{H_I}^3(t)\overline{H_0}(t) \qquad (187)\\
&\quad+\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)+\overline{H_0}^2(t)\overline{H_I}^2(t)-\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)+\overline{H_0}^3(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}^3(t) \qquad (188)\\
&\quad+\overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}(t)-\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)+2\overline{H_I}^2(t)\,\overline{H_0}^2(t)-2\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}^2(t) \qquad (189)\\
&\quad-3\overline{H_0}^2(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)-\overline{H_0}(t)\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big\rangle_{\overline{H_0}(t)}\ \text{(rewriting (186))} \qquad (190)\\
&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big(\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\overline{H_I}(t)-\overline{H_I}(t)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big)+\Big(\big(\overline{H_I}(t)\,\overline{H_0}(t)\big)\overline{H_I}^2(t)-\overline{H_I}^2(t)\,\big(\overline{H_I}(t)\,\overline{H_0}(t)\big)\Big) \qquad (191)\\
&\quad+\Big(\overline{H_0}(t)\overline{H_I}^3(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)\Big)+\big(\overline{H_0}(t)\big(\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\big)-\big(\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\big)\overline{H_0}(t)\big)+\Big(\overline{H_0}(t)\Big(\overline{H_0}(t)\overline{H_I}^2(t)\Big) \qquad (192)\\
&\quad-\Big(\overline{H_0}(t)\overline{H_I}^2(t)\Big)\overline{H_0}(t)\Big)+\Big(\overline{H_0}^3(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}^3(t)\Big)+\big(\big(\overline{H_I}(t)\overline{H_0}(t)\big)\big(\overline{H_0}(t)\overline{H_I}(t)\big)-\big(\overline{H_0}(t)\overline{H_I}(t)\big)\big(\overline{H_I}(t)\overline{H_0}(t)\big)\big) \qquad (193)\\
&\quad+2\overline{H_I}(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_I}(t)\big)\overline{H_0}(t)+3\overline{H_0}(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_I}(t)\big)\overline{H_0}(t)+\Big(\Big(\overline{H_I}^2(t)\overline{H_0}(t)\Big)\big(\overline{H_0}(t)\big) \qquad (194)\\
&\quad-\big(\overline{H_0}(t)\big)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big)\Big)\Big\rangle_{\overline{H_0}(t)}\ \text{(factorizing to introduce commutators)} \qquad (195)\\
&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_I}^3(t)\Big]+\big[\overline{H_0}(t)\,,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\big] \qquad (196)\\
&\quad+\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}^3(t)\,,\overline{H_I}(t)\Big]+\big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+2\overline{H_I}(t) \qquad (197)\\
&\quad\times\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}\ \text{(put the terms required in commutators).} \qquad (198)
\end{aligned}
$$

Summarizing we have that:

$$
\begin{aligned}
v_2(t)&=\Big\langle \overline{H_I}^2(t)\Big\rangle_{\overline{H_0}(t)}, \qquad (199)\\
v_3(t)&=\Big\langle \overline{H_I}^3(t)+\overline{H_I}(t)\,\big[\overline{H_0}(t)\,,\overline{H_I}(t)\big]\Big\rangle_{\overline{H_0}(t)}, \qquad (200)\\
v_4(t)&=\Big\langle \overline{H_I}^4(t)+\overline{H_I}(t)\Big(\big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}(t)\big]+\Big[\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\big[\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_I}(t)\big]+\big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\big]\Big)\Big\rangle_{\overline{H_0}(t)}, \qquad (201)\\
v_5(t)&=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\overline{H_0}(t)\,,\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\overline{H_0}(t)\,,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_I}^3(t)\Big]+\big[\overline{H_0}(t)\,,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\big] \qquad (202)\\
&\quad+\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}^3(t)\,,\overline{H_I}(t)\Big]+\big[\overline{H_I}(t)\overline{H_0}(t)\,,\overline{H_0}(t)\overline{H_I}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+2\overline{H_I}(t) \qquad (203)\\
&\quad\times\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}. \qquad (204)
\end{aligned}
$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_{\bar{S}}}(t) \equiv (\varepsilon_0(t)+R_0(t))\,|0\rangle\langle 0| + (\varepsilon_1(t)+R_1(t))\,|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \tag{205}$$

$$\overline{H_{\bar{I}}}(t) \equiv \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)), \tag{206}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{207}$$

$$= H_B. \tag{208}$$

In this case $\varepsilon_j(t)$, $R_j(t)$ for $j \in \{0,1\}$, $B_{10}^{\Re}(t)$, $B_{10}^{\Im}(t)$, $V_{10}^{\Re}(t)$ and $V_{10}^{\Im}(t)$ are scalars and the other operators are:

$$\sigma_x \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \tag{209}$$

$$\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{210}$$

$$\sigma_y \equiv -\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1| \tag{211}$$

$$\equiv \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \tag{212}$$

$$\sigma_z \equiv |1\rangle\langle 1| - |0\rangle\langle 0| \tag{213}$$

$$\equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{214}$$

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right) & \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \\ \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \tag{215}$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)\,v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right), \tag{216}$$

$$B_i^+(t)\,B_j^-(t) = \mathrm{e}^{\chi_{ij}(t)}\prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{217}$$

$$D(\pm v_{\mathbf{k}}(t)) \equiv \mathrm{e}^{\pm\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{218}$$

As we can see they verify the relationship $\sigma_x\sigma_y = \mathrm{i}\sigma_z$. The explicit form of $\overline{H_{\bar{I}}}^2(t)$ is:

$$\overline{H_{\bar{I}}}^2(t) = \left(\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\right)\left(\sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + V_{10}^{\Re}(t)(\sigma_x B_x(t)\right. \tag{219}$$

$$\left.+\sigma_y B_y(t)) + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\right) \tag{220}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i|\sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + V_{10}^{\Re}(t)\sum_i B_{iz}(t)|i\rangle\langle i|\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\sum_i B_{iz}(t)|i\rangle\langle i|\,(\sigma_x B_y(t) \tag{221}$$

$$-\sigma_y B_x(t)) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))\sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + \left(V_{10}^{\Re}(t)\right)^2(\sigma_x B_x(t) + \sigma_y B_y(t))(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{222}$$

$$+ V_{10}^{\Re}(t)V_{10}^{\Im}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t))(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + V_{10}^{\Im}(t) \tag{223}$$

$$\times V_{10}^{\Re}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + \left(V_{10}^{\Im}(t)\right)^2(\sigma_x B_y(t) - \sigma_y B_x(t))\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{224}$$

$$= \sum_i B_{iz}^2(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i(B_{iz}(t)\,B_x(t)\,|i\rangle\langle i|\sigma_x + B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i(B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_x - B_{iz}(t) \tag{225}$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2(\sigma_x^2 B_x^2(t) + \sigma_x\sigma_y B_x(t)B_y(t) + \sigma_y \tag{226}$$

$$\times\sigma_x B_y(t)B_x(t)+\sigma_y^2 B_y^2(t)\big)+V_{10}^{\Im}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\left(\sigma_x^2 B_y^2(t)+\sigma_y^2 B_x^2(t)\right. \tag{227}$$

$$-\sigma_x\sigma_y B_y(t)B_x(t)-\sigma_y\sigma_x B_x(t)B_y(t)\big)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\big(\sigma_x^2 B_y(t)B_x(t)+\sigma_x\sigma_y B_y^2(t)-\sigma_y\sigma_x B_x^2(t)-\sigma_y^2 B_x(t)B_y(t) \tag{228}$$

$$+\sigma_x^2 B_x(t)B_y(t)-\sigma_x\sigma_y B_x^2(t)+\sigma_y\sigma_x B_y^2(t)-\sigma_y^2 B_y(t)B_x(t)\big)\,, \tag{229}$$

$$\sigma_x\sigma_y=\mathrm{i}\sigma_z \text{ (by Pauli matrices properties)}, \tag{230}$$

$$\sigma_j^2=\mathbb{I} \text{ (for } j\in\{x,y,x\}), \tag{231}$$

$$\overline{H_I}^2(t)=\sum_i B_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i\left(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}(t)\sum_i\big(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t) \tag{232}$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y\big)+V_{10}^{\Re}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\big(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z \tag{233}$$

$$\times B_y(t)B_x(t)+B_y^2(t)\big)+V_{10}^{\Im}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\big(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_z \tag{234}$$

$$\times B_y(t)B_x(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)\big)\,. \tag{235}$$

To introduce the direct calculation of the expected values recall that the hamiltonian $\overline{H_0}(t)$ is a direct sum of the hamiltonians of two Hilbert spaces given by $\overline{H_{\bar S}}(t)$ and $\overline{H_{\bar B}}$, so we can write the hamiltonian $\overline{H_0}(t)$ as:

$$\overline{H_0}(t)=\overline{H_{\bar S}}(t)\otimes\mathbb{I}_{\bar B}+\mathbb{I}_{\bar S}\otimes\overline{H_{\bar B}}. \tag{236}$$

where $\mathbb{I}_{\bar B}$ and $\mathbb{I}_{\bar S}$ are the identity of the systems $\bar B$ and $\bar S$ respectively.
We can show that:

$$\begin{aligned}\left[\overline{H_{\bar S}}(t)\otimes\mathbb{I}_{\bar B},\mathbb{I}_{\bar S}\otimes\overline{H_{\bar B}}\right]&=\overline{H_{\bar S}}(t)\otimes\mathbb{I}_{\bar B}\cdot\mathbb{I}_{\bar S}\otimes\overline{H_{\bar B}}-\mathbb{I}_{\bar S}\otimes\overline{H_{\bar B}}\cdot\overline{H_{\bar S}}(t)\otimes\mathbb{I}_{\bar B} &(237)\\&=\overline{H_{\bar S}}(t)\,\mathbb{I}_{\bar S}\otimes\mathbb{I}_{\bar B}\overline{H_{\bar B}}-\mathbb{I}_{\bar S}\overline{H_{\bar S}}(t)\otimes\overline{H_{\bar B}}\mathbb{I}_{\bar B} &(238)\\&=\overline{H_{\bar S}}(t)\otimes\overline{H_{\bar B}}-\overline{H_{\bar S}}(t)\otimes\overline{H_{\bar B}}\text{ (by definition of identity operator)} &(239)\\&=0. &(240)\end{aligned}$$

Let's introduce the following partition functions $Z_{\bar S}(t)$ and $Z_{\bar B}$ related to the systems $\bar S$ and $\bar B$ respectively.:

$$Z_{\bar S}(t)\equiv\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right), \tag{241}$$

$$Z_{\bar B}\equiv\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right) \tag{242}$$

Using (9), (237) and $\mathrm{Tr}\,(\mathrm{A}\otimes\mathrm{B})=\mathrm{Tr}\,(\mathrm{A})\,\mathrm{Tr}\,(\mathrm{B})$ we can infer that the partition function $Z_0(t)$ can be factorized as:

$$\begin{aligned}Z_0(t)&=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0(t)}}\right). &(243)\\&=\mathrm{Tr}\left(\mathrm{e}^{-\beta\left(\overline{H_{\bar S}}(t)+\overline{H_{\bar B}}\right)}\right)\text{ (by (4))}, &(244)\\&=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\text{ (by (9))} &(245)\\&=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\otimes\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\text{ (because }\bar S\text{ and }\bar B\text{ are disjoint Hilbert spaces)} &(246)\\&=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\text{ (by Tr}(A\otimes B)\text{=Tr}(A)\text{Tr}(B)), &(247)\\&=Z_{\bar S}(t)\,Z_{\bar B}\text{ (by (241) and (242))}. &(248)\end{aligned}$$

For an operator $J(t)$ that can be factorized as $J(t)=S(t)\otimes B(t)$ with $S(t)\in\mathrm{gen}\left(\overline{H_{\bar S}}(t)\right)$ and $B(t)\in\mathrm{gen}\left(\overline{H_{\bar B}}\right)$, being $\mathrm{gen}\,(\mathrm{A})$ the vectorial space generated by the eigenvectors of the operator A, we calculate it's expected value respect to $\overline{H_0}(t)$ using a simple way as follows:

$$\langle J(t)\rangle_{\overline{H_0}(t)} = \frac{\mathrm{Tr}\left(J(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \text{ (by (5))} \tag{249}$$

$$= \frac{\mathrm{Tr}\left((S(t)\otimes B(t))\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\otimes\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by } J(t)=S(t)\otimes B(t) \text{ and } \mathrm{e}^{-\beta\overline{H_0}(t)}=\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\otimes\mathrm{e}^{-\beta\overline{H_{\bar B}}}\text{)} \tag{250}$$

$$= \frac{\mathrm{Tr}\left(\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\otimes\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (rearranging and factorizing)} \tag{251}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by Tr}(A\otimes B)\text{=Tr}(A)\text{Tr}(B)\text{)} \tag{252}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)}\frac{\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \tag{253}$$

$$= \langle S(t)\rangle_{\overline{H_{\bar S}}(t)}\,\langle B(t)\rangle_{\overline{H_{\bar B}}} \text{ (by (5)).} \tag{254}$$

The factorization of $\left\langle\overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)}$ in terms of expected values of elements from $\mathrm{gen}\left(\overline{H_{\bar S}}(t)\right)$ and $\mathrm{gen}\left(\overline{H_{\bar B}}\right)$ is:

$$\left\langle\overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)} = \sum_i \langle|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\,\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar B}}} + V_{10}^{\Re}(t)\sum_i\Big(\langle|i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \langle|i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}}(t)}\,\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}}\Big) \tag{255}$$

$$+V_{10}^{\Im}(t)\sum_i\Big(\langle|i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}} - \langle|i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}}\Big) + V_{10}^{\Re}(t)\sum_i\Big(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)} \tag{256}$$

$$\times\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} + \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\Big) + \left(V_{10}^{\Re}(t)\right)^2\Big(\langle B_x^2(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar B}}} \tag{257}$$

$$-\mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\,\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar B}}}\Big) + V_{10}^{\Im}(t)\sum_i\Big(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} - \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)} \tag{258}$$

$$\times\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\Big) + \left(V_{10}^{\Im}(t)\right)^2\Big(\langle B_y^2(t)\rangle_{\overline{H_{\bar B}}} + \left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar B}}} - \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\,\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)} \tag{259}$$

$$\times\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}\Big). \tag{260}$$

In order to obtain the expected values of $\left\langle\overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)}$ respect to the part related to the bath we need to calculate the following expected values that appear in the equation (255) and can be obtained using the bath and system terms. The expected values relevant for calculations are $\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_y(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar B}}}$, $\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$ and $\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$. Recalling the form of the hamiltonian $\overline{H_{\bar B}} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$ we can extend the result (248), introducing the notation:

$$A_1\otimes\cdots\otimes A_n \equiv \bigotimes_k A_k, \tag{261}$$

$$Z_{\mathbf{k}} \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) \tag{262}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{-1} \tag{263}$$

$$= f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{264}$$

with the creation $b_{\mathbf{k}}$ and annihilation $b_{\mathbf{k}}^\dagger$ operators defined in terms of their actions as:

$$b_{\mathbf{k}}\mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{265}$$

$$b_{\mathbf{k}}^\dagger\mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}+1}\mid j_{\mathbf{k}}+1\rangle. \tag{266}$$

being $|j_{\mathbf{k}}\rangle$ an eigenstate of $H_{\mathbf{k}} \equiv \omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$. With this notation we can write the partition function as:

$$Z_{\bar{B}} = \mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right), \tag{267}$$

$$\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}, \tag{268}$$

$$Z_{\bar{B}} = \mathrm{Tr}\left(\bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) \text{ (by (268))} \tag{269}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) \text{ (by } \mathrm{Tr}\left(A\otimes B\right) = \mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right)\text{)} \tag{270}$$

$$= \prod_{\mathbf{k}} Z_{\mathbf{k}} \text{ (by (268))}. \tag{271}$$

For a function $f\left(t\right)$ which can be factorized as:

$$f\left(t\right) \equiv \prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right). \tag{272}$$

with $f_{\mathbf{k}}\left(t\right) \in \mathrm{gen}\left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right)$, it's expected value is given by:

$$\left\langle f\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(f\left(t\right)\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \tag{273}$$

$$= \frac{\mathrm{Tr}\left(\prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \text{ (by (268) and (272))} \tag{274}$$

$$= \frac{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \tag{275}$$

$$= \frac{\prod_{\mathbf{k}} \mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\prod_{\mathbf{k}} \mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \tag{276}$$

$$= \prod_{\mathbf{k}} \frac{\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \tag{277}$$

$$= \prod_{\mathbf{k}} \left\langle f_{\mathbf{k}}\left(t\right)\right\rangle_{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}. \tag{278}$$

It means that for an operator that can be factorized in terms of functions generated by $\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$ for each $\mathbf{k}$ we only require to calculate the expected value respect to the Hilbert space where the operator belongs. This process lead us to the following explicit forms of the expected values relevant for our calculations:

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (215))}, \tag{279}$$

$$= \left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right)^{2} + \sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right)\left(\left(g_{i\mathbf{k}'}\right.\right.\right. \tag{280}$$

$$\left.\left.\left. - v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(t\right)\right)^{*} b_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by square expansion properties)}, \tag{281}$$

$$= \sum_{\mathbf{k}} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}\neq\mathbf{k}'} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad (282)$$

$$\times \left\langle \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))\, b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \, b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (278))}, \quad (283)$$

$$\left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (284)$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (285)$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \text{ (by (266))}, \quad (286)$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \mathrm{Tr}\left(|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (287)$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \cdot 0}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \text{ (by trace properties)}, \quad (288)$$

$$= 0, \quad (289)$$

$$\langle b_{\mathbf{k}} \rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (290)$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (291)$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \text{ (by (265))}, \quad (292)$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \mathrm{Tr}\left(|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (293)$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \cdot 0}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \text{ (by trace properties)}, \quad (294)$$

$$= 0, \quad (295)$$

$$\left\langle B_{iz}^2(t) \right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}\neq\mathbf{k}'} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \rangle_{\overline{H_{\bar{B}}}} \right) \quad (296)$$

$$\times \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left\langle b_{\mathbf{k}'}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \langle b_{\mathbf{k}'} \rangle_{\overline{H_{\bar{B}}}} \right) \quad (297)$$

$$= \sum_{\mathbf{k}} \left\langle \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}\neq\mathbf{k}'} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \cdot 0 + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \cdot 0)((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \cdot 0 \quad (298)$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \cdot 0) \text{ (by (284) and (290))} \quad (299)$$

$$= \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} \quad (300)$$

$$= \sum_{\mathbf{k}} \left\langle (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 \left(b_{\mathbf{k}}^{\dagger} \right)^2 + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + ((g_{i\mathbf{k}} \quad (301)$$

$$- v_{i\mathbf{k}}(t))^*)^2 \, b_{\mathbf{k}}^2 \right\rangle_{\overline{H_{\bar{B}}}} \quad (302)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 \left\langle \left(b_{\mathbf{k}}^{\dagger} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*)^2 \left\langle b_{\mathbf{k}}^2 \right\rangle_{\overline{H_{\bar{B}}}}, \quad (303)$$

$$\left\langle \left(b_{\mathbf{k}}^{\dagger} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger} \right)^2 \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (304)$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger}\right)^{2} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{305}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)} |j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (266) applied twice)} \tag{306}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)} \mathrm{Tr}\left(|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{307}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{308}$$

$$= 0, \tag{309}$$

$$\left\langle b_{\mathbf{k}}^{2}\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{2} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{310}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} |j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (265) applied twice)} \tag{311}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} \mathrm{Tr}\left(|j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{312}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{313}$$

$$= 0, \tag{314}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{315}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{316}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \text{ (now (265) and (266))} \tag{317}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}}+1} |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{318}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}}+1} \sqrt{j_{\mathbf{k}}+1} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{319}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} j_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(j_{\mathbf{k}}+1\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{320}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{321}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \mathrm{Tr}\left(|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{322}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \text{ (by properties of trace)} \tag{323}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}=0}^{\infty} \left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right), \tag{324}$$

$$\sum_{j_{\mathbf{k}}=0}^{\infty} x^{j_{\mathbf{k}}}(2j_{\mathbf{k}}+1) = \frac{1+x}{(1-x)^2}, \tag{325}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \left(1 - \mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \frac{\mathrm{e}^{-\beta\omega_{\mathbf{k}}}+1}{(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}})^2} \text{ (setting } x = \mathrm{e}^{-\beta\omega_{\mathbf{k}}} \text{ in (325) and by (315))}, \tag{326}$$

$$= \frac{1+\mathrm{e}^{-\beta\omega_{\mathbf{k}}}}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \tag{327}$$

$$= \frac{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}} \frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}+\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}}{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}} \frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}-\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}} \tag{328}$$

$$= \frac{\cosh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\sinh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{330}$$

$$\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \text{ (by (304), (310) and (330))}, \tag{331}$$

$$\langle B_{iz}(t) B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}}\right) \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{332}$$

$$= \frac{1}{2}\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}}\right) \left(\mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + \mathrm{e}^{\chi_{01}(t)} \right.\right. \tag{333}$$

$$\left.\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2} \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{334}$$

$$= \frac{1}{2}\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}}\right) \left(\mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + \mathrm{e}^{\chi_{01}(t)} \right.\right. \tag{335}$$

$$\left.\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (284) and (290))}, \tag{336}$$

$$\langle F(h)\rangle_{\overline{H_{\bar{B}}}} \equiv \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle\alpha| F(h) |\alpha\rangle \mathrm{d}^2\alpha \text{ (using the coherent representation with } N = \left(\mathrm{e}^{\beta\omega}-1\right)^{-1}), \tag{337}$$

$$D(\alpha_{\mathbf{k}}) \equiv \mathrm{e}^{\left(\frac{\alpha_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{\alpha_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)} \text{ (displacement operator definition)}, \tag{338}$$

$$|\alpha\rangle \equiv D(\alpha)|0\rangle \text{ (displacement operator properties)}, \tag{339}$$

$$\langle\alpha| \equiv \langle 0| D(-\alpha), \tag{340}$$

$$D(-\alpha)D(h)D(\alpha) \equiv D(h)\,\mathrm{e}^{h\alpha^* - h^*\alpha} \text{ (displacement operator properties)}, \tag{341}$$

$$D(0) \equiv \mathbb{I} \text{ (identity written in terms of the displacement operator)}, \tag{342}$$

$$D(-\alpha)D(0)D(\alpha) = D(0)\,\mathrm{e}^{0\cdot\alpha^* - 0^*\cdot\alpha} \tag{343}$$

$$= D(0) \tag{344}$$

$$= \mathbb{I}, \tag{345}$$

$$D(-\alpha)\, b^{\dagger} D(\alpha) = b^{\dagger} + \alpha^* \text{ (displacement operator properties)}, \tag{346}$$

$$D(-\alpha)\, b D(\alpha) = b + \alpha \text{ (displacement operator properties)}, \tag{347}$$

$$\langle D(h)\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)} \text{ (expected value displacement operator)}, \tag{348}$$

$$\left\langle b^{\dagger} D(h)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle \alpha \left| b^{\dagger} D(h) \right| \alpha \right\rangle \mathrm{d}^2\alpha \text{ (by (337))} \tag{349}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| D(-\alpha)\, b^{\dagger} D(h)\, D(\alpha) \right| 0 \right\rangle \mathrm{d}^2\alpha \text{ (by (339) and (340))} \tag{350}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| D(-\alpha)\, b^{\dagger} \mathbb{I} D(h)\, D(\alpha) \right| 0 \right\rangle \mathrm{d}^2\alpha \text{ (inserting identity operator)} \tag{351}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| \left(D(-\alpha)\, b^{\dagger} D(\alpha)\right) \left(D(-\alpha)\, D(h)\, D(\alpha)\right) \right| 0 \right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} \tag{352}$$

$$
\begin{aligned}
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|\left(b^\dagger+\alpha^*\right) D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (346) and (341))} && (353)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|b^\dagger D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|\alpha^* D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha && (354)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} 0D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\left|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|\alpha^* D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha && (355)\\
&= \frac{1}{\pi N}\int 0\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \left\langle 0\left|D\left(h\right)\right|0\right\rangle \mathrm{d}^2\alpha && (356)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \left\langle 0\left|D\left(h\right)\right|0\right\rangle \mathrm{d}^2\alpha && (357)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \left\langle 0|h\right\rangle \mathrm{d}^2\alpha \text{ (by (339))} && (358)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \mathrm{e}^{-\frac{|h|^2}{2}} \mathrm{d}^2\alpha \text{ (because } \left\langle 0|h\right\rangle\text{=}\mathrm{e}^{-\frac{|h|^2}{2}}\text{)}, && (359)\\
x &= \alpha^{\Re}\in\mathbb{R}, && (360)\\
y &= \alpha^{\Im}\in\mathbb{R}, && (361)\\
\alpha &= x+\mathrm{i}y, && (362)\\
\left\langle b^\dagger D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \mathrm{e}^{-\frac{|h|^2}{2}} \mathrm{d}^2\alpha && (363)\\
&= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(x-\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \text{ (by (360) and (361))} && (364)\\
&= -h^*\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}N && (365)\\
&= -h^*\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}N, && (366)\\
\left|h\right\rangle &= \mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}\left|n\right\rangle, && (367)\\
\left\langle bD\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle \alpha\left|bD\left(h\right)\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (by (340) and (337))} && (368)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|D\left(-\alpha\right)b\mathbb{I}D\left(h\right)D\left(\alpha\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (339) and (340))} && (369)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|\left(D\left(-\alpha\right)bD\left(\alpha\right)\right)\left(D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} && (370)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|\left(b+\alpha\right)D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (347) and (341))} && (371)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|bD\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0\left|\alpha D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha && (372)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|b\right|h\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha\ \left(D\left(h\right)\left|0\right\rangle=\left|h\right\rangle\right) && (373)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|b\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}\left|n\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \text{ (by (367))} && (374)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=1}^{\infty}\frac{h^n}{\sqrt{n!}}\sqrt{n}\left|n-1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \text{ (by (265))} && (375)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=1}^{\infty}\frac{h^n}{\sqrt{n!}}\sqrt{n}\delta_{0,n-1}\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \text{ (by } \left\langle n|n'\right\rangle=\delta_{nn'}\text{)} && (376)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\frac{h^1}{\sqrt{1!}}\sqrt{1}\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha && (377)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (because } \left\langle 0|h\right\rangle\text{=}\mathrm{e}^{-\frac{|h|^2}{2}}\text{)} && (378)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\left(\alpha+h\right)\mathrm{d}^2\alpha && (379)\\
&= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(x+\mathrm{i}y+h\right)\mathrm{d}x\mathrm{d}y && (380)
\end{aligned}
$$

$$= h\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}(N+1) \tag{381}$$
$$= h\left\langle D(h)\right\rangle_{\overline{H_{\bar{B}}}}(N+1), \tag{382}$$
$$\left\langle D(h)b\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|D(h)b\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (by (337))} \tag{383}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)D(h)\mathbb{I}bD(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (339) and (340))} \tag{384}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(D(-\alpha)D(h)D(\alpha)\right)\left(D(-\alpha)bD(\alpha)\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} \tag{385}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\mathrm{e}^{h\alpha^*-h^*\alpha}(b+\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (347) and (341))} \tag{386}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\mathrm{e}^{h\alpha^*-h^*\alpha}b\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\right|0\right\rangle \mathrm{d}^2\alpha \tag{387}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)b\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{388}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|D(h)0\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (265))} \tag{389}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{390}$$
$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}(x+\mathrm{i}y)\,\mathrm{d}x\mathrm{d}y \tag{391}$$
$$= hN\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)} \tag{392}$$
$$= hN\left\langle D(h)\right\rangle_B, \tag{393}$$
$$\left\langle D(h)b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|D(h)b^\dagger\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (by (337))} \tag{394}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)D(h)\mathbb{I}b^\dagger D(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (339) and (340))} \tag{395}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(D(-\alpha)D(h)D(\alpha)\right)\left(D(-\alpha)b^\dagger D(\alpha)\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} \tag{396}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\mathrm{e}^{h\alpha^*-h^*\alpha}\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (347) and (341))} \tag{397}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\mathrm{e}^{h\alpha^*-h^*\alpha}b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha^*D(h)\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \tag{398}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{399}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{400}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h\left|\sqrt{0+1}\right|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{401}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h\left|\sqrt{0+1}\right|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (340))}, \tag{402}$$
$$\left\langle h\right| = \mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(h^*)^n}{\sqrt{n!}}\left\langle n\right|, \tag{403}$$
$$\left\langle D(h)b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(-h^*)^n}{\sqrt{n!}}\left\langle n|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (403))} \tag{404}$$
$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\frac{(-h^*)^1}{\sqrt{1!}}\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by } \left\langle n|n'\right\rangle=\delta_{nn'}\text{)} \tag{405}$$
$$= \frac{1}{\pi N}\int (\alpha^*-h^*)\,\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{406}$$
$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}(x-\mathrm{i}y-h^*)\,\mathrm{d}x\mathrm{d}y \tag{407}$$
$$= -h^*\left\langle D(h)\right\rangle_B(N+1), \tag{408}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right.\right. \tag{409}$$

$$\left.\left.+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (replacing the definitions in (215))} \tag{410}$$

$$= \frac{1}{2}\left\langle\mathrm{e}^{\chi_{10}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)+\mathrm{e}^{\chi_{01}(t)}\right. \tag{411}$$

$$\left.\times\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{412}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\langle b_{\mathbf{k}}\right. \tag{413}$$

$$\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\Bigg)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right.\right.\right. \tag{414}$$

$$\left.\left.\left.-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right), \tag{415}$$

$$\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (348))}, \tag{416}$$

$$N_{\mathbf{k}} = \left(\mathrm{e}^{\beta\omega_{\mathbf{k}}}-1\right)^{-1}, \tag{417}$$

$$\langle b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} = \alpha_{\mathbf{k}}(N_{\mathbf{k}}+1)\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \text{ (by (382))}, \tag{418}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right\rangle_{\overline{H_{\bar{B}}}} = -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \text{ (by (366))}, \tag{419}$$

$$\left\langle\prod_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (416) and (278))}, \tag{420}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (278))} \tag{421}$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D(\alpha_{\mathbf{k}'})\rangle_{\overline{H_{\bar{B}}}} \text{ (by (278))} \tag{422}$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\prod_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \tag{423}$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (416))}, \tag{424}$$

$$\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} = \langle b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (278))} \tag{425}$$

$$= \alpha_{\mathbf{k}}(N_{\mathbf{k}}+1)\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D(\alpha_{\mathbf{k}'})\rangle_{\overline{H_{\bar{B}}}} \text{ (by (418))} \tag{426}$$

$$= \alpha_{\mathbf{k}}(N_{\mathbf{k}}+1)\prod_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \tag{427}$$

$$= \alpha_{\mathbf{k}}(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (416))}, \tag{428}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\langle b_{\mathbf{k}}\right. \tag{429}$$

$$\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\Bigg\rangle_{\overline{H_{\bar{B}}}}\Bigg)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\right. \tag{430}$$

$$\left.\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right)\ \text{(by (215))} \tag{431}$$

$$=\frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{432}$$

$$\left.\times\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,(-N_{\mathbf{k}}\right. \tag{433}$$

$$\left.\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1) \tag{434}$$

$$\left.\times\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{435}$$

$$=\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{436}$$

$$\left.\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,(-N_{\mathbf{k}}\right. \tag{437}$$

$$\left.\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \tag{438}$$

$$\left.\times(N_{\mathbf{k}}+1)\,\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{439}$$

$$=\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{10}(t)\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right. \tag{440}$$

$$\left.\times(N_{\mathbf{k}}+1)\,B_{10}(t)\right)+\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{01}(t)\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{441}$$

$$\left.\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,B_{01}(t)\right)\ \text{(by (215) using the definition } B_{ij}(t)\text{)} \tag{442}$$

$$=\frac{B_{10}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{443}$$

$$+\frac{B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{444}$$

$$=\frac{B_{10}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{445}$$

$$-\frac{B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{446}$$

$$=\frac{B_{10}(t)-B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{447}$$

$$\langle B_{iz}(t)B_{y}(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\,b_{\mathbf{k}}\right)\frac{B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{448}$$

$$=\frac{\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\left(B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}}+\frac{\left(B_{10}(t)-B_{01}(t)\right)}{2\mathrm{i}} \quad (449)$$

$$\times\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{(by the properties of expected value)} \quad (450)$$

$$=\frac{\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}}+\frac{\left(B_{10}(t)-B_{01}(t)\right)}{2\mathrm{i}}\cdot 0 \quad (451)$$

$$=\frac{\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \text{ (by (428) and (424))} \quad (452)$$

$$=\frac{\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \quad (453)$$

$$=\frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right. \quad (454)$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad (455)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{01}(t)}\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle_{\overline{H_{\bar{B}}}}-\mathrm{e}^{\chi_{10}(t)}\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\right. \quad (456)$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\mathrm{e}^{\chi_{01}(t)}\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad (457)$$

$$\left.\times\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\right) \quad (458)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\mathrm{e}^{\chi_{01}(t)}\left(-\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)-\mathrm{e}^{\chi_{10}(t)}\right. \quad (459)$$

$$\times\left(-\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\mathrm{e}^{\chi_{01}(t)} \quad (460)$$

$$\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*} \quad (461)$$

$$\left.\times\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \text{ (by (420), (428) and (424))} \quad (462)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(-\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{01}\left(t\right)\right)-\left(-\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{10}\left(t\right)\right)\right. \quad (463)$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)B_{01}(t)-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)B_{10}(t) \quad (464)$$

$$\left.\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\right) \text{ (by (215) using the definition } B_{ij}\left(t\right)) \quad (465)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{01}\left(t\right)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{10}\left(t\right)\right. \quad (466)$$

$$\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)B_{01}(t)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)B_{10}(t)\right) \quad (467)$$

$$=\frac{B_{10}(t)+B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right), \quad (468)$$

$$\langle B_{x}(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\frac{B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad (469)$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2} \tag{470}$$

$$\times\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by expected value properties and (428))} \tag{471}$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2}\cdot 0 \tag{472}$$

$$= \frac{1}{2}\left\langle\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{473}$$

$$= \frac{1}{2}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger\right.\right. \tag{474}$$

$$\left.\left.+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{475}$$

$$\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} = \alpha_{\mathbf{k}}N_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{476}$$

$$\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{477}$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{478}$$

$$= -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (477))} \tag{479}$$

$$= -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\prod_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{480}$$

$$= -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (416))}, \tag{481}$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{482}$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (476))} \tag{483}$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\prod_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{484}$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (416))}, \tag{485}$$

$$\left\langle B_x(t)B_{iz}(t)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\right. \tag{486}$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{487}$$

$$\left.\times\mathrm{e}^{\chi_{10}(t)}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \text{ (by (215))} \tag{488}$$

$$= \frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right. \tag{489}$$

$$+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{10}(t)} \tag{490}$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* \tag{491}$$

$$\times\left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\Bigg)\text{ (by (393) and (408))}\tag{492}$$

$$=\frac{1}{2}\sum_{\mathbf{k}}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)B_{10}(t)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\right.\tag{493}$$

$$\times(N_{\mathbf{k}}+1)B_{01}(t)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{10}(t)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\tag{494}$$

$$\left.\times N_{\mathbf{k}}B_{01}(t)\right)\tag{495}$$

$$=\frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)\right)\tag{496}$$

$$\left\langle B_{y}(t)\,B_{iz}(t)\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\tag{497}$$

$$=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\tag{498}$$

$$\times\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\text{ (by (393) and (408))}\tag{499}$$

$$=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\text{ (by (290) and (284))}\tag{500}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)-\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\right.\right.\tag{501}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\text{ (by (215))}\tag{502}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{01}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\right.\tag{503}$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\tag{504}$$

$$\left.\times\mathrm{e}^{\chi_{01}(t)}-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)\text{ (by expected value properties)}\tag{505}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{01}(t)}\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right.\tag{506}$$

$$-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)}\tag{507}$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\tag{508}$$

$$\left.\times\left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)\text{ (by (393) and (408))}\tag{509}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)B_{01}(t)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\right.\tag{510}$$

$$\times(N_{\mathbf{k}}+1)B_{10}(t)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{01}(t)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\tag{511}$$

$$\left.\times N_{\mathbf{k}}B_{10}(t)\right)\tag{512}$$

$$= \frac{B_{01}(t)+B_{10}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\right), \quad (513)$$

$$\mathrm{Var}_{\overline{H_{\bar{B}}}}(A) \equiv \left\langle A^2\right\rangle_{\overline{H_{\bar{B}}}} - \langle A\rangle^2_{\overline{H_{\bar{B}}}} \text{ (definition of variance)}, \quad (514)$$

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X) \text{ (properties of variance)}, \quad (515)$$

$$\langle B_x(t)\rangle_{\overline{H_{\bar{B}}}} = 0 \text{ (expected value of obtained in [2])}, \quad (516)$$

$$\langle B_y(t)\rangle_{\overline{H_{\bar{B}}}} = 0 \text{ (expected value of obtained in [2])}, \quad (517)$$

$$\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}(B_x(t)) + \langle B_x(t)\rangle^2_{\overline{H_{\bar{B}}}} \text{ (by (514))} \quad (518)$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)-B_{10}(t)-B_{01}(t)}{2}\right) \text{ (because } \langle B_x(t)\rangle_{\overline{H_{\bar{B}}}}=0) \quad (519)$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)-B_{10}(t)-B_{01}(t)\right) \quad (520)$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right) \text{ (by (515))} \quad (521)$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t)+B_{01}(t))^2\right) \text{ (by (514))} \quad (522)$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2+B_1^+(t)B_0^-(t)B_0^+(t)B_1^-(t)+B_0^+(t)B_1^-(t)B_1^+(t)B_0^-(t)+\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right. \quad (523)$$

$$\left.-(B_{10}(t)+B_{01}(t))^2\right) \quad (524)$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t)+B_{01}(t))^2\right) \text{ (by } B_j^{\pm}(t)B_j^{\mp}(t)=\mathbb{I}), \quad (525)$$

$$(D(h))^2 = D(h)\,D(h) \quad (526)$$

$$= D(h+h)\,\mathrm{e}^{\frac{1}{2}\left(\frac{h^*h-hh^*}{\omega^2}\right)} \text{ (by displacement operator properties)} \quad (527)$$

$$= D(2h), \quad (528)$$

$$\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \quad (529)$$

$$= \left\langle\prod_{\mathbf{k}}D\left(2\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (528))} \quad (530)$$

$$= \prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\,\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (416))} \quad (531)$$

$$= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\,\mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^2\left(\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^2 \quad (532)$$

$$= B_{ij}^2(t)\,|B_{ij}(t)|^2 \text{ (by (215))}, \quad (533)$$

$$\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t)+B_{01}(t))^2\right) \text{ (by (514))} \quad (534)$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2\langle\mathbb{I}\rangle_{\overline{H_{\bar{B}}}}+\left\langle\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t)+B_{01}(t))^2\right) \text{ (by expected value)} \quad (535)$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t)+B_{01}(t))^2\right) \quad (536)$$

$$= \frac{1}{4}\left(B_{10}^2(t)\left|B_{10}^2(t)\right|+2+B_{01}^2(t)\left|B_{01}^2(t)\right|-\left(B_{10}^2(t)+2B_{10}(t)\,B_{01}(t)+B_{01}^2(t)\right)\right) \text{ (by (533))} \quad (537)$$

$$= \frac{1}{4}\left(B_{10}^2(t)\left|B_{10}^2(t)\right|+2+B_{01}^2(t)\left|B_{10}^2(t)\right|-\left(B_{10}^2(t)+2\left|B_{10}^2(t)\right|+B_{01}^2(t)\right)\right) \quad (538)$$

$$= \frac{1}{4}\left(B_{10}^2(t)+B_{01}^2(t)-2\right)\left(\left|B_{10}^2(t)\right|-1\right), \quad (539)$$

$$\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}(B_y(t)) + \langle B_y(t)\rangle^2_{\overline{H_{\bar{B}}}} \quad (540)$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right) \text{ (by } \langle B_y(t)\rangle_{\overline{H_{\bar{B}}}}=0 \text{ and (215))} \quad (541)$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t) + B_{10}(t) - B_{01}(t)\right) \tag{542}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right) \tag{543}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{544}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - 2\mathbb{I} + \left(B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{545}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - 2\left\langle\mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)^2\right) \tag{546}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{01}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right) \text{ (by (533))} \tag{547}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{01}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + 2B_{01}(t)\,B_{10}(t) - B_{10}^2(t)\right) \tag{548}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{10}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + 2\,|B_{10}(t)|^2 - B_{10}^2(t)\right) \tag{549}$$

$$= -\frac{1}{4}\left(B_{10}^2(t) + B_{01}^2(t) + 2\right)\left(|B_{10}(t)|^2 - 1\right), \tag{550}$$

$$\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_x(t)\,\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{551}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\,\langle B_x(t)\rangle_{\overline{H_{\bar{B}}}} \tag{552}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\cdot 0 \text{ (by } \langle B_x(t)\rangle_{\overline{H_{\bar{B}}}} = 0) \tag{553}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{554}$$

$$= \left\langle \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (215))} \tag{555}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle\left(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)\right)\left(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\right)\big\rangle_{\overline{H_{\bar{B}}}} - \big\langle\left(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\right)\big\rangle_{\overline{H_{\bar{B}}}} \tag{556}$$

$$\times\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{557}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle B_1^+(t)\,B_0^-(t)\,B_0^+(t)B_1^-(t) - B_1^+(t)\,B_0^-(t)\,B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)\,B_0^+(t)\,B_1^-(t) - B_0^+(t)\,B_1^-(t) \tag{558}$$

$$\times B_1^+(t)B_0^-(t)\big\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{559}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\mathbb{I} - \left(B_1^+(t)\,B_0^-(t)\right)^2 + \left(B_0^+(t)\,B_1^-(t)\right)^2 - \mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{560}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{561}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{01}(t) + B_{10}(t)\right)\right) \tag{562}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}^2(t) - B_{10}^2(t)\right)\right) \tag{563}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\,|B_{10}(t)|^2 - B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + B_{10}^2(t)\right) \text{ (by (533))} \tag{564}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t) - B_{10}^2(t)\right)\left(|B_{10}(t)|^2 - 1\right), \tag{565}$$

$$\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (215))} \tag{566}$$

$$= \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_y(t)\,\frac{B_{10}(t) + B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{567}$$

$$= \frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t) + B_{01}(t)}{2}\,\langle B_y(t)\rangle_{\overline{H_{\bar{B}}}} \tag{568}$$

$$=\frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}}-\frac{B_{10}(t)+B_{01}(t)}{2}\cdot 0\ \text{(by } \langle B_y(t)\rangle_{\overline{H_{\bar{B}}}}=0\text{)} \tag{569}$$

$$=\frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{570}$$

$$=\frac{1}{2}\left\langle\frac{B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}}\ \text{(by (215))} \tag{571}$$

$$=\frac{1}{4\mathrm{i}}\langle\left(B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)\right)\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right)\rangle_{\overline{H_{\bar{B}}}}+\frac{(B_{10}(t)-B_{01}(t))}{4\mathrm{i}}\langle\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right)\rangle_{\overline{H_{\bar{B}}}} \tag{572}$$

$$=\frac{1}{4\mathrm{i}}\left\langle\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{(B_{10}(t)-B_{01}(t))\,(B_{10}(t)+B_{01}(t))}{4\mathrm{i}} \tag{573}$$

$$=\frac{1}{4\mathrm{i}}\left\langle\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} \tag{574}$$

$$=\frac{1}{4\mathrm{i}}\left\langle B_0^+(t)\,B_1^-(t)\,B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\,B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\,B_1^+(t)\,B_0^-(t)-B_1^+(t)\,B_0^-(t)\,B_0^+(t)\,B_1^-(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{575}$$

$$+\frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} \tag{576}$$

$$=\frac{1}{4\mathrm{i}}\left\langle\mathbb{I}+\left(B_0^+(t)\,B_1^-(t)\right)^2-\left(B_1^+(t)\,B_0^-(t)\right)^2-\mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}}+\frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} \tag{577}$$

$$=\frac{1}{4\mathrm{i}}\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2-\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} \tag{578}$$

$$=\frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\,|B_{10}(t)|^2-B_{10}^2(t)\,|B_{10}(t)|^2\right)+\frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}}\ \text{(by (533))} \tag{579}$$

$$=\frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\,|B_{10}(t)|^2-B_{10}^2(t)\,|B_{10}(t)|^2+B_{10}^2(t)-B_{01}^2(t)\right) \tag{580}$$

$$=\frac{1}{4\mathrm{i}}\left(B_{01}^2(t)-B_{10}^2(t)\right)\left(|B_{10}(t)|^2-1\right). \tag{581}$$

Summarizing the expected values obtained in the precedent lines we have:

$$\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar{B}}}}=\sum_{\mathbf{k}}|g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{582}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right), \tag{583}$$

$$\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}(t)+B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right), \tag{584}$$

$$\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)\right), \tag{585}$$

$$\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{01}(t)+B_{10}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\right), \tag{586}$$

$$\langle B_x^2(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4}\left(B_{10}^2(t)+B_{01}^2(t)-2\right)\left(|B_{10}^2(t)|-1\right), \tag{587}$$

$$\langle B_y^2(t)\rangle_{\overline{H_{\bar{B}}}}=-\frac{1}{4}\left(B_{10}^2(t)+B_{01}^2(t)+2\right)\left(|B_{10}(t)|^2-1\right), \tag{588}$$

$$\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4\mathrm{i}}\left(B_{01}^2(t)-B_{10}^2(t)\right)\left(|B_{10}(t)|^2-1\right), \tag{589}$$

$$\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4\mathrm{i}}\left(B_{01}^2(t)-B_{10}^2(t)\right)\left(|B_{10}(t)|^2-1\right). \tag{590}$$

The density matrix associated to $\rho_{\overline{S}}=\frac{\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}\equiv\sum\rho_{\overline{S},ij}|i\rangle\langle j|$ has the following element

$$\rho_{\overline{S},00}=\frac{1}{2}-\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}},\tag{591}$$

$$\rho_{\overline{S},01}=-\frac{B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}},\tag{592}$$

$$\rho_{\overline{S},10}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}},\tag{593}$$

$$\rho_{\overline{S},11}=\frac{1}{2}+\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}.\tag{594}$$

The expected values respect to the system $\overline{S}$ of relevance for calculating $\left\langle\overline{H_I}^2\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)}}$ are $\langle|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle|i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle|i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$ and $\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}$, we took account that $\sigma_x\sigma_y=\mathrm{i}\sigma_z$ and $\sigma_y\sigma_x=-\mathrm{i}\sigma_z$. The values needed for our calculation are:

$$\langle|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}-\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}},\tag{595}$$

$$\langle|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}+\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}},\tag{596}$$

$$\langle|0\rangle\langle 0|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}},\tag{597}$$

$$\langle|1\rangle\langle 1|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}},\tag{598}$$

$$\langle|0\rangle\langle 0|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}},\tag{599}$$

$$\langle |1\rangle\langle 1|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{600}$$

$$\langle \sigma_x|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{601}$$

$$\langle \sigma_x|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{602}$$

$$\langle \sigma_y|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{603}$$

$$\langle \sigma_y|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{604}$$

$$\langle \sigma_z\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}. \tag{605}$$

Our next step is to find $v_3\left(t\right)$, the commutator $\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]$ is a central point for our calculations and it is equal to:

$$\left[\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right] = \Big[\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{606}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\Big] \text{ (by (205) and (206))} \tag{607}$$

$$= \Big[\sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i| + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, \tag{608}$$

$$\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\Big] \text{ (introduced } \textstyle\sum \text{ to decrease the size)} \tag{609}$$

$$= \sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|\sum_{i'}B_{i'z}(t)|i'\rangle\langle i'| + \sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right) + \sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i| \tag{610}$$

$$\times V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right) + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i| + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{611}$$

$$\times V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right) + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right) - \sigma_y\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) \tag{612}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i - \sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right) - \sigma_y\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) \tag{613}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{614}$$

$$\times V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sum_{i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left|i'\right\rangle\left\langle i'\right|-\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{615}$$

$$+\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\left|i\right\rangle\left\langle i\right|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{616}$$

$$\times\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{617}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\left|i\right\rangle\left\langle i\right|-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{618}$$

$$+V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{619}$$

$$=\sum_{i,i'}(\varepsilon_i(t)+R_i(t))\delta_{ii'}B_{i'z}(t)\left|i\right\rangle\left\langle i'\right|+\sum_i(\varepsilon_i(t)+R_i(t))V_{10}^{\Re}(t)\left|i\right\rangle\left\langle i\right|\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+\sum_i(\varepsilon_i(t)+R_i(t))V_{10}^{\Im}(t)\left|i\right\rangle\left\langle i\right|(\sigma_x B_y(t)-\sigma_y \tag{620}$$

$$\times B_x(t))+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)\,\sigma_x\left|i\right\rangle\left\langle i\right|+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_x^2 B_x(t)+\sigma_x\sigma_y B_y(t)\right)+V_{10}^{\Im}(t) \tag{621}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_x^2 B_y(t)-\sigma_x\sigma_y B_x(t)\right)-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,\sigma_y\left|i\right\rangle\left\langle i\right|-V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) \tag{622}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\left(\sigma_y\sigma_x B_x(t)+\sigma_y^2 B_y(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_y\sigma_x B_y(t)-\sigma_y^2 B_x(t)\right)+\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\left|i\right\rangle\left\langle i\right| \tag{623}$$

$$+V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_{i,i'}(\varepsilon_i(t)+R_i(t))\delta_{ii'}B_{i'z}(t)\left|i\right\rangle\left\langle i'\right|-\Big(B_{10}^{\Re}(t) \tag{624}$$

$$\times V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sigma_x+\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sigma_y-V_{10}^{\Re}(t)\sum_i(\varepsilon_i(t)+R_i(t))\left(\sigma_x B_x(t)\right. \tag{625}$$

$$\left.+\sigma_y B_y(t)\right)\left|i\right\rangle\left\langle i\right|-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_x^2 B_x(t)+\sigma_y\sigma_x B_y(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_x\sigma_y B_x(t)\right. \tag{626}$$

$$\left.+\sigma_y^2 B_y(t)\right)-V_{10}^{\Re}(t)\sum_{\mathbf{k}}\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\left|i\right\rangle\left\langle i\right|-V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) \tag{627}$$

$$-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\left(\sigma_x^2 B_y(t)-\sigma_y\sigma_x B_x(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_x\sigma_y B_y(t)-\sigma_y^2 B_x(t)\right)-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y \tag{628}$$

$$\times B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{629}$$

$$=\sum_i(\varepsilon_i(t)+R_i(t))\left|i\right\rangle\left\langle i\right|V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i(\varepsilon_i(t)+R_i(t))\left|i\right\rangle\left\langle i\right|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{630}$$

$$\times\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(B_x(t)+\mathrm{i}\sigma_z B_y(t)\right)+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t) \tag{631}$$

$$\times\left(B_y(t)-\mathrm{i}\sigma_z B_x(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(-\mathrm{i}\sigma_z B_x(t)+B_y(t)\right) \tag{632}$$

$$-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\left(-\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{633}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sigma_y\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) \tag{634}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)-\sum_i B_{iz}(t)\left|i\right\rangle\left\langle i\right|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left|i\right\rangle\left\langle i\right|-V_{10}^{\Re}(t)\left(B_x(t)-\mathrm{i}\sigma_z B_y(t)\right) \tag{635}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\left(\mathrm{i}\sigma_z B_x(t)+B_y(t)\right)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right) \tag{636}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left|i\right\rangle\left\langle i\right|-V_{10}^{\Im}(t)\left(B_y(t)+\mathrm{i}\sigma_z B_x(t)\right)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{637}$$

$$+V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{638}$$

We will obtain a neat form of $\left[\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right]$ as we will see:

$$\left[\left|0\right\rangle\left\langle 0\right|,\sigma_x\right]=\left|0\right\rangle\left\langle 0\right|\left(\left|0\right\rangle\left\langle 1\right|+\left|1\right\rangle\left\langle 0\right|\right)-\left(\left|0\right\rangle\left\langle 1\right|+\left|1\right\rangle\left\langle 0\right|\right)\left|0\right\rangle\left\langle 0\right| \tag{639}$$

$$=\left|0\right\rangle\left\langle 1\right|-\left|1\right\rangle\left\langle 0\right| \tag{640}$$

$$=-\mathrm{i}\sigma_y, \tag{641}$$

$$
\begin{aligned}
[|0\rangle\langle 0|, \sigma_x] &= |0\rangle\langle 0| \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) - \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) |0\rangle\langle 0| && (642)\\
&= |0\rangle\langle 1| - |1\rangle\langle 0| && (643)\\
&= -\mathrm{i}\sigma_y, && (644)\\
[|1\rangle\langle 1|, \sigma_x] &= |1\rangle\langle 1| \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) - \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) |1\rangle\langle 1| && (645)\\
&= |1\rangle\langle 0| - |0\rangle\langle 1| && (646)\\
&= \mathrm{i}\sigma_y, && (647)\\
[|i\rangle\langle i|, \sigma_x] &= (-1)^{i+1}\, \mathrm{i}\sigma_y, && (648)\\
[|0\rangle\langle 0|, \sigma_y] &= |0\rangle\langle 0| \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) - \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) |0\rangle\langle 0| && (649)\\
&= \mathrm{i}|0\rangle\langle 1| + \mathrm{i}|1\rangle\langle 0| && (650)\\
&= \mathrm{i}\sigma_x, && (651)\\
[|1\rangle\langle 1|, \sigma_y] &= |1\rangle\langle 1| \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) - \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) |1\rangle\langle 1| && (652)\\
&= -\mathrm{i}|1\rangle\langle 0| - \mathrm{i}|0\rangle\langle 1| && (653)\\
&= -\mathrm{i}\sigma_x, && (654)\\
[|i\rangle\langle i|, \sigma_y] &= (-1)^{i}\, \mathrm{i}\sigma_x, && (655)
\end{aligned}
$$

$$
\begin{aligned}
\left[\overline{H_0}(t)\,, \overline{H_{\bar{I}}}(t)\right] &= \sum_{i,i'} (\varepsilon_i(t) + R_i(t))(\delta_{ii'} - \delta_{ii'})\, B_{i'z}(t)|i\rangle\langle i'| + V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t)) \left(\left(|i\rangle\langle i|\sigma_x - \sigma_x|i\rangle\langle i|\right) B_y(t) - \left(|i\rangle\langle i|\sigma_y\right.\right. && (656)\\
&\left.\left. -\sigma_y|i\rangle\langle i|\right) B_x(t)\right) + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t) \left(\sigma_x|i\rangle\langle i| - |i\rangle\langle i|\sigma_x\right) + V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t)\right. && (657)\\
&\left. -B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \left(\sigma_x^2 B_x(t) + \sigma_x\sigma_y B_y(t) - \sigma_x^2 B_x(t) - \sigma_y\sigma_x B_y(t)\right) + V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) && (658)\\
&\times \left(\sigma_x^2 B_y(t) - \sigma_x\sigma_y B_x(t) - \sigma_x^2 B_y(t) + \sigma_y\sigma_x B_x(t)\right) - \left(B_{10}^{\Re}(t)\, V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\, V_{10}^{\Re}(t)\right) \sum_i B_{iz}\,(t)\, \left(\sigma_y|i\rangle\langle i|\right. && (659)\\
&\left. -|i\rangle\langle i|\sigma_y\right) - V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \left(\sigma_y\sigma_x B_x(t) + \sigma_y^2 B_y(t) - \sigma_x\sigma_y B_x(t) - \sigma_y^2 B_y(t)\right) - V_{10}^{\Im}(t) && (660)\\
&\times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \left(\sigma_y\sigma_x B_y(t) - \sigma_y^2 B_x(t) - \sigma_x\sigma_y B_y(t) + \sigma_y^2 B_x(t)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_{iz}(t)\right. && (661)\\
&\left. -B_{iz}(t)\, b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) - \left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) + V_{10}^{\Im}(t) && (662)\\
&\times \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(\sigma_x B_y\,(t) - \sigma_y B_x\,(t)\right) - \left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) && (663)\\
&= V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t)) \left([|i\rangle\langle i|, \sigma_x]\, B_y(t) - [|i\rangle\langle i|, \sigma_y]\, B_x(t)\right) + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t) && (664)\\
&\times [\sigma_x, |i\rangle\langle i|] + V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) [\sigma_x, \sigma_y]\, B_y(t) + V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) && (665)\\
&\times [\sigma_y, \sigma_x]\, B_x(t) - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) [\sigma_y, |i\rangle\langle i|] - V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) && (666)\\
&\times [\sigma_y, \sigma_x]\, B_x(t) - V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) [\sigma_y, \sigma_x]\, B_y(t) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_{iz}(t)\right] |i\rangle\langle i| + V_{10}^{\Re}(t) && (667)\\
&\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x\,(t)\right] + \sigma_y \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y\,(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y\,(t)\right] - \sigma_y \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x\,(t)\right]\right), && (668)
\end{aligned}
$$

$$
\begin{aligned}
\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_{iz}(t)\right] &= \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t)\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t)\right)^* b_{\mathbf{k}'}\right)\right] && (669)\\
&= \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t)\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t)\right)^* b_{\mathbf{k}}\right) + \sum_{\mathbf{k}'\neq\mathbf{k}} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t)\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t)\right)^* b_{\mathbf{k}'}\right)\right] && (670)\\
&= \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right] + \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \sum_{\mathbf{k}'\neq\mathbf{k}} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^* b_{\mathbf{k}'}\right)\right] && (671)\\
&= \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t)\right) \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger\right] + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t)\right)^* \left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, b_{\mathbf{k}}\right] && (672)\\
&= \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t)\right) b_{\mathbf{k}}^\dagger \left[b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger\right] + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t)\right)^* \left[b_{\mathbf{k}}^\dagger, b_{\mathbf{k}}\right] b_{\mathbf{k}} && (673)
\end{aligned}
$$

$$= \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}, \tag{674}$$

$$\left[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)\right] = V_{10}^{\Im}\left(t\right) \sum_i \left(\varepsilon_i\left(t\right) + R_i\left(t\right)\right) (-1)^{i+1} \mathrm{i} \left(\sigma_y B_y(t) + \sigma_x B_x(t)\right) + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}\left(t\right) (-1)^{i} \mathrm{i}\sigma_y \tag{675}$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) \left(-1\right)^{i+1} \mathrm{i}\sigma_x + 2\mathrm{i} B_{10}^{\Re}(t) \left(\left(V_{10}^{\Re}(t)\right)^2 + \left(V_{10}^{\Im}(t)\right)^2\right) \sigma_z B_y(t) + 2\mathrm{i} B_{10}^{\Im}\left(t\right) \tag{676}$$

$$\times \left(\left(V_{10}^{\Re}(t)\right)^2 + \left(V_{10}^{\Im}(t)\right)^2\right) \sigma_z B_x(t) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{iz}(t)\right] |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] \sigma_x + \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right] \right. \tag{677}$$

$$\left. \times \sigma_y\right) + V_{10}^{\Im}\left(t\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right]\right) \tag{678}$$

$$= V_{10}^{\Im}\left(t\right) \mathrm{i} \left(\sigma_y B_y(t) + \sigma_x B_x(t)\right) \sum_i \left(\varepsilon_i\left(t\right) + R_i\left(t\right)\right) (-1)^{i+1} + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}\left(t\right) (-1)^{i} \mathrm{i}\sigma_y \tag{679}$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) \left(-1\right)^{i+1} \mathrm{i}\sigma_x + 2\mathrm{i} \left|V_{10}\left(t\right)\right|^2 \sigma_z \left(B_{10}^{\Re}(t)\, B_y(t) + B_{10}^{\Im}(t)\, B_x\left(t\right)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \tag{680}$$

$$\times \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}\left(t\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right]\right) + V_{10}^{\Im}\left(t\right) \tag{681}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right]\right) \tag{682}$$

$$= V_{10}^{\Im}\left(t\right) \mathrm{i} \left(\sigma_y B_y(t) + \sigma_x B_x(t)\right) \sum_i \left(\varepsilon_i\left(t\right) + R_i\left(t\right)\right) (-1)^{i+1} + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}\left(t\right) (-1)^{i} \mathrm{i}\sigma_y \tag{683}$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) \left(-1\right)^{i+1} \mathrm{i}\sigma_x + 2\mathrm{i} \left|V_{10}(t)\right|^2 \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} |i\rangle\langle i| \tag{684}$$

$$\times \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right) + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \tag{685}$$

$$\times \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right]\right) \tag{686}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i} \left(\sigma_y B_y(t) + \sigma_x B_x(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right) (-1)^{i+1} + 2\mathrm{i} \left|V_{10}(t)\right|^2 \sigma_z \left(B_{10}^{\Re}(t)\, B_y(t) + B_{10}^{\Im}(t)\, B_x(t)\right) + \sum_i B_{iz}(t) \tag{687}$$

$$\times (-1)^i \left(\left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \mathrm{i}\sigma_y + \left(B_{10}^{\Re}(t)\, V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\, V_{10}^{\Re}(t)\right) \mathrm{i}\sigma_x\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger}\right. \tag{688}$$

$$\left. - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right. \tag{689}$$

$$\left. - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right]\right) \tag{690}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i} \left(\sigma_y B_y(t) + \sigma_x B_x(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right) (-1)^{i+1} + 2\mathrm{i} \left|V_{10}(t)\right|^2 \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_i B_{iz}(t) \tag{691}$$

$$\times (-1)^i \left(\left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \left(|1\rangle\langle 0| - |0\rangle\langle 1|\right) + \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \mathrm{i} \left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \tag{692}$$

$$\times \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}\left(t\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right]\right) + V_{10}^{\Im}\left(t\right) \tag{693}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right]\right) \tag{694}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i} \left(\sigma_y B_y(t) + \sigma_x B_x(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right) (-1)^{i+1} + 2\mathrm{i} \left|V_{10}(t)\right|^2 \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_i B_{iz}(t) \tag{695}$$

$$\times (-1)^i \left(\left(\left(B_{10}(t)\, V_{10}(t)\right)^{\Re} + \mathrm{i} \left(B_{10}(t)\, V_{10}(t)\right)^{\Im}\right) |1\rangle\langle 0| + \left(\left(B_{10}(t)\, V_{10}(t)\right)^{\Re} - \mathrm{i} \left(B_{10}(t)\, V_{10}(t)\right)^{\Im}\right) |0\rangle\langle 1|\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \tag{696}$$

$$\times \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \tag{697}$$

$$\times \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right]\right) \tag{698}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i} \left(\sigma_y B_y(t) + \sigma_x B_x(t)\right) \sum_i \left(\varepsilon_i(t) + R_i(t)\right) (-1)^{i+1} + 2\mathrm{i} \left|V_{10}(t)\right|^2 \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_i B_{iz}(t) \tag{699}$$

$$\times (-1)^i \left(B_{10}(t)\, V_{10}(t)\, |1\rangle\langle 0| + B_{10}^{*}(t)\, V_{10}^{*}(t)\, |0\rangle\langle 1|\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \tag{700}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}\left(t\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\left(t\right)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\left(t\right)\right]\right), \tag{701}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{x}(t)\right]=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\right] \tag{702}$$

$$=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)}{2}\right]+\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},-\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2}\right] \tag{703}$$

$$=\frac{1}{2}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right]+\frac{1}{2}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)\right] \tag{704}$$

$$=\frac{1}{2}\left(\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right]+\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right]\right) \tag{705}$$

$$=\frac{1}{2}\left(\mathrm{e}^{\chi_{10}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\right. \tag{706}$$

$$\left.\times\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right), \tag{707}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{y}(t)\right]=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}\right] \tag{708}$$

$$=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)}{2\mathrm{i}}\right]+\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}\right] \tag{709}$$

$$=\frac{1}{2\mathrm{i}}\left(\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)\right]-\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right]\right) \tag{710}$$

$$=\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\chi_{01}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)-\mathrm{e}^{\chi_{10}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\right. \tag{711}$$

$$\left.\times\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right). \tag{712}$$

We will focus on the term $\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]$:

$$D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)=b_{\mathbf{k}}+\alpha_{\mathbf{k}}\ \text{(by properties of the displacement operator)}, \tag{713}$$

$$D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)=b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\ \text{(by properties of the displacement operator)}, \tag{714}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]=b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)-D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{715}$$

$$=b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)-D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(-\alpha_{\mathbf{k}}\right)D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\ \text{(introducing }\mathbb{I}=D\left(-\alpha\right)D\left(\alpha\right)\text{)} \tag{716}$$

$$=b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\left(D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right)-\left(D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(-\alpha_{\mathbf{k}}\right)\right)D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}} \tag{717}$$

$$=b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}\right)D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}} \tag{718}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}\right)\left(D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(-\alpha_{\mathbf{k}}\right)\right)D\left(\alpha_{\mathbf{k}}\right) \tag{719}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}-\alpha_{\mathbf{k}}\right)D\left(\alpha_{\mathbf{k}}\right) \tag{720}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^{2}\right)-\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^{2}\right)D\left(\alpha_{\mathbf{k}}\right) \tag{721}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)D\left(\alpha_{\mathbf{k}}\right) \tag{722}$$

$$=D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right) \tag{723}$$

$$=\left[D\left(\alpha_{\mathbf{k}}\right),b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right]+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right) \tag{724}$$

$$=-\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right), \tag{725}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]=\frac{1}{2}\left(\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right) \tag{726}$$

$$=\frac{1}{2}\left(\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right) \tag{727}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\alpha_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} D\left(\alpha_{\mathbf{k}}\right) b_{\mathbf{k}} + \alpha_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) + \alpha_{\mathbf{k}}^{*} D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\right) && (728)\\
&= \frac{D\left(\alpha_{\mathbf{k}}\right)}{2}\left(\alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \alpha_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) + \alpha_{\mathbf{k}}^{*}\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\right) && (729)\\
&= \frac{D\left(\alpha_{\mathbf{k}}\right)}{2}\left(\alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \left|\alpha_{\mathbf{k}}\right|^{2} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \left|\alpha_{\mathbf{k}}\right|^{2}\right) && (730)\\
&= D\left(\alpha_{\mathbf{k}}\right)\left(\alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \left|\alpha_{\mathbf{k}}\right|^{2}\right), && (731)\\
\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{i}^{+}(t) B_{j}^{-}(t)\right] &= \mathrm{e}^{\chi_{ij}(t)}\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right] \prod_{\mathbf{k}' \neq \mathbf{k}} D\left(\frac{v_{i\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) && (732)\\
&= \mathrm{e}^{\chi_{ij}(t)} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\left(\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) && (733)\\
&\quad \times \prod_{\mathbf{k}' \neq \mathbf{k}} D\left(\frac{v_{i\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) && (734)\\
&= \mathrm{e}^{\chi_{ij}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{i\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\left(\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) && (735)\\
&= B_{i}^{+}(t) B_{j}^{-}(t)\left(\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right), && (736)\\
v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t) &\equiv v_{ij\mathbf{k}}, && (737)\\
v_{ij\mathbf{k}} &= -v_{ji\mathbf{k}}, && (738)\\
\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{x}(t)\right] &= \frac{1}{2}\left(\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{1}^{+}(t) B_{0}^{-}(t)\right] + \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{0}^{+}(t) B_{1}^{-}(t)\right]\right) && (739)\\
&= \frac{1}{2}\left(B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) + B_{0}^{+}(t) B_{1}^{-}(t)\left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. && (740)\\
&\quad \left.\left. + \left|\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) && (741)\\
&= \frac{1}{2}\left(B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) + B_{0}^{+}(t) B_{1}^{-}(t)\left(-\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. && (742)\\
&\quad \left.\left. + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) && (743)\\
&= \frac{1}{2}\left(\left(B_{1}^{+}(t) B_{0}^{-}(t) - B_{0}^{+}(t) B_{1}^{-}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right) + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{1}^{+}(t) B_{0}^{-}(t) + B_{0}^{+}(t) B_{1}^{-}(t)\right)\right) && (744)\\
&= \frac{1}{2}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{1}^{+}(t) B_{0}^{-}(t) + B_{0}^{+}(t) B_{1}^{-}(t)\right) - \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right)\right) && (745)\\
&= \frac{1}{2}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(2B_{x}(t) + B_{10}(t) + B_{01}(t)\right) - \left(2\mathrm{i}B_{y}(t) - B_{10}(t) + B_{01}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right)\right) && (746)\\
&= \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{x}(t) + B_{10}^{\Re}\right) - \mathrm{i}\left(B_{y}(t) - B_{10}^{\Im}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right), && (747)\\
\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{y}(t)\right] &= \frac{1}{2\mathrm{i}}\left(\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{0}^{+}(t) B_{1}^{-}(t)\right] - \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{1}^{+}(t) B_{0}^{-}(t)\right]\right) && (748)\\
&= \frac{1}{2\mathrm{i}}\left(B_{0}^{+}(t) B_{1}^{-}(t)\left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) - B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. && (749)\\
&\quad \left.\left. + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) && (750)\\
&= \frac{1}{2\mathrm{i}}\left(B_{0}^{+}(t) B_{1}^{-}(t)\left(-\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) - B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. && (751)\\
&\quad \left.\left. + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) && (752)\\
&= \frac{1}{2\mathrm{i}}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t)\right) - \left(B_{0}^{+}(t) B_{1}^{-}(t) + B_{1}^{+}(t) B_{0}^{-}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right)\right) && (753)
\end{aligned}
$$

$$= \frac{1}{2\mathrm{i}}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2(2\mathrm{i}B_y(t) - B_{10}(t) + B_{01}(t)) - (2B_x(t) + B_{10}(t) + B_{01}(t))\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) \quad (754)$$

$$= \frac{1}{2\mathrm{i}}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(2\mathrm{i}B_y(t) - 2\mathrm{i}B_{10}^{\Im}(t)\right) - \left(2B_x(t) + 2B_{10}^{\Re}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) \quad (755)$$

$$= \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(B_y(t) - B_{10}^{\Im}(t)\right) + \mathrm{i}\left(B_x(t) + B_{10}^{\Re}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right). \quad (756)$$

The term that we will rewrite is defined as:

$$A_{T\mathbf{k}}(t) \equiv V_{10}^{\Re}(t)\left(\sigma_x\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right] + \sigma_y\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t)\left(\sigma_x\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right] - \sigma_y\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right]\right) \quad (757)$$

$$= \left(V_{10}^{\Re}(t)\sigma_x - V_{10}^{\Im}(t)\sigma_y\right)\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right] + \left(V_{10}^{\Re}(t)\sigma_y + V_{10}^{\Im}(t)\sigma_x\right)\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right] \quad (758)$$

$$= \left(V_{10}^{\Re}(t)(|1\rangle\langle 0| + |0\rangle\langle 1|) - V_{10}^{\Im}(t)(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|)\right)\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right] + \left(V_{10}^{\Re}(t)(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|) + V_{10}^{\Im}(t)(|1\rangle\langle 0| + |0\rangle\langle 1|)\right) \quad (759)$$

$$\times\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right] \quad (760)$$

$$= \left(|1\rangle\langle 0|\left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right) + |0\rangle\langle 1|\left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right] + \left(|1\rangle\langle 0|\left(-\mathrm{i}V_{10}^{\Re}(t) + V_{10}^{\Im}(t)\right) + |0\rangle\langle 1|\left(\mathrm{i}V_{10}^{\Re}(t)\right.\right. \quad (761)$$

$$\left.\left. + V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right] \quad (762)$$

$$= (|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t))\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right] - \mathrm{i}\left(|1\rangle\langle 0|\left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right) + |0\rangle\langle 1|\left(-V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right] \quad (763)$$

$$= (|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t))\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right] - \mathrm{i}\left(|1\rangle\langle 0|\left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right) - |0\rangle\langle 1|\left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right] \quad (764)$$

$$= (|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t))\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_x(t)\right] - \mathrm{i}(|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t))\left[b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, B_y(t)\right] \quad (765)$$

$$= (|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(B_x(t) + B_{10}^{\Re}(t)\right) - \mathrm{i}\left(B_y(t) - B_{10}^{\Im}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\right)b_{\mathbf{k}}\right) \quad (766)$$

$$- \mathrm{i}(|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(B_y(t) - B_{10}^{\Im}(t)\right) + \mathrm{i}\left(B_x(t) + B_{10}^{\Re}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right), \quad (767)$$

$$B_x(t) + B_{10}^{\Re}(t) = \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} + B_{10}^{\Re}(t) \quad (768)$$

$$= \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - 2B_{10}^{\Re}(t)}{2} + B_{10}^{\Re}(t) \quad (769)$$

$$= \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}, \quad (770)$$

$$B_y(t) - B_{10}^{\Im}(t) = \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} - B_{10}^{\Im}(t) \quad (771)$$

$$= \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + 2\mathrm{i}B_{10}^{\Im}(t)}{2\mathrm{i}} - B_{10}^{\Im}(t) \quad (772)$$

$$= \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2\mathrm{i}}, \quad (773)$$

$$A_{T\mathbf{k}}(t) = (|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}\right) - \mathrm{i}\left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2\mathrm{i}}\right)\right. \quad (774)$$

$$\left.\times\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) - \mathrm{i}(|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2\mathrm{i}}\right)\right. \quad (775)$$

$$\left. + \mathrm{i}\left(\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) \quad (776)$$

$$= (|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}\right) - \left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger\right.\right. \quad (777)$$

$$\left.\left. + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) - (|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2}\right) - \left(b_{\mathbf{k}}^\dagger\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} + b_{\mathbf{k}}\right.\right. \quad (778)$$

$$\times\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\Bigg)\left(\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)}{2}\right)\Bigg) \tag{779}$$

$$=(|1\rangle\langle 0|V_{10}(t)+|0\rangle\langle 1|V_{10}^{*}(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)}{2}\right)-\left(\frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right) \tag{780}$$

$$+(|1\rangle\langle 0|V_{10}(t)-|0\rangle\langle 1|V_{10}^{*}(t))\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)B_0^{-}(t)-B_0^{+}(t)B_1^{-}(t)}{2}\right)+\left(\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right) \tag{781}$$

$$=|1\rangle\langle 0|V_{10}\,(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}\,(t)\,B_0^{-}\,(t)+B_0^{+}\,(t)\,B_1^{-}\,(t)}{2}\right)-\left(\frac{B_0^{+}\,(t)\,B_1^{-}\,(t)-B_1^{+}\,(t)\,B_0^{-}\,(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right. \tag{782}$$

$$+\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)\,B_0^{-}(t)-B_0^{+}(t)\,B_1^{-}(t)}{2}\right)+\left(\frac{B_1^{+}(t)\,B_0^{-}(t)+B_0^{+}(t)\,B_1^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\Bigg)+|0\rangle\langle 1|V_{10}^{*}\,(t) \tag{783}$$

$$\times\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)\,B_0^{-}(t)+B_0^{+}(t)\,B_1^{-}(t)}{2}\right)-\left(\frac{B_0^{+}(t)\,B_1^{-}(t)-B_1^{+}(t)\,B_0^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)-\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right.\right. \tag{784}$$

$$\times\left(\frac{B_1^{+}\,(t)\,B_0^{-}\,(t)-B_0^{+}\,(t)\,B_1^{-}\,(t)}{2}\right)+\left(\frac{B_1^{+}\,(t)\,B_0^{-}\,(t)+B_0^{+}\,(t)\,B_1^{-}\,(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\Bigg)\Bigg) \tag{785}$$

$$=|1\rangle\langle 0|V_{10}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}B_1^{+}\,(t)\,B_0^{-}\,(t)+B_1^{+}\,(t)\,B_0^{-}\,(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)+|0\rangle\langle 1|V_{10}^{*}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}B_0^{+}\,(t)\,B_1^{-}\,(t)\right. \tag{786}$$

$$-B_0^{+}\,(t)\,B_1^{-}\,(t)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\Bigg) \tag{787}$$

$$=|1\rangle\langle 0|V_{10}\,(t)\,B_1^{+}(t)\,B_0^{-}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)+|0\rangle\langle 1|V_{10}^{*}\,(t)\,B_0^{+}(t)\,B_1^{-}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}-\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right.\right. \tag{788}$$

$$\left.\left.+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right). \tag{789}$$

Inserting the precedent term in the sum of $\left[\,\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right]$ help us to obtain:

$$\left[\,\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right]=V_{10}^{\Im}(t)\,\mathrm{i}\,(\sigma_y B_y(t)+\sigma_x B_x(t))\sum_i\left(\varepsilon_i(t)+R_i(t)\right)(-1)^{i+1}+2\mathrm{i}\,|V_{10}(t)|^{2}\,\sigma_z\left(B_{10}^{\Re}(t)\,B_y(t)+B_{10}^{\Im}(t)\,B_x(t)\right) \tag{790}$$

$$+\sum_i B_{iz}(t)(-1)^{i}(B_{10}(t)V_{10}(t)\,|1\rangle\langle 0|+B_{10}^{*}(t)V_{10}^{*}(t)\,|0\rangle\langle 1|)+\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}b_{\mathbf{k}}\right) \tag{791}$$

$$\times|i\rangle\langle i|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(|1\rangle\langle 0|V_{10}(t)\,B_1^{+}(t)\,B_0^{-}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)+|0\rangle\langle 1|V_{10}^{*}(t)\right. \tag{792}$$

$$\left.\times B_0^{+}\,(t)\,B_1^{-}\,(t)\left(\left|\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right|^{2}-\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)\right). \tag{793}$$

The term $\overline{H_{\bar{I}}}\,(t)\left[\,\overline{H_0}\,(t)\,,\overline{H_{\bar{I}}}\,(t)\right]$ is given by:

$$\overline{H_{\bar{I}}}(t)[\,\overline{H_0}(t),\overline{H_{\bar{I}}}(t)]=\left(\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\right)\left(\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\right. \tag{794}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{795}$$

$$\times\sum_i B_{iz}(t)|i\rangle\langle i|+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t))+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{796}$$

$$\times V_{10}^{\Im}(t)(B_y(t)-\mathrm{i}\sigma_z B_x(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{797}$$

$$\times V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t) \tag{798}$$

$$\times|i\rangle\langle i|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \tag{799}$$

$$\times\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y \tag{800}$$

$$\times B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V^{\Re}_{10}(t)(B_x(t)-\mathrm{i}\sigma_z B_y(t))\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+V^{\Re}_{10}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\,\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t) \tag{801}$$

$$+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)\sum_i(\varepsilon_i(t)+R_i(t)) \tag{802}$$

$$\times|i\rangle\langle i|-V^{\Im}_{10}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))+(\mathrm{i}\sigma_z B_y(t)-B_x(t))\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)V^{\Im}_{10}(t)-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\Big) \tag{803}$$

$$=\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{804}$$

$$+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\,\Big(B^{\Re}_{10}(t)\,V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\,V^{\Im}_{10}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\,\Big(B^{\Re}_{10}(t)\,V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\,V^{\Im}_{10}(t)\Big)\,V^{\Re}_{10}(t)\,(B_x(t) \tag{805}$$

$$+\mathrm{i}\sigma_z B_y\,(t))+\sum_i B_{iz}\,(t)\,|i\rangle\langle i|\sigma_x\,\Big(B^{\Re}_{10}\,(t)\,V^{\Re}_{10}(t)-B^{\Im}_{10}\,(t)\,V^{\Im}_{10}(t)\Big)\,V^{\Im}_{10}(t)\,(B_y\,(t)-\mathrm{i}\sigma_z B_x\,(t))-\sum_i B_{iz}\,(t)\,|i\rangle\langle i|\sigma_y\,\Big(B^{\Re}_{10}\,(t)\,V^{\Im}_{10}\,(t) \tag{806}$$

$$+B^{\Im}_{10}\,(t)\,V^{\Re}_{10}\,(t)\Big)\sum_i B_{iz}\,(t)\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|\,\Big(B^{\Re}_{10}\,(t)\,V^{\Im}_{10}\,(t)+B^{\Im}_{10}\,(t)\,V^{\Re}_{10}\,(t)\Big)\,V^{\Re}_{10}(t)\,(-\mathrm{i}\sigma_z B_x\,(t)+B_y\,(t))-\sum_i B_{iz}\,(t)\,|i\rangle\langle i| \tag{807}$$

$$\times\,\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)V^{\Im}_{10}(t)(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{808}$$

$$\times\,V^{\Re}_{10}\,(t)\,(\sigma_x\,B_x\,(t)+\sigma_y\,B_y\,(t))+\sum_i B_{iz}\,(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\,V^{\Im}_{10}\,(t)\,(\sigma_x\,B_y\,(t)-\sigma_y\,B_x\,(t))-\sum_i B_{iz}\,(t)\,|i\rangle\langle i|\sum_i B_{iz}\,(t)\,|i\rangle\langle i|\,\sigma_x \tag{809}$$

$$\times\,\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\,\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i| \tag{810}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-\sum_i B_{iz}\,(t)\,|i\rangle\langle i|V^{\Re}_{10}\,(t)\,(\sigma_x B_x\,(t)+\sigma_y B_y\,(t))\sum_i(\varepsilon_i\,(t)+R_i\,(t))\,|i\rangle\langle i|-\sum_i B_{iz}\,(t)\,|i\rangle\langle i|V^{\Re}_{10}\,(t)\,(B_x\,(t)-\mathrm{i}\sigma_z B_y\,(t)) \tag{811}$$

$$\times\,\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Re}_{10}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\,\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)-\sum_i B_{iz}(t)|i\rangle\langle i|V^{\Re}_{10}(t) \tag{812}$$

$$\times\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-\sum_i B_{iz}\,(t)\,|i\rangle\langle i|V^{\Im}_{10}\,(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i\,(t)+R_i\,(t))\,|i\rangle\langle i|-\sum_i B_{iz}\,(t)\,|i\rangle\langle i|V^{\Im}_{10}\,(t) \tag{813}$$

$$\times\,(B_y(t)+\mathrm{i}\sigma_z B_x(t))\,\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Im}_{10}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\,\Big(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\Big) \tag{814}$$

$$-\sum_i B_{iz}\,(t)\,|i\rangle\langle i|V^{\Im}_{10}\,(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}+V^{\Re}_{10}\,(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i\,(t)+R_i\,(t))\,|i\rangle\langle i|V^{\Re}_{10}\,(t)\,(\sigma_x B_x\,(t) \tag{815}$$

$$+\sigma_y B_y\,(t))+V^{\Re}_{10}\,(t)\,(\sigma_x B_x\,(t)+\sigma_y B_y\,(t))\sum_i(\varepsilon_i\,(t)+R_i\,(t))\,|i\rangle\langle i|V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V^{\Re}_{10}(t)\,(\sigma_x B_x\,(t)+\sigma_y B_y\,(t))\,\sigma_x \tag{816}$$

$$\times\,\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|+V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)V^{\Re}_{10}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t)) \tag{817}$$

$$+\,V^{\Re}_{10}\,(t)\,(\sigma_x B_x\,(t)+\sigma_y B_y\,(t))\,\sigma_x\,\Big(B^{\Re}_{10}\,(t)V^{\Re}_{10}\,(t)-B^{\Im}_{10}\,(t)V^{\Im}_{10}\,(t)\Big)\,V^{\Im}_{10}\,(t)\,(B_y\,(t)-\mathrm{i}\sigma_z B_x\,(t))-V^{\Re}_{10}\,(t)\,(\sigma_x B_x\,(t)+\sigma_y B_y\,(t))\,\sigma_y \tag{818}$$

$$\times\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)V^{\Re}_{10}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{819}$$

$$-\,V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)\,V^{\Im}_{10}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{820}$$

$$\times\sum_i B_{iz}\,(t)\,|i\rangle\langle i|+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}V^{\Re}_{10}\,(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{821}$$

$$\times\,V^{\Im}_{10}\,(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-V^{\Re}_{10}\,(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\,\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+V^{\Re}_{10}\,(t)\,(\sigma_x B_x(t) \tag{822}$$

$$+\sigma_y B_y\,(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\,\Big(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\Big)-V^{\Re}_{10}\,(t)\,(\sigma_x B_x\,(t)+\sigma_y B_y\,(t))\sum_i B_{iz}\,(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}\,(t) \tag{823}$$

$$\times\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V^{\Re}_{10}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{824}$$

$$\times\,\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Re}_{10}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\Big(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\Big)-V^{\Re}_{10}(t) \tag{825}$$

$$\times\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t) \tag{826}$$

$$+R_i(t))|i\rangle\langle i|-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V^{\Im}_{10}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{827}$$

$$\times\,V^{\Im}_{10}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)-V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{828}$$

$$+ V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_i(\varepsilon_i(t) + R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+ V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i| \tag{829}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t)) + V_{10}^{\Im}(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t))\,\sigma_x \Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) \sum_i B_{iz}\,(t)\,|i\rangle\langle i| + V_{10}^{\Im}(t)\,(\sigma_x B_y\,(t) \tag{830}$$

$$-\sigma_y B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\,V_{10}^{\Re}(t)(B_x\,(t)+\mathrm{i}\sigma_z B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) \tag{831}$$

$$\times V_{10}^{\Im}(t)\,(B_y(t) - \mathrm{i}\sigma_z B_x(t)) - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\,\sigma_y\,\Big(B_{10}^{\Re}\,(t)\,V_{10}^{\Im}\,(t) + B_{10}^{\Im}\,(t)\,V_{10}^{\Re}\,(t)\Big) \sum_i B_{iz}\,(t)\,|i\rangle\langle i| - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{832}$$

$$-\sigma_y B_x(t))\,\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\,V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t) + B_y(t)) - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big) \tag{833}$$

$$\times V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y\,(t) - B_x\,(t)) + V_{10}^{\Im}(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\,\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{834}$$

$$\times V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{835}$$

$$\times \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\,\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\,\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\,\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big) \tag{836}$$

$$- V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\,V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))\sum_i(\varepsilon_i(t) + R_i(t)) \tag{837}$$

$$\times\,|i\rangle\langle i| - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))V_{10}^{\Re}(t)\,(B_x(t) - \mathrm{i}\sigma_z B_y(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\,V_{10}^{\Re}(t) \tag{838}$$

$$\times\,(\mathrm{i}\sigma_z B_x\,(t) + B_y\,(t))\,\Big(B_{10}^{\Re}\,(t)\,V_{10}^{\Im}\,(t) + B_{10}^{\Im}\,(t)\,V_{10}^{\Re}\,(t)\Big) - V_{10}^{\Im}\,(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t))\,V_{10}^{\Re}\,(t)\,(\sigma_x B_x\,(t) + \sigma_y B_y\,(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{839}$$

$$-V_{10}^{\Im}\,(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t))\,V_{10}^{\Im}\,(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t))\sum_i(\varepsilon_i\,(t) + R_i\,(t))\,|i\rangle\langle i| - V_{10}^{\Im}\,(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t))\,V_{10}^{\Im}\,(t)\,(B_y\,(t) \tag{840}$$

$$+\mathrm{i}\sigma_z B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\,V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t) - B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big) \tag{841}$$

$$- V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{842}$$

$$= V_{10}^{\Re}(t)\sum_i(\varepsilon_i\,(t) + R_i\,(t))\,(|i\rangle\langle i|\sigma_x B_{iz}\,(t)\,B_x\,(t) + |i\rangle\langle i|\sigma_y B_{iz}\,(t)\,B_y\,(t)) + V_{10}^{\Im}(t)\sum_i(\varepsilon_i\,(t) + R_i\,(t))\,(|i\rangle\langle i|\sigma_x B_{iz}\,(t)\,B_y\,(t) - |i\rangle\langle i|\sigma_y \tag{843}$$

$$\times B_{iz}(t)\,B_x(t)) + \Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_{i\neq i'} B_{iz}(t)\,B_{i'z}(t)\,|i\rangle\langle i'| + \Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\,V_{10}^{\Re}\,(t)\sum_i(|i\rangle\langle i|B_{iz}\,(t) \tag{844}$$

$$\times B_x(t) + \mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)) + \Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\,V_{10}^{\Im}(t)\sum_i(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t) - |i\rangle\langle i|\sigma_y B_{iz}(t)\,B_x(t)) - \Big(B_{10}^{\Re}(t) \tag{845}$$

$$\times V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_{i\neq i'} B_{iz}(t)B_{i'z}(t)\,|i\rangle\langle i|\sigma_y|i'\rangle\langle i'| - \Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\,V_{10}^{\Re}(t)\sum_i(-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t) + B_{iz}(t) \tag{846}$$

$$\times B_y(t)\,|i\rangle\langle i|) + \Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\,V_{10}^{\Im}(t)\sum_i(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t) + |i\rangle\langle i|B_{iz}(t)\,B_x(t)) + \sum_{i,\mathbf{k}}|i\rangle\langle i|B_{iz}(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) \tag{847}$$

$$+ V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\Big(|i\rangle\langle i|\sigma_x B_{iz}\,(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + |i\rangle\langle i|\sigma_y B_{iz}(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t)\Big) + V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\Big(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - |i\rangle\langle i|\sigma_y B_{iz}\,(t) \tag{848}$$

$$\times\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t)\Big) - \Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i B_{iz}^2(t)\,|i\rangle\langle i|\sigma_x + \Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i B_{iz}^2\,(t)\,|i\rangle\langle i|\sigma_y - \sum_{i,\mathbf{k}}|i\rangle\langle i| \tag{849}$$

$$\times\, B_{iz}^2(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)\sum_{i,i'}(\varepsilon_{i'}(t) + R_{i'}(t))\big(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|B_{iz}(t)B_x(t) + |i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_y(t)\big) - V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t) \tag{850}$$

$$\times V_{10}^{\Im}(t)\Big)\sum_i(|i\rangle\langle i|B_{iz}(t)\,B_x(t) - \mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)) + V_{10}^{\Re}\,(t)\,\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\sum_i(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t) + |i\rangle\langle i| \tag{851}$$

$$\times B_{iz}(t)\,B_y(t)) - V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\Big(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\Big) - V_{10}^{\Im}(t)\sum_{i\neq i'}(\varepsilon_{i'}(t) + R_{i'}(t))\big(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'| \tag{852}$$

$$\times B_{iz}(t)\,B_y(t) - |i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)\,B_x(t)\big) - V_{10}^{\Im}(t)\,\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i(|i\rangle\langle i|B_{iz}(t)\,B_y(t) + \mathrm{i}|i\rangle\langle i|\sigma_z B_x(t)) + V_{10}^{\Im}(t) \tag{853}$$

$$\times\,\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t) - |i\rangle\langle i|B_{iz}(t)\,B_x(t)) - V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\Big(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - |i\rangle\langle i|\sigma_y \tag{854}$$

$$\times B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\Big) + \Big(V_{10}^{\Re}(t)\Big)^2\sum_i(\varepsilon_i(t) + R_i(t))\,\big(\sigma_x|i\rangle\langle i|\sigma_x B_x^2(t) + \sigma_x|i\rangle\langle i|\sigma_y B_x(t)\,B_y(t) + \sigma_y|i\rangle\langle i|\sigma_x B_y(t)\,B_x(t) + \sigma_y|i\rangle\langle i| \tag{855}$$

$$\times\sigma_y B_y^2(t)\big) + V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_i(\varepsilon_i(t) + R_i(t))\,\big(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_y(t) - \sigma_x|i\rangle\langle i|\sigma_y B_x^2(t) + \sigma_y|i\rangle\langle i|\sigma_x B_y^2(t) - \sigma_y|i\rangle\langle i|\sigma_y B_y(t)\,B_x(t)\big) \tag{856}$$

$$+ V_{10}^{\Re}(t)\,\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i(|i\rangle\langle i|B_x(t)\,B_{iz}(t) - \mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_{iz}(t)) + \Big(V_{10}^{\Re}(t)\Big)^2\,\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) \tag{857}$$

$$\times\Big(\sigma_x B_x^2(t)+\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)-\sigma_x B_y^2(t)\Big)+V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)V_{10}^{\Im}(t)\Big(B_x(t)\,B_y(t)-\mathrm{i}\sigma_z B_y^2(t) \tag{858}$$

$$-\mathrm{i}\sigma_z B_x^2(t)-B_y(t)\,B_x(t)\Big)-V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\sum_i\big(\mathrm{i}\sigma_z|i\rangle\!\langle i|B_x(t)\,B_{iz}(t)+|i\rangle\!\langle i|B_y(t)\,B_{iz}(t)\big)-V_{10}^{\Re}(t) \tag{859}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)V_{10}^{\Im}(t)\Big(-\sigma_y B_x^2(t)+\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B_y^2(t)\Big)-V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t) \tag{860}$$

$$\times V_{10}^{\Re}(t)\Big)V_{10}^{\Im}(t)\Big(-\sigma_y B_x(t)B_y(t)+\sigma_x B_y^2(t)-\sigma_x B_x^2(t)-\sigma_y B_y(t)B_x(t)\Big)+V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\big(\sigma_x|i\rangle\!\langle i|B_x(t)+\sigma_y|i\rangle\!\langle i|B_y(t)\big)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t) \tag{861}$$

$$+\Big(V_{10}^{\Re}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\Big)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}} \tag{862}$$

$$\times\Big(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\Big)-V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) \tag{863}$$

$$\times\sum_i\big(\sigma_x|i\rangle\!\langle i|\sigma_x B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\!\langle i|\sigma_x B_y(t)\,B_{iz}(t)\big)+V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i\big(\sigma_x|i\rangle\!\langle i|\sigma_y B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\!\langle i| \tag{864}$$

$$\times\sigma_y B_y(t)\,B_{iz}(t)\big)-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\Big(\sigma_x|i\rangle\!\langle i|B_x(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\sigma_y|i\rangle\!\langle i|B_y(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big)-\Big(V_{10}^{\Re}(t)\Big)^2\sum_i\big(\varepsilon_i(t)+R_i(t)\big)\big(|i\rangle\!\langle i|B_x^2(t) \tag{865}$$

$$-\mathrm{i}\sigma_z|i\rangle\!\langle i|B_y(t)\,B_x(t)+\mathrm{i}\sigma_z|i\rangle\!\langle i|B_x(t)\,B_y(t)+|i\rangle\!\langle i|B_y^2(t)\big)-\Big(V_{10}^{\Re}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\Big(\sigma_x B_x^2(t)+\sigma_y B_y(t)\,B_x(t) \tag{866}$$

$$-\sigma_y B_x(t)\,B_y(t)+\sigma_x B_y^2(t)\Big)+\Big(V_{10}^{\Re}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\Big(\sigma_y B_x^2(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B_y^2(t)\Big) \tag{867}$$

$$-\Big(V_{10}^{\Re}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_i\big(\varepsilon_i(t)+R_i(t)\big) \tag{868}$$

$$\times\big(|i\rangle\!\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\!\langle i|B_y^2(t)-\mathrm{i}\sigma_z|i\rangle\!\langle i|B_x^2(t)-|i\rangle\!\langle i|B_y(t)\,B_x(t)\big)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\Big(\sigma_x B_x(t) \tag{869}$$

$$\times B_y(t)+\sigma_y B_y^2(t)+\sigma_y B_x^2(t)-\sigma_x B_y(t)B_x(t)\Big)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\Big(\sigma_y B_x(t)B_y(t)-\sigma_x B_y^2(t)-\sigma_x B_x^2(t) \tag{870}$$

$$-\sigma_y B_y(t)\,B_x(t)\Big)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_x^2(t)-B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_i\big(\varepsilon_i(t) \tag{871}$$

$$+R_i(t)\big)\big(\sigma_x|i\rangle\!\langle i|\sigma_x B_y(t)\,B_x(t)-\sigma_y|i\rangle\!\langle i|\sigma_x B_x^2(t)+\sigma_x|i\rangle\!\langle i|\sigma_y B_y^2(t)-\sigma_y|i\rangle\!\langle i|\sigma_y B_x(t)\,B_y(t)\big)+\Big(V_{10}^{\Im}(t)\Big)^2\sum_i\big(\varepsilon_i(t)+R_i(t)\big)\big(\sigma_x|i\rangle\!\langle i| \tag{872}$$

$$\times\sigma_x B_y^2(t)-\sigma_y|i\rangle\!\langle i|\sigma_x B_x(t)\,B_y(t)-\sigma_x|i\rangle\!\langle i|\sigma_y B_y(t)\,B_x(t)+\sigma_y|i\rangle\!\langle i|\sigma_y B_x^2(t)\big)+V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i\big(|i\rangle\!\langle i|B_y(t) \tag{873}$$

$$\times B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\!\langle i|B_x(t)\,B_{iz}(t)\big)+V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)V_{10}^{\Re}(t)\big(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)+\sigma_y B_y^2(t)+\sigma_x B_x(t)B_y(t)\big) \tag{874}$$

$$+\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\big(B_y^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_x^2(t)\big)-V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big) \tag{875}$$

$$\times\sum_i\big(\mathrm{i}\sigma_z|i\rangle\!\langle i|B_y(t)B_{iz}(t)-|i\rangle\!\langle i|B_x(t)B_{iz}(t)\big)-V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)V_{10}^{\Re}(t)\big(-\sigma_y B_y(t)\,B_x(t)-\sigma_x B_x^2(t)+\sigma_x B_y^2(t) \tag{876}$$

$$-\sigma_y B_x(t)\,B_y(t)\big)-\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\big(-\sigma_y B_y^2(t)-\sigma_x B_x(t)\,B_y(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_y B_x^2(t)\big)+V_{10}^{\Im}(t) \tag{877}$$

$$\times\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\Big(\sigma_x|i\rangle\!\langle i|B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)-\sigma_y|i\rangle\!\langle i|B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\Big)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z \tag{878}$$

$$\times B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\Big)+\Big(V_{10}^{\Im}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+B_x(t) \tag{879}$$

$$\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\Big)-V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\sum_i\big(\sigma_x|i\rangle\!\langle i|\sigma_x B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\!\langle i|\sigma_x B_x(t)\,B_{iz}(t)\big)+V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) \tag{880}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i\big(\sigma_x|i\rangle\!\langle i|\sigma_y B_y(t)B_{iz}(t)-\sigma_y|i\rangle\!\langle i|\sigma_y B_x(t)B_{iz}(t)\big)-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\Big(\sigma_x|i\rangle\!\langle i|B_y(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sigma_y|i\rangle\!\langle i|B_x(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big) \tag{881}$$

$$-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_i\big(\varepsilon_i(t)+R_i(t)\big)\big(|i\rangle\!\langle i|B_y(t)B_x(t)+\mathrm{i}\sigma_z|i\rangle\!\langle i|B_x^2(t)+\mathrm{i}\sigma_z|i\rangle\!\langle i|B_y^2(t)-|i\rangle\!\langle i|B_x(t)B_y(t)\big)-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) \tag{882}$$

$$-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\big(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)-\sigma_y B_y^2(t)-\sigma_x B_x(t)B_y(t)\big)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\big(\sigma_y B_y(t)B_x(t) \tag{883}$$

$$+\sigma_x B_x^2(t)+\sigma_x B_y^2(t)-\sigma_y B_x(t)B_y(t)\big)-\sum_{\mathbf{k}}V_{10}^{\Im}(t)V_{10}^{\Re}(t)\,\omega_{\mathbf{k}}\Big(B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big) \tag{884}$$

$$-\Big(V_{10}^{\Im}(t)\Big)^2\sum_i\big(\varepsilon_i(t)+R_i(t)\big)\big(|i\rangle\!\langle i|B_y^2(t)+\mathrm{i}\sigma_z|i\rangle\!\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\!\langle i|B_y(t)\,B_x(t)+|i\rangle\!\langle i|B_x^2(t)\big)-\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) \tag{885}$$

$$-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\big(\sigma_x B_y^2(t)-\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)+\sigma_x B_x^2(t)\big)+\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\big(\sigma_y B_y^2(t) \tag{886}$$

$$+\sigma_x B_x(t)B_y(t)-\sigma_x B_y(t)B_x(t)+\sigma_y B_x^2(t)\big)-\Big(V_{10}^{\Im}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_y^2(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)B_y(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)B_x(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_x^2(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big). \tag{887}$$

Now let's obtain the form of $\overline{H_I}^3(t)$:

$$\overline{H_I}^3(t)=\Bigg(\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\Bigg)\Bigg(\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i\,(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x \tag{888}$$

$$+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t) \tag{889}$$

$$\times B_{iz}(t))+\Big(V_{10}^{\Re}(t)\Big)^2\big(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\big)+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) \tag{890}$$

$$+\Big(V_{10}^{\Im}(t)\Big)^2\,\big(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_z B_y\,(t)\,B_x\,(t)+\mathrm{i}\sigma_z B_x\,(t)\,B_y\,(t)\big)\Bigg) \tag{891}$$

$$=\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i\,(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t) \tag{892}$$

$$\times\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\sum_i|i\rangle\langle i| \tag{893}$$

$$\times B_{iz}(t)\Big(V_{10}^{\Re}(t)\Big)^2\big(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\big)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i| \tag{894}$$

$$\times B_x(t)B_{iz}(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\,\Big(V_{10}^{\Im}(t)\Big)^2\,\big(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)\big)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{895}$$

$$\times\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}\,(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y \tag{896}$$

$$\times B_y(t))\,V_{10}^{\Im}(t)\sum_i\,(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) \tag{897}$$

$$+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\Big(V_{10}^{\Re}\,(t)\Big)^2\,\big(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\big)+V_{10}^{\Re}\,(t) \tag{898}$$

$$\times\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Im}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,\Big(V_{10}^{\Im}(t)\Big)^2\,\big(B_y^2\,(t) \tag{899}$$

$$+B_x^2(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)\big)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Re}(t) \tag{900}$$

$$\times\sum_i\,(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Im}(t)\sum_i\,(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t) \tag{901}$$

$$|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Re}\,(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Im}\,(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{902}$$

$$\times\Big(V_{10}^{\Re}(t)\Big)^2\big(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\big)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y \tag{903}$$

$$\times|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\Big(V_{10}^{\Im}(t)\Big)^2\,\big(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_z B_y\,(t)\,B_x\,(t)+\mathrm{i}\sigma_z B_x\,(t)\,B_y\,(t)\big) \tag{904}$$

$$=\sum_i B_{iz}^3(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i\big(B_{iz}^2(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}^2(t)B_y(t)\,|i\rangle\langle i|\sigma_y\big)+V_{10}^{\Im}(t)\sum_i\big(B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_y\big) \tag{905}$$

$$+V_{10}^{\Re}(t)\sum_{i\neq i'}\big(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_x(t)B_{iz}(t)+|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)\big)+\Big(V_{10}^{\Re}(t)\Big)^2\sum_i\,\big(|i\rangle\langle i|B_{iz}(t)B_x^2(t)+\mathrm{i}|i\rangle\langle i|\sigma_z \tag{906}$$

$$\times B_{iz}(t)B_x(t)B_y(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)B_x(t)+|i\rangle\langle i|B_{iz}(t)B_y^2(t)\big)+V_{10}^{\Im}(t)\sum_{i\neq i'}\big(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)-|i'\rangle\langle i'|\sigma_y \tag{907}$$

$$\times|i\rangle\langle i|B_{i'z}(t)\,B_x(t)B_{iz}(t))+\Big(V_{10}^{\Im}(t)\Big)^2\sum_i\,\big(|i\rangle\langle i|B_{iz}(t)B_y^2(t)+|i\rangle\langle i|B_{iz}(t)B_x^2(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t) \tag{908}$$

$$\times B_x(t)B_y(t)\big)+V_{10}^{\Re}(t)\sum_i\big(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}^2(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}^2(t)\big)+\Big(V_{10}^{\Re}(t)\Big)^2\sum_i\big(B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_x+B_x(t)B_{iz}(t) \tag{909}$$

$$\times B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_x+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_x \tag{910}$$

$$-B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_x-B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+\Big(V_{10}^{\Re}(t)\Big)^2\sum_i\,\big(|i\rangle\langle i|B_x^2(t)B_{iz}(t) \tag{911}$$

$$+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y^2(t)B_{iz}(t)\big)+\Big(V_{10}^{\Re}(t)\Big)^3\,\big(\sigma_x B_x^3(t)+\sigma_y B_x^2(t)B_y(t)-\sigma_y B_x(t) \tag{912}$$

$$\times B_y(t)B_x(t)+\sigma_x B_x(t)B_y^2(t)+\sigma_y B_y(t)B_x^2(t)-\sigma_x B_y(t)B_x(t)B_y(t)+\sigma_x B_y^2(t)B_x(t)+\sigma_y B_y^3(t)\big)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i\big(|i\rangle\langle i|B_x(t) \tag{913}$$

$$\times B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_y^2(t)B_{iz}(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_y(t)B_x(t)B_{iz}(t)\big)+V_{10}^{\Re}(t)\Big(V_{10}^{\Im}(t)\Big)^2\big(\sigma_x B_x(t)B_y^2(t)+\sigma_x \tag{914}$$

$$\times B_x^3(t) - \sigma_y B_x(t)B_y(t)B_x(t) + \sigma_y B_x^2(t)B_y(t) + \sigma_y B_y^3(t) + \sigma_y B_y(t)B_x^2(t) + \sigma_x B_y^2(t)B_x(t) - \sigma_x B_y(t)B_x(t)B_y(t)\big) + V_{10}^{\Im}(t)\sum_i \big(\sigma_x |i\rangle\langle i| B_y(t) \tag{915}$$

$$\times B_{iz}^2(t) - \sigma_y |i\rangle\langle i| B_x(t)B_{iz}^2(t)\big) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x |i\rangle\langle i|\sigma_x B_y(t)B_{iz}(t)B_x(t) + \sigma_x |i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)B_y(t) - \sigma_y |i\rangle\langle i|\sigma_x B_y(t)B_{iz}(t)B_y(t) \tag{916}$$

$$-\sigma_x |i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)B_y(t)) + \left(V_{10}^{\Im}(t)\right)^2 (\sigma_x |i\rangle\langle i|\sigma_x B_y(t)B_{iz}(t)B_y(t) - \sigma_x |i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)B_x(t) - \sigma_y |i\rangle\langle i|\sigma_x B_x(t)B_{iz}(t)B_y(t) \tag{917}$$

$$+\sigma_y |i\rangle\langle i|\sigma_y B_x(t)B_{iz}(t)B_x(t)) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i \big(|i\rangle\langle i| B_y(t)B_x(t)B_{iz}(t) + \mathrm{i}\sigma_z |i\rangle\langle i| B_y^2(t)B_{iz}(t) + \mathrm{i}\sigma_z |i\rangle\langle i| B_x^2(t)B_{iz}(t) - |i\rangle\langle i| B_x(t)B_y(t) \tag{918}$$

$$\times B_{iz}(t)) + V_{10}^{\Im}(t)\left(V_{10}^{\Re}(t)\right)^2 (\sigma_x B_y(t)B_x^2(t) + \sigma_y B_y(t)B_x(t)B_y(t) - \sigma_y B_y^2(t)B_x(t) + \sigma_x B_y^3(t) - \sigma_y B_x^3(t) + \sigma_x B_x^2(t)B_y(t) - \sigma_x B_x(t)B_y(t) \tag{919}$$

$$\times B_x(t) - \sigma_y B_x(t)B_y^2(t)) + \left(V_{10}^{\Im}(t)\right)^2 \sum_i \big(|i\rangle\langle i| B_y^2(t)B_{iz}(t) - \mathrm{i}\sigma_z |i\rangle\langle i| B_y(t)B_x(t)B_{iz}(t) + \mathrm{i}\sigma_z |i\rangle\langle i| B_x(t)B_y(t)B_{iz}(t) + |i\rangle\langle i| B_x^2(t)B_{iz}(t)\big) \tag{920}$$

$$+\left(V_{10}^{\Im}(t)\right)^3 \big(\sigma_x B_y^3(t) + \sigma_x B_y(t)B_x^2(t) - \sigma_y B_y^2(t)B_x(t) + \sigma_y B_y(t)B_x(t)B_y(t) - \sigma_y B_x(t)B_y^2(t) - \sigma_y B_x^3(t) - \sigma_x B_x(t)B_y(t)B_x(t) + \sigma_x \tag{921}$$

$$\times B_x^2(t)B_y(t)\big)\,. \tag{922}$$

* n.dattani@cfa.harvard.edu
† edcchaparroso@unal.edu.co

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