A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$
$$H_S(t) = \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| + V_{01}(t)|0\rangle\langle 1|, \tag{2}$$
$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}(t)$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}(t). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i(t) \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$
$$V(t) = |0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V(t)}$ using the notation (6) will give us the following result:

$$
\begin{aligned}
e^{\pm V(t)} &= e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t))} && (10)\\
&= \mathbb{I} \pm \left(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t)\right) + \frac{\left(\pm\left(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t)\right)\right)^2}{2!} + ... && (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm \left(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t)\right) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2(t)}{2!} + ... && (12)\\
&= |0\rangle\langle 0|\left(\mathbb{I} \pm \hat{\varphi}_0(t) + \frac{\hat{\varphi}_0^2(t)}{2!} \pm ...\right) + |1\rangle\langle 1|\left(\mathbb{I} \pm \hat{\varphi}_1(t) + \frac{\hat{\varphi}_1^2(t)}{2!} \pm ...\right) && (13)\\
&= |0\rangle\langle 0| e^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1| e^{\pm\hat{\varphi}_1(t)} && (14)\\
&= |0\rangle\langle 0| e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{0\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1| e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{1\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0| B_0^{\pm}(t) + |1\rangle\langle 1| B_1^{\pm}(t), && (16)\\
B_i^{\pm}(t) &\equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
e^{r(X+Y)} = e^{rX}\, e^{rY}\, e^{-\frac{r^2}{2}[X,Y]}\, e^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\cdots. \tag{18}
$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i, j we can show making $r = 1$ in (18) the following result:

$$
\begin{aligned}
e^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\ldots && (19)\\
&= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}0}\ldots && (20)\\
&= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. && (21)
\end{aligned}
$$

By induction of this result we can write an expresion of $B_i^{\pm}(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$
\begin{aligned}
D(\pm v_{i\mathbf{k}}(t)) &\equiv e^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, && (22)\\
B_i^{\pm}(t) &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). && (23)
\end{aligned}
$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$
\overline{O(t)} \equiv e^{V(t)} O(t)\, e^{-V(t)}. \tag{24}
$$

We will use the following identities:

$$\begin{aligned}
\overline{|0\rangle\langle 0|\,(t)} &= e^{V(t)}|0\rangle\langle 0|e^{-V(t)} &(25)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(26)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(27)\\
&= |0\rangle\langle 0|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(28)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) &(29)\\
&= |0\rangle\langle 0|, &(30)\\
\overline{|1\rangle\langle 1|\,(t)} &= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(31)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(32)\\
&= |1\rangle\langle 1|B_1^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(33)\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) &(34)\\
&= B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) &(35)\\
&= |1\rangle\langle 1|, &(36)\\
\overline{|0\rangle\langle 1|\,(t)} &= e^{V(t)}|0\rangle\langle 1|e^{-V(t)} &(37)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(38)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(39)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(40)\\
&= |0\rangle\langle 1|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(41)\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) &(42)\\
&= |0\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t)\,, &(43)\\
\overline{|1\rangle\langle 0|\,(t)} &= e^{V(t)}|1\rangle\langle 0|e^{-V(t)} &(44)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(45)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(46)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(47)\\
&= |1\rangle\langle 0|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+\,(t)\,B_1^-\,(t) &(48)\\
&= |1\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t)\,, &(49)\\
\overline{b_{\mathbf{k}}\,(t)} &= e^{V(t)}b_{\mathbf{k}}e^{-V(t)} &(50)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(51)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| &(52)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,b_{\mathbf{k}}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,b_{\mathbf{k}}B_0^-(t) + |1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}B_1^-(t) &(53)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) &(54)\\
&= (|0\rangle\langle 0| + |1\rangle\langle 1|)\,b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} &(55)\\
&= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, &(56)\\
\overline{b_{\mathbf{k}}\,(t)}^{\dagger} &= e^{V(t)}b_{\mathbf{k}}^{\dagger}e^{-V(t)} &(57)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) &(58)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle 1| &(59)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) &(60)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) &(61)\\
&= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}. &(62)
\end{aligned}$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+}(t)\, b_{\mathbf{k}} B_i^{-}(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \tag{63}$$

$$B_i^{+}(t)\, b_{\mathbf{k}}^{\dagger} B_i^{-}(t) = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}. \tag{64}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|\,(t)} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{65}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|\,(t)} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{66}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|\,(t)} = V_{10}(t)\,|1\rangle\langle 0| B_1^{+}(t)\, B_0^{-}(t), \tag{67}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|\,(t)} = V_{01}(t)\,|0\rangle\langle 1| B_0^{+}(t)\, B_1^{-}(t), \tag{68}$$

$$\overline{\left(g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}}\right)(t)} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\right) + g_{i\mathbf{k}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{69}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{70}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{72}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{73}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{74}$$

$$= |0\rangle\langle 0| B_0^{+}(t)\,|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)|0\rangle\langle 0| B_0^{-}(t) \tag{75}$$

$$= |0\rangle\langle 0| B_0^{+}(t)\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_0^{-}(t) \tag{76}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{77}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{78}$$

$$= |1\rangle\langle 1| B_1^{+}(t)\,|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)|1\rangle\langle 1| B_1^{-}(t) \tag{79}$$

$$= |1\rangle\langle 1| B_1^{+}(t)\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_1^{-}(t) \tag{80}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{81}$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}(t)} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{82}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{+}(t)\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_0^{-}(t) + |1\rangle\langle 1| B_1^{+}(t)\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_1^{-}(t)\right) \tag{84}$$

$$= \omega_{\mathbf{k}} \sum_j |j\rangle\langle j| D\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}} \left(D\left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{90}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \omega_{\mathbf{k}}\left(|1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{91}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} - v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} - v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right) \tag{92}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right). \tag{93}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} \tag{94}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0|B_1^{+}(t)\,B_0^{-}(t) + V_{01}(t)\,|0\rangle\langle 1|B_0^{+}(t)\,B_1^{-}(t)\,, \tag{95}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} \tag{96}$$

$$= \overline{\sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} + \overline{\sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} \tag{97}$$

$$= \sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) + \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{98}$$

$$= \sum_{\mathbf{k},i}|i\rangle\langle i|\left(g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{99}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} \tag{100}$$

$$= \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right)\right) \tag{101}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{\mathbf{k}}\left(|1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right)\right). \tag{102}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_{j}\varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^{+}(t)\,B_{j'}^{-}(t) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}} + \frac{\left|v_{j\mathbf{k}}(t)\right|^{2}}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^{*}\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right). \tag{103}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}. \tag{104}$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{105}$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right), \tag{106}$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)\,v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right). \tag{107}$$

$\chi_{ij}(t)$ is an imaginary number so $e^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right) & e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)} \\ \frac{B_1^{+}(t)B_0^{-}(t) + B_0^{+}(t)B_1^{-}(t) - B_{10}(t) - B_{01}(t)}{2} & e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth(\beta\omega_{\mathbf{k}}/2)} \\ \frac{B_0^{+}(t)B_1^{-}(t) - B_1^{+}(t)B_0^{-}(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} & e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\chi_{ij}(t)} \end{pmatrix}, \tag{108}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{109}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{110}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)}. \tag{111}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \ldots \\ \sqrt{1} & 0 & 0 & \ldots & 0 & \ldots \\ 0 & \sqrt{2} & 0 & \ldots & 0 & \ldots \\ 0 & 0 & \sqrt{3} & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ldots \\ 0 & 0 & 0 & \ldots & \sqrt{n} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{112}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & \sqrt{2} & 0 & \ldots & 0 & \ldots \\ 0 & 0 & 0 & \sqrt{3} & \ldots & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{113}$$

So the product of the matrix representation of $b^{\dagger}$ and b with $-\beta$ is:

$$-\beta\omega b^{\dagger} b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & 1 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & 2 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ldots \\ 0 & 0 & 0 & \ldots & n & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{114}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega\, |j\rangle\langle j|, \tag{115}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|,, \tag{116}$$

$$e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{117}$$

The value of $\mathrm{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)$ is:

$$\begin{aligned}
\mathrm{Tr}\left(e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (118)\\
&= \sum_{j_{\mathbf{k}}}\left(e^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} && (119)\\
&= \frac{1}{1-e^{-\beta\omega_{\mathbf{k}}}} \quad \text{(by geometric series)} && (120)\\
&\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), && (121)\\
\mathrm{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (122)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (123)\\
&= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). && (124)
\end{aligned}$$

So the density matrix of the bath is:

$$\begin{aligned}
\rho_B &= \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} && (125)\\
&= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} && (126)\\
&= \prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. && (127)
\end{aligned}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{128}$$

$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. \tag{129}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_B}} = 0$ using the following property based on (128)-(129):

$$\begin{aligned}
\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Tr}\left(B_{iz}\left(t\right)\rho_B\right) && (130)\\
&= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\rho_B\right) && (131)\\
&= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\rho_B\right) && (132)\\
&= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) && (133)\\
&= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) && (134)\\
&= \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), && (135)\\
\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\quad\text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}\text{)} && (136)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) && (137)\\
&= 0, && (138)\\
\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\quad\text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} && (139)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) && (140)\\
&= 0. && (141)
\end{aligned}$$

we therefore find that:

$$\langle B_{iz}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0. \tag{142}$$

Another important expected value is $B\left(t\right)=\langle B^{\pm}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm}\left(t\right)=e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\begin{aligned}
\left\langle B^{\pm}\left(t\right)\right\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B^{\pm}\left(t\right)\right)=\mathrm{Tr}\left(B^{\pm}\left(t\right)\rho_B\right) && (143)\\
&= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) && (144)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\rho_B\right) && (145)\\
&= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\right\rangle. && (146)
\end{aligned}$$

Given that we can write a density operator as:

$$\rho=\int P\left(\alpha\right)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \tag{147}$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right)\mathrm{d}^2\alpha=1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P\left(\alpha\right)$ is given by:

$$\begin{aligned}
\langle A\rangle &= \mathrm{Tr}\left(A\rho\right) && (148)\\
&= \int P\left(\alpha\right)\langle\alpha\left|A\right|\alpha\rangle\mathrm{d}^2\alpha. && (149)
\end{aligned}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}}$ where $N = \left(e^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = \frac{1}{k_{\mathrm{B}} T}$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h)\rangle = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle\alpha|D(h)|\alpha\rangle \mathrm{d}^2\alpha \tag{150}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|D(-\alpha)D(h)D(\alpha)|0\rangle \mathrm{d}^2\alpha, \tag{151}$$

$$D(h)D(\alpha) = D(h+\alpha)e^{\frac{1}{2}(h\alpha^* - h^*\alpha)}, \tag{152}$$

$$D(-\alpha)(D(h)D(\alpha)) = D(-\alpha)D(h+\alpha)e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{153}$$

$$= D(h)e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))}e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{154}$$

$$= D(\alpha)e^{\frac{1}{2}\left(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2\right)}e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{155}$$

$$= D(\alpha)e^{(h\alpha^* - h^*\alpha)}, \tag{156}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|D(h)e^{(h\alpha^* - h^*\alpha)}|0\rangle \mathrm{d}^2\alpha \tag{157}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{(h\alpha^* - h^*\alpha)}\langle 0|D(h)|0\rangle \mathrm{d}^2\alpha \tag{158}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{(h\alpha^* - h^*\alpha)}\langle 0|h\rangle \mathrm{d}^2\alpha, \tag{159}$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{160}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{(h\alpha^* - h^*\alpha)}\langle 0|e^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha \tag{161}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{(h\alpha^* - h^*\alpha)}e^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{162}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int e^{-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha}\mathrm{d}^2\alpha, \tag{163}$$

$$\alpha = x + \mathrm{i}y, \tag{164}$$

$$\langle D(h)\rangle = \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{x^2+y^2}{N} + h(x-\mathrm{i}y) - h^*(x+\mathrm{i}y)}\mathrm{d}x\mathrm{d}y \tag{165}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}e^{-\frac{x^2}{N} + hx - h^*x}\mathrm{d}x\int_{-\infty}^{\infty}e^{-\frac{x^2}{N} + hx - h^*x}\mathrm{d}y, \tag{166}$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N}\left(x^2 - Nhx + Nh^*x\right) \tag{167}$$

$$= -\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 + \frac{N(h^* - h)^2}{4}, \tag{168}$$

$$-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^*y = -\frac{1}{N}\left(y^2 + \mathrm{i}Nhy + \mathrm{i}Nh^*y\right) \tag{169}$$

$$= -\frac{1}{N}\left(y^2 + \frac{\mathrm{i}N(h+h^*)}{2}\right) - \frac{N(h+h^*)^2}{4}, \tag{170}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x, \tag{171}$$

$$\langle D(h)\rangle = \frac{e^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}\mathrm{d}x\mathrm{d}y, \tag{172}$$

$$\begin{aligned}
\langle D\left(h\right)\rangle &= \frac{e^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty} e^{-\frac{\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}x\int_{-\infty}^{\infty} e^{-\frac{\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}y && (173)\\
&= \frac{e^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 && (174)\\
&= e^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}} && (175)\\
&= e^{-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}} && (176)\\
&= e^{-|h|^2\left(N+\frac{1}{2}\right)} && (177)\\
&= e^{-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)} && (178)\\
&= e^{-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)} && (179)\\
&= e^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}. && (180)
\end{aligned}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}=\coth\left(\frac{\beta\omega}{2}\right)$. So the value of (145) using (**??**) is given by:

$$B = e^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{181}$$

We will now force $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$. We will also introduce the bath renormalizing driving in $\overline{H_S}\left(t\right)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+\left(t\right)\sigma^+$ and $B_i^-\left(t\right)\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}\left(t\right)$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$.

A final form of the terms of the Hamiltonian $\overline{H}\left(t\right)$ is:

$$\begin{aligned}
\overline{H(t)} &= \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+\sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^*b_{\mathbf{k}}+\frac{\left|v_{j\mathbf{k}}(t)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}+g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) && (182)\\
&= \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_{jj'}(t)+\sum_j|j\rangle\langle j|B_{jz}(t)+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+(t)\,B_{j'}^-(t)-B_{jj'}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (183)\\
&\equiv \overline{H_{\bar{S}}\left(t\right)}+\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{B}}}. && (184)
\end{aligned}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\begin{aligned}
\langle B_i^+\left(t\right)B_j^-\left(t\right)\rangle &= B_{ij}\left(t\right) && (185)\\
&= \left\langle\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right\rangle && (186)\\
&= \left\langle\prod_{\mathbf{k}}\left(D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle && (187)
\end{aligned}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \right\rangle \tag{188}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \tag{189}$$

$$= \prod_{\mathbf{k}} e^{-\frac{1}{2}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \tag{190}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)}. \tag{191}$$

From the definition $B_{01}(t) = \langle B_0^{+}(t)\, B_1^{-}(t) \rangle$ using the displacement operator we have:

$$\langle B_0^{+}(t)\, B_1^{-}(t) \rangle = B_{01}(t) \tag{192}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \tag{193}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \tag{194}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \right\rangle \tag{195}$$

$$= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \tag{196}$$

$$= \prod_{\mathbf{k}} \left(e^{-\frac{1}{2}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \tag{197}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)}. \tag{198}$$

We can check:

$$\langle B_0^{+}(t)\, B_1^{-}(t) \rangle = B_{01}(t) \tag{199}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \tag{200}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)^{*}} \tag{201}$$

$$= \langle B_1^{+}(t)\, B_0^{-}(t) \rangle^{*} \tag{202}$$

$$= B_{10}^{*}(t)\,. \tag{203}$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0(t))\,|0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t))\,|1\rangle\langle 1| + V_{10}(t)\,B_{10}\sigma^{+} + V_{01}(t)\,B_{01}\sigma^{-}, \tag{204}$$

$$\overline{H_{\bar{I}}} \equiv V_{10}(t)\left(B_1^{+}(t)\,B_0^{-}(t) - B_{10}(t) \right)\sigma^{+} + V_{01}(t)\left(B_0^{+}(t)\,B_1^{-}(t) - B_{01}(t) \right)\sigma^{-} + |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t)\,, \tag{205}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{206}$$

$$= H_B. \tag{207}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (205) we can verify the condition $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{B}}}}=0$ in the following way:

$$\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle\sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)|n\rangle\langle n|+\sum_{j\neq j'}V_{jj'}\left(t\right)|j\rangle\langle j'|\left(B_{j}^{+}\left(t\right)B_{j'}^{-}\left(t\right)-B_{jj'}\left(t\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{208}$$

$$=\left\langle\sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\sum_{j\neq j'}V_{jj'}\left(t\right)|j\rangle\langle j'|\left(B_{j}^{+}\left(t\right)B_{j'}^{-}\left(t\right)-B_{jj'}\left(t\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{209}$$

$$=\sum_{n\mathbf{k}}\left(\left\langle\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}(t)B_{j}^{+}(t)B_{j'}^{-}(t)\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle V_{jj'}(t)B_{jj'}(t)\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{210}$$

$$=\sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)\left\langle b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)^{*}\left\langle b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}\left(t\right)\left(\left\langle B_{j}^{+}\left(t\right)B_{j'}^{-}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle B_{jj'}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{211}$$

$$=\sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)\left\langle b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\left(t\right)\right)^{*}\left\langle b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}\left(t\right)\left(B_{jj'}\left(t\right)-B_{jj'}\left(t\right)\right) \tag{212}$$

$$=0. \tag{213}$$

We used (142) and (**??**) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_{x}\left(t\right)=B_{x}^{\dagger}\left(t\right) \tag{214}$$

$$=\frac{B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}, \tag{215}$$

$$B_{y}\left(t\right)=B_{y}^{\dagger}\left(t\right) \tag{216}$$

$$=\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}, \tag{217}$$

$$B_{iz}\left(t\right)=B_{iz}^{\dagger}\left(t\right) \tag{218}$$

$$=\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right). \tag{219}$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}\left(t\right)=\sum_{j\in\{0,1\}}\left(\varepsilon_{j}\left(t\right)+R_{j}\left(t\right)\right)|j\rangle\langle j|+V_{10}\left(t\right)B_{10}\left(t\right)\sigma^{+}+V_{01}\left(t\right)B_{01}\left(t\right)\sigma^{-} \tag{220}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_{j}\left(t\right)+R_{j}\left(t\right)\right)|j\rangle\langle j|+V_{10}\left(t\right)B_{10}\left(t\right)\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}+V_{01}\left(t\right)B_{01}\left(t\right)\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2} \tag{221}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_{j}\left(t\right)+R_{j}\left(t\right)\right)|j\rangle\langle j|+V_{10}\left(t\right)\left(B_{10}^{\Re}\left(t\right)+\mathrm{i}B_{10}^{\Im}\left(t\right)\right)\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}+V_{01}\left(t\right)\left(B_{10}^{\Re}\left(t\right)-\mathrm{i}B_{10}^{\Im}\left(t\right)\right)\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2} \tag{222}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_{j}(t)+R_{j}(t)\right)|j\rangle\langle j|+B_{10}^{\Re}(t)\left(V_{10}(t)\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}+V_{01}(t)\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(V_{10}(t)\frac{\sigma_{x}+\mathrm{i}\sigma_{y}}{2}-V_{01}(t)\frac{\sigma_{x}-\mathrm{i}\sigma_{y}}{2}\right) \tag{223}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_{j}\left(t\right)+R_{j}\left(t\right)\right)|j\rangle\langle j|+B_{10}^{\Re}(t)\left(\sigma_{x}\frac{V_{10}(t)+V_{01}(t)}{2}+\mathrm{i}\sigma_{y}\frac{V_{10}(t)-V_{01}(t)}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\sigma_{x}\frac{V_{10}(t)-V_{01}(t)}{2}+\mathrm{i}\sigma_{y}\frac{V_{10}(t)+V_{01}(t)}{2}\right) \tag{224}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_{j}\left(t\right)+R_{j}\left(t\right)\right)|j\rangle\langle j|+B_{10}^{\Re}\left(t\right)\left(\sigma_{x}V_{10}^{\Re}\left(t\right)-\sigma_{y}V_{10}^{\Im}\left(t\right)\right)+\mathrm{i}B_{10}^{\Im}\left(t\right)\left(\mathrm{i}\sigma_{x}V_{10}^{\Im}\left(t\right)+\mathrm{i}\sigma_{y}V_{10}^{\Re}\left(t\right)\right) \tag{225}$$

$$=\left(\varepsilon_{0}\left(t\right)+R_{0}\left(t\right)\right)|0\rangle\langle 0|+\left(\varepsilon_{1}\left(t\right)+R_{1}\left(t\right)\right)|1\rangle\langle 1|+B_{10}^{\Re}\left(t\right)\left(\sigma_{x}V_{10}^{\Re}\left(t\right)-\sigma_{y}V_{10}^{\Im}\left(t\right)\right)+\mathrm{i}B_{10}^{\Im}\left(t\right)\left(\mathrm{i}\sigma_{x}V_{10}^{\Im}\left(t\right)+\mathrm{i}\sigma_{y}V_{10}^{\Re}\left(t\right)\right) \tag{226}$$

$$=\left(\varepsilon_{0}(t)+R_{0}(t)\right)|0\rangle\langle 0|+\left(\varepsilon_{1}(t)+R_{1}(t)\right)|1\rangle\langle 1|+\left(\sigma_{x}B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-\sigma_{y}B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right)-\left(\sigma_{x}B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)+\sigma_{y}B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) \tag{227}$$

$$=\left(\varepsilon_{0}(t)+R_{0}(t)\right)|0\rangle\langle 0|+\left(\varepsilon_{1}(t)+R_{1}(t)\right)|1\rangle\langle 1|+\sigma_{x}\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)-\sigma_{y}\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) \tag{228}$$

$$= \left(\varepsilon_0\left(t\right)+R_0\left(t\right)\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right)+R_1\left(t\right)\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right) - B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right) - \sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right) + B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right), \tag{229}$$

$$\overline{H_{\bar{I}}} = V_{10}\left(t\right)\left(\sigma^{+}B_1^{+}\left(t\right)B_0^{-}\left(t\right) - \sigma^{+}B_{10}\left(t\right)\right) + V_{01}\left(t\right)\left(\sigma^{-}B_0^{+}\left(t\right)B_1^{-}\left(t\right) - \sigma^{-}B_{01}\left(t\right)\right) + |0\rangle\langle 0|B_{0z}\left(t\right) + |1\rangle\langle 1|B_{1z}\left(t\right) \tag{230}$$

$$= |0\rangle\langle 0|B_{0z}\left(t\right) + |1\rangle\langle 1|B_{1z}\left(t\right) + \left(V_{10}^{\Re}\left(t\right)+\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{+}B_1^{+}\left(t\right)B_0^{-}\left(t\right) - \sigma^{+}B_{10}\left(t\right)\right) + \left(V_{10}^{\Re}\left(t\right)-\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{-}B_0^{+}\left(t\right)B_1^{-}\left(t\right) - \sigma^{-}B_{01}\left(t\right)\right) \tag{231}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma^{+}B_1^{+}\left(t\right)B_0^{-}\left(t\right) - \sigma^{+}B_{10}\left(t\right) + \sigma^{-}B_0^{+}\left(t\right)B_1^{-}\left(t\right) - \sigma^{-}B_{01}\left(t\right)\right) + \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma^{+}B_1^{+}\left(t\right)B_0^{-}\left(t\right) - \sigma^{+}B_{10}\left(t\right) - \sigma^{-}B_0^{+}\left(t\right)B_1^{-}\left(t\right) + \sigma^{-}B_{01}\left(t\right)\right) \tag{232}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}\left(t\right)B_0^{-}\left(t\right) - \frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}\left(t\right)B_1^{-}\left(t\right) - \frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{233}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}\left(t\right)B_0^{-}\left(t\right) - \frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}\left(t\right)B_1^{-}\left(t\right) - \frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{234}$$

$$+ \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}\left(t\right)B_0^{-}\left(t\right) - \frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}\left(t\right) - \frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}\left(t\right)B_1^{-}\left(t\right) + \frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{235}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) + B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2} + \mathrm{i}\sigma_y\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) - B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2}\right) \tag{236}$$

$$+ \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma_x\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) - B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2} + \mathrm{i}\sigma_y\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) + B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{237}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\mathrm{i}\sigma_x\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) - B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2} - \sigma_y\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) + B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{238}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^2\sigma_x\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) - B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2\mathrm{i}} - \sigma_y\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) + B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{239}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^2\sigma_x\frac{B_1^{+}B_0^{-}\left(t\right) - B_0^{+}B_1^{-}\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2\mathrm{i}} - \sigma_y\frac{B_1^{+}\left(t\right)B_0^{-}\left(t\right) + B_0^{+}\left(t\right)B_1^{-}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{240}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^2\sigma_x\left(-B_y\left(t\right)\right) - \sigma_y B_x\left(t\right)\right) \tag{241}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right) + \sigma_y B_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y\left(t\right) - \sigma_y B_x\left(t\right)\right). \tag{242}$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{243}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\left(t\right)\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\left(t\right)\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}\left(t\right)} = 0. \tag{244}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\left(t\right), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{245}$$

Then using the fact that $\overline{H_{\bar{S}}}\left(t\right)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{246}$$

So Eq. (244) becomes:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{247}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{248}$$

$$= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{249}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{250}$$

$$= 0 \quad (\text{by Eq. (244)}). \tag{251}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} \tag{252}$$

$$= 0. \tag{253}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{254}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left((\varepsilon_0(t)+R_0(t))|0\rangle\langle 0| + (\varepsilon_1(t)+R_1(t))|1\rangle\langle 1| + V_{10}(t)B_{10}(t)\sigma^+ + V_{01}(t)B_{01}(t)\sigma^-\right). \tag{255}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{256}$$

Let's define:

$$\varepsilon(t) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{257}$$

$$\eta \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{258}$$

The solutions of the equation (256) are:

$$\lambda = \beta\frac{-\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{259}$$

$$= \beta\frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{260}$$

$$= -\beta\frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{261}$$

The value of $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = \exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\exp\left(\frac{\eta(t)\beta}{2}\right) + \exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\exp\left(-\frac{\eta(t)\beta}{2}\right) \tag{262}$$

$$= 2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right). \tag{263}$$

Given that $v_{i\mathbf{k}}(t)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A(\varepsilon(t), \eta(t))$ to calculate $\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)}$ can lead to:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)} = \frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)} \tag{264}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\right)\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right) + 2\left(\frac{\beta}{2}\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\right)\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\sinh\left(\frac{\eta(t)\beta}{2}\right) \tag{265}$$

$$= -\beta\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\left(\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right)\right). \tag{266}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right) = 0. \tag{267}$$

The derivates included in the expression given are related to:

$$\langle B_0^+(t)B_1^-(t)\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v^*_{0\mathbf{k}}(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v^*_{1\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}\right)}\right) e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{268}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v^*_{1\mathbf{k}}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v^*_{0\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}\right)}\right)^* e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{269}$$

$$= \langle B_1^+(t)B_0^-(t)\rangle^*, \tag{270}$$

$$R_i(t) = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v^*_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} + g^*_{i\mathbf{k}}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{271}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}\frac{v^*_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - g^*_{i\mathbf{k}}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{272}$$

$$\langle B_0^+(t)B_1^-(t)\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v^*_{0\mathbf{k}}(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v^*_{1\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}\right)}\right) e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{273}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{v^*_{0\mathbf{k}}(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v^*_{1\mathbf{k}}(t)}{2\omega^2_{\mathbf{k}}}\right)\right) e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{274}$$

$$v^*_{0\mathbf{k}}(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v^*_{1\mathbf{k}}(t) = \left(v^{\Re}_{0\mathbf{k}}(t) - \mathrm{i}v^{\Im}_{0\mathbf{k}}(t)\right)\left(v^{\Re}_{1\mathbf{k}}(t) + \mathrm{i}v^{\Im}_{1\mathbf{k}}(t)\right) - \left(v^{\Re}_{0\mathbf{k}}(t) + \mathrm{i}v^{\Im}_{0\mathbf{k}}(t)\right)\left(v^{\Re}_{1\mathbf{k}}(t) - \mathrm{i}v^{\Im}_{1\mathbf{k}}(t)\right) \tag{275}$$

$$= \left(v^{\Re}_{0\mathbf{k}}(t)v^{\Re}_{1\mathbf{k}}(t) + \mathrm{i}v^{\Re}_{0\mathbf{k}}(t)v^{\Im}_{1\mathbf{k}}(t) - \mathrm{i}v^{\Im}_{0\mathbf{k}}(t)v^{\Re}_{1\mathbf{k}}(t) + v^{\Im}_{0\mathbf{k}}(t)v^{\Im}_{1\mathbf{k}}(t)\right) \tag{276}$$

$$- \left(v^{\Re}_{0\mathbf{k}}(t)v^{\Re}_{1\mathbf{k}}(t) - \mathrm{i}v^{\Re}_{0\mathbf{k}}(t)v^{\Im}_{1\mathbf{k}}(t) + \mathrm{i}v^{\Im}_{0\mathbf{k}}(t)v^{\Re}_{1\mathbf{k}}(t) + v^{\Im}_{0\mathbf{k}}(t)v^{\Im}_{1\mathbf{k}}(t)\right) \tag{277}$$

$$= 2\mathrm{i}\left(v^{\Re}_{0\mathbf{k}}(t)v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)v^{\Re}_{1\mathbf{k}}(t)\right), \tag{278}$$

$$
\begin{aligned}
|v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)|^2 &= (v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))^* && (279)\\
&= |v_{1\mathbf{k}}(t)|^2+|v_{0\mathbf{k}}(t)|^2-(v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)+v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)) && (280)\\
&= \left(v_{1\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Im}(t)\right)^2-\left(v_{1\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right)\left(v_{0\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right) && (281)\\
&-\left(v_{1\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right)\left(v_{0\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right) && (282)\\
&= \left(v_{1\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Im}(t)\right)^2-2\left(v_{1\mathbf{k}}^{\Re}(t)v_{0\mathbf{k}}^{\Re}(t)+v_{1\mathbf{k}}^{\Im}(t)v_{0\mathbf{k}}^{\Im}(t)\right) && (283)\\
&= \left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2. && (284)
\end{aligned}
$$

Rewriting in terms of real and imaginary parts.

$$
\begin{aligned}
R_i(t) &= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\right)\right) && (285)\\
&= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), && (286)\\
\langle B_0^+(t)B_1^-(t)\rangle &= \left(\prod_{\mathbf{k}}\exp\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (287)\\
&= \left(\prod_{\mathbf{k}}\exp\left(\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (288)\\
&= \left(\prod_{\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), && (289)
\end{aligned}
$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$
\begin{aligned}
\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\left(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (290)\\
&= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (291)\\
&= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, && (292)\\
\frac{\partial|B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (293)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) && (294)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}|B_{10}(t)|^2, && (295)\\
\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (296)\\
&= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} && (297)\\
&= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} && (298)
\end{aligned}
$$

$$= \frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{299}$$

$$= \frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Re}_{i\mathbf{k}}(t)-v^{\Re}_{i'\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{300}$$

$$= \frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t) - 4(\varepsilon(t)-\varepsilon_i(t)-R_i(t)) - \frac{4}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + \frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t) + 2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{301}$$

$$\left.+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{302}$$

From the equation (267) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}}{\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}} \tag{303}$$

$$= \frac{\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{2g^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}}{\frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + 2\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{304}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-g_{i\mathbf{k}}-g^{*}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}\right)(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t)))+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$= \frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t)))+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$= \frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$= \frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{308}$$

Separating (307) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2(\varepsilon_i(t)+R_i(t))-\varepsilon(t))+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{309}$$

$$v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}} = v^{\Re}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)) \tag{310}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{311}$$

$$v^{\Re}_{i\mathbf{k}}(t) = \frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{312}$$

$$v^{\Re}_{i\mathbf{k}}(t) = \frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{313}$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} = \frac{\partial\left(\varepsilon_{1}\left(t\right)+R_{1}\left(t\right)+\varepsilon_{0}\left(t\right)+R_{0}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{314}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}\left(t\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}\left(t\right)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{315}$$

$$= 2\frac{v_{i\mathbf{k}}^{\Im}\left(t\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{316}$$

$$\frac{\partial\left|B_{10}(t)\right|^{2}}{\partial v_{i\mathbf{k}}^{\Im}(t)} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{317}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-v_{0\mathbf{k}}^{\Re}\left(t\right)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{318}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\left|B_{10}\left(t\right)\right|^{2}, \tag{319}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} \tag{320}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} \tag{321}$$

$$= \frac{\varepsilon\left(t\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}(t)\right)\left(\varepsilon_{0}(t)+R_{0}(t)\right)-\left|V_{10}(t)\right|^{2}\left|B_{10}(t)\right|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\eta\left(t\right)} \tag{322}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{323}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-v_{i'\mathbf{k}}^{\Im}(t)\right)\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^{2}}\right)}{\eta\left(t\right)} \tag{324}$$

$$=\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\frac{4\left(\varepsilon_{i}(t)+R_{i}(t)\right)-2\varepsilon(t)-\frac{4\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}}{\eta(t)}+\frac{1}{\eta(t)}\left(2\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Im}(t)\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^{2}}\right). \tag{325}$$

From the equation (267) and replacing the derivatives obtained we have:

$$\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}=\tanh\left(\frac{\beta\eta(t)}{2}\right) \tag{326}$$

$$=\frac{2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{v_{i\mathbf{k}}^{\Im}(t)\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\right)+\frac{2}{\eta(t)}\left(\frac{g_{i\mathbf{k}}^{*\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}^{*\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. \tag{327}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v^{\Im}_{i\mathbf{k}}(t) - \mathrm{i}\left(g^{*}_{i\mathbf{k}} - g_{i\mathbf{k}}\right)\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \mathrm{i}\left(g^{*}_{i\mathbf{k}} - g_{i\mathbf{k}}\right)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right)\right) + 4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

$$= \frac{2\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g^{\Im}_{i\mathbf{k}}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right)\right) + 4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$= \frac{2\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{330}$$

$$= \frac{\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{331}$$

Separating (331) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{332}$$

$$v^{\Im}_{i\mathbf{k}} - g^{\Im}_{i\mathbf{k}} = v^{\Im}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) \tag{333}$$

$$+ 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{334}$$

$$v^{\Im}_{i\mathbf{k}}(t) = \frac{g^{\Im}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{335}$$

$$v^{\Im}_{i\mathbf{k}}(t) = \frac{g^{\Im}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{336}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t) = v^{\Re}_{i\mathbf{k}}(t) + \mathrm{i}v^{\Im}_{i\mathbf{k}}(t) \tag{337}$$

$$= \frac{g^{\Re}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{338}$$

$$+ \mathrm{i}\frac{g^{\Im}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\left|B_{10}\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{339}$$

$$= \frac{g_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{340}$$

Let's obtain the explicit form of $v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t)$ and $v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t)$, at first we have:

$$a_i(\omega_{\mathbf{k}},t)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)},\tag{341}$$

$$b_i(\omega_{\mathbf{k}},t)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\beta\omega_{\mathbf{k}}/2\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)}.\tag{342}$$

So the equation (337) written in explicit form is:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t),\tag{343}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t).\tag{344}$$

This system of equations has the following solutions:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)\tag{345}$$

$$=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+\left(g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)\right)b_0(\omega_{\mathbf{k}},t)\tag{346}$$

$$=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)\tag{347}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)(1-b_1(\omega_{\mathbf{k}},t)b_0(\omega_{\mathbf{k}},t))=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)\tag{348}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=\frac{g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)},\tag{349}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+\frac{g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}b_1(\omega_{\mathbf{k}},t)\tag{350}$$

$$=\frac{g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}.\tag{351}$$

For a shorter representation let's define:

$$r_i(\omega_{\mathbf{k}},t)=\frac{a_i(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)},\tag{352}$$

$$s_i(\omega_{\mathbf{k}},t)=\frac{a_{(i+1)\mathrm{mod}2}(\omega_{\mathbf{k}},t)\,b_{i\mathrm{mod}2}(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}.\tag{353}$$

So the variational parameter are:

$$\begin{pmatrix}v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\\ v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\end{pmatrix}\equiv\begin{pmatrix}r_0(\omega_{\mathbf{k}},t) & s_0(\omega_{\mathbf{k}},t)\\ r_1(\omega_{\mathbf{k}},t) & s_1(\omega_{\mathbf{k}},t)\end{pmatrix}\begin{pmatrix}g_0(\omega_{\mathbf{k}})\\ g_1(\omega_{\mathbf{k}})\end{pmatrix}.\tag{354}$$

Given that $v_{i\mathbf{k}}(\omega_{\mathbf{k}},t)\equiv g_i(\omega_{\mathbf{k}})\,F_i(\omega_{\mathbf{k}},t)$ then we can write:

$$F_0(\omega_{\mathbf{k}},t)=r_0(\omega_{\mathbf{k}},t)+\frac{g_1(\omega_{\mathbf{k}})}{g_0(\omega_{\mathbf{k}})}s_0(\omega_{\mathbf{k}},t)\tag{355}$$

$$F_1(\omega_{\mathbf{k}},t)=\frac{g_0(\omega_{\mathbf{k}})}{g_1(\omega_{\mathbf{k}})}r_1(\omega_{\mathbf{k}},t)+s_1(\omega_{\mathbf{k}},t)\tag{356}$$

In the continuous limit we have:

$$F_0(\omega,t)=r_0(\omega,t)+\frac{L_1(\omega)}{L_0(\omega)}s_0(\omega,t)\tag{357}$$

$$F_1(\omega,t)=\frac{L_0(\omega)}{L_1(\omega)}r_1(\omega,t)+s_1(\omega,t)\tag{358}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}$, where $\rho_B^{\mathrm{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$\begin{aligned}
\overline{\rho_T(0)} &\equiv e^{V(0)}\rho_T(0)\, e^{-V(0)} && (359)\\
&= \left(|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)\right)(\rho_S(0)\otimes\rho_B)\left(|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)\right), && (360)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } & |0\rangle\langle 0|0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (361)\\
&= |0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (362)\\
&= |0\rangle\langle 0| \otimes B_0^+(0)\,\rho_B B_0^-(0), && (363)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (364)\\
&= |1\rangle\langle 1|B_1^+(0)\,\rho_B B_1^-(0) && (365)\\
&= |1\rangle\langle 1| \otimes B_1^+(0)\,\rho_B B_1^-(0), && (366)\\
\text{for } \rho_S(0) = |0\rangle\langle 1|\text{ : } & |0\rangle\langle 0|B_0^+(0)\,|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (367)\\
&= |0\rangle\langle 1|B_0^+(0)\,\rho_B|1\rangle\langle 1|B_1^-(0) && (368)\\
&= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0)\,\rho_B B_1^-(0) && (369)\\
&= |0\rangle\langle 1| \otimes B_0^+(0)\,\rho_B B_1^-(0), && (370)\\
\text{for } \rho_S(0) = |1\rangle\langle 0|\text{ : } & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (371)\\
&= |1\rangle\langle 0| \otimes B_1^+(0)\,\rho_B B_0^-(0). && (372)
\end{aligned}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^\dagger(t)\, O(t)\, U(t), \tag{373}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\overline{H_{\bar{S}}}(t')\right). \tag{374}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\,\overline{\rho_S}(t)\,U(t)\text{, where} \tag{375}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}_T(t)\right). \tag{376}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \tag{377}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (**??**):

$$\begin{aligned}
\overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) && (378)\\
&= B_{0z}(t)\,|0\rangle\langle 0| + B_{1z}(t)\,|1\rangle\langle 1| + V_{10}^{\Re}(t)\,\sigma_x B_x(t) + V_{10}^{\Re}(t)\,\sigma_y B_y(t) + V_{10}^{\Im}(t)\,\sigma_x B_y(t) - V_{10}^{\Im}(t)\,\sigma_y B_x(t) && (379)\\
&= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right). && (380)
\end{aligned}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{381}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) \text{ -i} \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]\mathrm{d}t'. \tag{382}$$

Replacing the equation (382) in the equation (381) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]]\mathrm{d}t'. \tag{383}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdots]. \tag{384}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]] \cdots]. \tag{385}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\begin{aligned} \widetilde{\overline{\rho_S}}(t) &= (1 + W_1(t) + W_2(t) + ...) \overline{\rho_S}(0) & (386) \\ &= W(t) \overline{\rho_S}(0). & (387) \end{aligned}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]] \cdots]. \tag{388}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\begin{aligned} \frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) \overline{\rho_S}(0) & (389) \\ &= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} W(t) \overline{\rho_S}(0) & (390) \\ &= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} \widetilde{\overline{\rho_S}}(t). & (391) \end{aligned}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (389) becomes:

$$\begin{aligned} \frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= \dot{W}_2(t) \widetilde{\overline{\rho_S}}(t) & (392) \\ &= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B \left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]. & (393) \end{aligned}$$

Replacing $t_1 \rightarrow t-\tau$

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}\left(t\right),\left[\widetilde{\overline{H_{\bar{I}}}}\left(-\tau\right),\overline{\rho_S}\left(t\right)\rho_B\right]\right]. \tag{394}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}\left(t\right)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right) = U^{\dagger}\left(t\right)e^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}\left(t\right)e^{-\mathrm{i}H_B t}U\left(t\right). \tag{395}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right) = \sum_i C_i\left(t\right)\left(\widetilde{A_i}\left(t\right)\otimes\widetilde{B_i}\left(t\right)\right), \tag{396}$$

$$\widetilde{A_i}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) \tag{397}$$

$$= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} \tag{398}$$

$$= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathbb{I} \tag{399}$$

$$= U^{\dagger}\left(t\right)A_i U\left(t\right), \tag{400}$$

$$\widetilde{B_i}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) \tag{401}$$

$$= U^{\dagger}\left(t\right)U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} \tag{402}$$

$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} \tag{403}$$

$$= \mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}. \tag{404}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}\left(t\right),H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right),\mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (396) to replace it in the equation (393)

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}\left(t\right),\left[\widetilde{\overline{H_{\bar{I}}}}\left(t'\right),\widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' \tag{405}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j\left(t\right)\left(\widetilde{A_j}\left(t\right)\otimes\widetilde{B_j}\left(t\right)\right),\left[\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right),\widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' \tag{406}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right),\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\widetilde{\overline{\rho_S}}(t)\rho_B - \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\right]\mathrm{d}t' \tag{407}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\widetilde{\overline{\rho_S}}(t)\rho_B - \sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right) \tag{408}$$

$$- \sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right) + \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\Big)\mathrm{d}t'. \tag{409}$$

In order to calculate the correlation functions we define:

$$\mathscr{B}_{ij}\left(t,t'\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(t'\right)\rho_B\right). \tag{410}$$

An useful property is

$$\mathscr{B}^{*}_{ji}\left(t,t'\right) = \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right)^{\dagger} \tag{411}$$

$$= \mathrm{Tr}_B\left(\rho_B^{\dagger}\widetilde{B_i}^{\dagger}\left(t'\right)\widetilde{B_j}^{\dagger}\left(t\right)\right) \tag{412}$$

$$= \mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) \tag{413}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{414}$$

$$= \mathscr{B}_{ij}\left(t',t\right). \tag{415}$$

The correlation functions relevant that appear in the equation (409) are:

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) = \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\right\rangle_B \tag{416}$$

$$= \mathscr{B}_{ji}\left(t,t'\right) \tag{417}$$

$$= \mathscr{B}^*_{ij}\left(t',t\right) \tag{418}$$

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B\widetilde{B_i}\left(t'\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{419}$$

$$= \mathscr{B}_{ij}\left(t',t\right) \tag{420}$$

$$\mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\rho_B\widetilde{B_j}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) \tag{421}$$

$$= \mathscr{B}^*_{ij}\left(t',t\right) \tag{422}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{423}$$

$$= \mathscr{B}_{ij}\left(t',t\right) \tag{424}$$

The cyclic property of the trace was use widely in the development of equations (416) and (424). Replacing in (409)

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big) \tag{425}$$

$$-\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}t'. \tag{426}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \tag{427}$$

$$+\sum_{ij} C_i(t')C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{428}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \tag{429}$$

$$+\sum_{ij} C_i(t')C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{430}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \text{ (by permuting i and j because i,j}\in\text{J)} \tag{431}$$

$$+\sum_{ij} C_i(t')C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{432}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t') \tag{433}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{434}$$

$$= -\int_0^t \Big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\mathscr{B}_{ji}(t,t') - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\mathscr{B}_{ij}(t',t) \tag{435}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\mathscr{B}_{ij}(t',t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\mathscr{B}_{ji}(t,t')\big)\Big)\mathrm{d}t' \tag{436}$$

$$= -\int_0^t\left(\sum_{ij} C_j\left(t\right)C_i\left(t'\right)\left(\mathscr{B}_{ji}\left(t,t'\right)\left[\widetilde{A_j}\left(t\right),\widetilde{A_i}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \mathscr{B}_{ij}\left(t',t\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t'\right),\widetilde{A_j}\left(t\right)\right]\right)\right)\mathrm{d}t' \tag{437}$$

$$= -\int_0^t\Big(\sum_{ij} C_i(t)C_j(t')\big(\mathscr{B}_{ij}(t,t')\big[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)\big] + \mathscr{B}_{ji}(t',t)\big[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t'),\widetilde{A_i}(t)\big]\big)\Big)\mathrm{d}t' \text{ (exchanging i and j)} \tag{438}$$

$$= -\int_0^t\left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathscr{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \mathscr{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}t' \tag{439}$$

$$= -\int_0^t\left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathscr{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] - \mathscr{B}^*_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right)\right]\right)\right)\mathrm{d}t' \tag{440}$$

We could identify the following commutators in the equation deduced:

$$\mathscr{B}_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right) - \mathscr{B}_{ij}\left(t,t'\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right) = \mathscr{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\rho_s}\left(t\right)\right], \tag{441}$$

$$\mathscr{B}^*_{ij}\left(t,t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right) - \mathscr{B}^*_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right) = \mathscr{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]. \tag{442}$$

Returning to the Schroedinger picture we have:

$$U(t)\widetilde{A_i}(t)\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)U^\dagger(t) = U(t)\widetilde{A_i}(t)U^\dagger(t)U(t)\widetilde{A_j}(t')U^\dagger(t)U(t)\widetilde{\widetilde{\rho_S}}(t)U^\dagger(t), \tag{443}$$

$$= \left(U(t)\widetilde{A_i}(t)U^\dagger(t)\right)\left(U(t)\widetilde{A_j}(t')U^\dagger(t)\right)\left(U(t)\widetilde{\widetilde{\rho_S}}(t)U^\dagger(t)\right), \tag{444}$$

$$= A_i(t)\widetilde{A_j}(t',t)\overline{\rho_S}(t). \tag{445}$$

This procedure applying to the relevant commutators give us:

$$U(t)\left[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)\right]U^\dagger(t) = \left(U(t)\widetilde{A_i}(t)\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)U^\dagger(t) - U(t)\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t)U^\dagger(t)\right) \tag{446}$$

$$= A_i(t)\widetilde{A_j}(t',t)\overline{\rho_S}(t) - \widetilde{A_j}(t',t)\overline{\rho_S}(t)A_i \tag{447}$$

$$= \left[A_i(t),\widetilde{A_j}(t',t)\overline{\rho_S}(t)\right]. \tag{448}$$

Introducing this transformed commutators in the equation (440) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathscr{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}s C_i(t)C_j(t')\left(\mathscr{B}_{ij}(t,t')\left[A_i(t),\widetilde{A_j}(t',t)\overline{\rho_S}(t)\right] + \mathscr{B}^*_{ij}(t,t')\left[\overline{\rho_S}(t)\widetilde{A_j}(t',t),A_i\right]\right), \tag{449}$$

$$t' = t-\tau \text{ (Change of variables in the integration process)}, \tag{450}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)C_j(t')\left(\mathscr{B}_{ij}(t,t')\left[A_i(t),\widetilde{A_j}(t',t)\overline{\rho_S}(t)\right] + \mathscr{B}^*_{ij}(t,t')\left[\overline{\rho_S}(t)\widetilde{A_j}(t',t),A_i(t)\right]\right). \tag{451}$$

where $i,j\in\{1,2,3,4,5.6\}$ and $t'=t-\tau$.
Here $\widetilde{A_j}(t-\tau,t) = U(t)U^\dagger(t-\tau)A_j(t)U(t-\tau)U^\dagger(t)$ where $U(t)$ is given by (374). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. In order to write in a simplified way we define the following notation:

$$\mathscr{B}_{ij}(t,t') = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\widetilde{B_j}(t')\rho_B\right) \tag{452}$$

$$= \mathrm{Tr}_B\left(e^{iH_Bt}B_i(t)e^{-iH_Bt}e^{iH_Bt'}B_j(t')e^{-iH_Bt'}\rho_B\right) \tag{453}$$

$$e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!} \tag{454}$$

$$e^{-iH_Bt'}e^{-\beta H_B} = \sum_{m=0}^{\infty}\frac{(-iH_Bt')^m}{m!}\sum_{n=0}^{\infty}\frac{(-\beta H_B)^n}{n!} \tag{455}$$

$$= \sum_{m,n}\frac{(-iH_Bt')^m}{m!}\frac{(-\beta H_B)^n}{n!} \tag{456}$$

$$= \sum_{m,n}\frac{(-it')^m}{m!}\frac{(-\beta)^n}{n!}H_B^mH_B^n \tag{457}$$

$$= \sum_{m,n}\frac{(-it')^m}{m!}\frac{(-\beta)^n}{n!}H_B^nH_B^m \text{ (because the powers of a matrix commute)} \tag{458}$$

$$= \sum_{m,n}\frac{(-\beta)^n}{n!}H_B^n\frac{(-it')^m}{m!}H_B^m \tag{459}$$

$$= \sum_{m,n}\frac{(-\beta H_B)^n}{n!}\frac{(-it'H_B)^m}{m!} \tag{460}$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_B t')^m}{m!} \tag{461}$$

$$= e^{-\beta H_B} e^{-iH_B t'} \tag{462}$$

$$0 = e^{-iH_B t'} e^{-\beta H_B} - e^{-\beta H_B} e^{-iH_B t'} \text{ (then } e^{-iH_B t'} \text{ and } \rho_B \text{ commute)} \tag{463}$$

$$\mathscr{B}_{ij}(t,t') = \mathrm{Tr}_B \left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \rho_B e^{-iH_B t'} \right) \text{ (by permuting } e^{-iH_B t'} \text{ and } \rho_B) \tag{464}$$

$$= \mathrm{Tr}_B \left(\left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \right) \rho_B e^{-iH_B t'} \right) \text{ (by associative property)} \tag{465}$$

$$= \mathrm{Tr}_B \left(e^{-iH_B t'} \left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \right) \rho_B \right) \text{ (by cyclic property of the trace)} \tag{466}$$

$$= \mathrm{Tr}_B \left(\left(e^{-iH_B t'} e^{iH_B t} \right) B_i(t) \left(e^{-iH_B t} e^{iH_B t'} \right) B_j(t') \rho_B \right) \text{ (by associative property)} \tag{467}$$

$$[iH_B t, -iH_B t'] = iH_B t(-iH_B t') - (-iH_B t') iH_B t \tag{468}$$

$$= tt' H_B^2 - tt' H_B^2 \tag{469}$$

$$= 0 \text{ (so } iH_B t \text{ and } -iH_B t' \text{ commute)} \tag{470}$$

$$e^{-iH_B t'} e^{iH_B t} = e^{iH_B t - iH_B t'} \text{ (by the Zassenhaus formula because } iH_B t \text{ and } -iH_B t' \text{ commute)} \tag{471}$$

$$= e^{iH_B (t-t')} \tag{472}$$

$$= e^{iH_B \tau} \tag{473}$$

$$e^{iH_B t'} e^{-iH_B t} = e^{-iH_B t + iH_B t'} \text{ (by the Zassenhaus formula because } -iH_B t \text{ and } iH_B t' \text{ commute)} \tag{474}$$

$$= e^{iH_B (-t+t')} \tag{475}$$

$$= e^{-iH_B \tau} \tag{476}$$

$$\mathscr{B}_{ij}(t,t') = \mathrm{Tr}_B \left(e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(t') \rho_B \right) \tag{477}$$

$$B_i(t,\tau) \equiv e^{iH_B \tau} B_i(t) e^{-iH_B \tau} \tag{478}$$

$$\mathscr{B}_{ij}(t,t') = \mathrm{Tr}_B \left(e^{iH_B (t-t')} B_i(t) e^{-iH_B (t-t')} B_j(t') \rho_B \right) \tag{479}$$

$$t' = t - \tau \tag{480}$$

$$\mathscr{B}_{ij}(t,t') = \mathrm{Tr}_B \left(e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(t') \rho_B \right) \tag{481}$$

$$= \mathrm{Tr}_B \left(B_i(t,\tau) B_j(t',0) \rho_B \right) \tag{482}$$

Calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(t') \right\rangle_B = \mathrm{Tr}_B \left(B_{jz}(t,\tau) B_{jz}(t',0) \rho_B \right) \tag{483}$$

$$= \int \mathrm{d}^2\alpha P(\alpha) \left\langle \alpha \left| B_{jz}(t,\tau) B_{jz}(t',0) \right| \alpha \right\rangle \tag{484}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N} \right) \left\langle \alpha \left| B_{jz}(t,\tau) B_{jz}(t',0) \right| \alpha \right\rangle \mathrm{d}^2\alpha \tag{485}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N} \right) \left\langle \alpha \left| B_{jz}(t,\tau) B_{jz}(t',0) \right| \alpha \right\rangle \mathrm{d}^2\alpha, \tag{486}$$

$$B_{jz}(t,\tau) = \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), \tag{487}$$

$$B_{jz}(t',0) = \sum_{\mathbf{k}'} \left(\left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(t') \right) b_{\mathbf{k}'}^\dagger + \left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(t') \right)^* b_{\mathbf{k}'} \right), \tag{488}$$

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(t') \right\rangle_B = \mathrm{Tr}_B \left(B_{jz}(t,\tau) B_{jz}(t',0) \rho_B \right) \tag{489}$$

$$= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left(\left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(t') \right) b_{\mathbf{k}'}^\dagger + \left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(t') \right)^* b_{\mathbf{k}'} \right) \rho_B \right) \tag{490}$$

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(t') \right\rangle_B = \mathrm{Tr}_B \left(B_{jz}(t,\tau) B_{jz}(t',0) \rho_B \right) \tag{491}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'}(t'))b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}(t'))^{*}b_{\mathbf{k}'}\right)\rho_B\right) \tag{492}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'}(t'))b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}(t'))^{*}b_{\mathbf{k}'}\right)\rho_B\right) \tag{493}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t'))b_{\mathbf{k}}^{\dagger}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t'))^{*}b_{\mathbf{k}}\right)\rho_B\right), \tag{494}$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)=q_{j\mathbf{k}}(t) \tag{495}$$

$$\left\langle\widetilde{B_{jz}}(t)\widetilde{B_{jz}}(t')\right\rangle_B=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}(t')b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}(t')b_{\mathbf{k}'}\right)\rho_B\right) \tag{496}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}(t')b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}\right)\rho_B\right) \tag{497}$$

$$0=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}(t')b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}(t')b_{\mathbf{k}'}\right)\rho_B\right) \tag{498}$$

$$\left\langle\widetilde{B_{jz}}(t)\widetilde{B_{jz}}(t')\right\rangle_B=0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}(t')b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}\right)\rho_B\right) \tag{499}$$

$$=\mathrm{Tr}_B\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B \tag{500}$$

$$=\sum_{\mathbf{k}}\mathrm{Tr}_B\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{501}$$

$$+\mathrm{Tr}_B\left(q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{502}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{503}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right)+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right) \tag{504}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{505}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{506}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{507}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{508}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{509}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{510}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{511}$$

$$=\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right. \tag{512}$$

$$\left.+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{513}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right. \tag{514}$$

$$\left.+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{515}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\sum_{\mathbf{k}}\left(\left\langle 0\left|q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right)+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right)\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{516}$$

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$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \left\langle 0 \middle| \left|\alpha_{\mathbf{k}}\right|^{2} \middle| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \left\langle 0 \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \middle| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{526}$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \left\langle 0 \middle| \left|\alpha_{\mathbf{k}}\right|^{2} \middle| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right), \tag{527}$$

$$1 = \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \mathrm{d}^{2}\alpha_{\mathbf{k}}, \tag{528}$$

$$b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \left|0\right\rangle = 0, \tag{529}$$

$$b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \left|0\right\rangle = \left|0\right\rangle, \tag{530}$$

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \right\rangle_{B} = \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \left\langle 0 \middle| \left|\alpha_{\mathbf{k}}\right|^{2} \middle| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \left\langle 0 \middle| \left|\alpha_{\mathbf{k}}\right|^{2} \middle| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{531}$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{532}$$

$$= \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \left|\alpha_{\mathbf{k}}\right|^{2} \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \mathrm{d}^{2}\alpha_{\mathbf{k}} + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\frac{1}{\pi N} \int \left|\alpha_{\mathbf{k}}\right|^{2} \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \mathrm{d}^{2}\alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \right) \tag{533}$$

$$= \sum_{\mathbf{k}} \left(\left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \frac{1}{\pi N} \int \left|\alpha_{\mathbf{k}}\right|^{2} \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \mathrm{d}^{2}\alpha_{\mathbf{k}} + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{534}$$

$$\frac{1}{\pi N} \int_{0}^{2\pi} \int_{0}^{\infty} r^{2} \exp\left(-\frac{r^{2}}{N} \right) r \mathrm{d}r \mathrm{d}\theta = \frac{1}{\pi N} \int \left|\alpha_{\mathbf{k}}\right|^{2} \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^{2}}{N} \right) \mathrm{d}^{2}\alpha_{\mathbf{k}} \tag{535}$$

$$= N \tag{536}$$

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \right\rangle_{B} = \sum_{\mathbf{k}} \left(\left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) N + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \tag{537}$$

$$\left\langle\widetilde{B_{jz}}(t)\widetilde{B_{j'z}}(s)\right\rangle_B = \mathrm{Tr}_B\left(B_{jz}\left(t,\tau\right)B_{j'z}\left(s,0\right)\rho_B\right) \tag{538}$$

$$= \int \mathrm{d}^2\alpha P\left(\alpha\right)\left\langle\alpha\left|B_{jz}\left(t,\tau\right)B_{j'z}\left(s,0\right)\right|\alpha\right\rangle \tag{539}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha\right|^2}{N}\right)\left\langle\alpha\left|B_{jz}\left(t,\tau\right)B_{j'z}\left(s,0\right)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{540}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)b^\dagger_{\mathbf{k}'}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{541}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)b^\dagger_{\mathbf{k}'}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{542}$$

$$+\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)b^\dagger_{\mathbf{k}}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{543}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)b^\dagger_{\mathbf{k}}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{544}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)^*b^\dagger_{\mathbf{k}}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{545}$$

$$+ \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)b_{\mathbf{k}}b^\dagger_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{546}$$

$$= \sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{547}$$

$$+ \sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{548}$$

$$\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b^\dagger_{\mathbf{k}}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{549}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b^\dagger_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{550}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{551}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left|\alpha_{\mathbf{k}}\right|^2\mathrm{d}^2\alpha_{\mathbf{k}} \tag{552}$$

$$= N, \tag{553}$$

$$\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b^\dagger_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{554}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{555}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}+\alpha_{\mathbf{k}}b^\dagger_{\mathbf{k}}+b_{\mathbf{k}}\alpha^*_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{556}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{557}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left|\alpha_{\mathbf{k}}\right|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{558}$$

$$= N + 1, \tag{559}$$

$$\left\langle \widetilde{B_{jz}}(t)\widetilde{B_{j'z}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\left(s\right)\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\left(s\right)\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left(N+1\right) \tag{560}$$

$$= \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\left(s\right)\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)^* \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\left(s\right)\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) N + \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v \right. \tag{561}$$

$$= \sum_{\mathbf{k}} 2N \left(q_{j\mathbf{k}}\left(t\right) q^*_{j'\mathbf{k}}\left(s\right) e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Re} + \sum_{\mathbf{k}} q^*_{j\mathbf{k}}\left(t\right) q_{j'\mathbf{k}}\left(s\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{562}$$

$$D\left(h'\right) D\left(h\right) = \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) D\left(h'+h\right), \tag{563}$$

$$\left\langle D\left(h'\right) D\left(h\right)\right\rangle_B = \mathrm{Tr}_B\left(\exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) D\left(h'+h\right)\rho_B\right) \tag{564}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \mathrm{Tr}_B\left(D\left(h'+h\right)\rho_B\right) \tag{565}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \frac{1}{\pi N}\int \mathrm{d}^2\alpha P\left(\alpha\right)\left\langle \alpha\left|D\left(h'+h\right)\right|\alpha\right\rangle \tag{566}$$

$$= \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \exp\left(-\frac{\left|h+h'\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{567}$$

$$h' = h\exp\left(\mathrm{i}\omega\tau\right), \tag{568}$$

$$\left\langle D\left(h\exp\left(\mathrm{i}\omega\tau\right)\right) D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(hh^*\exp\left(\mathrm{i}\omega\tau\right) - h^*h\exp\left(-\mathrm{i}\omega\tau\right)\right)\right) \exp\left(-\frac{\left|h+h\exp\left(\mathrm{i}\omega\tau\right)\right|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{569}$$

$$\frac{1}{2}\left|h\right|^2\left(\exp(\mathrm{i}\omega\tau) - \exp(-\mathrm{i}\omega\tau)\right) = \frac{1}{2}\left(hh^*\exp\left(\mathrm{i}\omega\tau\right) - h^*h\exp\left(-\mathrm{i}\omega\tau\right)\right) \tag{570}$$

$$= \frac{1}{2}\left|h\right|^2\left(\cos\left(\omega\tau\right) + \mathrm{i}\sin\left(\omega\tau\right) - \cos\left(\omega\tau\right) + \mathrm{i}\sin\left(\omega\tau\right)\right) \tag{571}$$

$$= \frac{1}{2}\left|h\right|^2\left(2\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{572}$$

$$= \mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right), \tag{573}$$

$$-\frac{\left|h+h\exp\left(\mathrm{i}\omega\tau\right)\right|^2}{2} = -\left|h\right|^2\frac{\left|1+\exp\left(\mathrm{i}\omega\tau\right)\right|^2}{2} \tag{574}$$

$$= -\left|h\right|^2\frac{\left(1+2\cos\left(\omega\tau\right)+\cos^2\left(\omega\tau\right)\right)+\sin^2\left(\omega\tau\right)}{2} \tag{575}$$

$$= -\left|h\right|^2\frac{2+2\cos\left(\omega\tau\right)}{2} \tag{576}$$

$$= -\left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right), \tag{577}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\right\rangle_B = \exp\left(\mathrm{i}|h|^2\sin(\omega\tau)\right)\exp\left(-|h|^2(1+\cos(\omega\tau))\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{578}$$

$$= \exp\left(\mathrm{i}\left|h\right|^2 \sin\left(\omega\tau\right) - \left|h\right|^2 \left(1 + \cos\left(\omega\tau\right)\right) \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{579}$$

$$= \exp\left(-\left|h\right|^2 \left(-\mathrm{i}\sin\left(\omega\tau\right) + \cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\right) \exp\left(-\left|h\right|^2 \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{580}$$

$$= \left\langle D\left(h\right)\right\rangle_B \exp\left(-\phi\left(\tau\right)\right), \tag{581}$$

$$\exp\left(-\phi\left(\tau\right)\right) = \exp\left(-\left|h\right|^2 \left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right)\right), \tag{582}$$

$$\phi\left(\tau\right) = \left|h\right|^2 \left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right), \tag{583}$$

$$\left\langle D\left(h'\right) D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(h'h^* - h'^*h\right)\right) \exp\left(-\frac{\left|h + h'\right|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \tag{584}$$

$$h' = v\exp\left(i\omega\tau\right), \tag{585}$$

$$\left\langle \widetilde{B_1^+B_0^-}(t)\widetilde{B_1^+B_0^-}(s)\right\rangle_B = \left\langle B_1^+B_0^-\left(t,\tau\right)B_1^+B_0^-\left(s,0\right)\right\rangle_B \tag{586}$$

$$= \left\langle B_{10}\left(t,\tau\right)B_{10}\left(s,0\right)\right\rangle_B \tag{587}$$

$$= \mathrm{Tr}_B\left(B_{10}\left(t,\tau\right)B_{10}\left(s,0\right)\rho_B\right) \tag{588}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}(t)} - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(s)v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)v}{\omega_{\mathbf{k}}^2}\right.}\right.\right. \tag{589}$$

$$= \exp\left(\chi_{10}\left(t\right) + \chi_{10}\left(s\right)\right) \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{590}$$

$$= \exp\left(\chi_{10}\left(t\right) + \chi_{10}\left(s\right)\right)\prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{591}$$

$$= \exp\left(\chi_{10}\left(t\right) + \chi_{10}\left(s\right)\right)\prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)^* - \left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\right.\right.\right. \tag{592}$$

$$= \exp\left(\chi_{10}\left(t\right) + \chi_{10}\left(s\right)\right)\prod_{\mathbf{k}}\left(\exp\left(i\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)^*\right)^{\Im}\right)\exp\left(-\frac{\left|\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e\right.}{}\right.\right. \tag{593}$$

$$= \exp\left(\chi_{10}\left(t\right) + \chi_{10}\left(s\right)\right)\prod_{\mathbf{k}}\left(\exp\left(i\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)^*\right)^{\Im}\right)\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\right.\right.}{}\right.\right. \tag{594}$$

$$\left\langle \widetilde{B_0^+B_1^-}(t)\widetilde{B_0^+B_1^-}(s)\right\rangle_B = \exp\left(\chi_{01}\left(t\right) + \chi_{01}\left(s\right)\right)\prod_{\mathbf{k}}\left(\exp\left(i\left(\frac{v_{0\mathbf{k}}\left(t\right) - v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{0\mathbf{k}}\left(s\right) - v_{1\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)^*\right)^{\Im}\right)\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\right.\right.}{}\right.\right. \tag{595}$$

$$\left\langle D\left(h\right)b\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{\left|\alpha\right|^2}{2}\right)\left\langle \alpha\left|D\left(h\right)b\right|\alpha\right\rangle \tag{596}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \langle \alpha \left| D(-\alpha) D(h) b D(\alpha) \right| \alpha \rangle \tag{597}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \langle 0 \left| D(-\alpha) D(h) b D(\alpha) \right| 0 \rangle \tag{598}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \langle 0 \left| D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) \right| 0 \rangle \tag{599}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \langle 0 \left| D(-\alpha) D(h) D(\alpha) (b+\alpha) \right| 0 \rangle \tag{600}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 \left| D(h) (b+\alpha) \right| 0 \rangle \tag{601}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha | 0 \rangle \tag{602}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 \left| D(h) \alpha \right| 0 \rangle \tag{603}$$

$$= \frac{1}{\pi N} \int \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|h|^2}{2}\right) \mathrm{d}^2\alpha \tag{604}$$

$$= hN \langle D(h) \rangle_B , \tag{605}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| D(h) b^\dagger \right| \alpha \right\rangle \tag{606}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle 0 \left| D(-\alpha) D(h) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{607}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle 0 \left| D(-\alpha) D(h) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{608}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle 0 \left| D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) \right| 0 \right\rangle \tag{609}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle 0 \left| D(-\alpha) D(h) D(\alpha) \left(b^\dagger + \alpha^*\right) \right| 0 \right\rangle \tag{610}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \left\langle 0 \left| D(h) \left(b^\dagger + \alpha^*\right) \right| 0 \right\rangle \tag{611}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \left\langle 0 \left| D(h) b^\dagger \right| 0 \right\rangle + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \tag{612}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) | 1 \rangle + \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \tag{613}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle -h|1\rangle + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle, \tag{614}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right)\sum_n \frac{(-h^*)^n}{\sqrt{n!}}\langle n|, \tag{615}$$

$$\langle -h|1\rangle = \exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*), \tag{616}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|-h^*|^2}{2}\right)(-h^*) + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\exp\left(-\tfrac{|-h^*|^2}{2}\right) \tag{617}$$

$$= -h^*\langle D(h)\rangle_B (N+1), \tag{618}$$

$$\langle bD(h)\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle \alpha|bD(h)|\alpha\rangle \tag{619}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{620}$$

$$= h\langle D(h)\rangle_B (N+1), \tag{621}$$

$$\left\langle b^\dagger D(h)\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle \alpha\left|b^\dagger D(h)\right|\alpha\right\rangle \tag{622}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{623}$$

$$= -h^*\langle D(h)\rangle_B N, \tag{624}$$

The correlation functions can be found readily as:

$$B_1^+B_0^-\left(t,\tau\right)=\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}^2}\right)\right)\right), \tag{625}$$

$$B_0^+B_1^-\left(t,\tau\right)=\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*\left(t\right)v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)v_{1\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}^2}\right)\right)\right), \tag{626}$$

$$B_{10}\left(t\right)=\left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}^2}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{627}$$

$$B_x\left(t,\tau\right)=\frac{B_1^+B_0^-\left(t,\tau\right)+B_0^+B_1^-\left(t,\tau\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}, \tag{628}$$

$$B_y\left(t,\tau\right)=\frac{B_0^+B_1^-\left(t,\tau\right)-B_1^+B_0^-\left(t,\tau\right)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}, \tag{629}$$

$$B_{iz}\left(t,\tau\right)=\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{630}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s)\right\rangle_B=\langle B_{iz}(t,\tau)B_{jz}(s,0)\rangle_B \tag{631}$$

$$=\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(s\right)\right)b_{\mathbf{k}}^\dagger+\left(g_{j\mathbf{k}}-v_{jk}\left(s\right)\right)^*b_{\mathbf{k}}\right)\right\rangle_B \tag{632}$$

$$=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(s\right)\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(s\right)\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{633}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B=\langle B_x(t,\tau)B_x(s,0)\rangle_B \tag{634}$$

$$=\left\langle\left(\frac{B_1^+B_0^-\left(t,\tau\right)+B_0^+B_1^-\left(t,\tau\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\right)\left(\frac{B_1^+B_0^-\left(s,0\right)+B_0^+B_1^-\left(s,0\right)-B_{10}\left(s\right)-B_{01}\left(s\right)}{2}\right)\right\rangle_B \tag{635}$$

$$=\frac{1}{4}\left\langle\left(B_1^+B_0^-\left(t,\tau\right)+B_0^+B_1^-\left(t,\tau\right)-B_{10}\left(t\right)-B_{01}\left(t\right)\right)\left(B_1^+B_0^-\left(s,0\right)+B_0^+B_1^-\left(s,0\right)-B_{10}\left(s\right)-B_{01}\left(s\right)\right)\right\rangle_B \tag{636}$$

$$=\tfrac{1}{4}\Big\langle B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0)+B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0)-B_1^+B_0^-(t,\tau)B_{10}(s)-B_1^+B_0^-(\tau)B_{01}(s)+B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0)+B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0)-B_0^+ \tag{637}$$

$$-B_0^+B_1^-(t,\tau)B_{01}(s)-B_{10}(t)B_1^+B_0^-(s,0)-B_{10}(t)B_0^+B_1^-(s,0)+B_{10}(t)B_{10}(s)+B_{10}(t)B_{01}(s)-B_{01}(t)B_1^+B_0^-(s,0)-B_{01}(t)B_0^+B_1^-(s,0)+B_{01}(t)B_{10}(s)+B_{01}(\tag{638}$$

$$=\frac{1}{4}\langle B_1^+B_0^-(t,\tau)B_1^+B_0^-\left(s,0\right)+B_1^+B_0^-(t,\tau)B_0^+B_1^-\left(s,0\right)+B_0^+B_1^-(t,\tau)B_1^+B_0^-\left(s,0\right) \tag{639}$$

$$+B_0^+B_1^-(t,\tau)B_0^+B_1^-\left(s,0\right)\rangle-\frac{\left(B_{01}\left(t\right)+B_{10}\left(t\right)\right)\left(B_{01}\left(s\right)+B_{10}\left(s\right)\right)}{4}, \tag{640}$$

$$U_{10}\left(t,s\right)=\prod_{\mathbf{k}}\exp\left(i\left(\frac{\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)\right)^*\exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{641}$$

$$\left\langle B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0)\right\rangle_B=\exp(\chi_{10}(t)+\chi_{10}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{642}$$

$$\left\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0)\right\rangle_B=\exp(\chi_{01}(t)+\chi_{01}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{643}$$

$$\left\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0)\right\rangle_B=\left\langle\prod_{\mathbf{k}}\left(D\left(\tfrac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\tfrac{1}{2}\left(\tfrac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\left(D\left(\tfrac{v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\exp\left(\tfrac{1}{2}\left(\tfrac{v_{0\mathbf{k}}^*(s)v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)v_{1\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2}\right)\right)\right) \tag{644}$$

$$=\exp(\chi_{10}(t)+\chi_{01}(s))\left\langle\prod_{\mathbf{k}}\left(D\left(\tfrac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\prod_{\mathbf{k}}\left(D\left(\tfrac{v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{645}$$

$$=\exp(\chi_{10}(t)+\chi_{01}(s))\prod_{\mathbf{k}}\left\langle\left(D\left(\tfrac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\tfrac{v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{646}$$

$$=\exp(\chi_{10}(t)+\chi_{01}(s))U_{10}^*\left(t,s\right)\prod_{\mathbf{k}}\exp\left(-\tfrac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{647}$$

$$\left\langle B_0^+ B_1^-(t,\tau) B_1^+ B_0^-(s,0) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(s) v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s) v_{0\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2} \right) \right) \tag{648}$$

$$= \prod_{\mathbf{k}} \exp\left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(s) v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s) v_{0\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle D\left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right\rangle_B \tag{649}$$

$$= \exp\left(\chi_{01}(t) + \chi_{10}(s)\right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right\rangle_B \tag{650}$$

$$= \exp\left(\chi_{01}(t) + \chi_{10}(s)\right) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(- \frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{651}$$

$$\left\langle \widetilde{B_x}(t) \widetilde{B_x}(s) \right\rangle_B = \frac{1}{4} \langle B_1^+ B_0^-(t,\tau) B_1^+ B_0^-(s,0) + B_1^+ B_0^-(t,\tau) B_0^+ B_1^-(s,0) + B_0^+ B_1^-(t,\tau) B_1^+ B_0^-(s,0) = \tag{652}$$

$$+ B_0^+ B_1^-(t,\tau) B_0^+ B_1^-(s,0) \rangle - \frac{\left(B_{01}(t) + B_{10}(t)\right)\left(B_{01}(s) + B_{10}(s)\right)}{4}, \tag{653}$$

$$= \frac{1}{4} \left(\exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right. \tag{654}$$

$$+ \exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{655}$$

$$+ \exp\left(\chi_{01}(t) + \chi_{10}(s)\right) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(- \frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{656}$$

$$\left. + \exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{657}$$

$$- \left(\exp(\chi_{10}(t)) \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \left(\exp(\chi_{01}(s)) \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \tag{658}$$

$$= \frac{1}{2} \left(\left(\exp\left(\chi_{10}(t) + \chi_{10}(s)\right) \right)^{\Re} U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right. \tag{659}$$

$$\left. + \left(\exp\left(\chi_{10}(t) + \chi_{01}(s)\right) \right)^{\Re} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{660}$$

$$- \left(\exp(\chi_{10}(t)) \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \left(\exp(\chi_{01}(s)) \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \tag{661}$$

$$\left\langle \widetilde{B_y}(t) \widetilde{B_y}(s) \right\rangle_B = \langle B_y(t,\tau) B_y(s,0) \rangle_B \tag{662}$$

$$= \left\langle \left(\frac{B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \right) \left(\frac{B_0^+ B_1^-(s,0) - B_1^+ B_0^-(s,0) + B_{10}(s) - B_{01}(s)}{2\mathrm{i}} \right) \right\rangle \tag{663}$$

$$= -\frac{1}{4} \left\langle \left(B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t) \right) \left(B_0^+ B_1^-(s,0) - B_1^+ B_0^-(s,0) + B_{10}(s) - B_{01}(s) \right) \right\rangle \tag{664}$$

$$\tag{665}$$

$$= -\tfrac{1}{4}\Bigl\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) + B_0^+B_1^-(t,\tau)B_{10}(s) - B_0^+B_1^-(\tau)B_{01}(s) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0) \tag{666}$$

$$+B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0) - B_1^+B_0^-(t,\tau)B_{10}(s) + B_1^+B_0^-(t,\tau)B_{01}(s) + B_{10}(t)B_0^+B_1^-(s,0) - B_{10}(t)B_1^+B_0^-(s,0) + B_{10}(t)B_{10}(s) \tag{667}$$

$$-B_{10}(t)B_{01}(s) - B_{01}(t)B_0^+B_1^-(s,0) + B_{01}(t)B_1^+B_0^-(s,0) - B_{01}(t)B_{10}(s) + B_{01}(t)B_{01}(s)\Bigr\rangle \tag{668}$$

$$= -\tfrac{1}{4}\Bigl\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0) + B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0)\Bigr\rangle \tag{669}$$

$$+(B_{01}(t))^{\Im}(B_{10}(s))^{\Im} \tag{670}$$

$$= -\frac{1}{4}\Biggl(\exp(\chi_{01}(t)+\chi_{01}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\Bigl(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr) \tag{671}$$

$$-\exp(\chi_{01}(t)+\chi_{10}(s))U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\Bigl(-\tfrac{|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr) \tag{672}$$

$$-\exp(\chi_{10}(t)+\chi_{01}(s))U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\Bigl(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr) \tag{673}$$

$$+\exp(\chi_{10}(t)+\chi_{10}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\Bigl(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr)\Biggr) \tag{674}$$

$$+\Bigl(\exp(\chi_{01}(t))\Bigl(\exp\Bigl(-\tfrac{1}{2}\sum_{\mathbf{k}}\Bigl|\tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\Bigr|^2\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr)\Bigr)\Bigr)^{\Im}\Bigl(\exp(\chi_{10}(s))\Bigl(\exp\Bigl(-\tfrac{1}{2}\sum_{\mathbf{k}}\Bigl|\tfrac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Bigr|^2\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr)\Bigr)\Bigr)^{\Im} \tag{675}$$

$$= -\tfrac{1}{2}\Bigl((\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Re}U_{10}(t,s)\prod_{\mathbf{k}}\exp\Bigl(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr) \tag{676}$$

$$-(\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Re}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\Bigl(-\tfrac{|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr)\Bigr) \tag{677}$$

$$+\Bigl(\exp(\chi_{01}(t))\Bigl(\exp\Bigl(-\tfrac{1}{2}\sum_{\mathbf{k}}\Bigl|\tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\Bigr|^2\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr)\Bigr)\Bigr)^{\Im}\Bigl(\exp(\chi_{10}(s))\Bigl(\exp\Bigl(-\tfrac{1}{2}\sum_{\mathbf{k}}\Bigl|\tfrac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Bigr|^2\coth\bigl(\tfrac{\beta\omega_{\mathbf{k}}}{2}\bigr)\Bigr)\Bigr)\Bigr)^{\Im} \tag{678}$$

$$\tag{679}$$

$$\tag{680}$$

$$\Bigl\langle\widetilde{B_x}(t)\widetilde{B_y}(s)\Bigr\rangle_B = \langle B_x(t,\tau)B_y(s,0)\rangle_B \tag{681}$$

$$= \Biggl\langle\Biggl(\frac{B_1^+B_0^-(t,\tau)+B_0^+B_1^-(t,\tau)-B_{10}(t)-B_{01}(t)}{2}\Biggr)\Biggl(\frac{B_0^+B_1^-(s,0)-B_1^+B_0^-(s,0)+B_{10}(s)-B_{01}(s)}{2\mathrm{i}}\Biggr)\Biggr\rangle_B \tag{682}$$

$$= \frac{1}{4i}\bigl\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0) - B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0) + B_1^+B_0^-(t,\tau)B_{10}(s) - B_1^+B_0^-(t,\tau)B_{01}(s) \tag{683}$$

$$+B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) + B_0^+B_1^-(t,\tau)B_{10}(s) - B_0^+B_1^-(t,\tau)B_{01}(s) \tag{684}$$

$$-B_{10}(t)B_0^+B_1^-(s,0) + B_{10}(t)B_1^+B_0^-(s,0) - B_{10}(t)B_{10}(s) + B_{10}(t)B_{01}(s) \tag{685}$$

$$-B_{01}(t)B_0^+B_1^-(s,0) + B_{01}(t)B_1^+B_0^-(s,0) - B_{01}(t)B_{10}(s) + B_{01}(t)B_{01}(s)\bigr\rangle_B \tag{686}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_1^+ B_0^- \left(t,\tau\right) B_{10}\left(s\right) - B_1^+ B_0^- \left(t,\tau\right) B_{01}\left(s\right) \right. \tag{687}$$

$$+ B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_{10}\left(s\right) - B_0^+ B_1^- \left(t,\tau\right) B_{01}\left(s\right) \tag{688}$$

$$- B_{10}\left(t\right) B_0^+ B_1^- \left(s,0\right) + B_{10}\left(t\right) B_1^+ B_0^- \left(s,0\right) - B_{10}\left(t\right) B_{10}\left(s\right) + B_{10}\left(t\right) B_{01}\left(s\right) \tag{689}$$

$$\left. - B_{01}\left(t\right) B_0^+ B_1^- \left(s,0\right) + B_{01}\left(t\right) B_1^+ B_0^- \left(s,0\right) - B_{01}\left(t\right) B_{10}\left(s\right) + B_{01}\left(t\right) B_{01}\left(s\right) \right\rangle_B \tag{690}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) \right. \tag{691}$$

$$\left. - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) \right\rangle + \frac{1}{4i} \left(B_{10}\left(t\right) + B_{01}\left(t\right)\right) \left(B_{10}\left(s\right) - B_{01}\left(s\right)\right) \tag{692}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) \right. \tag{693}$$

$$\left. - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) \right\rangle + \frac{1}{4i} \left(B_{10}\left(t\right) + B_{01}\left(t\right)\right) \left(B_{10}\left(s\right) - B_{01}\left(s\right)\right) \tag{694}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) \right. \tag{695}$$

$$\left. - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) \right\rangle + \left(B_{10}\left(t\right)\right)^{\Re} \left(B_{10}\left(s\right)\right)^{\Im} \tag{696}$$

$$= \frac{1}{4i} \left(\exp(\chi_{10}(t)+\chi_{01}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \tag{697}$$

$$- \exp(\chi_{10}(t)+\chi_{10}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{698}$$

$$+ \exp(\chi_{01}(t)+\chi_{01}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{699}$$

$$\left. - \exp(\chi_{01}(t)+\chi_{10}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \tag{700}$$

$$= \tfrac{1}{4i} \left(2i(\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Im} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \tag{701}$$

$$\left. + 2i(\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im} U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \tag{702}$$

$$= \tfrac{1}{2} \left((\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Im} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \tag{703}$$

$$\left. + (\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im} U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \tag{704}$$

$$\left\langle \widetilde{B_y}(t)\,\widetilde{B_x}(s)\right\rangle_B = \left\langle \left(\frac{B_0^+B_1^-\,(t,\tau) - B_1^+B_0^-\,(t,\tau) + B_{10}\,(t) - B_{01}\,(t)}{2\mathrm{i}}\right)\left(\frac{B_1^+B_0^-\,(s,0) + B_0^+B_1^-\,(s,0) - B_{10}\,(s) - B_{01}\,(s)}{2}\right)\right\rangle_B \tag{705}$$

$$= \frac{1}{4\mathrm{i}}\left\langle \left(B_0^+B_1^-\,(t,\tau) - B_1^+B_0^-\,(t,\tau) + B_{10}\,(t) - B_{01}\,(t)\right)\left(B_1^+B_0^-\,(s,0) + B_0^+B_1^-\,(s,0) - B_{10}\,(s) - B_{01}\,(s)\right)\right\rangle_B \tag{706}$$

$$= \tfrac{1}{4\mathrm{i}}\Big\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_0^+B_1^-(t,\tau)B_{10}(s) - B_0^+B_1^-(t,\tau)B_{01}(s) - B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0) \tag{707}$$

$$-B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0) + B_1^+B_0^-(t,\tau)B_{10}(s) + B_1^+B_0^-(t,\tau)B_{01}(s) + B_{10}(t)B_1^+B_0^-(s,0) + B_{10}(t)B_0^+B_1^-(s,0) \tag{708}$$

$$-B_{10}(t)B_{10}(s) - B_{10}(t)B_{01}(s) - B_{01}(t)B_1^+B_0^-(s,0) - B_{01}(t)B_0^+B_1^-(s,0) + B_{01}(t)B_{10}(s) + B_{01}(t)B_{01}(s)\Big\rangle \tag{709}$$

$$= \tfrac{1}{4i}\Big\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0)\Big\rangle + (B_{10}(t))^{\Im}(B_{10}(s))^{\Re} \tag{710}$$

$$= \frac{1}{4i}\Bigg(\exp(\chi_{01}(t)+\chi_{10}(s))U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big) \tag{711}$$

$$+\exp(\chi_{01}(t)+\chi_{01}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big) \tag{712}$$

$$-\exp(\chi_{10}(t)+\chi_{10}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big) \tag{713}$$

$$-\exp(\chi_{10}(t)+\chi_{01}(s))U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big)\Bigg) + (B_{10}(t))^{\Im}(B_{10}(s))^{\Re} \tag{714}$$

$$= \tfrac{1}{4i}\Bigg(2i(\exp(\chi_{01}(t)+\chi_{10}(s)))^{\Im}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big) \tag{715}$$

$$+2i(\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im}U_{10}(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big)\Bigg) + (B_{10}(t))^{\Im}(B_{10}(s))^{\Re} \tag{716}$$

$$= \tfrac{1}{2}\Bigg((\exp(\chi_{01}(t)+\chi_{10}(s)))^{\Im}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big) \tag{717}$$

$$+(\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im}U_{10}(t,s)\prod_{\mathbf{k}}\exp\Big(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\big)\Big)\Bigg) + (B_{10}(t))^{\Im}(B_{10}(s))^{\Re} \tag{718}$$

$$\left\langle b^\dagger D\,(h)\right\rangle_B = -h^*\left\langle D\,(h)\right\rangle_B N \tag{719}$$

$$\left\langle bD\,(h)\right\rangle_B = h\left\langle D\,(h)\right\rangle_B (N+1) \tag{720}$$

$$\left\langle D\,(h)\,b^\dagger\right\rangle_B = -h^*\left\langle D\,(h)\right\rangle_B (N+1) \tag{721}$$

$$\left\langle D\,(h)\,b\right\rangle_B = h\left\langle D\,(h)\right\rangle_B N \tag{722}$$

$$\left\langle B_1^+ B_0^- \left(t,\tau\right)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right) b_{\mathbf{k}'}^\dagger \right\rangle_B = \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*\left(t\right) v_{0\mathbf{k}}\left(t\right) - v_{1\mathbf{k}}\left(t\right) v_{0\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}^2}\right)\right)\left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right. \tag{723}$$

$$= \exp\left(\chi_{10}\left(t\right)\right)\left\langle D\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right) b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(\right.}{\omega_{\mathbf{k}}}\right.\right.\right. \tag{724}$$

$$= \exp\left(\chi_{10}\left(t\right)\right) q_{i\mathbf{k}'}\left(s\right)\left\langle D\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{725}$$

$$= \exp\left(\chi_{10}\left(t\right)\right) q_{i\mathbf{k}'}\left(s\right)\left(-\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)\left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{726}$$

$$= q_{i\mathbf{k}'}\left(s\right)\left(-\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right)\exp\left(\chi_{10}\left(t\right)\right)\left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{727}$$

$$= -q_{i\mathbf{k}'}\left(s\right)\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right) B_{10}\left(t\right) \tag{728}$$

$$\left\langle B_1^+ B_0^-(t,\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)\right)\left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}\left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{729}$$

$$= q_{i\mathbf{k}'}^*\left(s\right)\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}\left(t\right), \tag{730}$$

$$\left\langle B_0^+ B_1^-(t,\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger\right\rangle_B = -q_{i\mathbf{k}'}(s)\left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{01}(t), \tag{731}$$

$$\left\langle B_0^+ B_1^-(t,\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* b_{\mathbf{k}'}\right\rangle_B = q_{i\mathbf{k}'}^*(s)\left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t), \tag{732}$$

$$\left\langle B_x(t,\tau) B_{iz}(s,0)\right\rangle_B = \left\langle\left(\frac{B_{1+}B_{0-}\left(\tau\right) + B_{0+}B_{1-}\left(\tau\right) - B_{10} - B_{10}^*}{2}\right)\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{733}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right) + B_{0+}B_{1-}\left(\tau\right) - B_{10} - B_{10}^*\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{734}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right) + B_{0+}B_{1-}\left(\tau\right)\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{735}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} + B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'} - v\right. \tag{736}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))\left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{10}(t) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))\left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + \right. \tag{737}$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^*\left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}(t) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^*\left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t)\Big) \tag{738}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))\left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{10}(t) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))\left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + \right. \tag{739}$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^*\left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}(t) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^*\left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t)\Big) \tag{740}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10}\left(t\right) + \left(\frac{v_{0\mathbf{k}'}\left(t\right) - v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right.\right.\right. \tag{741}$$

$$+ \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right)^* N_{\mathbf{k}'}\left(\left(\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10}\left(t\right) + \left(\frac{v_{0\mathbf{k}'}\left(t\right) - v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{01}\left(t\right.\right.\right.$$

$$\left\langle B_{iz}(t,\tau)B_x(s,0)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(\frac{B_1^{+}B_0^{-}(s,0)+B_0^{+}B_1^{-}(s,0)-B_{10}(s)-B}{2}\right. \tag{755}$$

$$=\sum_{\mathbf{k}'}\Big\langle\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(\frac{B_1^{+}B_0^{-}(s,0)+B_0^{+}B_1^{-}(s,0)-B_{10}(s)-B}{2} \tag{756}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Big\langle\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(B_1^{+}B_0^{-}(s,0)+B_0^{+}B_1^{-}(s,0)-B_{10}(s)- \tag{757}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_1^{+}B_0^{-}(s,0)+B_0^{+}B_1^{-}(s,0)\right)\right\rangle_B \tag{758}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}(s,0)+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}(s,0)+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B \tag{759}$$

$$+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}(s,0)\Big\rangle, \tag{760}$$

$$\Big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}(s,0)\Big\rangle_B=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\Big\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}(s,0)\Big\rangle_B \tag{761}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_B \tag{762}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^{*}(s)v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)v_{0\mathbf{k}'}^{*}(s)}{\omega_{\mathbf{k}'}^{2}}\right)\right)\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v}{\omega_{\mathbf{k}}^{2}}\right.\right.\right.\right. \tag{763}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{764}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{765}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\right\rangle_B e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{766}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left(-\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}\left\langle D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\right\rangle_B N\right. \tag{767}$$

$$=-\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right\rangle_B N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{768}$$

$$=-\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))B_{10}(s)N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{769}$$

$$\Big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}(s,0)\Big\rangle_B=-\left(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))B_{01}(s)N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{770}$$

$$\Big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}(s,0)\Big\rangle_B=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_1^{+}B_0^{-}(s,0)\right\rangle_B \tag{771}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right. \tag{772}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle b_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{773}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\left\langle D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}(s)-}{\omega_{\mathbf{k}}}\right.\right.\right. \tag{774}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^{*}(s)}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\left\langle D\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}(s)-}{\omega_{\mathbf{k}}}\right.\right.\right. \tag{775}$$

$$(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)B_{10}(s) \tag{776}$$

$$\left\langle B_y\left(t,\tau\right)B_{iz}\left(s,0\right)\right\rangle_B = \left\langle\!\!\left(\frac{B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(s\right)\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(s\right)\right)^*b_{\mathbf{k}'}\right)\!\!\right\rangle_B \tag{783}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\!\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}\left(t\right)-B_{01}\left(t\right)\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(s\right)\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(s\right)\right)^*b_{\mathbf{k}'}\right)\!\right\rangle_B \tag{784}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-\left(t,\tau\right)-B_1^+B_0^-\left(t,\tau\right)\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(s\right)\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(s\right)\right)^*b_{\mathbf{k}'}\right)\right\rangle_B \tag{785}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left\langle B_0^+B_1^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^\dagger-B_1^+B_0^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^\dagger+B_0^+B_1^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*b_{\mathbf{k}'}-B_1^+B_0^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*b_{\mathbf{k}'}\right\rangle, \tag{786}$$

$$\left\langle B_0^+B_1^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^\dagger\right\rangle_B = -\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\tfrac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}(t), \tag{787}$$

$$\left\langle B_0^+B_1^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*b_{\mathbf{k}'}\right\rangle_B = \big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*\left(\tfrac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t), \tag{788}$$

$$\left\langle B_1^+B_0^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^\dagger\right\rangle_B = -\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}(t), \tag{789}$$

$$\left\langle B_1^+B_0^-(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*b_{\mathbf{k}'}\right\rangle_B = \big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t), \tag{790}$$

$$\left\langle B_y(t,\tau)B_{iz}(s,0)\right\rangle_B = \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\tfrac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}(t)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}(t) \tag{791}$$

$$+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*\left(\tfrac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t)\Big) \tag{792}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\tfrac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}(t)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}(t) \tag{793}$$

$$+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*\left(\tfrac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t)\Big) \tag{794}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)^*\big(B_{10}(t)+B_{01}(t)\big)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\big(-B_{10}(t)-B_0 \tag{795}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)^*\big(B_{10}(t)+B_{01}(t)\big)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\big(B_{10}(t)+B_{01}(\tag{796}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)^*2\big(B_{10}(t)\big)^{\Re}-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)2\big(B_{10}(t)\big)^{\Re}\Big), \tag{797}$$

$$= \tfrac{(B_{10}(t))^{\Re}}{\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)^*-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\Big), \tag{798}$$

$$\left\langle B_{iz}\left(t,\tau\right)B_y\left(s,0\right)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^+B_1^-\left(s,0\right)-B_1^+B_0^-\left(s,0\right)+B_{10}\left(s\right)-B_{01}\left(s\right)}{2\mathrm{i}}\right.\right. \tag{799}$$

$$= \frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^-\left(s,0\right)-B_1^+B_0^-\left(s,0\right)+B_{10}\left(s\right)-B_{01}\right.\right. \tag{800}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^-\left(s,0\right)-B_1^+B_0^-\left(s,0\right)\right)\right\rangle_B \tag{801}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big\langle\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(s,0)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(s,0)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(s,0) \tag{802}$$

$$-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(s,0)\Big\rangle \tag{803}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\left(s,0\right)\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\left(s,0\right)\right\rangle\right. \tag{804}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\left(s,0\right)\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(s,0\right)\right\rangle\right\rangle \tag{805}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)\big\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-(s,0)\big\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)\big\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-(s,0)\big\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)^*\big\langle b_{\mathbf{k}'}B_0^+B_1^-(s,0)\big\rangle \tag{806}$$

$$+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*\Big(\Big(\tfrac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{01}(s)\Big)-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*\Big(\Big(\tfrac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{10}(s)\Big)\Big) \tag{815}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^* B_{01}(s)\,N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^* B_{10}(s)\,N_{\mathbf{k}'}\right)\right. \tag{816}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^*\left(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{01}(s)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^*\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}(s)\right)\right) \tag{817}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^*\left(B_{10}(s)+B_{01}(s)\right)N_{\mathbf{k}'}\right. \tag{818}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^*\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\left(B_{10}(s)+B_{01}(s)\right)\left(N_{\mathbf{k}'}+1\right)\right) \tag{819}$$

$$= \frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^* B_{10}^{\Re}(s)\,N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^*\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(s)\left(N_{\mathbf{k}'}+1\right.\right. \tag{820}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^*\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(s)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^* B_{10}^{\Re}(s)\,N_{\mathbf{k}'}\right. \tag{821}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^*\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(s)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^* B_{10}^{\Re}(s)\,N_{\mathbf{k}'}\right. \tag{822}$$

$$= \mathrm{i}B_{10}^{\Re}(s)\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^*\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^* N_{\mathbf{k}'}\right). \tag{823}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^* \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{824}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \frac{1}{2}\left(\left(\exp\left(\chi_{10}\left(t\right)+\chi_{10}\left(s\right)\right)\right)^{\Re} U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{825}$$

$$\left. + \left(\exp\left(\chi_{10}\left(t\right)+\chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}^*\left(t,s\right)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{826}$$

$$-\left(\exp(\chi_{10}(t))\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re}\left(\exp(\chi_{01}(s))\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re} \tag{827}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_y}(s)\right\rangle_B = -\frac{1}{2}\left((\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Re}U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{828}$$

$$\left. -(\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Re}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{829}$$

$$+\left(\exp(\chi_{01}(t))\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im}\left(\exp(\chi_{10}(s))\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im} \tag{830}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_y}(s)\right\rangle_B = \frac{1}{2}\left((\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Im}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{831}$$

$$\left. +(\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im}U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)+(B_{10}(t))^{\Re}(B_{10}(s))^{\Im} \tag{832}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_x}(s)\right\rangle_B = \frac{1}{2}\left((\exp(\chi_{01}(t)+\chi_{10}(s)))^{\Im}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{833}$$

$$\left. +(\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im}U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)+(B_{10}(t))^{\Im}(B_{10}(s))^{\Re} \tag{834}$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_x}\left(s\right)\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}} B_{01}^{\Im}(s)\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)^*-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{835}$$

$$\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_{iz}}\left(s\right)\right\rangle_B = \mathrm{i}B_{10}^{\Im}\left(t\right)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right))^* N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right))\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1) \tag{836}$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_y}\left(s\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(s\right)\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)^* N \tag{837}$$

$$\left\langle \widetilde{B_y}\left(t\right)\widetilde{B_{iz}}\left(s\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(t\right)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right))^* N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right))(N_{\mathbf{k}}+1)\,e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^* \tag{838}$$

$$\tag{839}$$

$$\tag{840}$$

$$\tag{841}$$

$$\tag{842}$$

$$\tag{843}$$

$$\tag{844}$$

$$\tag{845}$$

$$\tag{846}$$

$$\tag{847}$$

$$\tag{848}$$

The spectral density is defined in the usual way:

$$J_i\left(\omega\right) \equiv \sum_{\mathbf{k}} \left|g_{i\mathbf{k}}\right|^2 \delta\left(\omega - \omega_{\mathbf{k}}\right), \tag{849}$$

$$v_{i\mathbf{k}}\left(t\right) = g_{i\mathbf{k}} F_i\left(\omega_{\mathbf{k}}, t\right). \tag{850}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i\left(\omega\right)}{\omega} \mathrm{d}\omega. \tag{851}$$

The integral version of the correlation functions are given by:

$$\begin{aligned}
\chi_{10}\left(t\right) &= \int_0^\infty \frac{\sqrt{J_1^*\left(\omega\right) J_0\left(\omega\right)} F_1^*\left(\omega,t\right) F_0\left(\omega,t\right) - \sqrt{J_1\left(\omega\right) J_0^*\left(\omega\right)} F_1\left(\omega,t\right) F_0^*\left(\omega,t\right)}{2\omega^2} \mathrm{d}\omega \\
U_{10}\left(t,s\right) &= \exp\left(i \left(\int_0^\infty \frac{\left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)\right)\left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,s\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,s\right)\right)^* \exp\left(i\omega\tau\right)}{\omega^2} \mathrm{d}\omega \right)^{\Im} \\
B_{10}\left(t\right) &= \exp\left(\chi_{10}\left(t\right)\right) \exp\left(-\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)}{\omega} \right|^2 \coth\left(\frac{\beta\omega}{2}\right) \mathrm{d}\omega \right), \\
\xi^+\left(t,s\right) &= \exp\left(- \int_0^\infty \frac{\left|\left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)\right) \exp\left(i\omega\tau\right) + \sqrt{J_1\left(\omega\right)} F_1\left(\omega,s\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,s\right)\right|^2}{2\omega^2} \coth \right. \\
\xi^-\left(t,s\right) &= \exp\left(- \int_0^\infty \frac{\left|\left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)\right) \exp\left(i\omega\tau\right) - \left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,s\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,s\right)\right)\right|^2}{2\omega^2} \mathrm{co} \right. \\
Q\left(\omega,t\right) &= \frac{\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)}{\omega} \\
\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s) \right\rangle_B &= \frac{1}{2}\left(\left(\exp\left(\chi_{10}\left(t\right) + \chi_{10}\left(s\right)\right)\right)^{\Re} {}_{U_{10}(t,s)} \xi^+\left(t,s\right) + \left(\exp\left(\chi_{10}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) \right) {}_{-(B_{10}(t))^{\Re}(B_{01}(s))^{\Re}} \\
\left\langle \widetilde{B_y}(t)\widetilde{B_y}(s) \right\rangle_B &= -\frac{1}{2}\left(\left(\exp\left(\chi_{01}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}\left(t,s\right) \xi^+\left(t,s\right) - \left(\exp\left(\chi_{10}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) \right) {}_{+(B_{01}(t))^{\Im}(B_{10}(s}} \\
\left\langle \widetilde{B_x}(t)\widetilde{B_y}(s) \right\rangle_B &= \frac{1}{2}\left(\left(\exp\left(\chi_{10}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Im} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) + \left(\exp\left(\chi_{01}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Im} U_{10}\left(t,s\right) \xi^+\left(t,s\right) \right) + \left(B_{10}\left(t\right)\right)^{\Re} (B_{1 \\
\left\langle \widetilde{B_y}(t)\widetilde{B_x}(s) \right\rangle_B &= \frac{1}{2}\left(\left(\exp\left(\chi_{01}\left(t\right) + \chi_{10}\left(s\right)\right)\right)^{\Im} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) + \left(\exp\left(\chi_{01}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Im} U_{10}\left(t,s\right) \xi^+\left(t,s\right) \right) + \left(B_{10}\left(t\right)\right)^{\Im} (B_{1 \\
\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s) \right\rangle_B &= \int_0^\infty \left(\sqrt{J_i(\omega) J_j^*(\omega)} (1-F_i(\omega,t)) \left(1-F_j^*(\omega,s)\right) e^{\mathrm{i}\omega\tau} N(\omega) + \sqrt{J_i^*(\omega) J_j(\omega)} (1-F_i^*(\omega,t)) (1-F_j(\omega,s)) e^{-\mathrm{i}\omega\tau} (N(\omega)+1) \right) \mathrm{d}\omega, \\
\left\langle \widetilde{B_{iz}}\left(t\right) \widetilde{B_x}\left(s\right) \right\rangle_B &= \mathrm{i} B_{01}^{\Im}\left(s\right) \int_0^\infty \left(\sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,t\right)\right) Q^*\left(\omega,s\right) N\left(\omega\right) e^{\mathrm{i}\omega\tau} - \sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,t\right)\right) Q\left(\omega,s\right) e^{-\mathrm{i}\omega\tau} \left(N\left(\omega\right) + 1\right) \right) \\
\left\langle \widetilde{B_x}\left(t\right) \widetilde{B_{iz}}\left(s\right) \right\rangle_B &= \mathrm{i} B_{01}^{\Im}\left(t\right) \int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,s\right)\right) Q\left(\omega,t\right) N\left(\omega\right) e^{\mathrm{i}\omega\tau} - \sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,s\right)\right) Q^*\left(\omega,t\right) e^{-\mathrm{i}\omega\tau} \left(N\left(\omega\right) + 1\right) \right) \\
\left\langle \widetilde{B_{iz}}\left(t\right) \widetilde{B_y}\left(s\right) \right\rangle_B &= i B_{10}^{\Re}\left(s\right) \int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,s\right)\right) Q\left(\omega,s\right) \left(N\left(\omega\right) + 1\right) e^{-\mathrm{i}\omega\tau} - \sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,s\right)\right) Q^*\left(\omega,s\right) e^{\mathrm{i}\omega\tau} N\left(\omega\right) \right) \\
\left\langle \widetilde{B_y}\left(t\right) \widetilde{B_{iz}}\left(s\right) \right\rangle_B &= i B_{10}^{\Re}\left(t\right) \int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,s\right)\right) Q\left(\omega,t\right) N\left(\omega\right) e^{\mathrm{i}\omega\tau} - \sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,s\right)\right) Q^*\left(\omega,t\right) e^{-\mathrm{i}\omega\tau} \left(N\left(\omega\right) + 1\right) \right)
\end{aligned}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s)\right\rangle_B=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(s\right)\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(s\right)\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{852}$$

$$=\sum_{\mathbf{k}}\Big(\big(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}},t)\big)\big(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}},s)\big)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\big(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}},t)\big)^{*}\big(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}},s)\big)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\Big) \tag{853}$$

$$=\sum_{\mathbf{k}}\Big(g_{i\mathbf{k}}(1-F_i\left(\omega_{\mathbf{k}},t\right))\,g_{j\mathbf{k}}^{*}(1-F_j(\omega_{\mathbf{k}},s))^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+g_{i\mathbf{k}}^{*}(1-F_i\left(\omega_{\mathbf{k}},t\right))^{*}g_{j\mathbf{k}}(1-F_j(\omega_{\mathbf{k}},s))e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\Big) \tag{854}$$

$$\approx\int_0^{\infty}\Big(\sqrt{J_i(\omega)J_j^{*}(\omega)}(1-F_i(\omega,t))\big(1-F_j^{*}(\omega,s)\big)e^{\mathrm{i}\omega\tau}N(\omega)+\sqrt{J_i^{*}(\omega)J_j(\omega)}(1-F_i^{*}(\omega,t))(1-F_j(\omega,s))e^{-\mathrm{i}\omega\tau}(N(\omega)+1)\Big)\mathrm{d}\omega, \tag{855}$$

$$\chi_{10}\left(t\right)=\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}^{2}}\right) \tag{856}$$

$$=\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{1\mathbf{k}}^{*}F_1^{*}\left(\omega_{\mathbf{k}},t\right)g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}},t\right)-g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}},t\right)g_{0\mathbf{k}}^{*}F_0^{*}\left(\omega_{\mathbf{k}},t\right)}{\omega_{\mathbf{k}}^{2}}\right) \tag{857}$$

$$=\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{1\mathbf{k}}^{*}g_{0\mathbf{k}}F_1^{*}\left(\omega_{\mathbf{k}},t\right)F_0\left(\omega_{\mathbf{k}},t\right)-g_{1\mathbf{k}}g_{0\mathbf{k}}^{*}F_1\left(\omega_{\mathbf{k}},t\right)F_0^{*}\left(\omega_{\mathbf{k}},t\right)}{\omega_{\mathbf{k}}^{2}}\right) \tag{858}$$

$$\approx\int_0^{\infty}\frac{\sqrt{J_1^{*}\left(\omega\right)J_0\left(\omega\right)}F_1^{*}\left(\omega,t\right)F_0\left(\omega,t\right)-\sqrt{J_1\left(\omega\right)J_0^{*}\left(\omega\right)}F_1\left(\omega,t\right)F_0^{*}\left(\omega,t\right)}{2\omega^{2}}\mathrm{d}\omega, \tag{859}$$

$$U_{10}\left(t,s\right)=\prod_{\mathbf{k}}\exp\left(i\left(\frac{\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)\right)^{*}\exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^{2}}\right)^{\Im}\right) \tag{860}$$

$$=\exp\left(i\sum_{\mathbf{k}}\left(\frac{\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)\right)^{*}\exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^{2}}\right)^{\Im}\right) \tag{861}$$

$$=\exp\left(i\left(\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)\right)^{*}\exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^{2}}\right)^{\Im}\right) \tag{862}$$

$$=\exp\left(i\left(\sum_{\mathbf{k}}\frac{\left(g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}},t\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}},t\right)\right)\left(g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}},s\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}},s\right)\right)^{*}\exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^{2}}\right)^{\Im}\right) \tag{863}$$

$$\approx\exp\left(i\left(\int_0^{\infty}\frac{\left(\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t\right)\right)\left(\sqrt{J_1\left(\omega\right)}F_1\left(\omega,s\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,s\right)\right)^{*}\exp\left(i\omega\tau\right)}{\omega^{2}}\mathrm{d}\omega\right)^{\Im}\right) \tag{864}$$

(865)

$$B_{10}\left(t\right)=\left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{866}$$

$$=\exp\left(\chi_{10}\left(t\right)\right)\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}},t\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}},t\right)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{867}$$

$$\approx\exp\left(\chi_{10}\left(t\right)\right)\exp\left(-\frac{1}{2}\int_0^{\infty}\left|\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega,t\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega,t\right)}{\omega}\right|^{2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega\right) \tag{868}$$

(869)

$$\xi^{+}\left(t,s\right)=\prod_{\mathbf{k}}\exp\left(-\frac{|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{870}$$

$$=\exp\left(-\sum_{\mathbf{k}}\frac{|\left(v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{871}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda + \mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = 0. \tag{910}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S |\pm\rangle = \lambda_{\pm} |\pm\rangle$. Using this notation is possible to write $H_{\bar{S}} = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_i}(t)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\begin{aligned}\widetilde{A_i}(\tau) &= e^{\mathrm{i}\overline{H_{\bar{S}}}\tau} A_i e^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} &(911)\\ &= \sum_w e^{-\mathrm{i}w\tau} \mathscr{A}_i(w). &(912)\end{aligned}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.
In order to use the equation (912) to descompose the equation (374) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right)$ like:

$$\begin{aligned}U(t) &\equiv \mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) &(913)\\ &= \mathbb{I} + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1) H(t_2) ... H(t_n). &(914)\end{aligned}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \equiv \exp(-\mathrm{i}t H_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...\right)\right). \tag{915}$$

The terms can be found expanding $\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \tag{916}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{917}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{918}$$

In this case the Fourier decomposition using the expansion of $H_E(t)$ is:

$$\begin{aligned}\widetilde{A_i}(t) &= U^\dagger(t) A_i(t) U(t) &(919)\\ \widetilde{A_i}(t) &= e^{\mathrm{i}H_E(t)t} A_i(t) e^{-\mathrm{i}H_E(t)t} &(920)\\ &= \sum_{w(t)} e^{-\mathrm{i}w(t)t} \mathscr{A}_i(t, w(t)). &(921)\end{aligned}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau,t)$ using the Magnus expansion generates:

$$\begin{aligned}
\widetilde{A_j}(t-\tau,t) &= U(t)\,U^{\dagger}(t-\tau)\,A_j(t)\,U(t-\tau)\,U^{\dagger}(t) && (922)\\
&= e^{-\mathrm{i}tH_E(t)}e^{\mathrm{i}(t-\tau)H_E(t-\tau)}A_j(t)\,e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}e^{\mathrm{i}tH_E(t)} && (923)\\
&= e^{-\mathrm{i}tH_E(t)}\left(\sum_{w'(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j(t,w(t-\tau))\right)e^{\mathrm{i}tH_E(t)} && (924)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j(t,w(t-\tau),w'(t)) && (925)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j(t,w(t-\tau),w'(t)) && (926)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(t,w(t-\tau),w'(t)) && (927)
\end{aligned}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}(t-\tau)$ and $\overline{H_E}(t)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (912) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V}=\{|+\rangle,|-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta\in\mathrm{V}} \langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|. \tag{928}$$

Given that $[|+\rangle\langle+|,|-\rangle\langle-|]=0$, then using the Zassenhaus formula we obtain:

$$\begin{aligned}
e^{\mathrm{i}\overline{H_E}\tau} &= e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} && (929)\\
&= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} && (930)\\
&= \left(|-\rangle\langle-|+e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) && (931)\\
&= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|. && (932)
\end{aligned}$$

Calculating the transformation (912) directly using the previous relationship we find that:

$$\begin{aligned}
U^{\dagger}(\tau)\,A_i(\tau)U(\tau) &= \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in\mathrm{V}}\langle\alpha|A_i(\tau)|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) && (933)\\
&= \langle+|A_i(\tau)|+\rangle|+\rangle\langle+|+e^{\mathrm{i}\eta\tau}\langle+|A_i(\tau)|-\rangle|+\rangle\langle-|+e^{-\mathrm{i}\eta\tau}\langle-|A_i(\tau)|+\rangle|-\rangle\langle+|+\langle-|A_i(\tau)|-\rangle|-\rangle\langle-|. && (934)\\
&= \mathscr{A}_i(0)+\mathscr{A}_i(-w)\,e^{\mathrm{i}w\tau}+\mathscr{A}_i(w)\,e^{-\mathrm{i}w\tau} && (935)
\end{aligned}$$

Here $w=\lambda_+-\lambda-$. Comparing the RHS of the equations (912) and the explicit expression for $\widetilde{A_i}(\tau)$ in (920), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$\begin{aligned}
\mathscr{A}_i(0) &= \langle+|A_i(\tau)|+\rangle|+\rangle\langle+|+\langle-|A_i(\tau)|-\rangle|-\rangle\langle-|, && (936)\\
\mathscr{A}_i(-w) &= \langle+|A_i(\tau)|-\rangle|+\rangle\langle-|, && (937)\\
\mathscr{A}_i(w) &= \langle-|A_i(\tau)|+\rangle|-\rangle\langle+|. && (938)
\end{aligned}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau)=\widetilde{A_i}^{\dagger}(\tau)$ and $\widetilde{B_i}(\tau)=\widetilde{B_i}^{\dagger}(\tau)$ we can use the equation (912) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big] + \mathscr{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \tag{939}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{940}$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\Big[A_i, \overline{\rho_S}\left(t\right) e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w\left(t-\tau\right), w'\left(t\right)\right)\Big]\Big) \tag{941}$$

Given that $\mathscr{A}_j\left(w\left(t-\tau\right), w'\left(t\right)\right) = \mathscr{A}_j^\dagger\left(-w\left(t-\tau\right), -w'\left(t\right)\right)$ from the Fourier decomposition (912) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\mathscr{B}_{ij}\left(\tau\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B\right) \tag{942}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B\right). \tag{943}$$

Let's define:

$$\mathscr{A}_j\left(w\left(t-\tau\right), w'\left(t\right)\right) = \mathscr{A}_{jww'}\left(t-\tau, t\right) \tag{944}$$

The master equation can be re-written in the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\Big] \tag{945}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\Big[A_i, \overline{\rho_S}\left(t\right) e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau, t\right)\Big] \tag{946}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\Big] \tag{947}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\Big[A_i, \overline{\rho_S}\left(t\right) e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau, t\right)\Big] \tag{948}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\Big] \tag{949}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\Big[A_i, \overline{\rho_S}\left(t\right) e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau, t\right)\Big] \tag{950}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\Big] \tag{951}$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\Big[A_i, \overline{\rho_S}\left(t\right) e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau, t\right)\Big]\Big) \tag{952}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \textstyle\sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathrm{Tr}_B\Big(\Big[A_i,\widetilde{B_i}(\tau)\widetilde{B_j}(0)\rho_B e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\Big] \tag{953}$$

$$-\Big[A_i,\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\Big]\Big) \tag{954}$$

Given that if we define:

$$D_{ijww'}\left(t-\tau, t\right) = C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau, t\right) \tag{955}$$

then

$$D^\dagger_{ijww'}\left(t-\tau, t\right) = \left(C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau, t\right)\right)^\dagger \tag{956}$$

$$= \mathscr{B}^*_{ij}\left(\tau\right)C_i\left(t\right)C_j\left(t-\tau\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}^\dagger_{jww'}\left(t-\tau, t\right) \tag{957}$$

We used the fact that $C_i(t), C_j(t-\tau)$ are real. Now let's consider the following trace recalling that $\mathrm{Tr}(A)^* = \mathrm{Tr}\left(A^\dagger\right)$ so:

$$\mathrm{Tr}_B\left(\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\right) = \mathrm{Tr}_B\left(e^{-\mathrm{i}\tau H_B(\tau)}B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\right) \tag{958}$$

$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B e^{-\mathrm{i}\tau H_B(\tau)}\right) \text{ (by cyclic permutivity of trace)} \tag{959}$$

$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i e^{-\mathrm{i}\tau H_B(\tau)}\rho_B\right) \text{ (by commutativity of } e^{-\mathrm{i}\tau H_B(\tau)} \text{ and } \rho_B) \tag{960}$$

$$= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) \text{ (by definition of time evolution)} \tag{961}$$

$$= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) \tag{962}$$

$$= \mathrm{Tr}_B\left(\rho_B B_j\widetilde{B_i}(\tau)\right) \tag{963}$$

$$= \mathrm{Tr}_B\left(\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^\dagger\right) \text{ (by definition of adjoint)} \tag{964}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^* \tag{965}$$

$$= \mathscr{B}_{ij}^*(\tau) \tag{966}$$

So we can write the master equation like:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{967}$$

$$-\mathscr{B}_{ij}^*(\tau)\left[A_i,\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j^\dagger(w(t-\tau),w'(t))\right]\Big) \tag{968}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\left(\left[A_i, D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\right]\right) \tag{969}$$

Let's define the response matrix in the following way.

$$\mathscr{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau,t) \tag{970}$$

Then the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)\mathscr{D}_{ijww'}^\dagger(t)\right]\right) \tag{971}$$

If we extend the upper limit of integration to ∞ in the equation (970) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_S(t)}{\mathrm{d}t}e^V = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^V\right)}{\mathrm{d}t} \tag{972}$$

$$= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} \tag{973}$$

$$= -\mathrm{i}e^{-\mathrm{V}}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]e^V - \sum_{ijww'}\int_0^t \mathrm{d}\tau\Big(e^{-V}\left[A_i,D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right]e^V - e^{-V}\Big[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\Big]e^V\Big). \tag{974}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V} = e^{-V}\overline{A\mathbb{I}B}e^{V} \tag{975}$$
$$= e^{-V}\overline{A}e^{V}e^{-V}\bar{B}e^{V} \tag{976}$$
$$= \left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\bar{B}e^{V}\right) \tag{977}$$
$$= AB. \tag{978}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V} = e^{-V}\overline{(AB-BA)}e^{V} \tag{979}$$
$$= e^{-V}\overline{AB}e^{V} - e^{-V}\overline{BA}e^{V} \tag{980}$$
$$= AB - BA \tag{981}$$
$$= [A,B]. \tag{982}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\sum_{ijww'}\left(\left[A_i, \mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i, \overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]\right)e^{V} \tag{983}$$
$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right]e^{V} - \sum_{ijww'}\left(e^{-V}\left[A_i, \mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\left[A_i, \overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]e^{V}\right) \tag{984}$$
$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right), \rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V}, e^{-V}\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)e^{V}\right] - \left[e^{-V}A_ie^{V}, e^{-V}\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)e^{V}\right]\right) \tag{985}$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t), \rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V}, e^{-V}\mathscr{D}_{ijww'}(t)\,e^{V}e^{-V}\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V}, e^{-V}\overline{\rho_S}(t)e^{V}e^{-V}\mathscr{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right) \tag{986}$$
$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right), \rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V}, e^{-V}\mathscr{D}_{ijww'}\left(t\right)e^{V}\rho_S\left(t\right)\right] - \left[e^{-V}A_ie^{V}, \rho_S\left(t\right)e^{-V}\mathscr{D}^{\dagger}_{ijww'}\left(t\right)e^{V}\right]\right). \tag{987}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (971) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (971) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1,2,3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{988}$$

After performing the transformation (24) on the Hamiltonian (988) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)\,|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, \tag{989}$$
$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), \tag{990}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{991}$$

The Hamiltonian (989) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{992}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{993}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \tag{994}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \tag{995}$$

$$\lambda_{\pm} = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} \tag{996}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{997}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{998}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \tag{999}$$

$$\lambda_{\pm} = \frac{R \pm \eta}{2}. \tag{1000}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{1001}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{1002}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle$. Summarizing these results we can write:

$$\lambda_{+} = \frac{\epsilon + \eta}{2}, \tag{1003}$$
$$\lambda_{-} = \frac{\epsilon - \eta}{2}, \tag{1004}$$
$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \tag{1005}$$
$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, \tag{1006}$$
$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}, \tag{1007}$$
$$\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}. \tag{1008}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{1009}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{1010}$$

So the value of $\tan(\theta)$ using (1010) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}} + 1} \tag{1011}$$
$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{1012}$$
$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{1013}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (937)-(938) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\begin{aligned}
\langle+|\sigma_x|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1014)\\
&= 2\sin(\theta)\cos(\theta) && (1015)\\
&= \sin(2\theta)\,, && (1016)\\
\langle-|\sigma_x|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1017)\\
&= -2\sin(\theta)\cos(\theta) && (1018)\\
&= -\sin(2\theta)\,, && (1019)\\
\langle-|\sigma_x|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1020)\\
&= \cos^2(\theta) - \sin^2(\theta) && (1021)\\
&= \cos(2\theta)\,, && (1022)\\
\langle+|\sigma_y|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1023)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (1024)\\
&= 0, && (1025)\\
\langle-|\sigma_y|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1026)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (1027)\\
&= 0, && (1028)\\
\langle-|\sigma_y|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1029)\\
&= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) && (1030)\\
&= \mathrm{i}. && (1031)
\end{aligned}$$

$$\begin{aligned}
\langle+|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1032)\\
&= \cos(\theta)\cos(\theta) && (1033)\\
&= \cos^2(\theta)\,, && (1034)\\
\langle-|\frac{1+\sigma_z}{2}|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1035)\\
&= \sin(\theta)\sin(\theta) && (1036)\\
&= \sin^2(\theta)\,, && (1037)\\
\langle-|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1038)\\
&= -\sin(\theta)\cos(\theta) && (1039)\\
&= -\sin(\theta)\cos(\theta)\,. && (1040)
\end{aligned}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), \tag{1041}$$
$$A_1(\eta) = \cos(2\theta)\,|-\rangle\langle+|, \tag{1042}$$
$$A_2(0) = 0, \tag{1043}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{1044}$$
$$A_3(0) = \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-|, \tag{1045}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)\,|-\rangle\langle+|. \tag{1046}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (974) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (955) is equivalent to:

$$\mathscr{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau, t) \tag{1047}$$
$$= \int_0^t \mathrm{d}\tau C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_j\left(w(t-\tau), w'(t)\right) \tag{1048}$$
$$= \int_0^t \mathrm{d}\tau C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)} \mathscr{A}_j(w, w'). \tag{1049}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (415) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau). \tag{1050}$$

So the response matrix can be rewritten as:

$$\mathscr{D}_{ijww'}(t) = \left(\int_0^t \mathrm{d}\tau \Lambda'_{ij}(\tau)\, e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)}\right) \mathscr{A}_j(w, w') \tag{1051}$$

Let's define the response function like:

$$K_{ij}(w, w', t) = \int_0^t C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1052}$$
$$= \int_0^t \Lambda'_{ij}(\tau)\, e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1053}$$
$$= K_{ijww'}(t). \tag{1054}$$

Then we have the following equivalence:

$$\mathscr{D}_{ijww'}(t) = K_{ijww'}(t)\, \mathscr{A}_j(w, w') \tag{1055}$$
$$= K_{ijww'}(t)\, \mathscr{A}_{jww'} \tag{1056}$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]\right) \tag{1057}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i,K_{ijww'}\left(t\right)\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)K^{*}_{ijww'}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{1058}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(K_{ijww'}\left(t\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)\right] - K^{*}_{ijww'}\left(t\right)\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{1059}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t)+\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - \left(K^{\Re}_{ijww'}(t)-\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{1060}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}K^{\Re}_{ijww'}(t)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t)-\overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] - \mathrm{i}\sum_{ijww'}K^{\Im}_{ijww'}(t)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t)+\overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] \tag{1061}$$

Using the notation of the master equation (971), we can say that $C_1\left(t\right) = \frac{\Omega}{2} = C_2\left(t\right)$ and $C_3\left(t\right) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B\left(t\right) = B$. Taking the equations(824)-(848) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (415) are equal to:

$$\left\langle\widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left(\left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right)\left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right)^*\left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}} + 1\right)\right) \tag{1062}$$

$$= \sum_{\mathbf{k}} \left|g_{1\mathbf{k}} - v_{1\mathbf{k}}\right|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}} + 1\right)\right) \tag{1063}$$

$$\approx \int_0^\infty J_1(\omega)\left(1 - F_1(\omega)\right)^2 \left(e^{\mathrm{i}\omega\tau} N(\omega) + e^{-\mathrm{i}\omega\tau}\left(N(\omega) + 1\right)\right) \mathrm{d}\omega \tag{1064}$$

$$G_\pm(\omega,\tau) = e^{\mathrm{i}\omega\tau} N(\omega) + e^{-\mathrm{i}\omega\tau}\left(N(\omega) + 1\right) \tag{1065}$$

$$\left\langle\widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B \approx \int_0^\infty J_1(\omega)\left(1 - F_1(\omega)\right)^2 G_+(\omega, t)\,\mathrm{d}\omega \tag{1066}$$

$$\chi_{10}(t) = 0 \text{ (because } v_{0\mathbf{k}}(t) = 0 \text{ for all } \mathbf{k}) \tag{1067}$$

$$U_{10}(t,s) = \prod_{\mathbf{k}} \exp\left(i\left(\frac{\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\left(v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right)^* \exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{1068}$$

$$= \prod_{\mathbf{k}} \exp\left(i\left(\frac{v_{1\mathbf{k}}^2(t)\exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{1069}$$

$$= \prod_{\mathbf{k}} \exp\left(i\frac{v_{1\mathbf{k}}^2 \sin\left(\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2}\right) \tag{1070}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\Big(\textstyle\prod_{\mathbf{k}} \exp\Big(i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\Big)\prod_{\mathbf{k}}\exp\Big(-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)+\prod_{\mathbf{k}}\exp\Big(-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\Big)\prod_{\mathbf{k}}\exp\Big(-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\mathrm{cot} \tag{1071}$$

$$-\Big(\exp\Big(-\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\Big(\Big|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big|^2\Big)\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big)\Big(\exp\Big(-\tfrac{1}{2}\sum_{\mathbf{k}}\Big(\Big|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big|^2\Big)\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1072}$$

$$= \tfrac{1}{2}\Big(\textstyle\prod_{\mathbf{k}}\exp\Big(i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)+\prod_{\mathbf{k}}\exp\Big(-i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1073}$$

$$-\Big(\exp\Big(-\textstyle\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1074}$$

$$|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)\pm v_{1\mathbf{k}}|^2 = v_{1\mathbf{k}}^2|\exp(i\omega_{\mathbf{k}}\tau)\pm 1|^2 \tag{1075}$$

$$= v_{1\mathbf{k}}^2|\cos(\omega_{\mathbf{k}}\tau)+i\sin(\omega_{\mathbf{k}}\tau)\pm 1|^2 \tag{1076}$$

$$= v_{1\mathbf{k}}^2\left(\left(1\pm\cos\left(\omega_{\mathbf{k}}\tau\right)\right)^2 + \sin^2\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{1077}$$

$$= 2v_{1\mathbf{k}}^2\left(1\pm\cos\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{1078}$$

$$B \equiv \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{1079}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\Big(\exp\Big(\textstyle\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)+\exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1080}$$

$$-\Big(\exp\Big(-\textstyle\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1081}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right) + \cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{1082}$$

$$\approx \int_0^\infty \frac{J_1(\omega)F_1^2(\omega)}{\omega^2}\left(-\mathrm{i}\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega \tag{1083}$$

$$= \int_0^\infty \frac{J_1(\omega)F_1^2(\omega)}{\omega^2} G_+(\omega,\tau)\,\mathrm{d}\omega \tag{1084}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\Big(\exp\Big(\textstyle\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{2v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)+\exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{2v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big)-B^2 \tag{1085}$$

$$= \tfrac{1}{2}\Big(\exp\Big(\textstyle\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)+\exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big)-B^2 \tag{1086}$$

$$= \tfrac{1}{2}\Big(\exp\Big(-\textstyle\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\exp\Big(\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2\sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}-\frac{v_{1\mathbf{k}}^2\cos(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)+\exp\Big(-\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2\sin(\omega}{\omega_{\mathbf{k}}^2} \tag{1087}$$

$$= \frac{B^2}{2}\left(e^{-\phi(\tau)} + e^{\phi(\tau)} - 2\right) \tag{1088}$$

$$\Lambda_{11}'(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B\right) \tag{1129}$$
$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{1130}$$
$$= \frac{\Omega_r^2}{4}\left(\cosh\left(\phi(\tau)\right) - 1\right) \tag{1131}$$
$$\Lambda_{22}'(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B\right) \tag{1132}$$
$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} - e^{-\phi(\tau)}\right), \tag{1133}$$
$$\Lambda_{33}'(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)\left(1 - F(\omega)\right)^2 G_+(\tau), \tag{1134}$$
$$\Lambda_{32}'(\tau) = \frac{\Omega_r}{2}\int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)\left(1 - F(\omega)\right) \mathrm{i}G_-(\tau), \tag{1135}$$
$$\Lambda_{32}'(\tau) = -\Lambda_{23}'(\tau), \tag{1136}$$
$$\Lambda_{12}'(\tau) = \Lambda_{21}'(\tau) \tag{1137}$$
$$= \Lambda_{13}'(\tau) \tag{1138}$$
$$= \Lambda_{31}'(\tau) \tag{1139}$$
$$= 0. \tag{1140}$$

Finally taking the Hamiltonian (988) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that Det $\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, Tr $\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (337) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{1141}$$
$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{1142}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (451) under a time-independent approach providing similar results.
Given that the Hamiltonian of this system is time-independent, then $U(t)\,U^\dagger(t-\tau) = U(\tau)$. From the equation (971) and using the fact that

$$\widetilde{A_j}(t-\tau, t) = U(\tau)\,A_j U(-\tau) \tag{1143}$$
$$= \sum_w e^{\mathrm{i}w\tau}\mathscr{A}_j(-w) \tag{1144}$$
$$= \sum_w e^{-\mathrm{i}w\tau}\mathscr{A}_j(w). \tag{1145}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t}=-\mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}\left(t\right)\right]-\frac{1}{2}\sum_{ij}\sum_{w}\gamma_{ij}\left(w,t\right)\left[A_i,\mathscr{A}_j\left(w\right)\overline{\rho}_S\left(t\right)-\overline{\rho}_S\left(t\right)\mathscr{A}_j^{\dagger}\left(w\right)\right] \tag{1146}$$

$$-\sum_{ij}\sum_{w}S_{ij}\left(w,t\right)\left[A_i,\mathscr{A}_j\left(w\right)\overline{\rho}_S\left(t\right)+\overline{\rho}_S\left(t\right)\mathscr{A}_j^{\dagger}\left(w\right)\right]. \tag{1147}$$

where $\mathscr{A}_j^{\dagger}\left(w\right)=\mathscr{A}_j\left(-w\right)$, as we can see the equation (1147) contains the rates and energy shifts $\gamma_{ij}\left(w,t\right)=2K_{ij}^{\Re}\left(w,t\right)$ and $S_{ij}\left(w,t\right)=K_{ij}^{\Im}\left(w,t\right)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}\left(w,t\right)=\int_0^t\Lambda_{ij}'\left(\tau\right)e^{\mathrm{i}w\tau}\mathrm{d}\tau.$$

The fact $\mathscr{A}_j^{\dagger}\left(w\right)=\mathscr{A}_j\left(-w\right)$ can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{\mathrm{i}\overline{H_S}(t-\tau)}=e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)(t-\tau)} \tag{1148}$$

$$=e^{\mathrm{i}\lambda_+|+\rangle\langle+|(t-\tau)}e^{\mathrm{i}\lambda_-|-\rangle\langle-|(t-\tau)} \tag{1149}$$

$$=\left(|-\rangle\langle-|+e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|\right)\left(|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1150}$$

$$=e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|. \tag{1151}$$

Where $\lambda_+,\lambda_-$are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|,|-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (912) of (936)-(938) directly using the previous relationship we find that:

$$\widetilde{A_i(0)}(t-\tau)=\left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1152}$$

$$=\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|, \tag{1153}$$

$$\widetilde{A_i\left(w\right)}\left(t-\tau\right)=\left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|-\rangle|+\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1154}$$

$$=\langle+|A_i|-\rangle|+\rangle\langle-|e^{\mathrm{i}w(t-\tau)}, \tag{1155}$$

$$\widetilde{A_i\left(-w\right)}\left(t-\tau\right)=\left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle-|A_i|+\rangle|-\rangle\langle+|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1156}$$

$$=\langle-|A_i|+\rangle|-\rangle\langle+|e^{-\mathrm{i}w(t-\tau)}. \tag{1157}$$

Here $w=\lambda_+-\lambda-$. So we can see that for the equation (922) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w\neq w'$ then $A_j'\left(w,w'\right)=0$ so:

$$\widetilde{A_j}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_j(t)\,U(t-\tau)\,U^{\dagger}(t) \tag{1158}$$
$$= U(t)\left(\sum_{w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}A_j(w(t-\tau))\right)U^{\dagger}(t) \tag{1159}$$
$$= \sum_{w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}U(t)\,A_j(w(t-\tau))\,U^{\dagger}(t) \tag{1160}$$
$$= \sum_{w'(t),w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j(w(t-\tau),w'(t)) \tag{1161}$$
$$= \sum_{w'(t),w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_{jww'} \tag{1162}$$
$$== \sum_{w'(t),w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j(w)\,\delta_{ww'} \tag{1163}$$
$$= \sum_{w} e^{-\mathrm{i}(t-\tau)w}e^{\mathrm{i}tw}A_j(w) \tag{1164}$$
$$= \sum_{w} e^{\mathrm{i}\tau w}A_j(w) \tag{1165}$$
$$= U^{\dagger}(-\tau)\,A_jU(-\tau) \tag{1166}$$

So using now as reference the equation (1061) and $A'_j(w,w')=0$ we can deduce that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijw}K_{ij}^{\Re}(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t)-\overline{\rho_S}(t)A_j^{\dagger}(w)\right] - \mathrm{i}\sum_{ijw}K_{ij}^{\Im}(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t)+\overline{\rho_S}(t)A_j^{\dagger}(w)\right] \tag{1167}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (106) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{1168}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau)=B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (1168) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t)=0$, $V_{10}(t)=\frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (**??**) and (**??**) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)\,|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t), \tag{1169}$$
$$\overline{H_{\mathrm{I}}} = \frac{\Omega(t)}{2}\left(B_x\sigma_x + B_y\sigma_y\right). \tag{1170}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}})=1$ and using the equations (1129)-(1137) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^{\infty}\frac{J(\omega)}{\omega^2}G_{+}(\tau)\,\mathrm{d}\omega. \tag{1171}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (1171) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega. \tag{1172}$$

Writing the interaction Hamiltonian (1170) in the similar way to the equation (**??**) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (**??**) with $\delta'|1\rangle\langle 1|=\frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t). \tag{1173}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{1174}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1175}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{1176}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right). \tag{1177}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(451) is:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{1178}$$

$$\left.+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right). \tag{1179}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (1174) and (1177) on the equation (1179) we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{1180}$$

$$\left.+\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right). \tag{1181}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference $[3]$ we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (415) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right). \tag{1182}$$

Using the equation (971), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)\right), \rho_S\left(t\right)] - \frac{1}{2}\sum_{w}\gamma_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) - \rho_S\left(t\right)A_3^{\dagger}\left(w\right)\right] \tag{1183}$$

$$-\sum_{w} S_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) + \rho_S\left(t\right)A_3^{\dagger}\left(w\right)\right]\Bigg). \tag{1184}$$

The correlation functions are relevant if $F\left(\omega\right) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}\left(\tau\right) = \int_0^{\infty} \mathrm{d}\omega J\left(\omega\right) G_+\left(\tau\right), \tag{1185}$$

$$\Lambda_{33}\left(-\tau\right) = \int_0^{\infty} \mathrm{d}\omega J\left(\omega\right) G_+\left(-\tau\right). \tag{1186}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (1184) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)\right)] - \sum_{w}\left(K_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right)\right] + K_{33}^{*}\left(w,t\right)\left[\rho_S\left(t\right)A_3\left(w\right), A_3\right]\right). \tag{1187}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3\left(w\right)$, we have to remember that $H_S = \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{1188}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right), \tag{1189}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{1190}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{1191}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1192}$$

$$= \frac{1}{2}, \tag{1193}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{1194}$$

$$= \frac{1}{2}, \tag{1195}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1196}$$

$$= -\frac{1}{2}. \tag{1197}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{1198}$$

$$= \frac{\mathbb{I}}{2}, \tag{1199}$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle+| \tag{1200}$$

$$= \frac{1}{4}(\sigma_{\rm z} + {\rm i}\sigma_{\rm y}), \tag{1201}$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle-| \tag{1202}$$

$$= \frac{1}{4}(\sigma_z - {\rm i}\sigma_y). \tag{1203}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{{\rm d}\rho_S(t)}{{\rm d}t} = -{\rm i}\frac{\widetilde{\Omega}}{2}[\sigma_x, \rho_S(t))] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + {\rm i}\sigma_y)\rho_S(t)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - {\rm i}\sigma_y)\rho_S(t)\right] \tag{1204}$$

$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z + {\rm i}\sigma_y), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z - {\rm i}\sigma_y), \frac{\sigma_z}{2}\right]. \tag{1205}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega)\, G_+(\tau)\, e^{{\rm i}\widetilde{\Omega}\tau}{\rm d}\tau{\rm d}\omega \tag{1206}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{{\rm i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,{\rm e}^{-{\rm i}\tau\omega} + n(\omega)\, e^{{\rm i}\tau\omega}\right){\rm d}\tau{\rm d}\omega \tag{1207}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{{\rm i}\widetilde{\Omega}\tau}(n(\omega)+1)\,{\rm e}^{-{\rm i}\tau\omega}{\rm d}\tau{\rm d}\omega \tag{1208}$$

$$= \int_0^\infty \int_0^\infty J(\omega)(n(\omega)+1)\,{\rm e}^{{\rm i}\widetilde{\Omega}\tau - {\rm i}\tau\omega}{\rm d}\tau{\rm d}\omega \tag{1209}$$

$$= \int_0^\infty J(\omega)(n(\omega)+1)\,\pi\delta\left(\widetilde{\Omega} - \omega\right){\rm d}\omega \tag{1210}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right), \tag{1211}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega)\, G_+(\tau)\, e^{-{\rm i}\widetilde{\Omega}\tau}{\rm d}\tau{\rm d}\omega \tag{1212}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{-{\rm i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,{\rm e}^{-{\rm i}\tau\omega} + n(\omega)\, e^{{\rm i}\tau\omega}\right){\rm d}\tau{\rm d}\omega \tag{1213}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{-{\rm i}\widetilde{\Omega}\tau} n(\omega)\,{\rm e}^{{\rm i}\tau\omega}{\rm d}\tau{\rm d}\omega \tag{1214}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, n(\omega)\,{\rm e}^{-{\rm i}\widetilde{\Omega}\tau + {\rm i}\tau\omega}{\rm d}\tau{\rm d}\omega \tag{1215}$$

$$= \int_0^\infty J(\omega)\, n(\omega)\,\pi\delta\left(-\widetilde{\Omega} + \omega\right){\rm d}\omega \tag{1216}$$

$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right). \tag{1217}$$

Here we have used $\int_0^\infty {\rm d}s\, e^{\pm i\varepsilon s} = \pi\delta(\varepsilon) \pm {\rm i}\frac{\rm V.P.}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1204) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S(t)\right)] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S(t)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)\right]\right) \tag{1218}$$

$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S(t)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S(t)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right). \tag{1219}$$

This is the same result than the equation ($S17$), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x, \rho_S(t)\right)] - \frac{\pi}{8}J\left(\Omega(t)\right)\left((n\left(\Omega(t)\right)+1)[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)] + n\left(\Omega(t)\right)[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)]\right) \tag{1220}$$

$$- \frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right)+1\right)\left[\rho_S(t)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\Omega(t)\right)\left[\rho_S(t)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right). \tag{1221}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1222}$$

$$H_S(t) = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{1223}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right), \tag{1224}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b^\dagger_{u\mathbf{k}}b_{u\mathbf{k}}. \tag{1225}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} - f^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right) \tag{1226}$$

At first let's obtain $e^{\pm V}$ under the transformation (1226), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}}b^\dagger_{u\mathbf{k}} - f^*_{nu\mathbf{k}}b_{u\mathbf{k}}\right)$, so the equation (1226) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \tag{1227}$$
$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n| \hat{\varphi}_n\right)^2}{2!} + ... \tag{1228}$$
$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + ... \tag{1229}$$
$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + ... \tag{1230}$$
$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + ...\right) \tag{1231}$$
$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \tag{1232}$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*} b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}^{'}, \mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right) = e^{\pm\left(\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)} \tag{1233}$$
$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)}\right) \tag{1234}$$
$$= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) \tag{1235}$$
$$= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1236}$$
$$= \prod_u B_{nu\pm} \tag{1237}$$
$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1238}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V} A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1239}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1222):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1240}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1241}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1242}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1243}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1244}$$

$$= |0\rangle\langle 0|. \tag{1245}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1246}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1247}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1248}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, \tag{1249}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1250}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1251}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1252}$$

$$\textstyle\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1253}$$

$$\overline{\textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1254}$$

$$= \left(|0\rangle\langle 0| \textstyle\prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right)\left(\textstyle\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \textstyle\prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), \tag{1255}$$

$$= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, \tag{1256}$$

$$= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots \tag{1257}$$

$$= |0\rangle\langle 0| \left(\textstyle\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) \tag{1258}$$

$$+ |1\rangle\langle 1| \left(\textstyle\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots \tag{1259}$$

$$= |0\rangle\langle 0| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) \tag{1260}$$

$$+ |1\rangle\langle 1| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots \tag{1261}$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1262}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1263}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{1264}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) \tag{1265}$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{1266}$$

The transformed Hamiltonians of the equations (1223) to (1225) written in terms of (1240) to (1264) are:

$$\overline{H_S\left(t\right)}=\overline{\sum_{n}\varepsilon_{n}\left(t\right)\left|n\right\rangle\!\left\langle n\right|+\sum_{n\neq m}V_{nm}\left(t\right)\left|n\right\rangle\!\left\langle m\right|} \tag{1267}$$

$$=\overline{\sum_{n}\varepsilon_{n}\left(t\right)\left|n\right\rangle\!\left\langle n\right|}+\overline{\sum_{n\neq m}V_{nm}\left(t\right)\left|n\right\rangle\!\left\langle m\right|} \tag{1268}$$

$$=\sum_{n}\varepsilon_{n}\left(t\right)\left|n\right\rangle\!\left\langle n\right|+\sum_{n\neq m}V_{nm}\left(t\right)\left|n\right\rangle\!\left\langle m\right|\prod_{u}\left(B_{mu+}B_{nu-}\right) \tag{1269}$$

$$\overline{H_I}=\left(\sum_{n}\left|n\right\rangle\!\left\langle n\right|\prod_{u}B_{nu+}\right)\left(\sum_{nu\mathbf{k}}\left|n\right\rangle\!\left\langle n\right|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_{n}\left|n\right\rangle\!\left\langle n\right|\prod_{u}B_{nu-}\right) \tag{1270}$$

$$=\left(\sum_{n}\left|n\right\rangle\!\left\langle n\right|\prod_{u}B_{nu+}\right)\left(\sum_{u\mathbf{k}}\left|0\right\rangle\!\left\langle 0\right|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)+...\right)\left(\sum_{n}\left|n\right\rangle\!\left\langle n\right|\prod_{u}B_{nu-}\right) \tag{1271}$$

$$=\prod_{u}B_{0u+}\sum_{u\mathbf{k}}\left|0\right\rangle\!\left\langle 0\right|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_{u}B_{0u-}+\prod_{u}B_{1u+}\sum_{u\mathbf{k}}\left|1\right\rangle\!\left\langle 1\right|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_{u}B_{1u-}+\ldots \tag{1272}$$

$$=\sum_{u\mathbf{k}}\left|0\right\rangle\!\left\langle 0\right|\left(g_{0u\mathbf{k}}\prod_{u}B_{0u+}b_{u\mathbf{k}}^{\dagger}\prod_{u}B_{0u-}+g_{0u\mathbf{k}}^{*}\prod_{u}B_{0u+}b_{u\mathbf{k}}\prod_{u}B_{0u-}\right)+\sum_{u\mathbf{k}}\left|1\right\rangle\!\left\langle 1\right|\left(g_{1u\mathbf{k}}\prod_{u}B_{1u+}b_{u\mathbf{k}}^{\dagger}\prod_{u}B_{1u-}+g_{1u\mathbf{k}}^{*}\prod_{u}B_{1u+}b_{u\mathbf{k}}\prod_{u}B_{1u-}\right)+\ldots \tag{1273}$$

$$=\sum_{u\mathbf{k}}\left|0\right\rangle\!\left\langle 0\right|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\sum_{u\mathbf{k}}\left|1\right\rangle\!\left\langle 1\right|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\ldots \tag{1274}$$

$$=\sum_{nu\mathbf{k}}\left|n\right\rangle\!\left\langle n\right|\left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1275}$$

$$=\sum_{nu\mathbf{k}}\left|n\right\rangle\!\left\langle n\right|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1276}$$

$$\overline{H_B}=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}\left|n\right\rangle\!\left\langle n\right|\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1277}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H}=\sum_{n}\varepsilon_{n}(t)\left|n\right\rangle\!\left\langle n\right|+\sum_{n\neq m}V_{nm}(t)\left|n\right\rangle\!\left\langle m\right|\prod_{u}\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}\left|n\right\rangle\!\left\langle n\right|\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1278}$$

$$+\sum_{nu\mathbf{k}}\left|n\right\rangle\!\left\langle n\right|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1279}$$

Let’s define the following functions:

$$R_{n}\left(t\right)=\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1280}$$

$$B_{z,n}\left(t\right)=\sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)b_{u\mathbf{k}}^{\dagger}+\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)^{*}b_{u\mathbf{k}}\right) \tag{1281}$$

Using the previous functions we have that (1278) can be re-written in the following way:

$$\overline{H}=\sum_{n}\varepsilon_{n}(t)\left|n\right\rangle\!\left\langle n\right|+\sum_{n\neq m}V_{nm}(t)\left|n\right\rangle\!\left\langle m\right|\prod_{u}\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{n}R_{n}(t)\left|n\right\rangle\!\left\langle n\right|+\sum_{n}B_{z,n}(t)\left|n\right\rangle\!\left\langle n\right| \tag{1282}$$

$$\tag{1283}$$

Now in order to separate the elements of the hamiltonian (1283) let’s follow the references of the equations (**??**) and (**??**) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle\prod_u (B_{mu+}B_{nu-})\right\rangle_{\overline{H_0}}=\left\langle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1284}$$

$$= \left(\prod_{u\mathbf{k}}\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\left\langle\prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1285}$$

$$= \left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1286}$$

$$\equiv B_{nm} \tag{1287}$$

$$\left\langle\prod_u (B_{nu+}B_{mu-})\right\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1288}$$

$$= B^*_{nm} \tag{1289}$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+}B_{nu-}) - B_{nm} \tag{1290}$$

As we can see:

$$J^\dagger_{nm} = \left(\prod_u (B_{mu+}B_{nu-}) - B_{nm}\right)^\dagger \tag{1291}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B^*_{nm} \tag{1292}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{mn} \tag{1293}$$

$$= J_{mn} \tag{1294}$$

We can separate the Hamiltonian (1283) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)} = \sum_n (\varepsilon_n(t) + R_n)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|B_{nm} \tag{1295}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|J_{nm} + \sum_n B_{z,n}(t)|n\rangle\langle n|, \tag{1296}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \tag{1297}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right) + \langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H^2_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_B}}\right). \tag{1298}$$

Taking the equations (244)-(252) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right) = C\left(R_0, R_1, ..., R_{d-1}, B_{01}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^{\Re}_{nu\mathbf{k}}$ and $v^{\Im}_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1299}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1300}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}} v_{nu\mathbf{k}}^* + g_{nu\mathbf{k}}^* v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1301}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1302}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} + \mathrm{i} v_{nu\mathbf{k}}^{\Im} (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1303}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1304}$$

$$v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^* = \left(v_{mu\mathbf{k}}^{\Re} - \mathrm{i} v_{mu\mathbf{k}}^{\Im}\right)\left(v_{nu\mathbf{k}}^{\Re} + \mathrm{i} v_{nu\mathbf{k}}^{\Im}\right) - \left(v_{mu\mathbf{k}}^{\Re} + \mathrm{i} v_{mu\mathbf{k}}^{\Im}\right)\left(v_{nu\mathbf{k}}^{\Re} - \mathrm{i} v_{nu\mathbf{k}}^{\Im}\right) \tag{1305}$$

$$= \left(v_{mu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} + \mathrm{i} v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - \mathrm{i} v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} + v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}\right) \tag{1306}$$

$$- \left(v_{mu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - \mathrm{i} v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} + \mathrm{i} v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re} + v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}\right) \tag{1307}$$

$$= 2\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right) \tag{1308}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) v_{nu\mathbf{k}}^{\Re} + \mathrm{i} v_{nu\mathbf{k}}^{\Im} (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1309}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1310}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1311}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}} v_{mu\mathbf{k}}^* + v_{nu\mathbf{k}}^* v_{mu\mathbf{k}}) \tag{1312}$$

$$= \left(v_{mu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im}\right)^2 + \left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - \left(v_{nu\mathbf{k}}^{\Re} + \mathrm{i} v_{nu\mathbf{k}}^{\Im}\right)\left(v_{mu\mathbf{k}}^{\Re} - \mathrm{i} v_{mu\mathbf{k}}^{\Im}\right) \tag{1313}$$

$$- \left(v_{nu\mathbf{k}}^{\Re} - \mathrm{i} v_{nu\mathbf{k}}^{\Im}\right)\left(v_{mu\mathbf{k}}^{\Re} + \mathrm{i} v_{mu\mathbf{k}}^{\Im}\right) \tag{1314}$$

$$= \left(v_{mu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im}\right)^2 + \left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2\left(v_{nu\mathbf{k}}^{\Re} v_{mu\mathbf{k}}^{\Re} + v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Im}\right) \tag{1315}$$

$$= \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2 \tag{1316}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1317}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - \left(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*\right) v_{nu\mathbf{k}}^{\Re} - \mathrm{i} v_{nu\mathbf{k}}^{\Im} \left(g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1318}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re} - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}\right) \tag{1319}$$

$$B_{mn} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1320}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im} v_{mu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right)^2 + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1321}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1322}$$

$$= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1323}$$

$$= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1324}$$

$$\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1325}$$

$$= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1326}$$

$$= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1327}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) \tag{1328}$$

$$= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1329}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1330}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1331}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1332}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \tag{1333}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \tag{1334}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} \tag{1335}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1336}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1337}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1338}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1339}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1340}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1341}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1342}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1343}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1344}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1345}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1346}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1347}$$

Introducing this derivates in the equation (1322) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1348}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1349}$$

$$= 0 \tag{1350}$$

We can obtain the variational parameters:

$$- 2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1351}$$

$$= - \frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1352}$$

$$v^{\Re}_{nu\mathbf{k}} = \frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1353}$$

$$= \frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1354}$$

Let’s consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1355}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1356}$$

$$= 0 \tag{1357}$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1358}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1359}$$

$$v^{\Im}_{nu\mathbf{k}}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1360}$$

$$=\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1361}$$

$$v_{nu\mathbf{k}}=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}} \tag{1362}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1363}$$

$$\mathrm{i}\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1364}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1365}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1366}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1367}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1368}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1369}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1370}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^{V}\rho(0)e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1371}$$

$$0=\left(B_0^{+}|0\rangle\langle 0|B_0^{-}\right)\otimes\rho_B \tag{1372}$$

$$0=\rho(0) \tag{1373}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1374}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1375}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\,, \text{ where} \tag{1376}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1377}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1378}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1379}$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1380}$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1381}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1382}$$

$$= B_{n-}^{\dagger} B_{m+}^{\dagger} - B_n B_m \tag{1383}$$

$$= B_{n+}B_{m-} - B_n B_m \tag{1384}$$

$$= B_{nm} \tag{1385}$$

So we can say that the set of matrices (1378) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1378) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t)\,|n\rangle\langle n|, \tag{1386}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)\,|n\rangle\langle m| B_{nm} + V_{mn}(t)\,|m\rangle\langle n| B_{mn}\right) \tag{1387}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t)\, B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1388}$$

$$\left. + \Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t)\, B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1389}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1390}$$

$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1391}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1392}$$

$$\left. + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,x}\right) \tag{1393}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1394}$$

Now consider the following set of operators,

$$A_{1,nm}\left(t\right)=\sigma_{nm,x}\left(1-\delta_{mn}\right) \tag{1395}$$
$$A_{2,nm}\left(t\right)=\sigma_{nm,y}\left(1-\delta_{mn}\right) \tag{1396}$$
$$A_{3,nm}\left(t\right)=\delta_{mn}|n\rangle\langle m| \tag{1397}$$
$$A_{4,nm}\left(t\right)=A_{2,mn}\left(t\right) \tag{1398}$$
$$A_{5,nm}\left(t\right)=A_{1,nm}\left(t\right) \tag{1399}$$
$$B_{1,nm}\left(t\right)=B_{nm,x} \tag{1400}$$
$$B_{2,nm}\left(t\right)=B_{nm,y} \tag{1401}$$
$$B_{3,nm}\left(t\right)=B_{z,n}\left(t\right) \tag{1402}$$
$$B_{4,nm}\left(t\right)=B_{1,nm}\left(t\right) \tag{1403}$$
$$B_{5,nm}\left(t\right)=B_{2,nm}\left(t\right) \tag{1404}$$
$$C_{1,nm}\left(t\right)=\Re\left(V_{nm}\left(t\right)\right) \tag{1405}$$
$$C_{2,nm}\left(t\right)=C_{1,nm}\left(t\right) \tag{1406}$$
$$C_{3,nm}\left(t\right)=1 \tag{1407}$$
$$C_{4,nm}\left(t\right)=\Im\left(V_{nm}\left(t\right)\right) \tag{1408}$$
$$C_{5,nm}\left(t\right)=-\Im\left(V_{nm}\left(t\right)\right) \tag{1409}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}\left(t\right)$ as:

$$\overline{H_I}=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(A_{jp}\otimes B_{jp}\left(t\right)\right) \tag{1410}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1394).

We write the interaction Hamiltonian transformed under (1374) as:

$$\widetilde{H_I}\left(t\right)=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1411}$$
$$\widetilde{A_{jp}}\left(t\right)=U^{\dagger}\left(t\right)A_{jp}U\left(t\right) \tag{1412}$$
$$\widetilde{B_{jp}}\left(t\right)=\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\left(t\right)\left(\mathrm{t}\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{1413}$$

Taking as reference state ρ_B and truncating at second order in $H_I\left(t\right)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}\left(t\right)}{\mathrm{d}t}=-\int_0^t\mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right),\left[\widetilde{H_I}\left(s\right),\widetilde{\rho_S}\left(t\right)\rho_B\right]\right]\mathrm{d}s \tag{1414}$$

Replacing the equation (1411) in (1414) we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1415}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1416}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1417}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1418}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1419}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1420}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1421}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1422}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1423}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1424}$$

Here $s\to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1422) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1425}$$

$$= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1426}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1427}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1428}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1429}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1430}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1431}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) \tag{1432}$$

$$= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1433}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1434}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1435}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1436}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1437}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1438}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1439}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1415)and (1422) and using the equations (1425)-(1439) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}\left(t\right)}{\mathrm{d}t} = -\int_0^t\sum_{j,j',p,p'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)-\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\right)\right. \tag{1440}$$

$$\left.+C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right)-\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1441}$$

$$= -\int_0^t\sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[\widetilde{A_{jp}}\left(t\right),\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]+\Lambda_{j'p'jp}\left(-\tau\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right),\widetilde{A_{jp}}\left(t\right)\right]\right)\right) \tag{1442}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\int_0^t\sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right),A_{j'p'}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right]+\Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}\left(t\right)A_{j'p'}\left(t-\tau,t\right),A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau-\mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}\left(t\right)\right] \tag{1443}$$

For this case we used that $A_{jp}\left(t-\tau,t\right)=U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1444}$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{1445}$$

$$H_I = \left(\sum_{n=0} \mu_n(t)\,|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1446}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1447}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t)\,|n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1448}$$

At first let's obtain e^V under the transformation (1448), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{1449}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + \ldots \tag{1450}$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + \ldots \tag{1451}$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n|\left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \ldots\right) \tag{1452}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1453}$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+} \tag{1454}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1455}$$

$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1456}$$

$$= B_{\pm} \tag{1457}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_{-}$ because this form imposes that $e^{-V} e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+}\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{-}\right) \tag{1458}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1444):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1459}$$

$$= |0\rangle\langle 0|, \tag{1460}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1461}$$

$$= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, \tag{1462}$$

$$= |m\rangle\langle n|,\; m \neq 0,\; n \neq 0, \tag{1463}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1464}$$

$$= |0\rangle\langle m|B^{-}\; m \neq 0, \tag{1465}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1466}$$

$$= |0\rangle\langle m|B^{+}\; m \neq 0, \tag{1467}$$

$$\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1468}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B^{+}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B^{-} \tag{1469}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B^{+}b^{\dagger}_{\mathbf{k}}B^{-}\right)\left(B^{+}b_{\mathbf{k}}B^{-}\right) \tag{1470}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1471}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1472}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right) \tag{1473}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right) \tag{1474}$$

The transformed Hamiltonians of the equations (1445) to (1447) written in terms of (1459) to (1474) are:

$$\overline{H_{\bar{S}}\left(t\right)} = \overline{\sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|} \tag{1475}$$
$$= \overline{\sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n|} + \overline{\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|} \tag{1476}$$
$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \overline{\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n|} \tag{1477}$$
$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)\overline{|0\rangle\langle n|} + V_{n0}\left(t\right)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)\overline{|m\rangle\langle n|} \tag{1478}$$
$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)B^{-}|0\rangle\langle n| + V_{n0}\left(t\right)B^{+}|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1479}$$
$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1480}$$
$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(\left(\sum_{n=0}\mu_n\left(t\right)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1481}$$
$$= \left(\mu_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|B^{+}\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1482}$$
$$= \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}B^{+}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)B^{-} \tag{1483}$$
$$= \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1484}$$
$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1485}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1486}$$
$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1487}$$
$$+\sum_{n=0}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1488}$$
$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1489}$$
$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1490}$$
$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n\left(t\right) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1491}$$

Let's define the following functions:

$$R_n\left(t\right) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1492}$$
$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1493}$$
$$B_{z,n}\left(t\right) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n\left(t\right) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1494}$$

Using the previous functions we have that (1491) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n (t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n} (t) |0\rangle\langle n|B^- + V_{n0} (t) |n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0} V_{mn} (t) |m\rangle\langle n| \tag{1495}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0 (t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1496}$$

Now in order to separate the elements of the hamiltonian (1496) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian like:

$$\overline{H_S (t)} = \sum_{n=0} \varepsilon_n (t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n} (t) |0\rangle\langle n| + V_{n0} (t) |n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn} (t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \tag{1497}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0 (t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1} \left(V_{0n} (t) |0\rangle\langle n| \left(B^- - B\right) + V_{n0} (t) |n\rangle\langle 0| \left(B^+ - B\right)\right), \tag{1498}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{1499}$$

Here B is given by:

$$\begin{aligned} B &= \left\langle B^+ \right\rangle \\ &= \left\langle B^- \right\rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1500}$$

$$\sigma_{nm,y} = \mathrm{i} \left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1501}$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1502}$$

$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1503}$$

Using this set of hermitian operators to write the Hamiltonians (1445)-(1447)

$$\overline{H_S\left(t\right)} = \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1504}$$

$$= \varepsilon_0\left(t\right)|0\rangle\langle 0| + B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{0<m<n}\left(V_{mn}\left(t\right)|m\rangle\langle n| + V_{nm}\left(t\right)|n\rangle\langle m|\right) \tag{1505}$$

$$+ \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1506}$$

$$= \sum_{0<m<n}\left(\left(\Re\left(V_{mn}\left(t\right)\right) + \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)|m\rangle\langle n| + \left(\Re\left(V_{mn}\left(t\right)\right) - \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)|n\rangle\langle m|\right) + \varepsilon_0\left(t\right)|0\rangle\langle 0| \tag{1507}$$

$$+ B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1508}$$

$$= \sum_{0<m<n}\left(\left(\Re\left(V_{nm}\left(t\right)\right) + \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + \left(\Re\left(V_{nm}\left(t\right)\right) - \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1509}$$

$$+ B\sum_{n=1}\left(V_{0n}\left(t\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}\left(t\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1510}$$

$$= \sum_{0<m<n}\left(\Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,x} + \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}\right) + B\sum_{n=1}\left(\Re\left(V_{0n}\left(t\right)\right)\sigma_{0n,x} + \Im\left(V_{mn}\left(t\right)\right)\sigma_{0n,y}\right) \tag{1511}$$

$$+ \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1512}$$

$$\overline{H_I\left(t\right)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}\left(t\right)|n\rangle\langle 0|\left(B^+ - B\right)\right) \tag{1513}$$

$$= \sum_{n=1}\left(\left(\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\left(B^- - B\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + \left(\Re\left(V_{0n}\left(t\right)\right) - \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\left(B^+ - B\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1514}$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1515}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^- - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right) + \left(B^+ - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) - \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\right)\right. \tag{1516}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) - \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right) - \left(B^- - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\right)\right) \tag{1517}$$

$$+ \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1518}$$

$$= \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^+ + B^- - 2B\right)\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\left(B^- - B - B^+ + B\right)\Im\left(V_{0n}\left(t\right)\right)\right. \tag{1519}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B - B^- + B\right)\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\left(B - B^- + B - B^+\right)\Im\left(V_{0n}\left(t\right)\right)\right)\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1520}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\sigma_{0n,x}\left(B_x\Re\left(V_{0n}\left(t\right)\right) - B_y\Im\left(V_{0n}\left(t\right)\right)\right)\right. \tag{1521}$$

$$\left. + \sigma_{0n,y}\left(B_y\Re\left(V_{0n}\left(t\right)\right) + B_x\Im\left(V_{0n}\left(t\right)\right)\right)\right) \tag{1522}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_S+H_B}\right)}\right)\right)+\left\langle\overline{H_I}\right\rangle_{\overline{H_S+H_B}}+O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1523}$$

Taking the equations (244)-(252) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)=C\left(R_1,R_2,...,R_{d-1},B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (252) and using the chain rule we obtain that:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}}+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, &(1524)\\ &=0 &(1525)\end{aligned}$$

Let's recall the equations (1492) and (1494), we can write them in terms of the variational parameters

$$\begin{aligned}B &= \exp\left(-\left(1/2\right)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) &(1526)\\ R_n &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) &(1527)\end{aligned}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\begin{aligned}\frac{\partial B}{\partial v_{\mathbf{k}}} &= -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B &(1528)\\ \frac{\partial R_n}{\partial v_{\mathbf{k}}} &= \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1529)\end{aligned}$$

Introducing this derivates in the equation (1524) give us:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right)+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1530)\\ &= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right)-\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) &(1531)\end{aligned}$$

We can obtain the variational parameters:

$$\begin{aligned}v_{\mathbf{k}} &= \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1532)\\ &= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1533)\end{aligned}$$

Now taking $v_{\mathbf{k}}=g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}}=\frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1534}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$
\begin{aligned}
e^{V} \rho(0) e^{-V} = & \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+} \right) \left(|0\rangle\langle 0| \otimes \rho_B \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{-} \right) & (1535) \\
0 = & |0\rangle\langle 0| \otimes \rho_B & (1536) \\
0 = & \rho(0) & (1537)
\end{aligned}
$$

We transform any operator O into the interaction picture in the following way:

$$
\begin{aligned}
\widetilde{O} & \equiv U^{\dagger}(t) O U(t) & (1538) \\
U(t) & \equiv \mathcal{T}\exp\left(-\mathrm{i} \int_0^t \mathrm{d}t' \overline{H_S}(t') \right). & (1539)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\widetilde{\overline{\rho_S}}(t) & = U^{\dagger}(t) \overline{\rho_S}(t) U(t), \text{ where} & (1540) \\
\overline{\rho_S}(t) & = \mathrm{Tr}_B(\bar{\rho}(t)) & (1541)
\end{aligned}
$$

We can re-write the transformed interaction Hamiltonian operator like:

$$
\begin{aligned}
\overline{H_I(t)} = & B_{z,0}|0\rangle\langle 0| + \sum_{n=1} \left(\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n}|n\rangle\langle n| \right. & (1542) \\
& \left. + \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x} \right) & (1543)
\end{aligned}
$$

where

$$
\begin{aligned}
B_{z,0} & = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}} \right) & (1544) \\
B_{z,n} & = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}} \right) \text{ if} n \neq 0 & (1545)
\end{aligned}
$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1546}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1547}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1548}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1549}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1550}$$
$$B_{1n}(t) = B_x \tag{1551}$$
$$B_{2n}(t) = B_y \tag{1552}$$
$$B_{3n}(t) = B_{z,n} \tag{1553}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1554}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1555}$$
$$C_{10}(t) = 0 \tag{1556}$$
$$C_{20}(t) = 0 \tag{1557}$$
$$C_{40}(t) = 0 \tag{1558}$$
$$C_{50}(t) = 0 \tag{1559}$$
$$C_{30}(t) = 1 \tag{1560}$$
$$C_{1n}(t) = \Re\left(V_{0n}(t)\right) \tag{1561}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1562}$$
$$C_{3n}(t) = 1 \tag{1563}$$
$$C_{4n}(t) = \Im\left(V_{0n}(t)\right) \tag{1564}$$
$$C_{5n}(t) = -\Im\left(V_{0n}(t)\right) \tag{1565}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J} \sum_{n=1} C_{jn}(t)\left(A_{jn} \otimes B_{jn}(t)\right) \tag{1566}$$

Here $J = \{1, 2, 3, 4, 5\}$.
We write the interaction Hamiltonian transformed under (1538) as:

$$\widetilde{H_I}(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1567}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\, A_i U(t) \tag{1568}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{1569}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\, \rho_B\right]\right] \mathrm{d}s \tag{1570}$$

Replacing the equation (1567)in (1570)we can obtain:

$$
\begin{align}
\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1571}\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1572}\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1573}\\
&\left. -\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1574}\\
&= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1575}\\
&-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1576}\\
&-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1577}\\
&\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1578}
\end{align}
$$

In order to calculate the correlation functions we define:

$$
\begin{align}
\Lambda_{jnj'n'}(\tau) &= \left\langle \widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1579}\\
&= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1580}
\end{align}
$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1578) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right)=\left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1581}$$
$$=\left\langle\widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1582}$$
$$=\Lambda_{jnj'n'}(\tau) \tag{1583}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right)=\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1584}$$
$$=\left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1585}$$
$$=\left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1586}$$
$$=\Lambda_{j'n'jn}(-\tau) \tag{1587}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right)=\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1588}$$
$$=\left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1589}$$
$$=\left\langle\widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1590}$$
$$=\Lambda_{jnj'n'}(\tau) \tag{1591}$$
$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right)=\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1592}$$
$$=\left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1593}$$
$$=\left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1594}$$
$$=\Lambda_{j'n'jn}(-\tau) \tag{1595}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1571)and (1578) and using the equations (1581)-(1595) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t}=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)-\Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1596}$$

$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t)-\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1597}$$

$$=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t)\,,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\right]+\Lambda_{j'n'jn}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,,\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1598}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t}=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t)\,,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right]+\Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t)\,,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau-\mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1599}$$

For this case we used that $A_{jn}(t-\tau,t)=U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (451) as expected.

VIII. BIBLIOGRAPHY

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