A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

Nike Dattani*
Harvard-Smithsonian Center for Astrophysics

Camilo Chaparro Sogamoso†
National University of Colombia

I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right) = H_S\left(t\right) + H_I + H_B, \tag{1}$$

$$H_S\left(t\right) = \varepsilon_0\left(t\right)|0\rangle\langle 0| + \varepsilon_1\left(t\right)|1\rangle\langle 1| + V_{10}\left(t\right)|1\rangle\langle 0| + V_{01}\left(t\right)|0\rangle\langle 1|, \tag{2}$$

$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H\left(t\right)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}}\left(t\right) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}\left(t\right). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V = |0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6) will give us the following result:

$$
\begin{aligned}
e^{\pm V} &= e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)} && (10)\\
&= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + ... && (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + ... && (12)\\
&= |0\rangle\langle 0| \left(\mathbb{I} \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm ...\right) + |1\rangle\langle 1| \left(\mathbb{I} \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm ...\right) && (13)\\
&= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} && (14)\\
&= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_0^{\pm} + |1\rangle\langle 1|B_1^{\pm}, && (16)\\
B_i^{\pm} &\equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
e^{t(X+Y)} = e^{tX}\, e^{tY}\, e^{-\frac{t^2}{2}[X,Y]}\, e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \cdots. \tag{18}
$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}$ and i, j we can show making $t = 1$ in (18) the following result:

$$
\begin{aligned}
e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\ldots && (19)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}\,0}\ldots && (20)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. && (21)
\end{aligned}
$$

By induction of this result we can write an expresion of $B_i^{\pm}$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}})$:

$$
\begin{aligned}
D(\pm v_{i\mathbf{k}}) &\equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, && (22)\\
B_i^{\pm} &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right). && (23)
\end{aligned}
$$

this will help us to write operators O transformed in the variational frame as:

$$
\overline{O} \equiv \mathrm{e}^{V} O \mathrm{e}^{-V}. \tag{24}
$$

We will use the following identities:

$$\overline{|0\rangle\langle 0|} = \mathrm{e}^{V}|0\rangle\langle 0|\mathrm{e}^{-V} \tag{25}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{26}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{27}$$
$$= |0\rangle\langle 0|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{28}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+B_1^- \tag{29}$$
$$= |0\rangle\langle 0|, \tag{30}$$
$$\overline{|1\rangle\langle 1|} = \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{31}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+ + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{32}$$
$$= |1\rangle\langle 1|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{33}$$
$$= |1\rangle\langle 1|0\rangle\langle 0|B_1^+B_0^- + B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- \tag{34}$$
$$= B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- \tag{35}$$
$$= |1\rangle\langle 1|, \tag{36}$$
$$\overline{|0\rangle\langle 1|} = \mathrm{e}^{V}|0\rangle\langle 1|\mathrm{e}^{-V} \tag{37}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{38}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|B_1^+|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{39}$$
$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{40}$$
$$= |0\rangle\langle 1|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{41}$$
$$= |0\rangle\langle 1|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 1|1\rangle\langle 1|B_0^+B_1^- \tag{42}$$
$$= |0\rangle\langle 1|B_0^+B_1^-, \tag{43}$$
$$\overline{|1\rangle\langle 0|} = \mathrm{e}^{V}|1\rangle\langle 0|\mathrm{e}^{-V} \tag{44}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{45}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{46}$$
$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{47}$$
$$= |1\rangle\langle 0|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{48}$$
$$= |1\rangle\langle 0|B_1^+|0\rangle\langle 0|B_0^- + |1\rangle\langle 0|B_1^+|1\rangle\langle 1|B_1^- \tag{49}$$
$$= |1\rangle\langle 0|0\rangle\langle 0|B_1^+B_0^- + |1\rangle\langle 0|1\rangle\langle 1|B_1^+B_1^- \tag{50}$$
$$= |1\rangle\langle 0|B_1^+B_0^-, \tag{51}$$
$$\overline{b_{\mathbf{k}}} = \mathrm{e}^{V}b_{\mathbf{k}}\mathrm{e}^{-V} \tag{52}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{53}$$
$$= |0\rangle\langle 0|B_0^+b_{\mathbf{k}}B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+b_{\mathbf{k}}|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}B_1^-|1\rangle\langle 1| \tag{54}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+b_{\mathbf{k}}B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+b_{\mathbf{k}}B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+b_{\mathbf{k}}B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}B_1^- \tag{55}$$
$$= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{56}$$
$$= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{57}$$
$$= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{58}$$
$$\overline{b_{\mathbf{k}}}^{\dagger} = \mathrm{e}^{V}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V} \tag{59}$$
$$= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) \tag{60}$$
$$= |0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}B_1^-|1\rangle\langle 1| \tag{61}$$
$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+b_{\mathbf{k}}^{\dagger}B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+b_{\mathbf{k}}^{\dagger}B_0^- + |1\rangle\langle 1|1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}B_1^- \tag{62}$$
$$= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) \tag{63}$$
$$= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \tag{64}$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+} b_{\mathbf{k}} B_i^{-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{65}$$

$$B_i^{+} b_{\mathbf{k}}^{\dagger} B_i^{-} = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{66}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{67}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{68}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|} = V_{10}(t)\,|1\rangle\langle 0|B_1^{+}B_0^{-}, \tag{69}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|} = V_{01}(t)\,|0\rangle\langle 1|B_0^{+}B_1^{-}, \tag{70}$$

$$\overline{g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}}} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\right) + g_{\mathbf{ik}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{71}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{72}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{73}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{74}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0|B_0^{+} + |1\rangle\langle 1|B_1^{+}\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-} + |1\rangle\langle 1|B_1^{-}\right) \tag{75}$$

$$= |0\rangle\langle 0|B_0^{+}|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^{-} \tag{76}$$

$$= |0\rangle\langle 0|B_0^{+}\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)B_0^{-} \tag{77}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{78}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0|B_0^{+} + |1\rangle\langle 1|B_1^{+}\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-} + |1\rangle\langle 1|B_1^{-}\right) \tag{79}$$

$$= |1\rangle\langle 1|B_1^{+}|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^{-} \tag{80}$$

$$= |1\rangle\langle 1|B_1^{+}\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right)B_1^{-} \tag{81}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{82}$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+} + |1\rangle\langle 1|B_1^{+}\right)b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{-} + |1\rangle\langle 1|B_1^{-}\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|\prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1|\prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_0^{-} + |1\rangle\langle 1|B_1^{+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_1^{-}\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}' \neq \mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) + |1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}' \neq \mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{87}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{88}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} \right) \right) \tag{89}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{90}$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{91}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{92}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^{*} b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^{*} b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \tag{93}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*} b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*} b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right). \tag{94}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} \tag{95}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0| B_1^{+} B_0^{-} + V_{01}(t)\,|0\rangle\langle 1| B_0^{+} B_1^{-}, \tag{96}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}} \right)} \tag{97}$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}} \right)} \tag{98}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^{*} \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^{*} \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{99}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^{*} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^{*} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{100}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \tag{101}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*} b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*} b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right) \tag{102}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*} b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*} b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right). \tag{103}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \textstyle\sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_j^{+} B_{j'}^{-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^{*} b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^{*} \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right). \tag{104}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{105}$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{106}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right), \tag{107}$$

$$\chi_{ij} \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right). \tag{108}$$

χ_{ij} is an imaginary number so $e^{\chi_{ij}}$ is the phase associated to B_{ij}. Wwe can summarize these definitions with other that we will proof and use from now in the following matrix:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_{ij} \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{\chi_{ij}} \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2\mathrm{i}} & \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix}. \tag{109}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{110}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{111}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_{\mathrm{B}} T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)}. \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{113}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{114}$$

So the product of the matrix representation of $b^\dagger$ and b with $-\beta$ is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \, |j\rangle\langle j| \,, \tag{116}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \tag{117}$$

$$\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{118}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{119}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{120}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{121}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{122}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), \tag{123}$$

$$\mathrm{Tr}\left(\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{124}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{125}$$

$$= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{126}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{127}$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{128}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{129}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}} - 1\rangle, \tag{130}$$
$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} \mid j_{\mathbf{k}} + 1\rangle. \tag{131}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (130)-(131):

$$\begin{aligned}
\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Tr}\left(\rho_B B_{iz}\right) = \mathrm{Tr}\left(B_{iz}\rho_B\right) && (132)\\
&= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}}\right)\right)\rho_B\right) && (133)\\
&= \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}}\rho_B\right) && (134)\\
&= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) && (135)\\
&= \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^{*} b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) && (136)\\
&= \sum_{\mathbf{k}} \left(\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}}\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}} \left(\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}}\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), && (137)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \text{ (by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}\text{)} && (138)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}} + 1}\,|j_{\mathbf{k}} + 1\rangle\langle j_{\mathbf{k}}|\right) && (139)\\
&= 0, && (140)\\
\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \text{ (by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} && (141)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}} - 1\rangle\langle j_{\mathbf{k}}|\right) && (142)\\
&= 0. && (143)
\end{aligned}$$

we therefore find that:

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0. \tag{144}$$

Another important expected value is $B = \langle B^{\pm}\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm} = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\begin{aligned}
\left\langle B^{\pm}\right\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B_{\pm}\right) = \mathrm{Tr}\left(B_{\pm}\rho_B\right) && (145)\\
&= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) && (146)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\right)\rho_B\right) && (147)\\
&= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\right)\right\rangle. && (148)
\end{aligned}$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \tag{149}$$

where $P(\alpha)$ satisfies $\int P(\alpha)\,\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P(\alpha)$ is given by:

$$\begin{aligned}\langle A\rangle =&\mathrm{Tr}\left(A\rho\right) &(150)\\ =&\int P(\alpha)\,\langle\alpha\,|A|\,\alpha\rangle\mathrm{d}^2\alpha. &(151)\end{aligned}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega}-1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (151) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\begin{aligned}
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle\alpha|D(h)\,|\alpha\rangle\mathrm{d}^2\alpha &(152)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(-\alpha)\,D(h)\,D(\alpha)\,|0\rangle\mathrm{d}^2\alpha, &(153)\\
D(h)\,D(\alpha) &= D(h+\alpha)\,e^{\frac{1}{2}(h\alpha^*-h^*\alpha)}, &(154)\\
D(-\alpha)\,(D(h)\,D(\alpha)) &= D(-\alpha)\,D(h+\alpha)\,e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} &(155)\\
&= D(h)\,e^{\frac{1}{2}(-\alpha(h+\alpha)^*+\alpha^*(h+\alpha))}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} &(156)\\
&= D(\alpha)\,e^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^* h+|\alpha|^2\right)}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} &(157)\\
&= D(\alpha)\,e^{(h\alpha^*-h^*\alpha)}, &(158)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(h)\exp\left(h\alpha^*-h^*\alpha\right)|0\rangle\mathrm{d}^2\alpha &(159)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|D(h)\,|0\rangle\mathrm{d}^2\alpha &(160)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|h\rangle\mathrm{d}^2\alpha, &(161)\\
|\alpha\rangle &= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, &(162)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle\mathrm{d}^2\alpha &(163)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp\left(h\alpha^*-h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha &(164)\\
&= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha\right)\mathrm{d}^2\alpha, &(165)
\end{aligned}$$

$$\alpha = x + \mathrm{i}y, \tag{166}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{N}+h\left(x-\mathrm{i}y\right)-h^*\left(x+\mathrm{i}y\right)\right)\mathrm{d}x\mathrm{d}y \tag{167}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{N}+hx-h^*x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y\right)\mathrm{d}y, \tag{168}$$

$$-\frac{x^2}{N}+hx-h^*x = -\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) \tag{169}$$

$$= -\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2+\frac{N\left(h^*-h\right)^2}{4}, \tag{170}$$

$$-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y = -\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) \tag{171}$$

$$= -\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)-\frac{N\left(h+h^*\right)^2}{4}, \tag{172}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, \tag{173}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\exp\left(-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right)\mathrm{d}x, \tag{174}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(x+\frac{(Nh^*-Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}y \tag{175}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{176}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}\right) \tag{177}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}\right) \tag{178}$$

$$= \exp\left(-|h|^2\left(N+\frac{1}{2}\right)\right) \tag{179}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)\right) \tag{180}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)\right) \tag{181}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{182}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (147) using (182) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{183}$$

We will now force $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B^+\sigma^+$ and $B^-\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$
\begin{aligned}
\overline{H(t)} &= \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) && (184)\\
&= \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{jj'} + \sum_j |j\rangle\langle j| B_{jz} + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+ B_{j'}^- - B_{jj'} \right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (185)\\
&\equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. && (186)
\end{aligned}
$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$
\begin{aligned}
\langle B_i^+ B_j^- \rangle &= B_{ij} && (187)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle && (188)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle && (189)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle && (190)\\
&= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (191)\\
&= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (192)\\
&= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}. && (193)
\end{aligned}
$$

From the definition $B_{01} = \langle B_0^+ B_1^- \rangle$ using the displacement operator we have:

$$
\begin{aligned}
\langle B_0^+ B_1^- \rangle &= B_{01} && (194)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle && (195)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle && (196)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle && (197)\\
&= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) && (198)\\
&= \prod_{\mathbf{k}} \left(\exp\left(-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) && (199)\\
&= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}. && (200)
\end{aligned}
$$

We can check:

$$\langle B_0^+ B_1^- \rangle = B_{01} \tag{201}$$
$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{202}$$
$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)^*} \tag{203}$$
$$= \langle B_1^+ B_0^- \rangle^* \tag{204}$$
$$= B_{10}^*. \tag{205}$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0)|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)|1\rangle\langle 1| + V_{10}(t) B_{10}\sigma^+ + V_{01}(t) B_{01}\sigma^-, \tag{206}$$
$$\overline{H_{\bar{I}}} \equiv V_{10}(t)\left(B_1^+ B_0^- - B_{10}\right)\sigma^+ + V_{01}(t)\left(B_0^+ B_1^- - B_{01}\right)\sigma^- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, \tag{207}$$
$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{208}$$
$$= H_B. \tag{209}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (207) we can verify the condition $\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\right)|n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+ B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{210}$$
$$= \left\langle \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+ B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{211}$$
$$= \sum_{n\mathbf{k}}\left(\left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + \langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}(t) B_j^+ B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}} - \langle V_{jj'}(t) B_{jj'}\rangle_{\overline{H_{\bar{B}}}}\right) \tag{212}$$
$$= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(\left\langle B_j^+ B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}} - \langle B_{jj'}\rangle_{\overline{H_{\bar{B}}}}\right) \tag{213}$$
$$= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}} - v_{n\mathbf{k}})\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(B_{jj'} - B_{jj'}\right) \tag{214}$$
$$= 0. \tag{215}$$

We used (144) and (193) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_x = B_x^\dagger \tag{216}$$
$$= \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}, \tag{217}$$
$$B_y = B_y^\dagger \tag{218}$$
$$= \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2\mathrm{i}}, \tag{219}$$
$$B_{iz} = B_{iz}^\dagger \tag{220}$$
$$= \sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}}\right). \tag{221}$$

Writing the equations (206) and (207) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}(t)=\sum_{j\in\{0,1\}}\left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j|+V_{10}\left(t\right)B_{10}\sigma^{+}+V_{01}\left(t\right)B_{01}\sigma^{-} \tag{222}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j|+V_{10}\left(t\right)B_{10}\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}\left(t\right)B_{01}\frac{\sigma_x-\mathrm{i}\sigma_y}{2} \tag{223}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j|+V_{10}\left(t\right)\left(B_{10}^{\Re}\left(t\right)+\mathrm{i}B_{10}^{\Im}\left(t\right)\right)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\left(B_{10}^{\Re}\left(t\right)-\mathrm{i}B_{10}^{\Im}\left(t\right)\right)\frac{\sigma_x-\mathrm{i}\sigma_y}{2} \tag{224}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j|+B_{10}^{\Re}(t)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}-V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right) \tag{225}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_j(t)+R_j\right)|j\rangle\langle j|+B_{10}^{\Re}(t)\left(\sigma_x\frac{V_{10}(t)+V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)-V_{01}(t)}{2}\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\sigma_x\frac{V_{10}(t)-V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)+V_{01}(t)}{2}\right) \tag{226}$$

$$=\sum_{j\in\{0,1\}}\left(\varepsilon_j(t)+R_j\right)|j\rangle\langle j|+B_{10}^{\Re}(t)\left(\sigma_xV_{10}^{\Re}\left(t\right)-\sigma_yV_{10}^{\Im}\left(t\right)\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\mathrm{i}\sigma_xV_{10}^{\Im}(t)+\mathrm{i}\sigma_yV_{10}^{\Re}(t)\right) \tag{227}$$

$$=\left(\varepsilon_0(t)+R_0\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1\right)|1\rangle\langle 1|+B_{10}^{\Re}(t)\left(\sigma_xV_{10}^{\Re}\left(t\right)-\sigma_yV_{10}^{\Im}\left(t\right)\right)+\mathrm{i}B_{10}^{\Im}(t)\left(\mathrm{i}\sigma_xV_{10}^{\Im}(t)+\mathrm{i}\sigma_yV_{10}^{\Re}(t)\right) \tag{228}$$

$$=\left(\varepsilon_0(t)+R_0\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1\right)|1\rangle\langle 1|+\left(\sigma_xB_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-\sigma_yB_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)\right)-\left(\sigma_xB_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)+\sigma_yB_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right) \tag{229}$$

$$=\left(\varepsilon_0\left(t\right)+R_0\right)|0\rangle\langle 0|+\left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right)-\sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right) \tag{230}$$

$$=\left(\varepsilon_0(t)+R_0\right)|0\rangle\langle 0|+\left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right)-\sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right). \tag{231}$$

$$\overline{H_{\bar{I}}}=V_{10}\left(t\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}\right)+V_{01}\left(t\right)\left(\sigma^{-}B_0^{+}B_1^{-}-\sigma^{-}B_{01}\right)+|0\rangle\langle 0|B_{0z}+|1\rangle\langle 1|B_{1z} \tag{232}$$

$$=|0\rangle\langle 0|B_{0z}+|1\rangle\langle 1|B_{1z}+\left(V_{10}^{\Re}\left(t\right)+\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}\right)+\left(V_{10}^{\Re}\left(t\right)-\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{-}B_0^{+}B_1^{-}-\sigma^{-}B_{01}\right) \tag{233}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}+\sigma^{-}B_0^{+}B_1^{-}-\sigma^{-}B_{01}\right)+\mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}-\sigma^{-}B_0^{+}B_1^{-}+\sigma^{-}B_{01}\right) \tag{234}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}B_0^{-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}B_1^{-}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) \tag{235}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}B_0^{-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}B_1^{-}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) \tag{236}$$

$$+\mathrm{i}V_{10}^{\Im}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}B_0^{-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}B_1^{-}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) \tag{237}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_x\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}+\mathrm{i}\sigma_y\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2}\right) \tag{238}$$

$$+\mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2}+\mathrm{i}\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) \tag{239}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_xB_x+\sigma_yB_y)+V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2}-\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) \tag{240}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_xB_x+\sigma_yB_y)+V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) \tag{241}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_xB_x+\sigma_yB_y)+V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) \tag{242}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x+\sigma_yB_y\right)+V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^2\sigma_x\left(-B_y\right)-\sigma_yB_x\right) \tag{243}$$

$$=\sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x+\sigma_yB_y\right)+V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y-\sigma_yB_x\right). \tag{244}$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right) + \langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^{2}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{245}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = 0. \tag{246}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\,(t)\,,\overline{H_{\bar{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{247}$$

Then using the fact that $\overline{H_{\bar{S}}}\,(t)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{248}$$

So Eq. (246) becomes:

$$\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} &= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}} && (249)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (250)\\
&= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}} && (251)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (252)\\
&= 0 \quad (\text{by Eq. (246)}). && (253)
\end{aligned}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\begin{aligned}
\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} &= \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} && (254)\\
&= 0. && (255)
\end{aligned}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} = 0. \tag{256}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}\left(t\right)=-\beta\left(\left(\varepsilon_{0}\left(t\right)+R_{0}\right)\left|0\right\rangle\!\left\langle 0\right|+\left(\varepsilon_{1}\left(t\right)+R_{1}\right)\left|1\right\rangle\!\left\langle 1\right|+V_{10}\left(t\right)B_{10}\sigma^{+}+V_{01}\left(t\right)B_{01}\sigma^{-}\right). \tag{257}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}\left(t\right)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^{2}-\mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)+\mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)=0. \tag{258}$$

Let's define:

$$\varepsilon\left(t\right)\equiv\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), \tag{259}$$

$$\eta\equiv\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}. \tag{260}$$

The solutions of the equation (258) are:

$$\begin{aligned}\lambda&=\beta\frac{-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\pm\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}}{2} &(261)\\&=\beta\frac{-\varepsilon\left(t\right)\pm\eta\left(t\right)}{2} &(262)\\&=-\beta\frac{\varepsilon\left(t\right)\mp\eta\left(t\right)}{2}. &(263)\end{aligned}$$

The value of $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\begin{aligned}\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)&=\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right)+\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) &(264)\\&=2\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). &(265)\end{aligned}$$

Given that $v_{\mathbf{ik}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)=A\left(\varepsilon\left(t\right),\eta\left(t\right)\right)$ to calculate $\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)}{\partial v_{\mathbf{ik}}^{\Re}}$ can lead to:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)}{\partial v_{\mathbf{ik}}^{\Re}}&=\frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{\mathbf{ik}}^{\Re}} &(266)\\&=2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) &(267)\\&=-\beta\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). &(268)\end{aligned}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)=0. \tag{269}$$

The derivates included in the expression given are related to:

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{270}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)^*\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{271}$$

$$= \langle B_1^+ B_0^- \rangle^*, \tag{272}$$

$$R_i = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{273}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{274}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{275}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2\omega_{\mathbf{k}}^2}\left(v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{276}$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = \left(v_{0\mathbf{k}}^{\Re} - \mathrm{i}v_{0\mathbf{k}}^{\Im}\right)\left(v_{1\mathbf{k}}^{\Re} + \mathrm{i}v_{1\mathbf{k}}^{\Im}\right) - \left(v_{0\mathbf{k}}^{\Re} + \mathrm{i}v_{0\mathbf{k}}^{\Im}\right)\left(v_{1\mathbf{k}}^{\Re} - \mathrm{i}v_{1\mathbf{k}}^{\Im}\right) \tag{277}$$

$$= \left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Re} + \mathrm{i}v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im} - \mathrm{i}v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Im}\right) - \left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Re} - \mathrm{i}v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im} + \mathrm{i}v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Im}\right) \tag{278}$$

$$= 2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re}\right), \tag{279}$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = \left(v_{1\mathbf{k}} - v_{0\mathbf{k}}\right)\left(v_{1\mathbf{k}} - v_{0\mathbf{k}}\right)^* \tag{280}$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - \left(v_{1\mathbf{k}}v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}}\right) \tag{281}$$

$$= \left(v_{1\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im}\right)^2 + \left(v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{0\mathbf{k}}^{\Im}\right)^2 - \left(\left(v_{1\mathbf{k}}^{\Re} + \mathrm{i}v_{1\mathbf{k}}^{\Im}\right)\left(v_{0\mathbf{k}}^{\Re} - \mathrm{i}v_{0\mathbf{k}}^{\Im}\right) + \left(v_{1\mathbf{k}}^{\Re} - \mathrm{i}v_{1\mathbf{k}}^{\Im}\right)\left(v_{0\mathbf{k}}^{\Re} + \mathrm{i}v_{0\mathbf{k}}^{\Im}\right)\right) \tag{282}$$

$$= \left(v_{1\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im}\right)^2 + \left(v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{0\mathbf{k}}^{\Im}\right)^2 - 2\left(v_{1\mathbf{k}}^{\Re}v_{0\mathbf{k}}^{\Re} + v_{1\mathbf{k}}^{\Im}v_{0\mathbf{k}}^{\Im}\right) \tag{283}$$

$$= \left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2. \tag{284}$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re} - \mathrm{i}v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}^{\Re} + \mathrm{i}v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\right)\right) \tag{285}$$

$$= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2 + \left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - \mathrm{i}v_{i\mathbf{k}}^{\Im}\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{286}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} \exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{287}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re}\right)}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{288}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}v_{1\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re}\right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{289}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}}=\frac{\partial\left(\varepsilon_{1}\left(t\right)+R_{1}+\varepsilon_{0}\left(t\right)+R_{0}\right)}{\partial v^{\Re}_{i\mathbf{k}}} \tag{290}$$

$$=\frac{\partial\left(\left(\frac{\left(v^{\Re}_{i\mathbf{k}}\right)^{2}+\left(v^{\Im}_{i\mathbf{k}}\right)^{2}}{\omega_{\mathbf{k}}}-v^{\Re}_{i\mathbf{k}}\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}v^{\Im}_{i\mathbf{k}}\frac{g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v^{\Re}_{i\mathbf{k}}} \tag{291}$$

$$=\frac{2v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{292}$$

$$\frac{\partial|B_{10}|^{2}}{\partial v^{\Re}_{i\mathbf{k}}}=\frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)^{2}+\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)^{2}}{\omega^{2}_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v^{\Re}_{i\mathbf{k}}} \tag{293}$$

$$=-\frac{2\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)}{\omega^{2}_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)}{\partial v^{\Re}_{i\mathbf{k}}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)^{2}+\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)^{2}}{\omega^{2}_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{294}$$

$$=-\frac{2\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)}{\omega^{2}_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)}{\partial v^{\Re}_{i\mathbf{k}}}\left|B_{10}\right|^{2}, \tag{295}$$

$$\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}}=\frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v^{\Re}_{i\mathbf{k}}} \tag{296}$$

$$=\frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}{\partial v^{\Re}_{i\mathbf{k}}}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}{\partial v^{\Re}_{i\mathbf{k}}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} \tag{297}$$

$$=\frac{\varepsilon\left(t\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}\right)\left(\varepsilon_{0}(t)+R_{0}\right)-|V_{10}(t)|^{2}|B_{10}(t)|^{2}\right)}{\partial v^{\Re}_{i\mathbf{k}}}}{\eta\left(t\right)} \tag{298}$$

$$=\frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)}{\omega^{2}_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)}{\partial v^{\Re}_{i\mathbf{k}}}|B_{10}|^{2}|V_{10}\left(t\right)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{299}$$

$$=\frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Re}_{i\mathbf{k}}-v^{\Re}_{i'\mathbf{k}}\right)}{\omega^{2}_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{300}$$

$$=\frac{v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon\left(t\right)-4\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\left(t\right)}\right)+\frac{1}{\eta\left(t\right)}\left(-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon\left(t\right)\right. \tag{301}$$

$$\left.+2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{v^{\Re}_{i'\mathbf{k}}}{\omega^{2}_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{302}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}}}{\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}}} \tag{303}$$

$$=\frac{\frac{2v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{2g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})-\frac{2}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+2\frac{v^{\Re}_{i'\mathbf{k}}}{\omega^{2}_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{304}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}-g_{i\mathbf{k}}^{*}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$=\frac{\left(v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{308}$$

Separating (307) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{309}$$

$$v_{i\mathbf{k}}^{\Re}-g_{i\mathbf{k}}^{\Re}=v_{i\mathbf{k}}^{\Re}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{310}$$

$$v_{i\mathbf{k}}^{\Re}=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{311}$$

$$v_{i\mathbf{k}}^{\Re}=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}}\left|B_{10}\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{312}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Im}}=\frac{\partial\left(\varepsilon_1\left(t\right)+R_1+\varepsilon_0\left(t\right)+R_0\right)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{313}$$

$$=\frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^2+\left(v_{i\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{314}$$

$$=2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{315}$$

$$\frac{\partial\left|B_{10}\right|^2}{\partial v_{i\mathbf{k}}^{\Im}}=\frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^2+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}} \tag{316}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{317}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}\right|^{2}, \tag{318}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{319}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{320}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_{1}(t)+R_{1})(\varepsilon_{0}(t)+R_{0})-|V_{10}(t)|^{2}|B_{10}(t)|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{\eta(t)} \tag{321}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{322}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}-v_{i'\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{323}$$

$$= \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^{2}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{324}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}}} \tag{325}$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\left(g_{i\mathbf{k}}^{*}\right)^{\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\frac{\left(g_{i\mathbf{k}}^{*}\right)^{\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^{2}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. \tag{326}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v_{i\mathbf{k}}^{\Im}-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}B_{10}^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$= \frac{\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{330}$$

Separating (330) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (331)$$

$$v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}=v^{\Im}_{i\mathbf{k}}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2\left|V_{10}(t)B_{10}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i-\varepsilon(t)) \quad (332)$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (333)$$

$$v^{\Im}_{i\mathbf{k}}=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \quad (334)$$

$$v^{\Im}_{i\mathbf{k}}=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \quad (335)$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}})=v^{\Re}_{i\mathbf{k}}(\omega_{\mathbf{k}})+\mathrm{i}v^{\Im}_{i\mathbf{k}}(\omega_{\mathbf{k}}) \quad (336)$$

$$=\frac{g^{\Re}_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)v^{\Re}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\eta(t)\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (337)$$

$$+\mathrm{i}\frac{g^{\Im}_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (338)$$

$$=\frac{g_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \quad (339)$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0)\otimes\rho_B^{\mathrm{Thermal}}$, where $\rho_B^{\mathrm{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$
\begin{aligned}
\overline{\rho_T(0)} &\equiv e^V \rho_T(0)\, e^{-V} && (340)\\
&= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)(\rho_S(0)\otimes\rho_B)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right), && (341)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } & |0\rangle\langle 0|0\rangle B_0^+\langle 0|\rho_B|0\rangle\langle 0|B_0^- && (342)\\
&= |0\rangle B_0^+\langle 0|\rho_B|0\rangle\langle 0|B_0^- && (343)\\
&= |0\rangle\langle 0|\otimes B_0^+\rho_B B_0^-, && (344)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } & |1\rangle\langle 1|B_1^+|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^- && (345)\\
&= |1\rangle\langle 1|B_1^+\rho_B B_1^- && (346)\\
&= |1\rangle\langle 1|\otimes B_1^+\rho_B B_1^-, && (347)\\
\text{for } \rho_S(0) = |0\rangle\langle 1| : & |0\rangle\langle 0|B_0^+|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^- && (348)\\
&= |0\rangle\langle 1|B_0^+\rho_B|1\rangle\langle 1|B_1^- && (349)\\
&= |0\rangle\langle 1|1\rangle\langle 1|B_0^+\rho_B B_1^- && (350)\\
&= |0\rangle\langle 1|\otimes B_0^+\rho_B B_1^-, && (351)\\
\text{for } \rho_S(0) = |1\rangle\langle 0| : & |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^- && (352)\\
&= |1\rangle\langle 0|\otimes B_1^+\rho_B B_0^-. && (353)
\end{aligned}
$$

We transform any operator O into the interaction picture in the following way:

$$
\begin{aligned}
\widetilde{O}(t) &\equiv U^\dagger(t)\, O(t)\, U(t), && (354)\\
U(t) &\equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\overline{H_{\bar{S}}}(t')\right). && (355)
\end{aligned}
$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$
\begin{aligned}
\widetilde{\overline{\rho_S}}(t) &= U^\dagger(t)\,\overline{\rho_S}(t)\, U(t), \text{ where} && (356)\\
\overline{\rho_S}(t) &= \mathrm{Tr}_B\left(\bar{\rho}_T(t)\right). && (357)
\end{aligned}
$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$
\begin{pmatrix} A(t)\\ B(t)\\ C(t)\end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2}\\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z}\\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \tag{358}
$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (236):

$$
\begin{aligned}
\overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t)(\sigma_x B_y - \sigma_y B_x) && (359)\\
&= B_{0z}|0\rangle\langle 0| + B_{1z}|1\rangle\langle 1| + V_{10}^{\Re}(t)\,\sigma_x B_x + V_{10}^{\Re}(t)\,\sigma_y B_y + V_{10}^{\Im}(t)\,\sigma_x B_y - V_{10}^{\Im}(t)\,\sigma_y B_x && (360)\\
&= \sum_i C_i(t)\left(A_i\otimes B_i(t)\right). && (361)
\end{aligned}
$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\widetilde{H_{\bar{I}}}}(t), \widetilde{\widetilde{\rho_T}}(t)]. \tag{362}$$

This equation has the formal solution

$$\widetilde{\widetilde{\rho_T}}(t) = \overline{\rho_T}(0) \text{-i} \int_0^t [\widetilde{\widetilde{H_{\bar{I}}}}(s), \widetilde{\widetilde{\rho_T}}(s)]\mathrm{d}s. \tag{363}$$

Replacing the equation (363) in the equation (362) gives us:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\widetilde{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\widetilde{H_{\bar{I}}}}(t), [\widetilde{\widetilde{H_{\bar{I}}}}(s), \widetilde{\widetilde{\rho_T}}(s)]]\mathrm{d}s. \tag{364}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\widetilde{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\widetilde{H_{\bar{I}}}}(t_1), [\widetilde{\widetilde{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\widetilde{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdots]. \tag{365}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\widetilde{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\widetilde{H_{\bar{I}}}}(t_1), [\widetilde{\widetilde{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\widetilde{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]] \cdots]. \tag{366}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\widetilde{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\overline{\rho_S}(0) \tag{367}$$
$$= W(t)\overline{\rho_S}(0). \tag{368}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\widetilde{H_{\bar{I}}}}(t_1), [\widetilde{\widetilde{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\widetilde{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]] \cdots]. \tag{369}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{370}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} W(t)\overline{\rho_S}(0) \tag{371}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} \widetilde{\widetilde{\rho_S}}(t). \tag{372}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\widetilde{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (370) becomes:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\widetilde{\widetilde{\rho_S}}(t) \tag{373}$$
$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B \left[\widetilde{\widetilde{H_{\bar{I}}}}(t), \left[\widetilde{\widetilde{H_{\bar{I}}}}(t_1), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]. \tag{374}$$

Replacing $t_1 \rightarrow t-\tau$

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}\left(t\right), \left[\widetilde{\overline{H_{\bar{I}}}}\left(-\tau\right), \overline{\rho_S}\left(t\right)\rho_B\right]\right]. \tag{375}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}\left(t\right)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right). \tag{376}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right) = \sum_i C_i\left(t\right)\left(\widetilde{A_i}\left(t\right)\otimes\widetilde{B_i}\left(t\right)\right), \tag{377}$$

$$\widetilde{A_i}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) \tag{378}$$

$$= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} \tag{379}$$

$$= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathbb{I} \tag{380}$$

$$= U^{\dagger}\left(t\right)A_i U\left(t\right), \tag{381}$$

$$\widetilde{B_i}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) \tag{382}$$

$$= U^{\dagger}\left(t\right)U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} \tag{383}$$

$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} \tag{384}$$

$$= \mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}. \tag{385}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}\left(t\right), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (377) to replace it in the equation (374)

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}\left(t\right), \left[\widetilde{\overline{H_{\bar{I}}}}\left(s\right), \widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}s \tag{386}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j\left(t\right)\left(\widetilde{A_j}\left(t\right)\otimes\widetilde{B_j}\left(t\right)\right), \left[\sum_i C_i\left(s\right)\left(\widetilde{A_i}\left(s\right)\otimes\widetilde{B_i}\left(s\right)\right), \widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}s \tag{387}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\overline{\rho_S}}\left(t\right)\rho_B - \widetilde{\overline{\rho_S}}\left(t\right)\rho_B\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\right]\mathrm{d}s \tag{388}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) \tag{389}$$

$$-\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}s. \tag{390}$$

In order to calculate the correlation functions we define:

$$\mathscr{B}_{ij}\left(\tau\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B\right) \tag{391}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B\right). \tag{392}$$

The correlation functions relevant that appear in the equation (390) are:

$$
\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\rho_B\right) &= \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\right\rangle_B && (393)\\
&= \left\langle \widetilde{B_j}\left(\tau\right)\widetilde{B_i}\left(0\right)\right\rangle_B && (394)\\
&= \mathscr{B}_{ji}\left(\tau\right), && (395)\\
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B\widetilde{B_i}\left(s\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\rho_B\right) && (396)\\
&= \left\langle \widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right\rangle_B && (397)\\
&= \left\langle \widetilde{B_i}\left(-\tau\right)\widetilde{B_j}\left(0\right)\right\rangle_B && (398)\\
&= \mathscr{B}_{ij}\left(-\tau\right), && (399)\\
\mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\rho_B\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\rho_B\right) && (400)\\
&= \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\right\rangle_B && (401)\\
&= \left\langle \widetilde{B_j}\left(\tau\right)\widetilde{B_i}\left(0\right)\right\rangle_B && (402)\\
&= \mathscr{B}_{ji}\left(\tau\right), && (403)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\rho_B\right) && (404)\\
&= \left\langle \widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right\rangle_B && (405)\\
&= \left\langle \widetilde{B_i}\left(-\tau\right)\widetilde{B_j}\left(0\right)\right\rangle_B && (406)\\
&= \mathscr{B}_{ij}\left(-\tau\right). && (407)
\end{aligned}
$$

The cyclic property of the trace was use widely in the development of equations (393) and (407). Replacing in (390)

$$
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\Big(\mathscr{B}_{ij}(\tau)\widetilde{A_i}(t)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t) - \mathscr{B}_{ji}(-\tau)\widetilde{A_i}(t)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\Big) + C_i(t)C_j(s)\Big(\mathscr{B}_{ji}(-\tau)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\widetilde{A_i}(t) - \mathscr{B}_{ij}(\tau)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t)\Big)\Big)\mathrm{d}s \quad (408)
$$

$$
= -\int_0^t \sum_{ij}\left(C_i\left(t\right)C_j\left(s\right)\left(\mathscr{B}_{ij}\left(\tau\right)\left[\widetilde{A_i}\left(t\right), \widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\right] + \mathscr{B}_{ji}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right), \widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}s. \quad (409)
$$

We could identify the following commutators in the equation deduced:

$$
\begin{aligned}
\mathscr{B}_{ij}\left(\tau\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right) - \mathscr{B}_{ij}\left(\tau\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right) &= \mathscr{B}_{ij}\left(\tau\right)\left[\widetilde{A_i}\left(t\right), \widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_s}}\left(t\right)\right], && (410)\\
\mathscr{B}_{ji}\left(-\tau\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right) - \mathscr{B}_{ji}\left(-\tau\right)\widetilde{A_i}\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right) &= \mathscr{B}_{ji}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}, \widetilde{A_i}\left(t\right)\right]. && (411)
\end{aligned}
$$

Returning to the Schroedinger picture we have:

$$
\begin{aligned}
U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right) &= U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{A_j}\left(s\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right), && (412)\\
&= \left(U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{A_j}\left(s\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right)\right), && (413)\\
&= A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right). && (414)
\end{aligned}
$$

This procedure applying to the relevant commutators give us:

$$
\begin{aligned}
U\left(t\right)\left[\widetilde{A_i}\left(t\right), \widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\right]U^\dagger\left(t\right) &= \left(U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right) - U\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right) && (415)\\
&= A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right) - \widetilde{A_j}\left(s,t\right)\overline{\rho_S}\left(t\right)A_i && (416)\\
&= \left[A_i, \widetilde{A_j}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right]. && (417)
\end{aligned}
$$

Introducing this transformed commutators in the equation (409) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathscr{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}[H_S(t),\overline{\rho_S}(t)]-\sum_{ij}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}\,(t)\Big]+\mathscr{B}_{ji}(-\tau)\Big[\overline{\rho_S}\,(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \quad (418)$$

where $i,j\in\{1,2,3,4,5.6\}$.

Here $\widetilde{A_j}\,(s,t)=U\,(t)\,U^{\dagger}\,(s)\,A_jU\,(s)\,U^{\dagger}\,(t)$ where $U\,(t)$ is given by (355). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I\,(t)$.

Calculating the correlation functions allow us to obtain:

$$\Big\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\Big\rangle_B=\mathrm{Tr}_B\left(\widetilde{B_{jz}}\,(\tau)\,\widetilde{B_{jz}}\,(0)\,\rho_B\right) \quad (419)$$

$$=\int\mathrm{d}^2\alpha P\,(\alpha)\left\langle\alpha\left|\widetilde{B_{jz}}\,(\tau)\,\widetilde{B_{jz}}\,(0)\right|\alpha\right\rangle \quad (420)$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}\,(\tau)\,\widetilde{B_{jz}}\,(0)\right|\alpha\right\rangle\mathrm{d}^2\alpha \quad (421)$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}\,(\tau)\,\widetilde{B_{jz}}\,(0)\right|\alpha\right\rangle\mathrm{d}^2\alpha, \quad (422)$$

$$\widetilde{B_{jz}}\,(\tau)=\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})\,b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*\,b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \quad (423)$$

$$\widetilde{B_{jz}}\,(0)=\sum_{\mathbf{k}'}\left((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^*\,b_{\mathbf{k}'}\right), \quad (424)$$

$$\Big\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\Big\rangle_B=\mathrm{Tr}_B\left(\widetilde{B_{jz}}\,(\tau)\,\widetilde{B_{jz}}\,(0)\,\rho_B\right) \quad (425)$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\Big((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)\sum_{\mathbf{k}'}\Big((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^*b_{\mathbf{k}'}\Big)\rho_B\right) \quad (426)$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\Big((g_{j\mathbf{k}}-v_{j\mathbf{k}})\,b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)\Big((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^*b_{\mathbf{k}'}\Big)\,\rho_B\right) \quad (427)$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\Big((g_{j\mathbf{k}}-v_{j\mathbf{k}})\,b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)\Big((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}\Big)\rho_B\right), \quad (428)$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}}=p_{j\mathbf{k}} \quad (429)$$

$$\Big\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\Big\rangle_B=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}'}b_{\mathbf{k}'}^{\dagger}+p_{j\mathbf{k}'}^*b_{\mathbf{k}'}\right)\rho_B\right) \quad (430)$$

$$+\,\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^*b_{\mathbf{k}}\right)\rho_B\right) \quad (431)$$

$$=0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}\right)\rho_B\right) \tag{432}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left|p_{j\mathbf{k}}\right|^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left|p_{j\mathbf{k}}\right|^2b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B\right) \tag{433}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}p_{j\mathbf{k}}^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{434}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{435}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right)+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right)\right) \tag{436}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{437}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{438}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{439}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{440}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{441}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{442}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{443}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{444}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{445}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{446}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{447}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)\right. \tag{448}$$

$$\left.+\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)\right) \tag{449}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{450}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{451}$$

$$+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\tfrac{1}{\pi N}\int\exp\left(-\tfrac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{452}$$

$$1=\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{453}$$

$$b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left|0\right\rangle=0, \tag{454}$$

$$b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\left|0\right\rangle=\left|0\right\rangle, \tag{455}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{456}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{457}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{458}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{459}$$

$$\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int_0^{2\pi}\int_0^{\infty}r^2\exp\left(-\frac{r^2}{N}\right)r\mathrm{d}r\mathrm{d}\theta \tag{460}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{461}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{462}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{463}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}+\cos\left(\omega_{\mathbf{k}}\tau\right)-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{464}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(2+e^{\beta\omega_{\mathbf{k}}}-1\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{465}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(1+e^{\beta\omega_{\mathbf{k}}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{466}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}+e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}-e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{467}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{468}$$

$$=\sum_{\mathbf{k}}\left|g_{j\mathbf{k}}-v_{j\mathbf{k}}\right|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right), \tag{469}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\right\rangle_B = \int\mathrm{d}^2\alpha_{\mathbf{k}}P(\alpha_{\mathbf{k}})\left\langle\alpha_{\mathbf{k}}\left|\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\right|\alpha_{\mathbf{k}}\right\rangle \tag{470}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{j'z}}\left(0\right)\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{471}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{472}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{473}$$

$$+\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{474}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{475}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{476}$$

$$+\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{477}$$

$$=\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{478}$$

$$+\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{479}$$

$$
\begin{aligned}
\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (480)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\, D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (481)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (482)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)|\alpha_{\mathbf{k}}|^2\,\mathrm{d}^2\alpha_{\mathbf{k}} && (483)\\
&= N, && (484)\\
\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (485)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (486)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (487)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (488)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} && (489)\\
&= N+1, && (490)\\
\left\langle \widetilde{B_{jz}}(\tau)\,\widetilde{B_{j'z}}(0)\right\rangle_B &= \sum_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N+\sum_{\mathbf{k}}(g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}(g_{j'\mathbf{k}}-v_{j'\mathbf{k}})\,e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N+1) && (491)\\
&= \textstyle\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+N\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right), && (492)\\
D(h')\,D(h) &= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)D(h'+h), && (493)\\
\left\langle D(h')\,D(h)\right\rangle_B &= \mathrm{Tr}_B\left(\exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)D(h'+h)\,\rho_B\right) && (494)\\
&= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\mathrm{Tr}_B\left(D(h'+h)\,\rho_B\right) && (495)\\
&= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\frac{1}{\pi N}\int \mathrm{d}^2\alpha P(\alpha)\left\langle\alpha\left|D(h'+h)\right|\alpha\right\rangle && (496)\\
&= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), && (497)\\
h' &= h\exp(\mathrm{i}\omega\tau), && (498)\\
\left\langle D(h\exp(\mathrm{i}\omega\tau))\,D(h)\right\rangle_B &= \exp\left(\frac{1}{2}(hh^{*}\exp(\mathrm{i}\omega\tau)-h^{*}h\exp(-\mathrm{i}\omega\tau))\right)\exp\left(-\frac{|h+h\exp(\mathrm{i}\omega\tau)|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), && (499)\\
\frac{1}{2}|h|^2(\exp(\mathrm{i}\omega\tau)-\exp(-\mathrm{i}\omega\tau)) &= \frac{1}{2}\left(hh^{*}\exp(\mathrm{i}\omega\tau)-h^{*}h\exp(-\mathrm{i}\omega\tau)\right) && (500)\\
&= \frac{1}{2}|h|^2\left(\cos(\omega\tau)+\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)+\mathrm{i}\sin(\omega\tau)\right) && (501)\\
&= \frac{1}{2}|h|^2\left(2\mathrm{i}\sin(\omega\tau)\right) && (502)\\
&= \mathrm{i}\,|h|^2\sin(\omega\tau), && (503)\\
-\frac{|h+h\exp(\mathrm{i}\omega\tau)|^2}{2} &= -|h|^2\,\frac{|1+\exp(\mathrm{i}\omega\tau)|^2}{2} && (504)\\
&= -|h|^2\,\frac{\left(1+2\cos(\omega\tau)+\cos^2(\omega\tau)\right)+\sin^2(\omega\tau)}{2} && (505)
\end{aligned}
$$

$$= -\left|h\right|^2 \frac{2+2\cos\left(\omega\tau\right)}{2} \tag{506}$$

$$= -\left|h\right|^2 \left(1+\cos\left(\omega\tau\right)\right), \tag{507}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\right\rangle_B = \exp\left(\mathrm{i}|h|^2\sin(\omega\tau)\right)\exp\left(-|h|^2(1+\cos(\omega\tau))\coth\left(\tfrac{\beta\omega}{2}\right)\right) \tag{508}$$

$$= \exp\left(\mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right) - \left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{509}$$

$$= \exp\left(-\left|h\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-\left|h\right|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{510}$$

$$= \left\langle D\left(h\right)\right\rangle_B \exp\left(-\phi\left(\tau\right)\right), \tag{511}$$

$$\exp\left(-\phi\left(\tau\right)\right) = \exp\left(-\left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)\right), \tag{512}$$

$$\phi\left(\tau\right) = \left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right), \tag{513}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{514}$$

$$h' = v\exp\left(i\omega\tau\right), \tag{515}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_1^+B_0^-}\left(\tau\right)\widetilde{B_1^+B_0^-}\left(0\right)\rho_B\right) \tag{516}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_1^+B_0^-}\left(\tau\right)\widetilde{B_1^+B_0^-}\left(0\right)\rho_B\right) \tag{517}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{518}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{519}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{520}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\prod_{\mathbf{k}}\left(\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{521}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{522}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{523}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{524}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{525}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{526}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{527}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{528}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{529}$$

$$= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau+\pi)+\cos(\omega_{\mathbf{k}}\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{530}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)-\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{531}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{532}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{533}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}_B\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{534}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau+\pi)+\cos(\omega_{\mathbf{k}}\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{535}$$

$$= \left\langle |\widetilde{B_1^+B_0^-}(\tau)\,\widetilde{B_0^+B_1^-}(0)\right\rangle_B, \tag{536}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\rho_B\right), \tag{537}$$

$$\left\langle D(h)\,b\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle \alpha\left|D(h)\,b\right|\alpha\right\rangle \tag{538}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle \alpha\left|D(-\alpha)\,D(h)\,bD(\alpha)\right|\alpha\right\rangle \tag{539}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)\,D(h)\,bD(\alpha)\right|0\right\rangle \tag{540}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha)\right|0\right\rangle \tag{541}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)\,D(h)\,D(\alpha)\,(b+\alpha)\right|0\right\rangle \tag{542}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\left\langle 0\left|D(h)\,(b+\alpha)\right|0\right\rangle \tag{543}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)b|0\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\,\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \tag{544}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\left\langle 0\left|D(h)\,\alpha\right|0\right\rangle \tag{545}$$

$$= \frac{1}{\pi N}\int \alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{546}$$

$$= hN\left\langle D(h)\right\rangle_B, \tag{547}$$

$$\left\langle D(h)\,b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle \alpha\left|D(h)\,b^\dagger\right|\alpha\right\rangle \tag{548}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\right|0\right\rangle \tag{549}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\right|0\right\rangle \tag{550}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,b^\dagger D(\alpha)\right|0\right\rangle \tag{551}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle 0 \left| D(-\alpha) D(h) D(\alpha) \left(b^\dagger + \alpha^*\right) \right| 0 \right\rangle \tag{552}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \left\langle 0 \left| D(h) \left(b^\dagger + \alpha^*\right) \right| 0 \right\rangle \tag{553}$$

$$= \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \left\langle 0 \middle| D(h) b^\dagger \middle| 0 \right\rangle + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \tag{554}$$

$$= \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) | 1 \rangle + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \tag{555}$$

$$= \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \langle -h | 1 \rangle + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \alpha^* \langle 0 | D(h) | 0 \rangle, \tag{556}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n|, \tag{557}$$

$$\langle -h|1\rangle = \exp\left(-\frac{|-h^*|^2}{2}\right) (-h^*), \tag{558}$$

$$\left\langle D(h) b^\dagger \right\rangle_B = \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\tfrac{|-h^*|^2}{2}\right)(-h^*) + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \alpha^* \exp\left(-\tfrac{|-h^*|^2}{2}\right) \tag{559}$$

$$= -h^* \langle D(h) \rangle_B (N+1), \tag{560}$$

$$\langle b D(h) \rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \langle \alpha | b D(h) | \alpha \rangle \tag{561}$$

$$= \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\tfrac{|h|^2}{2}\right) h + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \alpha \exp\left(-\tfrac{|h|^2}{2}\right) \tag{562}$$

$$= h \langle D(h) \rangle_B (N+1), \tag{563}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| b^\dagger D(h) \right| \alpha \right\rangle \tag{564}$$

$$= \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\tfrac{|h|^2}{2}\right) h + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^*\alpha) \alpha \exp\left(-\tfrac{|h|^2}{2}\right) \tag{565}$$

$$= -h^* \langle D(h) \rangle_B N, \tag{566}$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle_B \tag{567}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right\rangle_B \tag{568}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right\rangle_B \tag{569}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{570}$$

$$= B_{10}. \tag{571}$$

The correlation functions can be found readily as:

$$\widetilde{B_1^+ B_0^-}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right), \tag{572}$$

$$\widetilde{B_0^+ B_1^-}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right), \tag{573}$$

$$\widetilde{B_x}(0) = \frac{B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}}{2}, \tag{574}$$

$$\widetilde{B_y}(0) = \frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}, \tag{575}$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{576}$$

$$B_{iz} = \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* b_{\mathbf{k}}\right), \tag{577}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \left\langle \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}} - v_{jk}\right)^* b_{\mathbf{k}}\right)\right\rangle_B \tag{578}$$

$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^*\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}} + 1\right), \tag{579}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \left\langle \frac{B_1^+B_0^-(\tau) + B_0^+B_1^-(\tau) - B_{10} - B_{01}}{2}\,\frac{B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}}{2}\right\rangle_B \tag{580}$$

$$= \frac{1}{4}\left\langle\left(B_1^+B_0^-(\tau) + B_0^+B_1^-(\tau) - B_{10} - B_{01}\right)\left(B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}\right)\right\rangle_B \tag{581}$$

$$= \tfrac{1}{4}\Big\langle B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- \tag{582}$$

$$-B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_{10}B_1^+B_0^- - B_{10}B_0^+B_1^- + B_{10}B_{10} + B_{10}B_{01} - B_{01}B_1^+B_0^- - B_{01}B_0^+B_1^- + B_{01}B_{10} + B_{01}B_{01}\Big\rangle \tag{583}$$

$$= \frac{1}{4}\Big\langle B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_1^+B_0^- \tag{584}$$

$$+B_0^+B_1^-(\tau)\,B_0^+B_1^- - B_0^+B_1^-(\tau)\,B_{10} - B_0^+B_1^-(\tau)\,B_{01}\Big\rangle, \tag{585}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{586}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right), \tag{587}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{588}$$

$$S = \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{589}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = U\exp(-\phi(\tau))S, \tag{590}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = U^*\exp(-\phi(\tau))S, \tag{591}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \exp(\phi(\tau))S, \tag{592}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = \left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B, \tag{593}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{594}$$

$$= U^{*1/2}S^{1/2}, \tag{595}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\Big\langle B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_1^+B_0^- \tag{596}$$

$$+B_0^+B_1^-(\tau)\,B_0^+B_1^- - B_0^+B_1^-(\tau)\,B_{10} - B_0^+B_1^-(\tau)\,B_{01}\Big\rangle, \tag{597}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\Big\langle B_1^+B_0^-(\tau)\,B_1^+B_0^- + B_1^+B_0^-(\tau)\,B_0^+B_1^- - B_1^+B_0^-(\tau)\,B_{10} - B_1^+B_0^-(\tau)\,B_{01} \tag{598}$$

$$+B_0^+B_1^-(\tau)\,B_1^+B_0^- + B_0^+B_1^-(\tau)\,B_0^+B_1^- - B_0^+B_1^-(\tau)\,B_{10} - B_0^+B_1^-(\tau)\,B_{01}\Big\rangle \tag{599}$$

$$= \frac{1}{4}\Big(U^*\exp(-\phi(\tau))S + \exp(\phi(\tau))S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau))S + U\exp(-\phi(\tau))S - B_{10}^{*2} - |B_{10}|^2\Big) \tag{600}$$

$$= \frac{1}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right)S + 2\exp\left(\phi\left(\tau\right)\right)S - 2\left(B_{10}^{2}\right)^{\Re} - 2\left|B_{10}\right|^{2}\right) \tag{601}$$

$$= \frac{1}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right)S + 2\exp\left(\phi\left(\tau\right)\right)S - 2\left(U^{*}\right)^{\Re}S - 2S\right) \tag{602}$$

$$= \frac{S}{2}\left(U^{\Re}\exp\left(-\phi\left(\tau\right)\right) + \exp\left(\phi\left(\tau\right)\right) - \left(U^{*}\right)^{\Re} - 1\right), \tag{603}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}}{2\mathrm{i}}\,\frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B \tag{604}$$

$$= -\frac{1}{4}\left\langle\left(B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}\right)\left(B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}\right)\right\rangle_B \tag{605}$$

$$= -\tfrac{1}{4}\langle B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_1^+B_0^-(\tau)B_0^+B_1^- + B_1^+B_0^-(\tau)B_1^+B_0^- \tag{606}$$

$$-B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01} + B_{10}B_0^+B_1^- - B_{10}B_1^+B_0^- + B_{10}B_{10} - B_{10}B_{01} - B_{01}B_0^+B_1^- + B_{01}B_1^+B_0^- - B_{01}B_{10} + B_{01}B_{01}\rangle \tag{607}$$

$$= -\frac{1}{4}\langle B_0^+B_1^-\left(\tau\right)B_0^+B_1^- - B_0^+B_1^-\left(\tau\right)B_1^+B_0^- + B_0^+B_1^-\left(\tau\right)B_{10} - B_0^+B_1^-\left(\tau\right)B_{01} \tag{608}$$

$$-B_1^+B_0^-\left(\tau\right)B_0^+B_1^- + B_1^+B_0^-\left(\tau\right)B_1^+B_0^- - B_1^+B_0^-\left(\tau\right)B_{10} + B_1^+B_0^-\left(\tau\right)B_{01}\rangle \tag{609}$$

$$= -\tfrac{1}{4}\langle B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_{01}B_{10} - B_{01}B_{01} - B_1^+B_0^-(\tau)B_0^+B_1^- + B_1^+B_0^-(\tau)B_1^+B_0^- - B_{10}B_{10} + B_{10}B_{01}\rangle \tag{610}$$

$$= -\frac{1}{4}\left(U\exp(-\phi\left(\tau\right))S - \exp\left(\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^{*}\exp\left(-\phi\left(\tau\right)\right)S + 2S - 2\left(U^{*}\right)^{\Re}S\right) \tag{611}$$

$$= -\frac{S}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right) - 2\exp\left(\phi\left(\tau\right)\right) + 2 - 2U^{\Re}\right) \tag{612}$$

$$= \frac{S}{2}\left(\exp\left(\phi\left(\tau\right)\right) - U^{\Re}\exp\left(-\phi\left(\tau\right)\right) - 1 + U^{\Re}\right), \tag{613}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_1^+B_0^-\left(\tau\right) + B_0^+B_1^-\left(\tau\right) - B_{10} - B_{01}}{2}\,\frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B \tag{614}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_1^+B_0^-\left(\tau\right) + B_0^+B_1^-\left(\tau\right) - B_{10} - B_{01}\right)\left(B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}\right)\right\rangle_B \tag{615}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10} \tag{616}$$

$$-B_0^+B_1^-(\tau)B_{01} - B_{10}B_0^+B_1^- + B_{10}B_1^+B_0^- - B_{10}B_{10} + B_{10}B_{01} - B_{01}B_0^+B_1^- + B_{01}B_1^+B_0^- - B_{01}B_{10} + B_{01}B_{01}\rangle \tag{617}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_1^+B_0^-(\tau)B_{10} - B_1^+B_0^-(\tau)B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_{10} \tag{618}$$

$$-B_0^+B_1^-(\tau)B_{01}\rangle \tag{619}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_{10}B_{10} - B_{10}B_{01} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- + B_{01}B_{10} - B_{01}B_{01}\rangle \tag{620}$$

$$= \frac{1}{4\mathrm{i}}\langle B_1^+B_0^-(\tau)B_0^+B_1^- - B_1^+B_0^-(\tau)B_1^+B_0^- + B_{10}B_{10} + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_1^+B_0^- - B_{01}B_{01}\rangle \tag{621}$$

$$= \frac{1}{4\mathrm{i}}\left(\exp\left(\phi\left(\tau\right)\right)S - U^{*}\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^{*}S - US\right) \tag{622}$$

$$= \frac{1}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S + U^{*}S - US\right) \tag{623}$$

$$= \frac{S}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi\left(\tau\right)\right) + U\exp\left(-\phi\left(\tau\right)\right) + U^{*} - U\right) \tag{624}$$

$$= \frac{S\left(U - U^{*}\right)}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{625}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{626}$$

$$= \frac{U^{\Im}S}{2}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right), \tag{627}$$

$$\left\langle\widetilde{B_y}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \left\langle\frac{B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}}{2\mathrm{i}}\,\frac{B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}}{2}\right\rangle_B \tag{628}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_0^+B_1^-\left(\tau\right) - B_1^+B_0^-\left(\tau\right) + B_{10} - B_{01}\right)\left(B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}\right)\right\rangle_B \tag{629}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_0^+B_1^-(\tau)B_1^+B_0^- + B_0^+B_1^-(\tau)B_0^+B_1^- - B_0^+B_1^-(\tau)B_{10} - B_0^+B_1^-(\tau)B_{01} - B_1^+B_0^-(\tau)B_1^+B_0^- - B_1^+B_0^-(\tau)B_0^+B_1^- \tag{630}$$

$$+B_1^+B_0^-(\tau)B_{10} + B_1^+B_0^-(\tau)B_{01} + B_{10}B_1^+B_0^- + B_{10}B_0^+B_1^- - B_{10}B_{10} - B_{10}B_{01} - B_{01}B_1^+B_0^- - B_{01}B_0^+B_1^- + B_{01}B_{10} + B_{01}B_{01}\rangle \tag{631}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_0^+ B_1^-\left(\tau\right) B_1^+ B_0^- + B_0^+ B_1^-\left(\tau\right) B_0^+ B_1^- - B_0^+ B_1^-\left(\tau\right) B_{10} - B_0^+ B_1^-\left(\tau\right) B_{01}\right. \tag{632}$$

$$\left. - B_1^+ B_0^-\left(\tau\right) B_1^+ B_0^- - B_1^+ B_0^-\left(\tau\right) B_0^+ B_1^- + B_1^+ B_0^-\left(\tau\right) B_{10} + B_1^+ B_0^-\left(\tau\right) B_{01}\right\rangle \tag{633}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_0^+ B_1^-\left(\tau\right) B_1^+ B_0^- + B_0^+ B_1^-\left(\tau\right) B_0^+ B_1^- - B_{01}B_{10} - B_{01}B_{01} - B_1^+ B_0^-\left(\tau\right) B_1^+ B_0^-\right. \tag{634}$$

$$\left. - B_1^+ B_0^-\left(\tau\right) B_0^+ B_1^- + B_{10}B_{10} + B_{10}B_{01}\right\rangle \tag{635}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_0^+ B_1^-\left(\tau\right) B_1^+ B_0^- + B_0^+ B_1^-\left(\tau\right) B_0^+ B_1^- - B_{01}B_{10} - B_{01}B_{01} - B_1^+ B_0^-\left(\tau\right) B_1^+ B_0^-\right. \tag{636}$$

$$\left. - B_1^+ B_0^-\left(\tau\right) B_0^+ B_1^- + B_{10}B_{10} + B_{10}B_{01}\right\rangle \tag{637}$$

$$= \tfrac{1}{4\mathrm{i}}\left\langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_{01}B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_{10}B_{10}\right\rangle \tag{638}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi\left(\tau\right)\right) S - U^*\exp\left(-\phi\left(\tau\right)\right) S + B_{10}^2 - B_{10}^{*2}\right) \tag{639}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp\left(-\phi\left(\tau\right)\right) S - U^*\exp\left(-\phi\left(\tau\right)\right) S + U^* S - US\right) \tag{640}$$

$$= \frac{S\left(U - U^*\right)}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{641}$$

$$= \frac{2\mathrm{i}U^{\Im} S}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{642}$$

$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'} + 1\right) B_{10}, \tag{643}$$

$$\left\langle B_1^+ B_0^-(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \prod_{\mathbf{k}} \exp\left(\tfrac{1}{2}\left(\tfrac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}\left\langle \prod_{\mathbf{k}}\left(D\left(\tfrac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle \tag{644}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}, \tag{645}$$

$$\left\langle B_0^+ B_1^-(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger\right\rangle_B = -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{01}. \tag{646}$$

$$\left\langle B_0^+ B_1^-(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}, \tag{647}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \tfrac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{01}\right. \tag{648}$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}\right) \tag{649}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'} + 1) B_{01}\right. \tag{650}$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*\left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}\right) \tag{651}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{01}\right)\right. \tag{652}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{01}\right)\right) \tag{653}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'} + 1)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{01}\right)\right. \tag{654}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{01}\right)\right) \tag{655}$$

$$= \tfrac{1}{2}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1)\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(B_{10} - B_{01}) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'}\left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)(B_{10} - B_{01})\right) \tag{656}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} 2\mathrm{i}B_{10}^{\Im}\left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{657}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} B_{10}^{\Im}\left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{658}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} B_{10}^{\Im}\left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right), \tag{659}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B = \left\langle \sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_B \tag{660}$$

$$= \sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_B \tag{661}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}\right)\right\rangle_B \tag{662}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_1^{+}B_0^{-}+B_0^{+}B_1^{-}\right)\right\rangle_B \tag{663}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}\right. \tag{664}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}\right\rangle, \tag{665}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}\right\rangle_B \tag{666}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_B \tag{667}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^{*}v_{0\mathbf{k}'}-v_{1\mathbf{k}'}v_{0\mathbf{k}'}^{*}}{\omega_{\mathbf{k}'}^{2}}\right)\right)\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_B \tag{668}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{669}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{670}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{671}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B N_{\mathbf{k}'}\right)e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{672}$$

$$= -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{673}$$

$$= -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{674}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^{*}v_{0\mathbf{k}'}-v_{1\mathbf{k}'}v_{0\mathbf{k}'}^{*}}{\omega_{\mathbf{k}'}^{2}}\right)\right)\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_B \tag{675}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{676}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \tag{677}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B \left\langle b_{\mathbf{k}'}^\dagger D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{678}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B \left(-\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B N_{\mathbf{k}'}\right) e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{679}$$

$$= -\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_B N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{680}$$

$$= -\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{681}$$

$$\left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_0^+ B_1^- \right\rangle_B = -\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B_{01} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{682}$$

$$\left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_1^+ B_0^- \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} B_1^+ B_0^- \right\rangle_B \tag{683}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left\langle b_{\mathbf{k}'} \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\right\rangle_B \tag{684}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle b_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{685}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{686}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) \left\langle D\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_B \tag{687}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B_{10}, \tag{688}$$

$$\left\langle \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+} B_1^- \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B_{01}, \tag{689}$$

$$\left\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \right\rangle_B = \frac{1}{2} \sum_{\mathbf{k}'} \left(-\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B_{10} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) B_{01} N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right. \tag{690}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \left(N_{\mathbf{k}'} + 1\right) B_{01} \right) \tag{691}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(B_{01} - B_{10}\right) + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right)\left(B_{10} - B_{01}\right) \right) \tag{692}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* \left(B_{01} - B_{10}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right)\left(B_{01} - B_{10}\right) \right) \tag{693}$$

$$= \mathrm{i} \sum_{\mathbf{k}'} B_{10}^{\Im} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) N_{\mathbf{k}'} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \left(N_{\mathbf{k}'} + 1\right) \right), \tag{694}$$

$$\left\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \right\rangle_B = \left\langle \left(\frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2\mathrm{i}}\right) \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{695}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}\right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{696}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau)\right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right) \right\rangle_B \tag{697}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}'} \left\langle B_0^+ B_1^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger - B_1^+ B_0^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + B_0^+ B_1^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} - B_1^+ B_0^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} \right\rangle, \tag{698}$$

$$\left\langle B_0^+ B_1^-(\tau) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger \right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{01}, \tag{699}$$

$$\left\langle B_0^+ B_1^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}, \tag{700}$$

$$\left\langle B_1^+ B_0^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger\right\rangle_B = -(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1) B_{10}, \tag{701}$$

$$\left\langle B_1^+ B_0^-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\right\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}, \tag{702}$$

$$\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right. \tag{703}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{01}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10}\right) \tag{704}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right. \tag{705}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{01}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}B_{10}\right) \tag{706}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{01})+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(-B_{10}-B_{01})\right) \tag{707}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{01})-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(B_{10}+B_{01})\right), \tag{708}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle \sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^+B_1^- - B_1^+B_0^- + B_{10}-B_{01}}{2\mathrm{i}}\right)\right\rangle_B \tag{709}$$

$$= \frac{1}{2\mathrm{i}}\left\langle \sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^- - B_1^+B_0^- + B_{10}-B_{01}\right)\right\rangle_B \tag{710}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^- - B_1^+B_0^-\right)\right\rangle_B \tag{711}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^- -(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^- +(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^- -(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-\right\rangle \tag{712}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right) \tag{713}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right) \tag{714}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle + e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right), \tag{715}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle_B = -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}N_{\mathbf{k}'}, \tag{716}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle_B = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{01}N_{\mathbf{k}'}, \tag{717}$$

$$\left\langle b_{\mathbf{k}'} B_1^+B_0^-\right\rangle_B = \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{10}, \tag{718}$$

$$\left\langle b_{\mathbf{k}'} B_0^+B_1^-\right\rangle_B = \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{01}, \tag{719}$$

$$\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{01}N_{\mathbf{k}'}\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}N_{\mathbf{k}'}\right)\right. \tag{720}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{01}\right)-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{10}\right)\right) \tag{721}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{01}N_{\mathbf{k}'}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}N_{\mathbf{k}'}\right)\right. \tag{722}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{01}\right)-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{10}\right)\right)\right) \tag{723}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left((-g_{i\mathbf{k}'}+v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{01}N_{\mathbf{k}'}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}N_{\mathbf{k}'}\right)\right. \tag{724}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{01}\right)-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{10}\right)\right)\right) \tag{725}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+)N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(B_{10}+)(N_{\mathbf{k}'}+1)\right) \tag{726}$$

$$= \frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}^{\Re}N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(N_{\mathbf{k}'}+1)\right) \tag{727}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}\right) \tag{728}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}\right) \tag{729}$$

$$= \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}N_{\mathbf{k}'}\right). \tag{730}$$

The correlation functions are equal to:

$$\left\langle\widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right\rangle_{B} = \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{731}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right), \tag{732}$$

$$\phi\left(\tau\right) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{733}$$

$$\left\langle\widetilde{B_{x}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \frac{\left|B_{10}\right|^{2}}{2}\left(U^{\Re}\exp\left(-\phi\left(\tau\right)\right)+\exp\left(\phi\left(\tau\right)\right)-U^{\Re}-1\right), \tag{734}$$

$$\left\langle\widetilde{B_{y}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = \frac{\left|B_{10}\right|^{2}}{2}\left(\exp\left(\phi\left(\tau\right)\right)-U^{\Re}\exp\left(-\phi\left(\tau\right)\right)-1+U^{\Re}\right), \tag{735}$$

$$\left\langle\widetilde{B_{x}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = \frac{U^{\Im}\left|B_{10}\right|^{2}}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right), \tag{736}$$

$$\left\langle\widetilde{B_{y}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \frac{U^{\Im}\left|B_{10}\right|^{2}}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right), \tag{737}$$

$$\left\langle\widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{738}$$

$$\left\langle\widetilde{B_{x}}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_{B} = \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{739}$$

$$\left\langle\widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{740}$$

$$\left\langle\widetilde{B_{y}}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_{B} = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}\right). \tag{741}$$

The spectral density is defined in the usual way:

$$J_{i}\left(\omega\right) \equiv \sum_{\mathbf{k}}\left|g_{i\mathbf{k}}\right|^{2}\delta\left(\omega-\omega_{\mathbf{k}}\right), \tag{742}$$

$$v_{i\mathbf{k}} = g_{i\mathbf{k}}F_{i}\left(\omega_{\mathbf{k}}\right). \tag{743}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_{i} = \int_{0}^{\infty}\frac{J_{i}\left(\omega\right)}{\omega}\mathrm{d}\omega. \tag{744}$$

The integral version of the correlation functions are given by:

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B=\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}})e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{745}$$

$$=\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{746}$$

$$=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}\left(1-F_i\left(\omega_{\mathbf{k}}\right)\right)g_{j\mathbf{k}}^*\left(1-F_j\left(\omega_{\mathbf{k}}\right)\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+g_{i\mathbf{k}}^*\left(1-F_i\left(\omega_{\mathbf{k}}\right)\right)^*g_{j\mathbf{k}}\left(1-F_j\left(\omega_{\mathbf{k}}\right)\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{747}$$

$$\approx\int_0^\infty\left(\sqrt{J_i(\omega)J_j^*(\omega)}(1-F_i(\omega))\left(1-F_j^*(\omega)\right)e^{\mathrm{i}\omega\tau}N(\omega)+\sqrt{J_i^*(\omega)J_j(\omega)}\left(1-F_i^*(\omega)\right)\left(1-F_j(\omega)\right)e^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega, \tag{748}$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{749}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right) \tag{750}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^*F_0^*\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}^*F_1^*\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\right) \tag{751}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^*F_0^*\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)g_{1\mathbf{k}}^*F_1^*\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\right) \tag{752}$$

$$=\exp\left(\sum_{\mathbf{k}}\frac{g_{0\mathbf{k}}^*g_{1\mathbf{k}}F_0^*\left(\omega_{\mathbf{k}}\right)F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}g_{1\mathbf{k}}^*F_0\left(\omega_{\mathbf{k}}\right)F_1^*\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}^2}\right) \tag{753}$$

$$\approx\exp\left(\int_0^\infty\frac{\sqrt{J_0^*\left(\omega\right)J_1\left(\omega\right)}F_0^*\left(\omega\right)F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)J_1^*\left(\omega\right)}F_0\left(\omega\right)F_1^*\left(\omega\right)}{\omega^2}\mathrm{d}\omega\right), \tag{754}$$

$$\phi\left(\tau\right)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{755}$$

$$=\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{756}$$

$$\approx\int_0^\infty\left|\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega\right)}{\omega}\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega, \tag{757}$$

$$B_{10}=\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{758}$$

$$=\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{759}$$

$$=\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^*F_1^*(\omega_{\mathbf{k}})-g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})g_{0\mathbf{k}}^*F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right)\right) \tag{760}$$

$$\approx\exp\left(-\frac{1}{2}\int_0^\infty\left|\frac{\sqrt{J_1(\omega)}F_1(\omega)-\sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega\right)\exp\left(\int_0^\infty\frac{1}{2}\left(\frac{\sqrt{J_0(\omega)J_1^*(\omega)}F_0(\omega)F_1^*(\omega)-\sqrt{J_0^*(\omega)J_1(\omega)}F_0^*(\omega)F_1(\omega)}{\omega^2}\right)\mathrm{d}\omega\right), \tag{761}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B=\frac{|B_{10}|^2}{2}\left(U^{\Re}\exp(-\phi(\tau))+\exp(\phi(\tau))-U^{\Re}-1\right), \tag{762}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B=\frac{|B_{10}|^2}{2}\left(\exp(\phi(\tau))-U^{\Re}\exp(-\phi(\tau))-1+U^{\Re}\right), \tag{763}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B=\frac{U^{\Im}|B_{10}|^2}{2}(\exp(-\phi(\tau))-1), \tag{764}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B=\frac{U^{\Im}|B_{10}|^2}{2}(\exp(-\phi(\tau))-1), \tag{765}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{766}$$

$$=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{767}$$

$$=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{768}$$

$$Q\left(\omega\right)=\sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega\right)\right)\left(\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega\right)}{\omega}\right)^*, \tag{769}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B\approx\mathrm{i}B_{10}^{\Im}\int_0^\infty\left(Q\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega, \tag{770}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{771}$$

$$=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^*\left(1-F_i^*(\omega_{\mathbf{k}})\right)\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}-g_{i\mathbf{k}}(1-F_i(\omega))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{772}$$

$$\approx \mathrm{i}B_{10}^{\Im}\int_{0}^{\infty}\left(Q^{*}\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega, \tag{773}$$

$$\left\langle\widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{y}}\left(0\right)\right\rangle_{B} = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right) \tag{774}$$

$$= \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}g_{i\mathbf{k}}^{*}\left(1-F_{i}^{*}\left(\omega_{\mathbf{k}}\right)\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}g_{i\mathbf{k}}\left(1-F_{i}\left(\omega_{\mathbf{k}}\right)\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right) \tag{775}$$

$$\approx \mathrm{i}B_{10}^{\Re}\int_{0}^{\infty}\left(e^{-\mathrm{i}\omega\tau}Q^{*}\left(\omega\right)\left(N\left(\omega\right)+1\right)-e^{\mathrm{i}\omega\tau}Q\left(\omega\right)N\left(\omega\right)\right)\mathrm{d}\omega, \tag{776}$$

$$\left\langle\widetilde{B_{y}}\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_{B} = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}\right) \tag{777}$$

$$= \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^{*}\left(1-F_{i}^{*}\left(\omega_{\mathbf{k}}\right)\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}\left(1-F_{i}\left(\omega_{\mathbf{k}}\right)\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}\right) \tag{778}$$

$$= \mathrm{i}B_{10}^{\Re}\int_{0}^{\infty}\left(e^{\mathrm{i}\omega\tau}Q^{*}\left(\omega\right)N\left(\omega\right)-e^{-\mathrm{i}\omega\tau}Q\left(\omega\right)\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega. \tag{779}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^{2}-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda+\mathrm{Det}\left(\overline{H_{\bar{S}}}\right)=0. \tag{780}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm}=\frac{\xi\pm\eta}{2}$ and they satisfy $H_{S}\left|\pm\right\rangle=\lambda_{\pm}\left|\pm\right\rangle$. Using this notation is possible to write $H_{\bar{S}}=\lambda_{+}|+\rangle\langle+|+\lambda_{-}|-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_{i}}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_{i}}\left(\tau\right)=e^{\mathrm{i}\overline{H_{\bar{S}}}\tau}A_{i}e^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} \tag{781}$$

$$=\sum_{w}e^{-\mathrm{i}w\tau}\mathscr{A}_{i}\left(w\right). \tag{782}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w\in\{0,\pm\eta\}$.
In order to use the equation (782) to descompose the equation (355) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U\left(t\right)\equiv\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ like:

$$U\left(t\right)\equiv\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right) \tag{783}$$

$$=\mathbb{I}+\sum_{n=1}^{\infty}\left(-\mathrm{i}\right)^{n}\int_{0}^{t}\mathrm{d}t_{1}\int_{0}^{t_{1}}\mathrm{d}t_{2}...\int_{0}^{t_{n-1}}\mathrm{d}t_{n}H\left(t_{1}\right)H\left(t_{2}\right)...H\left(t_{n}\right). \tag{784}$$

Here $0<t_{1}<t_{2}<...<t_{n-1}<t_{n}=t$ is a partition of the set $[0,t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_{E}\left(t\right)$ such that $\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)\equiv\mathrm{exp}\left(-\mathrm{i}tH_{E}\left(t\right)\right)$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_{E}\left(t\right)=H_{E}^{(0)}\left(t\right)+H_{E}^{(1)}\left(t\right)+H_{E}^{(2)}\left(t\right)+...$ so we can write:

$$U\left(t\right)=\mathrm{exp}\left(-\mathrm{i}t\left(H_{E}^{(0)}\left(t\right)+H_{E}^{(1)}\left(t\right)+H_{E}^{(2)}\left(t\right)+...\right)\right). \tag{785}$$

The terms can be found expanding $\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ and $U\left(t\right)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \tag{786}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{787}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{788}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}(t) = U^{\dagger}(t)\,A_i(t)\,U(t) \tag{789}$$

$$= e^{\mathrm{i}H_E(t)t} A_i(t)\, e^{-\mathrm{i}H_E(t)t} \tag{790}$$

$$= \sum_{w(t)} e^{-\mathrm{i}w(t)t} \mathscr{A}_i(w(t)). \tag{791}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau, t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau, t) = U(t)\,U^{\dagger}(t-\tau)\,A_j(t)\,U(t-\tau)\,U^{\dagger}(t) \tag{792}$$

$$= e^{-\mathrm{i}tH_E(t)} e^{\mathrm{i}(t-\tau)H_E(t-\tau)} A_j(t)\, e^{-\mathrm{i}(t-\tau)H_E(t-\tau)} e^{\mathrm{i}tH_E(t)} \tag{793}$$

$$= e^{-\mathrm{i}tH_E(t)} \left(\sum_{w'(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)} \mathscr{A}_j(w(t-\tau)) \right) e^{\mathrm{i}tH_E(t)} \tag{794}$$

$$= \sum_{w(t), w'(t-\tau)} e^{\mathrm{i}w'(t)t} e^{-\mathrm{i}(t-\tau)w(t-\tau)} \mathscr{A}_j(w(t-\tau), w'(t)) \tag{795}$$

$$= \sum_{w(t), w'(t-\tau)} e^{\mathrm{i}w'(t)t} e^{-\mathrm{i}(t-\tau)w(t-\tau)} \mathscr{A}_j(w(t-\tau), w'(t)) \tag{796}$$

$$= \sum_{w(t), w'(t-\tau)} e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_j(w(t-\tau), w'(t)) \tag{797}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}(t-\tau)$ and $\overline{H_E}(t)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (782) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V} = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta \in \mathrm{V}} \langle\alpha|A_i|\beta\rangle |\alpha\rangle\langle\beta|. \tag{798}$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{\mathrm{i}\overline{H_E}\tau} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{799}$$

$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau} e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} \tag{800}$$

$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{801}$$

$$= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|. \tag{802}$$

Calculating the transformation (782) directly using the previous relationship we find that:

$$U^{\dagger}(\tau)\,A_i(\tau)U(\tau) = \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in \mathrm{V}}\langle\alpha|A_i(\tau)|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{803}$$

$$= \langle+|A_i(\tau)|+\rangle|+\rangle\langle+| + e^{\mathrm{i}\eta\tau}\langle+|A_i(\tau)|-\rangle|+\rangle\langle-| + e^{-\mathrm{i}\eta\tau}\langle-|A_i(\tau)|+\rangle|-\rangle\langle+| + \langle-|A_i(\tau)|-\rangle|-\rangle\langle-|. \tag{804}$$

$$= \mathscr{A}_i(0) + \mathscr{A}_i(-w)\,e^{\mathrm{i}w\tau} + \mathscr{A}_i(w)\,e^{-\mathrm{i}w\tau} \tag{805}$$

Here $w = \lambda_+ - \lambda-$. Comparing the RHS of the equations (782) and the explicit expression for $\widetilde{A_i}(\tau)$ in (790), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$\mathscr{A}_i(0) = \langle+|A_i(\tau)|+\rangle|+\rangle\langle+| + \langle-|A_i(\tau)|-\rangle|-\rangle\langle-|, \tag{806}$$

$$\mathscr{A}_i(-w) = \langle+|A_i(\tau)|-\rangle|+\rangle\langle-|, \tag{807}$$

$$\mathscr{A}_i(w) = \langle-|A_i(\tau)|+\rangle|-\rangle\langle+|. \tag{808}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau) = \widetilde{A_i}^{\dagger}(\tau)$ and $\widetilde{B_i}(\tau) = \widetilde{B_i}^{\dagger}(\tau)$ we can use the equation (782) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big] + \mathscr{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \tag{809}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{810}$$

$$-\mathscr{B}_{ji}(-\tau)\left[A_i, \overline{\rho_S}(t)\,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\right]\Big) \tag{811}$$

Given that $\mathscr{A}_j(w(t-\tau),w'(t)) = \mathscr{A}_j^{\dagger}(-w(t-\tau),-w'(t))$ from the Fourier decomposition (782) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\mathscr{B}_{ij}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(s)\,\rho_B\right) \tag{812}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)\,\widetilde{B_j}(0)\,\rho_B\right). \tag{813}$$

Let's define:

$$\mathscr{A}_j(w(t-\tau),w'(t)) = \mathscr{A}_{jww'}(t-\tau,t) \tag{814}$$

The master equation can be re-written in the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \mathscr{B}_{ij}(\tau) \left[A_i, e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t) \overline{\rho_S}(t)\right] \tag{815}$$

$$+ \sum_{ijww'} \mathscr{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t)\right] \tag{816}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \mathscr{B}_{ij}(\tau) \left[A_i, e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t) \overline{\rho_S}(t)\right] \tag{817}$$

$$+ \sum_{ijww'} \mathscr{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{-\mathrm{i}\tau w(t-\tau)} e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t)\right] \tag{818}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \mathscr{B}_{ij}(\tau) \left[A_i, e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t) \overline{\rho_S}(t)\right] \tag{819}$$

$$+ \sum_{ijww'} \mathscr{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{-\mathrm{i}\tau w(t-\tau)} e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t)\right] \tag{820}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \mathscr{B}_{ij}(\tau) \left[A_i, e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t) \overline{\rho_S}(t)\right] \tag{821}$$

$$- \mathscr{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{-\mathrm{i}\tau w(t-\tau)} e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t)\right]\Big) \tag{822}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \textstyle\sum_{ijww'} \int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \mathrm{Tr}_B \Big(\left[A_i, \widetilde{B_i}(\tau) \widetilde{B_j}(0) \rho_B e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t) \overline{\rho_S}(t)\right] \tag{823}$$

$$- \left[A_i, \widetilde{B_j}(-\tau) \widetilde{B_i}(0) \rho_B \overline{\rho_S}(t) e^{-\mathrm{i}\tau w(t-\tau)} e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t)\right]\Big) \tag{824}$$

Given that if we define:

$$D_{ijww'}(t-\tau,t) = C_i(t) C_j(t-\tau) \mathscr{B}_{ij}(\tau) e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t) \tag{825}$$

then

$$D^{\dagger}_{ijww'}(t-\tau,t) = \left(C_i(t) C_j(t-\tau) \mathscr{B}_{ij}(\tau) e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_{jww'}(t-\tau,t)\right)^{\dagger} \tag{826}$$

$$= \mathscr{B}^*_{ij}(\tau) C_i(t) C_j(t-\tau) e^{-\mathrm{i}\tau w(t-\tau)} e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}^{\dagger}_{jww'}(t-\tau,t) \tag{827}$$

We used the fact that $C_i(t), C_j(t-\tau)$ are real. Now let's consider the following trace recalling that $\mathrm{Tr}(A)^* = \mathrm{Tr}\left(A^{\dagger}\right)$ so:

$$\mathrm{Tr}_B\left(\widetilde{B_j}(-\tau) \widetilde{B_i}(0) \rho_B\right) = \mathrm{Tr}_B\left(e^{-\mathrm{i}\tau H_E(\tau)} B_j e^{\mathrm{i}\tau H_E(\tau)} B_i \rho_B\right) \tag{828}$$

$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_E(\tau)} B_i \rho_B e^{-\mathrm{i}\tau H_E(\tau)}\right) \text{ (by cyclic permutivity of trace)} \tag{829}$$

$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_E(\tau)} B_i e^{-\mathrm{i}\tau H_E(\tau)} \rho_B\right) \text{ (by independence of Hilbert spaces)} \tag{830}$$

$$= \mathrm{Tr}_B\left(B_j \widetilde{B_i}(\tau) \rho_B\right) \text{ (by definition of time evolution)} \tag{831}$$

$$= \mathrm{Tr}_B\left(B_j \widetilde{B_i}(\tau) \rho_B\right) \tag{832}$$

$$= \mathrm{Tr}_B\left(\rho_B B_j \widetilde{B_i}(\tau)\right) \tag{833}$$

$$= \mathrm{Tr}_B\left(\left(\widetilde{B_i}(\tau) B_j \rho_B\right)^{\dagger}\right) \text{ (by definition of adjoint)} \tag{834}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau) B_j \rho_B\right)^{*} \tag{835}$$

$$= \mathscr{B}^*_{ij}(\tau) \tag{836}$$

So we can write the master equation like:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{837}$$

$$-\mathscr{B}^*_{ij}(\tau)\left[A_i, \overline{\rho_S}(t)\, e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j^\dagger\left(w\left(t-\tau\right), w'\left(t\right)\right)\right]\Big) \tag{838}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\left(\left[A_i, D_{ijww'}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i, \overline{\rho_S}\left(t\right)D^\dagger_{ijww'}\left(t-\tau,t\right)\right]\right) \tag{839}$$

Let's define the response matrix in the following way.

$$\mathscr{D}_{ijww'}\left(t\right) = \int_0^t \mathrm{d}\tau D_{ijww'}\left(t-\tau,t\right) \tag{840}$$

Then the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i, \mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i, \overline{\rho_S}\left(t\right)\mathscr{D}^\dagger_{ijww'}\left(t\right)\right]\right) \tag{841}$$

If we extend the upper limit of integration to ∞ in the equation (840) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_S(t)}{\mathrm{d}t}e^V = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^V\right)}{\mathrm{d}t} \tag{842}$$

$$= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} \tag{843}$$

$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]e^V - \sum_{ijww'}\int_0^t \mathrm{d}\tau\Big(e^{-V}\left[A_i, D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right]e^V - e^{-V}\left[A_i,\overline{\rho_S}(t)D^\dagger_{ijww'}(t-\tau,t)\right]e^V\Big). \tag{844}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^V = e^{-V}\overline{A\mathbb{I}B}e^V \tag{845}$$

$$= e^{-V}\overline{A}e^V e^{-V}\bar{B}e^V \tag{846}$$

$$= \left(e^{-V}\overline{A}e^V\right)\left(e^{-V}\bar{B}e^V\right) \tag{847}$$

$$= AB. \tag{848}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^V = e^{-V}\overline{(AB-BA)}e^V \tag{849}$$

$$= e^{-V}\overline{AB}e^V - e^{-V}\overline{BA}e^V \tag{850}$$

$$= AB - BA \tag{851}$$

$$= [A,B]. \tag{852}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]\right)e^{V} \tag{853}$$

$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^{V} - \sum_{ijww'}\left(e^{-V}\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]e^{V}\right) \tag{854}$$

$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)e^{V}\right]\right) \tag{855}$$

$$= -\mathrm{i}\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}(t)\,e^{V}e^{-V}\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}(t)e^{V}e^{-V}\mathscr{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right) \tag{856}$$

$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}\left(t\right)e^{V}\rho_S\left(t\right)\right] - \left[e^{-V}A_ie^{V},\rho_S\left(t\right)e^{-V}\mathscr{D}^{\dagger}_{ijww'}\left(t\right)e^{V}\right]\right). \tag{857}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (841) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (841) reproduces the results of the reference [1], for the latter case we have that $i,j\in\{1,2,3\}$ and $\omega\in(0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{858}$$

After performing the transformation (24) on the Hamiltonian (858) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)\,|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, \tag{859}$$

$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), \tag{860}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{861}$$

The Hamiltonian (859) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{862}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{863}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \tag{864}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix

$\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$
\begin{aligned}
0 &= \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, && (865)\\
\lambda_{\pm} &= \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} && (866)\\
&= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} && (867)\\
&= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} && (868)\\
\eta &= \sqrt{\epsilon^2 + \Omega_r^2}, && (869)\\
\lambda_{\pm} &= \frac{R \pm \eta}{2}. && (870)
\end{aligned}
$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$
\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{871}
$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$
\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{872}
$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$.
Summarizing these results we can write:

$$
\begin{aligned}
\lambda_+ &= \frac{\epsilon + \eta}{2}, && (873)\\
\lambda_- &= \frac{\epsilon - \eta}{2}, && (874)\\
|+\rangle &= \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, && (875)\\
|-\rangle &= \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, && (876)\\
\sin(\theta) &= \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}, && (877)\\
\cos(\theta) &= \frac{\epsilon + \eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}. && (878)
\end{aligned}
$$

This result is plausible because in the paper [1] we have that:

$$
\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{879}
$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{880}$$

So the value of $\tan(\theta)$ using (880) is equal to:

$$\begin{aligned}
\tan(\theta) &= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}+1} && (881)\\
&= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} && (882)\\
&= \frac{\Omega_r}{\epsilon+\eta}. && (883)
\end{aligned}$$

This proves our assertion.
Using this basis we can find the decomposition matrices using the equations (807)-(808) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ with$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\begin{aligned}
\langle+|\sigma_x|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (884)\\
&= 2\sin(\theta)\cos(\theta) && (885)\\
&= \sin(2\theta), && (886)\\
\langle-|\sigma_x|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (887)\\
&= -2\sin(\theta)\cos(\theta) && (888)\\
&= -\sin(2\theta), && (889)\\
\langle-|\sigma_x|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (890)\\
&= \cos^2(\theta) - \sin^2(\theta) && (891)\\
&= \cos(2\theta), && (892)\\
\langle+|\sigma_y|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (893)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (894)\\
&= 0, && (895)\\
\langle-|\sigma_y|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (896)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (897)\\
&= 0, && (898)\\
\langle-|\sigma_y|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (899)\\
&= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) && (900)\\
&= \mathrm{i}. && (901)
\end{aligned}$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{902}$$
$$= \cos(\theta)\cos(\theta) \tag{903}$$
$$= \cos^2(\theta), \tag{904}$$
$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{905}$$
$$= \sin(\theta)\sin(\theta) \tag{906}$$
$$= \sin^2(\theta), \tag{907}$$
$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{908}$$
$$= -\sin(\theta)\cos(\theta) \tag{909}$$
$$= -\sin(\theta)\cos(\theta). \tag{910}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)(|+\rangle\langle+| - |-\rangle\langle-|), \tag{911}$$
$$A_1(\eta) = \cos(2\theta)|-\rangle\langle+|, \tag{912}$$
$$A_2(0) = 0, \tag{913}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{914}$$
$$A_3(0) = \cos^2(\theta)|+\rangle\langle+| + \sin^2(\theta)|-\rangle\langle-|, \tag{915}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)|-\rangle\langle+|. \tag{916}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (844) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (825) is equivalent to:

$$\mathscr{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau,t) \tag{917}$$
$$= \int_0^t \mathrm{d}\tau C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_j(w(t-\tau), w'(t)) \tag{918}$$
$$= \int_0^t \mathrm{d}\tau C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)} \mathscr{A}_j(w, w'). \tag{919}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (392) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau). \tag{920}$$

So the response matrix can be rewritten as:

$$\mathscr{D}_{ijww'}(t) = \left(\int_0^t \mathrm{d}\tau \Lambda'_{ij}(\tau)\, e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)}\right) \mathscr{A}_j(w, w') \tag{921}$$

Let's define the response function like:

$$K_{ij}\left(w,w',t\right)=\int_0^t C_i\left(t\right)C_j\left(t-\tau\right)\Lambda_{ij}\left(\tau\right)e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{922}$$

$$=\int_0^t \Lambda'_{ij}\left(\tau\right)e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{923}$$

$$=K_{ijww'}\left(t\right). \tag{924}$$

Then we have the following equivalence:

$$\mathscr{D}_{ijww'}\left(t\right)=K_{ijww'}\left(t\right)\mathscr{A}_j\left(w,w'\right) \tag{925}$$

$$=K_{ijww'}\left(t\right)\mathscr{A}_{jww'} \tag{926}$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right]-\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]\right) \tag{927}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_i,K_{ijww'}\left(t\right)\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)\right]-\left[A_i,\overline{\rho_S}\left(t\right)K^{*}_{ijww'}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{928}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ijww'}\left(K_{ijww'}\left(t\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)\right]-K^{*}_{ijww'}\left(t\right)\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{929}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ijww'}\left(\left(K^{\Re}_{ijww'}\left(t\right)+\mathrm{i}K^{\Im}_{ijww'}\left(t\right)\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)\right]-\left(K^{\Re}_{ijww'}\left(t\right)-\mathrm{i}K^{\Im}_{ijww'}\left(t\right)\right)\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{930}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ijww'}K^{\Re}_{ijww'}\left(t\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)-\overline{\rho_S}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]-\mathrm{i}\sum_{ijww'}K^{\Im}_{ijww'}\left(t\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)+\overline{\rho_S}\left(t\right)\mathscr{A}^{\dagger}_{jww'} \tag{931}$$

Using the notation of the master equation (841), we can say that $C_1\left(t\right)=\frac{\Omega}{2}=C_2\left(t\right)$ and $C_3\left(t\right)=1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B\left(t\right)=B$. Taking the equations(731)-(741) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (392) are equal to:

$$\Lambda'_{11}\left(\tau\right)=\left(\frac{\Omega}{2}\right)^2\mathrm{Tr}_B\left(\widetilde{B_1}\left(\tau\right)\widetilde{B_1}\left(0\right)\rho_B\right) \tag{932}$$

$$=\frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}-2\right), \tag{933}$$

$$\Lambda'_{22}\left(\tau\right)=\left(\frac{\Omega}{2}\right)^2\mathrm{Tr}_B\left(\widetilde{B_2}\left(\tau\right)\widetilde{B_2}\left(0\right)\rho_B\right) \tag{934}$$

$$=\frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}\right), \tag{935}$$

$$\Lambda'_{33}\left(\tau\right)=\int_0^{\infty}\mathrm{d}\omega J\left(\omega\right)\left(1-F\left(\omega\right)\right)^2G_+\left(\tau\right), \tag{936}$$

$$\Lambda'_{32}\left(\tau\right)=\frac{\Omega_r}{2}\int_0^{\infty}\mathrm{d}\omega\frac{J\left(\omega\right)}{\omega}F\left(\omega\right)\left(1-F\left(\omega\right)\right)\mathrm{i}G_-\left(\tau\right), \tag{937}$$

$$\Lambda'_{32}\left(\tau\right)=-\Lambda'_{23}\left(\tau\right), \tag{938}$$

$$\Lambda'_{12}\left(\tau\right)=\Lambda'_{21}\left(\tau\right) \tag{939}$$

$$=\Lambda'_{13}\left(\tau\right) \tag{940}$$

$$=\Lambda'_{31}\left(\tau\right) \tag{941}$$

$$=0. \tag{942}$$

Finally taking the Hamiltonian (858) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (336) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{943}$$

$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{944}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (418) under a time-independent approach providing similar results.
Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (841) and using the fact that

$$\widetilde{A_j}(t-\tau, t) = U(\tau)A_jU(-\tau) \tag{945}$$

$$= \sum_w e^{\mathrm{i}w\tau}\mathscr{A}_j(-w) \tag{946}$$

$$= \sum_w e^{-\mathrm{i}w\tau}\mathscr{A}_j(w). \tag{947}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_w \gamma_{ij}(w,t)\left[A_i, \mathscr{A}_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)\mathscr{A}_j^\dagger(w)\right] \tag{948}$$

$$-\sum_{ij}\sum_w S_{ij}(w,t)\left[A_i, \mathscr{A}_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)\mathscr{A}_j^\dagger(w)\right]. \tag{949}$$

where $\mathscr{A}_j^\dagger(w) = \mathscr{A}_j(-w)$, as we can see the equation (949) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2K_{ij}^{\Re}(w,t)$ and $S_{ij}(w,t) = K_{ij}^{\Im}(w,t)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}(w,t) = \int_0^t \Lambda'_{ij}(\tau)e^{\mathrm{i}w\tau}\mathrm{d}\tau.$$

The fact $\mathscr{A}_j^\dagger(w) = \mathscr{A}_j(-w)$ can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{\mathrm{i}\overline{H_S}(t-\tau)} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)(t-\tau)} \tag{950}$$

$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|(t-\tau)}e^{\mathrm{i}\lambda_-|-\rangle\langle-|(t-\tau)} \tag{951}$$

$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{952}$$

$$= e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|. \tag{953}$$

Where $\lambda_+, \lambda_-$are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (782) of (806)-(808) directly using the previous relationship we find that:

$$\widetilde{A_i\left(0\right)}\left(t-\tau\right)=\left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right) \tag{954}$$

$$=\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|, \tag{955}$$

$$\widetilde{A_i\left(w\right)}\left(t-\tau\right)=\left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|-\rangle|+\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right) \tag{956}$$

$$=\langle+|A_i|-\rangle|+\rangle\langle-|e^{\mathrm{i}w(t-\tau)}, \tag{957}$$

$$\widetilde{A_i\left(-w\right)}\left(t-\tau\right)=\left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle-|A_i|+\rangle|-\rangle\langle+|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right) \tag{958}$$

$$=\langle-|A_i|+\rangle|-\rangle\langle+|e^{-\mathrm{i}w(t-\tau)}. \tag{959}$$

Here $w=\lambda_+-\lambda-$. So we can see that for the equation (792) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w\neq w'$ then $A'_j\left(w,w'\right)=0$ so:

$$\widetilde{A_j}\left(t-\tau,t\right)=U\left(t\right)U^\dagger\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^\dagger\left(t\right) \tag{960}$$

$$=U\left(t\right)\left(\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}A_j\left(w(t-\tau)\right)\right)U^\dagger\left(t\right) \tag{961}$$

$$=\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}U\left(t\right)A_j\left(w(t-\tau)\right)U^\dagger\left(t\right) \tag{962}$$

$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w(t-\tau),w'\left(t\right)\right) \tag{963}$$

$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_{jww'} \tag{964}$$

$$==\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w\right)\delta_{ww'} \tag{965}$$

$$=\sum_{w}e^{-\mathrm{i}(t-\tau)w}e^{\mathrm{i}tw}A_j\left(w\right) \tag{966}$$

$$=\sum_{w}e^{\mathrm{i}\tau w}A_j\left(w\right) \tag{967}$$

$$=U^\dagger\left(-\tau\right)A_jU\left(-\tau\right) \tag{968}$$

So using now as reference the equation (931) and $A'_j\left(w,w'\right)=0$ we can deduce that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ijw}K^{\Re}_{ij}\left(w,t\right)\left[A_i,A_j\left(w\right)\overline{\rho_S}\left(t\right)-\overline{\rho_S}\left(t\right)A^\dagger_j\left(w\right)\right]-\mathrm{i}\sum_{ijw}K^{\Im}_{ij}\left(w,t\right)\left[A_i,A_j\left(w\right)\overline{\rho_S}\left(t\right)+\overline{\rho_S}\left(t\right)A^\dagger_j\left(w\right)\right] \tag{969}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k\ll\omega_k$ then $f_k\approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z=0$. The Hamiltonian studied is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b^\dagger_{\mathbf{k}}+g^*_{\mathbf{k}}b_{\mathbf{k}}\right)\right)|1\rangle\langle1|+\frac{\Omega\left(t\right)}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}. \tag{970}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (970) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (225) and (236) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t), \tag{971}$$

$$\overline{H_{\mathrm{I}}} = \frac{\Omega(t)}{2}(B_x\sigma_x + B_y\sigma_y). \tag{972}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (932)-(939) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}G_+(\tau)\,\mathrm{d}\omega. \tag{973}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (973) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega. \tag{974}$$

Writing the interaction Hamiltonian (972) in the similar way to the equation (236) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (225) with $\delta'|1\rangle\langle 1|=\frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t). \tag{975}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{976}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{977}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{978}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right). \tag{979}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(418) is:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{980}$$

$$\left.+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right). \tag{981}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (976) and (979) on the equation (981) we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left([\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)]\Lambda_x(\tau)\right. \tag{982}$$

$$\left.+[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)]\Lambda_y(\tau) + [\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x]\Lambda_x(\tau) + [\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y]\Lambda_y(\tau)\right). \tag{983}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (392) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right). \tag{984}$$

Using the equation (841), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)), \rho_S(t)\right] - \frac{1}{2}\sum_w \gamma_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) - \rho_S(t)A_3^\dagger(w)\right] \tag{985}$$

$$-\sum_w S_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) + \rho_S(t)A_3^\dagger(w)\right]\Bigg). \tag{986}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(\tau), \tag{987}$$

$$\Lambda_{33}(-\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(-\tau). \tag{988}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (986) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t))\right] - \sum_w \left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^*(w,t)\left[\rho_S(t)A_3(w), A_3\right]\right). \tag{989}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{990}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle), \tag{991}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{992}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{993}$$

The elements of the decomposition matrices are:

$$\langle+|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{994}$$

$$= \frac{1}{2}, \tag{995}$$

$$\langle-|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{996}$$

$$= \frac{1}{2}, \tag{997}$$

$$\langle-|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{998}$$

$$= -\frac{1}{2}. \tag{999}$$

The decomposition matrices are

$$A_3\left(0\right) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{1000}$$

$$= \frac{\mathbb{I}}{2}, \tag{1001}$$

$$A_3\left(\eta\right) = -\frac{1}{2}|-\rangle\langle+| \tag{1002}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right), \tag{1003}$$

$$A_3\left(-\eta\right) = -\frac{1}{2}|+\rangle\langle-| \tag{1004}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right). \tag{1005}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{1006}$$

$$- K_{33}^{*}\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^{*}\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right]. \tag{1007}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)G_{+}\left(\tau\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1008}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1009}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1010}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)\left(n\left(\omega\right)+1\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1011}$$

$$= \int_0^{\infty} J\left(\omega\right)\left(n\left(\omega\right)+1\right)\pi\delta\left(\widetilde{\Omega}-\omega\right)\mathrm{d}\omega \tag{1012}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right), \tag{1013}$$

$$K_{33}\left(-\widetilde{\Omega}\right)=\int_0^\infty\int_0^\infty J(\omega)\,G_+(\tau)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1014}$$
$$=\int_0^\infty\int_0^\infty J(\omega)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega}+n(\omega)\,e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1015}$$
$$=\int_0^\infty\int_0^\infty J(\omega)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}n(\omega)\,\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1016}$$
$$=\int_0^\infty\int_0^\infty J(\omega)\,n(\omega)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau+\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1017}$$
$$=\int_0^\infty J(\omega)\,n(\omega)\,\pi\delta\left(-\widetilde{\Omega}+\omega\right)\mathrm{d}\omega \tag{1018}$$
$$=\pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right). \tag{1019}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s}=\pi\delta(\varepsilon)\pm\mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega=0$. Replacing in the equation (1006) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}=-\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x,\rho_S(t)\right)]-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\,\rho_S(t)\right]+n\left(\widetilde{\Omega}\right)\left[\sigma_z,(\sigma_z-\mathrm{i}\sigma_y)\,\rho_S(t)\right]\right) \tag{1020}$$
$$-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S(t)\,(\sigma_z+\mathrm{i}\sigma_y),\sigma_z\right]+n\left(\widetilde{\Omega}\right)\left[\rho_S(t)\,(\sigma_z-\mathrm{i}\sigma_y),\sigma_z\right]\right). \tag{1021}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}=-\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x,\rho_S(t)\right)]-\frac{\pi}{8}J(\Omega(t))\left(\!\left(n(\Omega(t))+1\right)[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\rho_S(t)]+n(\Omega(t))[\sigma_z,(\sigma_z-\mathrm{i}\sigma_y)\,\rho_S(t)]\right) \tag{1022}$$
$$-\frac{\pi}{8}J(\Omega(t))\left((n(\Omega(t))+1)\left[\rho_S(t)\,(\sigma_z+\mathrm{i}\sigma_y),\sigma_z\right]+n(\Omega(t))\left[\rho_S(t)\,(\sigma_z-\mathrm{i}\sigma_y),\sigma_z\right]\right). \tag{1023}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t)=H_S(t)+H_I+H_B, \tag{1024}$$
$$H_S(t)=\sum_n\varepsilon_n(t)\,|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|, \tag{1025}$$
$$H_I=\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger+g_{nu\mathbf{k}}^*b_{u\mathbf{k}}\right), \tag{1026}$$
$$H_B=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \tag{1027}$$

A. Variational Transformation

We consider the following operator:

$$V=\sum_{nu\mathbf{k}}|n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{1028}$$

At first let's obtain $e^{\pm V}$ under the transformation (1028), consider $\hat{\varphi}_n=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)$, so the equation (1028) can be written as $V=\sum_n|n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$e^{\pm V}=e^{\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n} \tag{1029}$$

$$=\mathbb{I}\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\left(\sum_n|n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!}+\ldots \tag{1030}$$

$$=\mathbb{I}\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\sum_n|n\rangle\langle n|\hat{\varphi}_n^2}{2!}+\ldots \tag{1031}$$

$$=\sum_n|n\rangle\langle n|\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\sum_n|n\rangle\langle n|\hat{\varphi}_n^2}{2!}+\ldots \tag{1032}$$

$$=\sum_n|n\rangle\langle n|\left(\mathbb{I}\pm\hat{\varphi}_n+\frac{\hat{\varphi}_n^2}{2!}+\ldots\right) \tag{1033}$$

$$=\sum_n|n\rangle\langle n|e^{\pm\hat{\varphi}_n} \tag{1034}$$

Given that $\left[f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}},f_{nu'\mathbf{k}'}b_{u'\mathbf{k}'}^{\dagger}-f_{nu'\mathbf{k}'}^{*}b_{u'\mathbf{k}'}\right]=0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u,u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right)=e^{\pm\left(\alpha_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-\alpha_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}}=\frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}=\prod_u e^{\pm\sum_{\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} \tag{1035}$$

$$=\prod_u\left(\prod_{\mathbf{k}}e^{\pm\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}\right) \tag{1036}$$

$$=\prod_u\left(\prod_{\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) \tag{1037}$$

$$=\prod_{u\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1038}$$

$$=\prod_u B_{nu\pm} \tag{1039}$$

$$B_{nu\pm}\equiv\prod_{\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1040}$$

As we can see $e^{-V}=\sum_n|n\rangle\langle n|\prod_u B_{nu-}$ and $e^{V}=\sum_n|n\rangle\langle n|\prod_u B_{nu+}$ this implies that $e^{-V}e^{V}=\mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V}=\left(\sum_n|n\rangle\langle n|\prod_u B_{nu+}\right)A\left(\sum_n|n\rangle\langle n|\prod_u B_{nu-}\right) \tag{1041}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1024):

$$
\begin{aligned}
\overline{|0\rangle\langle 0|} &= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), && (1042)\\
&= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, && (1043)\\
&= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, && (1044)\\
&= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} && (1045)\\
&= |0\rangle\langle 0| \prod_u \mathbb{I} && (1046)\\
&= |0\rangle\langle 0|. && (1047)\\
\overline{|m\rangle\langle n|} &= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), && (1048)\\
&= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, && (1049)\\
&= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, && (1050)\\
&= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right), \ m \neq n, && (1051)\\
&= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), && (1052)\\
&= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), && (1053)\\
&= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). && (1054)\\
\textstyle\prod_u \left(B_{mu+} B_{nu-}\right) &= \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). && (1055)\\
\overline{\textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} &= \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), && (1056)\\
&= \left(|0\rangle\langle 0| \textstyle\prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right)\left(\textstyle\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \textstyle\prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), && (1057)\\
&= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, && (1058)\\
&= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots && (1059)\\
&= |0\rangle\langle 0| \left(\textstyle\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) && (1060)\\
&+ |1\rangle\langle 1| \left(\textstyle\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots && (1061)\\
&= |0\rangle\langle 0| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) && (1062)\\
&+ |1\rangle\langle 1| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots && (1063)\\
&= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots && (1064)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1065)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) && (1066)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) && (1067)\\
&= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) && (1068)
\end{aligned}
$$

The transformed Hamiltonians of the equations (1025) to (1027) written in terms of (1042) to (1066) are:

$$\overline{H_S\left(t\right)} = \overline{\sum_{n}\varepsilon_{n}\left(t\right)\left|n\middle\rangle\middle\langle n\right| + \sum_{n\neq m}V_{nm}\left(t\right)\left|n\middle\rangle\middle\langle m\right|} \tag{1069}$$

$$= \overline{\sum_{n}\varepsilon_{n}\left(t\right)\left|n\middle\rangle\middle\langle n\right|} + \overline{\sum_{n\neq m}V_{nm}\left(t\right)\left|n\middle\rangle\middle\langle m\right|} \tag{1070}$$

$$= \sum_{n}\varepsilon_{n}\left(t\right)\left|n\middle\rangle\middle\langle n\right| + \sum_{n\neq m}V_{nm}\left(t\right)\left|n\middle\rangle\middle\langle m\right|\prod_{u}\left(B_{mu+}B_{nu-}\right) \tag{1071}$$

$$\overline{H_I} = \left(\sum_{n}\left|n\middle\rangle\middle\langle n\right|\prod_{u}B_{nu+}\right)\left(\sum_{nu\mathbf{k}}\left|n\middle\rangle\middle\langle n\right|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_{n}\left|n\middle\rangle\middle\langle n\right|\prod_{u}B_{nu-}\right) \tag{1072}$$

$$= \left(\sum_{n}\left|n\middle\rangle\middle\langle n\right|\prod_{u}B_{nu+}\right)\left(\sum_{u\mathbf{k}}\left|0\middle\rangle\middle\langle 0\right|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)+...\right)\left(\sum_{n}\left|n\middle\rangle\middle\langle n\right|\prod_{u}B_{nu-}\right) \tag{1073}$$

$$= \prod_{u}B_{0u+}\sum_{u\mathbf{k}}\left|0\middle\rangle\middle\langle 0\right|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_{u}B_{0u-}+\prod_{u}B_{1u+}\sum_{u\mathbf{k}}\left|1\middle\rangle\middle\langle 1\right|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_{u}B_{1u-}+\ldots \tag{1074}$$

$$=\sum_{u\mathbf{k}}\left|0\middle\rangle\middle\langle 0\right|\left(g_{0u\mathbf{k}}\prod_{u}B_{0u+}b_{u\mathbf{k}}^{\dagger}\prod_{u}B_{0u-}+g_{0u\mathbf{k}}^{*}\prod_{u}B_{0u+}b_{u\mathbf{k}}\prod_{u}B_{0u-}\right)+\sum_{u\mathbf{k}}\left|1\middle\rangle\middle\langle 1\right|\left(g_{1u\mathbf{k}}\prod_{u}B_{1u+}b_{u\mathbf{k}}^{\dagger}\prod_{u}B_{1u-}+g_{1u\mathbf{k}}^{*}\prod_{u}B_{1u+}b_{u\mathbf{k}}\prod_{u}B_{1u-}\right)+\ldots \tag{1075}$$

$$=\sum_{u\mathbf{k}}\left|0\middle\rangle\middle\langle 0\right|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\sum_{u\mathbf{k}}\left|1\middle\rangle\middle\langle 1\right|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\ldots \tag{1076}$$

$$= \sum_{nu\mathbf{k}}\left|n\middle\rangle\middle\langle n\right|\left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1077}$$

$$= \sum_{nu\mathbf{k}}\left|n\middle\rangle\middle\langle n\right|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1078}$$

$$\overline{H_B} = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}\left|n\middle\rangle\middle\langle n\right|\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1079}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H}=\sum_{n}\varepsilon_{n}(t)\left|n\middle\rangle\middle\langle n\right|+\sum_{n\neq m}V_{nm}(t)\left|n\middle\rangle\middle\langle m\right|\prod_{u}(B_{mu+}B_{nu-})+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}\left|n\middle\rangle\middle\langle n\right|\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1080}$$

$$+\sum_{nu\mathbf{k}}\left|n\middle\rangle\middle\langle n\right|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1081}$$

Let's define the following functions:

$$R_{n}\left(t\right) = \sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1082}$$

$$B_{z,n}\left(t\right) = \sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)b_{u\mathbf{k}}^{\dagger}+\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)^{*}b_{u\mathbf{k}}\right) \tag{1083}$$

Using the previous functions we have that (1080) can be re-written in the following way:

$$\overline{H}=\sum_{n}\varepsilon_{n}(t)\left|n\middle\rangle\middle\langle n\right|+\sum_{n\neq m}V_{nm}(t)\left|n\middle\rangle\middle\langle m\right|\prod_{u}(B_{mu+}B_{nu-})+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{n}R_{n}(t)\left|n\middle\rangle\middle\langle n\right|+\sum_{n}B_{z,n}(t)\left|n\middle\rangle\middle\langle n\right| \tag{1084}$$

$$\tag{1085}$$

Now in order to separate the elements of the hamiltonian (1085) let's follow the references of the equations (225) and (236) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \prod_{u}\left(B_{mu+}B_{nu-}\right)\right\rangle_{\overline{H_0}}=\left\langle \prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1086}$$

$$= \left(\prod_{u\mathbf{k}}\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\left\langle \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1087}$$

$$= \left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1088}$$

$$\equiv B_{nm} \tag{1089}$$

$$\left\langle \prod_{u}\left(B_{nu+}B_{mu-}\right)\right\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1090}$$

$$= B^*_{nm} \tag{1091}$$

Following the reference [4] we define:

$$J_{nm} = \prod_{u}\left(B_{mu+}B_{nu-}\right) - B_{nm} \tag{1092}$$

As we can see:

$$J^\dagger_{nm} = \left(\prod_{u}\left(B_{mu+}B_{nu-}\right) - B_{nm}\right)^\dagger \tag{1093}$$

$$= \prod_{u}\left(B_{nu+}B_{mu-}\right) - B^*_{nm} \tag{1094}$$

$$= \prod_{u}\left(B_{nu+}B_{mu-}\right) - B_{mn} \tag{1095}$$

$$= J_{mn} \tag{1096}$$

We can separate the Hamiltonian (1085) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}\left(t\right)} = \sum_{n}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} \tag{1097}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|J_{nm} + \sum_{n}B_{z,n}\left(t\right)|n\rangle\langle n|, \tag{1098}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b^\dagger_{u\mathbf{k}}b_{u\mathbf{k}} \tag{1099}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H^2_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_{B}}}\right). \tag{1100}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right) = C\left(R_0,R_1,...,R_{d-1},B_{01},...,B_{0(d-1)},...,B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^{\Re}_{nu\mathbf{k}}$ and $v^{\Im}_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}\left(t\right)}$ can be written like:

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1101}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1102}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}-\frac{g_{nu\mathbf{k}}v^*_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1103}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1104}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1105}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1106}$$

$$v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}=\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)-\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right) \tag{1107}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1108}$$

$$-\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1109}$$

$$=2\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right) \tag{1110}$$

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1111}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1112}$$

$$|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2=(v_{mu\mathbf{k}}-v_{nu\mathbf{k}})(v_{mu\mathbf{k}}-v_{nu\mathbf{k}})^* \tag{1113}$$

$$=|v_{mu\mathbf{k}}|^2+|v_{nu\mathbf{k}}|^2-(v_{nu\mathbf{k}}v^*_{mu\mathbf{k}}+v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}) \tag{1114}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1115}$$

$$-\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1116}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2\left(v^{\Re}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{nu\mathbf{k}}v^{\Im}_{mu\mathbf{k}}\right) \tag{1117}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2 \tag{1118}$$

$$R_n(t)=\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1119}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right) \tag{1120}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2g^{\Re}_{nu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-2g^{\Im}_{nu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \tag{1121}$$

$$B_{mn}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1122}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1123}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1124}$$

$$= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1125}$$

$$= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1126}$$

$$\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1127}$$

$$= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1128}$$

$$= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1129}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) \tag{1130}$$

$$= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1131}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1132}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1133}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1134}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \tag{1135}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \tag{1136}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^\dagger}{\partial a} \tag{1137}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1138}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1139}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1140}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1141}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1142}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1143}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1144}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1145}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1146}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1147}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1148}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1149}$$

Introducing this derivates in the equation (1124) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1150}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1151}$$

$$= 0 \tag{1152}$$

We can obtain the variational parameters:

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1153}$$

$$=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1154}$$

$$v^{\Re}_{nu\mathbf{k}}=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1155}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1156}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}}=\frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}}-g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1157}$$

$$\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}-\left(v^{\Im}_{nu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1158}$$

$$=0 \tag{1159}$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1160}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1161}$$

$$v_{nu\mathbf{k}}^{\Im}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1162}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1163}$$

$$v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im} \tag{1164}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1165}$$

$$\mathrm{i}\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1166}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1167}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1168}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1169}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1170}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1171}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1172}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^{V}\rho\left(0\right)e^{-V}=\left(\sum_{n}|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_{n}|n\rangle\langle n|B_{n+}\right) \tag{1173}$$

$$0=\left(B_0^{+}|0\rangle\langle 0|B_0^{-}\right)\otimes\rho_B \tag{1174}$$

$$0=\rho\left(0\right) \tag{1175}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1176}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1177}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\, \overline{\rho_S}(t)\, U(t), \text{ where} \tag{1178}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1179}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1180}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1181}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1182}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1183}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1184}$$
$$= B_{n-}^{\dagger}B_{m+}^{\dagger} - B_n B_m \tag{1185}$$
$$= B_{n+}B_{m-} - B_n B_m \tag{1186}$$
$$= B_{nm} \tag{1187}$$

So we can say that the set of matrices (1180) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1180) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)\, |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t)\, |n\rangle\langle n|, \tag{1188}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)\, |n\rangle\langle m| B_{nm} + V_{mn}(t)\, |m\rangle\langle n| B_{mn}\right) \tag{1189}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t)\, B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1190}$$
$$\left. + \Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t)\, B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1191}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1192}$$
$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1193}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1194}$$
$$\left. + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,x}\right) \tag{1195}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1196}$$

Now consider the following set of operators,

$$A_{1,nm}\left(t\right)=\sigma_{nm,x}\left(1-\delta_{mn}\right) \tag{1197}$$
$$A_{2,nm}\left(t\right)=\sigma_{nm,y}\left(1-\delta_{mn}\right) \tag{1198}$$
$$A_{3,nm}\left(t\right)=\delta_{mn}|n\rangle\langle m| \tag{1199}$$
$$A_{4,nm}\left(t\right)=A_{2,mn}\left(t\right) \tag{1200}$$
$$A_{5,nm}\left(t\right)=A_{1,nm}\left(t\right) \tag{1201}$$
$$B_{1,nm}\left(t\right)=B_{nm,x} \tag{1202}$$
$$B_{2,nm}\left(t\right)=B_{nm,y} \tag{1203}$$
$$B_{3,nm}\left(t\right)=B_{z,n}\left(t\right) \tag{1204}$$
$$B_{4,nm}\left(t\right)=B_{1,nm}\left(t\right) \tag{1205}$$
$$B_{5,nm}\left(t\right)=B_{2,nm}\left(t\right) \tag{1206}$$
$$C_{1,nm}\left(t\right)=\Re\left(V_{nm}\left(t\right)\right) \tag{1207}$$
$$C_{2,nm}\left(t\right)=C_{1,nm}\left(t\right) \tag{1208}$$
$$C_{3,nm}\left(t\right)=1 \tag{1209}$$
$$C_{4,nm}\left(t\right)=\Im\left(V_{nm}\left(t\right)\right) \tag{1210}$$
$$C_{5,nm}\left(t\right)=-\Im\left(V_{nm}\left(t\right)\right) \tag{1211}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}\left(t\right)$ as:

$$\overline{H_I}=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(A_{jp}\otimes B_{jp}\left(t\right)\right) \tag{1212}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1196).

We write the interaction Hamiltonian transformed under (1176) as:

$$\widetilde{H_I}\left(t\right)=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1213}$$
$$\widetilde{A_{jp}}\left(t\right)=U^{\dagger}\left(t\right)A_{jp}U\left(t\right) \tag{1214}$$
$$\widetilde{B_{jp}}\left(t\right)=\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\left(t\right)\left(\mathrm{t}\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{1215}$$

Taking as reference state ρ_B and truncating at second order in $H_I\left(t\right)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}\left(t\right)}{\mathrm{d}t}=-\int_0^t\mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right),\left[\widetilde{H_I}\left(s\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}s \tag{1216}$$

Replacing the equation (1213) in (1216) we can obtain:

$$
\begin{aligned}
\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \qquad (1217)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \qquad (1218)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \qquad (1219)\\
&\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \qquad (1220)\\
&= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \qquad (1221)\\
&-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \qquad (1222)\\
&-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \qquad (1223)\\
&\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \qquad (1224)
\end{aligned}
$$

In order to calculate the correlation functions we define:

$$
\begin{aligned}
\Lambda_{jpj'p'}(\tau) &= \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \qquad (1225)\\
&= \left\langle \widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \qquad (1226)
\end{aligned}
$$

Here $s\to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1224) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1227}$$
$$= \left\langle \widetilde{B_{jp}}(0)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1228}$$
$$= \Lambda_{jpj'p'}(\tau) \tag{1229}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\rho_B\widetilde{B_{j'p'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1230}$$
$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1231}$$
$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1232}$$
$$= \Lambda_{j'p'jp}(-\tau) \tag{1233}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\rho_B\widetilde{B_{jp}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) \tag{1234}$$
$$= \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1235}$$
$$= \left\langle \widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1236}$$
$$= \Lambda_{jpj'p'}(\tau) \tag{1237}$$
$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1238}$$
$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1239}$$
$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1240}$$
$$= \Lambda_{j'p'jp}(-\tau) \tag{1241}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1217)and (1224) and using the equations (1227)-(1241) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'} \Big(C_{jp}(t)\,C_{j'p'}(s)\Big(\Lambda_{jpj'p'}(\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'p'jp}(-\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\Big) \tag{1242}$$

$$+C_{jp}(t)\,C_{j'p'}(s)\Big(\Lambda_{j'p'jp}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jp}}(t)\Big)\Big)\,\mathrm{d}s \tag{1243}$$

$$= -\int_0^t \sum_{jj'pp'} \Big(C_{jp}(t)\,C_{j'p'}(s)\Big(\Lambda_{jpj'p'}(\tau)\Big[\widetilde{A_{jp}}(t)\,,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\Big] + \Lambda_{j'p'jp}(-\tau)\Big[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\,,\widetilde{A_{jp}}(t)\Big]\Big)\Big) \tag{1244}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'} \left(C_{jp}(t)\,C_{j'p'}(t-\tau)\left(\Lambda_{jpj'p'}(\tau)\left[A_{jp}(t)\,,A_{j'p'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'p'jp}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'p'}(t-\tau,t)\,,A_{jp}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1245}$$

For this case we used that $A_{jp}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jp}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1246}$$
$$H_S(t) = \sum_{n=0} \varepsilon_n(t)\, |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\, |n\rangle\langle m|, \tag{1247}$$
$$H_I = \left(\sum_{n=0} \mu_n(t)\, |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1248}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1249}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t)\, |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1250}$$

At first let's obtain e^V under the transformation (1250), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{1251}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + \ldots \tag{1252}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + \ldots \tag{1253}$$
$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \ldots\right) \tag{1254}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1255}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+} \tag{1256}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1257}$$
$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1258}$$
$$= B_{\pm} \tag{1259}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_{-}$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+}\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{-}\right) \tag{1260}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1246):

$$
\begin{aligned}
\overline{|0\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right), && (1261)\\
&= |0\rangle\langle 0|, && (1262)\\
\overline{|m\rangle\langle n|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right), && (1263)\\
&= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, && (1264)\\
&= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, && (1265)\\
\overline{|0\rangle\langle m|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right), && (1266)\\
&= |0\rangle\langle m|B^{-}\ m \neq 0, && (1267)\\
\overline{|m\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right) && (1268)\\
&= |0\rangle\langle m|B^{+}\ m \neq 0, && (1269)\\
\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right)\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right) && (1270)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}} B^{+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B^{-} && (1271)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B^{+} b_{\mathbf{k}}^{\dagger} B^{-}\right)\left(B^{+} b_{\mathbf{k}} B^{-}\right) && (1272)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1273)\\
&= |0\rangle\langle 0|\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1274)\\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right) && (1275)\\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n|\sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n|\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) && (1276)
\end{aligned}
$$

The transformed Hamiltonians of the equations (1247) to (1249) written in terms of (1261) to (1276) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1277}$$

$$= \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1278}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \overline{\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n|} \tag{1279}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)\overline{|0\rangle\langle n|} + V_{n0}(t)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0}V_{mn}(t)\overline{|m\rangle\langle n|} \tag{1280}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)B^-|0\rangle\langle n| + V_{n0}(t)B^+|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1281}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^- + V_{n0}(t)|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1282}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^+\right)\left(\left(\sum_{n=0}\mu_n(t)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^-\right) \tag{1283}$$

$$= \left(\mu_0(t)|0\rangle\langle 0| + \sum_{n=1}\mu_n(t)|n\rangle\langle n|B^+\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^-\right) \tag{1284}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}B^+\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)B^- \tag{1285}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1286}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1287}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^- + V_{n0}(t)|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1288}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1289}$$

$$+\sum_{n=0}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1290}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^- + V_{n0}(t)|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1291}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1292}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1293}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1294}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\right) \tag{1295}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1296}$$

Using the previous functions we have that (1293) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1297}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1298}$$

Now in order to separate the elements of the hamiltonian (1298) let's follow the references of the equations (236) and (225) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \tag{1299}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| \left(B^- - B\right) + V_{n0}(t) |n\rangle\langle 0| \left(B^+ - B\right)\right), \tag{1300}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{1301}$$

Here B is given by:

$$\begin{aligned} B &= \langle B^+ \rangle \\ &= \langle B^- \rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1302}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1303}$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1304}$$

$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1305}$$

Using this set of hermitian operators to write the Hamiltonians (1247)-(1249)

$$\overline{H_S\left(t\right)} = \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1306}$$

$$= \varepsilon_0\left(t\right)|0\rangle\langle 0| + B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{0<m<n}\left(V_{mn}\left(t\right)|m\rangle\langle n| + V_{nm}\left(t\right)|n\rangle\langle m|\right) \tag{1307}$$

$$+ \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1308}$$

$$= \sum_{0<m<n}\left(\left(\Re\left(V_{mn}\left(t\right)\right) + \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)|m\rangle\langle n| + \left(\Re\left(V_{mn}\left(t\right)\right) - \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)|n\rangle\langle m|\right) + \varepsilon_0\left(t\right)|0\rangle\langle 0| \tag{1309}$$

$$+ B\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right) + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1310}$$

$$= \sum_{0<m<n}\left(\left(\Re\left(V_{nm}\left(t\right)\right) + \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + \left(\Re\left(V_{nm}\left(t\right)\right) - \mathrm{i}\Im\left(V_{mn}\left(t\right)\right)\right)\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1311}$$

$$+ B\sum_{n=1}\left(V_{0n}\left(t\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}\left(t\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1312}$$

$$= \sum_{0<m<n}\left(\Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,x} + \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}\right) + B\sum_{n=1}\left(\Re\left(V_{0n}\left(t\right)\right)\sigma_{0n,x} + \Im\left(V_{mn}\left(t\right)\right)\sigma_{0n,y}\right) \tag{1313}$$

$$+ \varepsilon_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n| \tag{1314}$$

$$\overline{H_I\left(t\right)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|\left(B^{-} - B\right) + V_{n0}\left(t\right)|n\rangle\langle 0|\left(B^{+} - B\right)\right) \tag{1315}$$

$$= \sum_{n=1}\left(\left(\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\left(B^{-} - B\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + \left(\Re\left(V_{0n}\left(t\right)\right) - \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\left(B^{+} - B\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1316}$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1317}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^{-} - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right) + \left(B^{+} - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) - \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\right)\right. \tag{1318}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^{+} - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) - \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right) - \left(B^{-} - B\right)\left(\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\Im\left(V_{0n}\left(t\right)\right)\right)\right)\right) \tag{1319}$$

$$+ \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1320}$$

$$= \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^{+} + B^{-} - 2B\right)\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\left(B^{-} - B - B^{+} + B\right)\Im\left(V_{0n}\left(t\right)\right)\right. \tag{1321}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^{+} - B - B^{-} + B\right)\Re\left(V_{0n}\left(t\right)\right) + \mathrm{i}\left(B - B^{-} + B - B^{+}\right)\Im\left(V_{0n}\left(t\right)\right)\right)\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1322}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\sigma_{0n,x}\left(B_x\Re\left(V_{0n}\left(t\right)\right) - B_y\Im\left(V_{0n}\left(t\right)\right)\right)\right. \tag{1323}$$

$$\left. + \sigma_{0n,y}\left(B_y\Re\left(V_{0n}\left(t\right)\right) + B_x\Im\left(V_{0n}\left(t\right)\right)\right)\right) \tag{1324}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_S+H_B}\right)}\right)\right)+\left\langle\overline{H_I}\right\rangle_{\overline{H_S+H_B}}+O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1325}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)=C\left(R_1,R_2,...,R_{d-1},B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (254) and using the chain rule we obtain that:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}}+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, &(1326)\\ &= 0 &(1327)\end{aligned}$$

Let's recall the equations (1294) and (1296), we can write them in terms of the variational parameters

$$\begin{aligned}B &= \exp\left(-\left(1/2\right)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) &(1328)\\ R_n &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) &(1329)\end{aligned}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\begin{aligned}\frac{\partial B}{\partial v_{\mathbf{k}}} &= -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B &(1330)\\ \frac{\partial R_n}{\partial v_{\mathbf{k}}} &= \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1331)\end{aligned}$$

Introducing this derivates in the equation (1326) give us:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right)+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1332)\\ &= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right)-\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) &(1333)\end{aligned}$$

We can obtain the variational parameters:

$$\begin{aligned}v_{\mathbf{k}} &= \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1334)\\ &= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1335)\end{aligned}$$

Now taking $v_{\mathbf{k}}=g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}}=\frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1336}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+\right) (|0\rangle\langle 0| \otimes \rho_B) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^-\right) \tag{1337}$$
$$0 = |0\rangle\langle 0| \otimes \rho_B \tag{1338}$$
$$0 = \rho(0) \tag{1339}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^\dagger(t) O U(t) \tag{1340}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1341}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \tag{1342}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{1343}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle 0| + \sum_{n=1} (\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n}|n\rangle\langle n| \tag{1344}$$
$$+\Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x}) \tag{1345}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1346}$$
$$B_{z,n} = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \text{ if} n \neq 0 \tag{1347}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1348}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1349}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1350}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1351}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1352}$$
$$B_{1n}(t) = B_x \tag{1353}$$
$$B_{2n}(t) = B_y \tag{1354}$$
$$B_{3n}(t) = B_{z,n} \tag{1355}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1356}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1357}$$
$$C_{10}(t) = 0 \tag{1358}$$
$$C_{20}(t) = 0 \tag{1359}$$
$$C_{40}(t) = 0 \tag{1360}$$
$$C_{50}(t) = 0 \tag{1361}$$
$$C_{30}(t) = 1 \tag{1362}$$
$$C_{1n}(t) = \Re(V_{0n}(t)) \tag{1363}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1364}$$
$$C_{3n}(t) = 1 \tag{1365}$$
$$C_{4n}(t) = \Im(V_{0n}(t)) \tag{1366}$$
$$C_{5n}(t) = -\Im(V_{0n}(t)) \tag{1367}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn}\otimes B_{jn}(t)\right) \tag{1368}$$

Here $J = \{1, 2, 3, 4, 5\}$.
We write the interaction Hamiltonian transformed under (1340) as:

$$\widetilde{H_I}(t) = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1369}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\,A_i U(t) \tag{1370}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{1371}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t))$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1372}$$

Replacing the equation (1369)in (1372)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1373}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right),\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1374}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right. \tag{1375}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1376}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right. \tag{1377}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1378}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1379}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1380}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1381}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1382}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1380) are:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) &= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B && (1383)\\
&= \left\langle \widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B && (1384)\\
&= \Lambda_{jnj'n'}(\tau) && (1385)\\
\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) && (1386)\\
&= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B && (1387)\\
&= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B && (1388)\\
&= \Lambda_{j'n'jn}(-\tau) && (1389)\\
\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) && (1390)\\
&= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B && (1391)\\
&= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B && (1392)\\
&= \Lambda_{jnj'n'}(\tau) && (1393)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) && (1394)\\
&= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B && (1395)\\
&= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B && (1396)\\
&= \Lambda_{j'n'jn}(-\tau) && (1397)
\end{aligned}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1373)and (1380) and using the equations (1383)-(1397) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \Big(C_{jn}(t)\,C_{j'n'}(s)\Big(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\Big) \tag{1398}$$

$$+C_{jn}(t)\,C_{j'n'}(s)\Big(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\Big)\Big)\,\mathrm{d}s \tag{1399}$$

$$= -\int_0^t \sum_{j,j',n,n'} \Big(C_{jn}(t)\,C_{j'n'}(s)\Big(\Lambda_{jnj'n'}(\tau)\Big[\widetilde{A_{jn}}(t)\,,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\Big] + \Lambda_{j'n'jn}(-\tau)\Big[\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,,\widetilde{A_{jn}}(t)\Big]\Big)\Big) \tag{1400}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t)\,,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t)\,,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1401}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (418) as expected.

VIII. BIBLIOGRAPHY

[1] McCutcheon D P S, Dattani N S, Gauger E M, Lovett B W and Nazir A 2011 Phys. Rev. B 84 081305
[2] Dara P S McCutcheon and Ahsan Nazir 2010 New J. Phys. 12 113042
[3] Supplement: Theoretical model of phonon induced dephasing. A.J. Ramsay ey al 2009.
[4] Felix A Pollock et al 2013 New J. Phys. 15 075018

* n.dattani@cfa.harvard.edu
† edcchaparroso@unal.edu.co