A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| + V_{01}(t)|0\rangle\langle 1|, \tag{2}$$

$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}(t)$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}(t). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i(t) \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V(t) = |0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V(t)}$ using the notation (6) will give us the following result:

$$e^{\pm V(t)} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t))} \tag{10}$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)))^2}{2!} + ... \tag{11}$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2(t)}{2!} + ... \tag{12}$$

$$= |0\rangle\langle 0|\left(\mathbb{I} \pm \hat{\varphi}_0(t) + \frac{\hat{\varphi}_0^2(t)}{2!} \pm ...\right) + |1\rangle\langle 1|\left(\mathbb{I} \pm \hat{\varphi}_1(t) + \frac{\hat{\varphi}_1^2(t)}{2!} \pm ...\right) \tag{13}$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1(t)} \tag{14}$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{0\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{1\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} \tag{15}$$

$$= |0\rangle\langle 0|B_0^{\pm}(t) + |1\rangle\langle 1|B_1^{\pm}(t), \tag{16}$$

$$B_i^{\pm}(t) \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{17}$$

Let's recall the Zassenhaus formula:

$$e^{r(X+Y)} = e^{rX}\, e^{rY}\, e^{-\frac{r^2}{2}[X,Y]}\, e^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \cdots. \tag{18}$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}^{'}, \mathbf{k}$ and i, j we can show making $r = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} = e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]} \cdots \tag{19}$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}0} \cdots \tag{20}$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. \tag{21}$$

By induction of this result we can write an expresion of $B_i^{\pm}(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$D(\pm v_{i\mathbf{k}}(t)) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, \tag{22}$$

$$B_i^{\pm}(t) = \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). \tag{23}$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$\overline{O(t)} \equiv \mathrm{e}^{V(t)}O(t)\,\mathrm{e}^{-V(t)}. \tag{24}$$

We will use the following identities:

$$
\begin{aligned}
\overline{|0\rangle\langle 0|\,(t)} &= \mathrm{e}^{V(t)}|0\rangle\langle 0|\mathrm{e}^{-V(t)} \tag{25}\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{26}\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{27}\\
&= |0\rangle\langle 0|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{28}\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) \tag{29}\\
&= |0\rangle\langle 0|, \tag{30}\\
\overline{|1\rangle\langle 1|\,(t)} &= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{31}\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{32}\\
&= |1\rangle\langle 1|B_1^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{33}\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) \tag{34}\\
&= B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) \tag{35}\\
&= |1\rangle\langle 1|, \tag{36}\\
\overline{|0\rangle\langle 1|\,(t)} &= \mathrm{e}^{V(t)}|0\rangle\langle 1|\mathrm{e}^{-V(t)} \tag{37}\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{38}\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{39}\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{40}\\
&= |0\rangle\langle 1|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{41}\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) \tag{42}\\
&= |0\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t)\,, \tag{43}\\
\overline{|1\rangle\langle 0|\,(t)} &= \mathrm{e}^{V(t)}|1\rangle\langle 0|\mathrm{e}^{-V(t)} \tag{44}\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{45}\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{46}\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{47}\\
&= |1\rangle\langle 0|B_1^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{48}\\
&= |1\rangle\langle 0|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+\,(t)\,B_1^-\,(t) \tag{49}\\
&= |1\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t)\,, \tag{50}\\
\overline{b_{\mathbf{k}}\,(t)} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}\mathrm{e}^{-V(t)} \tag{51}\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{52}\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| \tag{53}\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t)\,b_{\mathbf{k}}B_0^-\,(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t)\,b_{\mathbf{k}}B_1^-\,(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\,b_{\mathbf{k}}B_0^-\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,b_{\mathbf{k}}B_1^-\,(t) \tag{54}\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{55}\\
&= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{56}\\
&= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{57}\\
\overline{b_{\mathbf{k}}\,(t)}^{\dagger} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V(t)} \tag{58}\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{59}\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle 1| \tag{60}\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t)b_{\mathbf{k}}^{\dagger}B_0^-\,(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t)b_{\mathbf{k}}^{\dagger}B_1^-\,(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)b_{\mathbf{k}}^{\dagger}B_0^-\,(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\,(t)b_{\mathbf{k}}^{\dagger}B_1^-\,(t) \tag{61}\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) \tag{62}\\
&= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}. \tag{63}
\end{aligned}
$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+}(t)\, b_{\mathbf{k}} B_i^{-}(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \tag{64}$$

$$B_i^{+}(t)\, b_{\mathbf{k}}^{\dagger} B_i^{-}(t) = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}. \tag{65}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|\,(t)} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{66}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|\,(t)} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{67}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|\,(t)} = V_{10}(t)\,|1\rangle\langle 0| B_1^{+}(t)\, B_0^{-}(t), \tag{68}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|\,(t)} = V_{01}(t)\,|0\rangle\langle 1| B_0^{+}(t)\, B_1^{-}(t), \tag{69}$$

$$\overline{\left(g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}}\right)(t)} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\right) + g_{\mathbf{ik}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{70}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{72}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{73}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{74}$$

$$= |0\rangle\langle 0| B_0^{+}(t)\,|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)|0\rangle\langle 0| B_0^{-}(t) \tag{75}$$

$$= |0\rangle\langle 0| B_0^{+}(t)\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_0^{-}(t) \tag{76}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{77}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{78}$$

$$= |1\rangle\langle 1| B_1^{+}(t)\,|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)|1\rangle\langle 1| B_1^{-}(t) \tag{79}$$

$$= |1\rangle\langle 1| B_1^{+}(t)\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_1^{-}(t) \tag{80}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{81}$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\,(t)} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{82}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{+}(t)\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_0^{-}(t) + |1\rangle\langle 1| B_1^{+}(t)\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_1^{-}(t)\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right. \tag{85}$$

$$\left. + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger}\right)\right) \tag{90}$$

$$= \omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger}\right)\right) \tag{91}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}}\left(|1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger}\right)\right) \tag{92}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} - v_{1\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} - v_{0\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right) \tag{93}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right)\right). \tag{94}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} \tag{95}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0| B_1^{+}(t)\, B_0^{-}(t) + V_{01}(t)\,|0\rangle\langle 1| B_0^{+}(t)\, B_1^{-}(t), \tag{96}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right)} \tag{97}$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right)} \tag{98}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{99}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^{*} \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^{*} \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{100}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \tag{101}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right)\right)\right) \tag{102}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\, b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger}\right)\right)\right). \tag{103}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'| B_j^{+}(t) B_{j'}^{-}(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^{*} b_{\mathbf{k}} + \frac{\left|v_{j\mathbf{k}}(t)\right|^{2}}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^{*} \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right). \tag{104}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H\left(t\right)}=\overline{H_{\bar{S}}}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{B}}}. \tag{105}$$

Let's define:

$$R_{i}\left(t\right)\equiv\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right), \tag{106}$$

$$B_{iz}\left(t\right)\equiv\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right), \tag{107}$$

$$\chi_{ij}\left(t\right)\equiv\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}\left(t\right)v_{j\mathbf{k}}\left(t\right)-v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}^{2}}\right). \tag{108}$$

$\chi_{ij}\left(t\right)$ is an imaginary number so $e^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}\left(t\right)$ as we'll will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix}B_{iz}\left(t\right) & B_{i}^{\pm}\left(t\right)\\ B_{x}\left(t\right) & B_{ij}\left(t\right)\\ B_{y}\left(t\right) & R_{i}\left(t\right)\end{pmatrix}\equiv\begin{pmatrix}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right) & e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\\ \frac{B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{10}^{*}(t)}{2} & e^{\chi_{ij}(t)}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\\ \frac{B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{10}^{*}(t)}{2\mathrm{i}} & \sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}(t)\right|^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\end{pmatrix}. \tag{109}$$

$$(\cdot)^{\Re}\equiv\Re(\cdot), \tag{110}$$

$$(\cdot)^{\Im}\equiv\Im(\cdot). \tag{111}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta=\frac{1}{k_{\mathrm{B}}T}$, considering the stationary bath state as reference written in the following way:

$$\rho_{B}=\frac{e^{-\beta H_{B}}}{\mathrm{Tr}\left(e^{-\beta H_{B}}\right)}. \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^{\dagger}=\begin{pmatrix}0 & 0 & 0 & \dots & 0 & \dots\\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots\\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots\\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots\\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}, \tag{113}$$

$$b=\begin{pmatrix}0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots\\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots\\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots\\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots\\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}. \tag{114}$$

So the product of the matrix representation of $b^\dagger$ and b with $-\beta$ is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \, |j\rangle\langle j| \, , \tag{116}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \tag{117}$$

$$\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{118}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{119}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{120}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{121}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{122}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), \tag{123}$$

$$\mathrm{Tr}\left(\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{124}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{125}$$

$$= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{126}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{127}$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{128}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{129}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\, |j_{\mathbf{k}} - 1\rangle, \tag{130}$$

$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} \mid j_{\mathbf{k}} + 1\rangle. \tag{131}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (130)-(131):

$$\begin{aligned}
\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Tr}\left(\rho_B B_{iz}(t)\right) = \mathrm{Tr}\left(B_{iz}(t)\,\rho_B\right) && (132)\\
&= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right)\right)\rho_B\right) && (133)\\
&= \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\rho_B\right) && (134)\\
&= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, \mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) && (135)\\
&= \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) && (136)\\
&= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, \mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right), && (137)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right) b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathbf{k}}^{\dagger}\text{)} && (138)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) && (139)\\
&= 0, && (140)\\
\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right) b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} && (141)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) && (142)\\
&= 0. && (143)
\end{aligned}$$

we therefore find that:

$$\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = 0. \tag{144}$$

Another important expected value is $B(t) = \langle B^{\pm}(t)\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm}(t) = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\begin{aligned}\left\langle B^{\pm}\left(t\right)\right\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B^{\pm}\left(t\right)\right) = \mathrm{Tr}\left(B^{\pm}\left(t\right)\rho_B\right) &(145)\\ &= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) &(146)\\ &= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\rho_B\right) &(147)\\ &= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\right\rangle. &(148)\end{aligned}$$

Given that we can write a density operator as:

$$\rho = \int P\left(\alpha\right)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \tag{149}$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right)\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P\left(\alpha\right)$ is given by:

$$\begin{aligned}\langle A\rangle =&\mathrm{Tr}\left(A\rho\right) &(150)\\ =&\int P\left(\alpha\right)\langle\alpha\left|A\right|\alpha\rangle\mathrm{d}^2\alpha. &(151)\end{aligned}$$

We are typically interested in thermal state density operators, for which it can be shown that $P\left(\alpha\right) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega}-1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_BT$.

Using the integral representation (151) we could obtain that the expected value for the displacement operator $D\left(h\right)$ with $h\in\mathbb{C}$ is equal to:

$$\begin{aligned}
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle\alpha|D(h)|\alpha\rangle \mathrm{d}^2\alpha && (152)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(-\alpha)D(h)D(\alpha)|0\rangle \mathrm{d}^2\alpha, && (153)\\
D(h)D(\alpha) &= D(h+\alpha)e^{\frac{1}{2}(h\alpha^*-h^*\alpha)}, && (154)\\
D(-\alpha)(D(h)D(\alpha)) &= D(-\alpha)D(h+\alpha)e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (155)\\
&= D(h)e^{\frac{1}{2}(-\alpha(h+\alpha)^*+\alpha^*(h+\alpha))}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (156)\\
&= D(\alpha)e^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^* h+|\alpha|^2\right)}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (157)\\
&= D(\alpha)e^{(h\alpha^*-h^*\alpha)}, && (158)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(h)\exp(h\alpha^*-h^*\alpha)|0\rangle \mathrm{d}^2\alpha && (159)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|0\rangle \mathrm{d}^2\alpha && (160)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|h\rangle \mathrm{d}^2\alpha, && (161)\\
|\alpha\rangle &= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, && (162)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha && (163)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha && (164)\\
&= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha\right)\mathrm{d}^2\alpha, && (165)
\end{aligned}$$

$$\alpha = x + \mathrm{i}y, \tag{166}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{N}+h\left(x-\mathrm{i}y\right)-h^*\left(x+\mathrm{i}y\right)\right)\mathrm{d}x\mathrm{d}y \tag{167}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{N}+hx-h^*x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y\right)\mathrm{d}y, \tag{168}$$

$$-\frac{x^2}{N}+hx-h^*x = -\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) \tag{169}$$

$$= -\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2+\frac{N\left(h^*-h\right)^2}{4}, \tag{170}$$

$$\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y = -\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) \tag{171}$$

$$= -\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)-\frac{N\left(h+h^*\right)^2}{4}, \tag{172}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, \tag{173}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\exp\left(-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right)\mathrm{d}x, \tag{174}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(x+\frac{(Nh^*-Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}y \tag{175}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{176}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}\right) \tag{177}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}\right) \tag{178}$$

$$= \exp\left(-|h|^2\left(N+\frac{1}{2}\right)\right) \tag{179}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)\right) \tag{180}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)\right) \tag{181}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{182}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (147) using (182) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{183}$$

We will now force $\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}\left(t\right)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+\left(t\right)\sigma^+$ and $B_i^-\left(t\right)\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}\left(t\right)$ and we subtract their expected value in order to satisfy $\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}\,(t)$ is:

$$\overline{H(t)} = \sum_{j}\varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}}|j\rangle\langle j|\Bigg((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \Bigg(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \tag{184}$$

$$= \sum_{j}\varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'}V_{jj'}(t)\,|j\rangle\langle j'|B_{jj'}(t) + \sum_{j}|j\rangle\langle j|B_{jz}(t) + \sum_{j\neq j'}V_{jj'}(t)\,|j\rangle\langle j'|\left(B_j^+(t)\,B_{j'}^-(t) - B_{jj'}(t)\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{185}$$

$$\equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}. \tag{186}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\left\langle B_i^+(t)\,B_j^-(t)\right\rangle = B_{ij}(t) \tag{187}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle \tag{188}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle \tag{189}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\right\rangle \tag{190}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \tag{191}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} \tag{192}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}. \tag{193}$$

From the definition $B_{01}(t) = \left\langle B_0^+(t)\,B_1^-(t)\right\rangle$ using the displacement operator we have:

$$\left\langle B_0^+(t)\,B_1^-(t)\right\rangle = B_{01}(t) \tag{194}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle \tag{195}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle \tag{196}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\right\rangle \tag{197}$$

$$= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right) \tag{198}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(-\frac{1}{2}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right) \tag{199}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}. \tag{200}$$

We can check:

$$\begin{aligned}
\left\langle B_0^+(t)\, B_1^-(t)\right\rangle &= B_{01}(t) && (201)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)} && (202)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)^*} && (203)\\
&= \left\langle B_1^+(t)\, B_0^-(t)\right\rangle^* && (204)\\
&= B_{10}^*(t)\,. && (205)
\end{aligned}$$

The parts of the splitted Hamiltonian are:

$$\begin{aligned}
\overline{H_{\bar{S}}(t)} &\equiv \left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1|+V_{10}(t)\,B_{10}\sigma^+ + V_{01}(t)\,B_{01}\sigma^-, && (206)\\
\overline{H_{\bar{I}}} &\equiv V_{10}(t)\left(B_1^+(t)\,B_0^-(t)-B_{10}(t)\right)\sigma^+ + V_{01}(t)\left(B_0^+(t)\,B_1^-(t)-B_{01}(t)\right)\sigma^- + |0\rangle\langle 0|B_{0z}(t)+|1\rangle\langle 1|B_{1z}(t)\,, && (207)\\
\overline{H_{\bar{B}}} &\equiv \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (208)\\
&= H_B. && (209)
\end{aligned}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (207) we can verify the condition $\langle \overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}}=0$ in the following way:

$$\begin{aligned}
\langle \overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} &= \left\langle \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right)|n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}(t)\,|j\rangle\langle j'|\left(B_j^+(t)\,B_{j'}^-(t)-B_{jj'}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (210)\\
&= \left\langle \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}(t)\,|j\rangle\langle j'|\left(B_j^+(t)\,B_{j'}^-(t)-B_{jj'}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (211)\\
&= \sum_{n\mathbf{k}}\left(\left\langle\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}(t)\,B_j^+(t)B_{j'}^-(t)\right\rangle_{\overline{H_{\bar{B}}}}-\langle V_{jj'}(t)\,B_{jj'}(t)\rangle_{\overline{H_{\bar{B}}}}\right) && (212)\\
&= \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(\left\langle B_j^+(t)\,B_{j'}^-(t)\right\rangle_{\overline{H_{\bar{B}}}}-\langle B_{jj'}(t)\rangle_{\overline{H_{\bar{B}}}}\right) && (213)\\
&= \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}(t)\right)^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(B_{jj'}(t)-B_{jj'}(t)\right) && (214)\\
&= 0. && (215)
\end{aligned}$$

We used (144) and (193) to evaluate the expression. Let's consider the following Hermitian combinations:

$$
\begin{aligned}
B_x(t) &= B_x^\dagger(t) && (216)\\
&= \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}, && (217)\\
B_y(t) &= B_y^\dagger(t) && (218)\\
&= \frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, && (219)\\
B_{iz}(t) &= B_{iz}^\dagger(t) && (220)\\
&= \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}} \right). && (221)
\end{aligned}
$$

Writing the equations (206) and (207) using the previous combinations we obtain that:

$$\overline{H_{\bar{S}}}(t) = \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right)|j\rangle\langle j| + V_{10}(t)\,B_{10}(t)\,\sigma^+ + V_{01}(t)\,B_{01}(t)\,\sigma^- \tag{222}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right)|j\rangle\langle j| + V_{10}(t)\,B_{10}(t)\,\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\,B_{01}(t)\,\frac{\sigma_x - \mathrm{i}\sigma_y}{2} \tag{223}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right)|j\rangle\langle j| + V_{10}(t)\left(B^{\Re}_{10}(t) + \mathrm{i}B^{\Im}_{10}(t)\right)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\left(B^{\Re}_{10}(t) - \mathrm{i}B^{\Im}_{10}(t)\right)\frac{\sigma_x - \mathrm{i}\sigma_y}{2} \tag{224}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right)|j\rangle\langle j| + B^{\Re}_{10}(t)\left(V_{10}(t)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t)\frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) + \mathrm{i}B^{\Im}_{10}(t)\left(V_{10}(t)\frac{\sigma_x + \mathrm{i}\sigma_y}{2} - V_{01}(t)\frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) \tag{225}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right)|j\rangle\langle j| + B^{\Re}_{10}(t)\left(\sigma_x\frac{V_{10}(t)+V_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t)-V_{01}(t)}{2}\right) + \mathrm{i}B^{\Im}_{10}(t)\left(\sigma_x\frac{V_{10}(t)-V_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{V_{10}(t)+V_{01}(t)}{2}\right) \tag{226}$$

$$= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right)|j\rangle\langle j| + B^{\Re}_{10}(t)\left(\sigma_x V^{\Re}_{10}(t) - \sigma_y V^{\Im}_{10}(t)\right) + \mathrm{i}B^{\Im}_{10}(t)\left(\mathrm{i}\sigma_x V^{\Im}_{10}(t) + \mathrm{i}\sigma_y V^{\Re}_{10}(t)\right) \tag{227}$$

$$= \left(\varepsilon_0(t) + R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right)|1\rangle\langle 1| + B^{\Re}_{10}(t)\left(\sigma_x V^{\Re}_{10}(t) - \sigma_y V^{\Im}_{10}(t)\right) + \mathrm{i}B^{\Im}_{10}(t)\left(\mathrm{i}\sigma_x V^{\Im}_{10}(t) + \mathrm{i}\sigma_y V^{\Re}_{10}(t)\right) \tag{228}$$

$$= \left(\varepsilon_0(t) + R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right)|1\rangle\langle 1| + \left(\sigma_x B^{\Re}_{10}(t)V^{\Re}_{10}(t) - \sigma_y B^{\Re}_{10}(t)V^{\Im}_{10}(t)\right) - \left(\sigma_x B^{\Im}_{10}(t)V^{\Im}_{10}(t) + \sigma_y B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{229}$$

$$= \left(\varepsilon_0(t) + R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right)|1\rangle\langle 1| + \sigma_x\left(B^{\Re}_{10}(t)\,V^{\Re}_{10}(t) - B^{\Im}_{10}(t)\,V^{\Im}_{10}(t)\right) - \sigma_y\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t) + B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right) \tag{230}$$

$$= \left(\varepsilon_0(t) + R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right)|1\rangle\langle 1| + \sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t) - B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right) - \sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t) + B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right). \tag{231}$$

$$\overline{H_{\bar{I}}} = V_{10}(t)\left(\sigma^+ B^+_1(t)\,B^-_0(t) - \sigma^+ B_{10}(t)\right) + V_{01}(t)\left(\sigma^- B^+_0(t)\,B^-_1(t) - \sigma^- B_{01}(t)\right) + |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) \tag{232}$$

$$= |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) + \left(V^{\Re}_{10}(t) + \mathrm{i}V^{\Im}_{10}(t)\right)\left(\sigma^+ B^+_1(t)\,B^-_0(t) - \sigma^+ B_{10}(t)\right) + \left(V^{\Re}_{10}(t) - \mathrm{i}V^{\Im}_{10}(t)\right)\left(\sigma^- B^+_0(t)\,B^-_1(t) - \sigma^- B_{01}(t)\right) \tag{233}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma^+ B^+_1(t)\,B^-_0(t) - \sigma^+ B_{10}(t) + \sigma^- B^+_0(t) B^-_1(t) - \sigma^- B_{01}(t)\right) + \mathrm{i}V^{\Im}_{10}(t)\left(\sigma^+ B^+_1(t) B^-_0(t) - \sigma^+ B_{10}(t) - \sigma^- B \right. \tag{234}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B^+_1(t)\,B^-_0(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B^+_0(t)\,B^-_1(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) \tag{235}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B^+_1(t)\,B^-_0(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B^+_0(t)\,B^-_1(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) \tag{236}$$

$$+ \mathrm{i}V^{\Im}_{10}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B^+_1(t)\,B^-_0(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B^+_0(t)\,B^-_1(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) \tag{237}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma_x\frac{B^+_1(t)\,B^-_0(t) + B^+_0(t)\,B^-_1(t) - B_{10}(t) - B_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{B^+_1(t)\,B^-_0(t) - B^+_0(t)\,B^-_1(t) - B_{10}(t}{2}\right. \tag{238}$$

$$+ \mathrm{i}V^{\Im}_{10}(t)\left(\sigma_x\frac{B^+_1(t)\,B^-_0(t) - B^+_0(t)\,B^-_1(t) - B_{10}(t) + B_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{B^+_1(t)B^-_0(t) + B^+_0(t)\,B^-_1(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{239}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V^{\Im}_{10}(t)\left(\mathrm{i}\sigma_x\frac{B^+_1(t)\,B^-_0(t) - B^+_0(t)\,B^-_1(t) - B_{10}(t) + B_{01}(t)}{2} - \sigma_y\frac{B^+_1(t)\,B^-_0(t) + B^+_0}{}\right. \tag{240}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V^{\Im}_{10}(t)\left(\mathrm{i}^2\sigma_x\frac{B^+_1(t)\,B^-_0(t) - B^+_0(t)\,B^-_1(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y\frac{B^+_1(t)\,B^-_0(t) + B^+_0}{}\right. \tag{241}$$

$$= \sum B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V^{\Im}_{10}(t)\left(\mathrm{i}^2\sigma_x\frac{B^+_1 B^-_0(t) - B^+_0 B^-_1(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y\frac{B^+_1(t)\,B^-_0(t) + B^+_0(t)\,B}{2}\right.$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right) + \langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{245}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}(t)\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}(t)\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{246}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}(t), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{247}$$

Then using the fact that $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{248}$$

So Eq. (246) becomes:

$$\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}(t)} &= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (249)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (250)\\
&= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (251)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} && (252)\\
&= 0 \quad \text{(by Eq. (246))}. && (253)
\end{aligned}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\begin{aligned}
\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} &= \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} && (254)\\
&= 0. && (255)
\end{aligned}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{256}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}\left(t\right)=-\beta\left(\left(\varepsilon_{0}\left(t\right)+R_{0}\left(t\right)\right)|0\rangle\langle0|+\left(\varepsilon_{1}\left(t\right)+R_{1}\left(t\right)\right)|1\rangle\langle1|+V_{10}\left(t\right)B_{10}\left(t\right)\sigma^{+}+V_{01}\left(t\right)B_{01}\left(t\right)\sigma^{-}\right). \tag{257}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}\left(t\right)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^{2}-\mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)+\mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)=0. \tag{258}$$

Let's define:

$$\varepsilon\left(t\right)\equiv\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), \tag{259}$$

$$\eta\equiv\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}. \tag{260}$$

The solutions of the equation (258) are:

$$\lambda=\beta\frac{-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\pm\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}}{2} \tag{261}$$

$$=\beta\frac{-\varepsilon\left(t\right)\pm\eta\left(t\right)}{2} \tag{262}$$

$$=-\beta\frac{\varepsilon\left(t\right)\mp\eta\left(t\right)}{2}. \tag{263}$$

The value of $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)=\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right)+\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) \tag{264}$$

$$=2\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{265}$$

Given that $v_{i\mathbf{k}}\left(t\right)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)=A\left(\varepsilon\left(t\right),\eta\left(t\right)\right)$ to calculate $\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}$ can lead to:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}\left(t\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}=\frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} \tag{266}$$

$$=2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) \tag{267}$$

$$=-\beta\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{268}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)=0. \tag{269}$$

The derivates included in the expression given are related to:

$$\langle B_0^+(t)\, B_1^-(t)\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{270}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^*\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{271}$$

$$= \langle B_1^+(t)\, B_0^-(t)\rangle^*, \tag{272}$$

$$R_i(t) = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{273}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{274}$$

$$\langle B_0^+(t)\, B_1^-(t)\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{275}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{v_{0\mathbf{k}}^*(t)\, v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\, v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{276}$$

$$v_{0\mathbf{k}}^*(t)\, v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\, v_{1\mathbf{k}}^*(t) = \left(v_{0\mathbf{k}}^{\Re}(t) - \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right) - \left(v_{0\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t) - \mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right) \tag{277}$$

$$= \big(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) - \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Im}(t)\big) - \big(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Re}(t) - \mathrm{i}v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t) + \mathrm{i}v_{0} \tag{278}$$

$$= 2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)\, v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\, v_{1\mathbf{k}}^{\Re}(t)\right), \tag{279}$$

$$|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2 = (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))\,(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))^* \tag{280}$$

$$= |v_{1\mathbf{k}}(t)|^2 + |v_{0\mathbf{k}}(t)|^2 - (v_{1\mathbf{k}}(t)\, v_{0\mathbf{k}}^*(t) + v_{1\mathbf{k}}^*(t)\, v_{0\mathbf{k}}(t)) \tag{281}$$

$$= \left(v_{1\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t)\right)^2 + \left(v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{0\mathbf{k}}^{\Im}(t)\right)^2 - \big(\big(v_{1\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\big)\big(v_{0\mathbf{k}}^{\Re}(t) - \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\big) + \big(v_{1\mathbf{k}}^{\Re}(t) - \mathrm{i}v \tag{282}$$

$$= \left(v_{1\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t)\right)^2 + \left(v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{0\mathbf{k}}^{\Im}(t)\right)^2 - 2\left(v_{1\mathbf{k}}^{\Re}(t)\, v_{0\mathbf{k}}^{\Re}(t) + v_{1\mathbf{k}}^{\Im}(t)\, v_{0\mathbf{k}}^{\Im}(t)\right) \tag{283}$$

$$= \left(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\right)^2. \tag{284}$$

Rewriting in terms of real and imaginary parts.

$$R_i(t) = \sum_{\mathbf{k}} \left(\frac{\left(v^{\Re}_{i\mathbf{k}}(t)\right)^2 + \left(v^{\Im}_{i\mathbf{k}}(t)\right)^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v^{\Re}_{i\mathbf{k}}(t) - \mathrm{i} v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} + g^*_{i\mathbf{k}} \frac{v^{\Re}_{i\mathbf{k}}(t) + \mathrm{i} v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \tag{285}$$

$$= \sum_{\mathbf{k}} \left(\frac{\left(v^{\Re}_{i\mathbf{k}}(t)\right)^2 + \left(v^{\Im}_{i\mathbf{k}}(t)\right)^2}{\omega_{\mathbf{k}}} - v^{\Re}_{i\mathbf{k}}(t) \frac{g_{i\mathbf{k}} + g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \mathrm{i} v^{\Im}_{i\mathbf{k}}(t) \frac{g^*_{i\mathbf{k}} - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{286}$$

$$\left\langle B^+_0(t) B^-_1(t) \right\rangle = \left(\prod_{\mathbf{k}} \exp\left(\frac{v^*_{0\mathbf{k}}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v^*_{1\mathbf{k}}(t)}{2\omega^2_{\mathbf{k}}} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{287}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{2\mathrm{i}\left(v^{\Re}_{0\mathbf{k}}(t) v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t) v^{\Re}_{1\mathbf{k}}(t) \right)}{2\omega^2_{\mathbf{k}}} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v^{\Re}_{1\mathbf{k}}(t) - v^{\Re}_{0\mathbf{k}}(t)\right)^2 + \left(v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)\right)^2}{\omega^2_{\mathbf{k}}} \coth \Bigg(\tag{288}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Re}_{0\mathbf{k}}(t) v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t) v^{\Re}_{1\mathbf{k}}(t) \right)}{\omega^2_{\mathbf{k}}} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v^{\Re}_{1\mathbf{k}}(t) - v^{\Re}_{0\mathbf{k}}(t)\right)^2 + \left(v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)\right)^2}{\omega^2_{\mathbf{k}}} \coth \Bigg(\beta \tag{289}$$

Calculating the derivates respect to $\alpha^{\Re}_{i\mathbf{k}}$ and $\alpha^{\Im}_{i\mathbf{k}}$ we have:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{290}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - \mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{291}$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{292}$$

$$\frac{\partial|B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{293}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{294}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}|B_{10}(t)|^2, \tag{295}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{296}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} - 4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{297}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \tag{298}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{299}$$

$$= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Re}(t)-v_{i'\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{300}$$

$$= \frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\varepsilon(t)\right. \tag{301}$$

$$\left.+2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^2}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{302}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}} \tag{303}$$

$$=\frac{\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{2g_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^2}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{304}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}-g_{i\mathbf{k}}^{*}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$=\frac{\left(2v_{i\mathbf{k}}^{\Re}(t)-2g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$=\frac{\left(v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{v_{i\mathbf{k}}^{\Re}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{308}$$

Separating (307) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}^{\Re}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Re}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{309}$$

$$v_{i\mathbf{k}}^{\Re}(t)-g_{i\mathbf{k}}^{\Re}=v_{i\mathbf{k}}^{\Re}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{310}$$

$$v_{i\mathbf{k}}^{\Re}\left(t\right)=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{311}$$

$$v_{i\mathbf{k}}^{\Re}\left(t\right)=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{312}$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)} = \frac{\partial\left(\varepsilon_1(t) + R_1(t) + \varepsilon_0(t) + R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} \tag{313}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - \mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} \tag{314}$$

$$= 2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{315}$$

$$\frac{\partial\left|B_{10}(t)\right|^2}{\partial v_{i\mathbf{k}}^{\Im}(t)} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} \tag{316}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{317}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}\left|B_{10}(t)\right|^2, \tag{318}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{319}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} - 4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{320}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left(\left(\varepsilon_1(t)+R_1(t)\right)\left(\varepsilon_0(t)+R_0(t)\right)-\left|V_{10}(t)\right|^2\left|B_{10}(t)\right|^2\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\eta(t)} \tag{321}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{322}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-v_{i'\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^2}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{323}$$

$$= \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t) - 4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right) - \frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + \frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t) + 2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} + 4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^2}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{324}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}} \tag{325}$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t) - 4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right) - \frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + \frac{2}{\eta(t)}\left(\frac{\left(g_{i\mathbf{k}}^*\right)^{\Im}}{\omega_{\mathbf{k}}}\varepsilon(t) - 2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\frac{\left(g_{i\mathbf{k}}^*\right)^{\Im}}{\omega_{\mathbf{k}}} + 2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^2}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. \tag{326}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v^{\Im}_{i\mathbf{k}}(t)-\mathrm{i}\left(g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$=\frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

$$=\frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{4|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$=\frac{\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{330}$$

Separating (330) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)+2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{331}$$

$$v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}=v^{\Im}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right) \tag{332}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{333}$$

$$v^{\Im}_{i\mathbf{k}}(t)=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{334}$$

$$v^{\Im}_{i\mathbf{k}}(t)=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{335}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t) = v_{i\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{i\mathbf{k}}^{\Im}(t) \tag{336}$$

$$= \frac{g_{i\mathbf{k}}^{\Re}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

$$+\,\mathrm{i}\frac{g_{i\mathbf{k}}^{\Im}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{338}$$

$$= \frac{g_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{339}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}$, where $\rho_B^{\mathrm{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^{V(0)}\rho_T(0)\,e^{-V(0)} \tag{340}$$

$$= \left(|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)\right)(\rho_S(0) \otimes \rho_B)\left(|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)\right), \tag{341}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } |0\rangle\langle 0|0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{342}$$

$$= |0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{343}$$

$$= |0\rangle\langle 0| \otimes B_0^+(0)\,\rho_B B_0^-(0), \tag{344}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) \tag{345}$$

$$= |1\rangle\langle 1|B_1^+(0)\,\rho_B B_1^-(0) \tag{346}$$

$$= |1\rangle\langle 1| \otimes B_1^+(0)\,\rho_B B_1^-(0), \tag{347}$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1| : |0\rangle\langle 0|B_0^+(0)\,|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) \tag{348}$$

$$= |0\rangle\langle 1|B_0^+(0)\,\rho_B|1\rangle\langle 1|B_1^-(0) \tag{349}$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0)\,\rho_B B_1^-(0) \tag{350}$$

$$= |0\rangle\langle 1| \otimes B_0^+(0)\,\rho_B B_1^-(0), \tag{351}$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0| : |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) \tag{352}$$

$$= |1\rangle\langle 0| \otimes B_1^+(0)\,\rho_B B_0^-(0). \tag{353}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^\dagger(t)\,O(t)\,U(t), \tag{354}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right). \tag{355}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\,\overline{\rho_S}(t)\,U(t), \text{ where} \tag{356}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\overline{\rho}_T(t)). \tag{357}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \tag{358}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (236):

$$\begin{aligned} \overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) && (359) \\ &= B_{0z}(t)\,|0\rangle\langle 0| + B_{1z}(t)\,|1\rangle\langle 1| + V_{10}^{\Re}(t)\,\sigma_x B_x(t) + V_{10}^{\Re}(t)\,\sigma_y B_y(t) + V_{10}^{\Im}(t)\,\sigma_x B_y(t) - V_{10}^{\Im}(t)\,\sigma_y B_x(t) && (360) \\ &= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right). && (361) \end{aligned}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{362}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) \text{-i} \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]\mathrm{d}s. \tag{363}$$

Replacing the equation (363) in the equation (362) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_T}}(s)]]\mathrm{d}s. \tag{364}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdots]. \tag{365}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\,\rho_B]] \cdots]. \tag{366}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\begin{aligned} \widetilde{\overline{\rho_S}}(t) &= \left(1 + W_1(t) + W_2(t) + ...\right)\overline{\rho_S}(0) && (367) \\ &= W(t)\,\overline{\rho_S}(0). && (368) \end{aligned}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\,\rho_B]] \cdots]. \tag{369}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + \ldots\right)\overline{\rho_S}(0) \tag{370}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + \ldots\right) W(t)^{-1} W(t)\,\overline{\rho_S}(0) \tag{371}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + \ldots\right) W(t)^{-1}\,\widetilde{\overline{\rho_S}}(t). \tag{372}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\,\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (370) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\,\widetilde{\overline{\rho_S}}(t) \tag{373}$$

$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\,\rho_B\right]\right]. \tag{374}$$

Replacing $t_1 \to t - \tau$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(-\tau), \overline{\rho_S}(t)\,\rho_B\right]\right]. \tag{375}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}(t)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = U^{\dagger}(t)\,\mathrm{e}^{\mathrm{i}H_B t}\,\overline{H_{\bar{I}}}(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t). \tag{376}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = \sum_i C_i(t)\left(\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)\right), \tag{377}$$

$$\widetilde{A_i}(t) = U^{\dagger}(t)\,\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{378}$$

$$= U^{\dagger}(t)\,A_i U(t)\,\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} \tag{379}$$

$$= U^{\dagger}(t)\,A_i U(t)\,\mathbb{I} \tag{380}$$

$$= U^{\dagger}(t)\,A_i U(t), \tag{381}$$

$$\widetilde{B_i}(t) = U^{\dagger}(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{382}$$

$$= U^{\dagger}(t)\,U(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{383}$$

$$= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{384}$$

$$= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}. \tag{385}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.

Using the expression (377) to replace it in the equation (374)

$$
\begin{aligned}
\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{\widetilde{H_{\bar{I}}}}(t), \left[\widetilde{\widetilde{H_{\bar{I}}}}(s), \widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s && (386)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \left[\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s && (387)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B - \widetilde{\widetilde{\rho_S}}(t)\,\rho_B \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\right]\mathrm{d}s && (388)\\
&= -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) && (389)\\
&\quad - \sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}s. && (390)
\end{aligned}
$$

In order to calculate the correlation functions we define:

$$\mathscr{B}_{ij}(t,s) = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(s)\,\rho_B\right) \tag{391}$$

An useful property is

$$
\begin{aligned}
\mathscr{B}_{ji}^*(t,s) &= \mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right)^\dagger && (392)\\
&= \mathrm{Tr}_B\left(\rho_B^\dagger\widetilde{B_i}^\dagger(s)\,\widetilde{B_j}^\dagger(t)\right) && (393)\\
&= \mathrm{Tr}_B\left(\rho_B\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right) && (394)\\
&= \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) && (395)\\
&= \mathscr{B}_{ij}(s,t) && (396)
\end{aligned}
$$

The correlation functions relevant that appear in the equation (390) are:

$$
\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right) &= \left\langle\widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B && (397)\\
&= \mathscr{B}_{ji}(t,s) && (398)\\
&= \mathscr{B}_{ij}^*(s,t) && (399)\\
\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\rho_B\widetilde{B_i}(s)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) && (400)\\
&= \mathscr{B}_{ij}(s,t) && (401)\\
\mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\rho_B\widetilde{B_j}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right) && (402)\\
&= \mathscr{B}_{ij}^*(s,t) && (403)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) && (404)\\
&= \mathscr{B}_{ij}(s,t) && (405)
\end{aligned}
$$

The cyclic property of the trace was use widely in the development of equations (397) and (405). Replacing in (390)

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) \tag{406}$$

$$-\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}s. \tag{407}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s)\big) \tag{408}$$

$$+\sum_{ij} C_i(s)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t) - \widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{409}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s)\big) \tag{410}$$

$$+\sum_{ij} C_i(s)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t) - \widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{411}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s)\big) \text{ (by permuting i and j because i,j}\in\text{J)} \tag{412}$$

$$+\sum_{ij} C_i(s)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t) - \widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{413}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(s)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{B_j}(t)\rho_B\widetilde{B_i}(s) \tag{414}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\rho_B\widetilde{B_i}(s)\widetilde{B_j}(t) - \widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(s)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}s \tag{415}$$

$$= -\int_0^t\Big(\sum_{ij} C_j(t)C_i(s)\big(\widetilde{A_j}(t)\widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\mathscr{B}_{ji}(t,s) - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\mathscr{B}_{ij}(s,t) \tag{416}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(s)\widetilde{A_j}(t)\mathscr{B}_{ij}(s,t) - \widetilde{A_i}(s)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\mathscr{B}_{ji}(t,s)\big)\Big)\mathrm{d}s \tag{417}$$

$$= -\int_0^t\left(\sum_{ij} C_j(t)\,C_i(s)\left(\mathscr{B}_{ji}(t,s)\left[\widetilde{A_j}(t),\widetilde{A_i}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \mathscr{B}_{ij}(s,t)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_i}(s),\widetilde{A_j}(t)\right]\right)\right)\mathrm{d}s \tag{418}$$

$$= -\int_0^t\Big(\sum_{ij} C_i(t)C_j(s)\big(\mathscr{B}_{ij}(t,s)\big[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\widetilde{\rho_S}}(t)\big] + \mathscr{B}_{ji}(s,t)\big[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)\big]\big)\Big)\mathrm{d}s \text{ (exchanging i and j)} \tag{419}$$

$$= -\int_0^t\left(\sum_{ij} C_i(t)\,C_j(s)\left(\mathscr{B}_{ij}(t,s)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \mathscr{B}^*_{ij}(t,s)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_j}(s),\widetilde{A_i}(t)\right]\right)\right)\mathrm{d}s \tag{420}$$

$$= -\int_0^t\left(\sum_{ij} C_i(t)\,C_j(s)\left(\mathscr{B}_{ij}(t,s)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] - \mathscr{B}^*_{ij}(t,s)\left[\widetilde{A_i}(t),\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_j}(s)\right]\right)\right)\mathrm{d}s \tag{421}$$

We could identify the following commutators in the equation deduced:

$$\mathscr{B}_{ij}(t,s)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \mathscr{B}_{ij}(t,s)\,\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_i}(t) = \mathscr{B}_{ij}(t,s)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_s}}(t)\right], \tag{422}$$

$$\mathscr{B}^*_{ij}(t,s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_j}(s)\,\widetilde{A_i}(t) - \mathscr{B}^*_{ij}(t,s)\,\widetilde{A_i}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_j}(s) = \mathscr{B}^*_{ij}(t,s)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_j},\widetilde{A_i}(t)\right]. \tag{423}$$

Returning to the Schroedinger picture we have:

$$U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t) = U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\,U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\,U(t)\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t), \tag{424}$$

$$= \left(U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t)\right), \tag{425}$$

$$= A_i(t)\,\widetilde{A_j}(s,t)\,\overline{\rho_S}(t). \tag{426}$$

This procedure applying to the relevant commutators give us:

$$U(t)\left[\widetilde{A_i}(t),\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right]U^\dagger(t) = \left(U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t) - U(t)\,\widetilde{A_j}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right) \tag{427}$$

$$= A_i(t)\,\widetilde{A_j}(s,t)\,\overline{\rho_S}(t) - \widetilde{A_j}(s,t)\,\overline{\rho_S}(t)\,A_i \tag{428}$$

$$= \left[A_i(t),\widetilde{A_j}(s,t)\,\overline{\rho_S}(t)\right]. \tag{429}$$

Introducing this transformed commutators in the equation (421) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathscr{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right] - \sum_{ij}\int_0^t \mathrm{d}s C_i(t) C_j(s) \Big(\mathscr{B}_{ij}(t,s)\left[A_i\left(t\right), \widetilde{A_j}(s,t)\,\overline{\rho_S}\left(t\right)\right] + \mathscr{B}^*_{ij}(t,s)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}(s,t), A_i\right]\Big) \tag{430}$$

$$s = t - \tau \text{ (Change of variables in the integration process)} \tag{431}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \Big(\mathscr{B}_{ij}(t,t-\tau)\left[A_i\left(t\right), \widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}\left(t\right)\right] + \mathscr{B}^*_{ij}(t,t-\tau)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}(t-\tau,t), A_i\left(t\right)\right]\Big) \tag{432}$$

where $i, j \in \{1, 2, 3, 4, 5.6\}$.

Here $\widetilde{A_j}\left(t-\tau, t\right) = U\left(t\right) U^{\dagger}\left(t-\tau\right) A_j\left(t\right) U\left(t-\tau\right) U^{\dagger}\left(t\right)$ where $U\left(t\right)$ is given by (355). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I\left(t\right)$. In order to write in a simplified way we define the following notation:

$$\begin{aligned}
\mathscr{B}_{ij}\left(t,s\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B\right) && (433)\\
&= \mathrm{Tr}_B\left(e^{iH_Bt}B_i\left(t\right)e^{-iH_Bt}e^{iH_Bs}B_j\left(s\right)e^{-iH_Bs}\rho_B\right) && (434)\\
e^A &= \sum_{k=0}^{\infty}\frac{A^k}{k!} && (435)\\
e^{-iH_Bs}e^{-\beta H_B} &= \sum_{m=0}^{\infty}\frac{\left(-iH_Bs\right)^m}{m!}\sum_{n=0}^{\infty}\frac{\left(-\beta H_B\right)^n}{n!} && (436)\\
&= \sum_{m,n}\frac{\left(-iH_Bs\right)^m}{m!}\frac{\left(-\beta H_B\right)^n}{n!} && (437)\\
&= \sum_{m,n}\frac{\left(-is\right)^m}{m!}\frac{\left(-\beta\right)^n}{n!}H_B^mH_B^n && (438)\\
&= \sum_{m,n}\frac{\left(-is\right)^m}{m!}\frac{\left(-\beta\right)^n}{n!}H_B^nH_B^m \text{ (because the powers of a matrix commute)} && (439)\\
&= \sum_{m,n}\frac{\left(-\beta\right)^n}{n!}H_B^n\frac{\left(-is\right)^m}{m!}H_B^m && (440)\\
&= \sum_{m,n}\frac{\left(-\beta H_B\right)^n}{n!}\frac{\left(-isH_B\right)^m}{m!} && (441)\\
&= \sum_{n=0}^{\infty}\frac{\left(-\beta H_B\right)^n}{n!}\sum_{m=0}^{\infty}\frac{\left(-iH_Bs\right)^m}{m!} && (442)\\
&= e^{-\beta H_B}e^{-iH_Bs} && (443)\\
0 &= e^{-iH_Bs}e^{-\beta H_B}-e^{-\beta H_B}e^{-iH_Bs} \text{ (then } e^{-iH_Bs} \text{ and } \rho_B\text{commute)} && (444)\\
\mathscr{B}_{ij}\left(t,s\right) &= \mathrm{Tr}_B\left(e^{iH_Bt}B_i\left(t\right)e^{-iH_Bt}e^{iH_Bs}B_j\left(s\right)\rho_Be^{-iH_Bs}\right) \text{ (by permuting } e^{-iH_Bs} \text{ and } \rho_B\text{because they commute)} && (445)\\
&= \mathrm{Tr}_B\left(\left(e^{iH_Bt}B_i\left(t\right)e^{-iH_Bt}e^{iH_Bs}B_j\left(s\right)\right)\rho_Be^{-iH_Bs}\right) \text{ (by associative property)} && (446)\\
&= \mathrm{Tr}_B\left(e^{-iH_Bs}\left(e^{iH_Bt}B_i\left(t\right)e^{-iH_Bt}e^{iH_Bs}B_j\left(s\right)\right)\rho_B\right) \text{ (by cyclic property of the trace)} && (447)\\
&= \mathrm{Tr}_B\left(\left(e^{-iH_Bs}e^{iH_Bt}\right)B_i\left(t\right)\left(e^{-iH_Bt}e^{iH_Bs}\right)B_j\left(s\right)\rho_B\right) \text{ (by associative property)} && (448)\\
\left[iH_Bt,-iH_Bs\right] &= iH_Bt\left(-iH_Bs\right)-\left(-iH_Bs\right)iH_Bt && (449)\\
&= tsH_B^2-tsH_B^2 && (450)\\
&= 0 \text{ (so } iH_Bt \text{ and } -iH_Bs \text{ commute)} && (451)\\
e^{-iH_Bs}e^{iH_Bt} &= e^{iH_Bt-iH_Bs} \text{ (by the Zassenhaus formula because } iH_Bt \text{ and } -iH_Bs \text{ commute)} && (452)\\
&= e^{iH_B(t-s)} && (453)\\
&= e^{iH_B\tau} && (454)\\
e^{iH_Bs}e^{-iH_Bt} &= e^{-iH_Bt+iH_Bs} \text{ (by the Zassenhaus formula because } -iH_Bt \text{ and } iH_Bs \text{ commute)} && (455)\\
&= e^{iH_B(-t+s)} && (456)\\
&= e^{-iH_B\tau} && (457)\\
\mathscr{B}_{ij}\left(t,s\right) &= \mathrm{Tr}_B\left(e^{iH_B\tau}B_i\left(t\right)e^{-iH_B\tau}B_j\left(s\right)\rho_B\right) && (458)\\
B_i\left(t,\tau\right) &\equiv e^{iH_B\tau}B_i\left(t\right)e^{-iH_B\tau} && (459)\\
\mathscr{B}_{ij}\left(t,s\right) &= \mathrm{Tr}_B\left(e^{iH_B(t-s)}B_i\left(t\right)e^{-iH_B(t-s)}B_j\left(s\right)\rho_B\right) && (460)\\
s &= t-\tau && (461)\\
\mathscr{B}_{ij}\left(t,s\right) &= \mathrm{Tr}_B\left(e^{iH_B\tau}B_i\left(t\right)e^{-iH_B\tau}B_j\left(s\right)\rho_B\right) && (462)\\
&= \mathrm{Tr}_B\left(B_i\left(t,\tau\right)B_j\left(s,0\right)\rho_B\right) && (463)
\end{aligned}$$

Calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{jz}}(t)\widetilde{B_{jz}}(s) \right\rangle_B = \mathrm{Tr}_B \left(B_{jz}\left(t,\tau\right) B_{jz}\left(s,0\right) \rho_B \right) \tag{464}$$

$$= \int \mathrm{d}^2\alpha P\left(\alpha\right) \left\langle \alpha \left| B_{jz}\left(t,\tau\right) B_{jz}\left(s,0\right) \right| \alpha \right\rangle \tag{465}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha\right|^2}{N} \right) \left\langle \alpha \left| B_{jz}\left(t,\tau\right) B_{jz}\left(s,0\right) \right| \alpha \right\rangle \mathrm{d}^2\alpha \tag{466}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha\right|^2}{N} \right) \left\langle \alpha \left| B_{jz}\left(t,\tau\right) B_{jz}\left(s,0\right) \right| \alpha \right\rangle \mathrm{d}^2\alpha, \tag{467}$$

$$B_{jz}\left(t,\tau\right) = \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), \tag{468}$$

$$B_{jz}\left(s,0\right) = \sum_{\mathbf{k}'} \left(\left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\left(s\right)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\left(s\right)\right)^{*} b_{\mathbf{k}'} \right), \tag{469}$$

$$\left\langle \widetilde{B_{jz}}(t)\widetilde{B_{jz}}(s) \right\rangle_B = \mathrm{Tr}_B \left(B_{jz}\left(t,\tau\right) B_{jz}\left(s,0\right) \rho_B \right) \tag{470}$$

$$= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left(\left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\left(s\right)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\left(s\right)\right)^{*} b_{\mathbf{k}'} \right) \rho_B \right) \tag{471}$$

$$= \mathrm{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\left(s\right)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\left(s\right)\right)^{*} b_{\mathbf{k}'} \right) \rho_B \right) \tag{472}$$

$$+ \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right)^{*} b_{\mathbf{k}} \right) \rho_B \right), \tag{473}$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right) = q_{j\mathbf{k}}\left(t\right) \tag{474}$$

$$\left\langle \widetilde{B_{jz}}(t)\widetilde{B_{jz}}(s) \right\rangle_B = \mathrm{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(q_{j\mathbf{k}}\left(t\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}\left(t\right) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}'}\left(s\right) b_{\mathbf{k}'}^{\dagger} + q_{j\mathbf{k}'}^{*}\left(s\right) b_{\mathbf{k}'} \right) \rho_B \right) \tag{475}$$

$$+ \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}\left(t\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}\left(t\right) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}}\left(s\right) b_{\mathbf{k}}^{\dagger} + q_{j\mathbf{k}}^{*}\left(s\right) b_{\mathbf{k}} \right) \rho_B \right) \tag{476}$$

$$0 = \mathrm{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(q_{j\mathbf{k}}\left(t\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}\left(t\right) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}'}\left(s\right) b_{\mathbf{k}'}^{\dagger} + q_{j\mathbf{k}'}^{*}\left(s\right) b_{\mathbf{k}'} \right) \rho_B \right) \tag{477}$$

$$\left\langle \widetilde{B_{jz}}(t)\widetilde{B_{jz}}(s) \right\rangle_B = 0 + \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}\left(t\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}\left(t\right) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}}\left(s\right) b_{\mathbf{k}}^{\dagger} + q_{j\mathbf{k}}^{*}\left(s\right) b_{\mathbf{k}} \right) \rho_B \right) \tag{478}$$

$$= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}\left(t\right) q_{j\mathbf{k}}\left(s\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}\left(t\right) q_{j\mathbf{k}}^{*}\left(s\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}\left(t\right) q_{j\mathbf{k}}\left(s\right) b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}\left(t\right) q_{j\mathbf{k}}^{*}\left(s\right) b_{\mathbf{k}} b_{\mathbf{k}} e^{-\mathrm{i}\omega \right. \right. \tag{479}$$

$$= \left(\sum_{\mathbf{k}} \mathrm{Tr}_B\left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \rho_B \right) + \mathrm{Tr}_B\left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(s) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \rho_B \right) + \mathrm{Tr}_B\left(q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(s) b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \rho_B \right) + \mathrm{Tr}_B\left(q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}^{*}(s) b_{\mathbf{k}} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \rho_B \right) \right) \tag{480}$$

$$= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} q_{j\mathbf{k}}\left(t\right) q_{j\mathbf{k}}^{*}\left(s\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \rho_B \right) + \mathrm{Tr}_B \left(\sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}\left(t\right) q_{j\mathbf{k}}\left(s\right) b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \rho_B \right) \tag{481}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}\left(t\right) q_{j\mathbf{k}}^{*}\left(s\right) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \mathrm{Tr}_B \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \rho_B \right) + \sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}\left(t\right) q_{j\mathbf{k}}\left(s\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \mathrm{Tr}_B \left(b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \rho_B \right) \tag{482}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}\left(t\right) q_{j\mathbf{k}}^{*}\left(s\right) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N} \right) \left\langle \alpha_{\mathbf{k}} \left| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{483}$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}\left(t\right) q_{j\mathbf{k}}\left(s\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N} \right) \left\langle \alpha_{\mathbf{k}} \left| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{484}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t)\, q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| D(-\alpha_{\mathbf{k}})\, b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{485}$$

$$+ \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{486}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t)\, q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| D(-\alpha_{\mathbf{k}})\, b^{\dagger}_{\mathbf{k}} D(\alpha_{\mathbf{k}})\, D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{487}$$

$$+ \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}} D(\alpha_{\mathbf{k}})\, D(-\alpha_{\mathbf{k}})\, b^{\dagger}_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{488}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| \left(b^{\dagger}_{\mathbf{k}} + \alpha^{*}_{\mathbf{k}}\right)\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right) \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \Big(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \Big\langle 0 \Big| \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\left(b^{\dagger}_{\mathbf{k}} + \alpha^{*}_{\mathbf{k}}\right) \Big| 0 \Big\rangle \mathrm{d} \tag{489}$$

$$= \sum_{\mathbf{k}} \Big(q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} + b^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha^{*}_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} + q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \Big\langle 0 \Big| b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} + b^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha^{*}_{\mathbf{k}} + |\alpha_{\mathbf{k}} \tag{490}$$

$$= \sum_{\mathbf{k}} \Big(q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha^{*}_{\mathbf{k}} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{491}$$

$$+ q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \Big(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} + b_{\mathbf{k}} \alpha^{*}_{\mathbf{k}} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \tag{492}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{493}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{494}$$

$$+ \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right), \tag{495}$$

$$1 = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \mathrm{d}^{2}\alpha_{\mathbf{k}}, \tag{496}$$

$$b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}} |0\rangle = 0, \tag{497}$$

$$b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} |0\rangle = |0\rangle\,, \tag{498}$$

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \right\rangle_{B} = \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| |\alpha_{\mathbf{k}}|^{2} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{499}$$

$$+ \sum_{\mathbf{k}} q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) \left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \left\langle 0 \left| b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} \right| 0 \right\rangle \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{500}$$

$$= \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^{2} \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \mathrm{d}^{2}\alpha_{\mathbf{k}} + q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^{2} \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \mathrm{d}^{2}\alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \right) \tag{501}$$

$$= \sum_{\mathbf{k}} \left(\left(q_{j\mathbf{k}}(t) q^{*}_{j\mathbf{k}}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^{2} \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \mathrm{d}^{2}\alpha_{\mathbf{k}} + q^{*}_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \mathrm{d}^{2}\alpha_{\mathbf{k}} \right) \tag{502}$$

$$\frac{1}{\pi N} \int_{0}^{2\pi} \int_{0}^{\infty} r^{2} \exp\left(-\frac{r^{2}}{N}\right) r \,\mathrm{d}r \,\mathrm{d}\theta = \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^{2} \exp\left(-\frac{|\alpha_{\mathbf{k}}|^{2}}{N}\right) \mathrm{d}^{2}\alpha_{\mathbf{k}} \tag{503}$$

$$= N \tag{504}$$

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \right\rangle_{B} = \sum_{\mathbf{k}} \left(\left(q_{j\mathbf{k}}(t)\, q^{*}_{j\mathbf{k}}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^{*}_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(s)\, e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) N + q^{*}_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(s)\, e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \tag{505}$$

$$\left\langle\widetilde{B_{jz}}(t)\widetilde{B_{j'z}}(s)\right\rangle_B = \mathrm{Tr}_B\left(B_{jz}\left(t,\tau\right)B_{j'z}\left(s,0\right)\rho_B\right) \tag{506}$$

$$= \int \mathrm{d}^2\alpha P\left(\alpha\right)\left\langle\alpha\left|B_{jz}\left(t,\tau\right)B_{j'z}\left(s,0\right)\right|\alpha\right\rangle \tag{507}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha\right|^2}{N}\right)\left\langle\alpha\left|B_{jz}\left(t,\tau\right)B_{j'z}\left(s,0\right)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{508}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)^{*}b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{509}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}(s)\right)^{*}b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{510}$$

$$+ \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)^{*}b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{511}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t))^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}(s)\right)^{*}b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{512}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)^{*}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{513}$$

$$+ \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{514}$$

$$= \sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{515}$$

$$+ \sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)\right)^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\left(s\right)\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}, \tag{516}$$

$$\frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{517}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{518}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{519}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left|\alpha_{\mathbf{k}}\right|^2 \mathrm{d}^2\alpha_{\mathbf{k}} \tag{520}$$

$$= N, \tag{521}$$

$$\frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{522}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{523}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+\left|\alpha_{\mathbf{k}}\right|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{524}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\left|\alpha_{\mathbf{k}}\right|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{525}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|\left|\alpha_{\mathbf{k}}\right|^2\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N}\int \exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{526}$$

$$= N+1, \tag{527}$$

$$\left\langle \widetilde{B_{jz}}(t)\widetilde{B_{j'z}}(s)\right\rangle_B = \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))\, e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N+1) \tag{528}$$

$$= \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))\, e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v \tag{529}$$

$$= \sum_{\mathbf{k}} 2N \left(q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(s) e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Re} + \sum_{\mathbf{k}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(s) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{530}$$

$$D(h')\, D(h) = \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) D(h'+h), \tag{531}$$

$$\langle D(h')\, D(h)\rangle_B = \mathrm{Tr}_B\left(\exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) D(h'+h)\, \rho_B\right) \tag{532}$$

$$= \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \mathrm{Tr}_B\left(D(h'+h)\, \rho_B\right) \tag{533}$$

$$= \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \frac{1}{\pi N} \int \mathrm{d}^2\alpha P(\alpha) \langle \alpha | D(h'+h) | \alpha \rangle \tag{534}$$

$$= \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \exp\left(-\frac{|h+h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \tag{535}$$

$$h' = h\exp(\mathrm{i}\omega\tau), \tag{536}$$

$$\langle D(h\exp(\mathrm{i}\omega\tau))\, D(h)\rangle_B = \exp\left(\frac{1}{2}(hh^*\exp(\mathrm{i}\omega\tau) - h^*h\exp(-\mathrm{i}\omega\tau))\right) \exp\left(-\frac{|h+h\exp(\mathrm{i}\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \tag{537}$$

$$\frac{1}{2}|h|^2(\exp(\mathrm{i}\omega\tau) - \exp(-\mathrm{i}\omega\tau)) = \frac{1}{2}(hh^*\exp(\mathrm{i}\omega\tau) - h^*h\exp(-\mathrm{i}\omega\tau)) \tag{538}$$

$$= \frac{1}{2}|h|^2 (\cos(\omega\tau) + \mathrm{i}\sin(\omega\tau) - \cos(\omega\tau) + \mathrm{i}\sin(\omega\tau)) \tag{539}$$

$$= \frac{1}{2}|h|^2 (2\mathrm{i}\sin(\omega\tau)) \tag{540}$$

$$= \mathrm{i}\,|h|^2 \sin(\omega\tau), \tag{541}$$

$$-\frac{|h+h\exp(\mathrm{i}\omega\tau)|^2}{2} = -|h|^2 \frac{|1+\exp(\mathrm{i}\omega\tau)|^2}{2} \tag{542}$$

$$= -|h|^2 \frac{\left(1 + 2\cos(\omega\tau) + \cos^2(\omega\tau)\right) + \sin^2(\omega\tau)}{2} \tag{543}$$

$$= -|h|^2 \frac{2 + 2\cos(\omega\tau)}{2} \tag{544}$$

$$= -|h|^2 (1 + \cos(\omega\tau)), \tag{545}$$

$$\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\rangle_B = \exp\big(\mathrm{i}|h|^2\sin(\omega\tau)\big)\exp\big(-|h|^2(1+\cos(\omega\tau))\coth\big(\tfrac{\beta\omega}{2}\big)\big) \tag{546}$$

$$= \exp\left(\mathrm{i}\,|h|^2 \sin(\omega\tau) - |h|^2 (1+\cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{547}$$

$$= \exp\left(-|h|^2 \left(-\mathrm{i}\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\right) \exp\left(-|h|^2 \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{548}$$

$$= \langle D(h)\rangle_B \exp(-\phi(\tau)), \tag{549}$$

$$\exp(-\phi(\tau)) = \exp\left(-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin(\omega\tau)\right)\right), \tag{550}$$

$$\phi(\tau) = |h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin(\omega\tau)\right), \tag{551}$$

$$\langle D(h')\,D(h)\rangle_B = \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{552}$$

$$h' = v\exp(i\omega\tau), \tag{553}$$

$$\left\langle \widetilde{B_1^+B_0^-}(t)\widetilde{B_1^+B_0^-}(s)\right\rangle_B = \left\langle B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0)\right\rangle_B \tag{554}$$

$$= \langle B_{10}(t,\tau)B_{10}(s,0)\rangle_B \tag{555}$$

$$= \mathrm{Tr}_B\left(B_{10}(t,\tau)B_{10}(s,0)\,\rho_B\right) \tag{556}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}(t)} - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(s)v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)v}{\omega_{\mathbf{k}}^2}\right.}\right.\right. \tag{557}$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s))\,\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{558}$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s))\prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{559}$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s))\prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)^* - \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\right.\right.\right. \tag{560}$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s))\prod_{\mathbf{k}}\left(\exp\left(i\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)^*\right)^{\Im}\right)\exp\left(-\frac{|\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e}{}\right.\right. \tag{561}$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s))\prod_{\mathbf{k}}\left(\exp\left(i\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)^*\right)^{\Im}\right)\exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}}{}\right.\right. \tag{562}$$

$$\left\langle \widetilde{B_0^+B_1^-}(t)\widetilde{B_0^+B_1^-}(s)\right\rangle_B = \exp(\chi_{01}(t) + \chi_{01}(s))\prod_{\mathbf{k}}\left(\exp\left(i\left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\left(\frac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)^*\right)^{\Im}\right)\exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}}{}\right.\right. \tag{563}$$

$$\langle D(h)\,b\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\,|D(h)\,b|\,\alpha\rangle \tag{564}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\left|D\left(-\alpha\right)D\left(h\right)bD\left(\alpha\right)\right|\alpha\rangle \tag{565}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D\left(-\alpha\right)D\left(h\right)bD\left(\alpha\right)\right|0\rangle \tag{566}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)D\left(-\alpha\right)bD\left(\alpha\right)\right|0\rangle \tag{567}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)\left(b+\alpha\right)\right|0\rangle \tag{568}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\left|D\left(h\right)\left(b+\alpha\right)\right|0\rangle \tag{569}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)b|0\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \tag{570}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\left|D\left(h\right)\alpha\right|0\rangle \tag{571}$$

$$= \frac{1}{\pi N}\int \alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{572}$$

$$= hN\left\langle D\left(h\right)\right\rangle_B, \tag{573}$$

$$\left\langle D\left(h\right)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle\alpha\left|D\left(h\right)b^\dagger\right|\alpha\right\rangle \tag{574}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)b^\dagger D\left(\alpha\right)\right|0\right\rangle \tag{575}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)b^\dagger D\left(\alpha\right)\right|0\right\rangle \tag{576}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)D\left(-\alpha\right)b^\dagger D\left(\alpha\right)\right|0\right\rangle \tag{577}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{578}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\left\langle 0\left|D\left(h\right)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{579}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\left\langle 0\left|D(h)b^\dagger\right|0\right\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha^*|0\rangle \tag{580}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|1\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle \tag{581}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle -h|1\rangle + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle, \tag{582}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right)\sum_n \frac{(-h^*)^n}{\sqrt{n!}}\langle n|, \tag{583}$$

$$\langle -h|1\rangle = \exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*), \tag{584}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|-h^*|^2}{2}\right)(-h^*) + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\exp\left(-\tfrac{|-h^*|^2}{2}\right) \tag{585}$$

$$= -h^*\langle D(h)\rangle_B (N+1), \tag{586}$$

$$\langle bD(h)\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle \alpha|bD(h)|\alpha\rangle \tag{587}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{588}$$

$$= h\langle D(h)\rangle_B (N+1), \tag{589}$$

$$\left\langle b^\dagger D(h)\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle \alpha\left|b^\dagger D(h)\right|\alpha\right\rangle \tag{590}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h + \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{591}$$

$$= -h^*\langle D(h)\rangle_B N, \tag{592}$$

The correlation functions can be found readily as:

$$B_1^+ B_0^- (t,\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t)\, v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)\, v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right), \tag{593}$$

$$B_0^+ B_1^- (t,\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t)\, v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\, v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right), \tag{594}$$

$$B_{10}(t) = \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t)\, v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)\, v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right), \tag{595}$$

$$B_x(t,\tau) = \frac{B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t)}{2}, \tag{596}$$

$$B_y(t,\tau) = \frac{B_0^+ B_1^- (t,\tau) - B_1^+ B_0^- (t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, \tag{597}$$

$$B_{iz}(t,\tau) = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*\, b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), \tag{598}$$

$$\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s) \right\rangle_B = \langle B_{iz}(t,\tau) B_{jz}(s,0) \rangle_B \tag{599}$$

$$= \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))\, b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{jk}(s))^*\, b_{\mathbf{k}} \right) \right\rangle_B \tag{600}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))^*\, e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*\, (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))\, e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \tag{601}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s) \right\rangle_B = \langle B_x(t,\tau) B_x(s,0) \rangle_B \tag{602}$$

$$= \left\langle \left(\frac{B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left(\frac{B_1^+ B_0^- (s,0) + B_0^+ B_1^- (s,0) - B_{10}(s) - B_{01}(s)}{2} \right) \right\rangle_B \tag{603}$$

$$= \frac{1}{4} \left\langle \left(B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t) \right) \left(B_1^+ B_0^- (s,0) + B_0^+ B_1^- (s,0) - B_{10}(s) - B_{01}(s) \right) \right\rangle_B \tag{604}$$

$$= \tfrac{1}{4} \left\langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (s,0) + B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (s,0) - B_1^+ B_0^- (t,\tau) B_{10}(s) - B_1^+ B_0^- (\tau) B_{01}(s) + B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (s,0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (s,0) - B_0^+ \right. \tag{605}$$

$$- B_0^+ B_1^- (t,\tau) B_{01}(s) - B_{10}(t) B_1^+ B_0^- (s,0) - B_{10}(t) B_0^+ B_1^- (s,0) + B_{10}(t) B_{10}(s) + B_{10}(t) B_{01}(s) - B_{01}(t) B_1^+ B_0^- (s,0) - B_{01}(t) B_0^+ B_1^- (s,0) + B_{01}(t) B_{10}(s) + B_{01}(\tag{606}$$

$$= \frac{1}{4} \langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (s,0) + B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (s,0) + B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (s,0) \tag{607}$$

$$+ B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (s,0) \rangle - \frac{(B_{01}(t) + B_{10}(t))\,(B_{01}(s) + B_{10}(s))}{4}, \tag{608}$$

$$U_{10}(t,s) = \prod_{\mathbf{k}} \exp\left(i \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))\, (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \tag{609}$$

$$\left\langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (s,0) \right\rangle_B = \exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\tfrac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{610}$$

$$\left\langle B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (s,0) \right\rangle_B = \exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\tfrac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{611}$$

$$\left\langle B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (s,0) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\tfrac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\tfrac{1}{2} \left(\tfrac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \left(D\left(\tfrac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \exp\left(\tfrac{1}{2} \left(\tfrac{v_{0\mathbf{k}}^*(s) v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s) v_{1\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \right. \tag{612}$$

$$= \exp(\chi_{10}(t) + \chi_{01}(s)) \left\langle \prod_{\mathbf{k}} \left(D\left(\tfrac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right) \prod_{\mathbf{k}} \left(D\left(\tfrac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_B \tag{613}$$

$$= \exp(\chi_{10}(t) + \chi_{01}(s)) \prod_{\mathbf{k}} \left\langle \left(D\left(\tfrac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left(\tfrac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_B \tag{614}$$

$$= \exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\tfrac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{615}$$

$$\left\langle B_0^+ B_1^-(t,\tau) B_1^+ B_0^-(s,0)\right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)\right)\right) \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right) \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2}\right)\right) \tag{616}$$

$$= \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)\right) \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v_{0\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle D\left(\frac{v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right\rangle_B \tag{617}$$

$$= \exp\left(\chi_{01}(t)+\chi_{10}(s)\right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) D\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)\right\rangle_B \tag{618}$$

$$= \exp\left(\chi_{01}(t)+\chi_{10}(s)\right) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{619}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \frac{1}{4}\langle B_1^+ B_0^-(t,\tau) B_1^+ B_0^-(s,0) + B_1^+ B_0^-(t,\tau) B_0^+ B_1^-(s,0) + B_0^+ B_1^-(t,\tau) B_1^+ B_0^-(s,0) = \tag{620}$$

$$+ B_0^+ B_1^-(t,\tau) B_0^+ B_1^-(s,0)\rangle - \frac{\left(B_{01}(t)+B_{10}(t)\right)\left(B_{01}(s)+B_{10}(s)\right)}{4}, \tag{621}$$

$$= \frac{1}{4}\left(\exp(\chi_{10}(t)+\chi_{10}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{622}$$

$$+ \exp(\chi_{10}(t)+\chi_{01}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{623}$$

$$+ \exp\left(\chi_{01}(t)+\chi_{10}(s)\right) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{624}$$

$$+ \exp(\chi_{01}(t)+\chi_{01}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\Bigg) \tag{625}$$

$$-\left(\exp(\chi_{10}(t)) \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re} \left(\exp(\chi_{01}(s)) \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re} \tag{626}$$

$$= \frac{1}{2}\left(\left(\exp\left(\chi_{10}(t)+\chi_{10}(s)\right)\right)^{\Re} U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{627}$$

$$\left. + \left(\exp\left(\chi_{10}(t)+\chi_{01}(s)\right)\right)^{\Re} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{628}$$

$$-\left(\exp(\chi_{10}(t)) \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re} \left(\exp(\chi_{01}(s)) \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re} \tag{629}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_y}(s)\right\rangle_B = \langle B_y(t,\tau) B_y(s,0)\rangle_B \tag{630}$$

$$= \left\langle \left(\frac{B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\right)\left(\frac{B_0^+ B_1^-(s,0) - B_1^+ B_0^-(s,0) + B_{10}(s) - B_{01}(s)}{2\mathrm{i}}\right)\right\rangle \tag{631}$$

$$= -\frac{1}{4}\left\langle \left(B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)\right)\left(B_0^+ B_1^-(s,0) - B_1^+ B_0^-(s,0) + B_{10}(s) - B_{01}(s)\right)\right\rangle \tag{632}$$

$$\tag{633}$$

$$= -\tfrac{1}{4}\Big\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) + B_0^+B_1^-(t,\tau)B_{10}(s) - B_0^+B_1^-(\tau)B_{01}(s) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0) \tag{634}$$

$$+B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0) - B_1^+B_0^-(t,\tau)B_{10}(s) + B_1^+B_0^-(t,\tau)B_{01}(s) + B_{10}(t)B_0^+B_1^-(s,0) - B_{10}(t)B_1^+B_0^-(s,0) + B_{10}(t)B_{10}(s) \tag{635}$$

$$-B_{10}(t)B_{01}(s) - B_{01}(t)B_0^+B_1^-(s,0) + B_{01}(t)B_1^+B_0^-(s,0) - B_{01}(t)B_{10}(s) + B_{01}(t)B_{01}(s)\Big\rangle \tag{636}$$

$$= -\tfrac{1}{4}\Big\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0) + B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0)\Big\rangle \tag{637}$$

$$+(B_{01}(t))^{\Im}(B_{10}(s))^{\Im} \tag{638}$$

$$= -\frac{1}{4}\left(\exp(\chi_{01}(t)+\chi_{01}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{639}$$

$$-\exp(\chi_{01}(t)+\chi_{10}(s))U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{640}$$

$$-\exp(\chi_{10}(t)+\chi_{01}(s))U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)-(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{641}$$

$$+\exp(\chi_{10}(t)+\chi_{10}(s))U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\Bigg) \tag{642}$$

$$+\left(\exp(\chi_{01}(t))\left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left|\tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im}\left(\exp(\chi_{10}(s))\left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left|\tfrac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im} \tag{643}$$

$$= -\tfrac{1}{2}\left((\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Re}U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{644}$$

$$-(\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Re}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\tfrac{|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\exp(i\omega_{\mathbf{k}}\tau)+(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\Bigg) \tag{645}$$

$$+\left(\exp(\chi_{01}(t))\left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left|\tfrac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im}\left(\exp(\chi_{10}(s))\left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left|\tfrac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\tfrac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im} \tag{646}$$

$$\tag{647}$$

$$\tag{648}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(s)\right\rangle_B = \langle B_x(t,\tau)B_y(s,0)\rangle_B \tag{649}$$

$$= \left\langle\left(\frac{B_1^+B_0^-(t,\tau)+B_0^+B_1^-(t,\tau)-B_{10}(t)-B_{01}(t)}{2}\right)\left(\frac{B_0^+B_1^-(s,0)-B_1^+B_0^-(s,0)+B_{10}(s)-B_{01}(s)}{2\mathrm{i}}\right)\right\rangle_B \tag{650}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(s,0) - B_1^+B_0^-(t,\tau)B_1^+B_0^-(s,0) + B_1^+B_0^-(t,\tau)B_{10}(s) - B_1^+B_0^-(t,\tau)B_{01}(s)\right. \tag{651}$$

$$+B_0^+B_1^-(t,\tau)B_0^+B_1^-(s,0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(s,0) + B_0^+B_1^-(t,\tau)B_{10}(s) - B_0^+B_1^-(t,\tau)B_{01}(s) \tag{652}$$

$$-B_{10}(t)B_0^+B_1^-(s,0) + B_{10}(t)B_1^+B_0^-(s,0) - B_{10}(t)B_{10}(s) + B_{10}(t)B_{01}(s) \tag{653}$$

$$\left.-B_{01}(t)B_0^+B_1^-(s,0) + B_{01}(t)B_1^+B_0^-(s,0) - B_{01}(t)B_{10}(s) + B_{01}(t)B_{01}(s)\right\rangle_B \tag{654}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_1^+ B_0^- \left(t,\tau\right) B_{10}\left(s\right) - B_1^+ B_0^- \left(t,\tau\right) B_{01}\left(s\right) \right. \tag{655}$$

$$+ B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_{10}\left(s\right) - B_0^+ B_1^- \left(t,\tau\right) B_{01}\left(s\right) \tag{656}$$

$$- B_{10}\left(t\right) B_0^+ B_1^- \left(s,0\right) + B_{10}\left(t\right) B_1^+ B_0^- \left(s,0\right) - B_{10}\left(t\right) B_{10}\left(s\right) + B_{10}\left(t\right) B_{01}\left(s\right) \tag{657}$$

$$\left. - B_{01}\left(t\right) B_0^+ B_1^- \left(s,0\right) + B_{01}\left(t\right) B_1^+ B_0^- \left(s,0\right) - B_{01}\left(t\right) B_{10}\left(s\right) + B_{01}\left(t\right) B_{01}\left(s\right) \right\rangle_B \tag{658}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) \right. \tag{659}$$

$$\left. - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) \right\rangle + \frac{1}{4i} \left(B_{10}\left(t\right) + B_{01}\left(t\right)\right) \left(B_{10}\left(s\right) - B_{01}\left(s\right)\right) \tag{660}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) \right. \tag{661}$$

$$\left. - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) \right\rangle + \frac{1}{4i} \left(B_{10}\left(t\right) + B_{01}\left(t\right)\right) \left(B_{10}\left(s\right) - B_{01}\left(s\right)\right) \tag{662}$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) - B_1^+ B_0^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(t,\tau\right) B_0^+ B_1^- \left(s,0\right) \right. \tag{663}$$

$$\left. - B_0^+ B_1^- \left(t,\tau\right) B_1^+ B_0^- \left(s,0\right) \right\rangle + \left(B_{10}\left(t\right)\right)^{\Re} \left(B_{10}\left(s\right)\right)^{\Im} \tag{664}$$

$$= \frac{1}{4i} \left(\exp(\chi_{10}(t)+\chi_{01}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \tag{665}$$

$$- \exp(\chi_{10}(t)+\chi_{10}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{666}$$

$$+ \exp(\chi_{01}(t)+\chi_{01}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{667}$$

$$\left. - \exp(\chi_{01}(t)+\chi_{10}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \tag{668}$$

$$= \frac{1}{4i} \left(2i(\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Im} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \tag{669}$$

$$\left. + 2i(\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im} U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \tag{670}$$

$$= \frac{1}{2} \left((\exp(\chi_{10}(t)+\chi_{01}(s)))^{\Im} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \tag{671}$$

$$\left. + (\exp(\chi_{01}(t)+\chi_{01}(s)))^{\Im} U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \tag{672}$$

$$\left\langle \widetilde{B_y}(t)\, \widetilde{B_x}(s) \right\rangle_B = \left\langle \left(\frac{B_0^+ B_1^- \left(t,\tau\right) - B_1^+ B_0^- \left(t,\tau\right) + B_{10}\left(t\right) - B_{01}\left(t\right)}{2\mathrm{i}} \right) \left(\frac{B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(s,0\right) - B_{10}\left(s\right) - B_{01}\left(s\right)}{2} \right) \right\rangle_B \tag{673}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+ B_1^- \left(t,\tau\right) - B_1^+ B_0^- \left(t,\tau\right) + B_{10}\left(t\right) - B_{01}\left(t\right) \right) \left(B_1^+ B_0^- \left(s,0\right) + B_0^+ B_1^- \left(s,0\right) - B_{10}\left(s\right) - B_{01}\left(s\right) \right) \right\rangle_B \tag{674}$$

$$= \tfrac{1}{4\mathrm{i}} \big\langle B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (s,0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (s,0) - B_0^+ B_1^- (t,\tau) B_{10}(s) - B_0^+ B_1^- (t,\tau) B_{01}(s) - B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (s,0) \tag{675}$$

$$- B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (s,0) + B_1^+ B_0^- (t,\tau) B_{10}(s) + B_1^+ B_0^- (t,\tau) B_{01}(s) + B_{10}(t) B_1^+ B_0^- (s,0) + B_{10}(t) B_0^+ B_1^- (s,0) \tag{676}$$

$$- B_{10}(t) B_{10}(s) - B_{10}(t) B_{01}(s) - B_{01}(t) B_1^+ B_0^- (s,0) - B_{01}(t) B_0^+ B_1^- (s,0) + B_{01}(t) B_{10}(s) + B_{01}(t) B_{01}(s) \big\rangle \tag{677}$$

$$= \tfrac{1}{4i} \big\langle B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (s,0) + B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (s,0) - B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (s,0) - B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (s,0) \big\rangle + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \tag{678}$$

$$= \frac{1}{4i} \Bigg(\exp(\chi_{01}(t) + \chi_{10}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) + \left(v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \tag{679}$$

$$+ \exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \tag{680}$$

$$- \exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \tag{681}$$

$$- \exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) - \left(v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \Bigg) + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \tag{682}$$

$$= \tfrac{1}{4i} \Bigg(2i(\exp(\chi_{01}(t) + \chi_{10}(s)))^{\Im} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) - \left(v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \tag{683}$$

$$+ 2i(\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \Bigg) + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \tag{684}$$

$$= \tfrac{1}{2} \Bigg((\exp(\chi_{01}(t) + \chi_{10}(s)))^{\Im} U_{10}^*(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) - \left(v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \tag{685}$$

$$+ (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t,s) \prod_{\mathbf{k}} \exp\bigg(- \tfrac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\Big(\tfrac{\beta\omega_{\mathbf{k}}}{2}\Big) \bigg) \Bigg) + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \tag{686}$$

$$\left\langle b^\dagger D\left(h\right) \right\rangle_B = -h^* \left\langle D\left(h\right) \right\rangle_B N \tag{687}$$

$$\left\langle b D\left(h\right) \right\rangle_B = h \left\langle D\left(h\right) \right\rangle_B \left(N+1\right) \tag{688}$$

$$\left\langle D\left(h\right) b^\dagger \right\rangle_B = -h^* \left\langle D\left(h\right) \right\rangle_B \left(N+1\right) \tag{689}$$

$$\left\langle D\left(h\right) b \right\rangle_B = h \left\langle D\left(h\right) \right\rangle_B N \tag{690}$$

$$\left\langle B_1^+ B_0^- \left(t,\tau\right) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right) b_{\mathbf{k}'}^\dagger \right\rangle_B = \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*\left(t\right) v_{0\mathbf{k}}\left(t\right) - v_{1\mathbf{k}}\left(t\right) v_{0\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}^2}\right)\right) \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right) \left(g_{i\mathbf{k}'} - v_{i\right. \tag{691}$$

$$= \exp\left(\chi_{10}\left(t\right)\right) \left\langle D\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right) b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(\right.}{\omega_{\mathbf{k}}}\right.\right.\right. \tag{692}$$

$$= \exp\left(\chi_{10}\left(t\right)\right) q_{i\mathbf{k}'}\left(s\right) \left\langle D\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k}\neq\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{693}$$

$$= \exp\left(\chi_{10}\left(t\right)\right) q_{i\mathbf{k}'}\left(s\right) \left(-\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{694}$$

$$= q_{i\mathbf{k}'}\left(s\right) \left(-\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) \exp\left(\chi_{10}\left(t\right)\right) \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{695}$$

$$= -q_{i\mathbf{k}'}\left(s\right) \left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}\left(t\right) \tag{696}$$

$$\left\langle B_1^+ B_0^- (t,\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* b_{\mathbf{k}'} \right\rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)\right) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right.\right. \tag{697}$$

$$= q_{i\mathbf{k}'}^*\left(s\right) \left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}\left(t\right), \tag{698}$$

$$\left\langle B_0^+ B_1^- (t,\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger \right\rangle_B = -q_{i\mathbf{k}'}(s) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'} + 1) B_{01}(t), \tag{699}$$

$$\left\langle B_0^+ B_1^- (t,\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* b_{\mathbf{k}'} \right\rangle_B = q_{i\mathbf{k}'}^*(s) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t), \tag{700}$$

$$\left\langle B_x(t,\tau) B_{iz}(s,0) \right\rangle_B = \left\langle \left(\frac{B_{1+}B_{0-}\left(\tau\right) + B_{0+}B_{1-}\left(\tau\right) - B_{10} - B_{10}^*}{2}\right) \sum_{\mathbf{k}'} \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right) \right\rangle_B \tag{701}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle \left(B_{1+}B_{0-}\left(\tau\right) + B_{0+}B_{1-}\left(\tau\right) - B_{10} - B_{10}^*\right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right) \right\rangle_B \tag{702}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle \left(B_{1+}B_{0-}\left(\tau\right) + B_{0+}B_{1-}\left(\tau\right)\right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right) \right\rangle_B \tag{703}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} + B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'} - v\right. \tag{704}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'} + \right. \tag{705}$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}(t) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t) \Big) \tag{706}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (N_{\mathbf{k}'} + \right. \tag{707}$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}(t) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}(t) \Big) \tag{708}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right) \left(N_{\mathbf{k}'} + 1\right) \left(\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10}\left(t\right) + \left(\frac{v_{0\mathbf{k}'}\left(t\right) - v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right.\right.\right. \tag{709}$$

$$+ \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(s\right)\right)^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'}\left(t\right) - v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{10}\left(t\right) + \left(\frac{v_{0\mathbf{k}'}\left(t\right) - v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) B_{01}\left(t\right.\right.$$

$$\langle B_{iz}(t,\tau)B_x(s,0)\rangle_B = \Big\langle\sum_{\mathbf{k}'}\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(\frac{B_1^+B_0^-(s,0)+B_0^+B_1^-(s,0)-B_{10}(s)-B}{2} \tag{723}$$

$$= \sum_{\mathbf{k}'}\Big\langle\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\,b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(\frac{B_1^+B_0^-(s,0)+B_0^+B_1^-(s,0)-B_{10}(s)-B}{2} \tag{724}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big\langle\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\,b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\big(B_1^+B_0^-(s,0)+B_0^+B_1^-(s,0)-B_{10}(s)- \tag{725}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big\langle\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\,b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\big(B_1^+B_0^-(s,0)+B_0^+B_1^-(s,0)\big)\Big\rangle_B \tag{726}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(s,0)+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(s,0)+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B \tag{727}$$

$$+ (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(s,0)\big\rangle, \tag{728}$$

$$\big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(s,0)\big\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\big\langle b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(s,0)\big\rangle_B \tag{729}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\Big\langle b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\Big(D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\Big)\Big\rangle_B \tag{730}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\Big\langle b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}'}(s)v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)v^*_{0\mathbf{k}'}(s)}{\omega^2_{\mathbf{k}'}}\Big)\Big)\Big)\Big\rangle_B\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v}{\omega^2_{\mathbf{k}}} \tag{731}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\Big\rangle_B\Big\langle b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\Big\rangle_B \tag{732}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\Big\rangle_B\Big\langle b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\Big\rangle_B \tag{733}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\Big\rangle_B\Big\langle b^\dagger_{\mathbf{k}'}D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\Big\rangle_B e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{734}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\Big\rangle_B\Big(-\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)^*\Big\langle D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\Big\rangle_B N \tag{735}$$

$$= -\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)^*(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\Big\langle\prod_{\mathbf{k}}D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\Big\rangle_B N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{736}$$

$$= -\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)^*(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\,B_{10}(s)\,N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{737}$$

$$\big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))b^\dagger_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(s,0)\big\rangle_B = -\Big(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)^*(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))B_{01}(s)N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{738}$$

$$\big\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(s,0)\big\rangle_B = (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\big\langle b_{\mathbf{k}'}B_1^+B_0^-(s,0)\big\rangle_B \tag{739}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\Big(D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big) \tag{740}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\Big\langle b_{\mathbf{k}'}D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\Big\rangle_B\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\Big)\Big)\Big\rangle_B \tag{741}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\Big\langle D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\Big\rangle_B\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}}(s)-}{\omega_{\mathbf{k}}} \tag{742}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\Big(\frac{1}{2}\Big(\frac{v^*_{1\mathbf{k}}(s)v_{0\mathbf{k}}(s)-v_{1\mathbf{k}}(s)v^*_{0\mathbf{k}}(s)}{\omega^2_{\mathbf{k}}}\Big)\Big)\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\Big\langle D\Big(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\Big)\Big\rangle_B\Big\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\Big(D\Big(\frac{v_{1\mathbf{k}}(s)-}{\omega_{\mathbf{k}}} \tag{743}$$

$$= (g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{}(N_{\mathbf{k}'}+1)B_{\ldots}(s) \tag{744}$$

$$\left\langle B_y(t,\tau)\, B_{iz}(s,0)\right\rangle_B = \left\langle\!\!\left\langle\left(\frac{B_0^{+}B_1^{-}(t,\tau) - B_1^{+}B_0^{-}(t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)\right)^{*} b_{\mathbf{k}'}\right)\right\rangle\!\!\right\rangle_B \tag{751}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\!\!\left\langle\left(B_0^{+}B_1^{-}(t,\tau) - B_1^{+}B_0^{-}(t,\tau) + B_{10}(t) - B_{01}(t)\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)\right)^{*} b_{\mathbf{k}'}\right)\right\rangle\!\!\right\rangle_B \tag{752}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^{+}B_1^{-}(t,\tau) - B_1^{+}B_0^{-}(t,\tau)\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)\right)^{*} b_{\mathbf{k}'}\right)\right\rangle_B \tag{753}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big\langle B_0^{+}B_1^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^{\dagger}-B_1^{+}B_0^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^{\dagger}+B_0^{+}B_1^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}b_{\mathbf{k}'}-B_1^{+}B_0^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}b_{\mathbf{k}'}\Big\rangle, \tag{754}$$

$$\left\langle B_0^{+}B_1^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^{\dagger}\right\rangle_B = -\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}(N_{\mathbf{k}'}+1)B_{01}(t), \tag{755}$$

$$\left\langle B_0^{+}B_1^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}b_{\mathbf{k}'}\right\rangle_B = \big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}\left(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t), \tag{756}$$

$$\left\langle B_1^{+}B_0^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)b_{\mathbf{k}'}^{\dagger}\right\rangle_B = -\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\left(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}(N_{\mathbf{k}'}+1)B_{10}(t), \tag{757}$$

$$\left\langle B_1^{+}B_0^{-}(t,\tau)\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}b_{\mathbf{k}'}\right\rangle_B = \big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}\left(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t), \tag{758}$$

$$\left\langle B_y(t,\tau)B_{iz}(s,0)\right\rangle_B = \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\Big(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)^{*}(N_{\mathbf{k}'}+1)B_{01}(t)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)^{*}(N_{\mathbf{k}'}+1)B_{10}(t) \tag{759}$$

$$+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}\left(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}\left(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t)\Big) \tag{760}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\Big(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)^{*}(N_{\mathbf{k}'}+1)B_{01}(t)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)^{*}(N_{\mathbf{k}'}+1)B_{10}(t) \tag{761}$$

$$+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}\left(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}\left(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t)\Big) \tag{762}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)^{*}\big(B_{10}(t)+B_{01}(t)\big)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)\big(-B_{10}(t)-B_0 \tag{763}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)^{*}\big(B_{10}(t)+B_{01}(t)\big)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)\big(B_{10}(t)+B_{01}(\tag{764}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)^{*}2\big(B_{10}(t)\big)^{\Re}-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)2\big(B_{10}(t)\big)^{\Re}\Big), \tag{765}$$

$$= \tfrac{(B_{10}(t))^{\Re}}{\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big(\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)^{*}-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s)\big)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\Big)\Big), \tag{766}$$

$$\left\langle B_{iz}(t,\tau)\, B_y(s,0)\right\rangle\!\!\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^{+}B_1^{-}(s,0)-B_1^{+}B_0^{-}(s,0)+B_{10}(s)-B_{01}(s)}{2\mathrm{i}}\right.\right. \tag{767}$$

$$= \frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^{+}B_1^{-}(s,0)-B_1^{+}B_0^{-}(s,0)+B_{10}(s)-B_{01}\right.\right. \tag{768}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^{+}B_1^{-}(s,0)-B_1^{+}B_0^{-}(s,0)\right)\right\rangle_B \tag{769}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big\langle\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}(s,0)-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}(s,0)+\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}(s,0) \tag{770}$$

$$-\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}(s,0)\Big\rangle \tag{771}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_0^{+}B_1^{-}(s,0)\right\rangle - e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_1^{+}B_0^{-}(s,0)\right\rangle\right. \tag{772}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left\langle b_{\mathbf{k}'}B_0^{+}B_1^{-}(s,0)\right\rangle - e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left\langle b_{\mathbf{k}'}B_1^{+}B_0^{-}(s,0)\right\rangle\right\rangle \tag{773}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\Big\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)\big\langle b_{\mathbf{k}'}^{\dagger}B_0^{+}B_1^{-}(s,0)\big\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)\big\langle b_{\mathbf{k}'}^{\dagger}B_1^{+}B_0^{-}(s,0)\big\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\big(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\big)^{*}\big\langle b_{\mathbf{k}'}B_0^{+}B_1^{-}(s,0)\big\rangle \tag{774}$$

$$+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}\left(\left(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{01}(s)\right)-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^{*}\left(\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{10}(s)\right)\Bigg) \tag{783}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}B_{01}(s)\,N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}(s)\,N_{\mathbf{k}'}\right)\right. \tag{784}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{01}(s)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{10}(s)\right)\right) \tag{785}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}(s)+B_{01}(s)\right)N_{\mathbf{k}'}\right. \tag{786}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)\left(B_{10}(s)+B_{01}(s)\right)(N_{\mathbf{k}'}+1)\right) \tag{787}$$

$$= \frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}(s)\,N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(s)\,(N_{\mathbf{k}'}+1\right. \tag{788}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(s)\,(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}(s)\,N_{\mathbf{k}'}\right. \tag{789}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(s)\,(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}(s)\,N_{\mathbf{k}'}\right. \tag{790}$$

$$= \mathrm{i}B_{10}^{\Re}(s)\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t)\right)\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)^{*}N_{\mathbf{k}'}\right). \tag{791}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^* \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{792}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \frac{1}{2}\left(\left(\exp\left(\chi_{10}\left(t\right)+\chi_{10}\left(s\right)\right)\right)^{\Re} U_{10}(t,s) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{793}$$

$$\left.+\left(\exp\left(\chi_{10}\left(t\right)+\chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}^*\left(t,s\right) \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{794}$$

$$-\left(\exp(\chi_{10}(t))\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re}\left(\exp(\chi_{01}(s))\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)^{\Re} \tag{795}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_y}(s)\right\rangle_B = -\tfrac{1}{2}\left(\left(\exp(\chi_{01}(t)+\chi_{01}(s))\right)^{\Re}U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{796}$$

$$\left.-\left(\exp(\chi_{10}(t)+\chi_{01}(s))\right)^{\Re}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{797}$$

$$+\left(\exp(\chi_{01}(t))\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im}\left(\exp(\chi_{10}(s))\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\right)^{\Im} \tag{798}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_y}(s)\right\rangle_B = \tfrac{1}{2}\left(\left(\exp(\chi_{10}(t)+\chi_{01}(s))\right)^{\Im}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{799}$$

$$\left.+\left(\exp(\chi_{01}(t)+\chi_{01}(s))\right)^{\Im}U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)+(B_{10}(t))^{\Re}(B_{10}(s))^{\Im} \tag{800}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\left(\left(\exp(\chi_{01}(t)+\chi_{10}(s))\right)^{\Im}U_{10}^*(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)-\left(v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right. \tag{801}$$

$$\left.+\left(\exp(\chi_{01}(t)+\chi_{01}(s))\right)^{\Im}U_{10}(t,s)\prod_{\mathbf{k}}\exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right)+v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)+(B_{10}(t))^{\Im}(B_{10}(s))^{\Re} \tag{802}$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_x}\left(s\right)\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}} B_{01}^{\Im}(s)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\frac{v_{1\mathbf{k}}(s)-v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{803}$$

$$\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_{iz}}\left(s\right)\right\rangle_B = \mathrm{i}B_{10}^{\Im}\left(t\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right)\right)^* N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1) \tag{804}$$

$$\left\langle \widetilde{B_{iz}}\left(t\right)\widetilde{B_y}\left(s\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(s\right)\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(s\right)-v_{0\mathbf{k}}\left(s\right)}{\omega_{\mathbf{k}}}\right)^* N \tag{805}$$

$$\left\langle \widetilde{B_y}\left(t\right)\widetilde{B_{iz}}\left(s\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(t\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right)\right)^* N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(s\right)\right)(N_{\mathbf{k}}+1)\,e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^* \tag{806}$$

$$\tag{807}$$

$$\tag{808}$$

$$\tag{809}$$

$$\tag{810}$$

$$\tag{811}$$

$$\tag{812}$$

$$\tag{813}$$

$$\tag{814}$$

$$\tag{815}$$

$$\tag{816}$$

The spectral density is defined in the usual way:

$$J_i\left(\omega\right) \equiv \sum_{\mathbf{k}} \left|g_{i\mathbf{k}}\right|^2 \delta\left(\omega - \omega_{\mathbf{k}}\right), \tag{817}$$

$$v_{i\mathbf{k}}\left(t\right) = g_{i\mathbf{k}} F_i\left(\omega_{\mathbf{k}}, t\right). \tag{818}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i\left(\omega\right)}{\omega} \mathrm{d}\omega. \tag{819}$$

The integral version of the correlation functions are given by:

$$
\begin{aligned}
\chi_{10}\left(t\right) &= \int_0^\infty \frac{\sqrt{J_1^*\left(\omega\right) J_0\left(\omega\right)} F_1^*\left(\omega,t\right) F_0\left(\omega,t\right) - \sqrt{J_1\left(\omega\right) J_0^*\left(\omega\right)} F_1\left(\omega,t\right) F_0^*\left(\omega,t\right)}{2\omega^2} \mathrm{d}\omega \\
U_{10}\left(t,s\right) &= \exp\left(i \left(\int_0^\infty \frac{\left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)\right) \left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,s\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,s\right)\right)^* \exp\left(i\omega\tau\right)}{\omega^2} \mathrm{d}\omega \right)^{\Im} \\
B_{10}\left(t\right) &= \exp\left(\chi_{10}\left(t\right)\right) \exp\left(-\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)}{\omega} \right|^2 \coth\left(\frac{\beta\omega}{2}\right) \mathrm{d}\omega \right), \\
\xi^+\left(t,s\right) &= \exp\left(- \int_0^\infty \frac{\left| \left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)\right) \exp\left(i\omega\tau\right) + \sqrt{J_1\left(\omega\right)} F_1\left(\omega,s\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,s\right) \right|^2}{2\omega^2} \coth \right. \\
\xi^-\left(t,s\right) &= \exp\left(- \int_0^\infty \frac{\left| \left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)\right) \exp\left(i\omega\tau\right) - \left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega,s\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,s\right)\right) \right|^2}{2\omega^2} \mathrm{co} \right. \\
Q\left(\omega,t\right) &= \frac{\sqrt{J_1\left(\omega\right)} F_1\left(\omega,t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega,t\right)}{\omega} \\
\left\langle \widetilde{B_x}(t)\widetilde{B_x}(s) \right\rangle_B &= \frac{1}{2} \left(\left(\exp\left(\chi_{10}\left(t\right) + \chi_{10}\left(s\right)\right)\right)^{\Re} {}_{U_{10}(t,s)} \xi^+\left(t,s\right) + \left(\exp\left(\chi_{10}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) \right) {}_{-(B_{10}(t))^{\Re}(B_{01}(s))^{\Re}} \\
\left\langle \widetilde{B_y}(t)\widetilde{B_y}(s) \right\rangle_B &= -\frac{1}{2} \left(\left(\exp\left(\chi_{01}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}\left(t,s\right) \xi^+\left(t,s\right) - \left(\exp\left(\chi_{10}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Re} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) \right) {}_{+(B_{01}(t))^{\Im}(B_{10}(s}} \\
\left\langle \widetilde{B_x}(t)\widetilde{B_y}(s) \right\rangle_B &= \frac{1}{2} \left(\left(\exp\left(\chi_{10}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Im} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) + \left(\exp\left(\chi_{01}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Im} U_{10}\left(t,s\right) \xi^+\left(t,s\right) \right) + \left(B_{10}\left(t\right)\right)^{\Re} (B_{1} \\
\left\langle \widetilde{B_y}(t)\widetilde{B_x}(s) \right\rangle_B &= \frac{1}{2} \left(\left(\exp\left(\chi_{01}\left(t\right) + \chi_{10}\left(s\right)\right)\right)^{\Im} U_{10}^*\left(t,s\right) \xi^-\left(t,s\right) + \left(\exp\left(\chi_{01}\left(t\right) + \chi_{01}\left(s\right)\right)\right)^{\Im} U_{10}\left(t,s\right) \xi^+\left(t,s\right) \right) + \left(B_{10}\left(t\right)\right)^{\Im} (B_{1} \\
\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s) \right\rangle_B &= \int_0^\infty \left(\sqrt{J_i(\omega) J_j^*(\omega)} (1-F_i(\omega,t)) \left(1-F_j^*(\omega,s)\right) e^{\mathrm{i}\omega\tau} N(\omega) + \sqrt{J_i^*(\omega) J_j(\omega)} (1-F_i^*(\omega,t)) (1-F_j(\omega,s)) e^{-\mathrm{i}\omega\tau} (N(\omega)+1) \right) \mathrm{d}\omega, \\
\left\langle \widetilde{B_{iz}}\left(t\right) \widetilde{B_x}\left(s\right) \right\rangle_B &= \mathrm{i} B_{01}^{\Im}\left(s\right) \int_0^\infty \left(\sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,t\right)\right) Q^*\left(\omega,s\right) N\left(\omega\right) e^{\mathrm{i}\omega\tau} - \sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,t\right)\right) Q\left(\omega,s\right) e^{-\mathrm{i}\omega\tau} \left(N\left(\omega\right) + 1\right) \right) \\
\left\langle \widetilde{B_x}\left(t\right) \widetilde{B_{iz}}\left(s\right) \right\rangle_B &= \mathrm{i} B_{01}^{\Im}\left(t\right) \int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,s\right)\right) Q\left(\omega,t\right) N\left(\omega\right) e^{\mathrm{i}\omega\tau} - \sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,s\right)\right) Q^*\left(\omega,t\right) e^{-\mathrm{i}\omega\tau} \left(N\left(\omega\right) + 1\right) \right) \\
\left\langle \widetilde{B_{iz}}\left(t\right) \widetilde{B_y}\left(s\right) \right\rangle_B &= i B_{10}^{\Re}\left(s\right) \int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,s\right)\right) Q\left(\omega,s\right) \left(N\left(\omega\right) + 1\right) e^{-\mathrm{i}\omega\tau} - \sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,s\right)\right) Q^*\left(\omega,s\right) e^{\mathrm{i}\omega\tau} N\left(\omega\right) \right) \\
\left\langle \widetilde{B_y}\left(t\right) \widetilde{B_{iz}}\left(s\right) \right\rangle_B &= i B_{10}^{\Re}\left(t\right) \int_0^\infty \left(\sqrt{J_i^*\left(\omega\right)} \left(1 - F_i^*\left(\omega,s\right)\right) Q\left(\omega,t\right) N\left(\omega\right) e^{\mathrm{i}\omega\tau} - \sqrt{J_i\left(\omega\right)} \left(1 - F_i\left(\omega,s\right)\right) Q^*\left(\omega,t\right) e^{-\mathrm{i}\omega\tau} \left(N\left(\omega\right) + 1\right) \right)
\end{aligned}
$$

$$\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{jz}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^* \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(s\right)\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left(N_{\mathbf{k}} + 1\right), \tag{820}$$

$$= \sum_{\mathbf{k}} \Big((g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t))(g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}}, s))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t))^* (g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}}, s)) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}} + 1)\Big) \tag{821}$$

$$= \sum_{\mathbf{k}} \Big(g_{i\mathbf{k}}(1 - F_i\left(\omega_{\mathbf{k}}, t\right))\, g^*_{j\mathbf{k}}(1 - F_j(\omega_{\mathbf{k}}, s))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g^*_{i\mathbf{k}}(1 - F_i\left(\omega_{\mathbf{k}}, t\right))^* g_{j\mathbf{k}}(1 - F_j(\omega_{\mathbf{k}}, s)) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}} + 1)\Big) \tag{822}$$

$$\approx \int_0^\infty \left(\sqrt{J_i(\omega) J_j^*(\omega)}(1 - F_i(\omega, t))\big(1 - F_j^*(\omega, s)\big) e^{\mathrm{i}\omega\tau} N(\omega) + \sqrt{J_i^*(\omega) J_j(\omega)}(1 - F_i^*(\omega, t))(1 - F_j(\omega, s)) e^{-\mathrm{i}\omega\tau}(N(\omega) + 1)\right) \mathrm{d}\omega, \tag{823}$$

$$\chi_{10}\left(t\right) = \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v^*_{1\mathbf{k}}\left(t\right) v_{0\mathbf{k}}\left(t\right) - v_{1\mathbf{k}}\left(t\right) v^*_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}^2} \right) \tag{824}$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g^*_{1\mathbf{k}} F^*_1\left(\omega_{\mathbf{k}}, t\right) g_{0\mathbf{k}} F_0\left(\omega_{\mathbf{k}}, t\right) - g_{1\mathbf{k}} F_1\left(\omega_{\mathbf{k}}, t\right) g^*_{0\mathbf{k}} F^*_0\left(\omega_{\mathbf{k}}, t\right)}{\omega_{\mathbf{k}}^2} \right) \tag{825}$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g^*_{1\mathbf{k}} g_{0\mathbf{k}} F^*_1\left(\omega_{\mathbf{k}}, t\right) F_0\left(\omega_{\mathbf{k}}, t\right) - g_{1\mathbf{k}} g^*_{0\mathbf{k}} F_1\left(\omega_{\mathbf{k}}, t\right) F^*_0\left(\omega_{\mathbf{k}}, t\right)}{\omega_{\mathbf{k}}^2} \right) \tag{826}$$

$$\approx \int_0^\infty \frac{\sqrt{J^*_1\left(\omega\right) J_0\left(\omega\right)} F^*_1\left(\omega, t\right) F_0\left(\omega, t\right) - \sqrt{J_1\left(\omega\right) J^*_0\left(\omega\right)} F_1\left(\omega, t\right) F^*_0\left(\omega, t\right)}{2\omega^2} \mathrm{d}\omega, \tag{827}$$

$$U_{10}\left(t, s\right) = \prod_{\mathbf{k}} \exp\left(i \left(\frac{\left(v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)\right)^* \exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \tag{828}$$

$$= \exp\left(i \sum_{\mathbf{k}} \left(\frac{\left(v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)\right)^* \exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \tag{829}$$

$$= \exp\left(i \left(\sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)\right)\left(v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)\right)^* \exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \tag{830}$$

$$= \exp\left(i \left(\sum_{\mathbf{k}} \frac{\left(g_{1\mathbf{k}} F_1\left(\omega_{\mathbf{k}}, t\right) - g_{0\mathbf{k}} F_0\left(\omega_{\mathbf{k}}, t\right)\right)\left(g_{1\mathbf{k}} F_1\left(\omega_{\mathbf{k}}, s\right) - g_{0\mathbf{k}} F_0\left(\omega_{\mathbf{k}}, s\right)\right)^* \exp\left(i\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \tag{831}$$

$$\approx \exp\left(i \left(\int_0^\infty \frac{\left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega, t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega, t\right)\right)\left(\sqrt{J_1\left(\omega\right)} F_1\left(\omega, s\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega, s\right)\right)^* \exp\left(i\omega\tau\right)}{\omega^2} \mathrm{d}\omega \right)^{\Im} \right) \tag{832}$$

$$\tag{833}$$

$$B_{10}\left(t\right) = \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2} \left(\frac{v^*_{1\mathbf{k}}\left(t\right) v_{0\mathbf{k}}\left(t\right) - v_{1\mathbf{k}}\left(t\right) v^*_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right), \tag{834}$$

$$= \exp\left(\chi_{10}\left(t\right)\right) \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1\left(\omega_{\mathbf{k}}, t\right) - g_{0\mathbf{k}} F_0\left(\omega_{\mathbf{k}}, t\right)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{835}$$

$$\approx \exp\left(\chi_{10}\left(t\right)\right) \exp\left(-\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1\left(\omega\right)} F_1\left(\omega, t\right) - \sqrt{J_0\left(\omega\right)} F_0\left(\omega, t\right)}{\omega} \right|^2 \coth\left(\frac{\beta\omega}{2} \right) \mathrm{d}\omega \right) \tag{836}$$

$$\tag{837}$$

$$\xi^+\left(t, s\right) = \prod_{\mathbf{k}} \exp\left(-\frac{\left|\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{838}$$

$$= \exp\left(-\sum_{\mathbf{k}} \frac{\left|\left(v_{1\mathbf{k}}\left(t\right) - v_{0\mathbf{k}}\left(t\right)\right)\exp\left(i\omega_{\mathbf{k}}\tau\right) + v_{1\mathbf{k}}\left(s\right) - v_{0\mathbf{k}}\left(s\right)\right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{839}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda + \mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = 0. \tag{878}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S\left|\pm\right\rangle = \lambda_{\pm}\left|\pm\right\rangle$. Using this notation is possible to write $H_{\bar{S}} = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\begin{aligned}
\widetilde{A_i}\left(\tau\right) &= e^{\mathrm{i}\overline{H_{\bar{S}}}\tau} A_i e^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} && (879)\\
&= \sum_{w} e^{-\mathrm{i}w\tau} \mathscr{A}_i\left(w\right). && (880)
\end{aligned}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.
In order to use the equation (880) to descompose the equation (355) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U\left(t\right) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ like:

$$\begin{aligned}
U\left(t\right) &\equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right) && (881)\\
&= \mathbb{I} + \sum_{n=1}^{\infty} \left(-\mathrm{i}\right)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H\left(t_1\right) H\left(t_2\right) ... H\left(t_n\right). && (882)
\end{aligned}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E\left(t\right)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right) \equiv \exp\left(-\mathrm{i}tH_E\left(t\right)\right)$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E\left(t\right) = H_E^{(0)}\left(t\right) + H_E^{(1)}\left(t\right) + H_E^{(2)}\left(t\right) + ...$ so we can write:

$$U\left(t\right) = \exp\left(-\mathrm{i}t\left(H_E^{(0)}\left(t\right) + H_E^{(1)}\left(t\right) + H_E^{(2)}\left(t\right) + ...\right)\right). \tag{883}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ and $U\left(t\right)$ then equating the terms of the same power. The lowest terms are:

$$\begin{aligned}
H_E^{(0)}\left(t\right) &= \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}\left(t'\right)\mathrm{d}t', && (884)\\
H_E^{(1)}\left(t\right) &= -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}\left(t'\right), \overline{H_{\bar{S}}}\left(t''\right)\right], && (885)\\
H_E^{(2)}\left(t\right) &= \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}\left(t'\right), \overline{H_{\bar{S}}}\left(t''\right)\right], \overline{H_{\bar{S}}}\left(t'''\right)\right] + \left[\left[\overline{H_{\bar{S}}}\left(t'''\right), \overline{H_{\bar{S}}}\left(t''\right)\right], \overline{H_{\bar{S}}}\left(t'\right)\right]\right). && (886)
\end{aligned}$$

In this case the Fourier decomposition using the expansion of $H_E\left(t\right)$ is:

$$\begin{aligned}
\widetilde{A_i}\left(t\right) &= U^{\dagger}\left(t\right) A_i\left(t\right) U\left(t\right) && (887)\\
\widetilde{A_i}\left(t\right) &= e^{\mathrm{i}H_E(t)t} A_i\left(t\right) e^{-\mathrm{i}H_E(t)t} && (888)\\
&= \sum_{w(t)} e^{-\mathrm{i}w(t)t} \mathscr{A}_i\left(w\left(t\right)\right). && (889)
\end{aligned}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau,t)$ using the Magnus expansion generates:

$$
\begin{aligned}
\widetilde{A_j}(t-\tau,t) &= U(t)\,U^{\dagger}(t-\tau)\,A_j(t)\,U(t-\tau)\,U^{\dagger}(t) && (890)\\
&= e^{-\mathrm{i}tH_E(t)}e^{\mathrm{i}(t-\tau)H_E(t-\tau)}A_j(t)\,e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}e^{\mathrm{i}tH_E(t)} && (891)\\
&= e^{-\mathrm{i}tH_E(t)}\left(\sum_{w'(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j(w(t-\tau))\right)e^{\mathrm{i}tH_E(t)} && (892)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j(w(t-\tau),w'(t)) && (893)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j(w(t-\tau),w'(t)) && (894)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t)) && (895)
\end{aligned}
$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}(t-\tau)$ and $\overline{H_E}(t)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (880) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V}=\{|+\rangle,|-\rangle\}$ in the following way:

$$
A_i = \sum_{\alpha,\beta\in\mathrm{V}} \langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|. \tag{896}
$$

Given that $[|+\rangle\langle+|,|-\rangle\langle-|]=0$, then using the Zassenhaus formula we obtain:

$$
\begin{aligned}
e^{\mathrm{i}\overline{H_E}\tau} &= e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} && (897)\\
&= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} && (898)\\
&= \left(|-\rangle\langle-|+e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) && (899)\\
&= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|. && (900)
\end{aligned}
$$

Calculating the transformation (880) directly using the previous relationship we find that:

$$
\begin{aligned}
U^{\dagger}(\tau)\,A_i(\tau)U(\tau) &= \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in\mathrm{V}}\langle\alpha|A_i(\tau)|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) && (901)\\
&= \langle+|A_i(\tau)|+\rangle|+\rangle\langle+|+e^{\mathrm{i}\eta\tau}\langle+|A_i(\tau)|-\rangle|+\rangle\langle-|+e^{-\mathrm{i}\eta\tau}\langle-|A_i(\tau)|+\rangle|-\rangle\langle+|+\langle-|A_i(\tau)|-\rangle|-\rangle\langle-|. && (902)\\
&= \mathscr{A}_i(0)+\mathscr{A}_i(-w)\,e^{\mathrm{i}w\tau}+\mathscr{A}_i(w)\,e^{-\mathrm{i}w\tau} && (903)
\end{aligned}
$$

Here $w=\lambda_+-\lambda-$. Comparing the RHS of the equations (880) and the explicit expression for $\widetilde{A_i}(\tau)$ in (888), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$
\begin{aligned}
\mathscr{A}_i(0) &= \langle+|A_i(\tau)|+\rangle|+\rangle\langle+|+\langle-|A_i(\tau)|-\rangle|-\rangle\langle-|, && (904)\\
\mathscr{A}_i(-w) &= \langle+|A_i(\tau)|-\rangle|+\rangle\langle-|, && (905)\\
\mathscr{A}_i(w) &= \langle-|A_i(\tau)|+\rangle|-\rangle\langle+|. && (906)
\end{aligned}
$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau)=\widetilde{A_i}^{\dagger}(\tau)$ and $\widetilde{B_i}(\tau)=\widetilde{B_i}^{\dagger}(\tau)$ we can use the equation (880) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ij}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big]+\mathscr{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \tag{907}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{908}$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right)\right]\Big) \tag{909}$$

Given that $\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right)=\mathscr{A}_j^{\dagger}\left(-w\left(t-\tau\right),-w'\left(t\right)\right)$ from the Fourier decomposition (880) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\mathscr{B}_{ij}\left(\tau\right)=\mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B\right) \tag{910}$$

$$=\mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B\right). \tag{911}$$

Let's define:

$$\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right)=\mathscr{A}_{jww'}\left(t-\tau,t\right) \tag{912}$$

The master equation can be re-written in the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{913}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \tag{914}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{915}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \tag{916}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{917}$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \tag{918}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{919}$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right]\Big) \tag{920}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\textstyle\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\mathrm{Tr}_B\Big(\Big[A_i,\widetilde{B_i}(\tau)\widetilde{B_j}(0)\rho_B e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\Big] \tag{921}$$

$$-\Big[A_i,\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\Big]\Big) \tag{922}$$

Given that if we define:

$$D_{ijww'}\left(t-\tau,t\right)=C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right) \tag{923}$$

then

$$D_{ijww'}^{\dagger}\left(t-\tau,t\right)=\left(C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right)^{\dagger} \tag{924}$$

$$=\mathscr{B}_{ij}^{*}\left(\tau\right)C_i\left(t\right)C_j\left(t-\tau\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}^{\dagger}\left(t-\tau,t\right) \tag{925}$$

We used the fact that $C_i(t), C_j(t-\tau)$ are real. Now let's consider the following trace recalling that $\mathrm{Tr}(A)^* = \mathrm{Tr}(A^\dagger)$ so:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\right) &= \mathrm{Tr}_B\left(e^{-\mathrm{i}\tau H_B(\tau)}B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\right) &(926)\\
&= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B e^{-\mathrm{i}\tau H_B(\tau)}\right) \text{ (by cyclic permutivity of trace)} &(927)\\
&= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i e^{-\mathrm{i}\tau H_B(\tau)}\rho_B\right) \text{ (by commutativity of } e^{-\mathrm{i}\tau H_B(\tau)} \text{ and } \rho_B) &(928)\\
&= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) \text{ (by definition of time evolution)} &(929)\\
&= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) &(930)\\
&= \mathrm{Tr}_B\left(\rho_B B_j\widetilde{B_i}(\tau)\right) &(931)\\
&= \mathrm{Tr}_B\left(\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^\dagger\right) \text{ (by definition of adjoint)} &(932)\\
&= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^* &(933)\\
&= \mathscr{B}_{ij}^*(\tau) &(934)
\end{aligned}$$

So we can write the master equation like:

$$\begin{aligned}
\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} &= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] &(935)\\
&-\mathscr{B}_{ij}^*(\tau)\left[A_i,\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j^\dagger(w(t-\tau),w'(t))\right]\Big) &(936)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\left(\left[A_i, D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\right]\right) &(937)
\end{aligned}$$

Let's define the response matrix in the following way.

$$\mathscr{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau,t) \tag{938}$$

Then the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)\mathscr{D}_{ijww'}^\dagger(t)\right]\right) \tag{939}$$

If we extend the upper limit of integration to ∞ in the equation (938) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$\begin{aligned}
e^{-V}\frac{\mathrm{d}\overline{\rho}_S(t)}{\mathrm{d}t}e^V &= \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^V\right)}{\mathrm{d}t} &(940)\\
&= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} &(941)\\
&= -\mathrm{i}e^{-\mathrm{V}}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]e^V - \sum_{ijww'}\int_0^t \mathrm{d}\tau\Big(e^{-V}\left[A_i,D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right]e^V - e^{-V}\Big[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\Big]e^V\Big). &(942)
\end{aligned}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V} = e^{-V}\overline{A\mathbb{I}B}e^{V} \tag{943}$$
$$= e^{-V}\overline{A}e^{V}e^{-V}\bar{B}e^{V} \tag{944}$$
$$= \left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\bar{B}e^{V}\right) \tag{945}$$
$$= AB. \tag{946}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V} = e^{-V}\overline{(AB-BA)}e^{V} \tag{947}$$
$$= e^{-V}\overline{AB}e^{V} - e^{-V}\overline{BA}e^{V} \tag{948}$$
$$= AB - BA \tag{949}$$
$$= [A,B]. \tag{950}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}_{ijww'}^{\dagger}\left(t\right)\right]\right)e^{V} \tag{951}$$
$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]e^{V} - \sum_{ijww'}\left(e^{-V}\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right]e^{V} - e^{-V}\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}_{ijww'}^{\dagger}\left(t\right)\right]e^{V}\right) \tag{952}$$
$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}\left(t\right)\mathscr{D}_{ijww'}^{\dagger}\left(t\right)e^{V}\right]\right) \tag{953}$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}(t)\,e^{V}e^{-V}\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}(t)e^{V}e^{-V}\mathscr{D}_{ijww'}^{\dagger}(t)e^{V}\right]\right) \tag{954}$$
$$= -\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho_S\left(t\right)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}\left(t\right)e^{V}\rho_S\left(t\right)\right] - \left[e^{-V}A_ie^{V},\rho_S\left(t\right)e^{-V}\mathscr{D}_{ijww'}^{\dagger}\left(t\right)e^{V}\right]\right). \tag{955}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (939) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (939) reproduces the results of the reference [1], for the latter case we have that $i,j \in \{1,2,3\}$ and $\omega \in (0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{956}$$

After performing the transformation (24) on the Hamiltonian (956) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)\,|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, \tag{957}$$
$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), \tag{958}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{959}$$

The Hamiltonian (957) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\left|1\middle\rangle\middle\langle 1\right| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)\left|1\middle\rangle\middle\langle 1\right| - \frac{\delta}{2}|0\rangle\langle 0| \tag{960}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{961}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \tag{962}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \tag{963}$$

$$\lambda_{\pm} = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} \tag{964}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{965}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{966}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \tag{967}$$

$$\lambda_{\pm} = \frac{R \pm \eta}{2}. \tag{968}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{969}$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{970}$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$.
Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon+\eta}{2}, \tag{971}$$
$$\lambda_- = \frac{\epsilon-\eta}{2}, \tag{972}$$
$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \tag{973}$$
$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, \tag{974}$$
$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}, \tag{975}$$
$$\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}. \tag{976}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{977}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{978}$$

So the value of $\tan(\theta)$ using (978) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}+1} \tag{979}$$
$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{980}$$
$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{981}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (905)-(906) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{982}$$
$$= 2\sin(\theta)\cos(\theta) \tag{983}$$
$$= \sin(2\theta)\,, \tag{984}$$
$$\langle -|\sigma_x|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{985}$$
$$= -2\sin(\theta)\cos(\theta) \tag{986}$$
$$= -\sin(2\theta)\,, \tag{987}$$
$$\langle -|\sigma_x|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{988}$$
$$= \cos^2(\theta) - \sin^2(\theta) \tag{989}$$
$$= \cos(2\theta)\,, \tag{990}$$
$$\langle +|\sigma_y|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{991}$$
$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{992}$$
$$= 0, \tag{993}$$
$$\langle -|\sigma_y|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{994}$$
$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{995}$$
$$= 0, \tag{996}$$
$$\langle -|\sigma_y|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{997}$$
$$= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) \tag{998}$$
$$= \mathrm{i}. \tag{999}$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1000}$$
$$= \cos(\theta)\cos(\theta) \tag{1001}$$
$$= \cos^2(\theta)\,, \tag{1002}$$
$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{1003}$$
$$= \sin(\theta)\sin(\theta) \tag{1004}$$
$$= \sin^2(\theta)\,, \tag{1005}$$
$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1006}$$
$$= -\sin(\theta)\cos(\theta) \tag{1007}$$
$$= -\sin(\theta)\cos(\theta)\,. \tag{1008}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), \tag{1009}$$
$$A_1(\eta) = \cos(2\theta)\,|-\rangle\langle+|, \tag{1010}$$
$$A_2(0) = 0, \tag{1011}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{1012}$$
$$A_3(0) = \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-|, \tag{1013}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)\,|-\rangle\langle+|. \tag{1014}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (942) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (923) is equivalent to:

$$\mathscr{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau, t) \tag{1015}$$
$$= \int_0^t \mathrm{d}\tau C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathscr{A}_j\left(w(t-\tau), w'(t)\right) \tag{1016}$$
$$= \int_0^t \mathrm{d}\tau C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)} \mathscr{A}_j(w, w'). \tag{1017}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (396) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau). \tag{1018}$$

So the response matrix can be rewritten as:

$$\mathscr{D}_{ijww'}(t) = \left(\int_0^t \mathrm{d}\tau \Lambda'_{ij}(\tau)\, e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)}\right) \mathscr{A}_j(w, w') \tag{1019}$$

Let's define the response function like:

$$K_{ij}(w, w', t) = \int_0^t C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau)\, e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1020}$$
$$= \int_0^t \Lambda'_{ij}(\tau)\, e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1021}$$
$$= K_{ijww'}(t). \tag{1022}$$

Then we have the following equivalence:

$$\mathscr{D}_{ijww'}(t) = K_{ijww'}(t)\, \mathscr{A}_j(w, w') \tag{1023}$$
$$= K_{ijww'}(t)\, \mathscr{A}_{jww'} \tag{1024}$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right)\right]\right) \tag{1025}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i,K_{ijww'}\left(t\right)\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)K^{*}_{ijww'}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{1026}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(K_{ijww'}\left(t\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}\left(t\right)\right] - K^{*}_{ijww'}\left(t\right)\left[A_i,\overline{\rho_S}\left(t\right)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{1027}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t)+\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - \left(K^{\Re}_{ijww'}(t)-\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{1028}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'} K^{\Re}_{ijww'}(t)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t)-\overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] - \mathrm{i}\sum_{ijww'} K^{\Im}_{ijww'}(t)\left[A_i,\mathscr{A}_{jww'}\overline{\rho_S}(t)+\overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] \tag{1029}$$

Using the notation of the master equation (939), we can say that $C_1\left(t\right) = \frac{\Omega}{2} = C_2\left(t\right)$ and $C_3\left(t\right) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B\left(t\right) = B$. Taking the equations(792)-(816) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (396) are equal to:

$$\left\langle\widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left((g_{1\mathbf{k}} - v_{1\mathbf{k}}) (g_{1\mathbf{k}} - v_{1\mathbf{k}})^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{1\mathbf{k}} - v_{1\mathbf{k}})^* (g_{1\mathbf{k}} - v_{1\mathbf{k}}) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \tag{1030}$$

$$= \sum_{\mathbf{k}} |g_{1\mathbf{k}} - v_{1\mathbf{k}}|^2 \left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \tag{1031}$$

$$\approx \int_0^\infty J_1(\omega) (1 - F_1(\omega))^2 \left(e^{\mathrm{i}\omega\tau} N(\omega) + e^{-\mathrm{i}\omega\tau} (N(\omega) + 1) \right) \mathrm{d}\omega \tag{1032}$$

$$G_\pm(\omega, \tau) = e^{\mathrm{i}\omega\tau} N(\omega) + e^{-\mathrm{i}\omega\tau} (N(\omega) + 1) \tag{1033}$$

$$\left\langle\widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B \approx \int_0^\infty J_1(\omega) (1 - F_1(\omega))^2 G_+(\omega, t) \, \mathrm{d}\omega \tag{1034}$$

$$\chi_{10}(t) = 0 \text{ (because } v_{0\mathbf{k}}(t) = 0 \text{ for all } \mathbf{k}) \tag{1035}$$

$$U_{10}(t, s) = \prod_{\mathbf{k}} \exp \left(i \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \tag{1036}$$

$$= \prod_{\mathbf{k}} \exp \left(i \left(\frac{v_{1\mathbf{k}}^2(t) \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \tag{1037}$$

$$= \prod_{\mathbf{k}} \exp \left(i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right) \tag{1038}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\Big(\textstyle\prod_{\mathbf{k}} \exp\Big(i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\Big) \prod_{\mathbf{k}} \exp\Big(-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big) + \prod_{\mathbf{k}} \exp\Big(-i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\Big) \prod_{\mathbf{k}} \exp\Big(-\frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2} \mathrm{cot} \tag{1039}$$

$$-\Big(\exp\Big(-\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\Big(\Big|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big|^2\Big)\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big)\Big(\exp\Big(-\tfrac{1}{2}\sum_{\mathbf{k}}\Big(\Big|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\Big|^2\Big)\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1040}$$

$$= \tfrac{1}{2}\Big(\textstyle\prod_{\mathbf{k}}\exp\Big(i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big) + \prod_{\mathbf{k}}\exp\Big(-i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1041}$$

$$-\Big(\exp\Big(-\textstyle\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1042}$$

$$|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)\pm v_{1\mathbf{k}}|^2 = v_{1\mathbf{k}}^2|\exp(i\omega_{\mathbf{k}}\tau)\pm 1|^2 \tag{1043}$$

$$= v_{1\mathbf{k}}^2|\cos(\omega_{\mathbf{k}}\tau)+i\sin(\omega_{\mathbf{k}}\tau)\pm 1|^2 \tag{1044}$$

$$= v_{1\mathbf{k}}^2 \left((1 \pm \cos(\omega_{\mathbf{k}}\tau))^2 + \sin^2(\omega_{\mathbf{k}}\tau) \right) \tag{1045}$$

$$= 2v_{1\mathbf{k}}^2 (1 \pm \cos(\omega_{\mathbf{k}}\tau)) \tag{1046}$$

$$B \equiv \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{1047}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\Big(\exp\Big(\textstyle\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big) + \exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}}\exp(i\omega_{\mathbf{k}}\tau)-v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1048}$$

$$-\Big(\exp\Big(-\textstyle\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) \tag{1049}$$

$$\phi(\tau) = \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \tag{1050}$$

$$\approx \int_0^\infty \frac{J_1(\omega) F_1^2(\omega)}{\omega^2} \left(-\mathrm{i}\sin(\omega\tau) + \cos(\omega\tau) \coth\left(\frac{\beta\omega}{2} \right) \right) \mathrm{d}\omega \tag{1051}$$

$$= \int_0^\infty \frac{J_1(\omega) F_1^2(\omega)}{\omega^2} G_+(\omega, \tau) \, \mathrm{d}\omega \tag{1052}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\Big(\exp\Big(\textstyle\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{2v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big) + \exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{2v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) - B^2 \tag{1053}$$

$$= \tfrac{1}{2}\Big(\exp\Big(\textstyle\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big) + \exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\Big) - B^2 \tag{1054}$$

$$= \tfrac{1}{2}\Big(\exp\Big(-\textstyle\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\exp\Big(\sum_{\mathbf{k}} i\frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2 \cos(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big) + \exp\Big(-\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\Big(\frac{\beta\omega_{\mathbf{k}}}{2}\Big)\Big)\exp\Big(\sum_{\mathbf{k}} -i\frac{v_{1\mathbf{k}}^2 \sin(\omega}{\omega_{\mathbf{k}}^2} \tag{1055}$$

$$= \frac{B^2}{2} \left(e^{-\phi(\tau)} + e^{\phi(\tau)} - 2 \right) \tag{1056}$$

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{1097}$$
$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1098}$$
$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{1099}$$
$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right), \tag{1100}$$
$$\Lambda'_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)(1 - F(\omega))^2 G_+(\tau), \tag{1101}$$
$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)(1 - F(\omega))\mathrm{i}G_-(\tau), \tag{1102}$$
$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \tag{1103}$$
$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \tag{1104}$$
$$= \Lambda'_{13}(\tau) \tag{1105}$$
$$= \Lambda'_{31}(\tau) \tag{1106}$$
$$= 0. \tag{1107}$$

Finally taking the Hamiltonian (956) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that Det $\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, Tr $\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (336) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{1108}$$
$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{1109}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (432) under a time-independent approach providing similar results.
Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (939) and using the fact that

$$\widetilde{A_j}(t-\tau, t) = U(\tau)A_jU(-\tau) \tag{1110}$$
$$= \sum_w e^{\mathrm{i}w\tau}\mathscr{A}_j(-w) \tag{1111}$$
$$= \sum_w e^{-\mathrm{i}w\tau}\mathscr{A}_j(w). \tag{1112}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_w \gamma_{ij}(w,t)\left[A_i, \mathscr{A}_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)\mathscr{A}_j^\dagger(w)\right] \tag{1113}$$
$$-\sum_{ij}\sum_w S_{ij}(w,t)\left[A_i, \mathscr{A}_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)\mathscr{A}_j^\dagger(w)\right]. \tag{1114}$$

where $\mathscr{A}_j^\dagger(w) = \mathscr{A}_j(-w)$, as we can see the equation (1114) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2K_{ij}^{\Re}(w,t)$ and $S_{ij}(w,t) = K_{ij}^{\Im}(w,t)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}(w,t) = \int_0^t \Lambda'_{ij}(\tau)\, e^{\mathrm{i}w\tau}\mathrm{d}\tau.$$

The fact $\mathscr{A}_j^\dagger(w) = \mathscr{A}_j(-w)$ can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$\begin{aligned}
e^{\mathrm{i}\overline{H_S}(t-\tau)} &= e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)(t-\tau)} && (1115)\\
&= e^{\mathrm{i}\lambda_+|+\rangle\langle+|(t-\tau)}e^{\mathrm{i}\lambda_-|-\rangle\langle-|(t-\tau)} && (1116)\\
&= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) && (1117)\\
&= e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|. && (1118)
\end{aligned}$$

Where $\lambda_+, \lambda_-$are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (880) of (904)-(906) directly using the previous relationship we find that:

$$\begin{aligned}
\widetilde{A_i(0)}(t-\tau) &= \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) && (1119)\\
&= \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|, && (1120)\\
\widetilde{A_i(w)}(t-\tau) &= \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|-\rangle|+\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) && (1121)\\
&= \langle+|A_i|-\rangle|+\rangle\langle-|e^{\mathrm{i}w(t-\tau)}, && (1122)\\
\widetilde{A_i(-w)}(t-\tau) &= \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle-|A_i|+\rangle|-\rangle\langle+|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) && (1123)\\
&= \langle-|A_i|+\rangle|-\rangle\langle+|e^{-\mathrm{i}w(t-\tau)}. && (1124)
\end{aligned}$$

Here $w = \lambda_+ - \lambda-$. So we can see that for the equation (890) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w \neq w'$ then $A'_j(w,w') = 0$ so:

$$\begin{aligned}
\widetilde{A_j}(t-\tau,t) &= U(t)\,U^\dagger(t-\tau)\,A_j(t)\,U(t-\tau)\,U^\dagger(t) && (1125)\\
&= U(t)\left(\sum_{w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}A_j(w(t-\tau))\right)U^\dagger(t) && (1126)\\
&= \sum_{w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}U(t)\,A_j(w(t-\tau))\,U^\dagger(t) && (1127)\\
&= \sum_{w'(t),w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j(w(t-\tau),w'(t)) && (1128)\\
&= \sum_{w'(t),w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_{jww'} && (1129)\\
&== \sum_{w'(t),w(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j(w)\,\delta_{ww'} && (1130)\\
&= \sum_{w} e^{-\mathrm{i}(t-\tau)w}e^{\mathrm{i}tw}A_j(w) && (1131)\\
&= \sum_{w} e^{\mathrm{i}\tau w}A_j(w) && (1132)\\
&= U^\dagger(-\tau)\,A_jU(-\tau) && (1133)
\end{aligned}$$

So using now as reference the equation (1029) and $A'_j(w,w')=0$ we can deduce that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijw}K^{\Re}_{ij}(w,t)\Big[A_i,A_j(w)\overline{\rho_S}(t)-\overline{\rho_S}(t)A^{\dagger}_j(w)\Big]-\mathrm{i}\sum_{ijw}K^{\Im}_{ij}(w,t)\Big[A_i,A_j(w)\overline{\rho_S}(t)+\overline{\rho_S}(t)A^{\dagger}_j(w)\Big] \tag{1134}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z=0$. The Hamiltonian studied is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}+g^{*}_{\mathbf{k}}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega\left(t\right)}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{1135}$$

If $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then $B\left(\tau\right)=B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (1135) using the Hamiltonian of the equation (1) we can say that $\delta=\varepsilon_1\left(t\right)$, $\varepsilon_0\left(t\right)=0$, $V_{10}\left(t\right)=\frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then, in this case and using the equation (225) and (236) we obtain the following transformed Hamiltonians:

$$\overline{H_S}=\left(\delta+R_1\right)|1\rangle\langle 1|+\frac{B\sigma_x}{2}\Omega\left(t\right), \tag{1136}$$

$$\overline{H_{\mathrm{I}}}=\frac{\Omega\left(t\right)}{2}\left(B_x\sigma_x+B_y\sigma_y\right). \tag{1137}$$

In this case $R_1=\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}=g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right)=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta+R_1=\delta'$. If $F\left(\omega_{\mathbf{k}}\right)=1$ and using the equations (1097)-(1104) we can deduce that the only terms that survive are $\Lambda_{11}\left(\tau\right)$ and $\Lambda_{22}\left(\tau\right)$. The phonon propagator for this case is:

$$\phi\left(\tau\right)=\int_0^{\infty}\frac{J\left(\omega\right)}{\omega^2}G_+\left(\tau\right)\mathrm{d}\omega. \tag{1138}$$

Writing $G_+\left(\tau\right)=\coth\left(\frac{\beta\omega}{2}\right)\cos\left(\omega\tau\right)-i\sin\left(\omega\tau\right)$ so (1138) can be written as:

$$\phi\left(\tau\right)=\int_0^{\infty}\frac{J\left(\omega\right)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos\left(\omega\tau\right)-i\sin\left(\omega\tau\right)\right)\mathrm{d}\omega. \tag{1139}$$

Writing the interaction Hamiltonian (1137) in the similar way to the equation (236) allow us to to write $A_1=\sigma_x$, $A_2=\sigma_y$, $B_1\left(t\right)=B_x$, $B_2\left(t\right)=B_y$ and $C_1\left(t\right)=\frac{\Omega(t)}{2}=C_2\left(t\right)$. Now taking the equation (225) with $\delta'|1\rangle\langle 1|=\frac{\delta'}{2}\sigma_z+\frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S}=\frac{\delta'}{2}\sigma_z+\frac{B\sigma_x}{2}\Omega\left(t\right). \tag{1140}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}\left(\tau\right)=\mathrm{Tr}_B\left(\widetilde{B_1}\left(\tau\right)\widetilde{B_1}\left(0\right)\rho_B\right) \tag{1141}$$

$$=\frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}-2\right), \tag{1142}$$

$$\Lambda_{22}\left(\tau\right)=\mathrm{Tr}_B\left(\widetilde{B_2}\left(\tau\right)\widetilde{B_2}\left(0\right)\rho_B\right) \tag{1143}$$

$$=\frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}\right). \tag{1144}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(432) is:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau \Big(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\Big[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\Big] \tag{1145}$$

$$+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\Big[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\Big]\Big). \tag{1146}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (1141) and (1144) on the equation (1146) we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{1147}$$

$$\left.+\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right). \tag{1148}$$

As we can see $\Big[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\Big]^\dagger = \Big[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\Big]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (396) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right). \tag{1149}$$

Using the equation (939), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)), \rho_S(t)\right] - \frac{1}{2}\sum_w \gamma_{33}(w,t)\Big[A_3, A_3(w)\rho_S(t) - \rho_S(t)A_3^\dagger(w)\Big] \tag{1150}$$

$$-\sum_w S_{33}(w,t)\Big[A_3, A_3(w)\rho_S(t) + \rho_S(t)A_3^\dagger(w)\Big]\Bigg). \tag{1151}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)G_+(\tau), \tag{1152}$$

$$\Lambda_{33}(-\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)G_+(-\tau). \tag{1153}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (1151) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t))\right] - \sum_w \left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^*(w,t)\left[\rho_S(t)A_3(w), A_3\right]\right). \tag{1154}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{1155}$$
$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle), \tag{1156}$$
$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{1157}$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{1158}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1159}$$
$$= \frac{1}{2}, \tag{1160}$$
$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{1161}$$
$$= \frac{1}{2}, \tag{1162}$$
$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1163}$$
$$= -\frac{1}{2}. \tag{1164}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \tag{1165}$$
$$= \frac{\mathbb{I}}{2}, \tag{1166}$$
$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle +| \tag{1167}$$
$$= \frac{1}{4}(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}), \tag{1168}$$
$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle -| \tag{1169}$$
$$= \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y). \tag{1170}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}[\sigma_x, \rho_S(t))] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)\right] \tag{1171}$$
$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right]. \tag{1172}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega)\, G_+(\tau)\, e^{\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{1173}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{\mathrm{i}\widetilde{\Omega}\tau} \left((n(\omega)+1)\, \mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega)\, e^{\mathrm{i}\tau\omega} \right) \mathrm{d}\tau \mathrm{d}\omega \tag{1174}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{\mathrm{i}\widetilde{\Omega}\tau} (n(\omega)+1)\, \mathrm{e}^{-\mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1175}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, (n(\omega)+1)\, \mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1176}$$

$$= \int_0^\infty J(\omega)\, (n(\omega)+1)\, \pi\delta\left(\widetilde{\Omega} - \omega\right) \mathrm{d}\omega \tag{1177}$$

$$= \pi J\left(\widetilde{\Omega}\right) \left(n\left(\widetilde{\Omega}\right) + 1 \right), \tag{1178}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega)\, G_+(\tau)\, e^{-\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{1179}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{-\mathrm{i}\widetilde{\Omega}\tau} \left((n(\omega)+1)\, \mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega)\, e^{\mathrm{i}\tau\omega} \right) \mathrm{d}\tau \mathrm{d}\omega \tag{1180}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, e^{-\mathrm{i}\widetilde{\Omega}\tau} n(\omega)\, \mathrm{e}^{\mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1181}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, n(\omega)\, \mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau + \mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1182}$$

$$= \int_0^\infty J(\omega)\, n(\omega)\, \pi\delta\left(-\widetilde{\Omega} + \omega\right) \mathrm{d}\omega \tag{1183}$$

$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right). \tag{1184}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s} = \pi\delta(\varepsilon) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1171) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S(t)\right)] - \frac{\pi}{8} J\left(\widetilde{\Omega}\right) \left(\left(n\left(\widetilde{\Omega}\right) + 1 \right) \left[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\, \rho_S(t)\right] + n\left(\widetilde{\Omega}\right) \left[\sigma_z, (\sigma_z - \mathrm{i}\sigma_y)\, \rho_S(t)\right] \right) \tag{1185}$$

$$- \frac{\pi}{8} J\left(\widetilde{\Omega}\right) \left(\left(n\left(\widetilde{\Omega}\right) + 1 \right) \left[\rho_S(t)\, (\sigma_z + \mathrm{i}\sigma_y), \sigma_z\right] + n\left(\widetilde{\Omega}\right) \left[\rho_S(t)\, (\sigma_z - \mathrm{i}\sigma_y), \sigma_z\right] \right). \tag{1186}$$

This is the same result than the equation ($S17$), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x, \rho_S(t)\right)] - \frac{\pi}{8} J(\Omega(t))\, ((n(\Omega(t)) + 1)[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)] + n(\Omega(t))[\sigma_z, (\sigma_z - \mathrm{i}\sigma_y)\, \rho_S(t)]) \tag{1187}$$

$$- \frac{\pi}{8} J(\Omega(t))\, ((n(\Omega(t)) + 1)\, [\rho_S(t)\, (\sigma_z + \mathrm{i}\sigma_y), \sigma_z] + n(\Omega(t))\, [\rho_S(t)\, (\sigma_z - \mathrm{i}\sigma_y), \sigma_z]). \tag{1188}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1189}$$
$$H_S(t) = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|, \tag{1190}$$
$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right), \tag{1191}$$
$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}. \tag{1192}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{1193}$$

At first let's obtain $e^{\pm V}$ under the transformation (1193), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)$, so the equation (1193) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$\begin{aligned}
e^{\pm V} &= e^{\pm\sum_n |n\rangle\langle n|\hat{\varphi}_n} &(1194)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!} + \ldots &(1195)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + \ldots &(1196)\\
&= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + \ldots &(1197)\\
&= \sum_n |n\rangle\langle n|\left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \ldots\right) &(1198)\\
&= \sum_n |n\rangle\langle n|e^{\pm\hat{\varphi}_n} &(1199)
\end{aligned}$$

Given that $\left[f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}, f_{nu'\mathbf{k}'}b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*}b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D(\pm\alpha_{nu\mathbf{k}}) = e^{\pm\left(\alpha_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$\begin{aligned}
e^{\pm\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} &= \prod_u e^{\pm\sum_{\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} &(1200)\\
&= \prod_u\left(\prod_{\mathbf{k}} e^{\pm\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}\right) &(1201)\\
&= \prod_u\left(\prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}})\right) &(1202)\\
&= \prod_{u\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}) &(1203)\\
&= \prod_u B_{nu\pm} &(1204)\\
B_{nu\pm} &\equiv \prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}) &(1205)
\end{aligned}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1206}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1189):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1207}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1208}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1209}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1210}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1211}$$

$$= |0\rangle\langle 0|. \tag{1212}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1213}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1214}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1215}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, \tag{1216}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1217}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1218}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1219}$$

$$\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1220}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1221}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right) \left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), \tag{1222}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, \tag{1223}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots \tag{1224}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) \tag{1225}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots \tag{1226}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) \tag{1227}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots \tag{1228}$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1229}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1230}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{1231}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) \tag{1232}$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{1233}$$

The transformed Hamiltonians of the equations (1190) to (1192) written in terms of (1207) to (1231) are:

$$
\begin{aligned}
\overline{H_S\left(t\right)} &= \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} && (1234)\\
&= \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} && (1235)\\
&= \sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|\prod_u\left(B_{mu+}B_{nu-}\right) && (1236)\\
\overline{H_I} &= \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) && (1237)\\
&= \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right) + \ldots\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) && (1238)\\
&= \prod_u B_{0u+}\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{0u-}+\prod_u B_{1u+}\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{1u-}+\ldots && (1239)\\
&= \sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\prod_u B_{0u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{0u-}+g_{0u\mathbf{k}}^{*}\prod_u B_{0u+}b_{u\mathbf{k}}\prod_u B_{0u-}\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\prod_u B_{1u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{1u-}+g_{1u\mathbf{k}}^{*}\prod_u B_{1u+}b_{u\mathbf{k}}\prod_u B_{1u-}\right)+\ldots && (1240)\\
&= \sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\ldots && (1241)\\
&= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1242)\\
&= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1243)\\
\overline{H_B} &= \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) && (1244)
\end{aligned}
$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$
\begin{aligned}
\overline{H} &= \sum_n \varepsilon_n(t)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_u\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) && (1245)\\
&+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1246)
\end{aligned}
$$

Let's define the following functions:

$$
\begin{aligned}
R_n\left(t\right) &= \sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1247)\\
B_{z,n}\left(t\right) &= \sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)b_{u\mathbf{k}}^{\dagger}+\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)^{*}b_{u\mathbf{k}}\right) && (1248)
\end{aligned}
$$

Using the previous functions we have that (1245) can be re-written in the following way:

$$
\overline{H}=\sum_n \varepsilon_n(t)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_u\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_n R_n(t)|n\rangle\langle n|+\sum_n B_{z,n}(t)|n\rangle\langle n| \quad (1249)
$$

$$(1250)$$

Now in order to separate the elements of the hamiltonian (1250) let's follow the references of the equations (225) and (236) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle\prod_{u}\left(B_{mu+}B_{nu-}\right)\right\rangle_{\overline{H_0}}=\left\langle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*}+\alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1251}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*}+\alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right)\left\langle\prod_{u\mathbf{k}}D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1252}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1253}$$

$$\equiv B_{nm} \tag{1254}$$

$$\left\langle\prod_{u}\left(B_{nu+}B_{mu-}\right)\right\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{nu\mathbf{k}}^{*}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v_{mu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1255}$$

$$=B_{nm}^{*} \tag{1256}$$

Following the reference [4] we define:

$$J_{nm}=\prod_{u}\left(B_{mu+}B_{nu-}\right)-B_{nm} \tag{1257}$$

As we can see:

$$J_{nm}^{\dagger}=\left(\prod_{u}\left(B_{mu+}B_{nu-}\right)-B_{nm}\right)^{\dagger} \tag{1258}$$

$$=\prod_{u}\left(B_{nu+}B_{mu-}\right)-B_{nm}^{*} \tag{1259}$$

$$=\prod_{u}\left(B_{nu+}B_{mu-}\right)-B_{mn} \tag{1260}$$

$$=J_{mn} \tag{1261}$$

We can separate the Hamiltonian (1250) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}\left(t\right)}=\sum_{n}\left(\varepsilon_{n}\left(t\right)+R_{n}\right)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} \tag{1262}$$

$$\overline{H_{\bar{I}}}=\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|J_{nm}+\sum_{n}B_{z,n}\left(t\right)|n\rangle\langle n|, \tag{1263}$$

$$\overline{H_{\bar{B}}}=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} \tag{1264}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A\leq A_{\mathrm{B}}\equiv-\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right)+\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}+O\left(\left\langle\overline{H_{\bar{I}}^{2}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_{B}}}\right). \tag{1265}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right)=C\left(R_{0},R_{1},...,R_{d-1},B_{01},...,B_{0(d-1)},...,B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}$. So our approach will be based on the derivation respect to $v_{nu\mathbf{k}}^{\Re}$ and $v_{nu\mathbf{k}}^{\Im}$. The Hamiltonian $\overline{H_{\bar{S}}\left(t\right)}$ can be written like:

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1266}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1267}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\frac{g_{nu\mathbf{k}}v_{nu\mathbf{k}}^*+g_{nu\mathbf{k}}^*v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1268}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1269}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^*\right)v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(g_{nu\mathbf{k}}^*-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1270}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1271}$$

$$v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*=\left(v_{mu\mathbf{k}}^{\Re}-\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right)\left(v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right)-\left(v_{mu\mathbf{k}}^{\Re}+\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right)\left(v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right) \tag{1272}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-\mathrm{i}v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Im}\right) \tag{1273}$$

$$-\left(v_{mu\mathbf{k}}^{\Re}v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}+\mathrm{i}v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Im}\right) \tag{1274}$$

$$=2\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}\right) \tag{1275}$$

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^*\right)v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(g_{nu\mathbf{k}}^*-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1276}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1277}$$

$$\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2=\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)^* \tag{1278}$$

$$=\left|v_{mu\mathbf{k}}\right|^2+\left|v_{nu\mathbf{k}}\right|^2-\left(v_{nu\mathbf{k}}v_{mu\mathbf{k}}^*+v_{nu\mathbf{k}}^*v_{mu\mathbf{k}}\right) \tag{1279}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}\right)^2+\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-\left(v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right)\left(v_{mu\mathbf{k}}^{\Re}-\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right) \tag{1280}$$

$$-\left(v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}\right)\left(v_{mu\mathbf{k}}^{\Re}+\mathrm{i}v_{mu\mathbf{k}}^{\Im}\right) \tag{1281}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}\right)^2+\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-2\left(v_{nu\mathbf{k}}^{\Re}v_{mu\mathbf{k}}^{\Re}+v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Im}\right) \tag{1282}$$

$$=\left(v_{mu\mathbf{k}}^{\Re}-v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}-v_{nu\mathbf{k}}^{\Im}\right)^2 \tag{1283}$$

$$R_n(t)=\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1284}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^*\right)v_{nu\mathbf{k}}^{\Re}-\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(g_{nu\mathbf{k}}^*-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1285}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{nu\mathbf{k}}^{\Im}\right)^2-2g_{nu\mathbf{k}}^{\Re}v_{nu\mathbf{k}}^{\Re}-2g_{nu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}\right) \tag{1286}$$

$$B_{mn}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v_{mu\mathbf{k}}^*v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^*\right)}{2\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1287}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Im}v_{mu\mathbf{k}}^{\Re}-v_{mu\mathbf{k}}^{\Im}v_{nu\mathbf{k}}^{\Re}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{mu\mathbf{k}}^{\Re}-v_{nu\mathbf{k}}^{\Re}\right)^2+\left(v_{mu\mathbf{k}}^{\Im}-v_{nu\mathbf{k}}^{\Im}\right)^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1288}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1289}$$

$$= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1290}$$

$$= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1291}$$

$$\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1292}$$

$$= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1293}$$

$$= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1294}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) \tag{1295}$$

$$= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1296}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1297}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{-\mathrm{i} v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1298}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\mathrm{i} v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1299}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \tag{1300}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \tag{1301}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^\dagger}{\partial a} \tag{1302}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1303}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1304}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1305}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1306}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1307}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1308}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1309}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1310}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1311}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1312}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1313}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1314}$$

Introducing this derivates in the equation (1289) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1315}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1316}$$

$$= 0 \tag{1317}$$

We can obtain the variational parameters:

$$
\begin{aligned}
& -2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) && (1318)\\
&=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) && (1319)
\end{aligned}
$$

$$
v_{nu\mathbf{k}}^{\Re}=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1320}
$$

$$
=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1321}
$$

Let’s consider the imaginary part of the variation parameters

$$
\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{nu\mathbf{k}}^{\Im}} &= \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v_{nu\mathbf{k}}^{\Im}-g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}\right)+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}-\left(v_{nu\mathbf{k}}^{\Im}-v_{mu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. && (1322)\\
&\quad\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}-\left(v_{nu\mathbf{k}}^{\Im}-v_{mu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) && (1323)\\
&=0 && (1324)
\end{aligned}
$$

Rearranging we obtain

$$
\begin{aligned}
& -2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) && (1325)\\
&= -2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) && (1326)\\
v_{nu\mathbf{k}}^{\Im} &= \frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} && (1327)\\
&= \frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1328)\\
v_{nu\mathbf{k}} &= v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im} && (1329)\\
&= \frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1330)\\
& \mathrm{i}\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1331)\\
&= \frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1332)\\
& -\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1333)\\
& -\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1334)\\
&= \frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1335)\\
& -\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1336)\\
&= \frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1337)
\end{aligned}
$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$
\begin{aligned}
e^{V}\rho(0)\,e^{-V} &= \left(\sum_n |n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_n |n\rangle\langle n|B_{n+}\right) && (1338)\\
0 &= \left(B_0^{+}|0\rangle\langle 0|B_0^{-}\right)\otimes\rho_B && (1339)\\
0 &= \rho(0) && (1340)
\end{aligned}
$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1341}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1342}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\, \overline{\rho_S}(t)\, U(t)\,, \text{ where} \tag{1343}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1344}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1345}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1346}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1347}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1348}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1349}$$
$$= B_{n-}^{\dagger} B_{m+}^{\dagger} - B_n B_m \tag{1350}$$
$$= B_{n+} B_{m-} - B_n B_m \tag{1351}$$
$$= B_{nm} \tag{1352}$$

So we can say that the set of matrices (1345) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1345) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t)\, |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t)\, |n\rangle\langle n|, \tag{1353}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m} \left(V_{nm}(t)\, |n\rangle\langle m| B_{nm} + V_{mn}(t)\, |m\rangle\langle n| B_{mn}\right) \tag{1354}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m} \left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t)\, B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1355}$$
$$\left. + \Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t)\, B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1356}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m} \left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right) \sigma_{nm,y} \frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1357}$$
$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right) \sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1358}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m} \left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x} B_{nm,x} - \Im\left(V_{nm}(t)\right) \sigma_{nm,x} B_{nm,y} + \Re\left(V_{nm}(t)\right) \sigma_{nm,y} B_{nm,y}\right. \tag{1359}$$
$$\left. + \Im\left(V_{nm}(t)\right) \sigma_{nm,y} B_{nm,x}\right) \tag{1360}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1361}$$

Now consider the following set of operators,

$$A_{1,nm}\left(t\right)=\sigma_{nm,x}\left(1-\delta_{mn}\right) \tag{1362}$$
$$A_{2,nm}\left(t\right)=\sigma_{nm,y}\left(1-\delta_{mn}\right) \tag{1363}$$
$$A_{3,nm}\left(t\right)=\delta_{mn}|n\rangle\langle m| \tag{1364}$$
$$A_{4,nm}\left(t\right)=A_{2,mn}\left(t\right) \tag{1365}$$
$$A_{5,nm}\left(t\right)=A_{1,nm}\left(t\right) \tag{1366}$$
$$B_{1,nm}\left(t\right)=B_{nm,x} \tag{1367}$$
$$B_{2,nm}\left(t\right)=B_{nm,y} \tag{1368}$$
$$B_{3,nm}\left(t\right)=B_{z,n}\left(t\right) \tag{1369}$$
$$B_{4,nm}\left(t\right)=B_{1,nm}\left(t\right) \tag{1370}$$
$$B_{5,nm}\left(t\right)=B_{2,nm}\left(t\right) \tag{1371}$$
$$C_{1,nm}\left(t\right)=\Re\left(V_{nm}\left(t\right)\right) \tag{1372}$$
$$C_{2,nm}\left(t\right)=C_{1,nm}\left(t\right) \tag{1373}$$
$$C_{3,nm}\left(t\right)=1 \tag{1374}$$
$$C_{4,nm}\left(t\right)=\Im\left(V_{nm}\left(t\right)\right) \tag{1375}$$
$$C_{5,nm}\left(t\right)=-\Im\left(V_{nm}\left(t\right)\right) \tag{1376}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}\left(t\right)$ as:

$$\overline{H_I}=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(A_{jp}\otimes B_{jp}\left(t\right)\right) \tag{1377}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1361).

We write the interaction Hamiltonian transformed under (1341) as:

$$\widetilde{H_I}\left(t\right)=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1378}$$
$$\widetilde{A_{jp}}\left(t\right)=U^{\dagger}\left(t\right)A_{jp}U\left(t\right) \tag{1379}$$
$$\widetilde{B_{jp}}\left(t\right)=\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\left(t\right)\left(\mathrm{t}\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{1380}$$

Taking as reference state ρ_B and truncating at second order in $H_I\left(t\right)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}\left(t\right)}{\mathrm{d}t}=-\int_0^t\mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right),\left[\widetilde{H_I}\left(s\right),\widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}s \tag{1381}$$

Replacing the equation (1378) in (1381) we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}\left(t\right)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right),\left[\widetilde{H_I}\left(s\right),\widetilde{\rho_S}\left(t\right)\rho_B\right]\right]\mathrm{d}s \tag{1382}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right),\left[\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right),\widetilde{\rho_S}\left(t\right)\rho_B\right]\right]\mathrm{d}s \tag{1383}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right),\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\rho_S}\left(t\right)\rho_B\right. \tag{1384}$$

$$\left.-\widetilde{\rho_S}\left(t\right)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\right]\mathrm{d}s \tag{1385}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\rho_S}\left(t\right)\rho_B\right. \tag{1386}$$

$$-\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\widetilde{\rho_S}\left(t\right)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right) \tag{1387}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\rho_S}\left(t\right)\rho_B\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1388}$$

$$\left.+\widetilde{\rho_S}\left(t\right)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\sum_{j\in J,p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1389}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}\left(\tau\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1390}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1391}$$

Here $s\to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}\left(t\right)$ and elements related to the system given by $\widetilde{A_{jp}}\left(t\right)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(t\right)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}\left(t\right)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}\left(t\right)\right)$. The correlation functions relevant of the master equation (1389) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1392}$$

$$= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1393}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1394}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1395}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1396}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1397}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1398}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) \tag{1399}$$

$$= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1400}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1401}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1402}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1403}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1404}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1405}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1406}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1382)and (1389) and using the equations (1392)-(1406) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}\left(t\right)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right) - \Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\right)\right. \tag{1407}$$

$$\left. + C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right) - \Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1408}$$

$$= -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[\widetilde{A_{jp}}\left(t\right),\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right),\widetilde{A_{jp}}\left(t\right)\right]\right)\right) \tag{1409}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right),A_{j'p'}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}\left(t\right)A_{j'p'}\left(t-\tau,t\right),A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}\left(t\right)\right] \tag{1410}$$

For this case we used that $A_{jp}\left(t-\tau,t\right) = U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1411}$$
$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1412}$$
$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1413}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1414}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1415}$$

At first let's obtain e^V under the transformation (1415), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{1416}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + ... \tag{1417}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + ... \tag{1418}$$
$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + ...\right) \tag{1419}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1420}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+} \tag{1421}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1422}$$
$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1423}$$
$$= B_{\pm} \tag{1424}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_{-}$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+}\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{-}\right) \tag{1425}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1411):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1426}$$

$$= |0\rangle\langle 0|, \tag{1427}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1428}$$

$$= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, \tag{1429}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1430}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1431}$$

$$= |0\rangle\langle m|B^{-}\ m \neq 0, \tag{1432}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1433}$$

$$= |0\rangle\langle m|B^{+}\ m \neq 0, \tag{1434}$$

$$\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1435}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B^{+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B^{-} \tag{1436}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B^{+}b_{\mathbf{k}}^{\dagger}B^{-}\right)\left(B^{+}b_{\mathbf{k}}B^{-}\right) \tag{1437}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1438}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1439}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right) \tag{1440}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1441}$$

The transformed Hamiltonians of the equations (1412) to (1414) written in terms of (1426) to (1441) are:

$$
\begin{aligned}
\overline{H_{\bar{S}}\left(t\right)} &= \overline{\sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|} && (1442)\\
&= \overline{\sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n|} + \overline{\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|} && (1443)\\
&= \sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n| + \overline{\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n|} && (1444)\\
&= \sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)\overline{|0\rangle\langle n|} + V_{n0}\left(t\right)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)\overline{|m\rangle\langle n|} && (1445)\\
&= \sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)B^{-}|0\rangle\langle n| + V_{n0}\left(t\right)B^{+}|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| && (1446)\\
&= \sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| && (1447)\\
\overline{H_{I}} &= \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(\left(\sum_{n=0}\mu_{n}\left(t\right)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) && (1448)\\
&= \left(\mu_{0}\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\mu_{n}\left(t\right)|n\rangle\langle n|B^{+}\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) && (1449)\\
&= \mu_{0}\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_{n}\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}B^{+}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)B^{-} && (1450)\\
&= \mu_{0}\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_{n}\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1451)\\
\overline{H_{B}} &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) && (1452)
\end{aligned}
$$

Joining this terms allow us to write:

$$
\begin{aligned}
\overline{H} &= \sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| && (1453)\\
&+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) && (1454)\\
&+\sum_{n=0}\mu_{n}\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) - \sum_{n=1}\mu_{n}\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} && (1455)\\
&= \sum_{n=0}\varepsilon_{n}\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| && (1456)\\
&+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_{n}\left(t\right)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_{0}\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) && (1457)\\
&+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_{n}\left(t\right) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) && (1458)
\end{aligned}
$$

Let's define the following functions:

$$
\begin{aligned}
R_{n}\left(t\right) &= \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_{n}\left(t\right)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1459)\\
&= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_{n}\left(t\right)g_{\mathbf{k}}\right) && (1460)\\
B_{z,n}\left(t\right) &= \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_{n}\left(t\right) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) && (1461)
\end{aligned}
$$

Using the previous functions we have that (1458) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^- + V_{n0}(t)|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1462}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1} R_n|n\rangle\langle n| + \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1463}$$

Now in order to separate the elements of the hamiltonian (1463) let's follow the references of the equations (236) and (225) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| + \sum_{n=1} R_n|n\rangle\langle n| \tag{1464}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^+ - B\right)\right), \tag{1465}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{1466}$$

Here B is given by:

$$\begin{aligned} B &= \left\langle B^+\right\rangle \\ &= \left\langle B^-\right\rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1467}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1468}$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1469}$$

$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1470}$$

Using this set of hermitian operators to write the Hamiltonians (1412)-(1414)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1471}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \tag{1472}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1473}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \tag{1474}$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1475}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1476}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1477}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1478}$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1479}$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^+ - B\right)\right) \tag{1480}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\left(B^- - B\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\left(B^+ - B\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1481}$$

$$+ \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1482}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + \left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\right)\right. \tag{1483}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - \left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\right)\right) \tag{1484}$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1485}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^+ + B^- - 2B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B^- - B - B^+ + B\right)\Im(V_{0n}(t))\right. \tag{1486}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B - B^- + B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B - B^- + B - B^+\right)\Im(V_{0n}(t))\right)\right) + \sum_{n=1} B_{z,n}|n\rangle\langle n| \tag{1487}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1488}$$

$$+ \sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1489}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_S+H_B}\right)}\right)\right)+\left\langle\overline{H_I}\right\rangle_{\overline{H_S+H_B}}+O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1490}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)=C\left(R_1,R_2,...,R_{d-1},B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (254) and using the chain rule we obtain that:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}}+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, &(1491)\\ &=0 &(1492)\end{aligned}$$

Let's recall the equations (1459) and (1461), we can write them in terms of the variational parameters

$$\begin{aligned}B &= \exp\left(-\left(1/2\right)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) &(1493)\\ R_n &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) &(1494)\end{aligned}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\begin{aligned}\frac{\partial B}{\partial v_{\mathbf{k}}} &= -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B &(1495)\\ \frac{\partial R_n}{\partial v_{\mathbf{k}}} &= \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1496)\end{aligned}$$

Introducing this derivates in the equation (1491) give us:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right)+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1497)\\ &= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right)-\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) &(1498)\end{aligned}$$

We can obtain the variational parameters:

$$\begin{aligned}v_{\mathbf{k}} &= \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1499)\\ &= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1500)\end{aligned}$$

Now taking $v_{\mathbf{k}}=g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1501}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho(0)e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)(|0\rangle\langle 0| \otimes \rho_B)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1502}$$
$$0 = |0\rangle\langle 0| \otimes \rho_B \tag{1503}$$
$$0 = \rho(0) \tag{1504}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\,OU(t) \tag{1505}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}(t')\right). \tag{1506}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\,,\text{ where} \tag{1507}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{1508}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle 0| + \sum_{n=1}\left(\Re(V_{0n}(t))\,B_x\sigma_{0n,x} + \Re(V_{0n}(t))\,B_y\sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right. \tag{1509}$$
$$\left.+\Im(V_{0n}(t))\,B_x\sigma_{0n,y} - \Im(V_{0n}(t))\,B_y\sigma_{0n,x}\right) \tag{1510}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}}\mu_0(t)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1511}$$
$$B_{z,n} = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\ \text{if} n \neq 0 \tag{1512}$$

Now consider the following set of operators:

$$
\begin{aligned}
A_{1n}(t) &= \sigma_{0n,x} && (1513)\\
A_{2n}(t) &= \sigma_{0n,y} && (1514)\\
A_{3n}(t) &= |n\rangle\langle n| && (1515)\\
A_{4n}(t) &= A_{2n}(t) && (1516)\\
A_{5n}(t) &= A_{1n}(t) && (1517)\\
B_{1n}(t) &= B_x && (1518)\\
B_{2n}(t) &= B_y && (1519)\\
B_{3n}(t) &= B_{z,n} && (1520)\\
B_{4n}(t) &= B_{1n}(t) && (1521)\\
B_{5n}(t) &= B_{2n}(t) && (1522)\\
C_{10}(t) &= 0 && (1523)\\
C_{20}(t) &= 0 && (1524)\\
C_{40}(t) &= 0 && (1525)\\
C_{50}(t) &= 0 && (1526)\\
C_{30}(t) &= 1 && (1527)\\
C_{1n}(t) &= \Re\left(V_{0n}(t)\right) && (1528)\\
C_{2n}(t) &= C_{1n}(t) && (1529)\\
C_{3n}(t) &= 1 && (1530)\\
C_{4n}(t) &= \Im\left(V_{0n}(t)\right) && (1531)\\
C_{5n}(t) &= -\Im\left(V_{0n}(t)\right) && (1532)
\end{aligned}
$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$
\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn}\otimes B_{jn}(t)\right) \tag{1533}
$$

Here $J = \{1,2,3,4,5\}$.

We write the interaction Hamiltonian transformed under (1505) as:

$$
\begin{aligned}
\widetilde{H_I}(t) &= \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) && (1534)\\
\widetilde{A_i}(t) &= U^{\dagger}(t)\,A_i U(t) && (1535)\\
\widetilde{B_i}(t) &= \mathrm{e}^{\mathrm{i}H_B t}B_i(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} && (1536)
\end{aligned}
$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\overline{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1537}
$$

Replacing the equation (1534)in (1537)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1538}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1539}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1540}$$

$$\left. -\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1541}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1542}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1543}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1544}$$

$$\left. +\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1545}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1546}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1547}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1545) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1548}$$
$$= \left\langle \widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1549}$$
$$= \Lambda_{jnj'n'}(\tau) \tag{1550}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1551}$$
$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1552}$$
$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1553}$$
$$= \Lambda_{j'n'jn}(-\tau) \tag{1554}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1555}$$
$$= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1556}$$
$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1557}$$
$$= \Lambda_{jnj'n'}(\tau) \tag{1558}$$
$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1559}$$
$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1560}$$
$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1561}$$
$$= \Lambda_{j'n'jn}(-\tau) \tag{1562}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1538)and (1545) and using the equations (1548)-(1562) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1563}$$

$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1564}$$

$$= -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t), \widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s), \widetilde{A_{jn}}(t)\right]\right)\right) \tag{1565}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t), A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t), A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] \tag{1566}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (432) as expected.

VIII. BIBLIOGRAPHY

[1] McCutcheon D P S, Dattani N S, Gauger E M, Lovett B W and Nazir A 2011 Phys. Rev. B 84 081305
[2] Dara P S McCutcheon and Ahsan Nazir 2010 New J. Phys. 12 113042
[3] Supplement: Theoretical model of phonon induced dephasing. A.J. Ramsay ey al 2009.
[4] Felix A Pollock et al 2013 New J. Phys. 15 075018

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