# A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between


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We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| + V_{01}(t)|0\rangle\langle 1|, \tag{2}$$

$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

## I. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $\mathrm{e}^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}(t)$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}(t). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i(t) \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V(t) = |0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $\mathrm{e}^{\pm V(t)}$ using the notation (6) will give us the following result:

$$
\begin{aligned}
\mathrm{e}^{\pm V(t)} &= \mathrm{e}^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t))} && (10)\\
&= \mathbb{I} \pm \left(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)\right) + \frac{\left(\pm\left(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)\right)\right)^2}{2!} + ... && (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm \left(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)\right) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2(t)}{2!} + ... && (12)\\
&= |0\rangle\langle 0|\left(\mathbb{I} \pm \hat{\varphi}_0(t) + \frac{\hat{\varphi}_0^2(t)}{2!} \pm ...\right) + |1\rangle\langle 1|\left(\mathbb{I} \pm \hat{\varphi}_1(t) + \frac{\hat{\varphi}_1^2(t)}{2!} \pm ...\right) && (13)\\
&= |0\rangle\langle 0|\mathrm{e}^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1|\mathrm{e}^{\pm\hat{\varphi}_1(t)} && (14)\\
&= |0\rangle\langle 0|\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{0\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1|\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{1\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_0^{\pm}(t) + |1\rangle\langle 1|B_1^{\pm}(t), && (16)\\
B_i^{\pm}(t) &\equiv \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\,\mathrm{e}^{rY}\,\mathrm{e}^{-\frac{r^2}{2}[X,Y]}\,\mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\,\mathrm{e}^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\cdots. \qquad (18)
$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}, i, j$ we can show plugging $r = 1$ in (18) the following result:

$$
\begin{aligned}
\mathrm{e}^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\cdots && (19)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{-\frac{1}{2}0}\cdots && (20)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathbb{I} && (21)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. && (22)
\end{aligned}
$$

By induction of this result we can write an expresion of $B_i^{\pm}(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$
\begin{aligned}
D(\pm v_{i\mathbf{k}}(t)) &\equiv \mathrm{e}^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, && (23)\\
B_i^{\pm}(t) &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). && (24)
\end{aligned}
$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$
\overline{O(t)} \equiv \mathrm{e}^{V(t)}O(t)\,\mathrm{e}^{-V(t)}. \qquad (25)
$$

We will use the following identities:

$$\begin{aligned}
\overline{|0\rangle\langle 0|\,(t)} &= \mathrm{e}^{V(t)}|0\rangle\langle 0|\mathrm{e}^{-V(t)} && (26)\\
&= \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (27)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (28)\\
&= |0\rangle\langle 0|B_0^+(t)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (29)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,B_1^-(t) && (30)\\
&= |0\rangle\langle 0|, && (31)\\
\overline{|1\rangle\langle 1|\,(t)} &= \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (32)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (33)\\
&= |1\rangle\langle 1|B_1^+(t)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (34)\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,B_0^-(t) + B_1^+(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) && (35)\\
&= B_1^+(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) && (36)\\
&= |1\rangle\langle 1|, && (37)\\
\overline{|0\rangle\langle 1|\,(t)} &= \mathrm{e}^{V(t)}|0\rangle\langle 1|\mathrm{e}^{-V(t)} && (38)\\
&= \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (39)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\,|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (40)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (41)\\
&= |0\rangle\langle 1|B_0^+(t)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (42)\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_0^+(t)\,B_0^-(t) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+(t)\,B_1^-(t) && (43)\\
&= |0\rangle\langle 1|B_0^+(t)\,B_1^-(t)\,, && (44)\\
\overline{|1\rangle\langle 0|\,(t)} &= \mathrm{e}^{V(t)}|1\rangle\langle 0|\mathrm{e}^{-V(t)} && (45)\\
&= \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (46)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\,|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (47)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (48)\\
&= |1\rangle\langle 0|0\rangle\langle 0|B_1^+(t)\,B_0^-(t) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+(t)\,B_1^-(t) && (49)\\
&= |1\rangle\langle 0|B_1^+(t)\,B_0^-(t)\,, && (50)\\
\overline{b_{\mathbf{k}}(t)} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}\mathrm{e}^{-V(t)} && (51)\\
&= \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (52)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| && (53)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,b_{\mathbf{k}}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,b_{\mathbf{k}}B_0^-(t) + |1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}B_1^-(t) && (54)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (55)\\
&= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} && (56)\\
&= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, && (57)\\
\overline{b_{\mathbf{k}}(t)}^{\dagger} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V(t)} && (58)\\
&= \left(|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)\right) && (59)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle 1| && (60)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) && (61)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) && (62)\\
&= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. && (63)
\end{aligned}$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+}\left(t\right)b_{\mathbf{k}}B_i^{-}\left(t\right) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}, \tag{64}$$

$$B_i^{+}\left(t\right)b_{\mathbf{k}}^{\dagger}B_i^{-}\left(t\right) = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}. \tag{65}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0\left(t\right)|0\rangle\langle 0|\left(t\right)} = \varepsilon_0\left(t\right)|0\rangle\langle 0|, \tag{66}$$

$$\overline{\varepsilon_1\left(t\right)|1\rangle\langle 1|\left(t\right)} = \varepsilon_1\left(t\right)|1\rangle\langle 1|, \tag{67}$$

$$\overline{V_{10}\left(t\right)|1\rangle\langle 0|\left(t\right)} = V_{10}\left(t\right)|1\rangle\langle 0|B_1^{+}\left(t\right)B_0^{-}\left(t\right), \tag{68}$$

$$\overline{V_{01}\left(t\right)|0\rangle\langle 1|\left(t\right)} = V_{01}\left(t\right)|0\rangle\langle 1|B_0^{+}\left(t\right)B_1^{-}\left(t\right), \tag{69}$$

$$\overline{\left(g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}\right)(t)} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\right)+g_{\mathbf{ik}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{70}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{72}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{73}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{74}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0|B_0^{+}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{-}\left(t\right)\right) \tag{75}$$

$$= |0\rangle\langle 0|B_0^{+}\left(t\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^{-}\left(t\right) \tag{76}$$

$$= |0\rangle\langle 0|B_0^{+}\left(t\right)\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_0^{-}\left(t\right) \tag{77}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)+g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right), \tag{78}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0|B_0^{+}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{-}\left(t\right)\right) \tag{79}$$

$$= |1\rangle\langle 1|B_1^{+}\left(t\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^{-}\left(t\right) \tag{80}$$

$$= |1\rangle\langle 1|B_1^{+}\left(t\right)\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_1^{-}\left(t\right) \tag{81}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)+g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right), \tag{82}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}(t)} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{-}\left(t\right)\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}\left(t\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_1^{-}\left(t\right)\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\sum_{j}|j\rangle\langle j|D\left(\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{j\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{j\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\mathbb{I}+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left( |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (92)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \quad (93)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left( v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left( v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right). \quad (94)$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)|0\rangle\langle 0|} + \overline{\varepsilon_1(t)|1\rangle\langle 1|} + \overline{V_{10}(t)|1\rangle\langle 0|} + \overline{V_{01}(t)|0\rangle\langle 1|} \quad (95)$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| B_1^+(t) B_0^-(t) + V_{01}(t)|0\rangle\langle 1| B_0^+(t) B_1^-(t), \quad (96)$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \quad (97)$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \quad (98)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (99)$$

$$= \sum_{\mathbf{k},i} |i\rangle\langle i| \left( g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (100)$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \quad (101)$$

$$= \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left( v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left( v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) \right) \quad (102)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left( |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left( v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left( v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) \right). \quad (103)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t)|j\rangle\langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left( \left( g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left( g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} + \frac{\left| v_{j\mathbf{k}}(t) \right|^2}{\omega_{\mathbf{k}}} - \left( g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right). \quad (104)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}. \quad (105)$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (106)$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}} \left( \left( g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left( g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^* b_{\mathbf{k}} \right), \quad (107)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right). \quad (108)$$

$\chi_{ij}(t)$ is an imaginary number so $\mathrm{e}^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right) & \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \\ \frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \tag{109}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{110}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{111}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{\mathrm{e}^{-\beta H_B}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_B}\right)}. \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{113}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{114}$$

So the product of the matrix representation of $b^{\dagger}$ and $b$ with $-\beta$ is:

$$-\beta\omega b^{\dagger}b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega\,|j\rangle\langle j|, \tag{116}$$

So the density matrix $\rho_B$ written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all $i, j$.

$$\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, , \tag{117}$$

$$\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{118}$$

The value of $\mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)$ is:

$$\begin{aligned}
\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (119)\\
&= \sum_{j_{\mathbf{k}}} \left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} && (120)\\
&= \frac{1}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \quad \text{(by geometric series)} && (121)\\
&\equiv f_{\text{Bose}-\text{Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), && (122)\\
\mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (123)\\
&= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (124)\\
&= \prod_{\mathbf{k}} f_{\text{Bose}-\text{Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). && (125)
\end{aligned}$$

So the density matrix of the bath is:

$$\begin{aligned}
\rho_B &= \frac{\mathrm{e}^{-\beta H_B}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_B}\right)} && (126)\\
&= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose}-\text{Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} && (127)\\
&= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose}-\text{Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. && (128)
\end{aligned}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{129}$$

$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. \tag{130}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (129)-(130):

$$\begin{aligned}
\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} =& \mathrm{Tr}\left(B_{iz}\left(t\right)\rho_B\right) && (131)\\
=& \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\rho_B\right) && (132)\\
=& \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\rho_B\right) && (133)\\
=& \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) && (134)\\
=& \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) && (135)\\
=& \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), && (136)\\
\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) =& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\ \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}\text{)} && (137)\\
=& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) && (138)\\
=& 0, && (139)\\
\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) =& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\ \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} && (140)\\
=& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) && (141)\\
=& 0. && (142)
\end{aligned}$$

we therefore find that:

$$\langle B_{iz}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0. \tag{143}$$

Another important expected value is $B\left(t\right)=\langle B^{\pm}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm}\left(t\right)=\;^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\begin{aligned}
\left\langle B^{\pm}\left(t\right)\right\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B^{\pm}\left(t\right)\right)=\mathrm{Tr}\left(B^{\pm}\left(t\right)\rho_B\right) && (144)\\
&= \mathrm{Tr}\left(\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) && (145)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\rho_B\right) && (146)\\
&= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\right\rangle. && (147)
\end{aligned}$$

Given that we can write a density operator as:

$$\rho=\int P\left(\alpha\right)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \tag{148}$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right)\mathrm{d}^2\alpha=1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P\left(\alpha\right)$ is given by:

$$\begin{aligned}
\langle A\rangle &= \mathrm{Tr}\left(A\rho\right) && (149)\\
&= \int P\left(\alpha\right)\langle\alpha\left|A\right|\alpha\rangle\mathrm{d}^2\alpha. && (150)
\end{aligned}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N}\mathrm{e}^{-\frac{|\alpha|^2}{N}}$ where $N = \left(\mathrm{e}^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency $\omega$ at inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$.

Using the integral representation (150) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$
\begin{aligned}
\langle D(h)\rangle &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle\alpha|D(h)|\alpha\rangle\mathrm{d}^2\alpha && (151)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(-\alpha)D(h)D(\alpha)|0\rangle\mathrm{d}^2\alpha, && (152)\\
D(h)D(\alpha) &= D(h+\alpha)\mathrm{e}^{\frac{1}{2}(h\alpha^*-h^*\alpha)}, && (153)\\
D(-\alpha)(D(h)D(\alpha)) &= D(-\alpha)D(h+\alpha)\mathrm{e}^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (154)\\
&= D(h)\mathrm{e}^{\frac{1}{2}(-\alpha(h+\alpha)^*+\alpha^*(h+\alpha))}\mathrm{e}^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (155)\\
&= D(\alpha)\mathrm{e}^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^*h+|\alpha|^2\right)}\mathrm{e}^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (156)\\
&= D(\alpha)\mathrm{e}^{(h\alpha^*-h^*\alpha)}, && (157)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(h)\mathrm{e}^{(h\alpha^*-h^*\alpha)}|0\rangle\mathrm{d}^2\alpha && (158)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^*-h^*\alpha)}\langle 0|D(h)|0\rangle\mathrm{d}^2\alpha && (159)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^*-h^*\alpha)}\langle 0|h\rangle\mathrm{d}^2\alpha, && (160)\\
|\alpha\rangle &= \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, && (161)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^*-h^*\alpha)}\langle 0|\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle\mathrm{d}^2\alpha && (162)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^*-h^*\alpha)}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha && (163)\\
&= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha, && (164)\\
\alpha &= x+\mathrm{i}y, && (165)\\
\langle D(h)\rangle &= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N}+h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\mathrm{d}x\mathrm{d}y && (166)\\
&= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{N}+hx-h^*x}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{N}+hx-h^*x}\mathrm{d}y, && (167)\\
-\frac{x^2}{N}+hx-h^*x &= -\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) && (168)\\
&= -\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2+\frac{N(h^*-h)^2}{4}, && (169)\\
-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y &= -\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) && (170)\\
&= -\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)-\frac{N(h+h^*)^2}{4}, && (171)\\
\sqrt{2\pi}\sigma &= \int_{-\infty}^{\infty}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x, && (172)
\end{aligned}
$$

$$\langle D(h)\rangle=\frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}\mathrm{d}x\mathrm{d}y, \tag{173}$$

$$\begin{aligned}\langle D\left(h\right)\rangle&=\frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}y &(174)\\
&=\frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 &(175)\\
&=\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}} &(176)\\
&=\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}} &(177)\\
&=\mathrm{e}^{-|h|^2\left(N+\frac{1}{2}\right)} &(178)\\
&=\mathrm{e}^{-|h|^2\left(\frac{1}{\mathrm{e}^{\beta\omega}-1}+\frac{1}{2}\right)} &(179)\\
&=\mathrm{e}^{-\frac{|h|^2}{2}\left(\frac{\mathrm{e}^{\beta\omega}+1}{\mathrm{e}^{\beta\omega}-1}\right)} &(180)\\
&=\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}. &(181)\end{aligned}$$

In the last line we used $\frac{\mathrm{e}^{\beta\omega}+1}{\mathrm{e}^{\beta\omega}-1}=\coth\left(\frac{\beta\omega}{2}\right)$. So the value of (146) using (181) is given by:

$$B=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{182}$$

We will now force $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$. We will also introduce the bath renormalizing driving in $\overline{H_S}\left(t\right)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+\left(t\right)\sigma^+$ and $B_i^-\left(t\right)\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}\left(t\right)$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$.

A final form of the terms of the Hamiltonian $\overline{H}\left(t\right)$ is:

$$\begin{aligned}\overline{H(t)}&=\textstyle\sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+\sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^*b_{\mathbf{k}}+\frac{\left|v_{j\mathbf{k}}(t)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}+g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) &(183)\\
&=\sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_{jj'}(t)+\sum_j|j\rangle\langle j|B_{jz}(t)+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+(t)\,B_{j'}^-(t)-B_{jj'}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} &(184)\\
&\equiv\overline{H_{\bar{S}}\left(t\right)}+\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{B}}}. &(185)\end{aligned}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\begin{aligned}\langle B_i^+\left(t\right)B_j^-\left(t\right)\rangle&=B_{ij}\left(t\right) &(186)\\
&=\left\langle\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right\rangle &(187)\\
&=\left\langle\prod_{\mathbf{k}}\left(D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle &(188)\end{aligned}$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \tag{189}$$

$$= \prod_{\mathbf{k}} \left\langle D\left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \mathrm{e}^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{190}$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{-\frac{1}{2}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \mathrm{e}^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{191}$$

$$= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \tag{192}$$

From the definition $B_{01}(t) = \langle B_0^+(t)\, B_1^-(t) \rangle$ using the displacement operator we have:

$$\langle B_0^+(t)\, B_1^-(t) \rangle = B_{01}(t) \tag{193}$$

$$= \left\langle \prod_{\mathbf{k}} D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D\left( -\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \tag{194}$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D\left( -\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \tag{195}$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \tag{196}$$

$$= \prod_{\mathbf{k}} \left( \left\langle D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \mathrm{e}^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \tag{197}$$

$$= \prod_{\mathbf{k}} \left( \mathrm{e}^{-\frac{1}{2}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \mathrm{e}^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \tag{198}$$

$$= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \tag{199}$$

We can check:

$$\langle B_0^+(t)\, B_1^-(t) \rangle = B_{01}(t) \tag{200}$$

$$= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{201}$$

$$= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left( \frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)^*} \tag{202}$$

$$= \langle B_1^+(t)\, B_0^-(t) \rangle^* \tag{203}$$

$$= B_{10}^*(t). \tag{204}$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}(t)} \equiv (\varepsilon_0(t) + R_0(t))\, |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t))\, |1\rangle\langle 1| + V_{10}(t)\, B_{10}\sigma^+ + V_{01}(t)\, B_{01}\sigma^-, \tag{205}$$

$$\overline{H_{\bar{I}}} \equiv V_{10}(t) \left( B_1^+(t)\, B_0^-(t) - B_{10}(t) \right) \sigma^+ + V_{01}(t) \left( B_0^+(t)\, B_1^-(t) - B_{01}(t) \right) \sigma^- + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t)\,, \tag{206}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{207}$$

$$= H_B. \tag{208}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, $H_B$, before transforming to the variational frame.
For the Hamiltonian (206) we can verify the condition $\langle \overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\begin{aligned}
\left\langle \overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{B}}}} =& \left\langle \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right) |n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}\left(t\right) |j\rangle\langle j'| \left(B_{j}^{+}\left(t\right) B_{j'}^{-}\left(t\right) - B_{jj'}\left(t\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (209)\\
=& \left\langle \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right) |n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}\left(t\right) |j\rangle\langle j'| \left(B_{j}^{+}\left(t\right) B_{j'}^{-}\left(t\right) - B_{jj'}\left(t\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} && (210)\\
=& \sum_{n\mathbf{k}} \left(\left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| \left(\left\langle V_{jj'}(t) B_{j}^{+}(t) B_{j'}^{-}(t)\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle V_{jj'}(t) B_{jj'}(t)\right\rangle_{\overline{H_{\bar{B}}}}\right) && (211)\\
=& \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) \left\langle b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} \left\langle b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}\left(t\right) \left(\left\langle B_{j}^{+}\left(t\right) B_{j'}^{-}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_{jj'}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}\right) && (212)\\
=& \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) \left\langle b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} \left\langle b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}\left(t\right) \left(B_{jj'}\left(t\right) - B_{jj'}\left(t\right)\right) && (213)\\
=& 0. && (214)
\end{aligned}$$

We used (143) and (199) to evaluate the expression. Let's consider the following Hermitian combinations:

$$\begin{aligned}
B_x\left(t\right) &= B_x^{\dagger}\left(t\right) && (215)\\
&= \frac{B_1^{+}\left(t\right) B_0^{-}\left(t\right) + B_0^{+}\left(t\right) B_1^{-}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}, && (216)\\
B_y\left(t\right) &= B_y^{\dagger}\left(t\right) && (217)\\
&= \frac{B_0^{+}\left(t\right) B_1^{-}\left(t\right) - B_1^{+}\left(t\right) B_0^{-}\left(t\right) + B_{10}\left(t\right) - B_{01}\left(t\right)}{2\mathrm{i}}, && (218)\\
B_{iz}\left(t\right) &= B_{iz}^{\dagger}\left(t\right) && (219)\\
&= \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right). && (220)
\end{aligned}$$

Writing the equations (205) and (206) using the previous combinations we obtain that:

$$\begin{aligned}
\overline{H_{\bar{S}}}\left(t\right) =& \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right) |j\rangle\langle j| + V_{10}\left(t\right) B_{10}\left(t\right) \sigma^{+} + V_{01}\left(t\right) B_{01}\left(t\right) \sigma^{-} && (221)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right) |j\rangle\langle j| + V_{10}\left(t\right) B_{10}\left(t\right) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}\left(t\right) B_{01}\left(t\right) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} && (222)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right) |j\rangle\langle j| + V_{10}\left(t\right) \left(B_{10}^{\Re}\left(t\right) + \mathrm{i}B_{10}^{\Im}\left(t\right)\right) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}\left(t\right) \left(B_{10}^{\Re}\left(t\right) - \mathrm{i}B_{10}^{\Im}\left(t\right)\right) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} && (223)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) + \mathrm{i}B_{10}^{\Im}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2}\right) && (224)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right)|j\rangle\langle j| + B_{10}^{\Re}(t)\left(\sigma_x \frac{V_{10}(t)+V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t)-V_{01}(t)}{2}\right) + \mathrm{i}B_{10}^{\Im}(t)\left(\sigma_x \frac{V_{10}(t)-V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t)+V_{01}(t)}{2}\right) && (225)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right) + R_j\left(t\right)\right) |j\rangle\langle j| + B_{10}^{\Re}\left(t\right) \left(\sigma_x V_{10}^{\Re}\left(t\right) - \sigma_y V_{10}^{\Im}\left(t\right)\right) + \mathrm{i}B_{10}^{\Im}\left(t\right) \left(\mathrm{i}\sigma_x V_{10}^{\Im}\left(t\right) + \mathrm{i}\sigma_y V_{10}^{\Re}\left(t\right)\right) && (226)\\
=& \left(\varepsilon_0\left(t\right) + R_0\left(t\right)\right) |0\rangle\langle 0| + \left(\varepsilon_1\left(t\right) + R_1\left(t\right)\right) |1\rangle\langle 1| + B_{10}^{\Re}\left(t\right) \left(\sigma_x V_{10}^{\Re}\left(t\right) - \sigma_y V_{10}^{\Im}\left(t\right)\right) + \mathrm{i}B_{10}^{\Im}\left(t\right) \left(\mathrm{i}\sigma_x V_{10}^{\Im}\left(t\right) + \mathrm{i}\sigma_y V_{10}^{\Re}\left(t\right)\right) && (227)\\
=& \left(\varepsilon_0(t) + R_0(t)\right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right) |1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t)\, V_{10}^{\Im}(t)\right) - \left(\sigma_x B_{10}^{\Im}(t)\, V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t)\, V_{10}^{\Re}(t)\right) && (228)\\
=& \left(\varepsilon_0(t) + R_0(t)\right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) - \sigma_y \left(B_{10}^{\Re}(t)\, V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\, V_{10}^{\Re}(t)\right) && (229)
\end{aligned}$$

$$= \left(\varepsilon_0(t) + R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \tag{230}$$

$$\overline{H_{\bar{I}}}(t) = V_{10}(t)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t)\right) + V_{01}(t)\left(\sigma^- B_0^+(t)\,B_1^-(t) - \sigma^- B_{01}(t)\right) + |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) \tag{231}$$

$$= |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) + \left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t)\right) + \left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^- B_0^+(t)\,B_1^-(t) - \sigma^- B_{01}(t)\right) \tag{232}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t) + \sigma^- B_0^+(t) B_1^-(t) - \sigma^- B_{01}(t)\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) - \sigma^- B_0^+(t) B_1^-(t) + \sigma^- B_{01}(t)\right) \tag{233}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_0^+(t)\,B_1^-(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{01}(t)\right) \tag{234}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_0^+(t)\,B_1^-(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{01}(t)\right) \tag{235}$$

$$+ \mathrm{i}V_{10}^{\Im}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2} B_{10}(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_0^+(t)\,B_1^-(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2} B_{01}(t)\right) \tag{236}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2}\right) \tag{237}$$

$$+ \mathrm{i}V_{10}^{\Im}(t)\left(\sigma_x \frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{238}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x \frac{B_1^+(t)B_0^-(t) - B_0^+(t)B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} - \sigma_y \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{239}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x \frac{B_1^+(t)B_0^-(t) - B_0^+(t)B_1^-(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{240}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x \frac{B_1^+ B_0^-(t) - B_0^+ B_1^-(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{241}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\left(-B_y(t)\right) - \sigma_y B_x(t)\right) \tag{242}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right). \tag{243}$$

## II. FREE-ENERGY MINIMIZATION

The true free energy $E_{\text{Free}}(t)$ is bounded by the Bogoliubov inequality:

$$E_{\text{Free}}(t) \leq E_{\text{Free,B}}(t) \equiv -\frac{1}{\beta}\ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right)\right) + \left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right). \tag{244}$$

We will optimize the set of variational parameters $\{v_{\mathbf{k}}(t)\}$ in order to minimize $E_{\text{Free,B}}(t)$ (i.e. to make it as close to the true free energy $E_{\text{Free}}(t)$ as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{\mathbf{k}}(t)\}$:

$$\frac{\partial E_{\text{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{245}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}(t), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = \mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}. \tag{246}$$

Then using the fact that $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) = \text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right). \tag{247}$$

So Eq. (245) becomes:

$$\frac{\partial E_{\text{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{248}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{249}$$

$$= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{250}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{251}$$

$$= 0 \quad \text{(by Eq. (245))}. \tag{252}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of $H_B$ that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} \tag{253}$$

$$= 0. \tag{254}$$

This means we need to impose:

$$\frac{\partial \text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{255}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left(\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1| + V_{10}(t)B_{10}(t)\sigma^+ + V_{01}(t)B_{01}(t)\sigma^-\right). \tag{256}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \text{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{257}$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{258}$$

$$\eta \equiv \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{259}$$

The solutions of the equation (257) are:

$$\lambda = \beta\frac{-\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{260}$$

$$= \beta\frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{261}$$

$$= -\beta\frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{262}$$

The value of $\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a $2\times 2$ matrix):

$$\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)=\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{\frac{\eta(t)\beta}{2}}+\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{-\frac{\eta(t)\beta}{2}} \tag{263}$$

$$=2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{264}$$

Given that $v_{i\mathbf{k}}\left(t\right)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)=A\left(\varepsilon\left(t\right),\eta\left(t\right)\right)$ to calculate $\frac{\partial\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}=\frac{\partial\left(2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} \tag{265}$$

$$=2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\,\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\sinh\left(\frac{\eta(t)\,\beta}{2}\right) \tag{266}$$

$$=-\beta\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{267}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)=0. \tag{268}$$

The derivates included in the expression given are related to:

$$\left\langle B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)\right\rangle=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{269}$$

$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\right)^{*}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{270}$$

$$=\left\langle B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right\rangle^{*}, \tag{271}$$

$$R_{i}\left(t\right)=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right) \tag{272}$$

$$=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^{2}}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right), \tag{273}$$

$$\left\langle B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)\right\rangle=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{274}$$

$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{2\omega_{\mathbf{k}}^{2}}}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{275}$$

$$v_{0\mathbf{k}}^{*}\left(t\right)v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)v_{1\mathbf{k}}^{*}\left(t\right)=\left(v_{0\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{1\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{1\mathbf{k}}^{\Im}\left(t\right)\right)-\left(v_{0\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{1\mathbf{k}}^{\Im}\left(t\right)\right) \tag{276}$$

$$=\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)\right) \tag{277}$$

$$-\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)\right) \tag{278}$$

$$=2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)\right), \tag{279}$$

$$\begin{aligned}
\left|v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right|^2 &= \left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)^* && (280)\\
&= \left|v_{1\mathbf{k}}(t)\right|^2+\left|v_{0\mathbf{k}}(t)\right|^2-\left(v_{1\mathbf{k}}(t)\,v_{0\mathbf{k}}^*(t)+v_{1\mathbf{k}}^*(t)\,v_{0\mathbf{k}}(t)\right) && (281)\\
&= \left(v_{1\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Im}(t)\right)^2-2\left(v_{1\mathbf{k}}^{\Re}(t)\,v_{0\mathbf{k}}^{\Re}(t)+v_{1\mathbf{k}}^{\Im}(t)\,v_{0\mathbf{k}}^{\Im}(t)\right) && (282)\\
&= \left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2. && (283)
\end{aligned}$$

Rewriting in terms of real and imaginary parts.

$$\begin{aligned}
R_i(t) &= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\right)\right) && (284)\\
&= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\,\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\,\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), && (285)\\
\left\langle B_0^+(t)\,B_1^-(t)\right\rangle &= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) && (286)\\
&= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{2\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) && (287)\\
&= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), && (288)
\end{aligned}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\begin{aligned}
\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\left(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (289)\\
&= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\,\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\,\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (290)\\
&= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, && (291)\\
\frac{\partial\left|B_{10}(t)\right|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (292)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (293)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\left|B_{10}(t)\right|^2, && (294)\\
\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (295)\\
&= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} && (296)\\
&= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-\left|V_{10}(t)\right|^2\left|B_{10}(t)\right|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} && (297)
\end{aligned}$$

$$=\frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}(t)-R_{i}(t)\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\omega^{2}_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\left|B_{10}(t)\,V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{298}$$

$$=\frac{\varepsilon\left(t\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Re}_{i\mathbf{k}}(t)-v^{\Re}_{i'\mathbf{k}}(t)\right)}{\omega^{2}_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{299}$$

$$=\frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon\left(t\right)-4\left(\varepsilon\left(t\right)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\left(t\right)}\right)+\frac{1}{\eta\left(t\right)}\left(-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon\left(t\right)+2\left(\varepsilon\left(t\right)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{300}$$

$$\left.+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^{2}_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{301}$$

From the equation (268) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}}{\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}} \tag{302}$$

$$=\frac{\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{2g^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}}{\frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))-\frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^{2}_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{303}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-g_{i\mathbf{k}}-g^{*}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))-\frac{4|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}\right)(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)))+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{304}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))-\frac{4|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)))+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))-\frac{4|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t))+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$=\frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t))+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{307}$$

Separating (306) such that the terms with $v_{i\mathbf{k}}\left(t\right)$ are located at one side of the equation permit us to write:

$$\frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))-\frac{2|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2\left(\varepsilon_{i}(t)+R_{i}(t)\right)-\varepsilon(t))+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{308}$$

$$v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}=v^{\Re}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right) \tag{309}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta\left(t\right)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{310}$$

$$v^{\Re}_{i\mathbf{k}}(t)=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{311}$$

$$v^{\Re}_{i\mathbf{k}}(t)=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{312}$$

The imaginary part can be found in the following way:

$$\begin{aligned}
\frac{\partial \varepsilon\left(t\right)}{\partial v^{\Im}_{i\mathbf{k}}\left(t\right)} =& \frac{\partial\left(\varepsilon_1\left(t\right)+R_1\left(t\right)+\varepsilon_0\left(t\right)+R_0\left(t\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}\left(t\right)} && (313)\\
=& \frac{\partial\left(\left(\frac{\left(v^{\Re}_{i\mathbf{k}}(t)\right)^2+\left(v^{\Im}_{i\mathbf{k}}(t)\right)^2}{\omega_{\mathbf{k}}}-v^{\Re}_{i\mathbf{k}}\left(t\right)\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}v^{\Im}_{i\mathbf{k}}\left(t\right)\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}\left(t\right)} && (314)\\
=& 2\frac{v^{\Im}_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, && (315)\\
\frac{\partial\left|B_{10}(t)\right|^2}{\partial v^{\Im}_{i\mathbf{k}}(t)} =& \frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(\mathrm{v}^{\Re}_{1\mathbf{k}}(t)-\mathrm{v}^{\Re}_{0\mathbf{k}}(t)\right)^2+\left(\mathrm{v}^{\Im}_{1\mathbf{k}}(t)-\mathrm{v}^{\Im}_{0\mathbf{k}}(t)\right)^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v^{\Im}_{i\mathbf{k}}\left(t\right)} && (316)\\
=& -\frac{2\left(v^{\Im}_{1\mathbf{k}}\left(t\right)-v^{\Im}_{0\mathbf{k}}\left(t\right)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(v^{\Im}_{1\mathbf{k}}\left(t\right)-v^{\Im}_{0\mathbf{k}}\left(t\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}\left(t\right)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)^2+\left(v^{\Im}_{1\mathbf{k}}(t)-v^{\Im}_{0\mathbf{k}}(t)\right)^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (317)\\
=& -\frac{2\left(v^{\Im}_{1\mathbf{k}}\left(t\right)-v^{\Im}_{0\mathbf{k}}\left(t\right)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(v^{\Im}_{1\mathbf{k}}\left(t\right)-v^{\Im}_{0\mathbf{k}}\left(t\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}\left(t\right)}\left|B_{10}\left(t\right)\right|^2, && (318)\\
\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}\left(t\right)} =& \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v^{\Re}_{i\mathbf{k}}\left(t\right)} && (319)\\
=& \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} && (320)\\
=& \frac{\varepsilon\left(t\right)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|B_{10}(t)|^2|V_{10}(t)|^2\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{\eta\left(t\right)} && (321)\\
=& \frac{\varepsilon\left(t\right)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|B_{10}(t)V_{10}(t)|^2\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{\eta\left(t\right)} && (322)\\
=& \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Im}_{1\mathbf{k}}(t)-v^{\Im}_{0\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(v^{\Im}_{1\mathbf{k}}(t)-v^{\Im}_{0\mathbf{k}}(t)\right)}{\partial v^{\Im}_{i\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} && (323)\\
\delta_{1i}-\delta_{0i} =& \frac{\partial\left(v^{\Im}_{1\mathbf{k}}(t)-v^{\Im}_{0\mathbf{k}}(t)\right)}{\partial v^{\Im}_{i\mathbf{k}}} && (324)\\
\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}\left(t\right)} =& \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-v^{\Im}_{i'\mathbf{k}}(t)\right)|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega^2_{\mathbf{k}}}\right)}{\eta\left(t\right)} && (325)\\
=& \frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\frac{4(\varepsilon_i(t)+R_i(t))-2\varepsilon(t)-\frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}}{\eta(t)}+\frac{1}{\eta(t)}\left(2\frac{g^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}+4\frac{v^{\Im}_{i'\mathbf{k}}(t)|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega^2_{\mathbf{k}}}\right). && (326)
\end{aligned}$$

From the equation (268) and replacing the derivatives obtained we have:

$$\begin{aligned}
\frac{\frac{\partial\varepsilon(t)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{\frac{\partial\eta(t)}{\partial v^{\Im}_{i\mathbf{k}}(t)}} =& \tanh\left(\frac{\beta\eta(t)}{2}\right) && (327)\\
=& \frac{2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{v^{\Im}_{i\mathbf{k}}(t)\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\right)+\frac{2}{\eta(t)}\left(\frac{g^{*\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g^{*\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. && (328)
\end{aligned}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v^{\Im}_{i\mathbf{k}}(t)-\mathrm{i}\left(g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$=\frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{330}$$

$$=\frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{331}$$

$$=\frac{\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{332}$$

Separating (332) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{333}$$

$$v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}=v^{\Im}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right) \tag{334}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{335}$$

$$v^{\Im}_{i\mathbf{k}}(t)=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{336}$$

$$v^{\Im}_{i\mathbf{k}}(t)=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{337}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t)=v^{\Re}_{i\mathbf{k}}(t)+\mathrm{i}v^{\Im}_{i\mathbf{k}}(t) \tag{338}$$

$$=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{339}$$

$$+\mathrm{i}\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{340}$$

$$=\frac{g_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{341}$$

Let's obtain the explicit form of $v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)$ and $v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)$, at first we have:

$$a_i\left(\omega_{\mathbf{k}},t\right)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{342}$$

$$b_i\left(\omega_{\mathbf{k}},t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}.\tag{343}$$

So the equation (338) written in explicit form is:

$$v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right),\tag{344}$$

$$v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)=g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right).\tag{345}$$

This system of equations has the following solutions:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{346}$$

$$=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+\left(g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{347}$$

$$=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{348}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)(1-b_1(\omega_{\mathbf{k}},t)b_0(\omega_{\mathbf{k}},t))=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{349}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=\frac{g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{350}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+\frac{g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}b_1\left(\omega_{\mathbf{k}},t\right)\tag{351}$$

$$=\frac{g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{352}$$

For a shorter representation let's define:

$$r_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_i\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{353}$$

$$s_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_{(i+1)\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)b_{i\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{354}$$

So the variational parameters are given by:

$$\begin{pmatrix}v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\\ v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\end{pmatrix}\equiv\begin{pmatrix}r_0\left(\omega_{\mathbf{k}},t\right) & s_0\left(\omega_{\mathbf{k}},t\right)\\ r_1\left(\omega_{\mathbf{k}},t\right) & s_1\left(\omega_{\mathbf{k}},t\right)\end{pmatrix}\begin{pmatrix}g_0\left(\omega_{\mathbf{k}}\right)\\ g_1\left(\omega_{\mathbf{k}}\right)\end{pmatrix}.\tag{355}$$

Given that $v_{i\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\equiv g_i\left(\omega_{\mathbf{k}}\right)F_i\left(\omega_{\mathbf{k}},t\right)$ then we can write:

$$F_0\left(\omega_{\mathbf{k}},t\right)=r_0\left(\omega_{\mathbf{k}},t\right)+\frac{g_1\left(\omega_{\mathbf{k}}\right)}{g_0\left(\omega_{\mathbf{k}}\right)}s_0\left(\omega_{\mathbf{k}},t\right)\tag{356}$$

$$F_1\left(\omega_{\mathbf{k}},t\right)=\frac{g_0\left(\omega_{\mathbf{k}}\right)}{g_1\left(\omega_{\mathbf{k}}\right)}r_1\left(\omega_{\mathbf{k}},t\right)+s_1\left(\omega_{\mathbf{k}},t\right)\tag{357}$$

## III. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$, where $\rho_B^{\text{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$
\begin{aligned}
\overline{\rho_T(0)} &\equiv e^{V(0)} \rho_T(0) e^{-V(0)} && (358)\\
&= \left( |0\rangle\langle 0| B_0^+(0) + |1\rangle\langle 1| B_1^+(0) \right) (\rho_S(0) \otimes \rho_B) \left( |0\rangle\langle 0| B_0^-(0) + |1\rangle\langle 1| B_1^-(0) \right), && (359)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|: \quad & |0\rangle\langle 0|0\rangle B_0^+(0) \langle 0| \rho_B |0\rangle\langle 0| B_0^-(0) && (360)\\
&= |0\rangle B_0^+(0) \langle 0| \rho_B |0\rangle\langle 0| B_0^-(0) && (361)\\
&= |0\rangle\langle 0| \otimes B_0^+(0) \rho_B B_0^-(0), && (362)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|: \quad & |1\rangle\langle 1| B_1^+(0) |1\rangle\langle 1| \rho_B |1\rangle\langle 1| B_1^-(0) && (363)\\
&= |1\rangle\langle 1| B_1^+(0) \rho_B B_1^-(0) && (364)\\
&= |1\rangle\langle 1| \otimes B_1^+(0) \rho_B B_1^-(0), && (365)\\
\text{for } \rho_S(0) = |0\rangle\langle 1|: \quad & |0\rangle\langle 0| B_0^+(0) |0\rangle\langle 1| \rho_B |1\rangle\langle 1| B_1^-(0) && (366)\\
&= |0\rangle\langle 1| B_0^+(0) \rho_B |1\rangle\langle 1| B_1^-(0) && (367)\\
&= |0\rangle\langle 1|1\rangle\langle 1| B_0^+(0) \rho_B B_1^-(0) && (368)\\
&= |0\rangle\langle 1| \otimes B_0^+(0) \rho_B B_1^-(0), && (369)\\
\text{for } \rho_S(0) = |1\rangle\langle 0|: \quad & |1\rangle\langle 1| B_1^+(0) |1\rangle\langle 0| \rho_B |0\rangle\langle 0| B_0^-(0) && (370)\\
&= |1\rangle\langle 0| \otimes B_1^+(0) \rho_B B_0^-(0). && (371)
\end{aligned}
$$

We transform any operator $O$ into the interaction picture in the following way:

$$
\begin{aligned}
\widetilde{O}(t) &\equiv U^\dagger(t) O(t) U(t), && (372)\\
U(t) &\equiv \mathcal{T}\exp\left( -\mathrm{i} \int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t') \right). && (373)
\end{aligned}
$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$
\begin{aligned}
\widetilde{\overline{\rho_S}}(t) &= U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} && (374)\\
\overline{\rho_S}(t) &= \mathrm{Tr}_B\left( \bar{\rho}_T(t) \right). && (375)
\end{aligned}
$$

In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$
\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \tag{376}
$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (243):

$$
\begin{aligned}
\overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left( \sigma_x B_x(t) + \sigma_y B_y(t) \right) + V_{10}^{\Im}(t) \left( \sigma_x B_y(t) - \sigma_y B_x(t) \right) && (377)\\
&= B_{0z}(t) |0\rangle\langle 0| + B_{1z}(t) |1\rangle\langle 1| + V_{10}^{\Re}(t) \sigma_x B_x(t) + V_{10}^{\Re}(t) \sigma_y B_y(t) + V_{10}^{\Im}(t) \sigma_x B_y(t) - V_{10}^{\Im}(t) \sigma_y B_x(t) && (378)\\
&= \sum_i C_i(t) \left( A_i \otimes B_i(t) \right). && (379)
\end{aligned}
$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{380}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) \text{ -i} \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]\mathrm{d}t'. \tag{381}$$

Replacing the equation (381) in the equation (380) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]]\mathrm{d}t'. \tag{382}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdots]. \tag{383}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]] \cdots]. \tag{384}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\overline{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\, \overline{\rho_S}(0) \tag{385}$$

$$= W(t)\, \overline{\rho_S}(0). \tag{386}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]] \cdots]. \tag{387}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) \overline{\rho_S}(0) \tag{388}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} W(t)\, \overline{\rho_S}(0) \tag{389}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1}\, \widetilde{\overline{\rho_S}}(t). \tag{390}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (388) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\, \widetilde{\overline{\rho_S}}(t) \tag{391}$$

$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B \left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]. \tag{392}$$

Replacing $t_1 \to t-\tau$

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}\left(t\right),\left[\widetilde{\overline{H_{\bar{I}}}}\left(-\tau\right),\overline{\rho_S}\left(t\right)\rho_B\right]\right]. \tag{393}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}\left(t\right)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right). \tag{394}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right) = \sum_i C_i\left(t\right)\left(\widetilde{A_i}\left(t\right)\otimes\widetilde{B_i}\left(t\right)\right), \tag{395}$$

$$\begin{aligned}
\widetilde{A_i}\left(t\right) &= U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) &(396)\\
&= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} &(397)\\
&= U^{\dagger}\left(t\right)A_i U\left(t\right)\mathbb{I} &(398)\\
&= U^{\dagger}\left(t\right)A_i U\left(t\right), &(399)\\
\widetilde{B_i}\left(t\right) &= U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) &(400)\\
&= U^{\dagger}\left(t\right)U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} &(401)\\
&= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} &(402)\\
&= \mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}. &(403)
\end{aligned}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}\left(t\right),H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right),\mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (395) to replace it in the equation (392)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}\left(t\right),\left[\widetilde{\overline{H_{\bar{I}}}}\left(t'\right),\widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' &(404)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j\left(t\right)\left(\widetilde{A_j}\left(t\right)\otimes\widetilde{B_j}\left(t\right)\right),\left[\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right),\widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' &(405)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right),\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\widetilde{\overline{\rho_S}}(t)\rho_B-\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\right]\mathrm{d}t' &(406)\\
&= -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\widetilde{\overline{\rho_S}}(t)\rho_B-\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right) &(407)\\
&\quad -\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)+\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\Big)\mathrm{d}t'. &(408)
\end{aligned}$$

In order to calculate the correlation functions we define:

$$\mathcal{B}_{ij}\left(t,t'\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(t'\right)\rho_B\right). \tag{409}$$

An useful property is

$$\begin{aligned}
\mathcal{B}^*_{ji}\left(t,t'\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right)^{\dagger} &(410)\\
&= \mathrm{Tr}_B\left(\rho_B^{\dagger}\widetilde{B_i}^{\dagger}\left(t'\right)\widetilde{B_j}^{\dagger}\left(t\right)\right) &(411)\\
&= \mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) &(412)\\
&= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) &(413)\\
&= \mathcal{B}_{ij}\left(t',t\right). &(414)
\end{aligned}$$

The correlation functions relevant that appear in the equation (408) are:

$$
\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) &= \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\right\rangle_B \qquad (415)\\
&= \mathcal{B}_{ji}\left(t,t'\right) \qquad (416)\\
&= \mathcal{B}^*_{ij}\left(t',t\right) \qquad (417)\\
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B\widetilde{B_i}\left(t'\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \qquad (418)\\
&= \mathcal{B}_{ij}\left(t',t\right) \qquad (419)\\
\mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\rho_B\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) \qquad (420)\\
&= \mathcal{B}^*_{ij}\left(t',t\right) \qquad (421)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \qquad (422)\\
&= \mathcal{B}_{ij}\left(t',t\right) \qquad (423)
\end{aligned}
$$

The cyclic property of the trace was use widely in the development of equations (415) and (423). Replacing in (408)

$$
\begin{aligned}
\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} =& -\textstyle\int_0^t \mathrm{Tr}_B\big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big) \qquad (424)\\
&-\textstyle\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\big)\mathrm{d}t'. \qquad (425)\\
=& -\textstyle\int_0^t \mathrm{Tr}_B\big(\sum_{ji} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \qquad (426)\\
&+\textstyle\sum_{ij} C_i(t')C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\big)\mathrm{d}t' \qquad (427)\\
=& -\textstyle\int_0^t \mathrm{Tr}_B\big(\sum_{ji} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \qquad (428)\\
&+\textstyle\sum_{ij} C_i(t')C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\big)\mathrm{d}t' \qquad (429)\\
=& -\textstyle\int_0^t \mathrm{Tr}_B\big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\big) \text{ (by permuting i and j because i,j}\in\text{J)} \qquad (430)\\
&+\textstyle\sum_{ij} C_i(t')C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\big)\mathrm{d}t' \qquad (431)\\
=& -\textstyle\int_0^t \mathrm{Tr}_B\big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t') \qquad (432)\\
&+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\big)\big)\mathrm{d}t' \qquad (433)\\
=& -\textstyle\int_0^t \big(\sum_{ij} C_j(t)C_i(t')\big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\mathcal{B}_{ji}(t,t')-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\mathcal{B}_{ij}(t',t) \qquad (434)\\
&+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\mathcal{B}_{ij}(t',t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\mathcal{B}_{ji}(t,t')\big)\big)\mathrm{d}t' \qquad (435)\\
=& -\int_0^t \left(\sum_{ij} C_j\left(t\right)C_i\left(t'\right)\left(\mathcal{B}_{ji}\left(t,t'\right)\left[\widetilde{A_j}\left(t\right),\widetilde{A_i}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]+\mathcal{B}_{ij}\left(t',t\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t'\right),\widetilde{A_j}\left(t\right)\right]\right)\right)\mathrm{d}t' \qquad (436)\\
=& -\textstyle\int_0^t \big(\sum_{ij} C_i(t)C_j(t')\big(\mathcal{B}_{ij}(t,t')\big[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)\big]+\mathcal{B}_{ji}(t',t)\big[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t'),\widetilde{A_i}(t)\big]\big)\big)\mathrm{d}t' \text{ (exchanging i and j)} \qquad (437)\\
=& -\int_0^t \left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]+\mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}t' \qquad (438)\\
=& -\int_0^t \left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]-\mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right)\right]\right)\right)\mathrm{d}t' \qquad (439)
\end{aligned}
$$

We could identify the following commutators in the equation deduced:

$$
\begin{aligned}
\mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)-\mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right) &= \mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\rho_s}\left(t\right)\right], \qquad (440)\\
\mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right) &= \mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]. \qquad (441)
\end{aligned}
$$

Returning to the Schroedinger picture we have:

$$U\left(t\right)\widetilde{A_{i}}\left(t\right)\widetilde{A_{j}}\left(t'\right)\widetilde{\widetilde{\rho_{S}}}\left(t\right)U^{\dagger}\left(t\right)=U\left(t\right)\widetilde{A_{i}}\left(t\right)U^{\dagger}\left(t\right)U\left(t\right)\widetilde{A_{j}}\left(t'\right)U^{\dagger}\left(t\right)U\left(t\right)\widetilde{\widetilde{\rho_{S}}}\left(t\right)U^{\dagger}\left(t\right), \tag{442}$$

$$=\left(U\left(t\right)\widetilde{A_{i}}\left(t\right)U^{\dagger}\left(t\right)\right)\left(U\left(t\right)\widetilde{A_{j}}\left(t'\right)U^{\dagger}\left(t\right)\right)\left(U\left(t\right)\widetilde{\widetilde{\rho_{S}}}\left(t\right)U^{\dagger}\left(t\right)\right), \tag{443}$$

$$=A_{i}\left(t\right)\widetilde{A_{j}}\left(t',t\right)\overline{\rho_{S}}\left(t\right). \tag{444}$$

This procedure applying to the relevant commutators give us:

$$U\left(t\right)\left[\widetilde{A_{i}}\left(t\right),\widetilde{A_{j}}\left(t'\right)\widetilde{\widetilde{\rho_{S}}}\left(t\right)\right]U^{\dagger}\left(t\right)=\left(U\left(t\right)\widetilde{A_{i}}\left(t\right)\widetilde{A_{j}}\left(t'\right)\widetilde{\widetilde{\rho_{S}}}\left(t\right)U^{\dagger}\left(t\right)-U\left(t\right)\widetilde{A_{j}}\left(t'\right)\widetilde{\widetilde{\rho_{S}}}\left(t\right)\widetilde{A_{i}}\left(t\right)U^{\dagger}\left(t\right)\right) \tag{445}$$

$$=A_{i}\left(t\right)\widetilde{A_{j}}\left(t',t\right)\overline{\rho_{S}}\left(t\right)-\widetilde{A_{j}}\left(t',t\right)\overline{\rho_{S}}\left(t\right)A_{i} \tag{446}$$

$$=\left[A_{i}\left(t\right),\widetilde{A_{j}}\left(t',t\right)\overline{\rho_{S}}\left(t\right)\right]. \tag{447}$$

Introducing this transformed commutators in the equation (439) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathcal{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\sum_{ij}\int_0^t\mathrm{d}sC_i(t)\,C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[A_i\left(t\right),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]+\mathcal{B}_{ij}^{*}\left(t,t'\right)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}\left(t',t\right),A_i\right]\right), \tag{448}$$

$$t'=t-\tau\ \text{(Change of variables in the integration process)}, \tag{449}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ij}\int_0^t\mathrm{d}\tau C_i(t)\,C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[A_i(t),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]+\mathcal{B}_{ij}^{*}\left(t,t'\right)\left[\overline{\rho_S}(t)\widetilde{A_j}\left(t',t\right),A_i\left(t\right)\right]\right). \tag{450}$$

where $i,j\in\{1,2,3,4,5.6\}$ and $t'=t-\tau$.
Here $\widetilde{A_j}\left(t-\tau,t\right)=U\left(t\right)U^{\dagger}\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$ where $U\left(t\right)$ is given by (373). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I\left(t\right)$. In order to write in a simplified way we define the following notation:

$$\mathcal{B}_{ij}\left(t,t'\right)=\mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(t'\right)\rho_B\right) \tag{451}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{iH_Bt}B_i\left(t\right)\mathrm{e}^{-iH_Bt}\mathrm{e}^{iH_Bt'}B_j\left(t'\right)\mathrm{e}^{-iH_Bt'}\rho_B\right) \tag{452}$$

$$\mathrm{e}^{A}=\sum_{k=0}^{\infty}\frac{A^k}{k!} \tag{453}$$

$$\mathrm{e}^{-iH_Bt'}\mathrm{e}^{-\beta H_B}=\sum_{m=0}^{\infty}\frac{\left(-iH_Bt'\right)^m}{m!}\sum_{n=0}^{\infty}\frac{\left(-\beta H_B\right)^n}{n!} \tag{454}$$

$$=\sum_{m,n}\frac{\left(-iH_Bt'\right)^m}{m!}\frac{\left(-\beta H_B\right)^n}{n!} \tag{455}$$

$$=\sum_{m,n}\frac{\left(-it'\right)^m}{m!}\frac{\left(-\beta\right)^n}{n!}H_B^mH_B^n \tag{456}$$

$$=\sum_{m,n}\frac{\left(-it'\right)^m}{m!}\frac{\left(-\beta\right)^n}{n!}H_B^nH_B^m\ \text{(because the powers of a matrix commute)} \tag{457}$$

$$=\sum_{m,n}\frac{\left(-\beta\right)^n}{n!}H_B^n\frac{\left(-it'\right)^m}{m!}H_B^m \tag{458}$$

$$=\sum_{m,n}\frac{\left(-\beta H_B\right)^n}{n!}\frac{\left(-it'H_B\right)^m}{m!} \tag{459}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_B t')^m}{m!} && (460)\\
&= \mathrm{e}^{-\beta H_B} \mathrm{e}^{-iH_B t'} && (461)\\
0 &= \mathrm{e}^{-iH_B t'} \mathrm{e}^{-\beta H_B} - \mathrm{e}^{-\beta H_B} \mathrm{e}^{-iH_B t'} \text{ (then } \mathrm{e}^{-iH_B t'} \text{ and } \rho_B \text{ commute)} && (462)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left( \mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \rho_B \mathrm{e}^{-iH_B t'} \right) \text{ (by permuting } \mathrm{e}^{-iH_B t'} \text{ and } \rho_B\text{)} && (463)\\
&= \mathrm{Tr}_B \left( \left( \mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \right) \rho_B \mathrm{e}^{-iH_B t'} \right) \text{ (by associative property)} && (464)\\
&= \mathrm{Tr}_B \left( \mathrm{e}^{-iH_B t'} \left( \mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \right) \rho_B \right) \text{ (by cyclic property of the trace)} && (465)\\
&= \mathrm{Tr}_B \left( \left( \mathrm{e}^{-iH_B t'} \mathrm{e}^{iH_B t} \right) B_i(t) \left( \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} \right) B_j(t') \rho_B \right) \text{ (by associative property)} && (466)\\
[iH_B t, -iH_B t'] &= iH_B t(-iH_B t') - (-iH_B t') iH_B t && (467)\\
&= tt' H_B^2 - tt' H_B^2 && (468)\\
&= 0 \text{ (so } iH_B t \text{ and } -iH_B t' \text{ commute)} && (469)\\
\mathrm{e}^{-iH_B t'} \mathrm{e}^{iH_B t} &= \mathrm{e}^{iH_B t - iH_B t'} \text{ (by the Zassenhaus formula because } iH_B t \text{ and } -iH_B t' \text{ commute)} && (470)\\
&= \mathrm{e}^{iH_B (t-t')} && (471)\\
&= \mathrm{e}^{iH_B \tau} && (472)\\
\mathrm{e}^{iH_B t'} \mathrm{e}^{-iH_B t} &= \mathrm{e}^{-iH_B t + iH_B t'} \text{ (by the Zassenhaus formula because } -iH_B t \text{ and } iH_B t' \text{ commute)} && (473)\\
&= \mathrm{e}^{iH_B (-t+t')} && (474)\\
&= \mathrm{e}^{-iH_B \tau} && (475)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left( \mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} B_j(t') \rho_B \right) && (476)\\
B_i(t,\tau) &\equiv \mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} && (477)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left( \mathrm{e}^{iH_B (t-t')} B_i(t) \mathrm{e}^{-iH_B (t-t')} B_j(t') \rho_B \right) && (478)\\
t' &= t - \tau && (479)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left( \mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} B_j(t') \rho_B \right) && (480)\\
&= \mathrm{Tr}_B \left( B_i(t,\tau) B_j(t',0) \rho_B \right) && (481)
\end{aligned}
$$

For the following results $i, j \in \{3, 6\}$, calculating the correlation functions allow us to obtain:

$$
\begin{aligned}
\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B &= \mathrm{Tr}_B \left( B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \rho_B \right) && (482)\\
&= \int \mathrm{d}^2\alpha P(\alpha) \left\langle \alpha \left| B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \right| \alpha \right\rangle && (483)\\
&= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle \alpha \left| B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \right| \alpha \right\rangle \mathrm{d}^2\alpha, && (484)\\
q_{j\mathbf{k}}(t) &= g_{j\mathrm{mod}2\mathbf{k}} - v_{j\mathrm{mod}2\mathbf{k}}(t) && (485)\\
B_{j\mathrm{mod}2z}(t,\tau) &= \sum_{\mathbf{k}} \left( q_{j\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), && (486)\\
B_{j\mathrm{mod}2z}(t',0) &= \sum_{\mathbf{k}'} \left( q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^{\dagger} + q_{j\mathbf{k}'}^{*}(t') b_{\mathbf{k}'} \right), && (487)\\
\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B &= \mathrm{Tr}_B \left( B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \rho_B \right) && (488)\\
&= \mathrm{Tr}_B \left( \sum_{\mathbf{k}} \left( q_{j\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left( q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^{\dagger} + q_{j\mathbf{k}'}^{*}(t') b_{\mathbf{k}'} \right) \rho_B \right) && (489)
\end{aligned}
$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{490}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{491}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_B\right), \tag{492}$$

$$\left\langle\widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{493}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_B\right) \tag{494}$$

$$0=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{495}$$

$$\left\langle\widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B=0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_B\right) \tag{496}$$

$$=\mathrm{Tr}_B\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\left(b_{\mathbf{k}}^{\dagger}\right)^2\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{2}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B \tag{497}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{498}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right)+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right) \tag{499}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{500}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{501}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{502}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{503}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{504}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{505}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{506}$$

$$=\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right. \tag{507}$$

$$\left.+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{508}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right. \tag{509}$$

$$\left.+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{510}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\sum_{\mathbf{k}}\left(\left\langle0\left|q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right)+q_{j\mathbf{k}}^{*}(t)\,q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right)\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{511}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\left\langle0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle+\left\langle0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{512}$$

$$+\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\left\langle0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|0\right\rangle+\left\langle0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{513}$$

$$1=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{514}$$

$$b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left|0\right\rangle=0, \tag{515}$$

$$b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\left|0\right\rangle=\left|0\right\rangle, \tag{516}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}} \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left(\left(q_{j\mathbf{k}}(t)\, q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle \right. \tag{517}$$

$$\left. + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|0\right\rangle\right)\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{518}$$

$$= \frac{1}{\pi N}\int\sum_{\mathbf{k}}\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(\left(q_{j\mathbf{k}}(t)q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)|\alpha_{\mathbf{k}}|^2 + q^*_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{519}$$

$$\int_0^{2\pi}\int_0^{+\infty} r^2\mathrm{e}^{-\frac{r^2}{N}} r\mathrm{d}r\mathrm{d}\theta = \int |\alpha_{\mathbf{k}}|^2\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\mathrm{d}^2\alpha_{\mathbf{k}} \tag{520}$$

$$= \pi N^2 \tag{521}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}}\left(\left(q_{j\mathbf{k}}(t)\, q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)N + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{522}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j'\mathrm{mod}2z}}(t')\right\rangle_B = \mathrm{Tr}_B\left(B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0)\,\rho_B\right) \tag{523}$$

$$= \int \mathrm{d}^2\alpha P(\alpha)\left\langle\alpha\left|B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0)\right|\alpha\right\rangle \tag{524}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\left|B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0)\right|\alpha\right\rangle\mathrm{d}^2\alpha \tag{525}$$

$$= \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}'} + q^*_{j'\mathbf{k}'}(t')\, b_{\mathbf{k}'}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{526}$$

$$= \langle\alpha|\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}'} + q^*_{j'\mathbf{k}'}(t')b_{\mathbf{k}'}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{527}$$

$$+ \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}} + q^*_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{528}$$

$$= \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}}(t')\, b^\dagger_{\mathbf{k}} + q^*_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{529}$$

$$= \left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\, b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{530}$$

$$+ \left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{531}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{532}$$

$$+ \sum_{\mathbf{k}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{533}$$

$$\left\langle b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right\rangle_B = \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\, D(-\alpha_{\mathbf{k}})\, b^\dagger_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{534}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{535}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + \alpha_{\mathbf{k}}b^\dagger_{\mathbf{k}} + b_{\mathbf{k}}\alpha^*_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{536}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{537}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{538}$$

$$= N+1, \tag{539}$$

$$\left\langle b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right\rangle_B = \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{540}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}|\alpha_{\mathbf{k}}|^2\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{541}$$

$$= N, \tag{542}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j'\mathrm{mod}2z}}\left(t'\right)\right\rangle_{B} = \sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N + q_{j\mathbf{k}}^{*}\left(t\right)q_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N+1\right)\right) \tag{543}$$

$$= \sum_{\mathbf{k}}2N\left(q_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Re} + \sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{544}$$

$$D\left(h'\right)D\left(h\right) = \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}D\left(h'+h\right), \tag{545}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_{B} = \mathrm{Tr}_{B}\left(\mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}D\left(h'+h\right)\rho_{B}\right) \tag{546}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\mathrm{Tr}_{B}\left(D\left(h'+h\right)\rho_{B}\right) \tag{547}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\frac{1}{\pi N}\int \mathrm{d}^{2}\alpha P\left(\alpha\right)\left\langle\alpha\left|D\left(h'+h\right)\right|\alpha\right\rangle \tag{548}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\mathrm{e}^{-\frac{\left|h+h'\right|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{549}$$

$$h' = h\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{550}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_{B} = \mathrm{e}^{\frac{1}{2}\left(hh^{*}\mathrm{e}^{\mathrm{i}\omega\tau}-h^{*}h\mathrm{e}^{-\mathrm{i}\omega\tau}\right)}\mathrm{e}^{-\frac{\left|h+h\mathrm{e}^{\mathrm{i}\omega\tau}\right|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{551}$$

$$\frac{1}{2}\left|h\right|^{2}\left(\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\right) = \frac{1}{2}\left(hh^{*}\mathrm{e}^{\mathrm{i}\omega\tau}-h^{*}h\mathrm{e}^{-\mathrm{i}\omega\tau}\right) \tag{552}$$

$$= \frac{1}{2}\left|h\right|^{2}\left(\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)-\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{553}$$

$$= \frac{1}{2}\left|h\right|^{2}\left(2\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{554}$$

$$= \mathrm{i}\left|h\right|^{2}\sin\left(\omega\tau\right), \tag{555}$$

$$-\frac{\left|h+h\mathrm{e}^{\mathrm{i}\omega\tau}\right|^{2}}{2} = -\left|h\right|^{2}\frac{\left|1+\mathrm{e}^{\mathrm{i}\omega\tau}\right|^{2}}{2} \tag{556}$$

$$= -\left|h\right|^{2}\frac{\left(1+2\cos\left(\omega\tau\right)+\cos^{2}\left(\omega\tau\right)\right)+\sin^{2}\left(\omega\tau\right)}{2} \tag{557}$$

$$= -\left|h\right|^{2}\frac{2+2\cos\left(\omega\tau\right)}{2} \tag{558}$$

$$= -\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right), \tag{559}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_{B} = \mathrm{e}^{\mathrm{i}\left|h\right|^{2}\sin\left(\omega\tau\right)}\mathrm{e}^{-\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)} \tag{560}$$

$$= \mathrm{e}^{\mathrm{i}\left|\mathrm{h}\right|^{2}\sin\left(\omega\tau\right)-\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)} \tag{561}$$

$$= \mathrm{e}^{-\left|h\right|^{2}\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)}\mathrm{e}^{-\left|h\right|^{2}\coth\left(\frac{\beta\omega}{2}\right)} \tag{562}$$

$$= \left\langle D\left(h\right)\right\rangle_{B}\mathrm{e}^{-\phi\left(\tau\right)}, \tag{563}$$

$$\mathrm{e}^{-\phi\left(\tau\right)} = \mathrm{e}^{-\left|h\right|^{2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)}, \tag{564}$$

$$\phi\left(\tau\right) = \left|h\right|^{2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right), \tag{565}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_{B} = \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\mathrm{e}^{-\frac{\left|h+h'\right|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{566}$$

$$h' = v\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{567}$$

$$m_{\mathbf{k}}\left(t\right) = \frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}, \tag{568}$$

$$\Gamma_{\mathbf{k}}\left(t\right) = \frac{v_{1\mathbf{k}}^{*}\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}^{2}} \tag{569}$$

$$\left\langle \widetilde{B_{1}^{+}B_{0}^{-}}\left(t\right)\widetilde{B_{1}^{+}B_{0}^{-}}\left(t'\right)\right\rangle_{B} = \left\langle B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B} \tag{570}$$

$$= \left\langle B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\right\rangle_{B} \tag{571}$$

$$= \mathrm{Tr}_{B}\left(B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\rho_{B}\right) \tag{572}$$

$$
\begin{aligned}
&= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')} \mathrm{Tr}_B \left( \prod_{\mathbf{k}} \left( D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right) \prod_{\mathbf{k}} \left( D\left(m_{\mathbf{k}}(t')\right)\right) \rho_B \right) && (573)\\
&= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')} \mathrm{Tr}_B \left( \prod_{\mathbf{k}} \left( D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega\tau}\right) D\left(m_{\mathbf{k}}(t')\right)\right) \rho_B \right) && (574)\\
&= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau} m^*_{\mathbf{k}}(t') - \left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}\right)^* m_{\mathbf{k}}(t')\right) - \frac{|m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}+m_{\mathbf{k}}(t')|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (575)\\
&= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')} \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau} m^*_{\mathbf{k}}(t')\right)^{\Im} - \frac{\left|\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\mathrm{e}^{\mathrm{i}\omega\tau}+\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (576)\\
&= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')} \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau} m^*_{\mathbf{k}}(t')\right)^{\Im} - \frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (577)\\
\left\langle \widetilde{B_0^+B_1^-}(t)\widetilde{B_0^+B_1^-}(t') \right\rangle_B &= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')} \prod_{\mathbf{k}} \left( \mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau} m^*_{\mathbf{k}}(t')\right)^{\Im} - \frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) && (578)\\
\left\langle D(h)\,b\right\rangle_B &= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle \alpha \left| D(h)\,b \right| \alpha \right\rangle && (579)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle \alpha \left| D(-\alpha)\,D(h)\,bD(\alpha) \right| \alpha \right\rangle && (580)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle 0 \left| D(-\alpha)\,D(h)\,bD(\alpha) \right| 0 \right\rangle && (581)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle 0 \left| D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha) \right| 0 \right\rangle && (582)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle 0 \left| D(-\alpha)\,D(h)\,D(\alpha)\,(b+\alpha) \right| 0 \right\rangle && (583)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h)\,(b+\alpha) \right| 0 \right\rangle && (584)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h)\,b \right| 0 \right\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h)\,\alpha \right| 0 \right\rangle && (585)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h)\,\alpha \right| 0 \right\rangle && (586)\\
&= \frac{1}{\pi N}\int \alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} \mathrm{d}^2\alpha && (587)\\
&= hN \left\langle D(h)\right\rangle_B , && (588)\\
\left\langle D(h)\,b^\dagger\right\rangle_B &= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle \alpha \left| D(h)\,b^\dagger \right| \alpha \right\rangle && (589)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle 0 \left| D(-\alpha)\,D(h)\,b^\dagger D(\alpha) \right| 0 \right\rangle && (590)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle 0 \left| D(-\alpha)\,D(h)\,b^\dagger D(\alpha) \right| 0 \right\rangle && (591)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle 0 \left| D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,b^\dagger D(\alpha) \right| 0 \right\rangle && (592)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle 0 \left| D(-\alpha)\,D(h)\,D(\alpha)\left(b^\dagger+\alpha^*\right) \right| 0 \right\rangle && (593)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h)\left(b^\dagger+\alpha^*\right) \right| 0 \right\rangle && (594)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h)\,b^\dagger \right| 0 \right\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h)\,\alpha^* \right| 0 \right\rangle && (595)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \left\langle 0 \left| D(h) \right| 1 \right\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \left\langle 0 \left| D(h) \right| 0 \right\rangle && (596)
\end{aligned}
$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \langle -h|1\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \langle 0\left|D\left(h\right)\right|0\rangle , \tag{597}$$

$$\langle -h| = \mathrm{e}^{-\frac{|-h^*|^2}{2}} \sum_n \frac{\left(-h^*\right)^n}{\sqrt{n!}} \langle n| \tag{598}$$

$$\langle -h|1\rangle = \mathrm{e}^{-\frac{|-h^*|^2}{2}} \sum_n \frac{\left(-h^*\right)^n}{\sqrt{n!}} \langle n|1\rangle \tag{599}$$

$$\langle -h|1\rangle = \mathrm{e}^{-\frac{|-h^*|^2}{2}} \left(-h^*\right), \tag{600}$$

$$\left\langle D\left(h\right)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \mathrm{e}^{-\frac{|-h^*|^2}{2}} \left(-h^*\right) + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \mathrm{e}^{-\frac{|-h^*|^2}{2}} \tag{601}$$

$$\left\langle D\left(h\right)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \mathrm{e}^{-\frac{|-h^*|^2}{2}} \left(-h^*\right) + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^* \mathrm{e}^{-\frac{|-h^*|^2}{2}} \tag{602}$$

$$= -h^* \left\langle D\left(h\right)\right\rangle_B \left(N+1\right), \tag{603}$$

$$\left\langle bD\left(h\right)\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle \alpha\left|bD\left(h\right)\right|\alpha\right\rangle \tag{604}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} h + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} \tag{605}$$

$$= h\left\langle D\left(h\right)\right\rangle_B \left(N+1\right), \tag{606}$$

$$\left\langle b^\dagger D\left(h\right)\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left\langle \alpha\left|b^\dagger D\left(h\right)\right|\alpha\right\rangle \tag{607}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} h + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{h\alpha^*-h^*\alpha} \mathrm{e}^{-\frac{|h|^2}{2}} \tag{608}$$

$$= -h^* \left\langle D\left(h\right)\right\rangle_B N. \tag{609}$$

The correlation functions can be found readily as:

$$B_1^+B_0^-\left(t,\tau\right) = \prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), \tag{610}$$

$$B_0^+B_1^-\left(t,\tau\right) = \prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), \tag{611}$$

$$B_{10}\left(t\right) = \mathrm{e}^{\chi_{10}(t)}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{612}$$

$$B_x\left(t,\tau\right) = \frac{B_1^+B_0^-\left(t,\tau\right) + B_0^+B_1^-\left(t,\tau\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}, \tag{613}$$

$$B_y\left(t,\tau\right) = \frac{B_0^+B_1^-\left(t,\tau\right) - B_1^+B_0^-\left(t,\tau\right) + B_{10}\left(t\right) - B_{01}\left(t\right)}{2\mathrm{i}}, \tag{614}$$

$$B_{i\mathrm{mod}2z}\left(t,\tau\right) = \sum_{\mathbf{k}}\left(q_{i\mathbf{k}}\left(t\right)b_{\mathbf{k}}^\dagger\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{i\mathbf{k}}^*\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{615}$$

$$\left\langle \widetilde{B_{i\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j\mathrm{mod}2z}}\left(t'\right)\right\rangle_B = \left\langle B_{i\mathrm{mod}2z}\left(t,\tau\right)B_{j\mathrm{mod}2z}\left(t',0\right)\right\rangle_B \tag{616}$$

$$= \left\langle \sum_{\mathbf{k}}\left(q_{i\mathbf{k}}\left(t\right)b_{\mathbf{k}}^\dagger\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{i\mathbf{k}}^*\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*\left(t'\right)b_{\mathbf{k}}\right)\right\rangle_B \tag{617}$$

$$= \sum_{\mathbf{k}} q_{i\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^*\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + \sum_{\mathbf{k}} q_{i\mathbf{k}}^*\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{618}$$

$$\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_x}\left(t'\right)\right\rangle_B = \left\langle B_x\left(t,\tau\right)B_x\left(t',0\right)\right\rangle_B \tag{619}$$

$$=\left\langle\left(\frac{B_1^+B_0^-(t,\tau)+B_0^+B_1^-(t,\tau)-B_{10}(t)-B_{01}(t)}{2}\right)\left(\frac{B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_B \tag{620}$$

$$=\frac{1}{4}\left\langle\left(B_1^+B_0^-(t,\tau)+B_0^+B_1^-(t,\tau)-B_{10}(t)-B_{01}(t)\right)\left(B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')\right)\right\rangle_B \tag{621}$$

$$=\frac{1}{4}\left\langle B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)\,B_{10}(t')-B_1^+B_0^-(\tau)\,B_{01}(t')\right. \tag{622}$$

$$+B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)-B_0^+B_1^-(t,\tau)\,B_{10}(t')-B_0^+B_1^-(t,\tau)\,B_{01}(t') \tag{623}$$

$$-B_{10}(t)B_1^+B_0^-(t',0)-B_{10}(t)B_0^+B_1^-(t',0)+B_{10}(t)\,B_{10}(t')+B_{10}(t)\,B_{01}(t')-B_{01}(t)\,B_1^+B_0^-(t',0) \tag{624}$$

$$\left.-B_{01}(t)\,B_0^+B_1^-(t',0)+B_{01}(t)\,B_{10}(t')+B_{01}(t)\,B_{01}(t')\right\rangle \tag{625}$$

$$=\frac{1}{4}\left\langle B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0)+B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right. \tag{626}$$

$$\left.+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle-\frac{(B_{01}(t)+B_{10}(t))\,(B_{01}(t')+B_{10}(t'))}{4}, \tag{627}$$

$$U_{10}(t,t')=\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)m_{\mathbf{k}}^*(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Im}}, \tag{628}$$

$$\left\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle_B=\left\langle\Pi_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau})\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right)\Pi_{\mathbf{k}}\left(D(-m_{\mathbf{k}}(t'))\mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B \tag{629}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}(t')\right)\right)\right\rangle_B \tag{630}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\prod_{\mathbf{k}}\left\langle\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(-m_{\mathbf{k}}(t')\right)\right)\right\rangle_B \tag{631}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{632}$$

$$\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\right\rangle_B=\left\langle\prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right)\prod_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B \tag{633}$$

$$=\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle D\left(-m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(m_{\mathbf{k}}(t')\right)\right\rangle_B \tag{634}$$

$$=\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\left\langle D\left(-m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(m_{\mathbf{k}}(t')\right)\right\rangle_B \tag{635}$$

$$=\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{636}$$

$$\left\langle B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)\right\rangle_B=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{637}$$

$$\left\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle_B=\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{638}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{4}\left\langle B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0)+B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right. \tag{639}$$

$$\left.+B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right\rangle-\frac{(B_{01}(t)+B_{10}(t))\,(B_{01}(t')+B_{10}(t'))}{4}, \tag{640}$$

$$=\frac{1}{4}\left(2U_{10}(t,t')\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{641}$$

$$\left.+2U_{10}^*(t,t')\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{642}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{643}$$

$$=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{644}$$

$$+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{645}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^{2}\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{646}$$

$$\left\langle\widetilde{B_{y}}(t)\widetilde{B_{y}}(t')\right\rangle_{B}=\left\langle B_{y}\left(t,\tau\right)B_{y}\left(t',0\right)\right\rangle_{B} \tag{647}$$

$$=\frac{1}{2}\Bigg(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{648}$$

$$+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{649}$$

$$=\left\langle\left(\frac{B_{0}^{+}B_{1}^{-}(t,\tau)-B_{1}^{+}B_{0}^{-}(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\left(\frac{B_{0}^{+}B_{1}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_{B} \tag{650}$$

$$=-\frac{1}{4}\left\langle\left(B_{0}^{+}B_{1}^{-}(t,\tau)-B_{1}^{+}B_{0}^{-}(t,\tau)+B_{10}(t)-B_{01}(t)\right)\left(B_{0}^{+}B_{1}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t',0)+B_{10}(t')-B_{01}(t')\right)\right\rangle_{B} \tag{651}$$

$$=-\frac{1}{4}\Big\langle B_{0}^{+}B_{1}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{0}^{+}B_{1}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t,\tau)B_{10}(t')-B_{0}^{+}B_{1}^{-}(\tau)B_{01}(t')-B_{1}^{+}B_{0}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0) \tag{652}$$

$$+B_{1}^{+}B_{0}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t,\tau)B_{10}(t')+B_{1}^{+}B_{0}^{-}(t,\tau)B_{01}(t')+B_{10}(t)B_{0}^{+}B_{1}^{-}(t',0)-B_{10}(t)B_{1}^{+}B_{0}^{-}(t',0) \tag{653}$$

$$+B_{10}(t)B_{10}(t')-B_{10}(t)B_{01}(t')-B_{01}(t)B_{0}^{+}B_{1}^{-}(t',0)+B_{01}(t)B_{1}^{+}B_{0}^{-}(t',0)-B_{01}(t)B_{10}(t')+B_{01}(t)B_{01}(t')\Big\rangle \tag{654}$$

$$=-\frac{1}{4}\left\langle B_{0}^{+}B_{1}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{0}^{+}B_{1}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)+B_{1}^{+}B_{0}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)\right\rangle \tag{655}$$

$$+\left(B_{01}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Im} \tag{656}$$

$$=-\frac{1}{4}\Bigg(2\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{657}$$

$$-2\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{658}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{659}$$

$$=-\frac{1}{2}\Bigg(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{660}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{661}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{662}$$

$$\left\langle\widetilde{B_{x}}(t)\widetilde{B_{y}}(t')\right\rangle_{B}=\left\langle B_{x}(t,\tau)B_{y}(t',0)\right\rangle_{B} \tag{663}$$

$$=\left\langle\left(\frac{B_{1}^{+}B_{0}^{-}(t,\tau)+B_{0}^{+}B_{1}^{-}(t,\tau)-B_{10}(t)-B_{01}(t)}{2}\right)\left(\frac{B_{0}^{+}B_{1}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_{B} \tag{664}$$

$$=\frac{1}{4i}\Big\langle B_{1}^{+}B_{0}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)+B_{1}^{+}B_{0}^{-}(t,\tau)B_{10}(t')-B_{1}^{+}B_{0}^{-}(t,\tau)B_{01}(t')+B_{01}(t)B_{1}^{+}B_{0}^{-}(t',0) \tag{665}$$

$$+B_{0}^{+}B_{1}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{0}^{+}B_{1}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t,\tau)B_{10}(t')-B_{0}^{+}B_{1}^{-}(t,\tau)B_{01}(t')+B_{01}(t)B_{01}(t') \tag{666}$$

$$-B_{10}(t)B_{0}^{+}B_{1}^{-}(t',0)+B_{10}(t)B_{1}^{+}B_{0}^{-}(t',0)-B_{10}(t)B_{10}(t')+B_{10}(t)B_{01}(t')-B_{01}(t)B_{0}^{+}B_{1}^{-}(t',0)-B_{01}(t)B_{10}(t')\Big\rangle_{B} \tag{667}$$

$$=\frac{1}{4i}\Big\langle B_{1}^{+}B_{0}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)+B_{1}^{+}B_{0}^{-}(t,\tau)B_{10}(t')-B_{1}^{+}B_{0}^{-}(t,\tau)B_{01}(t') \tag{668}$$

$$+B_{0}^{+}B_{1}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{0}^{+}B_{1}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t,\tau)B_{10}(t')-B_{0}^{+}B_{1}^{-}(t,\tau)B_{01}(t') \tag{669}$$

$$-B_{10}(t)B_{0}^{+}B_{1}^{-}(t',0)+B_{10}(t)B_{1}^{+}B_{0}^{-}(t',0)-B_{10}(t)B_{10}(t')+B_{10}(t)B_{01}(t')-B_{01}(t)B_{0}^{+}B_{1}^{-}(t',0) \tag{670}$$

$$+B_{01}(t)B_{1}^{+}B_{0}^{-}(t',0)-B_{01}(t)B_{10}(t')+B_{01}(t)B_{01}(t')\Big\rangle_{B} \tag{671}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\right\rangle \tag{672}$$

$$+ \frac{1}{4i}\left(B_{10}(t) + B_{01}(t)\right)\left(B_{10}(t') - B_{01}(t')\right) \tag{673}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right. \tag{674}$$

$$\left. - B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right\rangle + \frac{1}{4i}\left(B_{10}(t) + B_{01}(t)\right)\left(B_{10}(t') - B_{01}(t')\right) \tag{675}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right. \tag{676}$$

$$\left. - B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right\rangle + \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{677}$$

$$= \frac{1}{4i}\left(\left(e^{\chi_{10}(t)+\chi_{01}(t')} - e^{\chi_{01}(t)+\chi_{10}(t')}\right)U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{678}$$

$$\left. + \left(e^{\chi_{01}(t)+\chi_{01}(t')} - e^{\chi_{10}(t)+\chi_{10}(t')}\right)U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{679}$$

$$+ \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{680}$$

$$= \frac{1}{2}\left(\left(e^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{681}$$

$$\left. + \left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{682}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B = \left\langle \left(\frac{B_0^+B_1^-(t,\tau) - B_1^+B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2i}\right)\left(\frac{B_1^+B_0^-(t',0) + B_0^+B_1^-(t',0) - B_{10}(t') - B_{01}(t')}{2}\right)\right\rangle_B \tag{683}$$

$$= \frac{1}{4i}\left\langle \left(B_0^+B_1^-(t,\tau) - B_1^+B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)\right)\left(B_1^+B_0^-(t',0) + B_0^+B_1^-(t',0) - B_{10}(t') - B_{01}(t')\right)\right\rangle_B \tag{684}$$

$$= \frac{1}{4i}\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_0^+B_1^-(t,\tau)B_{10}(t') - B_0^+B_1^-(t,\tau)B_{01}(t') + B_{01}(t)B_{01}(t')\right. \tag{685}$$

$$- B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) - B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) + B_1^+B_0^-(t,\tau)\,B_{10}(t') + B_1^+B_0^-(t,\tau)\,B_{01}(t') \tag{686}$$

$$+ B_{10}(t)\,B_0^+B_1^-(t',0) - B_{10}(t)\,B_{10}(t') - B_{10}(t)\,B_{01}(t') - B_{01}(t)\,B_1^+B_0^-(t',0) - B_{01}(t)\,B_0^+B_1^-(t',0) \tag{687}$$

$$\left. + B_{01}(t)\,B_{10}(t') + B_{10}(t)\,B_1^+B_0^-(t',0)\right\rangle \tag{688}$$

$$= \frac{1}{4i}\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle \tag{689}$$

$$+ \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{690}$$

$$= \frac{1}{4i}\left(e^{\chi_{01}(t)+\chi_{10}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{691}$$

$$+ e^{\chi_{01}(t)+\chi_{01}(t')}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{692}$$

$$- e^{\chi_{10}(t)+\chi_{10}(t')}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{693}$$

$$\left. - e^{\chi_{10}(t)+\chi_{01}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{694}$$

$$= \frac{1}{4i}\left(2i\left(e^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{695}$$

$$\left. + 2i\left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{696}$$

$$= (B_{10}(t))^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re} + \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im} U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{697}$$

$$\left. + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im} U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{698}$$

$$\left\langle b^{\dagger} D\left(h\right)\right\rangle_{B} = -h^{*}\left\langle D\left(h\right)\right\rangle_{B} N \tag{699}$$

$$\left\langle b D\left(h\right)\right\rangle_{B} = h\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{700}$$

$$\left\langle D\left(h\right) b^{\dagger}\right\rangle_{B} = -h^{*}\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{701}$$

$$\left\langle D\left(h\right) b\right\rangle_{B} = h\left\langle D\left(h\right)\right\rangle_{B} N \tag{702}$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right) = q_{j\mathbf{k}}\left(t\right) \tag{703}$$

$$\left\langle B_{1}^{+}B_{0}^{-}\left(t,\tau\right) q_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} = \prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right) q_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} \tag{704}$$

$$= \mathrm{e}^{\chi_{10}(t)}\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) q_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{705}$$

$$= \mathrm{e}^{\chi_{10}(t)} q_{i\mathbf{k}'}\left(t'\right)\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{706}$$

$$= \mathrm{e}^{\chi_{10}(t)} q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{707}$$

$$= q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{708}$$

$$= -q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right) B_{10}\left(t\right) \tag{709}$$

$$\left\langle B_{1}^{+}B_{0}^{-}\left(t,\tau\right) q_{i\mathbf{k}'}^{*}\left(t'\right) b_{\mathbf{k}'}\right\rangle_{B} = q_{i\mathbf{k}'}^{*}\left(t'\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'}\left\langle\prod_{\mathbf{k}} D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle \tag{710}$$

$$= q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}\left(t\right), \tag{711}$$

$$\left\langle B_{0}^{+}B_{1}^{-}(t,\tau) q_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} = -q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right) B_{01}\left(t\right), \tag{712}$$

$$\left\langle B_{0}^{+}B_{1}^{-}(t,\tau) q_{i\mathbf{k}'}^{*}\left(t'\right) b_{\mathbf{k}'}\right\rangle_{B} = q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}\left(t\right), \tag{713}$$

$$q_{i\mathbf{k}'}\left(0\right) = g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \tag{714}$$

$$\left\langle B_{x}\left(t,\tau\right) B_{i \bmod 2z}\left(t',0\right)\right\rangle_{B} = \left\langle\left(\frac{B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}}{2}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}\left(0\right) b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right) b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{715}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}\right)\left(q_{i\mathbf{k}'}\left(0\right) b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right) b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{716}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)\right)\left(q_{i\mathbf{k}'}\left(0\right) b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right) q_{i\mathbf{k}'}^{*}\left(0\right) b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{717}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle B_{1+}B_{0-}\left(\tau\right) q_{i\mathbf{k}'}\left(0\right) b_{\mathbf{k}'}^{\dagger}+B_{0+}B_{1-}\left(\tau\right) q_{i\mathbf{k}'}\left(0\right) b_{\mathbf{k}'}^{\dagger}+B_{1+}B_{0-}\left(\tau\right) q_{i\mathbf{k}'}^{*}\left(0\right) b_{\mathbf{k}'}\right. \tag{718}$$

$$\left. +B_{0+}B_{1-}\left(\tau\right) q_{i\mathbf{k}'}^{*}\left(0\right) b_{\mathbf{k}'}\right\rangle_{B} \tag{719}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right) B_{10}\left(t\right)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right) B_{01}\left(t\right)\right. \tag{720}$$

$$\left. +q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}\left(t\right)\right) \tag{721}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right) B_{10}\left(t\right)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right) B_{01}\left(t\right)\right. \tag{722}$$

$$\left. +q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}\left(t\right)\right) \tag{723}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*} B_{10}\left(t\right)+\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*} B_{01}\left(t\right)\right)\right. \tag{724}$$

$$\left. +q_{i\mathbf{k}'}^{*}\left(t'\right) N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{01}\left(t\right)\right)\right) \tag{725}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*} B_{10}\left(t\right)-\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*} B_{01}\left(t\right)\right)\right. \tag{726}$$

$$\left. +q_{i\mathbf{k}'}^{*}\left(t'\right) N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{01}\left(t\right)\right)\right) \tag{727}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(B_{10}\left(t\right)-B_{01}\left(t\right)\right)+q_{i\mathbf{k}'}^{*}\left(t'\right) N_{\mathbf{k}'} m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(B_{10}\left(t\right)\right.\right. \tag{728}$$

$$-B_{01}(t)))\tag{729}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}2\mathrm{i}B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right)\tag{730}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right)\tag{731}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)(m_{\mathbf{k}'}(t))^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right),\tag{732}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)(m_{\mathbf{k}'}(t))^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\tag{733}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_x(t',0)\right\rangle_B=\left\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_B\tag{734}$$

$$=\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_B\tag{735}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')\right)\right\rangle_B\tag{736}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)\right)\right\rangle_B\tag{737}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)+q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\right.\tag{738}$$

$$\left.+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)\right\rangle,\tag{739}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\right\rangle_B=q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\right\rangle_B\tag{740}$$

$$=q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B\tag{741}$$

$$=q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D(m_{\mathbf{k}'}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}'}(t')}{2}}\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B\tag{742}$$

$$=q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D(m_{\mathbf{k}'}(t'))\right\rangle_B\tag{743}$$

$$=q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D(m_{\mathbf{k}'}(t'))\right\rangle_B\tag{744}$$

$$=q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}D(m_{\mathbf{k}'}(t'))\right\rangle_B\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\tag{745}$$

$$=q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left(-m_{\mathbf{k}'}^{*}(t')\langle D(m_{\mathbf{k}'}(t'))\rangle_B N_{\mathbf{k}'}\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\tag{746}$$

$$=-m_{\mathbf{k}'}^{*}(t')q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}}D(m_{\mathbf{k}}(t'))\right\rangle_B N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\tag{747}$$

$$=-m_{\mathbf{k}'}^{*}(t')q_{i\mathbf{k}'}(t)B_{10}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau},\tag{748}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)\right\rangle_B=m_{\mathbf{k}'}^{*}(t')q_{i\mathbf{k}'}(t)B_{01}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau},\tag{749}$$

$$\left\langle q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\right\rangle_B=q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_1^+B_0^-(t',0)\right\rangle_B\tag{750}$$

$$=q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B\tag{751}$$

$$=q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\langle b_{\mathbf{k}'}D(m_{\mathbf{k}'}(t'))\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}(D(m_{\mathbf{k}}(t')))\right\rangle_B\tag{752}$$

$$=q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\langle D(m_{\mathbf{k}'}(t'))\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}(D(m_{\mathbf{k}}(t')))\right\rangle_B\tag{753}$$

$$=q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\langle D(m_{\mathbf{k}'}(t'))\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}(D(m_{\mathbf{k}}(t')))\right\rangle_B\tag{754}$$

$$= q^*_{i\mathbf{k}'}\,(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right) B_{10}\left(t'\right), \tag{755}$$

$$\left\langle (q_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+} B_1^-\left(t',0\right)\right\rangle_B = q^*_{i\mathbf{k}'}\,(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-m_{\mathbf{k}'}\left(t'\right)\right)\left(N_{\mathbf{k}'}+1\right) B_{01}\left(t'\right), \tag{756}$$

$$\left\langle B_{i\mathrm{mod}2z}\,(t,\tau) B_x\left(t',0\right)\right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'}\Big(-m^*_{\mathbf{k}'}\left(t'\right) q_{i\mathbf{k}'}(t) B_{10}\left(t'\right) N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(-m^*_{\mathbf{k}'}\left(t'\right)\right) q_{i\mathbf{k}'}(t) B_{01}\left(t'\right) N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{757}$$

$$+ q^*_{i\mathbf{k}'}(t) e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}\left(t'\right)(N_{\mathbf{k}'}+1) B_{10}\left(t'\right) + q^*_{i\mathbf{k}'}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-m_{\mathbf{k}'}\left(t'\right)\right)(N_{\mathbf{k}'}+1) B_{01}\left(t'\right)\Big) \tag{758}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}\,(t)\, N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m^*_{\mathbf{k}'}\left(t'\right)\left(B_{01}\left(t'\right) - B_{10}\left(t'\right)\right) + q^*_{i\mathbf{k}'}\,(t)\, m_{\mathbf{k}'}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{759}$$

$$\times\left(N_{\mathbf{k}'}+1\right)\left(B_{10}\left(t'\right) - B_{01}\left(t'\right)\right)\Big) \tag{760}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}\,(t)\, N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m^*_{\mathbf{k}'}\left(t'\right)\left(B_{01}\left(t'\right) - B_{10}\left(t'\right)\right) - q^*_{i\mathbf{k}'}\,(t)\, m_{\mathbf{k}'}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{761}$$

$$\times\left(N_{\mathbf{k}'}+1\right)\left(B_{01}\left(t'\right) - B_{10}\left(t'\right)\right)\Big) \tag{762}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} B_{01}^{\Im}\left(t'\right)\Big(q_{i\mathbf{k}'}\,(t)\, N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m^*_{\mathbf{k}'}\left(t'\right) - q^*_{i\mathbf{k}'}\,(t)\, m_{\mathbf{k}'}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\Big), \tag{763}$$

$$\left\langle B_y\,(t,\tau)\, B_{i\mathrm{mod}2z}\left(t',0\right)\right\rangle_B = \left\langle\left(\frac{B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}\left(t'\right) b^\dagger_{\mathbf{k}'} + q^*_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}\right)\right\rangle_B \tag{764}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}\,(t) - B_{01}\,(t)\right)\left(q_{i\mathbf{k}'}\left(t'\right) b^\dagger_{\mathbf{k}'} + q^*_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}\right)\right\rangle_B \tag{765}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+ B_1^-\,(t,\tau) - B_1^+ B_0^-\,(t,\tau)\right)\left(q_{i\mathbf{k}'}\left(t'\right) b^\dagger_{\mathbf{k}'} + q^*_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}\right)\right\rangle_B \tag{766}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle B_0^+ B_1^-(t,\tau)\, q_{i\mathbf{k}'}\left(t'\right) b^\dagger_{\mathbf{k}'} - B_1^+ B_0^-(t,\tau)\, q_{i\mathbf{k}'}\left(t'\right) b^\dagger_{\mathbf{k}'} + B_0^+ B_1^-(t,\tau)\, q^*_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'} \tag{767}$$

$$- B_1^+ B_0^-\,(t,\tau)\, q^*_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}\Big\rangle, \tag{768}$$

$$\left\langle B_0^+ B_1^-(t,\tau) q_{i\mathbf{k}'}\left(t'\right) b^\dagger_{\mathbf{k}'}\right\rangle_B = -q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\,(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'}+1\right) B_{01}\,(t), \tag{769}$$

$$\left\langle B_0^+ B_1^-(t,\tau) q^*_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}\right\rangle_B = q^*_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\,(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{01}\,(t), \tag{770}$$

$$\left\langle B_1^+ B_0^-(t,\tau) q_{i\mathbf{k}'}\left(t'\right) b^\dagger_{\mathbf{k}'}\right\rangle_B = -q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\,(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\left(N_{\mathbf{k}'}+1\right) B_{10}\,(t), \tag{771}$$

$$\left\langle B_1^+ B_0^-(t,\tau) q^*_{i\mathbf{k}'}\left(t'\right) b_{\mathbf{k}'}\right\rangle_B = q^*_{i\mathbf{k}'}\left(t'\right) m_{\mathbf{k}'}\,(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} N_{\mathbf{k}'} B_{10}\,(t), \tag{772}$$

$$\left\langle B_y\,(t,\tau)\, B_{i\mathrm{mod}2z}\left(t',0\right)\right\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}(B_{01}(t) + B_{10}(t))\Big(q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1) - q^*_{i\mathbf{k}'}\left(t'\right) m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} N_{\mathbf{k}'}\Big) \tag{773}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(q^*_{i\mathbf{k}'}\left(t'\right)(N_{\mathbf{k}'}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m^*_{\mathbf{k}'}(t) - q^*_{i\mathbf{k}'}\left(t'\right) N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t)\Big)\left(B_{10}\,(t) + B_{01}\,(t)\right), \tag{774}$$

$$= \frac{2}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}\left(t'\right)(N_{\mathbf{k}'}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m^*_{\mathbf{k}'}(t)(B_{10}(t))^{\Re} - q^*_{i\mathbf{k}'}\left(t'\right) N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t)(B_{10}(t))^{\Re}\Big), \tag{775}$$

$$= \frac{(B_{10}\,(t))^{\Re}}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}\left(t'\right)(N_{\mathbf{k}'}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m^*_{\mathbf{k}'}(t) - q^*_{i\mathbf{k}'}\left(t'\right) N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t)\Big), \tag{776}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau) B_y\left(t',0\right)\right\rangle_B = \left\langle \sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t) b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + q^*_{i\mathbf{k}'}(t) b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^+ B_1^-\left(t',0\right) - B_1^+ B_0^-\left(t',0\right) + B_{10}\left(t'\right) - B_{01}\left(t'\right)}{2\mathrm{i}}\right)\right\rangle_B \tag{777}$$

$$= \frac{1}{2\mathrm{i}}\Big\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)\, b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + q^*_{i\mathbf{k}'}(t) b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\Big(B_0^+ B_1^-\left(t',0\right) - B_1^+ B_0^-\left(t',0\right) + B_{10}\left(t'\right) \tag{778}$$

$$- B_{01}\left(t'\right)\Big)\Big\rangle_B \tag{779}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)\, b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + q^*_{i\mathbf{k}'}(t)\, b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+ B_1^-\left(t',0\right) - B_1^+ B_0^-\left(t',0\right)\right)\right\rangle_B \tag{780}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle q_{i\mathbf{k}'}\,(t)\, b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_0^+ B_1^-\left(t',0\right) - q_{i\mathbf{k}'}\,(t)\, b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_1^+ B_0^-\left(t',0\right) \tag{781}$$

$$+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)-q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\Big\rangle \tag{782}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}\left(t',0\right)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}\left(t',0\right)\Big\rangle \tag{783}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}\left(t',0\right)\right\rangle-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle\Big\rangle \tag{784}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}(t',0)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}(t',0)\Big\rangle+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}(t)\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}(t',0)\rangle \tag{785}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle \tag{786}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}(t',0)\Big\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}(t',0)\Big\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}(t)\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}(t',0)\rangle \tag{787}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle\Big) \tag{788}$$

$$\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}(t',0)\Big\rangle_{B}=-m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}, \tag{789}$$

$$\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}(t',0)\Big\rangle_{B}=m_{\mathbf{k}'}^{*}\left(t'\right)B_{01}\left(t'\right)N_{\mathbf{k}'}, \tag{790}$$

$$\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}(t',0)\right\rangle_{B}=m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right), \tag{791}$$

$$\left\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}(t',0)\right\rangle_{B}=-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right), \tag{792}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_{y}(t',0)\right\rangle_{B}=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-\left(-m_{\mathbf{k}'}^{*}\left(t'\right)\right)B_{01}\left(t'\right)N_{\mathbf{k}'}\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \tag{793}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)\right)-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\Big) \tag{794}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-q_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}^{*}\left(t'\right)\right)B_{01}\left(t'\right)N_{\mathbf{k}'}+q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \tag{795}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(q_{i\mathbf{k}'}^{*}\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)-q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\right)\Big) \tag{796}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(B_{10}\left(t'\right)+B_{01}\left(t'\right)\right)\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big) \tag{797}$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}^{\Re}\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^{\Re}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big) \tag{798}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^{\Re}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}^{\Re}\left(t'\right)N_{\mathbf{k}'}\Big) \tag{799}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^{\Re}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}^{\Re}\left(t'\right)N_{\mathbf{k}'}\Big) \tag{800}$$

$$=\mathrm{i}B_{10}^{\Re}\left(t'\right)\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\Big). \tag{801}$$

The correlation functions are equal to:

$$\Big\langle\widetilde{B_{i\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j\mathrm{mod}2z}}\left(t'\right)\Big\rangle_{B}=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)^{*}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{802}$$

$$\Big\langle\widetilde{B_{x}}\left(t\right)\widetilde{B_{x}}\left(t'\right)\Big\rangle_{B}=\frac{1}{2}\Bigg(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{803}$$

$$+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{804}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{805}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B=-\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(\mathrm{v}_{1\mathbf{k}}(\mathrm{t})-\mathrm{v}_{0\mathbf{k}}(\mathrm{t}))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')-\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{806}$$

$$\left.-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{807}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}-\frac{\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im} \tag{808}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{809}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{810}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{811}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{812}$$

$$\left\langle\widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_x}(t')\right\rangle_B=\mathrm{i}\sum_{\mathbf{k}}B_{01}^{\Im}(t')\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^{*}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{813}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_{i\mathrm{mod}2z}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^{*}N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{814}$$

$$\left\langle\widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_y}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t')\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{815}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_{i\mathrm{mod}2z}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^{*}N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))(N_{\mathbf{k}}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\right). \tag{816}$$

Let's consider the following expression related to the sum of coupling constants for a bath over all the frequences:

$$L_i(\omega)\equiv\sum_{\mathbf{k}}g_{i\mathbf{k}}\sqrt{\delta(\omega-\omega_{\mathbf{k}})}. \tag{817}$$

Under this definition we have the following expression for a function $f(\omega)\in L^2$:

$$\int_0^\infty f(\omega)L_i(\omega)L_j^{*}(\omega)\,\mathrm{d}\omega\approx\int_0^\infty f(\omega)\sum_{\mathbf{k}}g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sum_{\mathbf{k}'}g_j(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega \tag{818}$$

$$=\int_0^\infty f(\omega)\sum_{\mathbf{k},\mathbf{k}'}g_i(\omega_{\mathbf{k}})g_j(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega, \tag{819}$$

$$\int_0^\infty f(\omega)\sum_{\mathbf{k}}g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\mathrm{d}\omega=\sum_{\mathbf{k}}\int_0^\infty f(\omega)g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\mathrm{d}\omega. \tag{820}$$

Now we will approach to the function $\sqrt{\delta(\omega-\omega_{\mathbf{k}})}$ using the normal distribution, so:

$$\delta\left(\omega-\omega_{\mathbf{k}}\right)=\lim_{\sigma\to0^{+}}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\sigma^{2}}} \tag{821}$$

$$\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}=\lim_{\sigma\to0^{+}}\sqrt{\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\sigma^{2}}}} \tag{822}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{4\sigma^{2}}} \tag{823}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\left(\sqrt{2}\sigma\right)^{2}}} \tag{824}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right). \tag{825}$$

So we can obtain that:

$$\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega=\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega \tag{826}$$

$$=\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{827}$$

$$=\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{828}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega \tag{829}$$

$$=\sum_{\mathbf{k}}\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)\left(\lim_{\sigma\to0^{+}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega\right) \tag{830}$$

$$=\sum_{\mathbf{k}}\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\ \text{(with } f\left(\omega\right)\text{,}g_{i}\left(\omega\right)\in L^{2}\text{)} \tag{831}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\ \text{(with } f\left(\omega\right)\text{,}g_{i}\left(\omega\right)\in L^{2}\text{)} \tag{832}$$

$$=0\,\text{(because the sum }\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\text{ is finite).} \tag{833}$$

Then we can proof the following:

$$\int_{0}^{\infty}f(\omega)L_{i}(\omega)L_{j}^{*}\left(\omega\right)\mathrm{d}\omega\approx\int_{0}^{\infty}f\left(\omega\right)\sum_{\mathbf{k}}g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sum_{\mathbf{k}'}g_{j}^{*}\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{834}$$

$$=\int_{0}^{\infty}f\left(\omega\right)\sum_{\mathbf{k},\mathbf{k}'}g_{i}\left(\omega_{\mathbf{k}}\right)g_{j}^{*}\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{835}$$

$$=\sum_{\mathbf{k}\neq\mathbf{k}'}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega+\sum_{\mathbf{k}}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{836}$$

$$=0+\sum_{\mathbf{k}}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{837}$$

$$=\sum_{\mathbf{k}}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{838}$$

$$=\sum_{\mathbf{k}}f(\omega_{\mathbf{k}})g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}}) \tag{839}$$

if $i=j$ we recover the spectral density defined in the usual way when we integrate for a function $f\left(\omega\right)$ that belongs to the set $L^{2}$:

$$\sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) = \sum_{\mathbf{k}} \int_0^{\infty} f(\omega) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) \delta(\omega - \omega_{\mathbf{k}}) \mathrm{d}\omega \tag{840}$$
$$= \int_0^{\infty} f(\omega) J_{ii}(\omega) \mathrm{d}\omega \tag{841}$$
$$= \int_0^{\infty} f(\omega) |L_i(\omega)|^2 \mathrm{d}\omega. \tag{842}$$

where

$$J_{ii}(\omega) = \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \tag{843}$$
$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t). \tag{844}$$

In this case $g_i(\omega)$ and $v_i(\omega, t)$ are the continuous version of $g_i(\omega_{\mathbf{k}})$ and $v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t)$ respectively.
The integral version of the correlation functions can be obtained as follows:

$$\left\langle \widetilde{B_{iz}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t')) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \tag{845}$$
$$= \sum_{\mathbf{k}} \left( g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}}, t)) g_{j\mathbf{k}}^* (1 - F_j(\omega_{\mathbf{k}}, t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}, t))^* g_{j\mathbf{k}} (1 - F_j(\omega_{\mathbf{k}}, t')) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \tag{846}$$
$$\approx \int_0^{\infty} \left( L_i(\omega) L_j^*(\omega) (1 - F_i(\omega, t)) \left(1 - F_j^*(\omega, t')\right) \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) + L_i^*(\omega) L_j(\omega) \left(1 - F_i^*(\omega, t)\right) \left(1 - F_j(\omega, t')\right) \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega) + 1) \right) \mathrm{d}\omega, \tag{847}$$
$$\chi_{10}(t) = \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \tag{848}$$
$$= \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \tag{849}$$
$$= \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{g_{1\mathbf{k}}^* g_{0\mathbf{k}} F_1^*(\omega_{\mathbf{k}}, t) F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} g_{0\mathbf{k}}^* F_1(\omega_{\mathbf{k}}, t) F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \tag{850}$$
$$\approx \int_0^{\infty} \frac{L_0(\omega) L_1^*(\omega) F_1^*(\omega, t) F_0(\omega, t) - L_1(\omega) L_0^*(\omega) F_1(\omega, t) F_0^*(\omega, t)}{2\omega^2} \mathrm{d}\omega, \tag{851}$$
$$U_{10}(t, t') = \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\left( \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{852}$$
$$= \mathrm{e}^{\mathrm{i} \sum_{\mathbf{k}} \left( \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{853}$$
$$= \mathrm{e}^{\mathrm{i}\left( \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{854}$$
$$= \mathrm{e}^{\mathrm{i}\left( \sum_{\mathbf{k}} \frac{(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t))(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{855}$$
$$\approx \mathrm{e}^{\mathrm{i}\left( \int_0^{\infty} \frac{(L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t))(L_1(\omega) F_1(\omega, t') - L_0(\omega) F_0(\omega, t'))^* \mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2} \mathrm{d}\omega \right)^{\Im}}, \tag{856}$$
$$B_{10}(t) = \left( \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left( \frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \left( \mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right), \tag{857}$$
$$= \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{858}$$
$$\approx \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\frac{1}{2} \int_0^{\infty} \left| \frac{L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t)}{\omega} \right|^2 \coth\left( \frac{\beta\omega}{2} \right) \mathrm{d}\omega} \tag{859}$$
$$\xi^+(t, t') = \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{860}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{861}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{862}$$

$$\approx\mathrm{e}^{-\int_0^\infty\frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}+L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t')|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{863}$$

$$\xi^-(t,t')=\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{864}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{865}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t'))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{866}$$

$$\approx\mathrm{e}^{-\int_0^\infty\frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}-(L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t'))|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{867}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}(t,t')\xi^+(t,t')+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*(t,t')\,\xi^-(t,t')\right)-(B_{10}(t))^{\Re}(B_{01}(t'))^{\Re} \tag{868}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B=-\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}(t,t')\,\xi^+(t,t')-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*(t,t')\,\xi^-(t,t')\right)+(B_{01}(t))^{\Im}(B_{10}(t'))^{\Im} \tag{869}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*(t,t')\,\xi^-(t,t')+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\,\xi^+(t,t')\right)+(B_{10}(t))^{\Re}(B_{10}(t'))^{\Im} \tag{870}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^*(t,t')\,\xi^-(t,t')+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\,\xi^+(t,t')\right)+(B_{10}(t))^{\Im}(B_{10}(t'))^{\Re} \tag{871}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B=\mathrm{i}B_{01}^{\Im}(t')\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{872}$$

$$=\mathrm{i}B_{01}^{\Im}(t')\sum_{\mathbf{k}}\left(-g_{i\mathbf{k}}^*\left(1-F_i(\omega_{\mathbf{k}},t)\right)^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right. \tag{873}$$

$$\left.+g_{i\mathbf{k}}\left(1-F_i(\omega_{\mathbf{k}},t)\right)\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{874}$$

$$Q(\omega,t)=\frac{L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t)}{\omega} \tag{875}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B\approx\mathrm{i}B_{01}^{\Im}(t')\int_0^\infty\left(L_i(\omega)\left(1-F_i(\omega,t)\right)Q^*(\omega,t')\,N(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}-L_i^*(\omega)\left(1-F_i^*(\omega,t)\right)Q(\omega,t')\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega, \tag{876}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_{iz}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{877}$$

$$\approx\mathrm{i}B_{01}^{\Im}(t)\int_0^\infty\left(L_i^*(\omega)\left(1-F_i^*(\omega,t')\right)Q(\omega,t)\,N(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}-L_i(\omega)\left(1-F_i(\omega,t')\right)Q^*(\omega,t)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{878}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_y}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t')\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right), \tag{879}$$

$$\approx\mathrm{i}B_{10}^{\Re}(t')\int_0^\infty\left(L_i^*(\omega)\left(1-F_i^*(\omega,t')\right)Q(\omega,t')\left(N(\omega)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-L_i(\omega)\left(1-F_i(\omega,t')\right)Q^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{880}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_{iz}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\right) \tag{881}$$

$$\approx i B_{10}^{\Re}(t)\int_0^\infty\left(L_i^*(\omega)\left(1-F_i^*(\omega,t')\right)Q(\omega,t)\,N(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}-L_i(\omega)\left(1-F_i(\omega,t')\right)Q^*(\omega,t)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega. \tag{882}$$

The integral version of $F_0(\omega,t)$ and $F_1(\omega,t)$ are:

$$a_i(\omega_{\mathbf{k}},t)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{883}$$

$$b_i\left(\omega_{\mathbf{k}},t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{884}$$

$$r_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_i\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{885}$$

$$s_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_{(i+1)\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)b_{i\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{886}$$

$$\begin{aligned}F_0\left(\omega,t\right)&=r_0\left(\omega_{\mathbf{k}},t\right)+\frac{g_1\left(\omega_{\mathbf{k}}\right)}{g_0\left(\omega_{\mathbf{k}}\right)}s_0\left(\omega_{\mathbf{k}},t\right)&&(887)\\&\approx r_0\left(\omega,t\right)+\frac{g_1\left(\omega\right)}{g_0\left(\omega\right)}s_0\left(\omega,t\right)&&(888)\\&=r_0\left(\omega,t\right)+\frac{L_1\left(\omega\right)}{L_0\left(\omega\right)}s_0\left(\omega,t\right),&&(889)\\F_1\left(\omega,t\right)&=\frac{g_0\left(\omega_{\mathbf{k}}\right)}{g_1\left(\omega_{\mathbf{k}}\right)}r_1\left(\omega_{\mathbf{k}},t\right)+s_1\left(\omega_{\mathbf{k}},t\right)&&(890)\\&\approx\frac{g_0\left(\omega\right)}{g_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right)&&(891)\\&=\frac{L_0\left(\omega\right)}{L_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right).&&(892)\end{aligned}$$

The expressions showed are well defined because the relevant products present in the correlations functions are of the form:

$$\begin{aligned}\int_0^{\infty}f(\omega)L_j(\omega)F_j(\omega,t)L_i^*(\omega)F_i^*(\omega,t)\,\mathrm{d}\omega&=\int_0^{\infty}f(\omega)L_j(\omega)\left(r_j(\omega,t)+\frac{L_i(\omega)}{L_j(\omega)}s_j(\omega,t)\right)L_i^*(\omega)\left(r_i^*(\omega,t)+\frac{L_j^*(\omega)}{L_i^*(\omega)}s_i^*(\omega,t)\right)\mathrm{d}\omega&&(893)\\&=\int_0^{\infty}f\left(\omega\right)\left(L_j\left(\omega\right)r_j\left(\omega,t\right)+L_i\left(\omega\right)s_j\left(\omega,t\right)\right)\left(L_i^*\left(\omega\right)r_i^*\left(\omega,t\right)+L_j^*\left(\omega\right)s_i^*\left(\omega,t\right)\right)\mathrm{d}\omega&&(894)\\&=\int_0^{\infty}f\left(\omega\right)\Big(L_j\left(\omega\right)L_i^*\left(\omega\right)r_j\left(\omega,t\right)r_i^*\left(\omega,t\right)+\left|L_j\left(\omega\right)\right|^2r_j\left(\omega,t\right)s_i^*\left(\omega,t\right)&&(895)\\&\quad+\left|L_i\left(\omega\right)\right|^2s_j\left(\omega,t\right)r_i^*\left(\omega,t\right)+L_i\left(\omega\right)L_j^*\left(\omega\right)s_j\left(\omega,t\right)s_i^*\left(\omega,t\right)\Big)\mathrm{d}\omega.&&(896)\end{aligned}$$

here $f\left(\omega\right)\in L^2$. As we could proof these integral are convergent.
So the integral version of the correlation functions $\mathcal{B}_{ij}\left(t,t'\right)$ is can be written in a neater form as:

$$\mathcal{B}\left(t,t'\right)\equiv\begin{pmatrix}\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{13}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{16}\left(t,t'\right)\\\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{23}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{26}\left(t,t'\right)\\\mathcal{B}_{31}\left(t,t'\right)&\mathcal{B}_{32}\left(t,t'\right)&\mathcal{B}_{33}\left(t,t'\right)&\mathcal{B}_{32}\left(t,t'\right)&\mathcal{B}_{31}\left(t,t'\right)&\mathcal{B}_{36}\left(t,t'\right)\\\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{23}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{26}\left(t,t'\right)\\\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{13}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{16}\left(t,t'\right)\\\mathcal{B}_{61}\left(t,t'\right)&\mathcal{B}_{62}\left(t,t'\right)&\mathcal{B}_{63}\left(t,t'\right)&\mathcal{B}_{62}\left(t,t'\right)&\mathcal{B}_{61}\left(t,t'\right)&\mathcal{B}_{66}\left(t,t'\right)\end{pmatrix},\tag{897}$$

$$\mathcal{B}_{11}(t,t')=\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}^{\Re}\left(t\right)B_{01}^{\Re}\left(t'\right),\tag{898}$$

$$\mathcal{B}_{22}(t,t')=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)+B_{01}^{\Im}(t)B_{10}^{\Im}\left(t'\right),\tag{899}$$

$$\mathcal{B}_{12}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right),\tag{900}$$

$$\mathcal{B}_{21}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Im}\left(t\right)B_{10}^{\Re}\left(t'\right),\tag{901}$$

$$\mathcal{B}_{ij}\left(t,t'\right)=\int_{0}^{\infty}\left(P_{i}\left(\omega,t\right)P_{j}^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_{i}^{*}\left(\omega,t\right)P_{j}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega,i,j\in\left\{3,6\right\},\tag{902}$$

$$\mathcal{B}_{i1}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t'\right)\int_{0}^{\infty}\left(P_{i}\left(\omega,t\right)Q_{10}^{*}\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}^{*}\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega,i\in\left\{3,6\right\},\tag{903}$$

$$\mathcal{B}_{1i}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t\right)\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega,i\in\left\{3,6\right\},\tag{904}$$

$$\mathcal{B}_{i2}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t'\right)\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega,i\in\left\{3,6\right\},\tag{905}$$

$$\mathcal{B}_{2i}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t\right)\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega,i\in\left\{3,6\right\},\tag{906}$$

$$\zeta_{ij}\left(t,t'\right)=\mathrm{e}^{\mathrm{i}\Im\left(\int_{0}^{\infty}\frac{\left(L_{i}(\omega)F_{i}(\omega,t)-L_{j}(\omega)F_{j}(\omega,t)\right)\left(L_{i}(\omega)F_{i}(\omega,t')-L_{j}(\omega)F_{j}(\omega,t')\right)^{*}\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^{2}}\mathrm{d}\omega\right)},\tag{907}$$

$$\xi_{ij}^{\pm}\left(t,t'\right)=\mathrm{e}^{-\int_{0}^{\infty}\frac{\left|\left(L_{i}(\omega)F_{i}(\omega,t)-L_{j}(\omega)F_{j}(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm\left(L_{i}(\omega)F_{i}(\omega,t')-L_{j}(\omega)F_{j}(\omega,t')\right)\right|^{2}}{2\omega^{2}}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega},\tag{908}$$

$$P_{i}\left(\omega,t\right)=L_{i\mathrm{mod}2}\left(\omega\right)\left(1-F_{i\mathrm{mod}2}\left(\omega,t\right)\right),\tag{909}$$

$$Q_{ij}(\omega,t)=\frac{L_{i}\left(\omega\right)F_{1}\left(\omega,t\right)-L_{j}\left(\omega\right)F_{j}\left(\omega,t\right)}{\omega},\tag{910}$$

$$a_{i}\left(\omega,t\right)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)-\frac{2|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)},\tag{911}$$

$$b_{i}\left(\omega,t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)-\frac{2|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)},\tag{912}$$

$$r_{i}\left(\omega,t\right)=\frac{a_{i}\left(\omega,t\right)}{1-b_{0}\left(\omega,t\right)b_{1}\left(\omega,t\right)},\tag{913}$$

$$s_{i}\left(\omega,t\right)=\frac{a_{(i+1)\mathrm{mod}2}\left(\omega,t\right)b_{i\mathrm{mod}2}\left(\omega,t\right)}{1-b_{0}\left(\omega,t\right)b_{1}\left(\omega,t\right)},\tag{914}$$

$$F_{0}\left(\omega,t\right)=r_{0}\left(\omega,t\right)+\frac{L_{1}\left(\omega\right)}{L_{0}\left(\omega\right)}s_{0}\left(\omega,t\right),\tag{915}$$

$$F_{1}\left(\omega,t\right)=\frac{L_{0}\left(\omega\right)}{L_{1}\left(\omega\right)}r_{1}\left(\omega,t\right)+s_{1}\left(\omega,t\right).\tag{916}$$

The time-dependence of the system operators $\widetilde{A_{i}}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_{i}}\left(\tau\right)=e^{\mathrm{i}\overline{H_{\bar{S}}}\tau}A_{i}\left(t\right)\mathrm{e}^{-\mathrm{i}\overline{H_{\bar{S}}}\tau}\tag{917}$$

$$=\sum_{w}\mathrm{e}^{-\mathrm{i}w\tau}A_{i}\left(w\right).\tag{918}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w\in\{0,\pm\eta\}$.

In order to use the equation (918) to descompose the equation (373) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U\left(t\right)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \tag{919}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1) H(t_2) ... H(t_n). \tag{920}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0,t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_{\text{eff}}(t)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \equiv \exp(-\mathrm{i}tH_{\text{eff}}(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_{\text{eff}}(t) = H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + ...\right)\right). \tag{921}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \tag{922}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{923}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{924}$$

We can summarize that:

$$\widetilde{O}(t) \equiv U^\dagger(t) O(t) U(t), \tag{925}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_T}(t')\right) \tag{926}$$

$$= \exp\left(-\mathrm{i}\overline{H_{T,\text{eff}}}(t)\right), \text{ where} \tag{927}$$

$$H_X^{\text{eff}}(t) \equiv \frac{1}{t}\int_0^t H_X(t')\mathrm{d}t' - \frac{\mathrm{i}}{2t}\int_0^t\int_0^{t'}\left[H_X(t'), H_X(t'')\right]\mathrm{d}t'\mathrm{d}t'' + \frac{1}{6t}\int_0^t\int_0^{t'}\int_0^{t''}\left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t''' + \cdots. \tag{928}$$

In order to show the explicit form of the matrices present in the RHS of the equation (918) for a general $2 \times 2$ matrix in a given time let's write the matrix $A_i(t)$ in the base $W(t) = \left\{\left|\overline{H_{\bar{S},\text{eff},1}}(t)\right\rangle, \left|\overline{H_{\bar{S},\text{eff},0}}(t)\right\rangle\right\}$, formed by the time-dependent eigenvectors of $\overline{H_{\bar{S},\text{eff}}}(t)$ in the following way:

$$A_i(t) = \sum_{j,j'} \left\langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \left| A_i(t) \right| \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \right\rangle \left|\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\right\rangle\!\left\langle\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right|. \tag{929}$$

Let's obtain $U^\dagger(t') A_i(t) U(t')$ in explicit form:

$$U^\dagger(t')A_i(t)U(t') = U^\dagger(t')\left(\sum_{j,j'}\left\langle\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\left|A_i(t)\right|\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\right\rangle\!\left\langle\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right|\right)U(t') \tag{930}$$

$$= \sum_{j,j'}\left\langle\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\left|A_i(t)\right|\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right\rangle U^\dagger(t')\left|\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\right\rangle\!\left\langle\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right| U(t') \tag{931}$$

$$
\begin{aligned}
&= \sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t-\tau) \left| A_i(t) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t-\tau) \right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\lambda_j(t-\tau)} \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t-\tau) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t-\tau) \right| \mathrm{e}^{-\mathrm{i}(t-\tau)\lambda_{j'}(t-\tau)} && (932)\\
&= \sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t-\tau) \left| A_i(t) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t-\tau) \right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\left(\lambda_j(t-\tau)-\lambda_{j'}(t-\tau)\right)} \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t-\tau) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t-\tau) \right|, && (933)\\
M_{jj'}(t-\tau) &= \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},j}}\,(t-\tau) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j'}}\,(t-\tau) \right|, && (934)\\
U^\dagger(t')A_i(t)U(t') &= M_{00}(t-\tau) + M_{01}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)(\lambda_0(t-\tau)-\lambda_1(t-\tau))} + M_{10}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)(\lambda_1(t-\tau)-\lambda_0(t-\tau))} + M_{11}(t-\tau)\,, && (935)\\
w(t-\tau) &= \lambda_1\,(t-\tau) - \lambda_0\,(t-\tau)\,, && (936)\\
U^\dagger(t')A_i(t)U(t') &= M_{00}\,(t-\tau) + M_{01}\,(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} + M_{10}\mathrm{e}^{\mathrm{i}(t-\tau)w(t-\tau)} + M_{11} && (937)\\
&= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i\,(w\,(t-\tau)) && (938)\\
&= A_i\,(0) + A_i\,(w\,(t-\tau))\,\mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} + A_i\,(-w\,(t-\tau))\,\mathrm{e}^{\mathrm{i}(t-\tau)w(t-\tau)}. && (939)
\end{aligned}
$$

By direct comparison we obtain that:

$$
\begin{aligned}
A_i\,(w\,(t-\tau)) &= \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\mathrm{eff},1}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}\,(t-\tau) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}\,(t-\tau) \right|, && (940)\\
A_i\,(-w\,(t-\tau)) &= \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\mathrm{eff},0}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}\,(t-\tau) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}\,(t-\tau) \right|, && (941)\\
A_i\,(0) &= \sum_{j} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\mathrm{eff},j}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},j}}\,(t-\tau) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j}}\,(t-\tau) \right|. && (942)
\end{aligned}
$$

These matrix have the following property $A_i\,(w\,(t-\tau)) = A_i^\dagger\,(-w\,(t-\tau))$. Now in order to perform the double Fourier decomposition let's recall:

$$
\widetilde{A_i}\,(t,t') \equiv U\,(t)\,U^\dagger\,(t')\,A_i\,(t)\,U\,(t')\,U^\dagger\,(t)\,. \tag{943}
$$

In this case the decomposition can be written as:

$$
\begin{aligned}
\widetilde{A_i}\,(t,t-\tau) &\equiv U\,(t)\,U^\dagger\,(t-\tau)\,A_i\,(t)\,U\,(t-\tau)\,U^\dagger\,(t) && (944)\\
&= U\,(t)\left(\sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i\,(w\,(t-\tau))\right) U^\dagger\,(t)\,. && (945)
\end{aligned}
$$

Now writting $A_i\,(w\,(t-\tau))$ in terms of the eigenstates of $\overline{H_{\bar{S},\mathrm{eff}}}\,(t)$ we find:

$$
A_i\,(w\,(t-\tau)) = \sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}\,(t) \left| A_i\,(w\,(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}\,(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},j}}\,(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j'}}\,(t) \right|. \tag{946}
$$

Then the time evolution is given by:

$$\widetilde{A_i}(t,t-\tau) = U(t)\left(\sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i(w(t-\tau))\right) U^\dagger(t) \tag{947}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} U(t) A_i(w(t-\tau)) U^\dagger(t) \tag{948}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} U(t) \left(\sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right| \right) U^\dagger(t) \tag{949}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right\rangle U(t) \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right| U^\dagger(t) \tag{950}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right\rangle \mathrm{e}^{-\mathrm{i}t\lambda_j(t)} \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right| \mathrm{e}^{\mathrm{i}t\lambda_{j'}(t)}, \tag{951}$$

$$w'(t) = \lambda_1(t) - \lambda_0(t), \tag{952}$$

$$\widetilde{A_i}(t,t-\tau) = \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big( \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{953}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{954}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{955}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big) \tag{956}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{w'(t)} \mathrm{e}^{\mathrm{i}tw'(t)} A_{iww'}(t-\tau,t) \tag{957}$$

$$= \sum_{w(t-\tau),w'(t)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \mathrm{e}^{\mathrm{i}tw'(t)} A_{iww'}(t-\tau,t) \tag{958}$$

$$= \sum_{w(t-\tau),w'(t)} \mathrm{e}^{\mathrm{i}\tau w(t-\tau)} \mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} A_{iww'}(t-\tau,t). \tag{959}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big( \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{960}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{961}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{962}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big) \tag{963}$$

$$= \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(0) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| + \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(0) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{964}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{965}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{966}$$

$$+ \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big( \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{967}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{968}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{969}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big) \tag{970}$$

$$+ e^{\mathrm{i}(t-\tau)w(t-\tau)} \Big( \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{971}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{972}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{973}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big). \tag{974}$$

Directly we can find that the decomposition matrices are:

$$A_{i0w'}(t-\tau,t) = \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(0) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right|, \tag{975}$$

$$A_{iww'}(t-\tau,t) = \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right|, \tag{976}$$

$$A_{iw(-w')}\left(t-\tau,t\right)=\left\langle\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\left|A_i\left(w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},0}}\left(t-\tau\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},1}}\left(t-\tau\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},0}}\left(t-\tau\right)\right|, \tag{977}$$

$$A_{iw0}\left(t-\tau,t\right)=\sum_j\left\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\left|A_i\left(w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right|, \tag{978}$$

$$A_{i00}\left(t-\tau,t\right)=\sum_j\left\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_i\left(0\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right|, \tag{979}$$

$$A_{i0(-w')}\left(t-\tau,t\right)=\left\langle\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\left|A_i\left(0\right)\right|\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\right|, \tag{980}$$

$$A_{i(-w)0}\left(t-\tau,t\right)=\sum_j\left\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_i\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right|, \tag{981}$$

$$A_{i(-w)w'}\left(t-\tau,t\right)=\left\langle\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\left|A_i\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\right|, \tag{982}$$

$$A_{i(-w)(-w')}\left(t-\tau,t\right)=\left\langle\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\left|A_i\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},0}}\left(t-\tau\right)\right|. \tag{983}$$

Let's prove that $A_{jww'}\left(t-\tau,t\right)=A^{\dagger}_{j(-w)(-w')}\left(t-\tau,t\right)$:

$$\left(\left\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_i\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j'}}\left(t-\tau\right)\right|\right)^{\dagger}=\left\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_i\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\right\rangle^{*}\left|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right| \tag{984}$$

$$=\left\langle\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\left|A_i^{\dagger}\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right| \tag{985}$$

$$=\left\langle\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\left|A_i\left(w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right|. \tag{986}$$

It can be seen that the index $-w$ and $-w'$ change to the functions $w$ and $w'$.

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}\left(\tau\right)=\widetilde{A_i}^{\dagger}\left(\tau\right)$ and $\widetilde{B_i}\left(\tau\right)=\widetilde{B_i}^{\dagger}\left(\tau\right)$ we can use the equation (918) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ij}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big]+\mathcal{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \tag{987}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\Big[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\overline{\rho_S}(t)\Big] \tag{988}$$

$$-\mathcal{B}_{ji}\left(-\tau\right)\Big[A_i,\overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)\Big]\Big). \tag{989}$$

Now let's consider the following trace recalling that $\mathrm{Tr}\left(A\right)^{*}=\mathrm{Tr}\left(A^{\dagger}\right)$ so:

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(-\tau\right)\widetilde{B_i}\left(0\right)\rho_B\right)=\mathrm{Tr}_B\left(\mathrm{e}^{-\mathrm{i}\tau H_B(\tau)}B_j\mathrm{e}^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\right) \tag{990}$$

$$=\mathrm{Tr}_B\left(B_j\mathrm{e}^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\mathrm{e}^{-\mathrm{i}\tau H_B(\tau)}\right)\ \text{(by cyclic permutivity of trace)} \tag{991}$$

$$=\mathrm{Tr}_B\left(B_j\mathrm{e}^{\mathrm{i}\tau H_B(\tau)}B_i\mathrm{e}^{-\mathrm{i}\tau H_B(\tau)}\rho_B\right)\ \text{(by commutativity of } \mathrm{e}^{-\mathrm{i}\tau H_B(\tau)} \text{ and } \rho_B\text{)} \tag{992}$$

$$=\mathrm{Tr}_B\left(B_j\widetilde{B_i}\left(\tau\right)\rho_B\right)\ \text{(by definition of time evolution)} \tag{993}$$

$$=\mathrm{Tr}_B\left(B_j\widetilde{B_i}\left(\tau\right)\rho_B\right) \tag{994}$$

$$=\mathrm{Tr}_B\left(\rho_B B_j\widetilde{B_i}\left(\tau\right)\right) \tag{995}$$

$$=\mathrm{Tr}_B\left(\left(\widetilde{B_i}\left(\tau\right)B_j\rho_B\right)^{\dagger}\right)\ \text{(by definition of adjoint)} \tag{996}$$

$$=\mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)B_j\rho_B\right)^{*} \tag{997}$$

$$=\mathcal{B}^{*}_{ij}\left(\tau\right) \tag{998}$$

Given that $A_{jww'}\left(t-\tau,t\right)=A^{\dagger}_{j(-w)(-w')}\left(t-\tau,t\right)$ and $w\left(t-\tau\right),w'\left(t\right)$ belong to the set of differences of eigenvalues of $\overline{H^{\text{eff}}_{\bar{S}}}\left(t-\tau\right)$ and $\overline{H^{\text{eff}}_{\bar{S}}}\left(t\right)$ denoted by $J_t$ and $J_{t-\tau}$ respectively that depends of the time we can take an application where $w\left(t-\tau\right)\rightarrow-w\left(t-\tau\right)$ and $w'\left(t\right)\rightarrow-w'\left(t\right)$ such that the sum:

$$\sum_{ww'}\int_0^t \mathrm{d}\tau \mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)=\sum_{ww'}\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j(-w)(-w')}\left(t-\tau,t\right) \tag{999}$$

$$=\sum_{ww'}\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right). \tag{1000}$$

is invariant because if $\left(w\left(t-\tau\right),w'\left(t\right)\right)\in J_{t-\tau}\times J_t$ then $\left(-w\left(t-\tau\right),-w'\left(t\right)\right)\in J_{t-\tau}\times J_t$ where $J_t$ denotes the set of differences of eigenvalues at time $t$. So the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t)\,,\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)\,C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\left[A_i(t)\,,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\,\overline{\rho_S}(t)\right] \tag{1001}$$

$$+\mathcal{B}^{*}_{ij}\left(\tau\right)\left[\overline{\rho_S}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right),A_i\left(t\right)\right]\Big) \tag{1002}$$

With the definition:

$$L_{ijww'}\left(t,t'\right)\equiv\int_0^t C_i\left(t\right)C_j\left(t'\right)\mathcal{B}_{ij}\left(\tau\right)\mathrm{e}^{\mathrm{i}\tau w\left(t'\right)}\mathrm{e}^{-\mathrm{i}t\left(w\left(t'\right)-w'(t)\right)}A_{jww'}\left(t,t'\right)\mathrm{d}\tau. \tag{1003}$$

We can show that:

$$L^{\dagger}_{ijww'}\left(t,t'\right)=\int_0^t\Big(C_i\left(t\right)C_j\left(t'\right)\mathcal{B}_{ij}\left(t,t'\right)\mathrm{e}^{\mathrm{i}\tau w\left(t'\right)}\mathrm{e}^{-\mathrm{i}t\left(w\left(t'\right)-w'(t)\right)}A_{jww'}\left(t,t'\right)\mathrm{d}\tau\Big)^{\dagger} \tag{1004}$$

$$=\int_0^t C^{*}_i\left(t\right)C^{*}_j\left(t'\right)\mathcal{B}^{*}_{ij}\left(t,t'\right)\mathrm{e}^{-\mathrm{i}\tau w^{*}\left(t'\right)}\mathrm{e}^{\mathrm{i}t\left(w\left(t'\right)-w'(t)\right)^{*}}A^{\dagger}_{jww'}\left(t,t'\right)\mathrm{d}\tau \tag{1005}$$

$$=\int_0^t C_i\left(t\right)C_j\left(t'\right)\mathcal{B}^{*}_{ij}\left(\tau\right)\mathrm{e}^{-\mathrm{i}\tau w\left(t'\right)}\mathrm{e}^{\mathrm{i}t\left(w\left(t'\right)-w'(t)\right)}A^{\dagger}_{jww'}\left(t,t'\right)\mathrm{d}\tau. \tag{1006}$$

So we can write the master equation as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\left[A_i,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{1007}$$

$$-\mathcal{B}^{*}_{ij}\left(\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right)\right]\Big) \tag{1008}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijww'}\Big(\left[A_i,L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left[\overline{\rho_{\bar{S}}}\left(t\right)L^{\dagger}_{ijww'}\left(t\right),A_i\right]\Big). \tag{1009}$$

If we extend the upper limit of integration to $\infty$ in the equation (1006) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$\mathrm{e}^{-V}\overline{AB}\mathrm{e}^{V}=\mathrm{e}^{-V}\overline{A\mathbb{I}B}\mathrm{e}^{V} \tag{1010}$$

$$=\mathrm{e}^{-V}\overline{A}\mathrm{e}^{V}\mathrm{e}^{-V}\bar{B}\mathrm{e}^{V} \tag{1011}$$

$$=\left(\mathrm{e}^{-V}\overline{A}\mathrm{e}^{V}\right)\left(\mathrm{e}^{-V}\bar{B}\mathrm{e}^{V}\right) \tag{1012}$$

$$=AB. \tag{1013}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$\mathrm{e}^{-V}\overline{[A,B]}\mathrm{e}^{V} = \mathrm{e}^{-V}\overline{(AB-BA)}\mathrm{e}^{V} \tag{1014}$$
$$= \mathrm{e}^{-V}\overline{AB}\mathrm{e}^{V} - \mathrm{e}^{-V}\overline{BA}\mathrm{e}^{V} \tag{1015}$$
$$= AB - BA \tag{1016}$$
$$= [A,B]. \tag{1017}$$

So we will obtain that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}\mathrm{e}^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{S}}\left(t\right)\right]\mathrm{e}^{V} - \mathrm{e}^{-V}\sum_{ijww'}\left(\left[A_i, L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right] - \left[A_i, \overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right)\right]\right)\mathrm{e}^{V} \tag{1018}$$
$$= -\mathrm{i}\mathrm{e}^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{S}}\left(t\right)\right]\mathrm{e}^{V} - \sum_{ijww'}\left(\mathrm{e}^{-V}\left[A_i, L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]\mathrm{e}^{V} - \mathrm{e}^{-V}\left[A_i, \overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right)\right]\mathrm{e}^{V}\right) \tag{1019}$$
$$= -\mathrm{i}\left[H\left(t\right),\rho\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i, \mathrm{e}^{-V}L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\mathrm{e}^{V}\right] - \left[A_i, \mathrm{e}^{-V}\overline{\rho_{S}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right)\mathrm{e}^{V}\right]\right) \tag{1020}$$
$$= -\mathrm{i}\left[H(t),\rho(t)\right] - \sum_{ijww'}\left(\left[A_i, \mathrm{e}^{-V}L_{ijww'}\left(t\right)\mathrm{e}^{V}\mathrm{e}^{-V}\overline{\rho_{S}}(t)\mathrm{e}^{V}\right] - \left[\mathrm{e}^{-V}A_i\mathrm{e}^{V}, \mathrm{e}^{-V}\overline{\rho_{S}}(t)\mathrm{e}^{V}\mathrm{e}^{-V}L_{ijww'}\left(t\right)\mathrm{e}^{V}\right]\right) \tag{1021}$$
$$= -\mathrm{i}\left[H\left(t\right),\rho\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i, L_{ijww'}\left(t\right)\rho\left(t\right)\right] - \left[A_i, \rho\left(t\right)L_{ijww'}\left(t\right)\right]\right). \tag{1022}$$

Our master equation in the variationally optimized frame is:

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i, L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right] + \left[\overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right), A_i\right]\right), \tag{1023}$$
$$\dot{\rho}\left(t\right) = -\mathrm{i}\left[H\left(t\right),\rho\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i, L_{ijww'}\left(t\right)\rho\left(t\right)\right] + \left[\rho\left(t\right)L_{ijww'}^{\dagger}\left(t\right), A_i\right]\right). \tag{1024}$$

## IV. LIMIT CASES

In order to show the plausibility of the master equation (1023) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

### A. Time-dependent VPQME for 2LS with real-valued system Hamiltonian and real-valued uniform coupling

This hamiltonian has as particular feature that the coupling constants are real, so we know that $g_{\mathbf{k}} = g_{\mathbf{k}}^{*}$ then:

$$H_T\left(t\right) = H_S\left(t\right) + H_I + H_B, \tag{1025}$$
$$H_S\left(t\right) = \sum_i \varepsilon_i\left(t\right)|i\rangle\langle i| + \sum_{i\neq j} V_{ij}\left(t\right)|i\rangle\langle j|, \tag{1026}$$
$$H_I = \sum_i |i\rangle\langle i|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right), \tag{1027}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1028}$$

The transformed hamiltonian is:

$$\overline{H_{\bar{S}}}\left(t\right) \equiv \sum_i \left(\varepsilon_i\left(t\right) + R_i\left(t\right)\right)|i\rangle\langle i| + \sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right) - B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right) - \sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right) + B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right), \tag{1029}$$

We can summarize the principal results of the elements of the variational parameters and the transformed hamiltonians as:

$$\overline{H_{\bar{S}}}\left(t\right) \equiv \sum_{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)|i\rangle\langle i|+\sigma_{x}B_{10}\left(t\right)V_{10}\left(t\right)-\sigma_{y}B_{10}\left(t\right)V_{10}\left(t\right), \tag{1030}$$

$$R_{i}(t) = \int_{0}^{\infty}\frac{J(\omega)}{\omega}\left(F_{i}^{2}\left(\omega,t\right)-2F_{i}\left(\omega,t\right)\right)\mathrm{d}\omega, \tag{1031}$$

$$\chi_{ij}\left(t\right) = 0, \tag{1032}$$

$$B_{ij}(t) = \mathrm{e}^{-\frac{1}{2}\int_{0}^{\infty}\frac{J(\omega)\left(F_{i}(\omega,t)-F_{j}(\omega,t)\right)^{2}}{\omega^{2}}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{1033}$$

$$F_{i}(\omega,t) = \frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}\left(t\right)-\varepsilon\left(t\right)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{F_{i'}(\omega,t)g(\omega)}{\omega}B_{10}^{2}\left(t\right)V_{10}^{2}\left(t\right)\coth\left(\beta\omega/2\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)-\frac{2B_{10}^{2}(t)V_{10}^{2}(t)\coth(\beta\omega/2)}{\omega}\right)}, \tag{1034}$$

$$\eta\left(t\right) \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}, \tag{1035}$$

$$\varepsilon\left(t\right) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), \tag{1036}$$

$$J\left(\omega\right) \equiv \sum_{\mathbf{k}}\left|g_{\mathbf{k}}\right|^{2}\delta\left(\omega-\omega_{\mathbf{k}}\right). \tag{1037}$$

The Fourier decomposition remains without change:

$$L_{ijww'}\left(t,t'\right) \equiv \int_{0}^{t}C_{i}\left(t\right)C_{j}\left(t'\right)\mathcal{B}_{ij}\left(t,t'\right)\mathrm{e}^{\mathrm{i}\tau w\left(t'\right)}\mathrm{e}^{-\mathrm{i}t\left(w\left(t'\right)-w'\left(t\right)\right)}A_{jww'}\left(t,t'\right)\mathrm{d}\tau, \tag{1038}$$

$$t' = t-\tau, \tag{1039}$$

$$A \equiv \begin{pmatrix}\sigma_{x} & \sigma_{y} & \frac{I-\sigma_{z}}{2} & \sigma_{x} & \sigma_{y} & \frac{I+\sigma_{z}}{2}\end{pmatrix}, \tag{1040}$$

$$C\left(t\right) \equiv \begin{pmatrix}V_{10}\left(t\right) & V_{10}\left(t\right) & 1 & 0 & 0 & 1\end{pmatrix}, \tag{1041}$$

$$A_{j00}(t,t') = \sum_{i}\left\langle\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\left|A_{j0}\left(t'\right)\right|\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\right\rangle\left\langle\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\right|, \tag{1042}$$

$$A_{j0w'}(t,t') = \left\langle\overline{H_{\bar{S},\mathrm{eff},0}}\left(t\right)\left|A_{j0}\left(t'\right)\right|\overline{H_{\bar{S},\mathrm{eff},1}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\mathrm{eff},0}}\left(t\right)\right\rangle\left\langle\overline{H_{\bar{S},\mathrm{eff},1}}\left(t\right)\right|, \tag{1043}$$

$$A_{jw0}(t,t') = \sum_{i}\left\langle\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\left|A_{jw}\left(t'\right)\right|\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\right\rangle\left\langle\overline{H_{\bar{S},\mathrm{eff},i}}\left(t\right)\right|, \tag{1044}$$

$$A_{jww'}(t,t') = \left\langle\overline{H_{\bar{S},\mathrm{eff},0}}\left(t\right)\left|A_{jw}\left(t'\right)\right|\overline{H_{\bar{S},\mathrm{eff},1}}\left(t\right)\right\rangle\left|\overline{H_{\bar{S},\mathrm{eff},0}}\left(t\right)\right\rangle\left\langle\overline{H_{\bar{S},\mathrm{eff},1}}\left(t\right)\right|, \tag{1045}$$

$$A_{jw(-w')}(t,t') = \left\langle\overline{H_{\bar{S},\mathrm{eff},1}}\left(t\right)\left|A_{jw}\left(t'\right)\right|\overline{H_{\bar{S},\mathrm{eff},0}}\left(t-\tau\right)\right\rangle\left|\overline{H_{\bar{S},\mathrm{eff},1}}\left(t\right)\right\rangle\left\langle\overline{H_{\bar{S},\mathrm{eff},0}}\left(t\right)\right|, \tag{1046}$$

$$A_{j(-w)(-w')}(t,t') = A_{jww'}^{\dagger}(t,t') \tag{1047}$$

$$A_{j0}\left(t'\right) = \sum_{i}\left\langle\overline{H_{\bar{S},\mathrm{eff},i}}\left(t'\right)\left|A_{j}\left(t\right)\right|\overline{H_{\bar{S},\mathrm{eff},i}}\left(t'\right)\right\rangle\left|\overline{H_{\bar{S},\mathrm{eff},i}}\left(t'\right)\right\rangle\left\langle\overline{H_{\bar{S},\mathrm{eff},i}}\left(t'\right)\right|, \tag{1048}$$

$$A_{jw}\left(t'\right) = \left\langle\overline{H_{\bar{S},\mathrm{eff},0}}\left(t'\right)\left|A_{j}\left(t\right)\right|\overline{H_{\bar{S},\mathrm{eff},1}}\left(t'\right)\right\rangle\left|\overline{H_{\bar{S},\mathrm{eff},0}}\left(t'\right)\right\rangle\left\langle\overline{H_{\bar{S},\mathrm{eff},1}}\left(t'\right)\right|, \tag{1049}$$

$$A_{j(-w)}\left(t'\right) = A_{jw}^{\dagger}\left(t'\right). \tag{1050}$$

The effective hamiltonian is:

$$H_{X}^{\mathrm{eff}}(t) \equiv \frac{1}{t}\int_{0}^{t}H_{X}(t')\mathrm{d}t'-\frac{\mathrm{i}}{2t}\int_{0}^{t}\int_{0}^{t'}\left[H_{X}\left(t'\right),H_{X}(t'')\right]\mathrm{d}t'\mathrm{d}t''+\frac{1}{6t}\int_{0}^{t}\int_{0}^{t'}\int_{0}^{t''}\left(\left[\left[\overline{H_{\bar{S}}}(t'),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}(t''')\right]+\left[\left[\overline{H_{\bar{S}}}\left(t'''\right),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}\left(t'\right)\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t'''+\cdots, \tag{1051}$$

The correlation functions are:

$$\mathcal{B}\left(t,t'\right)\equiv\begin{pmatrix}\mathcal{B}_{11}\left(t,t'\right) & 0 & 0 & 0 & \mathcal{B}_{11}\left(t,t'\right) & 0\\ 0 & \mathcal{B}_{22}\left(t,t'\right) & \mathcal{B}_{23}\left(t,t'\right) & \mathcal{B}_{22}\left(t,t'\right) & 0 & \mathcal{B}_{26}\left(t,t'\right)\\ 0 & \mathcal{B}_{32}\left(t,t'\right) & \mathcal{B}_{33}\left(t,t'\right) & \mathcal{B}_{32}\left(t,t'\right) & 0 & \mathcal{B}_{36}\left(t,t'\right)\\ 0 & \mathcal{B}_{22}\left(t,t'\right) & \mathcal{B}_{23}\left(t,t'\right) & \mathcal{B}_{22}\left(t,t'\right) & 0 & \mathcal{B}_{26}\left(t,t'\right)\\ \mathcal{B}_{11}\left(t,t'\right) & 0 & 0 & 0 & \mathcal{B}_{11}\left(t,t'\right) & 0\\ 0 & \mathcal{B}_{62}\left(t,t'\right) & \mathcal{B}_{63}\left(t,t'\right) & \mathcal{B}_{62}\left(t,t'\right) & 0 & \mathcal{B}_{66}\left(t,t'\right)\end{pmatrix}, \tag{1052}$$

$$v_{i\mathbf{k}}^{*}\left(t\right)=v_{i\mathbf{k}}\left(t\right), \tag{1053}$$

$$\chi_{10}\left(t\right)=\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}^{*}\left(t\right)v_{j\mathbf{k}}\left(t\right)-v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}^{*}\left(t\right)}{2\omega_{\mathbf{k}}^{2}}\right) \tag{1054}$$

$$=\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}\left(t\right)-v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}\left(t\right)}{2\omega_{\mathbf{k}}^{2}}\right) \tag{1055}$$

$$=0, \tag{1056}$$

$$B_{10}\left(t\right)=B_{10}^{*}\left(t\right), \tag{1057}$$

$$Q_{ij}(\omega,t)=Q_{ij}^{*}(\omega,t), \tag{1058}$$

$$\zeta_{ij}\left(t,t'\right)=\mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty\frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)^{*}\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1059}$$

$$=\mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty\frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1060}$$

$$=\zeta_{ji}\left(t,t'\right), \tag{1061}$$

$$\mathcal{B}_{11}(t,t')=\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}\left(t'\right)}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}\left(t'\right)}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}^{\Re}\left(t\right)B_{01}^{\Re}\left(t'\right) \tag{1062}$$

$$=\frac{1}{2}\left(\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}\left(t\right)B_{01}\left(t'\right) \tag{1063}$$

$$=\frac{1}{2}\left(\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}\left(t\right)B_{01}\left(t'\right), \tag{1064}$$

$$\mathcal{B}_{22}(t,t')=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}\left(t'\right)}\right)\zeta_{10}(t,t')\xi_{10}^{+}(t,t')-\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}\left(t'\right)}\right)\zeta_{10}^{*}(t,t')\xi_{10}^{-}(t,t')\right)+B_{01}^{\Im}(t)B_{10}^{\Im}(t') \tag{1065}$$

$$=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right) \tag{1066}$$

$$=-\frac{1}{2}\left(\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right), \tag{1067}$$

$$\mathcal{B}_{12}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}\left(t'\right)}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}\left(t'\right)}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) \tag{1068}$$

$$=\frac{1}{2}\left(\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) \tag{1069}$$

$$=\frac{1}{2}\left(0\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+0\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)0 \tag{1070}$$

$$=0, \tag{1071}$$

$$\mathcal{B}_{21}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}\left(t'\right)}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}\left(t'\right)}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Im}\left(t\right)B_{10}^{\Re}\left(t'\right) \tag{1072}$$

$$=\frac{1}{2}\left(\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+0B_{10}^{\Re}\left(t'\right) \tag{1073}$$

$$=0, \tag{1074}$$

$$\mathcal{B}_{i2}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t'\right)\int_0^\infty\left(P_i^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\} \tag{1075}$$

$$=\mathrm{i}B_{10}\left(t'\right)\int_0^\infty\left(P_i\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\} \tag{1076}$$

$$=\mathrm{i}B_{10}\left(t'\right)\int_0^\infty P_i\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{1077}$$

$$\mathcal{B}_{2i}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t\right)\int_0^\infty\left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\left\{3,6\right\}, \tag{1078}$$

$$=\mathrm{i}B_{10}\left(t\right)\int_0^\infty\left(P_i\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\left\{3,6\right\}, \tag{1079}$$

$$=\mathrm{i}B_{10}\left(t\right)\int_0^\infty P_i\left(\omega,t'\right)Q_{10}\left(\omega,t\right)\left(N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\left\{3,6\right\}, \tag{1080}$$

$$P_i\left(\omega,t\right)=P_i^*\left(\omega,t\right), \tag{1081}$$

$$\mathcal{B}_{ij}\left(t,t'\right)=\int_0^\infty\left(P_i\left(\omega,t\right)P_j^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_i^*\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\left\{3,6\right\}, \tag{1082}$$

$$=\int_0^\infty\left(P_i\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_i\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\left\{3,6\right\}, \tag{1083}$$

$$=\int_0^\infty P_i\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}\left(N\left(\omega\right)+\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\left\{3,6\right\}, \tag{1084}$$

$$\mathcal{B}_{i1}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t'\right)\int_0^\infty\left(P_i\left(\omega,t\right)Q_{10}^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i^*\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1085}$$

$$=\mathrm{i}0\int_0^\infty\left(P_i\left(\omega,t\right)Q_{10}^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i^*\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1086}$$

$$=0, i\in\left\{3,6\right\}, \tag{1087}$$

$$\mathcal{B}_{1i}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t\right)\int_0^\infty\left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1088}$$

$$=\mathrm{i}0\int_0^\infty\left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1089}$$

$$=0, i\in\left\{3,6\right\}, \tag{1090}$$

$$\zeta_{ij}\left(t,t'\right)=\mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty\frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i\left(\omega,t'\right)-L_j(\omega)F_j\left(\omega,t'\right)\right)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)}, \tag{1091}$$

$$\xi_{ij}^{\pm}\left(t,t'\right)=\mathrm{e}^{-\int_0^\infty\frac{\left|\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm\left(L_i(\omega)F_i\left(\omega,t'\right)-L_j(\omega)F_j\left(\omega,t'\right)\right)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{1092}$$

$$P_i\left(\omega,t\right)=L_{i\mathrm{mod}2}\left(\omega\right)\left(1-F_{i\mathrm{mod}2}\left(\omega,t\right)\right), \tag{1093}$$

$$Q_{ij}(\omega,t)=\frac{L_i\left(\omega\right)F_1\left(\omega,t\right)-L_j\left(\omega\right)F_j\left(\omega,t\right)}{\omega}. \tag{1094}$$

### B. Time-independent variational quantum master equation

At first let's show that the master equation (1023) reproduces the results of the reference [1], for the latter case we have that $i,j\in\{1,2,3\}$ and $\omega\in(0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{1095}$$

After performing the transformation (25) on the Hamiltonian (1095) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S}=\left(\delta+R\right)|1\rangle\langle 1|+\frac{\Omega_r}{2}\sigma_x, \tag{1096}$$

$$\overline{H_I}=B_z|1\rangle\langle 1|+\frac{\Omega}{2}\left(B_x\sigma_x+B_y\sigma_y\right), \tag{1097}$$

$$H_B=\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{1098}$$

The Hamiltonian (1096) differs from the transformed Hamiltonian $H_S$ of the reference written like $H_S = \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$, where $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ (this base for the Pauli matrix is different from the base assumed in PRB2011 which is $\sigma_z' = |1\rangle\langle 1| - |0\rangle\langle 0|$) by a term proportional to the identity given by $-\frac{\delta}{2}\mathbb{I}$ which is independent of the variational parameters, this can be seen in the following way, with $\epsilon = \delta + R$:

$$
\begin{aligned}
(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} &= (\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| - \frac{\delta}{2}|1\rangle\langle 1| && (1099)\\
&= \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1100)\\
&= \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1101)\\
&= \frac{R}{2}|1\rangle\langle 1| + \left(\frac{\delta}{2} + \frac{R}{2}\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1102)\\
&= \frac{R}{2}|1\rangle\langle 1| + \frac{R}{2}|0\rangle\langle 0| + \left(\frac{\delta}{2} + \frac{R}{2}\right)|1\rangle\langle 1| - \frac{R}{2}|0\rangle\langle 0| - \frac{\delta}{2}|0\rangle\langle 0| && (1103)\\
&= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) && (1104)\\
&= \frac{R}{2}\mathbb{I} - \frac{\delta + R}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) && (1105)\\
&= \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) && (1106)\\
&= \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\sigma_z. && (1107)
\end{aligned}
$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$
\begin{aligned}
0 &= \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, && (1108)\\
\lambda_\pm &= \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} && (1109)\\
&= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} && (1110)\\
&= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} && (1111)\\
\eta &= \sqrt{\epsilon^2 + \Omega_r^2}, && (1112)\\
\lambda_\pm &= \frac{R \pm \eta}{2}. && (1113)
\end{aligned}
$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$
\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \quad (1114)
$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{1115}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle$. Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon+\eta}{2}, \tag{1116}$$

$$\lambda_- = \frac{\epsilon-\eta}{2}, \tag{1117}$$

$$|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle\,, \tag{1118}$$

$$|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle\,, \tag{1119}$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}, \tag{1120}$$

$$\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}. \tag{1121}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{1122}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{1123}$$

So the value of $\tan(\theta)$ using (1123) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}} + 1} \tag{1124}$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{1125}$$

$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{1126}$$

This proves our assertion.
Using this basis we can find the decomposition matrices using the equations (940)-(942) and the fact that $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$ and $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\begin{aligned}
\langle+|\sigma_x|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1127)\\
&= 2\sin(\theta)\cos(\theta) && (1128)\\
&= \sin(2\theta)\,, && (1129)\\
\langle-|\sigma_x|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1130)\\
&= -2\sin(\theta)\cos(\theta) && (1131)\\
&= -\sin(2\theta)\,, && (1132)\\
\langle-|\sigma_x|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1133)\\
&= \cos^2(\theta) - \sin^2(\theta) && (1134)\\
&= \cos(2\theta)\,, && (1135)\\
\langle+|\sigma_y|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1136)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (1137)\\
&= 0, && (1138)\\
\langle-|\sigma_y|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1139)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (1140)\\
&= 0, && (1141)\\
\langle-|\sigma_y|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1142)\\
&= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) && (1143)\\
&= \mathrm{i}. && (1144)
\end{aligned}$$

$$\begin{aligned}
\langle+|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1145)\\
&= \cos(\theta)\cos(\theta) && (1146)\\
&= \cos^2(\theta)\,, && (1147)\\
\langle-|\frac{1+\sigma_z}{2}|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1148)\\
&= \sin(\theta)\sin(\theta) && (1149)\\
&= \sin^2(\theta)\,, && (1150)\\
\langle-|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1151)\\
&= -\sin(\theta)\cos(\theta) && (1152)\\
&= -\sin(\theta)\cos(\theta)\,. && (1153)
\end{aligned}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$\begin{aligned}
A_1(0) &= \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), && (1154)\\
A_1(\eta) &= \cos(2\theta)\,|-\rangle\langle+|, && (1155)
\end{aligned}$$

$$A_2(0) = 0, \tag{1156}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{1157}$$
$$A_3(0) = \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-|, \tag{1158}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)\,|-\rangle\langle+|. \tag{1159}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (1023) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (1003) is equivalent to:

$$L_{ijww'}(t, t-\tau) \equiv \int_0^t C_i(t)\, C_j(t-\tau)\, \mathcal{B}_{ij}(t, t-\tau)\, \mathrm{e}^{\mathrm{i}\tau w(t-\tau)} \mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} A_{jww'}(t, t-\tau)\, \mathrm{d}\tau, \tag{1160}$$
$$= \int_0^t C_i(t)\, C_j(t-\tau)\, \mathcal{B}_{ij}(\tau)\, \mathrm{e}^{\mathrm{i}\tau w} \mathrm{e}^{-\mathrm{i}t\left(w-w'\right)} A_j(w, w')\, \mathrm{d}\tau. \tag{1161}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda_{ij}(\tau)$ relate with the correlation functions defined in the equation (414) in the following way:

$$\Lambda_{ij}(\tau) = C_i(t)\, C_j(t-\tau)\, \mathcal{B}_{ij}(\tau). \tag{1162}$$

So the response matrix can be rewritten as:

$$L_{ijww'}(t, t-\tau) = \left(\int_0^t \mathrm{d}\tau \Lambda_{ij}(\tau)\, \mathrm{e}^{\mathrm{i}\tau w} \mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}\right) A_j(w, w'). \tag{1163}$$

Let's define the response function like:

$$K_{ij}(w, w', t) = \int_0^t C_i(t)\, C_j(t-\tau)\, \mathcal{B}_{ij}(\tau)\, \mathrm{e}^{\mathrm{i}w\tau} \mathrm{e}^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1164}$$
$$= \int_0^t \Lambda_{ij}(\tau)\, \mathrm{e}^{\mathrm{i}w\tau} \mathrm{e}^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1165}$$
$$= K_{ijww'}(t). \tag{1166}$$

Then we have the following equivalence:

$$L_{ijww'}(t) = K_{ijww'}(t)\, A_j(w, w') \tag{1167}$$
$$= K_{ijww'}(t)\, A_{jww'}. \tag{1168}$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \left(\left[A_i, L_{ijww'}(t)\, \overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\, L^\dagger_{ijww'}(t)\right]\right) \tag{1169}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \left(\left[A_i, K_{ijww'}(t)\, A_{jww'}\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\, K^*_{ijww'}(t)\, A^\dagger_{jww'}\right]\right) \tag{1170}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \left(K_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho_S}(t)\right] - K^*_{ijww'}(t)\left[A_i, \overline{\rho_S}(t)\, A^\dagger_{jww'}\right]\right) \tag{1171}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} \left(\left(K^{\Re}_{ijww'}(t) + \mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i, A_{jww'}\overline{\rho_S}(t)\right] - \left(K^{\Re}_{ijww'}(t) - \mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i, \overline{\rho_S}(t) A^\dagger_{jww'}\right]\right) \tag{1172}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} K^{\Re}_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho_S}(t) - \overline{\rho_S}(t) A^\dagger_{jww'}\right] - \mathrm{i}\sum_{ijww'} K^{\Im}_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho_S}(t) + \overline{\rho_S}(t) A^\dagger_{jww'}\right] \tag{1173}$$

For the time-independent PRB2011 we have the following correlations obtained from the general model, we take account from the fact that $L_0(\omega) = 0$, $\Im(L_1(\omega)) = 0$, $\Im(F_1(\omega)) = 0$, $F_0(\omega) = 0$ and $\int_0^\infty |L_1(\omega)|^2 f(\omega)\,\mathrm{d}\omega = \int_0^\infty J(\omega) f(\omega)\,\mathrm{d}\omega$ for $f(\omega) \in L^2$. We can drop the time vector $(t, t')$ and instead write the correlation functions as function of $\tau$, we will drop $t$ and $t'$ from the expressions that contain them given the time independence of the hamiltonian:

$$\chi_{ij}(t) = \int_0^\infty \frac{L_1^*(\omega) L_0(\omega) F_1^*(\omega,t) F_0(\omega,t) - L_1(\omega) L_0^*(\omega) F_1(\omega,t) F_0^*(\omega,t)}{2\omega^2}\mathrm{d}\omega, \tag{1174}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^+(t,t') + \Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^*(t,t')\,\xi_{10}^-(t,t')\right) - B_{10}^{\Re}(t)\,B_{01}^{\Re}(t'), \tag{1175}$$

$$= \frac{1}{2}\left(\mathrm{e}^{\chi_{10}+\chi_{10}}\zeta_{10}\xi_{10}^+ + \mathrm{e}^{\chi_{10}+\chi_{01}}\zeta_{10}^*\xi_{10}^-\right) - B^2, \tag{1176}$$

$$\chi_{ij} = \int_0^\infty \frac{L_1(\omega) L_0(\omega) F_1(\omega) F_0(\omega) - L_1(\omega) L_0(\omega) F_1(\omega) F_0(\omega)}{2\omega^2}\mathrm{d}\omega, \tag{1177}$$

$$= 0 \tag{1178}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\mathrm{e}^0\zeta_{10}\xi_{10}^+ + \mathrm{e}^0\zeta_{10}^*\xi_{10}^-\right) - B^2, \tag{1179}$$

$$\zeta_{10} = \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{(L_1(\omega)F_1(\omega)-L_0(\omega)F_0(\omega))(L_1(\omega)F_1(\omega)-L_0(\omega)F_0(\omega))^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1180}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{(L_1(\omega)F_1(\omega))(L_1(\omega)F_1(\omega))^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1181}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{J(\omega)F^2(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1182}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}(\cos(\omega\tau)+\mathrm{i}\sin(\omega\tau))\mathrm{d}\omega\right)} \tag{1183}$$

$$= \mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega} \tag{1184}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\xi_{10}^+ + \mathrm{e}^{-\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\xi_{10}^-\right) - B^2, \tag{1185}$$

$$\xi_{10}^\pm = \mathrm{e}^{-\int_0^\infty \frac{\left|(L_1(\omega)F_1(\omega)-L_0(\omega)F_0(\omega))\mathrm{e}^{\mathrm{i}\omega\tau}\pm L_1(\omega)F_1(\omega)\mp L_0(\omega)F_0(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1186}$$

$$= \mathrm{e}^{-\int_0^\infty \frac{\left|L_1(\omega)F_1(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}\pm L_1(\omega)F_1(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1187}$$

$$= -\int_0^\infty \frac{J(\omega)F^2(\omega)\left|\mathrm{e}^{\mathrm{i}\omega\tau}\pm 1\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega \tag{1188}$$

$$\left|\mathrm{e}^{\mathrm{i}\omega\tau}\pm 1\right|^2 = 2\left(1\pm\cos(\omega\tau)\right) \tag{1189}$$

$$\xi_{10}^\pm = \mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1\pm\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1190}$$

$$\mathcal{B}_{11}(\tau) = -B^2 + \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1+\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right. \tag{1191}$$

$$\left. + \mathrm{e}^{-\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1-\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right) \tag{1192}$$

$$= -B^2 + \frac{\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}}{2}\left(\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right. \tag{1193}$$

$$\left. + \mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(-\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)+\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right) \tag{1194}$$

$$B = \mathrm{e}^{-\frac{1}{2}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{1195}$$

$$G_+(\omega) = \mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + \mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right) \tag{1196}$$

$$= \left(\cos\left(\omega\tau\right) + \mathrm{i}\sin\left(\omega\tau\right)\right) N\left(\omega\right) + \left(\cos\left(\omega\tau\right) - \mathrm{i}\sin\left(\omega\tau\right)\right)\left(N\left(\omega\right)+1\right) \tag{1197}$$

$$= \cos\left(\omega\tau\right)\left(2N\left(\omega\right)+1\right) - \mathrm{i}\sin\left(\omega\tau\right) \tag{1198}$$

$$= \cos\left(\omega\tau\right)\left(\frac{2}{\mathrm{e}^{\beta\omega}-1}+1\right) - \mathrm{i}\sin\left(\omega\tau\right) \tag{1199}$$

$$= \cos\left(\omega\tau\right)\left(\frac{1+\mathrm{e}^{\beta\omega}}{\mathrm{e}^{\beta\omega}-1}\right) - \mathrm{i}\sin\left(\omega\tau\right) \tag{1200}$$

$$= \cos\left(\omega\tau\right)\left(\frac{\mathrm{e}^{-\beta\omega/2}+\mathrm{e}^{\beta\omega/2}}{-\mathrm{e}^{-\beta\omega/2}+\mathrm{e}^{\beta\omega/2}}\right) - \mathrm{i}\sin\left(\omega\tau\right) \tag{1201}$$

$$= \cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right), \tag{1202}$$

$$\phi\left(\tau\right) = \int_0^\infty \frac{J\left(\omega\right)F^2\left(\omega\right)}{\omega^2} G_+\left(\omega,\tau\right)\mathrm{d}\omega \tag{1203}$$

$$= \int_0^\infty \frac{J\left(\omega\right)F^2\left(\omega\right)}{\omega^2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right)\mathrm{d}\omega, \tag{1204}$$

$$\mathcal{B}_{11}(\tau) = -B^2 + \frac{\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}}{2}\left(\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right. \tag{1205}$$

$$\left. + \mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(-\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)+\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right) \tag{1206}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{-\phi(\tau)} + \mathrm{e}^{\phi(\tau)} - 2\right) \tag{1207}$$

$$\mathcal{B}_{22}(\tau) = -\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\xi_{10}^{+}(t,t') - \Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}(t,t')\xi_{10}^{-}(t,t')\right) + B_{01}^{\Im}(t)B_{10}^{\Re}(t'), \tag{1208}$$

$$= -\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}+\chi_{01}}\right)\zeta_{10}\xi_{10}^{+} - \left(\mathrm{e}^{\chi_{10}+\chi_{01}}\right)\zeta_{10}^{*}\xi_{10}^{-}\right), \tag{1209}$$

$$= \frac{1}{2}\left(\zeta_{10}^{*}\xi_{10}^{-} - \zeta_{10}\xi_{10}^{+}\right), \tag{1210}$$

$$= \frac{1}{2}\left(\mathrm{e}^{-\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1-\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right. \tag{1211}$$

$$\left. - \mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1+\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right) \tag{1212}$$

$$= \frac{1}{2}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\left(\mathrm{e}^{\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right. \tag{1213}$$

$$\left. - \mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right) \tag{1214}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} - \mathrm{e}^{-\phi(\tau)}\right), \tag{1215}$$

$$\mathcal{B}_{12}(\tau) = \frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right) + \Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right) + B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) \tag{1216}$$

$$= 0, \tag{1217}$$

$$\mathcal{B}_{21}(\tau) = \frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^{*}(t,t')\,\xi_{10}^{-}(t,t') + \Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^{+}(t,t')\right) + B_{10}^{\Im}(t)\,B_{10}^{\Re}(t'), \tag{1218}$$

$$= 0, \tag{1219}$$

$$\mathcal{B}_{ij}\left(t,t'\right) = \int_0^\infty \left(P_i\left(\omega,t\right)P_j^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right) + P_i^{*}\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j \in \{3,6\}, \tag{1220}$$

$$\mathcal{B}_{66}\left(t,t'\right) = \int_0^\infty \left(P_6\left(\omega,t\right)P_6^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right) + P_6^{*}\left(\omega,t\right)P_6\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1221}$$

$$P_6(\omega,t) = L_{6\mathrm{mod}2}(\omega)(1-F_{6\mathrm{mod}2}(\omega,t)), \tag{1222}$$
$$= L_0(\omega)(1-F_0(\omega,t)), \tag{1223}$$
$$= 0, \tag{1224}$$
$$\mathcal{B}_{66}(\tau) = 0, \tag{1225}$$
$$\mathcal{B}_{36}(t,t') = \int_0^\infty \left(P_3(\omega,t)P_6^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)P_6(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1226}$$
$$= 0, \tag{1227}$$
$$\mathcal{B}_{63}(t,t') = \int_0^\infty \left(P_6(\omega,t)P_3^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_6^*(\omega,t)P_3(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1228}$$
$$= 0, \tag{1229}$$
$$\mathcal{B}_{33}(t,t') = \int_0^\infty \left(P_3(\omega,t)P_3^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)P_3(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1230}$$
$$= \int_0^\infty \left(P_3(\omega,t)P_3^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)P_3(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1231}$$
$$P_3(\omega,t) = L_{3\mathrm{mod}2}(\omega)(1-F_{3\mathrm{mod}2}(\omega,t)), \tag{1232}$$
$$= L_1(\omega)(1-F_1(\omega,t)), \tag{1233}$$
$$P_3(\omega,t)P_3^*(\omega,t') = L_1(\omega)(1-F_1(\omega))L_1^*(\omega)(1-F_1(\omega)), \tag{1234}$$
$$= |L_1(\omega)|^2(1-F_1(\omega))^2 \tag{1235}$$
$$\mathcal{B}_{33}(t,t') = \int_0^\infty |L_1(\omega)|^2(1-F_1(\omega))^2\left(\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + \mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1236}$$
$$= \int_0^\infty J(\omega)(1-F(\omega))^2 G_+(\omega)\mathrm{d}\omega \tag{1237}$$
$$\mathcal{B}_{i1}(t,t') = \mathrm{i}B_{01}^{\Im}(t')\int_0^\infty \left(P_i(\omega,t)Q_{10}^*(\omega,t')N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_i^*(\omega,t)Q_{10}(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega, i\in\{3,6\} \tag{1238}$$
$$= 0, \tag{1239}$$
$$\mathcal{B}_{1i}(t,t') = \mathrm{i}B_{01}^{\Im}(t)\int_0^\infty \left(P_i^*(\omega,t')Q_{10}(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_i(\omega,t')Q_{10}^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{1240}$$
$$= 0, \tag{1241}$$
$$\mathcal{B}_{62}(t,t') = \mathrm{i}B_{10}^{\Re}(t')\int_0^\infty \left(P_6^*(\omega,t')Q_{10}(\omega,t')(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} - P_6(\omega,t')Q_{10}^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{1242}$$
$$= 0, \tag{1243}$$
$$\mathcal{B}_{26}(t,t') = \mathrm{i}B_{10}^{\Re}(t)\int_0^\infty \left(P_6^*(\omega,t')Q_{10}(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_6(\omega,t')Q_{10}^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1244}$$
$$= 0, \tag{1245}$$
$$\mathcal{B}_{32}(t,t') = \mathrm{i}B_{10}^{\Re}(t')\int_0^\infty \left(P_3^*(\omega,t')Q_{10}(\omega,t')(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} - P_3(\omega,t')Q_{10}^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{1246}$$
$$= \mathrm{i}B\int_0^\infty \left(P_3^*(\omega)Q_{10}(\omega)(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} - P_3(\omega)Q_{10}^*(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega, \tag{1247}$$
$$Q_{ij}(\omega,t) = \frac{L_i(\omega)F_j(\omega,t) - L_i(\omega)F_j(\omega,t)}{\omega}, \tag{1248}$$
$$Q_{10}(\omega,t) = \frac{L_1(\omega)F_1(\omega,t)}{\omega}, \tag{1249}$$
$$\mathcal{B}_{32}(t,t') = \mathrm{i}B\int_0^\infty \Bigg(L_1^*(\omega)(1-F_1(\omega,t))\frac{L_1(\omega)F_1(\omega,t)}{\omega}(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} \tag{1250}$$
$$-L_1(\omega)(1-F_1(\omega,t))\frac{L_1^*(\omega)F_1(\omega,t)}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\Bigg)\mathrm{d}\omega \tag{1251}$$

$$
\begin{aligned}
&= \mathrm{i}B \int_0^\infty |L_1(\omega)|^2 \left( (1 - F_1(\omega)) \frac{F_1(\omega)}{\omega} (N(\omega)+1) \mathrm{e}^{-\mathrm{i}\omega\tau} - (1 - F_1(\omega)) \frac{F_1(\omega)}{\omega} \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) \right) \mathrm{d}\omega && (1252) \\
&= \mathrm{i}B \int_0^\infty J(\omega) (1 - F(\omega,t)) \frac{F(\omega)}{\omega} \big( (N(\omega)+1) \mathrm{e}^{-\mathrm{i}\omega\tau} - \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) \big) \mathrm{d}\omega && (1253) \\
&= \mathrm{i}B \int_0^\infty J(\omega) (1 - F(\omega,t)) \frac{F(\omega)}{\omega} G_-(\omega) \mathrm{d}\omega && (1254) \\
\mathcal{B}_{23}(t,t') &= \mathrm{i}B_{10}^{\Re}(t) \int_0^\infty \big( P_3^*(\omega,t') Q_{10}(\omega,t) N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - P_3(\omega,t') Q_{10}^*(\omega,t) \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega)+1) \big) \mathrm{d}\omega && (1255) \\
&= \mathrm{i}B \int_0^\infty \big( P_3^*(\omega,t') Q_{10}(\omega,t) N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - P_3(\omega,t') Q_{10}^*(\omega,t) \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega)+1) \big) \mathrm{d}\omega && (1256) \\
&= \mathrm{i}B \int_0^\infty \left( L_1^*(\omega)(1-F_1(\omega,t)) \frac{L_1(\omega)F_1(\omega,t)}{\omega} N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - L_1(\omega)(1-F_1(\omega,t)) \frac{L_1^*(\omega)F_1(\omega,t)}{\omega} \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega)+1) \right) \mathrm{d}\omega && (1257) \\
&= \mathrm{i}B \int_0^\infty J(\omega) (1 - F_1(\omega,t)) \frac{F_1(\omega,t)}{\omega} \big( N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega)+1) \big) \mathrm{d}\omega && (1258) \\
&= -\mathrm{i}B \int_0^\infty J(\omega) (1 - F_1(\omega,t)) \frac{F_1(\omega,t)}{\omega} \big( -N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} + \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega)+1) \big) \mathrm{d}\omega && (1259) \\
&= -\mathcal{B}_{32}(t,t') && (1260) \\
\zeta_{ij}(t,t') &= \mathrm{e}^{\mathrm{i}\Im\left( \int_0^\infty \frac{(L_i(\omega)F_i(\omega,t) - L_j(\omega)F_j(\omega,t))(L_i(\omega)F_i(\omega,t') - L_j(\omega)F_j(\omega,t'))^* \mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2} \mathrm{d}\omega \right)}, && (1261) \\
\xi_{ij}^{\pm}(t,t') &= \mathrm{e}^{-\int_0^\infty \frac{\left| (L_i(\omega)F_i(\omega,t) - L_j(\omega)F_j(\omega,t)) \mathrm{e}^{\mathrm{i}\omega\tau} \pm L_i(\omega)F_i(\omega,t') \mp L_j(\omega)F_j(\omega,t') \right|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) \mathrm{d}\omega}, && (1262) \\
P_i(\omega,t) &= L_{i\mathrm{mod}2}(\omega) (1 - F_{i\mathrm{mod}2}(\omega,t)), && (1263) \\
Q_{ij}(\omega,t) &= \frac{L_i(\omega) F_j(\omega,t) - L_i(\omega) F_j(\omega,t)}{\omega}, && (1264)
\end{aligned}
$$

$$
\mathcal{B}(\tau) \equiv \begin{pmatrix}
\mathcal{B}_{11}(\tau) & 0 & 0 & 0 & \mathcal{B}_{11}(\tau) & 0 \\
0 & \mathcal{B}_{22}(\tau) & \mathcal{B}_{23}(\tau) & \mathcal{B}_{22}(\tau) & 0 & 0 \\
0 & -\mathcal{B}_{23}(\tau) & \mathcal{B}_{33}(\tau) & -\mathcal{B}_{23}(\tau) & 0 & 0 \\
0 & \mathcal{B}_{22}(\tau) & \mathcal{B}_{23}(\tau) & \mathcal{B}_{22}(\tau) & 0 & 0 \\
\mathcal{B}_{11}(\tau) & 0 & 0 & 0 & \mathcal{B}_{11}(\tau) & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{1265}
$$

The correlation functions as we can see in PRB2011 can be obtained using the following definition:

$$
\Lambda_{ij}(\tau) \equiv C_i C_j \mathcal{B}_{ij}(\tau) \tag{1266}
$$

Also the matrix C(t) can be decomposed as:

$$
\begin{aligned}
C_1 = C_2 &= \frac{\Omega}{2}, && (1267) \\
C_3 = C_6 &= 1, && (1268) \\
C_4 = C_4 &= 0 && (1269)
\end{aligned}
$$

Let's recall that $\Omega_r = B\Omega$. So the correlation functions $\Lambda$ are:

$$
\begin{aligned}
\Lambda_{11}(\tau) &= C_1 C_1 \mathcal{B}_{11}(\tau) && (1270) \\
&= \left(\frac{\Omega}{2}\right)^2 \frac{B^2}{2} \left( \mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2 \right) && (1271)
\end{aligned}
$$

$$= \frac{(\Omega B)^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{1272}$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1273}$$

$$\Lambda_{22}(\tau) = C_2 C_2 \mathcal{B}_{22}(\tau) \tag{1274}$$

$$= \left(\frac{\Omega}{2}\right)^2 \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} - \mathrm{e}^{-\phi(\tau)}\right) \tag{1275}$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} - \mathrm{e}^{-\phi(\tau)}\right), \tag{1276}$$

$$\Lambda_{33}(\tau) = C_3 C_3 \mathcal{B}_{33}(\tau) \tag{1277}$$

$$= (1)^2 \int_0^\infty J(\omega)\,(1 - F(\omega))^2\, G_+(\omega)\,\mathrm{d}\omega \tag{1278}$$

$$= \int_0^\infty J(\omega)\,(1 - F(\omega))^2\, G_+(\omega)\,\mathrm{d}\omega, \tag{1279}$$

$$\Lambda_{23}(\tau) = C_2 C_3 \mathcal{B}_{23}(\tau) \tag{1280}$$

$$= \frac{\Omega}{2} 1\mathrm{i}B \int_0^\infty J(\omega)\,(1 - F(\omega, t))\, \frac{F(\omega)}{\omega} G_-(\omega)\,\mathrm{d}\omega \tag{1281}$$

$$= \mathrm{i}\frac{\Omega_r}{2} \int_0^\infty J(\omega)\,(1 - F(\omega, t))\, \frac{F(\omega)}{\omega} G_-(\omega)\,\mathrm{d}\omega, \tag{1282}$$

$$\Lambda_{12}(\tau) = \Lambda_{13}(\tau) = \Lambda_{16}(\tau) \tag{1283}$$

$$= \Lambda_{21}(\tau) = \Lambda_{26}(\tau) \tag{1284}$$

$$= \Lambda_{31}(\tau) = \Lambda_{36}(\tau) = 0. \tag{1285}$$

Now let's define:

$$K_{ijw}(t) = \int_0^t C_i C_j \mathcal{B}_{ij}(\tau)\, \mathrm{e}^{\mathrm{i}w\tau} \mathrm{d}\tau. \tag{1286}$$

So

$$L_{ijw}(t) = A_{jw} K_{ijw}(t). \tag{1287}$$

Now for a time-independent hamiltonian is possible to show that for the decomposition matrix $A_j(w(t)) = A_j(w)$:

$$U^\dagger(t)\, A_j(t)\, U(t) = \sum_w \mathrm{e}^{-\mathrm{i}wt} A_j(w). \tag{1288}$$

It means that a decomposition matrix of $\widetilde{A_j}(t)$ associated to the eigenvector under evolution for the same time-independent hamiltonian $U(t)\, A_j(w)\, U^\dagger(t)$ generates the same decomposition matrix multiplied by a phase $\mathrm{e}^{\mathrm{i}wt}$. It means that the decomposition matrix $A_{jww'}$ for a time-independent hamiltonian fulfill $A_{jww'} = A_j(w)\,\delta_{ww'}$ so only if $w = w'$ then the response function is relevant for taking account and it's equal to:

$$K_{ijww}(t) = \int_0^t C_i C_j \mathcal{B}_{ij}(\tau)\, \mathrm{e}^{\mathrm{i}w\tau} \mathrm{d}\tau \tag{1289}$$

$$\equiv K_{ijw}(t). \tag{1290}$$

Finally taking the Hamiltonian (1095) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that Det $\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, Tr $\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (338) we have that:

$$f_k = \frac{g_k\left(1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1-\frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon-\frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{1291}$$

$$= \frac{g_k\left(1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1-\frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{1292}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (450) under a time-independent approach providing similar results.

The master equation for this special case is:

$$L_{ijww'}(t) = \delta_{ww'}A_{jw}K_{ijw}(t), \tag{1293}$$

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijww'}\left(\left[A_i, L_{ijww'}(t)\overline{\rho_{\bar{S}}}(t)\right] + \left[\overline{\rho_{\bar{S}}}(t)L^\dagger_{ijww'}(t), A_i\right]\right) \tag{1294}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijw}\left(\left[A_i, L_{ijw}(t)\overline{\rho_{\bar{S}}}(t)\right] + \left[\overline{\rho_{\bar{S}}}(t)L^\dagger_{ijw}(t), A_i\right]\right) \tag{1295}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijw}\left(\left[A_i, A_{jw}K_{ijw}(t)\overline{\rho_{\bar{S}}}(t)\right] + \left[\overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}K^*_{ijw}(t), A_i\right]\right) \tag{1296}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijw}\left(\left(K^{\Re}_{ijw}(t) + \mathrm{i}K^{\Im}_{ijw}(t)\right)\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t)\right] + \left(K^{\Re}_{ijw}(t) - \mathrm{i}K^{\Im}_{ijw}(t)\right)\left[\overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}, A_i\right]\right) \tag{1297}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijw}\left(K^{\Re}_{ijw}(t)\left(\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t)\right] + \left[\overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}, A_i\right]\right) + \mathrm{i}K^{\Im}_{ijw}(t)\left(\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t)\right] - \left[\overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}, A_i\right]\right)\right) \tag{1298}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijw}\left(K^{\Re}_{ijw}(t)\left(\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t)\right] - \left[A_i, \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right]\right) + \mathrm{i}K^{\Im}_{ijw}(t)\left(\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t)\right] + \left[A_i, \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right]\right)\right) \tag{1299}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijw}\left(K^{\Re}_{ijw}(t)\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t) - \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right] + \mathrm{i}K^{\Im}_{ijw}(t)\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t) + \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right]\right) \tag{1300}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \sum_{ijw}K^{\Re}_{ijw}(t)\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t) - \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right] - \mathrm{i}\sum_{ijw}K^{\Im}_{ijw}(t)\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t) + \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right], \tag{1301}$$

$$\gamma_{ij}(w,t) \equiv 2K^{\Re}_{ijw}(t) \tag{1302}$$

$$S_{ij}(w,t) \equiv K^{\Im}_{ijw}(t) \tag{1303}$$

$$A_j(\omega) \equiv A_{jw} \tag{1304}$$

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \frac{1}{2}\sum_{ijw}\gamma_{ij}(w,t)\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t) - \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right] - \mathrm{i}\sum_{ijw}S_{ij}(w,t)\left[A_i, A_{jw}\overline{\rho_{\bar{S}}}(t) + \overline{\rho_{\bar{S}}}(t)A^\dagger_{jw}\right] \tag{1305}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_{\bar{S}}}(t)\right] - \frac{1}{2}\sum_{ijw}\gamma_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho_{\bar{S}}}(t) - \overline{\rho_{\bar{S}}}(t)A^\dagger_j(w)\right] - \mathrm{i}\sum_{ijw}S_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho_{\bar{S}}}(t) + \overline{\rho_{\bar{S}}}(t)A^\dagger_j(w)\right]. \tag{1306}$$

### C. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b^\dagger_{\mathbf{k}} + g^*_{\mathbf{k}}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}. \tag{1307}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so $B$ is independent of the time. In order to reproduce the Hamiltonian of the equation (1307) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (1096) and (1097) we obtain ther following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t), \tag{1308}$$

$$\overline{H_{\mathrm{I}}} = \frac{\Omega(t)}{2}(B_x\sigma_x + B_y\sigma_y). \tag{1309}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (1270)-(1285) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}G_+(\tau)\,\mathrm{d}\omega. \tag{1310}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (1310) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega. \tag{1311}$$

Writing the interaction Hamiltonian (1309) in the similar way to the equation (1097) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (1096) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t). \tag{1312}$$

As we can see the function $B$ is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{1313}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1314}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{1315}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right). \tag{1316}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(450) is:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{1317}$$

$$\left.+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right). \tag{1318}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (1313) and (1316) on the equation (1318) we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{1319}$$

$$\left.+\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right). \tag{1320}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

### D. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (414) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*} b_{\mathbf{k}}\right). \tag{1321}$$

Using the equation (1023), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)), \rho_S(t)\right] - \frac{1}{2}\sum_{w} \gamma_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) - \rho_S(t)A_3^{\dagger}(w)\right] \tag{1322}$$

$$-\sum_{w} S_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) + \rho_S(t)A_3^{\dagger}(w)\right]\Bigg). \tag{1323}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^{\infty} \mathrm{d}\omega J(\omega) G_+(\tau), \tag{1324}$$

$$\Lambda_{33}(-\tau) = \int_0^{\infty} \mathrm{d}\omega J(\omega) G_+(-\tau). \tag{1325}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (1323) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t))\right] - \sum_{w}\left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^{*}(w,t)\left[\rho_S(t)A_3(w), A_3\right]\right). \tag{1326}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{1327}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle), \tag{1328}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{1329}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{1330}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1331}$$

$$= \frac{1}{2}, \tag{1332}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{1333}$$

$$= \frac{1}{2}, \tag{1334}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1335}$$

$$= -\frac{1}{2}. \tag{1336}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{1337}$$

$$= \frac{\mathbb{I}}{2}, \tag{1338}$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle+| \tag{1339}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right), \tag{1340}$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle-| \tag{1341}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right). \tag{1342}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S(t))\right] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S(t)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)\right] \tag{1343}$$

$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right]. \tag{1344}$$

Calculating the response functions extending the upper limit of $\tau$ to $\infty$, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega)\, G_+(\tau)\, \mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1345}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, \mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n(\omega)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega)\,\mathrm{e}^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1346}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\, \mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\left(n(\omega)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1347}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\left(n(\omega)+1\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1348}$$

$$= \int_0^\infty J(\omega)\left(n(\omega)+1\right)\pi\delta\left(\widetilde{\Omega} - \omega\right)\mathrm{d}\omega \tag{1349}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right), \tag{1350}$$

$$K_{33}\left(-\widetilde{\Omega}\right)=\int_0^\infty\int_0^\infty J\left(\omega\right)G_+\left(\tau\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1351}$$

$$=\int_0^\infty\int_0^\infty J\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}+n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1352}$$

$$=\int_0^\infty\int_0^\infty J\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1353}$$

$$=\int_0^\infty\int_0^\infty J\left(\omega\right)n\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau+\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1354}$$

$$=\int_0^\infty J\left(\omega\right)n\left(\omega\right)\pi\delta\left(-\widetilde{\Omega}+\omega\right)\mathrm{d}\omega \tag{1355}$$

$$=\pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right). \tag{1356}$$

Here we have used $\int_0^\infty \mathrm{d}s\,\mathrm{e}^{\pm i\varepsilon s}=\pi\delta\left(\varepsilon\right)\pm\mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3\left(0\right)$ because the spectral density $J\left(\omega\right)$ is equal to zero when $\omega=0$. Replacing in the equation (1343) lead us to obtain:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t}=-\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x,\rho_S\left(t\right)\right)]-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z,\left(\sigma_z+\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]+n\left(\widetilde{\Omega}\right)\left[\sigma_z,\left(\sigma_z-\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{1357}$$

$$-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z+\mathrm{i}\sigma_y\right),\sigma_z\right]+n\left(\widetilde{\Omega}\right)\left[\rho_S\left(t\right)\left(\sigma_z-\mathrm{i}\sigma_y\right),\sigma_z\right]\right). \tag{1358}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}=-\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x,\rho_S(t)\right)]-\frac{\pi}{8}J\left(\Omega(t)\right)(\!(n\left(\Omega(t)\right)+1)[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\rho_S(t)]+n\left(\Omega(t)\right)[\sigma_z,\left(\sigma_z-\mathrm{i}\sigma_y\right)\rho_S(t)]) \tag{1359}$$

$$-\frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z+\mathrm{i}\sigma_y\right),\sigma_z\right]+n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z-\mathrm{i}\sigma_y\right),\sigma_z\right]\right). \tag{1360}$$

## V. FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving the free energy of the system using the variational parameters optimization we consider the generalization in [5] of the inequality (244), at first we consider the convex function of the form:

$$f_N\left(x\right)=\mathrm{e}^x-1-\frac{x}{1!}-\cdots-\frac{x^{2N-1}}{(2N-1)!}. \tag{1361}$$

with $N\in\mathbb{N}^*$. By the Jensen inequality we can proof that for an arbitrary constant $\alpha$:

$$\left\langle \mathrm{e}^{x-\langle x\rangle}\right\rangle\geq 1+\mathrm{e}^{-\alpha}\left(\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^2\right\rangle-\alpha^2}{2!}+\cdots+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{2N-1}\right\rangle-\alpha^{2N-1}}{(2N-1)!}\right) \tag{1362}$$

$$=1+\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^k\right\rangle-\alpha^k}{k!}. \tag{1363}$$

For $N=3$ we can obtain the initial step to get the third Bogoliubov inequality, the RHS of (1363) is written explicitly as follows:

$$1+\mathrm{e}^{-\alpha}\sum_{k=2}^{5}\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{k}\right\rangle-\alpha^{k}}{k!}=1+\mathrm{e}^{-\alpha}\left(\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{2}\right\rangle-\alpha^{2}}{2!}+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{3}\right\rangle-\alpha^{3}}{3!}\right. \tag{1364}$$

$$\left.+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{4}\right\rangle-\alpha^{4}}{4!}+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{5}\right\rangle-\alpha^{5}}{5!}\right). \tag{1365}$$

We consider the partition functions of $\overline{H}$ and $\overline{H_0}$ respect to $\overline{H_0}$ as:

$$
\begin{aligned}
Z &= \left\langle \mathrm{e}^{-\beta\overline{H}}\right\rangle_{\overline{H_0}} && (1366)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}}\right), && (1367)\\
\overline{H} &= \overline{H_{\overline{I}}}+\overline{H_0}, && (1368)\\
Z_0 &= \mathrm{e}^{-\beta\left\langle\overline{H}\right\rangle_{\overline{H_0}}} && (1369)\\
&= \mathrm{e}^{-\beta\left\langle\overline{H_{\overline{I}}}+\overline{H_0}\right\rangle_{\overline{H_0}}} && (1370)\\
&= \mathrm{e}^{-\beta\left\langle\overline{H_{\overline{I}}}\right\rangle_{\overline{H_0}}-\beta\left\langle\overline{H_0}\right\rangle_{\overline{H_0}}} && (1371)\\
&= \mathrm{e}^{0}\mathrm{e}^{-\beta\left\langle\overline{H_0}\right\rangle_{\overline{H_0}}} && (1372)\\
&= \mathrm{e}^{-\beta\left\langle\overline{H_0}\right\rangle_{\overline{H_0}}} && (1373)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}}\right). && (1374)
\end{aligned}
$$

Taking the Quantum Bogoliubov inequality from [5]:

$$
\begin{aligned}
Z &\geq Z_0\mathrm{e}^{-\left\langle\overline{H_{\overline{I}}}\right\rangle_{\overline{H_0}}}\left(1+F_M\left(\overrightarrow{u}\right)+F_N\left(\overrightarrow{v}-\overrightarrow{u}\right)\right), && (1375)\\
F_N\left(\overrightarrow{u}\right) &= \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\frac{u_k}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. && (1376)
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{H_{\overline{I}D}} &= \sum_n\left\langle n\left|\overline{H_{\overline{I}}}\right|n\right\rangle|n\rangle\langle n| \quad \text{(with } |n\rangle \text{ is an eigenstate of } \overline{H_0}\text{)}, && (1377)\\
\overline{H_0}\,|n\rangle &= E_{0,n}\,|n\rangle, && (1378)\\
Z_0 &= \sum_n \mathrm{e}^{-\beta E_{0,n}}, && (1379)\\
u_k &= \left\langle\left(\overline{H_{\overline{I}D}}-\langle\overline{H_{\overline{I}}}\rangle_{\overline{H_0}}\right)^k\right\rangle_{\overline{H_0}} && (1380)\\
&= \frac{1}{Z_0}\sum_n \mathrm{e}^{-\beta E_{0,n}}\left(\left\langle n\left|\overline{H_{\overline{I}}}\right|n\right\rangle-\langle\overline{H_{\overline{I}}}\rangle_{\overline{H_0}}\right)^k, && (1381)\\
v_k &= \frac{1}{Z_0}\sum_n \mathrm{e}^{-\beta E_{0,n}}\left\langle n\left|\left(\overline{H_0}-E_{0,n}+\overline{H_{\overline{I}}}-\langle\overline{H_{\overline{I}}}\rangle_{\overline{H_0}}\right)^k\right|n\right\rangle. && (1382)
\end{aligned}
$$

By construction $\langle\overline{H_{\overline{I}}}\rangle_{\overline{H_0}}=0$, so we arrive to:

$$Z \geq Z_0 \left(1 + F_M \left(\overrightarrow{u}\right) + F_N \left(\overrightarrow{v} - \overrightarrow{u}\right)\right), \tag{1383}$$

$$u_k = \left\langle \left(\overline{H_{\overline{I}D}}\right)^k \right\rangle_{\overline{H_0}}, \tag{1384}$$

$$v_k = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^k \right| n \right\rangle. \tag{1385}$$

$$\overline{H_{\overline{I}D}} = \sum_n \left\langle n \left| \overline{H_{\overline{I}}} \right| n \right\rangle |n\rangle\!\langle n| \quad \text{(with } |n\rangle \text{ is an eigenstate of } \overline{H_0}\text{)},$$

As we can see the expression (1383) was written in terms of the expected value of an operator, we want to do the same for (1385) in order to write that expressions in a short form, following this we obtained:

$$\left(\overline{H_0} - E_{0,n}\right)|n\rangle = \overline{H_0}\,|n\rangle - E_{0,n}\,|n\rangle \tag{1386}$$
$$= E_{0,n}\,|n\rangle - E_{0,n}\,|n\rangle \tag{1387}$$
$$= 0, \tag{1388}$$
$$\langle n| \left(\overline{H_0} - E_{0,n}\right) = \langle n|\,\overline{H_0} - \langle n|\,E_{0,n} \tag{1389}$$
$$= \langle n|\,E_{0,n} - \langle n|\,E_{0,n} \tag{1390}$$
$$= 0. \tag{1391}$$

At first we calculated $v_1$ like:

$$v_1 = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1392}$$
$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_0} - E_{0,n} \right| n \right\rangle + \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1393}$$
$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| E_{0,n} - E_{0,n} \right| n \right\rangle + \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1394}$$
$$= 0 + \left\langle \overline{H_{\overline{I}}} \right\rangle_{\overline{H_0}} \tag{1395}$$
$$= 0. \tag{1396}$$

For $k \geq 2$ and $k \in N$ we calculated:

$$v_k = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^k \right| n \right\rangle \tag{1397}$$
$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right) \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^{k-2} \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right) \right| n \right\rangle \tag{1398}$$
$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right) \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^{k-2} \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right) \right| n \right\rangle \tag{1399}$$
$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \left(E_{0,n} - E_{0,n} + \overline{H_{\overline{I}}}\right) \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^{k-2} \left(E_{0,n} - E_{0,n} + \overline{H_{\overline{I}}}\right) \right| n \right\rangle \tag{1400}$$
$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^{k-2} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1401}$$

We will obtain the explicit form of $v_2$ and $v_3$ using (1401):

$$v_2 = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^{2-2} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1402}$$
$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left(\overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}}\right)^{0} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1403}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \mathbb{1} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1404}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}}^2 \right| n \right\rangle \tag{1405}$$

$$= \left\langle \overline{H_{\overline{I}}}^2 \right\rangle_{\overline{H_0}}, \tag{1406}$$

$$v_3 = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}} \right)^{3-2} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1407}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}} \right)^{1} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1408}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}} \right) \overline{H_{\overline{I}}} \right| n \right\rangle, \tag{1409}$$

$$\overline{H_0} \left| n \right\rangle = E_{0,n} \left| n \right\rangle, \tag{1410}$$

$$\left\langle n \right| \overline{H_0} = \left\langle n \right| E_{0,n}, \tag{1411}$$

$$v_3 = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}} \right) \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1412}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}}\, \overline{H_0}\, \overline{H_{\overline{I}}} - \overline{H_{\overline{I}}} E_{0,n} \overline{H_{\overline{I}}} + \overline{H_{\overline{I}}}\, \overline{H_{\overline{I}}}\, \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1413}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}}\, \overline{H_0}\, \overline{H_{\overline{I}}} + \overline{H_{\overline{I}}}^3 - \overline{H_{\overline{I}}} E_{0,n} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1414}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}}\, \overline{H_0}\, \overline{H_{\overline{I}}} + \overline{H_{\overline{I}}}^3 - \overline{H_{\overline{I}}}\, \overline{H_{\overline{I}}} E_{0,n} \right| n \right\rangle \tag{1415}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}}\, \overline{H_0}\, \overline{H_{\overline{I}}} + \overline{H_{\overline{I}}}^3 - \overline{H_{\overline{I}}}\, \overline{H_{\overline{I}}}\, \overline{H_0} \right| n \right\rangle \tag{1416}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}}^3 + \overline{H_{\overline{I}}} \left( \overline{H_0}\, \overline{H_{\overline{I}}} - \overline{H_{\overline{I}}}\, \overline{H_0} \right) \right| n \right\rangle \tag{1417}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}}^3 + \overline{H_{\overline{I}}} \left[ \overline{H_0}, \overline{H_{\overline{I}}} \right] \right| n \right\rangle \tag{1418}$$

$$= \left\langle \overline{H_{\overline{I}}}^3 + \overline{H_{\overline{I}}} \left[ \overline{H_0}, \overline{H_{\overline{I}}} \right] \right\rangle_{\overline{H_0}}. \tag{1419}$$

In general we have:

$$v_k = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \overline{H_0} - E_{0,n} + \overline{H_{\overline{I}}} \right)^{k-2} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1420}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \overline{H_0} + \overline{H_{\overline{I}}} - E_{0,n} \right)^{k-2} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1421}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \overline{H} - E_{0,n} \right)^{k-2} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1422}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_{\overline{I}}} \left( \sum_j^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j} E_{0,n}^j \right) \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1423}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_{0,n}} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}\, \overline{H}^{k-2-j} \overline{H_{\overline{I}}} E_{0,n}^j \right| n \right\rangle \tag{1424}$$

$$= \frac{1}{Z_0}\sum_n \mathrm{e}^{-\beta E_{0,n}} \sum_{j=0}^{k-2}(-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_I}\,\overline{H}^{k-2-j}\overline{H_I}\,\overline{H_0}^j \right| n \right\rangle \tag{1425}$$

$$= \sum_{j=0}^{k-2}(-1)^j \binom{k-2}{j} \frac{1}{Z_0}\sum_n \mathrm{e}^{-\beta E_{0,n}} \left\langle n \left| \overline{H_I}\,\overline{H}^{k-2-j}\overline{H_I}\,\overline{H_0}^j \right| n \right\rangle \tag{1426}$$

$$= \sum_{j=0}^{k-2}(-1)^j \binom{k-2}{j} \left\langle \overline{H_I}\,\overline{H}^{k-2-j}\overline{H_I}\,\overline{H_0}^j \right\rangle_{\overline{H_0}} \tag{1427}$$

$$= \left\langle \sum_{j=0}^{k-2}(-1)^j \binom{k-2}{j} \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^{k-2-j}\overline{H_I}\,\overline{H_0}^j \right\rangle_{\overline{H_0}}. \tag{1428}$$

The formula (1428) is well defined taking as example $k = 2, 3$.

$$v_2 = \left\langle \sum_{j=0}^{2-2}(-1)^j \binom{2-2}{j} \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^{2-2-j}\overline{H_I}\,\overline{H_0}^j \right\rangle_{\overline{H_0}} \tag{1429}$$

$$= (-1)^0 \left\langle \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^0 \overline{H_I}\,\overline{H_0}^0 \right\rangle_{\overline{H_0}} \tag{1430}$$

$$= \left\langle \overline{H_I}^2 \right\rangle_{\overline{H_0}}, \tag{1431}$$

$$v_3 = \left\langle \sum_{j=0}^{3-2}(-1)^j \binom{3-2}{j} \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^{3-2-j}\overline{H_I}\,\overline{H_0}^j \right\rangle_{\overline{H_0}} \tag{1432}$$

$$= \left\langle \sum_{j=0}^{1}(-1)^j \binom{1}{j} \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^{1-j}\overline{H_I}\,\overline{H_0}^j \right\rangle_{\overline{H_0}} \tag{1433}$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^{1-0}\overline{H_I}\,\overline{H_0}^0 + (-1)^1 \binom{1}{1} \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^{1-1}\overline{H_I}\,\overline{H_0}^1 \right\rangle_{\overline{H_0}} \tag{1434}$$

$$= \left\langle \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)\overline{H_I}\,\mathbb{I} - \overline{H_I}\mathbb{I}\overline{H_I}\,\overline{H_0} \right\rangle_{\overline{H_0}} \tag{1435}$$

$$= \left\langle \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)\overline{H_I} - \overline{H_I}\,\overline{H_I}\,\overline{H_0} \right\rangle_{\overline{H_0}} \tag{1436}$$

$$= \left\langle \overline{H_I}^3 + \overline{H_I}\,\overline{H_0}\,\overline{H_I} - \overline{H_I}\,\overline{H_I}\,\overline{H_0} \right\rangle_{\overline{H_0}} \tag{1437}$$

$$= \left\langle \overline{H_I}^3 + \overline{H_I}\left(\overline{H_0}\,\overline{H_I} - \overline{H_I}\,\overline{H_0}\right) \right\rangle_{\overline{H_0}} \tag{1438}$$

$$= \left\langle \overline{H_I}^3 + \overline{H_I}\left[\overline{H_0}, \overline{H_I}\right] \right\rangle_{\overline{H_0}}. \tag{1439}$$

So we summarize:

$$\overline{H_I}_D = \sum_n \left\langle n \left| \overline{H_I} \right| n \right\rangle |n\rangle\langle n|, \tag{1440}$$

$$u_k = \left\langle \left(\overline{H_I}_D\right)^k \right\rangle_{\overline{H_0}}, \tag{1441}$$

$$v_k = \sum_{j=0}^{k-2}(-1)^j \binom{k-2}{j} \left\langle \overline{H_I}\left(\overline{H_I}+\overline{H_0}\right)^{k-2-j}\overline{H_I}\,\overline{H_0}^j \right\rangle_{\overline{H_0}}. \tag{1442}$$

We know that $\langle \overline{H_I} \rangle_{\overline{H_0}} = 0$ by construction, then obtained finally:

$$Z \geq Z_0 \left(1 + F_M\left(\overrightarrow{u}\right) + F_N\left(\overrightarrow{v} - \overrightarrow{u}\right)\right), \tag{1443}$$

The free energy is defined as:

$$E_{\text{free}} = -\frac{1}{\beta}\ln\left(Z\right). \tag{1444}$$

It is well known that the function $f\left(x\right) = \ln\left(x\right)$ is monotonic and increasing so we can transform (1443):

$$Z \geq Z_0\left(1 + F_M\left(\overrightarrow{u}\right) + F_N\left(\overrightarrow{v} - \overrightarrow{u}\right)\right), \tag{1445}$$

$$E_{\text{free},1} = -\frac{1}{\beta}\ln\left(Z_0\right), \tag{1446}$$

$$E_{\text{free}} \leq E_{\text{free},1} - \frac{1}{\beta}\ln\left(1 + F_M\left(\overrightarrow{u}\right) + F_N\left(\overrightarrow{v} - \overrightarrow{u}\right)\right) \tag{1447}$$

$$\equiv E_{\text{free,MNs}}. \tag{1448}$$

here $E_{\text{free,MNs}}$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of MN order. In our approach we will set $N = M$, so our quantum Bogoliubov inequality of N order is:

$$E_{\text{free}} \leq E_{\text{free},1} - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{u}\right) + F_N\left(\overrightarrow{v} - \overrightarrow{u}\right)\right) \tag{1449}$$

$$= E_{\text{free,NN}}. \tag{1450}$$

A weaker form of the inequality (1450) making $\overrightarrow{u} = 0$ is:

$$E_{\text{free}} \leq E_{\text{free},1} - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{v}\right)\right) \tag{1451}$$

$$\equiv E_{\text{free,N}}. \tag{1452}$$

Now we consider the expected value of the form:

$$G\left(t\right) \equiv \left\langle \mathrm{e}^{-t\beta\overline{H_{\overline{I}}}}\right\rangle_{\overline{H_0}}, \tag{1453}$$

$$G^{(j)}\left(t\right) = \frac{\mathrm{d}^j}{\mathrm{d}t^j}\left\langle \mathrm{e}^{-t\beta\overline{H_{\overline{I}}}}\right\rangle_{\overline{H_0}} \tag{1454}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j}\mathrm{e}^{-t\beta\overline{H_{\overline{I}}}}\right\rangle_{\overline{H_0}} \tag{1455}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\mathbb{I} - \frac{t\beta\overline{H_{\overline{I}}}}{1!} + \frac{t^2\beta^2\overline{H_{\overline{I}}}^2}{2!} - \dots\right)\right\rangle_{\overline{H_0}} \tag{1456}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\sum_{k=0}^{\infty}\frac{\left(-t\beta\overline{H_{\overline{I}}}\right)^k}{k!}\right)\right\rangle_{\overline{H_0}} \tag{1457}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\sum_{k=0}^{\infty}\frac{\left(-1\right)^k t^k\beta^k\overline{H_{\overline{I}}}^k}{k!}\right)\right\rangle_{\overline{H_0}}, \tag{1458}$$

$$G^{(j)}\left(0\right) = \left(-1\right)^j\beta^j\left\langle \overline{H_{\overline{I}}}^j\right\rangle_{\overline{H_0}}, \tag{1459}$$

$$\beta^j\left\langle \overline{H_{\overline{I}}}^j\right\rangle_{\overline{H_0}} = \left(-1\right)^j G^{(j)}\left(0\right). \tag{1460}$$

So we obtain that:

$$E_{\mathrm{free,N}} = E_{\mathrm{free,1}} - \frac{1}{\beta}\ln\left(1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right). \tag{1461}$$

From the fact that $\alpha$ is a free parameter then we can minimize the expression:

$$h(\alpha) = \ln\left(1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right). \tag{1462}$$

This process leads is to:

$$h'(\alpha) = \frac{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\alpha^{k-j}}{k!}\right)}{1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\alpha^{k-j}}{k!}} \tag{1463}$$

$$= 0, \tag{1464}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right) \tag{1465}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,(k-j)\,\alpha^{k-j-1}}{k!} \tag{1466}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,(k-j)\,\alpha^{k-j-1}}{k!} - \sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right) \tag{1467}$$

$$= \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\left(\frac{(-1)^j\binom{k-1}{j}G^{(j)}(0)\,\alpha^{k-j-1}}{(k-1)!} - \frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right). \tag{1468}$$

Then $\alpha$ fullfils the following algebraic equation:

$$\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j G^{(j)}(0)}{\alpha^j}\left(\frac{\binom{k}{j}\alpha^k}{k!} - \frac{\binom{k-1}{j}\alpha^{k-1}}{(k-1)!}\right) = 0. \tag{1469}$$

## VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1470}$$

$$H_S(t) = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{1471}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right), \tag{1472}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}. \tag{1473}$$

### A. Variational Transformation

We consider the following operator:

$$V=\sum_{nu\mathbf{k}}|n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{1474}$$

At first let's obtain $e^{\pm V}$ under the transformation (1474), consider $\hat{\varphi}_n=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)$, so the equation (1474) can be written as $V=\sum_n|n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$e^{\pm V}=e^{\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n} \tag{1475}$$
$$=\mathbb{I}\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\left(\sum_n|n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!}+... \tag{1476}$$
$$=\mathbb{I}\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\sum_n|n\rangle\langle n|\hat{\varphi}_n^2}{2!}+... \tag{1477}$$
$$=\sum_n|n\rangle\langle n|\pm\sum_n|n\rangle\langle n|\hat{\varphi}_n+\frac{\sum_n|n\rangle\langle n|\hat{\varphi}_n^2}{2!}+... \tag{1478}$$
$$=\sum_n|n\rangle\langle n|\left(\mathbb{I}\pm\hat{\varphi}_n+\frac{\hat{\varphi}_n^2}{2!}+...\right) \tag{1479}$$
$$=\sum_n|n\rangle\langle n|e^{\pm\hat{\varphi}_n} \tag{1480}$$

Given that $\left[f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}},f_{nu'\mathbf{k}'}b_{u'\mathbf{k}'}^{\dagger}-f_{nu'\mathbf{k}'}^{*}b_{u'\mathbf{k}'}\right]=0$ for all $\mathbf{k}'$, $\mathbf{k}$ and $u,u'$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right)=e^{\pm\left(\alpha_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-\alpha_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}$ in the same way than (24) with $\alpha_{nu\mathbf{k}}=\frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}=\prod_u e^{\pm\sum_{\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} \tag{1481}$$
$$=\prod_u\left(\prod_{\mathbf{k}}e^{\pm\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}-f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}\right) \tag{1482}$$
$$=\prod_u\left(\prod_{\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) \tag{1483}$$
$$=\prod_{u\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1484}$$
$$=\prod_u B_{nu\pm} \tag{1485}$$
$$B_{nu\pm}\equiv\prod_{\mathbf{k}}D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1486}$$

As we can see $e^{-V}=\sum_n|n\rangle\langle n|\prod_u B_{nu-}$ and $e^{V}=\sum_n|n\rangle\langle n|\prod_u B_{nu+}$ this implies that $e^{-V}e^{V}=\mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V}=\left(\sum_n|n\rangle\langle n|\prod_u B_{nu+}\right)A\left(\sum_n|n\rangle\langle n|\prod_u B_{nu-}\right) \tag{1487}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1470):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1488}$$
$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1489}$$
$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1490}$$
$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1491}$$
$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1492}$$
$$= |0\rangle\langle 0|. \tag{1493}$$
$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1494}$$
$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1495}$$
$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1496}$$
$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, \tag{1497}$$
$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1498}$$
$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1499}$$
$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1500}$$
$$\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1501}$$
$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1502}$$
$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right)\left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), \tag{1503}$$
$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, \tag{1504}$$
$$= |0\rangle\langle 0| \prod_u B_{0u+}\left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+}\left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots \tag{1505}$$
$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) \tag{1506}$$
$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots \tag{1507}$$
$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) \tag{1508}$$
$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots \tag{1509}$$
$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1510}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1511}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{1512}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) \tag{1513}$$
$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{1514}$$

The transformed Hamiltonians of the equations (1471) to (1473) written in terms of (1488) to (1512) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1515}$$
$$= \overline{\sum_n \varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1516}$$
$$= \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) \tag{1517}$$
$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1518}$$
$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}\right) + \ldots\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1519}$$
$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}\right)\prod_u B_{0u-} + \prod_u B_{1u+}\sum_{u\mathbf{k}} |1\rangle\langle 1|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{1u\mathbf{k}}^* b_{u\mathbf{k}}\right)\prod_u B_{1u-} + \ldots \tag{1520}$$
$$= \sum_{u\mathbf{k}} |0\rangle\langle 0|\left(g_{0u\mathbf{k}}\prod_u B_{0u+}b_{u\mathbf{k}}^\dagger \prod_u B_{0u-} + g_{0u\mathbf{k}}^* \prod_u B_{0u+}b_{u\mathbf{k}}\prod_u B_{0u-}\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1|\left(g_{1u\mathbf{k}}\prod_u B_{1u+}b_{u\mathbf{k}}^\dagger \prod_u B_{1u-} + g_{1u\mathbf{k}}^* \prod_u B_{1u+}b_{u\mathbf{k}}\prod_u B_{1u-}\right) + \ldots \tag{1521}$$
$$= \sum_{u\mathbf{k}} |0\rangle\langle 0|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{0u\mathbf{k}}^*\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{1u\mathbf{k}}^*\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1522}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}}\right) + g_{nu\mathbf{k}}^*\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1523}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1524}$$
$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{1525}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right)\right) \tag{1526}$$
$$+ \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1527}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^*\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1528}$$
$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left(\left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right) b_{u\mathbf{k}}^\dagger + \left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right)^* b_{u\mathbf{k}}\right) \tag{1529}$$

Using the previous functions we have that (1526) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n R_n(t)|n\rangle\langle n| + \sum_n B_{z,n}(t)|n\rangle\langle n| \tag{1530}$$
$$\tag{1531}$$

Now in order to separate the elements of the hamiltonian (1531) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle\prod_u\left(B_{mu+}B_{nu-}\right)\right\rangle_{\overline{H_0}}=\left\langle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1532}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\left\langle\prod_{u\mathbf{k}}D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1533}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1534}$$

$$\equiv B_{nm} \tag{1535}$$

$$\left\langle\prod_u\left(B_{nu+}B_{mu-}\right)\right\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1536}$$

$$=B^*_{nm} \tag{1537}$$

Following the reference [4] we define:

$$J_{nm}=\prod_u\left(B_{mu+}B_{nu-}\right)-B_{nm} \tag{1538}$$

As we can see:

$$J^\dagger_{nm}=\left(\prod_u\left(B_{mu+}B_{nu-}\right)-B_{nm}\right)^\dagger \tag{1539}$$

$$=\prod_u\left(B_{nu+}B_{mu-}\right)-B^*_{nm} \tag{1540}$$

$$=\prod_u\left(B_{nu+}B_{mu-}\right)-B_{mn} \tag{1541}$$

$$=J_{mn} \tag{1542}$$

We can separate the Hamiltonian (1531) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}\left(t\right)}=\sum_n\left(\varepsilon_n\left(t\right)+R_n\right)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} \tag{1543}$$

$$\overline{H_{\bar{I}}}=\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|J_{nm}+\sum_n B_{z,n}\left(t\right)|n\rangle\langle n|, \tag{1544}$$

$$\overline{H_{\bar{B}}}=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b^\dagger_{u\mathbf{k}}b_{u\mathbf{k}} \tag{1545}$$

### B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A\leq A_{\mathrm{B}}\equiv-\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right)+\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}+O\left(\left\langle\overline{H^2_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_B}}\right). \tag{1546}$$

Taking the equations (245)-(253) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right)=C\left(R_0,R_1,...,R_{d-1},B_{01},...,B_{0(d-1)},...,B_{(d-2)(d-1)}\right)$, where each $R_i$ and $B_{kj}$ depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}}=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^{\Re}_{nu\mathbf{k}}$ and $v^{\Im}_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}\left(t\right)}$ can be written like:

$$\overline{H_{\bar{S}}(t)}=\sum_n \left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1547}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1548}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\frac{g_{nu\mathbf{k}}v^*_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1549}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1550}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}\right)v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\left(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1551}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1552}$$

$$v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}=\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)-\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right) \tag{1553}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1554}$$

$$-\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1555}$$

$$=2\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right) \tag{1556}$$

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{\left(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}\right)v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\left(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1557}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1558}$$

$$\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2=\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)\left(v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right)^* \tag{1559}$$

$$=\left|v_{mu\mathbf{k}}\right|^2+\left|v_{nu\mathbf{k}}\right|^2-\left(v_{nu\mathbf{k}}v^*_{mu\mathbf{k}}+v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}\right) \tag{1560}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1561}$$

$$-\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1562}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2\left(v^{\Re}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{nu\mathbf{k}}v^{\Im}_{mu\mathbf{k}}\right) \tag{1563}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2 \tag{1564}$$

$$R_n(t)=\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1565}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-\left(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}\right)v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\left(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1566}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2g^{\Re}_{nu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-2g^{\Im}_{nu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \tag{1567}$$

$$B_{mn}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\left(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}\right)}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1568}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1569}$$

Then we can obtain using the chain rule that:

$$
\begin{aligned}
\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1570)\\
&= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1571)\\
&= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1572)\\
\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1573)\\
&= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1574)\\
&= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1575)
\end{aligned}
$$

Given that:

$$
\begin{aligned}
\ln B_{mn} &= \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_{u} \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) && (1576)\\
&= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u} \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1577)\\
&= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1578)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1579)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1580)\\
\frac{\partial \ln B_{mn}}{\partial a} &= \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} && (1581)\\
\frac{\partial B_{mn}}{\partial a} &= B_{mn} \frac{\partial \ln B_{mn}}{\partial a} && (1582)\\
\frac{\partial B_{mn}}{\partial a} &= \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} && (1583)
\end{aligned}
$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} \tag{1584}$$

$$= B_{mn} \left( \frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} - \left(v_{nu\mathbf{k}}^{\Re} - v_{mu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1585}$$

$$= B_{mn} \left( \frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1586}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Re}} = \left( \frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Re}} \right)^{\dagger} \tag{1587}$$

$$= \left( B_{mn} \left( \frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \tag{1588}$$

$$= B_{nm} \left( \frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1589}$$

$$\frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} \tag{1590}$$

$$= B_{mn} \left( \frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} - \left(v_{nu\mathbf{k}}^{\Im} - v_{mu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1591}$$

$$= B_{mn} \left( \frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1592}$$

$$\frac{\partial B_{nm}}{\partial v_{nu\mathbf{k}}^{\Im}} = \left( \frac{\partial B_{mn}}{\partial v_{nu\mathbf{k}}^{\Im}} \right)^{\dagger} \tag{1593}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1594}$$

$$= B_{nm} \left( \frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re} + \left(v_{mu\mathbf{k}}^{\Im} - v_{nu\mathbf{k}}^{\Im}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \tag{1595}$$

Introducing this derivates in the equation (1570) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{nu\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n} \left( 2 \frac{v_{nu\mathbf{k}}^{\Re} - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n<m} \left( \frac{\partial A_{\mathrm{B}}}{\partial B_{nm}} B_{nm} \left( \frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right. \tag{1596}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}} B_{mn} \left( \frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im} + \left(v_{mu\mathbf{k}}^{\Re} - v_{nu\mathbf{k}}^{\Re}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \tag{1597}$$

$$= 0 \tag{1598}$$

We can obtain the variational parameters:

$$
\begin{aligned}
& -2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) && (1599)\\
&=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) && (1600)\\
v_{nu\mathbf{k}}^{\Re}&=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} && (1601)\\
&=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1602)
\end{aligned}
$$

Let’s consider the imaginary part of the variation parameters

$$
\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{nu\mathbf{k}}^{\Im}}&=\frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v_{nu\mathbf{k}}^{\Im}-g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}\right)+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}-\left(v_{nu\mathbf{k}}^{\Im}-v_{mu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. && (1603)\\
&\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}-\left(v_{nu\mathbf{k}}^{\Im}-v_{mu\mathbf{k}}^{\Im}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) && (1604)\\
&=0 && (1605)
\end{aligned}
$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1606}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1607}$$

$$v^{\Im}_{nu\mathbf{k}}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1608}$$

$$=\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1609}$$

$$v_{nu\mathbf{k}}=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}} \tag{1610}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1611}$$

$$\mathrm{i}\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1612}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1613}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1614}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1615}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1616}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1617}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1618}$$

### C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^{V}\rho(0)\,e^{-V}=\left(\sum_n |n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_n |n\rangle\langle n|B_{n+}\right) \tag{1619}$$

$$0=\left(B_0^{+}|0\rangle\langle 0|B_0^{-}\right)\otimes\rho_B \tag{1620}$$

$$0=\rho(0) \tag{1621}$$

We transform any operator $O$ into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1622}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1623}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\, \overline{\rho_S}(t)\, U(t)\,, \text{ where} \tag{1624}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{1625}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1626}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1627}$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1628}$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1629}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1630}$$

$$= B_{n-}^{\dagger}B_{m+}^{\dagger} - B_n B_m \tag{1631}$$

$$= B_{n+}B_{m-} - B_n B_m \tag{1632}$$

$$= B_{nm} \tag{1633}$$

So we can say that the set of matrices (1626) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1626) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)\, |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t)\, |n\rangle\langle n|, \tag{1634}$$

$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)\, |n\rangle\langle m| B_{nm} + V_{mn}(t)\, |m\rangle\langle n| B_{mn}\right) \tag{1635}$$

$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t)\, B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1636}$$

$$\left. + \Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t)\, B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1637}$$

$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right) \sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1638}$$

$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right) \sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1639}$$

$$= \sum_n B_{z,n}(t)\, |n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}(t)\right) \sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}(t)\right) \sigma_{nm,y}B_{nm,y}\right. \tag{1640}$$

$$\left. + \Im\left(V_{nm}(t)\right) \sigma_{nm,y}B_{nm,x}\right) \tag{1641}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1642}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1-\delta_{mn}) \tag{1643}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1-\delta_{mn}) \tag{1644}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1645}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1646}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1647}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1648}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1649}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1650}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1651}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1652}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1653}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1654}$$
$$C_{3,nm}(t) = 1 \tag{1655}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1656}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1657}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J, p\in P} C_{jp}(t)\left(A_{jp}\otimes B_{jp}(t)\right) \tag{1658}$$

Here $J=\{1,2,3,4,5\}$and $P$ the set defined in (1642).

We write the interaction Hamiltonian transformed under (1622) as:

$$\widetilde{H_I}(t) = \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1659}$$
$$\widetilde{A_{jp}}(t) = U^{\dagger}(t)\,A_{jp}U(t) \tag{1660}$$
$$\widetilde{B_{jp}}(t) = \mathrm{e}^{\mathrm{i}H_B t}B_{jp}(t)\,(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{1661}$$

Taking as reference state $\rho_B$ and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1662}$$

Replacing the equation (1659) in (1662) we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1663}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1664}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1665}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1666}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1667}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1668}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1669}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1670}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1671}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1672}$$

Here $s\rightarrow t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1670) are:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) &= \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B && (1673)\\
&= \left\langle\widetilde{B_{jp}}(0)\,\widetilde{B_{j'p'}}(0)\right\rangle_B && (1674)\\
&= \Lambda_{jpj'p'}(\tau) && (1675)\\
\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\rho_B\widetilde{B_{j'p'}}(s)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) && (1676)\\
&= \left\langle\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B && (1677)\\
&= \left\langle\widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B && (1678)\\
&= \Lambda_{j'p'jp}(-\tau) && (1679)\\
\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\rho_B\widetilde{B_{jp}}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) && (1680)\\
&= \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B && (1681)\\
&= \left\langle\widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B && (1682)\\
&= \Lambda_{jpj'p'}(\tau) && (1683)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) && (1684)\\
&= \left\langle\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B && (1685)\\
&= \left\langle\widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B && (1686)\\
&= \Lambda_{j'p'jp}(-\tau) && (1687)
\end{aligned}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1663)and (1670) and using the equations (1673)-(1687) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'}\Big(C_{jp}(t)\,C_{j'p'}(s)\Big(\Lambda_{jpj'p'}(\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)-\Lambda_{j'p'jp}(-\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\Big) \tag{1688}$$

$$+C_{jp}(t)\,C_{j'p'}(s)\Big(\Lambda_{j'p'jp}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{A_{jp}}(t)-\Lambda_{jpj'p'}(\tau)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jp}}(t)\Big)\Big)\,\mathrm{d}s \tag{1689}$$

$$= -\int_0^t \sum_{jj'pp'}\Big(C_{jp}(t)\,C_{j'p'}(s)\Big(\Lambda_{jpj'p'}(\tau)\Big[\widetilde{A_{jp}}(t),\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\Big]+\Lambda_{j'p'jp}(-\tau)\Big[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s),\widetilde{A_{jp}}(t)\Big]\Big)\Big) \tag{1690}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'}\left(C_{jp}(t)\,C_{j'p'}(t-\tau)\left(\Lambda_{jpj'p'}(\tau)\left[A_{jp}(t),A_{j'p'}(t-\tau,t)\,\overline{\rho_S}(t)\right]+\Lambda_{j'p'jp}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'p'}(t-\tau,t),A_{jp}(t)\right]\right)\right)\mathrm{d}\tau-\mathrm{i}\left[H_S(t),\overline{\rho_S}(t)\right] \tag{1691}$$

For this case we used that $A_{jp}(t-\tau,t)=U(t)\,U^{\dagger}(t-\tau)\,A_{jp}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation.

## VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1692}$$
$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1693}$$
$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1694}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1695}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

### A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1696}$$

At first let's obtain $e^V$ under the transformation (1696), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \tag{1697}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n| \hat{\varphi}\right)^2}{2!} + ... \tag{1698}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + ... \tag{1699}$$
$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + ...\right) \tag{1700}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1701}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+} \tag{1702}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (24):

$$e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1703}$$
$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1704}$$
$$= B_{\pm} \tag{1705}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_{-}$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+}\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{-}\right) \tag{1706}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1692):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1707}$$

$$= |0\rangle\langle 0|, \tag{1708}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1709}$$

$$= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, \tag{1710}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1711}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1712}$$

$$= |0\rangle\langle m|B^{-}\ m \neq 0, \tag{1713}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1714}$$

$$= |0\rangle\langle m|B^{+}\ m \neq 0, \tag{1715}$$

$$\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1716}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B^{+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B^{-} \tag{1717}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B^{+}b_{\mathbf{k}}^{\dagger}B^{-}\right)\left(B^{+}b_{\mathbf{k}}B^{-}\right) \tag{1718}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1719}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1720}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right) \tag{1721}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1722}$$

The transformed Hamiltonians of the equations (1693) to (1695) written in terms of (1707) to (1722) are:

$$\overline{H_{\bar{S}}\left(t\right)} = \overline{\sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|} \tag{1723}$$

$$= \overline{\sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n|} + \overline{\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|} \tag{1724}$$

$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \overline{\sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n| + V_{n0}\left(t\right)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n|} \tag{1725}$$

$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)\overline{|0\rangle\langle n|} + V_{n0}\left(t\right)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)\overline{|m\rangle\langle n|} \tag{1726}$$

$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)B^{-}|0\rangle\langle n| + V_{n0}\left(t\right)B^{+}|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1727}$$

$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1728}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(\left(\sum_{n=0}\mu_n\left(t\right)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1729}$$

$$= \left(\mu_0\left(t\right)|0\rangle\langle 0| + \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|B^{+}\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1730}$$

$$= \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}B^{+}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)B^{-} \tag{1731}$$

$$= \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1732}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1733}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1734}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1735}$$

$$+\sum_{n=0}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n\left(t\right)|n\rangle\langle n|\sum_{\mathbf{k}}2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1736}$$

$$= \sum_{n=0}\varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}\left(t\right)|0\rangle\langle n|B^{-} + V_{n0}\left(t\right)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}\left(t\right)|m\rangle\langle n| \tag{1737}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0\left(t\right)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1738}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n\left(t\right) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1739}$$

Let's define the following functions:

$$R_n\left(t\right) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1740}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1741}$$

$$B_{z,n}\left(t\right) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n\left(t\right) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1742}$$

Using the previous functions we have that (1739) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n|B^- + V_{n0}(t)\,|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1743}$$

$$+\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n|n\rangle\langle n| + \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1744}$$

Now in order to separate the elements of the hamiltonian (1744) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t)\,|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| + \sum_{n=1} R_n|n\rangle\langle n| \tag{1745}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)\,|n\rangle\langle 0|\left(B^+ - B\right)\right), \tag{1746}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{1747}$$

Here B is given by:

$$\begin{aligned} B &= \left\langle B^+ \right\rangle \\ &= \left\langle B^- \right\rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1748}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1749}$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1750}$$

$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1751}$$

Using this set of hermitian operators to write the Hamiltonians (1693)-(1695)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1752}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \tag{1753}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1754}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \tag{1755}$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1756}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1757}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1758}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1759}$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1760}$$

$$\overline{H_I(t)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^+ - B\right)\right) \tag{1761}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\left(B^- - B\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\left(B^+ - B\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1762}$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1763}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + \left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\right)\right. \tag{1764}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - \left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\right)\right) \tag{1765}$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1766}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^+ + B^- - 2B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B^- - B - B^+ + B\right)\Im(V_{0n}(t))\right. \tag{1767}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B - B^- + B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B - B^- + B - B^+\right)\Im(V_{0n}(t))\right)\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \tag{1768}$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1769}$$

$$+ \sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1770}$$

### B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right)+\langle\overline{H_I}\rangle_{\overline{H_S+H_B}}+O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1771}$$

Taking the equations (245)-(253) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)=C\left(R_1,R_2,...,R_{d-1},B\right)$, where each $R_i$ and $B$ depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (253) and using the chain rule we obtain that:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}}+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, &(1772)\\ &=0 &(1773)\end{aligned}$$

Let's recall the equations (1740) and (1742), we can write them in terms of the variational parameters

$$\begin{aligned}B &= \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) &(1774)\\ R_n &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) &(1775)\end{aligned}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\begin{aligned}\frac{\partial B}{\partial v_{\mathbf{k}}} &= -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B &(1776)\\ \frac{\partial R_n}{\partial v_{\mathbf{k}}} &= \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1777)\end{aligned}$$

Introducing this derivates in the equation (1772) give us:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right)+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1778)\\ &= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right)-\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) &(1779)\end{aligned}$$

We can obtain the variational parameters:

$$\begin{aligned}v_{\mathbf{k}} &= \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1780)\\ &= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1781)\end{aligned}$$

Now taking $v_{\mathbf{k}}=g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1782}$$

### C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_{B}$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho\left(0\right)e^{-V}=\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(|0\rangle\langle 0|\otimes\rho_{B}\right)\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1783}$$

$$0=|0\rangle\langle 0|\otimes\rho_{B} \tag{1784}$$

$$0=\rho\left(0\right) \tag{1785}$$

We transform any operator $O$ into the interaction picture in the following way:

$$\widetilde{O}\equiv U^{\dagger}\left(t\right)OU\left(t\right) \tag{1786}$$

$$U\left(t\right)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{S}}\left(t'\right)\right). \tag{1787}$$

Therefore:

$$\widetilde{\overline{\rho_{S}}}\left(t\right)=U^{\dagger}\left(t\right)\overline{\rho_{S}}\left(t\right)U\left(t\right),\text{ where} \tag{1788}$$

$$\overline{\rho_{S}}\left(t\right)=\mathrm{Tr}_{B}\left(\bar{\rho}\left(t\right)\right) \tag{1789}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_{I}\left(t\right)}=B_{z,0}|0\rangle\langle 0|+\sum_{n=1}\left(\Re\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,x}+\Re\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,y}+B_{z,n}|n\rangle\langle n|\right. \tag{1790}$$

$$\left.+\Im\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,y}-\Im\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,x}\right) \tag{1791}$$

where

$$B_{z,0}=\sum_{\mathbf{k}}g_{\mathbf{k}}\mu_{0}\left(t\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1792}$$

$$B_{z,n}=\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_{n}\left(t\right)-\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\ \text{if}n\neq 0 \tag{1793}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1794}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1795}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1796}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1797}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1798}$$
$$B_{1n}(t) = B_x \tag{1799}$$
$$B_{2n}(t) = B_y \tag{1800}$$
$$B_{3n}(t) = B_{z,n} \tag{1801}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1802}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1803}$$
$$C_{10}(t) = 0 \tag{1804}$$
$$C_{20}(t) = 0 \tag{1805}$$
$$C_{40}(t) = 0 \tag{1806}$$
$$C_{50}(t) = 0 \tag{1807}$$
$$C_{30}(t) = 1 \tag{1808}$$
$$C_{1n}(t) = \Re\left(V_{0n}(t)\right) \tag{1809}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1810}$$
$$C_{3n}(t) = 1 \tag{1811}$$
$$C_{4n}(t) = \Im\left(V_{0n}(t)\right) \tag{1812}$$
$$C_{5n}(t) = -\Im\left(V_{0n}(t)\right) \tag{1813}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn} \otimes B_{jn}(t)\right) \tag{1814}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1786) as:

$$\widetilde{H_I}(t) = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1815}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\, A_i U(t) \tag{1816}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{1817}$$

Taking as reference state $\rho_B$ and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1818}$$

Replacing the equation (1815)in (1818)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1819}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1820}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right),\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1821}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1822}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1823}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1824}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1825}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1826}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle\widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1827}$$

$$= \left\langle\widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1828}$$

Here $s\to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1826) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1829}$$

$$= \left\langle \widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1830}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1831}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1832}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1833}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1834}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1835}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1836}$$

$$= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1837}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1838}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1839}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1840}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1841}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1842}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1843}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1819)and (1826) and using the equations (1829)-(1843) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1844}$$

$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1845}$$

$$= -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t)\,,\widetilde{A_{j'n'}}(s)\,\widetilde{\overline{\rho_S}}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,,\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1846}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t)\,,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t)\,,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1847}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (450) as expected.

## VIII. BIBLIOGRAPHY


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