Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality. Let´s define the partition functions of $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ given by $Z\left(t\right)$ and $Z_0\left(t\right)$ respectively as:

$$Z\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}\left(t\right)}\right), \tag{1}$$

$$Z_0\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}_0\left(t\right)}\right). \tag{2}$$

where the transformed hamiltonians $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ are defined as:

$$\overline{H}\left(t\right) \equiv \overline{H_{\bar{I}}}\left(t\right) + \overline{H_0}\left(t\right), \tag{3}$$

$$\overline{H_0}\left(t\right) \equiv \overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}. \tag{4}$$

For any operator $A\left(t\right)$ we define the expected value respect to $\overline{H_0}\left(t\right)$ as:

$$\left\langle A\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} \equiv \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}\left(t\right)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}\left(t\right)}\right)}. \tag{5}$$

The terms $\overline{H_{\bar{S}}}\left(t\right)$, $\overline{H_{\bar{B}}}$ and $\overline{H_{\bar{I}}}\left(t\right)$ are related to the variational transformation performed in [1,2], this transformation allowed us to construct $\overline{H_{\bar{I}}}\left(t\right)$ such that $\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} = 0$. The diagonalization of $\overline{H_0}\left(t\right)$ in terms of it's eigenstates and eigenvalues such that $\overline{H_0}\left(t\right)|n\rangle = E_{0,n}\left(t\right)|n\rangle$, being $|n\rangle$ an eigenstate of $\overline{H_0}\left(t\right)$ with eigenvalue $E_{0,n}\left(t\right)$ is $\overline{H_0}\left(t\right) = \sum_n E_{0,n}\left(t\right)|n\rangle\langle n|$, with $\langle n|n'\rangle = \delta_{nn'}$, so a simple form of $\mathrm{e}^{-\beta\overline{H_0}\left(t\right)}$ can be found as follows:

$$\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\mathrm{e}^{rY}\mathrm{e}^{-\frac{r^2}{2}[X,Y]}\mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\cdots \text{ (Zassenhaus formula)}, \tag{6}$$

$$\mathrm{e}^{X+Y} = \mathrm{e}^{X}\mathrm{e}^{Y}\mathrm{e}^{-\frac{r^2}{2}0}\mathrm{e}^{\frac{r^3}{6}(2[Y,0]+[X,0])}\cdots \text{ (setting } r=1 \text{ and } [X,Y]=0 \text{ in (6))} \tag{7}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}\mathbb{I} \tag{8}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}, \tag{9}$$

$$\mathrm{e}^{-\beta\overline{H_0}\left(t\right)} = \mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by the diagonalization of } \overline{H_0}\left(t\right)) \tag{10}$$

$$
\begin{aligned}
\mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\langle n|} &= \prod_n \mathrm{e}^{-\beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by (9) and } [|n\rangle\langle n|, |n'\rangle\langle n'|] = 0) && (11)\\
&= \prod_n \sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}(t)\,|n\rangle\langle n|\right)^j}{j!} \text{ (by the exponential formula)} && (12)\\
&= \prod_n \left(\mathbb{I} + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j |n\rangle\langle n|}{j!}\right) \text{ (using } (aA)^j = a^j A^j \text{ and } (|n\rangle\langle n|)^2 = |n\rangle\langle n|) && (13)\\
&= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j |n\rangle\langle n|}{j!}\right) && (14)\\
&= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left(\sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j}{j!}\right)\right) && (15)\\
&= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \text{ (by the exponential formula)} && (16)\\
&= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right). && (17)
\end{aligned}
$$

We will prove by induction a neat form for (17), we will show that:

$$
\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \mathbb{I} + \sum_{j=1}^{n} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \tag{18}
$$

For $n = 1$ the formula is trivial, in the case $n = 2$ we obtain that:

$$
\begin{aligned}
\prod_{j=1}^{2} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) &= \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2|\right) && (19)\\
&= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1| + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2| \text{ (by } \langle i|j\rangle = \delta_{ij}) && (20)\\
&= \mathbb{I} + \sum_{j=1}^{2} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. && (21)
\end{aligned}
$$

It is our case base, our induction step is (18), in the case $n + 1$ we will have:

$$
\begin{aligned}
\prod_{j=1}^{n+1} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) &= \left(\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right)\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) && (22)\\
&= \left(\mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) \text{ (by induction step)} && (23)\\
&= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1| + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| \text{ (by } \langle i|j\rangle = \delta_{ij}) && (24)\\
&= \mathbb{I} + \sum_{j=1}^{n+1} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. && (25)
\end{aligned}
$$

By mathematical induction we proved that (18) is true for all $n \in \mathbb{N}$. Given that the resolution of the identity is $\mathbb{I} = \sum_n |n\rangle\langle n|$ so we find that:

$$
\begin{aligned}
\mathrm{e}^{-\beta\overline{H_0}(t)} &= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right) && (26)\\
&= \mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| && (27)\\
&= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \sum_n |n\rangle\langle n| && (28)\\
&= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \mathbb{I} \text{ (by the resolution of the identity)} && (29)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|. && (30)
\end{aligned}
$$

The partition function $Z_0\left(t\right)$ is equal to:

$$
\begin{aligned}
Z_0\left(t\right) &= \mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) && (31)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(|n\rangle\langle n|\right) && (32)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}. && (33)
\end{aligned}
$$

The explicit form of the average value $\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)}$ can be found from the partition function $Z_0\left(t\right)$:

$$
\begin{aligned}
\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)} &= \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{Z_0\left(t\right)} && (34)\\
&= \frac{\mathrm{Tr}\left(\sum_n A\left(t\right)\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta\overline{H_0}(t)}\right)} && (35)\\
&= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)} && (36)\\
&= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} && (37)\\
&= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}}. && (38)
\end{aligned}
$$

At first we show a double sequence of inequalities of order M, N which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$
Z\left(t\right) \geq Z_0\left(t\right)\mathrm{e}^{-\left\langle\overline{H_I}(t)\right\rangle_{\overline{H_0}(t)}}\left(1 + F_M\left(\overrightarrow{u}\left(t\right);\alpha\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right);\alpha\right)\right). \tag{39}
$$

where the funcion $F_N\left(\overrightarrow{u}\left(t\right);\alpha\right)$ is defined as:

$$
F_N\left(\overrightarrow{w}\left(t\right);\alpha\right) \equiv \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{w_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. \tag{40}
$$

In this case α is a parameter that can be optimized, $\beta \equiv \frac{1}{\mathrm{k_B}T}$, $\overrightarrow{w}\left(t\right)$ is a vector such that $\overrightarrow{w}\left(t\right) = \left(w_1, w_2, ...\right)$ and $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian $\overline{H_{I}}_D\left(t\right)$ respect to the basis of eigenstates of $\overline{H_0}\left(t\right)$ as:

$$\overline{H_{\overline{I}D}}(t) \equiv \sum_n \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n|. \tag{41}$$

We will prove an important property related to $\overline{H_{\overline{I}D}}(t)$, which is a Hamiltonian written as a linear combination of a set of ortonormal operators. Let's consider a vector space R with two operations $+$ and $\cdot$, if there exist $a, b \in R$ such that $a \cdot b = 0$ and $b \cdot a = 0$ then for any $k \in \mathbb{N}$ we have $(a+b)^k = a^k + b^k$ where $a^k = a^{k-1} \cdot a$ is a recursive definition of the power of an element written in terms of $\cdot$. At first we prove that this result yields for any $k \in \mathbb{N}$ by induction, the case $k = 1$ is trivial so we will focus on the case $k = 2$, we have that:

$$\begin{aligned}
(a+b)^2 &= (a+b) \cdot (a+b) && (42)\\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b && (43)\\
&= a^2 + a \cdot b + b \cdot a + b^2 && (44)\\
&= a^2 + 0 + 0 + b^2 \text{ (because } a \cdot b = b \cdot a = 0) && (45)\\
&= a^2 + b^2. && (46)
\end{aligned}$$

This is the base case. By induction step we will consider that $(a+b)^k = a^k + b^k$ with $k \geq 2$, now for $k+1$ we will have that:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)^k \cdot (a+b) && (47)\\
&= \left(a^k + b^k\right) \cdot (a+b) \text{ (by induction step)} && (48)\\
&= a^k \cdot a + a^k \cdot b + b^k \cdot a + b^k \cdot b && (49)\\
&= a^{k+1} + a^{k-1} \cdot a \cdot b + b^{k-1} \cdot b \cdot a + b^{k+1} \text{ (by recursive definition of } a^k) && (50)\\
&= a^{k+1} + a^{k-1} \cdot (a \cdot b) + b^{k-1} \cdot (b \cdot a) + b^{k+1} \text{ (by associativity on } R) && (51)\\
&= a^{k+1} + a^{k-1} \cdot (0) + b^{k-1} \cdot (0) + b^{k+1} \text{ (because } a \cdot b = b \cdot a = 0) && (52)\\
&= a^{k+1} + b^{k+1}. && (53)
\end{aligned}$$

By the principle of mathematical induction we can conclude that the proposition is true for all $k \in \mathbb{N}$. Now we will extend the result, let $a_1, ..., a_n \in R$ such that $a_i \cdot a_j = 0$ for all $i \neq j$ then $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$. The case $n = 1$ is trivial as well so we will focus on $n = 2$, this case was proved in the precedent lines so it will be our base case. By induction step we will consider that $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$ with $n \geq 2$, now for $n+1$ we will have that:

$$\begin{aligned}
a_{n+1} \cdot (a_1 + ... + a_n) &= a_{n+1} \cdot a_1 + ... + a_{n+1} \cdot a_n && (54)\\
&= 0 \text{ (because } a_i \cdot a_j = 0 \text{ for all } i \neq j \text{)}, && (55)\\
(a_1 + ... + a_n + a_{n+1})^k &= ((a_1 + ... + a_n) + a_{n+1})^k && (56)\\
&= (a_1 + ... + a_n)^k + a_{n+1}^k \text{ (by (47) and (55))} && (57)\\
&= a_1^k + ... + a_n^k + a_{n+1}^k \text{ (by inductive step)}. && (58)
\end{aligned}$$

So we can conclude by mathematical induction that the proposition is true for all $n \in \mathbb{N}$. We can prove the following property for $\left(\overline{H_{\overline{I}D}}(t)\right)^k$:

$$\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n|\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle |n'\rangle\langle n'| = \left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle |n\rangle\langle n|n'\rangle\langle n'| \tag{59}$$

$$= \left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle |n\rangle\langle n'|\delta_{nn'}, \tag{60}$$

$$\left(\overline{H_{I}}_{D}(t)\right)^{k} = \left(\sum_{n}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n|\right)^{k} \text{ (by (41))} \tag{61}$$

$$= \sum_{n}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n|\right)^{k} \text{ (by (58) and (60))}, \tag{62}$$

$$(aA)^{k} = a^{k}A^{k} \text{ (by the property of the power of a matrix)}, \tag{63}$$

$$(|n\rangle\langle n|)^{k} = |n\rangle\langle n| \text{ (because } |n\rangle\langle n| \text{ is a projector and } k\in\mathbb{N}^{*}), \tag{64}$$

$$\left(\overline{H_{I}}_{D}(t)\right)^{k} = \sum_{n}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle\right)^{k} |n\rangle\langle n| \text{ (by (63) and (64))}. \tag{65}$$

The vectors $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ are defined as $\overrightarrow{u}(t) \equiv (u_1, u_2, ...)$ and $\overrightarrow{v}(t) \equiv (v_1, v_2, ...)$. We can define the elements of $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ in terms of the matrix $\overline{H_{I}}_{D}(t)$:

$$u_k(t) \equiv \left\langle\left(\overline{H_{I}}_{D}(t) - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k}\right\rangle_{\overline{H_0}(t)} \tag{66}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\sum_n\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle |n\rangle\langle n| - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k} |n\rangle\langle n|\right)}{Z_0(t)} \text{ (by (38))}, \tag{67}$$

$$\left(\sum_n\langle n|\overline{H_{I}}(t)|n\rangle |n\rangle\langle n| - \langle\overline{H_{I}}(t)\rangle_{\overline{H_0}(t)}\right)^{k} = \sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left(\sum_n\langle n|\overline{H_{I}}(t)|n\rangle |n\rangle\langle n|\right)^{j}\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k-j} \text{ (by binomial theorem)} \tag{68}$$

$$= \sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left(\sum_n\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle^{j} |n\rangle\langle n|\right)\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k-j} \text{ (by (65))} \tag{69}$$

$$= \sum_n\left(\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle^{j}\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k-j}\right)|n\rangle\langle n| \tag{70}$$

$$= \sum_n\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k} |n\rangle\langle n|, \tag{71}$$

$$= \sum_n\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k} |n\rangle\langle n|, \tag{72}$$

$$u_k(t) = \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\sum_{n'}\left(\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k} |n'\rangle\langle n'|n\rangle\langle n|\right)}{Z_0(t)} \tag{73}$$

$$= \frac{\sum_{nn'} \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k} |n'\rangle\langle n|\delta_{nn'}\right)}{Z_0(t)} \tag{74}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k}\mathrm{Tr}\left(|n\rangle\langle n|\right)}{Z_0(t)} \tag{75}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k} 1}{Z_0(t)} \tag{76}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k}}{Z_0(t)}, \tag{77}$$

$$v_k(t) \equiv \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{I}}(t) - \left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_0}(t)}\right)^{k}\right|n\right\rangle}{Z_0(t)}. \tag{78}$$

By construction $\left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} = 0$, so we summarize the double inequality that generalizes the Bogoliubov inequality and it's coefficients as:

$$
\begin{aligned}
Z\left(t\right) &\geq Z_0\left(t\right)\left(1 + F_M\left(\overrightarrow{u}\left(t\right)\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right)\right)\right), && (79)\\
Z\left(t\right) &= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}\left(t\right)}\right), && (80)\\
Z_0\left(t\right) &= \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}, && (81)\\
F_N\left(\overrightarrow{u}\left(t\right)\right) &= \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k \frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}, && (82)\\
u_k\left(t\right) &= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle^k}{Z_0\left(t\right)}, && (83)\\
v_k\left(t\right) &= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right)^k\right|n\right\rangle}{Z_0\left(t\right)}. && (84)
\end{aligned}
$$

As we can see the expression (83) was written in shorter terms, we want to do the same for (84) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for $v_k\left(t\right)$:

$$
\begin{aligned}
\left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right)\left|n\right\rangle &= \overline{H_0}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (85)\\
&= E_{0,n}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (86)\\
&= 0, && (87)\\
\left\langle n\right|\left(\overline{H_0}\left(t\right) - E_{0,n}\right) &= \left\langle n\right|\overline{H_0}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) && (88)\\
&= \left\langle n\right|E_{0,n}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) && (89)\\
&= 0. && (90)
\end{aligned}
$$

At first we calculated $v_1\left(t\right)$ using the definition (84) :

$$
\begin{aligned}
v_1\left(t\right) &= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right|n\right\rangle && (91)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right|n\right\rangle + \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle && (92)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|\overline{H_0}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle\overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} && (93)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle\overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} && (94)\\
&= 0 + \left\langle\overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} \quad \text{(by construction } \left\langle\overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} = 0\text{)} && (95)\\
&= 0. && (96)
\end{aligned}
$$

For $k \geq 2$ and $k \in \mathbb{N}$ we calculated:

$$v_k(t)= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^k\right|n\right\rangle \tag{97}$$

$$= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^{k-2}\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\right|n\right\rangle \tag{98}$$

$$= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^{k-2}\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\right|n\right\rangle \tag{99}$$

$$= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(E_{0,n}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^{k-2}\left(E_{0,n}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\right|n\right\rangle \tag{100}$$

$$= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle . \tag{101}$$

In general we can write a formula for $v_k\left(t\right)$ that implies an expected value of a dependent expression of $\overline{H_{\overline{I}}}\left(t\right)$ and $\overline{H_0}\left(t\right)$:

$$v_k\left(t\right) = \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle \tag{102}$$

$$= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_0}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)-E_{0,n}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle \tag{103}$$

$$= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H}\left(t\right)-E_{0,n}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle \tag{104}$$

$$= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\overline{H}^{k-2-j}\left(t\right)E_{0,n}^j\left(t\right)\right)\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle \tag{105}$$

$$= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\overline{H}^{k-2-j}\left(t\right)\overline{H_{\overline{I}}}\left(t\right)E_{0,n}^j\left(t\right)\right|n\right\rangle \tag{106}$$

$$= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\overline{H}^{k-2-j}\left(t\right)\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^j\left(t\right)\right|n\right\rangle \tag{107}$$

$$= \sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\overline{H}^{k-2-j}\left(t\right)\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^j\left(t\right)\right|n\right\rangle \tag{108}$$

$$= \sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\left\langle\overline{H_{\overline{I}}}\left(t\right)\ \overline{H}^{k-2-j}\left(t\right)\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{109}$$

$$= \sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^{k-2-j}\overline{H_{\overline{I}}}\left(t\right)\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0}(t)} . \tag{110}$$

The formula (110) is well defined taking as example $k=2,3$.

$$v_2\left(t\right) = \left\langle\sum_{j=0}^{2-2}(-1)^j\binom{2-2}{j}\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^{2-2-j}\overline{H_{\overline{I}}}\left(t\right)\ \overline{H_0}\left(t\right)^j\right\rangle_{\overline{H_0}(t)} \tag{111}$$

$$= (-1)^0\left\langle\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_{\overline{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^0\overline{H_{\overline{I}}}\left(t\right)\ \overline{H_0}^0\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{112}$$

$$= \left\langle\overline{H_{\overline{I}}}^2\left(t\right)\right\rangle_{\overline{H_0}(t)} , \tag{113}$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{3-2-j} \overline{H_I}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{114}$$

$$= \left\langle \sum_{j=0}^{1} (-1)^j \binom{1}{j} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{1-j} \overline{H_I}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{115}$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^1 \overline{H_I}(t)\, \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^0 \overline{H_I}(t)\, \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \tag{116}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \mathbb{I} - \overline{H_I}(t)\, \mathbb{I} \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{117}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{118}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t)\, \overline{H_0}(t)\, \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{119}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) \left(\overline{H_0}(t)\, \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_0}(t)\right) \right\rangle_{\overline{H_0}(t)} \tag{120}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) \left[\overline{H_0}(t), \overline{H_I}(t)\right] \right\rangle_{\overline{H_0}(t)} \text{ (because } \left[\overline{H_0}(t), \overline{H_I}(t)\right] = \overline{H_0}(t)\, \overline{H_I}(t) - \overline{H_I}(t)\, \overline{H_0}(t)\text{).} \tag{121}$$

So we summarize:

$$\overline{H_I}_D(t) = \sum_n \left\langle n \left| \overline{H_I}(t) \right| n \right\rangle |n\rangle\langle n|, \tag{122}$$

$$u_k(t) = \left\langle \left(\overline{H_I}_D(t)\right)^k \right\rangle_{\overline{H_0}(t)}, \tag{123}$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_I}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{124}$$

The free energy is defined as:

$$E_{\text{free}}(t) \equiv -\frac{1}{\beta} \ln\left(Z(t)\right). \tag{125}$$

We define the free energy at first order as:

$$E_{\text{free},1}(t) \equiv -\frac{1}{\beta} \ln\left(Z_0(t)\right). \tag{126}$$

It is well-known that the function $f(x) = -\ln(x)$ is a decreasing function so we can transform (39):

$$E_{\text{free}}(t) \le -\frac{1}{\beta} \ln\left(Z_0(t) \left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right)\right) \tag{127}$$

$$= -\frac{1}{\beta} \ln\left(Z_0(t)\right) - \frac{1}{\beta} \ln\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) \tag{128}$$

$$= E_{\text{free},1}(t) - \frac{1}{\beta} \ln\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) \tag{129}$$

$$\equiv E_{\text{free,MN}}(t). \tag{130}$$

here $E_{\text{free,MN}}(t)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of M, N order. In our approach we will set $N = M$, so the inequality (130) of N, N order is:

$$E_{\text{free}}(t) \le E_{\text{free},1}(t) - \frac{1}{\beta} \ln\left(1 + F_N\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) \tag{131}$$

$$= E_{\text{free,NN}}(t). \tag{132}$$

A weaker form of the inequality (132) is obtained making $\overrightarrow{u}(t)=0$ as suggest [3]:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{v}(t)\right)\right) \tag{133}$$
$$\equiv E_{\text{free,N}}(t). \tag{134}$$

The algebraic equation associated with $\alpha_{\text{opt}}(t)$ such that $E_{\text{free,N}}(t)$ is closer to $E_{\text{free}}(t)$ follows from the fact that in the optimal parameter $\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha}|_{\alpha_{\text{opt}(t)}}=0$, calculating this derivate we have:

$$\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{v}(t)\right)\right)\right) \tag{135}$$
$$= -\frac{1}{\beta}\frac{\frac{\partial}{\partial\alpha}\left(F_N\left(\overrightarrow{v}(t)\right)\right)}{1+F_N\left(\overrightarrow{v}(t)\right)} \tag{136}$$
$$=0. \tag{137}$$

The precedent equation is equivalent to make the numerator equal to 0:

$$\frac{\partial F_N\left(\overrightarrow{v}(t)\right)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{138}$$
$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\partial}{\partial\alpha}\frac{\alpha^i}{i!}\ \text{(by product rule)} \tag{139}$$
$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} \tag{140}$$
$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{141}$$
$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{j=0}^{2N-2-k}\frac{\alpha^j}{j!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right)\ \text{(setting } j=i-1\text{)} \tag{142}$$
$$= \mathrm{e}^{-\alpha}\left(-\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha^{2N-1-k}}{(2N-1-k)!}\right)\ \text{(performing the difference)} \tag{143}$$
$$=0. \tag{144}$$

Then the optimal value $\alpha_{\text{opt}}(t)$ will sastisfy the following equation:

$$G\left(\alpha_{\text{opt}}(t)\right) \equiv \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \tag{145}$$
$$=0. \tag{146}$$

The elements presented are the required to find variational parameters of the system using the inequality (134) and the self consistent equation (SCE) (145) to a particular order required.

II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2(t), v_3(t), v_4(t), v_5(t)$ using the (124), we have already $v_2(t), v_3(t)$ and the form of $v_4(t)$ and $v_5(t)$ is given by:

$$v_4\,(t) = \sum_{j=0}^{4-2} (-1)^j \binom{4-2}{j} \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right)^{4-2-j} \overline{H_I}\,(t)\,\overline{H_0}^{j}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{147}$$

$$= \sum_{j=0}^{2} (-1)^j \binom{2}{j} \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right)^{2-j} \overline{H_I}\,(t)\,\overline{H_0}^{j}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{148}$$

$$= \left\langle \overline{H_I}\,(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}\,(t)\,\overline{H_0}^{0}(t) \right\rangle_{\overline{H_0(t)}} - 2 \left\langle \overline{H_I}\,(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^1 \overline{H_I}\,(t)\,\overline{H_0}^{1}(t) \right\rangle_{\overline{H_0(t)}} + \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) \right.\right. \tag{149}$$

$$\left.\left. + \overline{H_0}\,(t)\right)^0 \overline{H_I}\,(t)\,\overline{H_0}^{2}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{150}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^2 \overline{H_I}\,(t)\,\mathbb{I} \right\rangle_{\overline{H_0(t)}} - 2 \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right)^1 \overline{H_I}\,(t)\,\overline{H_0}^{1}(t) \right\rangle_{\overline{H_0(t)}} + \left\langle \overline{H_I}^{2}(t)\,\overline{H_0}^{2}(t) \right\rangle_{\overline{H_0(t)}} \tag{151}$$

$$= \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right)^2 \overline{H_I}\,(t) \right\rangle_{\overline{H_0(t)}} - 2 \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right) \overline{H_I}\,(t)\; \overline{H_0}\,(t) \right\rangle_{\overline{H_0(t)}} + \left\langle \overline{H_I}^{2}\,(t)\,\overline{H_0}^{2}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{152}$$

$$= \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right)^2 \overline{H_I}\,(t) - 2\overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right) \overline{H_I}\,(t)\,\overline{H_0}\,(t) + \overline{H_I}^{2}\,(t)\,\overline{H_0}^{2}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{153}$$

$$= \left\langle \overline{H_I}\,(t) \left(\overline{H_I}^{2}\,(t) + \overline{H_I}\,(t)\,\overline{H_0}(t) + \overline{H_0}(t)\,\overline{H_I}\,(t) + \overline{H_0}^{2}\,(t) \right) \overline{H_I}\,(t) - 2\overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}(t)\right) \overline{H_I}\,(t)\,\overline{H_0}(t) + \overline{H_I}^{2}\,(t) \right. \tag{154}$$

$$\left. \times \overline{H_0}^{2}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{155}$$

$$= \left\langle \overline{H_I}^{4}\,(t) + \overline{H_I}^{2}\,(t)\,\overline{H_0}\,(t)\,\overline{H_I}\,(t) + \overline{H_I}\,(t)\,\overline{H_0}\,(t)\,\overline{H_I}^{2}(t) + \overline{H_I}\,(t)\,\overline{H_0}^{2}\,(t)\,\overline{H_I}\,(t) - 2\overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right) \overline{H_I}\,(t)\,\overline{H_0}\,(t) \right. \tag{156}$$

$$\left. + \overline{H_I}^{2}(t)\,\overline{H_0}^{2}(t) \right\rangle_{\overline{H_0(t)}} \tag{157}$$

$$= \left\langle \overline{H_I}^{4}(t) + \overline{H_I}^{2}(t)\overline{H_0}(t)\,\overline{H_I}(t) + \overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^{2}(t) + \overline{H_I}(t)\overline{H_0}^{2}(t)\,\overline{H_I}(t) - 2\overline{H_I}^{3}(t)\overline{H_0}(t) + \overline{H_I}^{2}(t)\overline{H_0}^{2}(t) - 2\overline{H_I}\,(t) \right. \tag{158}$$

$$\left. \times \overline{H_0}\,(t)\,\overline{H_I}\,(t)\,\overline{H_0}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{159}$$

$$= \left\langle \overline{H_I}^{4}(t) + \overline{H_I}^{2}(t)\overline{H_0}(t)\,\overline{H_I}(t) + \overline{H_I}\,(t)\overline{H_0}(t)\overline{H_I}^{2}(t) + \overline{H_I}\,(t)\overline{H_0}^{2}(t)\,\overline{H_I}\,(t) - \overline{H_I}^{3}(t)\overline{H_0}(t) - \overline{H_I}^{3}(t)\overline{H_0}(t) + \overline{H_I}^{2}(t) \right. \tag{160}$$

$$\left. \times \overline{H_0}^{2}\,(t) - \overline{H_I}\,(t)\,\overline{H_0}\,(t)\,\overline{H_I}\,(t)\,\overline{H_0}\,(t) - \overline{H_I}\,(t)\,\overline{H_0}\,(t)\,\overline{H_I}\,(t)\,\overline{H_0}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{161}$$

$$= \left\langle \overline{H_I}^{4}(t) + \overline{H_I}^{2}(t)\overline{H_0}(t)\overline{H_I}\,(t) - \overline{H_I}^{3}(t)\overline{H_0}(t) + \overline{H_I}\,(t)\overline{H_0}(t)\,\overline{H_I}^{2}(t) - \overline{H_I}^{3}(t)\overline{H_0}(t) + \overline{H_I}\,(t)\overline{H_0}^{2}(t)\,\overline{H_I}\,(t) - \overline{H_I}\,(t) \right. \tag{162}$$

$$\left. \times \overline{H_0}\,(t)\,\overline{H_I}\,(t)\,\overline{H_0}\,(t) + \overline{H_I}^{2}\,(t)\,\overline{H_0}^{2}\,(t) - \overline{H_I}\,(t)\,\overline{H_0}\,(t)\,\overline{H_I}\,(t)\,\overline{H_0}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{163}$$

$$= \left\langle \overline{H_I}^{4}(t) + \overline{H_I}\,(t) \left(\left(\left(\overline{H_I}\,(t)\overline{H_0}(t)\right)\overline{H_I}\,(t) - \overline{H_I}\,(t)\left(\overline{H_I}(t)\overline{H_0}(t)\right)\right) + \left(\overline{H_0}(t)\overline{H_I}^{2}(t) - \overline{H_I}^{2}(t)\overline{H_0}(t)\right) + \left(\overline{H_0}(t)\left(\overline{H_0}(t) \right.\right.\right.\right. \tag{164}$$

$$\left.\left.\left.\left. \times \overline{H_I}\,(t)\right) - \left(\overline{H_0}\,(t)\,\overline{H_I}\,(t)\right)\overline{H_0}\,(t)\right) + \left(\left(\overline{H_I}\,(t)\,\overline{H_0}\,(t)\right)\overline{H_0}\,(t) - \overline{H_0}\,(t)\left(\overline{H_I}\,(t)\,\overline{H_0}\,(t)\right)\right)\right)\right\rangle_{\overline{H_0(t)}} \tag{165}$$

$$= \left\langle \overline{H_I}^{4}(t) + \overline{H_I}\,(t) \left(\left[\,\overline{H_I}\,(t)\overline{H_0}\,(t), \overline{H_I}\,(t)\right] + \left[\overline{H_0}\,(t), \overline{H_I}^{2}(t)\right] + \left[\,\overline{H_0}\,(t), \overline{H_0}\,(t)\overline{H_I}\,(t)\right] + \left[\,\overline{H_I}\,(t)\overline{H_0}\,(t), \overline{H_0}\,(t)\right] \right) \right\rangle_{\overline{H_0(t)}}, \tag{166}$$

$$v_5\,(t) = \sum_{j=0}^{5-2} (-1)^j \binom{5-2}{j} \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right)^{5-2-j} \overline{H_I}\,(t)\; \overline{H_0}\,(t)^{j} \right\rangle_{\overline{H_0(t)}} \tag{167}$$

$$= \sum_{j=0}^{3} (-1)^j \binom{3}{j} \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right)^{3-j} \overline{H_I}\,(t)\; \overline{H_0}\,(t)^{j} \right\rangle_{\overline{H_0(t)}} \tag{168}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^3 \overline{H_I}\,(t)\overline{H_0}^{0}(t) - 3\overline{H_I}\,(t)\left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^2 \overline{H_I}\,(t)\overline{H_0}(t) - \overline{H_I}\,(t)\left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^0 \overline{H_I}\,(t)\,\overline{H_0}^{3}(t) \right. \tag{169}$$

$$\left. + 3\overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right) \overline{H_I}\,(t)\,\overline{H_0}^{2}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{170}$$

$$= \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^3 \overline{H_I}\,(t) - 3\overline{H_I}\,(t)\left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^2 \overline{H_I}\,(t)\overline{H_0}(t) + 3\overline{H_I}\,(t)\left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)\overline{H_I}\,(t)\overline{H_0}^{2}(t) - \overline{H_I}\,(t) \right. \tag{171}$$

$$\left. \times \overline{H_I}\,(t)\,\overline{H_0}^{3}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{172}$$

$$= \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^3 \overline{H_I}\,(t) - 3\overline{H_I}\,(t)\left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^2 \overline{H_I}\,(t)\,\overline{H_0}(t) + 3\overline{H_I}\,(t)\left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)\overline{H_I}\,(t)\,\overline{H_0}^{2}(t) \right. \tag{173}$$

$$\left. - \overline{H_I}\,(t)\,\overline{H_I}\,(t)\,\overline{H_0}^{3}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{174}$$

$$= \left\langle \overline{H_I}\,(t) \left(\overline{H_I}\,(t) + \overline{H_0}(t)\right)^3 \overline{H_I}\,(t) - 3\overline{H_I}\,(t) \left(\overline{H_I}^{2}(t) + \overline{H_I}\,(t)\overline{H_0}(t) + \overline{H_0}(t)\overline{H_I}\,(t) + \overline{H_0}^{2}(t) \right) \overline{H_I}\,(t)\overline{H_0}(t) + 3\overline{H_I}\,(t) \right. \tag{175}$$

$$\left. \times \left(\overline{H_I}\,(t) + \overline{H_0}\,(t)\right) \overline{H_I}\,(t)\,\overline{H_0}^{2}\,(t) - \overline{H_I}\,(t)\,\overline{H_I}\,(t)\,\overline{H_0}^{3}\,(t) \right\rangle_{\overline{H_0(t)}} \tag{176}$$

$$=\Big\langle \overline{H_I}(t)\Big(\overline{H_I}^3(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}(t) \tag{177}$$

$$\times\overline{H_0}^2(t)+\overline{H_0}^3(t)\Big)\,\overline{H_I}(t)-3\overline{H_I}(t)\Big(\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^2(t)\Big)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}(t)\,\big(\overline{H_I}(t) \tag{178}$$

$$+\overline{H_0}(t)\big)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} \tag{179}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}^3(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^3(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t) \tag{180}$$

$$\times\,\overline{H_0}^2(t)\,\overline{H_I}^2(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}^4(t)\,\overline{H_0}(t)-3\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-3\overline{H_I}(t) \tag{181}$$

$$\times\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)-3\overline{H_I}(t)\overline{H_0}^2(t)\,\overline{H_I}(t)\overline{H_0}(t)+3\overline{H_I}^3(t)\overline{H_0}^2(t)+3\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}^2(t)-\overline{H_I}^2(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0}(t)} \tag{182}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_0}(t)\,\overline{H_I}^3(t)+\overline{H_0}^2(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t) \tag{183}$$

$$+\,\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}(t)-3\overline{H_I}^3(t)\overline{H_0}(t)-3\overline{H_0}^2(t)\,\overline{H_I}(t)\overline{H_0}(t)-\overline{H_I}(t) \tag{184}$$

$$\times\overline{H_0}^3(t)+3\overline{H_I}^2(t)\,\overline{H_0}^2(t)+3\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-3\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big\rangle_{\overline{H_0}(t)} \tag{185}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}^3(t)-\overline{H_0}(t)\overline{H_I}^2(t) \tag{186}$$

$$\times\,\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}(t)-\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)+\overline{H_0}^3(t)\,\overline{H_I}(t) \tag{187}$$

$$-\,\overline{H_I}(t)\,\overline{H_0}^3(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)-\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)+2\overline{H_I}^2(t)\,\overline{H_0}^2(t)-2\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_0}(t)\,\overline{H_I}(t) \tag{188}$$

$$\times\overline{H_0}^2(t)-3\overline{H_0}^2(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)+\overline{H_0}^2(t)\,\overline{H_I}^2(t)\Big)\Big\rangle_{\overline{H_0}(t)} \tag{189}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big(\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\overline{H_I}(t)-\overline{H_I}(t)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big)+\Big(\big(\overline{H_I}(t)\,\overline{H_0}(t)\big)\overline{H_I}^2(t)-\overline{H_I}^2(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)\big)\Big) \tag{190}$$

$$+\,\Big(\overline{H_0}(t)\overline{H_I}^3(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)\Big)+\big(\overline{H_0}(t)\big(\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\big)-\big(\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\big)\overline{H_0}(t)\big)+\Big(\overline{H_0}(t)\Big(\overline{H_0}(t)\overline{H_I}^2(t)\Big) \tag{191}$$

$$-\,\Big(\overline{H_0}(t)\overline{H_I}^2(t)\Big)\overline{H_0}(t)\Big)+\Big(\overline{H_0}^3(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}^3(t)\Big)+\big(\big(\overline{H_I}(t)\overline{H_0}(t)\big)\big(\overline{H_0}(t)\overline{H_I}(t)\big)-\big(\overline{H_0}(t)\overline{H_I}(t)\big)\big(\overline{H_I}(t)\overline{H_0}(t)\big)\big) \tag{192}$$

$$+\,3\overline{H_0}(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_I}(t)\big)\overline{H_0}(t)+2\overline{H_I}(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_I}(t)\big)\overline{H_0}(t)+\Big(\Big(\overline{H_I}^2(t)\overline{H_0}(t)\Big)\big(\overline{H_0}(t)\big) \tag{193}$$

$$-\,\big(\overline{H_0}(t)\big)\,\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big)\Big)\Big\rangle_{\overline{H_0}(t)} \tag{194}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\,\Big(\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_I}^3(t)\Big]+\big[\overline{H_0}(t)\,,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\big] \tag{195}$$

$$+\,\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}^3(t)\,,\overline{H_I}(t)\Big]+\big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+2\overline{H_I}(t) \tag{196}$$

$$\times\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}. \tag{197}$$

Summarizing we have that:

$$v_2(t)=\Big\langle \overline{H_I}^2(t)\Big\rangle_{\overline{H_0}(t)}, \tag{198}$$

$$v_3(t)=\Big\langle \overline{H_I}^3(t)+\overline{H_I}(t)\,\big[\overline{H_0}(t)\,,\overline{H_I}(t)\big]\Big\rangle_{\overline{H_0}(t)}, \tag{199}$$

$$v_4(t)=\Big\langle \overline{H_I}^4(t)+\overline{H_I}(t)\Big(\big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}(t)\big]+\Big[\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\big[\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_I}(t)\big]+\big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\big]\Big)\Big\rangle_{\overline{H_0}(t)}, \tag{200}$$

$$v_5(t)=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_I}^3(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}^2(t)\Big] \tag{201}$$

$$+\,\big[\overline{H_0}(t)\,,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\big]+\Big[\overline{H_0}^3(t)\,,\overline{H_I}(t)\Big]+\big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t) \tag{202}$$

$$+2\overline{H_I}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}. \tag{203}$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_{\bar{S}}}\left(t\right) \equiv \left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \quad (204)$$

$$\overline{H_{\bar{I}}}\left(t\right) \equiv \sum_i B_{iz}\left(t\right)|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right)+V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y\left(t\right)-\sigma_y B_x\left(t\right)\right), \quad (205)$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \quad (206)$$

$$= H_B. \quad (207)$$

In this case $\varepsilon_j\left(t\right)$, $R_j\left(t\right)$ for $j\in\{0,1\}$, $B_{10}^{\Re}(t)$, $B_{10}^{\Im}(t)$, $V_{10}^{\Re}(t)$ and $V_{10}^{\Im}(t)$ are scalars and the other operators are:

$$\sigma_x \equiv |1\rangle\langle 0|+|0\rangle\langle 1| \quad (208)$$

$$\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (209)$$

$$\sigma_y \equiv -\mathrm{i}|1\rangle\langle 0|+\mathrm{i}|0\rangle\langle 1| \quad (210)$$

$$\equiv \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad (211)$$

$$\sigma_z \equiv |1\rangle\langle 1|-|0\rangle\langle 0| \quad (212)$$

$$\equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (213)$$

$$\begin{pmatrix} B_{iz}\left(t\right) & B_i^{\pm}\left(t\right) \\ B_x\left(t\right) & B_i\left(t\right) \\ B_y\left(t\right) & B_{ij}\left(t\right) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right) & \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \\ \frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \quad (214)$$

$$\chi_{ij}\left(t\right) \equiv \sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}\left(t\right)v_{j\mathbf{k}}\left(t\right)-v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}^2}\right), \quad (215)$$

$$B_i^{+}\left(t\right)B_j^{-}\left(t\right) = \mathrm{e}^{\chi_{ij}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right), \quad (216)$$

$$D\left(\pm v_{\mathbf{k}}\left(t\right)\right) \equiv \mathrm{e}^{\pm\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (217)$$

As we can see they verify the relationship $\sigma_x\sigma_y=\mathrm{i}\sigma_z$. The explicit form of $\overline{H_{\bar{I}}}^2\left(t\right)$ is:

$$\overline{H_{\bar{I}}}^2\left(t\right) = \sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i B_{iz}(t)|i\rangle\langle i|(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)\sum_i B_{iz}(t)\,|i\rangle\langle i|\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)+V_{10}^{\Re}(t) \quad (218)$$

$$\times\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right)\sum_i B_{iz}\left(t\right)|i\rangle\langle i|+\left(V_{10}^{\Re}(t)\right)^2\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right)^2+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right) \quad (219)$$

$$\times\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))(\sigma_x B_x(t) \quad (220)$$

$$+\sigma_y B_y\left(t\right))+\left(V_{10}^{\Im}\left(t\right)\right)^2\left(\sigma_x B_y\left(t\right)-\sigma_y B_x\left(t\right)\right)^2 \quad (221)$$

$$= \sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i\left(B_{iz}(t)\,B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}(t)\sum_i\left(B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)\right. \quad (222)$$

$$\left.\times B_x(t)|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2\left(\sigma_x^2B_x^2(t)+\sigma_x\sigma_y B_x(t)B_y(t)+\sigma_y\right. \quad (223)$$

$$\left.\times\sigma_x B_y(t)B_x(t)+\sigma_y^2B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\left(\sigma_x^2B_y^2(t)+\sigma_y^2B_x^2(t)\right. \quad (224)$$

$$\left.-\sigma_x\sigma_y B_y(t)B_x(t)-\sigma_y\sigma_x B_x(t)B_y(t)\right)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\left(\sigma_x^2B_y(t)B_x(t)+\sigma_x\sigma_y B_y^2(t)-\sigma_y\sigma_x B_x^2(t)-\sigma_y^2B_x(t)B_y(t)\right. \quad (225)$$

$$\left.+\sigma_x^2B_x(t)B_y(t)-\sigma_x\sigma_y B_x^2(t)+\sigma_y\sigma_x B_y^2(t)-\sigma_y^2B_y(t)B_x(t)\right), \quad (226)$$

$$\sigma_x\sigma_y = \mathrm{i}\sigma_z \text{ (by Pauli matrices properties)}, \tag{227}$$

$$\overline{H_I}^2(t) = \sum_i B_{iz}^2(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i (B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x + B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i (B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x - B_{iz}(t) \tag{228}$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i (\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2 (B_x^2(t) + \mathrm{i}\sigma_z B_x(t)B_y(t) - \mathrm{i}\sigma_z \tag{229}$$

$$\times B_y(t)B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t)\sum_i (\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_y^2(t) + B_x^2(t) - \mathrm{i}\sigma_z \tag{230}$$

$$\times B_y(t)B_x(t) + \mathrm{i}\sigma_z B_x(t)B_y(t)). \tag{231}$$

To introduce the direct calculation of the expected values recall that the hamiltonian $\overline{H_0}(t)$ is a direct sum of the hamiltonians of two Hilbert spaces given by $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$, so we can write in general the hamiltonian $\overline{H_0}(t)$ as:

$$\overline{H_0}(t) = \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} + \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}. \tag{232}$$

where $\mathbb{I}_{\bar{B}}$ and $\mathbb{I}_{\bar{S}}$ are the identity of the systems $\bar{B}$ and $\bar{S}$ respectively.
We can show that:

$$\begin{aligned}
\left[\overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}}, \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}\right] &= \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} \cdot \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} \cdot \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} && (233)\\
&= \overline{H_{\bar{S}}}(t)\,\mathbb{I}_{\bar{S}} \otimes \mathbb{I}_{\bar{B}}\overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}}\overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}}\mathbb{I}_{\bar{B}} && (234)\\
&= \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} - \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} \text{ (by definition of identity operator)} && (235)\\
&= 0. && (236)
\end{aligned}$$

Let's introduce the following partition functions $Z_{\bar{S}}(t)$ and $Z_{\bar{B}}$ related to the systems $\bar{S}$ and $\bar{B}$ respectively.:

$$Z_{\bar{S}}(t) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right), \tag{237}$$

$$Z_{\bar{B}} \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \tag{238}$$

Using (9), (233) and $\mathrm{Tr}\,(\mathrm{A} \otimes \mathrm{B}) = \mathrm{Tr}\,(\mathrm{A})\,\mathrm{Tr}\,(\mathrm{B})$ we can infer that the partition function $Z_0(t)$ can be factorized as:

$$\begin{aligned}
Z_0(t) &= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0(t)}}\right). && (239)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}\right)}\right) \text{ (by (4))}, && (240)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by (9))} && (241)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)} \otimes \mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (because } \bar{S} \text{ and } \bar{B} \text{ are disjoint Hilbert spaces)} && (242)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by Tr}(A \otimes B)\text{=Tr}(A)\text{Tr}(B)), && (243)\\
&= Z_{\bar{S}}(t)\,Z_{\bar{B}} \text{ (by (237) and (238))}. && (244)
\end{aligned}$$

For an operator $J(t)$ that can be factorized as $J(t) = S(t) \otimes B(t)$ with $S(t) \in \mathrm{gen}\left(\overline{H_{\bar{S}}}(t)\right)$ and $B(t) \in \mathrm{gen}\left(\overline{H_{\bar{B}}}\right)$, being $\mathrm{gen}\,(\mathrm{A})$ the vectorial space generated by the eigenvectors of the operator A, we calculate it's expected value respect to $\overline{H_0}(t)$ using a simple way as follows:

$$\langle J(t)\rangle_{\overline{H_0}(t)} = \frac{\mathrm{Tr}\left(J(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \text{ (by (5))} \tag{245}$$

$$= \frac{\mathrm{Tr}\left((S(t)\otimes B(t))\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\otimes\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by } J(t)=S(t)\otimes B(t) \text{ and } \mathrm{e}^{-\beta\overline{H_0}(t)}=\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\otimes\mathrm{e}^{-\beta\overline{H_{\bar B}}}\text{)} \tag{246}$$

$$= \frac{\mathrm{Tr}\left(\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\otimes\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (rearranging and factorizing)} \tag{247}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by Tr}(A\otimes B)\text{=Tr}(A)\text{Tr}(B)\text{)} \tag{248}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)}\frac{\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \tag{249}$$

$$= \langle S(t)\rangle_{\overline{H_{\bar S}}(t)}\,\langle B(t)\rangle_{\overline{H_{\bar B}}} \text{ (by (5)).} \tag{250}$$

The factorization of $\left\langle \overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)}$ in terms of expected values of elements from gen $\left(\overline{H_{\bar S}}(t)\right)$ and gen $\left(\overline{H_{\bar B}}\right)$ is:

$$\left\langle \overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)} = \sum_i \langle |i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar B}}} + V_{10}^{\Re}(t)\sum_i\Big(\langle |i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \langle |i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}}\Big) \tag{251}$$

$$+V_{10}^{\Im}(t)\sum_i\Big(\langle |i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}} - \langle |i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}}\Big) + V_{10}^{\Re}(t)\sum_i\Big(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)} \tag{252}$$

$$\times\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} + \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\Big) + \left(V_{10}^{\Re}(t)\right)^2\Big(\langle B_x^2(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar B}}} \tag{253}$$

$$-\mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\,\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \langle B_y^2(t)\rangle_{\overline{H_{\bar B}}}\Big) + V_{10}^{\Im}(t)\sum_i\Big(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} - \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)} \tag{254}$$

$$\times\,\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\Big) + \left(V_{10}^{\Im}(t)\right)^2\Big(\langle B_y^2(t)\rangle_{\overline{H_{\bar B}}} + \langle B_x^2(t)\rangle_{\overline{H_{\bar B}}} - \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\,\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)} \tag{255}$$

$$\times\,\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}\Big). \tag{256}$$

In order to obtain the expected values of $\left\langle \overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)}$ respect to the part related to the bath we need to calculate the following expected values that appear in the equation (231) and can be obtained using the factorization of (251). The expected values relevant for calculations are $\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_y(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x^2(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_y^2(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$ and $\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$. Recalling the form of the hamiltonian $\overline{H_{\bar B}} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$ we can extend the result (244), introducing the notation:

$$A_1\otimes\cdots\otimes A_n \equiv \bigotimes_k A_k, \tag{257}$$

$$Z_{\mathbf{k}} \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) \tag{258}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{-1} \tag{259}$$

$$= f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{260}$$

with the creation $b_{\mathbf{k}}$ and annihilation $b_{\mathbf{k}}^\dagger$ operators defned in terms of their actions as:

$$b_{\mathbf{k}}\mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{261}$$

$$b_{\mathbf{k}}^\dagger\mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}+1}\mid j_{\mathbf{k}}+1\rangle. \tag{262}$$

being $|j_{\mathbf{k}}\rangle$ an eigenstate of $H_{\mathbf{k}} \equiv \omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$. With this notation we can write the partition function as:

$$Z_{\bar{B}} = \mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right), \tag{263}$$

$$\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}, \tag{264}$$

$$Z_{\bar{B}} = \mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)\ \text{(by (264))} \tag{265}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)\ \left(\text{by } \mathrm{Tr}\left(A\otimes B\right) = \mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right)\right) \tag{266}$$

$$= \prod_{\mathbf{k}} Z_{\mathbf{k}}\ \text{(by (264))}. \tag{267}$$

For a function $f\left(t\right)$ which can be factorized as:

$$f\left(t\right) \equiv \prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right). \tag{268}$$

with $f_{\mathbf{k}}\left(t\right) \in \mathrm{gen}\left(\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)$, it's expected value is given by:

$$\left\langle f\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(f\left(t\right)\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \tag{269}$$

$$= \frac{\mathrm{Tr}\left(\prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}\ \text{(by (264) and (268))} \tag{270}$$

$$= \frac{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{271}$$

$$= \frac{\prod_{\mathbf{k}}\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\prod_{\mathbf{k}}\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{272}$$

$$= \prod_{\mathbf{k}}\frac{\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{273}$$

$$= \prod_{\mathbf{k}}\left\langle f_{\mathbf{k}}\left(t\right)\right\rangle_{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}. \tag{274}$$

It means that for an operator that can be factorized in terms of functions generated by $\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}$ for each $\mathbf{k}$ we only require to calculate the expected value respect to the Hilbert space where the operator belongs. This process lead us to the following explicit forms of the expected values relevant for our calculations:

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}}\ \text{(by (214))}, \tag{275}$$

$$= \left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)^{2}+\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(\left(g_{i\mathbf{k}'}\right.\right.\right. \tag{276}$$

$$\left.\left.\left.-v_{i\mathbf{k}'}(t)\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^{*}b_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}}\ \text{(by square expansion properties)}, \tag{277}$$

$$= \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{278}$$

$$\times \left\langle \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t) \right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t) \right)^{*} b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (274))}, \tag{279}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \tag{280}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \tag{281}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \text{ (by (262))}, \tag{282}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \mathrm{Tr}\left(|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \tag{283}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \text{ (by trace properties)}, \tag{284}$$

$$= 0, \tag{285}$$

$$\left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \tag{286}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \tag{287}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \text{ (by (261))}, \tag{288}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \mathrm{Tr}\left(|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \tag{289}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \text{ (by trace properties)}, \tag{290}$$

$$= 0, \tag{291}$$

$$\left\langle B_{iz}^{2}(t) \right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{292}$$

$$\times \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t) \right) \left\langle b_{\mathbf{k}'}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t) \right)^{*} \left\langle b_{\mathbf{k}'} \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{293}$$

$$= \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) \cdot 0 + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \cdot 0 \right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t) \right) \cdot 0 \right. \tag{294}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t) \right)^{*} \cdot 0 \right) \text{ (by (280) and (286))} \tag{295}$$

$$= \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{296}$$

$$= \sum_{\mathbf{k}} \left\langle \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{2} \left(b_{\mathbf{k}}^{\dagger} \right)^{2} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(\left(g_{i\mathbf{k}} \right. \right. \right. \tag{297}$$

$$\left. \left. \left. - v_{i\mathbf{k}}(t) \right)^{*} \right)^{2} b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{298}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{2} \left\langle \left(b_{\mathbf{k}}^{\dagger} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left| g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right|^{2} \left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \right)^{2} \left\langle b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}}, \tag{299}$$

$$\left\langle \left(b_{\mathbf{k}}^{\dagger} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger} \right)^{2} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}} \right)} \tag{300}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger}\right)^{2} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{301}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)}|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (262) applied twice)} \tag{302}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)}\mathrm{Tr}\left(|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{303}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{304}$$

$$= 0, \tag{305}$$

$$\left\langle b_{\mathbf{k}}^{2}\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{2}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{306}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)}|j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (261) applied twice)} \tag{307}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)}\mathrm{Tr}\left(|j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{308}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{309}$$

$$= 0, \tag{310}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right)\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{311}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{312}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \text{ (now (261) and (262))} \tag{313}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\sqrt{j_{\mathbf{k}}}|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}\sqrt{j_{\mathbf{k}}+1}|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{314}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\sqrt{j_{\mathbf{k}}}\sqrt{j_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}\sqrt{j_{\mathbf{k}}+1}\sqrt{j_{\mathbf{k}}+1}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{315}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}j_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(j_{\mathbf{k}}+1\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{316}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{317}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right)\mathrm{Tr}\left(|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{318}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right) \text{ (by properties of trace)} \tag{319}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\sum_{j_{\mathbf{k}}=0}^{\infty} \left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right), \tag{320}$$

$$\sum_{j_{\mathbf{k}}=0}^{\infty} x^{j_{\mathbf{k}}}(2j_{\mathbf{k}}+1)= \frac{1+x}{(1-x)^2}, \tag{321}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\frac{\mathrm{e}^{-\beta\omega_{\mathbf{k}}}+1}{(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}})^2} \text{ (setting } x=\mathrm{e}^{-\beta\omega_{\mathbf{k}}} \text{ in (321) and by (311))}, \tag{322}$$

$$= \frac{1+\mathrm{e}^{-\beta\omega_{\mathbf{k}}}}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \tag{323}$$

$$= \frac{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}\frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}+\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}}{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}\frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}-\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}} \tag{324}$$

$$= \frac{\cosh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\sinh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{325}$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{326}$$

$$\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar{B}}}}=\sum_{\mathbf{k}}\left|g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \text{ (by (300), (306) and (326))}, \tag{327}$$

$$\left\langle B_{iz}(t)\,B_x(t)\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle\sum_{\mathbf{k}}\Big((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\Big)\,\frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{328}$$

$$= \frac{1}{2}\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\right.\right. \tag{329}$$

$$\left.\left.\times\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}-\frac{B_{10}(t)+B_{01}(t)}{2}\left\langle\sum_{\mathbf{k}}\Big((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\Big)\right\rangle_{\overline{H_{\bar{B}}}} \tag{330}$$

$$= \frac{1}{2}\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\right.\right. \tag{331}$$

$$\left.\left.\times\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (280) and (286))}, \tag{332}$$

$$\left\langle F(h)\right\rangle_{\overline{H_{\bar{B}}}}\equiv\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\right|F(h)\left|\alpha\right\rangle\mathrm{d}^2\alpha \text{ (using the coherent representation with } N=\left(\mathrm{e}^{\beta\omega}-1\right)^{-1}\text{)}, \tag{333}$$

$$D(\alpha_{\mathbf{k}})\equiv\mathrm{e}^{\left(\frac{\alpha_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{\alpha_{\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \text{ (displacement operator definition)}, \tag{334}$$

$$\left|\alpha\right\rangle\equiv D(\alpha)\left|0\right\rangle \text{ (displacement operator properties)}, \tag{335}$$

$$\left\langle\alpha\right|\equiv\left\langle 0\right|D(-\alpha), \tag{336}$$

$$D(-\alpha)D(h)D(\alpha)\equiv D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha} \text{ (displacement operator properties)}, \tag{337}$$

$$D(0)\equiv\mathbb{I} \text{ (identity written in terms of the displacement operator)}, \tag{338}$$

$$D(-\alpha)D(0)D(\alpha)=D(0)\,\mathrm{e}^{0\cdot\alpha^*-0^*\cdot\alpha} \tag{339}$$

$$=D(0) \tag{340}$$

$$=\mathbb{I}, \tag{341}$$

$$D(-\alpha)\,b^{\dagger}D(\alpha)=b^{\dagger}+\alpha^*, \tag{342}$$

$$D(-\alpha)\,bD(\alpha)=b+\alpha, \tag{343}$$

$$\left\langle D(h)\right\rangle_{\overline{H_{\bar{B}}}}=\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{344}$$

$$\left\langle b^{\dagger}D(h)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\left|b^{\dagger}D(h)\right|\alpha\right\rangle\mathrm{d}^2\alpha \text{ (definition of expected value)} \tag{345}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,b^{\dagger}D(h)\,D(\alpha)\right|0\right\rangle\mathrm{d}^2\alpha \text{ (by (335) and (336))} \tag{346}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,b^{\dagger}\mathbb{I}D(h)\,D(\alpha)\right|0\right\rangle\mathrm{d}^2\alpha \text{ (inserting identity operator)} \tag{347}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\Big(D(-\alpha)\,b^{\dagger}D(\alpha)\Big)\left(D(-\alpha)\,D(h)\,D(\alpha)\right)\right|0\right\rangle\mathrm{d}^2\alpha \text{ (by associative property)} \tag{348}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(b^\dagger+\alpha^*\right)D(h)\,e^{h\alpha^*-h^*\alpha}\right|0\right\rangle d^2\alpha \text{ (by (342) and (337))} \tag{349}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|b^\dagger D(h)\,e^{h\alpha^*-h^*\alpha}\right|0\right\rangle d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha^* D(h)\,e^{h\alpha^*-h^*\alpha}\right|0\right\rangle d^2\alpha \tag{350}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}0D(h)\,e^{h\alpha^*-h^*\alpha}|0\rangle d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha^* D(h)\,e^{h\alpha^*-h^*\alpha}\right|0\right\rangle d^2\alpha \tag{351}$$

$$= \frac{1}{\pi N}\int 0d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*\langle 0|D(h)|0\rangle d^2\alpha \tag{352}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*\langle 0|D(h)|0\rangle d^2\alpha \tag{353}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*\langle 0|h\rangle d^2\alpha \text{ (by (335))} \tag{354}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^* e^{-\frac{|h|^2}{2}}d^2\alpha \text{ (because } \langle 0|h\rangle{=}e^{-\frac{|h|^2}{2}}\text{)}, \tag{355}$$

$$x=\alpha^{\Re}\in\mathbb{R}, \tag{356}$$

$$y=\alpha^{\Im}\in\mathbb{R}, \tag{357}$$

$$\alpha=x+iy, \tag{358}$$

$$\left\langle b^\dagger D(h)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^* e^{-\frac{|h|^2}{2}}d^2\alpha \tag{359}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\left(x^2+y^2\right)/N}e^{h(x-iy)-h^*(x+iy)}(x-iy)\,dx dy \text{ (by (356) and (357))} \tag{360}$$

$$= -h^* e^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}N \tag{361}$$

$$= -h^*\langle D(h)\rangle_{\overline{H_{\bar{B}}}}N, \tag{362}$$

$$|h\rangle = e^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle, \tag{363}$$

$$\langle bD(h)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle\alpha|bD(h)|\alpha\rangle d^2\alpha \text{ (definition of expected value)} \tag{364}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|D(-\alpha)\,b\mathbb{I}D(h)\,D(\alpha)|0\rangle d^2\alpha \text{ (by (335) and (336))} \tag{365}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|(D(-\alpha)\,bD(\alpha))\,(D(-\alpha)\,D(h)\,D(\alpha))|0\rangle d^2\alpha \text{ (by associative property)} \tag{366}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|(b+\alpha)\,D(h)\,e^{h\alpha^*-h^*\alpha}\right|0\right\rangle d^2\alpha \text{ (by (343) and (337))} \tag{367}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|bD(h)\,e^{h\alpha^*-h^*\alpha}\right|0\right\rangle d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha D(h)\,e^{h\alpha^*-h^*\alpha}\right|0\right\rangle d^2\alpha \tag{368}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\langle 0|b|h\rangle d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle d^2\alpha \tag{369}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\langle 0|\,be^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle d^2\alpha \text{ (by (363))} \tag{370}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\langle 0|\,e^{-\frac{|h|^2}{2}}\sum_{n=1}^{\infty}\frac{h^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle d^2\alpha \text{ (by (261))} \tag{371}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}e^{-\frac{|h|^2}{2}}\sum_{n=1}^{\infty}\frac{h^n}{\sqrt{n!}}\sqrt{n}\delta_{0,n-1}d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle d^2\alpha \text{ (by } \langle n|n'\rangle{=}\delta_{nn'}\text{)} \tag{372}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}e^{-\frac{|h|^2}{2}}\frac{h^1}{\sqrt{1!}}\sqrt{1}d^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle d^2\alpha \tag{373}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}e^{-\frac{|h|^2}{2}}hd^2\alpha+\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha e^{-\frac{|h|^2}{2}}d^2\alpha \text{ (because } \langle 0|h\rangle{=}e^{-\frac{|h|^2}{2}}\text{)} \tag{374}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}e^{-\frac{|h|^2}{2}}(\alpha+h)\,d^2\alpha \tag{375}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\left(x^2+y^2\right)/N}e^{h(x-iy)-h^*(x+iy)}(x+iy+h)\,dx dy \tag{376}$$

$$= h\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}(N+1) \tag{377}$$

$$= h\left\langle D(h)\right\rangle_{\overline{H_{\bar{B}}}}(N+1), \tag{378}$$

$$\left\langle D(h)\,b\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|D(h)\,b\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (definition of expected value)} \tag{379}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,\mathbb{I}D(h)\,bD(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (335) and (336))} \tag{380}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|(D(-\alpha)\,D(h)\,D(\alpha))\,(D(-\alpha)\,bD(\alpha))\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} \tag{381}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}(b+\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (343) and (337))} \tag{382}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}b\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\right|0\right\rangle \mathrm{d}^2\alpha \tag{383}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{384}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|D(h)\,0\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (261))} \tag{385}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{386}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}(x+\mathrm{i}y)\,\mathrm{d}x\mathrm{d}y \tag{387}$$

$$= hN\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)} \tag{388}$$

$$= hN\left\langle D(h)\right\rangle_B, \tag{389}$$

$$\left\langle D(h)\,b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|D(h)\,b^\dagger\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (definition of expected value)} \tag{390}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,D(h)\,\mathbb{I}b^\dagger D(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (335) and (336))} \tag{391}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|(D(-\alpha)\,D(h)\,D(\alpha))\left(D(-\alpha)\,b^\dagger D(\alpha)\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} \tag{392}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (343) and (337))} \tag{393}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha^* D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \tag{394}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{395}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{396}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h\left|\sqrt{0+1}\right|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{397}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h\left|\sqrt{0+1}\right|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (336))}, \tag{398}$$

$$\left\langle h\right| = \mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(h^*)^n}{\sqrt{n!}}\left\langle n\right|, \tag{399}$$

$$\left\langle D(h)\,b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(-h^*)^n}{\sqrt{n!}}\left\langle n|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (399))} \tag{400}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\frac{(-h^*)^1}{\sqrt{1!}}\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by } \left\langle n|n'\right\rangle=\delta_{nn'}\text{)} \tag{401}$$

$$= \frac{1}{\pi N}\int (\alpha^*-h^*)\,\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{402}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}(x-\mathrm{i}y-h^*)\,\mathrm{d}x\mathrm{d}y \tag{403}$$

$$= -h^*\left\langle D(h)\right\rangle_B(N+1), \tag{404}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right.\right. \tag{405}$$

$$\left.\left.+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (replacing the definitions in (214))} \tag{406}$$

$$= \frac{1}{2}\left\langle\mathrm{e}^{\chi_{10}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)+\mathrm{e}^{\chi_{01}(t)}\right. \tag{407}$$

$$\left.\times\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{408}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\langle b_{\mathbf{k}}\right. \tag{409}$$

$$\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\Bigg)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right.\right.\right. \tag{410}$$

$$\left.\left.\left.-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right), \tag{411}$$

$$\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (344))}, \tag{412}$$

$$N_{\mathbf{k}} = \left(\mathrm{e}^{\beta\omega_{\mathbf{k}}}-1\right)^{-1}, \tag{413}$$

$$\langle b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} = \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (378))}, \tag{414}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} = -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (362))}, \tag{415}$$

$$\left\langle\prod_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (412) and (274))}, \tag{416}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (274))} \tag{417}$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}'}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (274))} \tag{418}$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\prod_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \tag{419}$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (412))}, \tag{420}$$

$$\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} = \langle b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (274))} \tag{421}$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}'}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (414))} \tag{422}$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\prod_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \tag{423}$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (412))}, \tag{424}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\langle b_{\mathbf{k}}\right. \tag{425}$$

$$\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\Bigg\rangle_{\overline{H_{\bar{B}}}}\Bigg)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\right. \tag{426}$$

$$\left.\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right)\quad\text{(by (214))} \tag{427}$$

$$=\frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{428}$$

$$\times\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,(-N_{\mathbf{k}}\right. \tag{429}$$

$$\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1) \tag{430}$$

$$\left.\times\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{431}$$

$$=\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{432}$$

$$\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,(-N_{\mathbf{k}}\right. \tag{433}$$

$$\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \tag{434}$$

$$\left.\times(N_{\mathbf{k}}+1)\,\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{435}$$

$$=\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{10}(t)\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right. \tag{436}$$

$$\times(N_{\mathbf{k}}+1)\,B_{10}(t)\Bigg)+\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{01}(t)\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{437}$$

$$\left.\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,B_{01}(t)\right)\quad\text{(by (214) using the defintion } B_{ij}(t)\text{)} \tag{438}$$

$$=\frac{B_{10}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{439}$$

$$+\frac{B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{440}$$

$$=\frac{B_{10}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{441}$$

$$-\frac{B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{442}$$

$$=\frac{B_{10}(t)-B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{443}$$

$$\langle B_{iz}(t)B_{y}(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\,b_{\mathbf{k}}\right)\frac{B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{444}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}} \right) \left(B_0^{+}(t) B_1^{-}(t) - B_1^{+}(t) B_0^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} + \frac{(B_{10}(t) - B_{01}(t))}{2\mathrm{i}} \tag{445}$$

$$\times \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by the properties of expected value)} \tag{446}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}} \right) \left(B_0^{+}(t) B_1^{-}(t) - B_1^{+}(t) B_0^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} + \frac{(B_{10}(t) - B_{01}(t))}{2\mathrm{i}} \cdot 0 \tag{447}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}} \right) \left(B_0^{+}(t) B_1^{-}(t) - B_1^{+}(t) B_0^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \tag{448}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} b_{\mathbf{k}} \right) \left(\mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \tag{449}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right. \tag{450}$$

$$\left. + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} b_{\mathbf{k}} \mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} b_{\mathbf{k}} \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{451}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, \mathrm{e}^{\chi_{01}(t)} \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} - \mathrm{e}^{\chi_{10}(t)} \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \right. \tag{452}$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \mathrm{e}^{\chi_{01}(t)} \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{453}$$

$$\left. \times \mathrm{e}^{\chi_{10}(t)} \, (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \right) \tag{454}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, \mathrm{e}^{\chi_{01}(t)} \left(- \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) - \mathrm{e}^{\chi_{10}(t)} \right. \tag{455}$$

$$\times \left(- \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \mathrm{e}^{\chi_{01}(t)} \tag{456}$$

$$\times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \tag{457}$$

$$\left. \times \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) \quad \text{(by (416), (424) and (420))} \tag{458}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(- \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{01}(t) \right) - \left(- \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{10}(t) \right) \right. \tag{459}$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \tag{460}$$

$$\left. \times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \right) \tag{461}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{01}(t) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{10}(t) \right. \tag{462}$$

$$\left. - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \right) \tag{463}$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right), \tag{464}$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} = \left\langle \frac{B_1^{+}(t)\, B_0^{-}(t) + B_0^{+}(t)\, B_1^{-}(t) - B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{465}$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2} \tag{466}$$

$$\times \left\langle \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by expected value properties and (424))} \tag{467}$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2}\cdot 0 \tag{468}$$

$$= \frac{1}{2}\left\langle \left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{469}$$

$$= \frac{1}{2}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger\right.\right. \tag{470}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{471}$$

$$\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} = \alpha_{\mathbf{k}}N_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{472}$$

$$\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{473}$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{474}$$

$$= -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (473))} \tag{475}$$

$$= -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\prod_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{476}$$

$$= -\alpha_{\mathbf{k}}^*\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\alpha_{\mathbf{k}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (412))}, \tag{477}$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{478}$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (472))} \tag{479}$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\prod_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{480}$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\alpha_{\mathbf{k}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (412))}, \tag{481}$$

$$\left\langle B_x(t)B_{iz}(t)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\right. \tag{482}$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{483}$$

$$\left.\times\mathrm{e}^{\chi_{10}(t)}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \text{ (by (214))} \tag{484}$$

$$= \frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right. \tag{485}$$

$$+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{10}(t)} \tag{486}$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* \tag{487}$$

$$\times\left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\Bigg)\quad\text{(by (389) and (404))}\tag{488}$$

$$=\frac{1}{2}\sum_{\mathbf{k}}\Bigg(-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)B_{10}(t)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\tag{489}$$

$$\times(N_{\mathbf{k}}+1)B_{01}(t)+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{10}(t)+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\tag{490}$$

$$\times N_{\mathbf{k}}B_{01}(t)\Bigg)\tag{491}$$

$$=\frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)\right)\tag{492}$$

$$\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\tag{493}$$

$$=\left\langle\frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\tag{494}$$

$$\times\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\quad\text{(by (389) and (404))}\tag{495}$$

$$=\left\langle\frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\quad\text{(by (286) and (280))}\tag{496}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)-\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger\right.\right.\tag{497}$$

$$\left.\left.+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\quad\text{(by (214))}\tag{498}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\Bigg((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\mathrm{e}^{\chi_{01}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}-\mathrm{e}^{\chi_{10}(t)}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\tag{499}$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\tag{500}$$

$$\times\mathrm{e}^{\chi_{01}(t)}-\mathrm{e}^{\chi_{10}(t)}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\Bigg)\quad\text{(by expected value properties)}\tag{501}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\Bigg((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\mathrm{e}^{\chi_{01}(t)}\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\tag{502}$$

$$-\mathrm{e}^{\chi_{10}(t)}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)}\tag{503}$$

$$\times(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)-\mathrm{e}^{\chi_{10}(t)}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\tag{504}$$

$$\times\left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\Bigg)\quad\text{(by (389) and (404))}\tag{505}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\Bigg((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)B_{01}(t)+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\tag{506}$$

$$\times(N_{\mathbf{k}}+1)B_{10}(t)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{01}(t)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\tag{507}$$

$$\times N_{\mathbf{k}}B_{10}(t)\Bigg)\tag{508}$$

$$
\begin{aligned}
&= \frac{B_{01}(t)+B_{10}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}(N_{\mathbf{k}}+1)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\right) && (509)\\
\mathrm{Var}_{\overline{H_{\bar{B}}}}(A) &\equiv \left\langle A^2\right\rangle_{\overline{H_{\bar{B}}}}-\left\langle A\right\rangle^2_{\overline{H_{\bar{B}}}} \ \text{(definition of variance)}, && (510)\\
\mathrm{Var}\left(aX+b\right) &= a^2\mathrm{Var}\left(X\right) \ \text{(properties of variance)}, && (511)\\
\left\langle B_x(t)\right\rangle_{\overline{H_{\bar{B}}}} &= 0 \ \text{(expected value of obtained in [2])}, && (512)\\
\left\langle B_y(t)\right\rangle_{\overline{H_{\bar{B}}}} &= 0 \ \text{(expected value of obtained in [2])}, && (513)\\
\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_x(t)\right)+\left\langle B_x(t)\right\rangle^2_{\overline{H_{\bar{B}}}} \ \text{(by (510))} && (514)\\
&= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)}{2}\right) \ \left(\text{because } \left\langle B_x(t)\right\rangle_{\overline{H_{\bar{B}}}}=0\right) && (515)\\
&= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)\right) && (516)\\
&= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right) \ \text{(by (511))} && (517)\\
&= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \ \text{(by (510))} && (518)\\
&= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2+B_1^+(t)B_0^-(t)B_0^+(t)B_1^-(t)+B_0^+(t)B_1^-(t)B_1^+(t)B_0^-(t)+\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right. && (519)\\
&\left.-\left(B_{10}(t)+B_{01}(t)\right)^2\right) && (520)\\
&= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right), && (521)\\
\left(D(h)\right)^2 &= D(h)D(h) && (522)\\
&= D(h+h)\,\mathrm{e}^{\frac{1}{2}\left(\frac{h^*h-hh^*}{\omega^2}\right)} \ \text{(by displacement operator properties)} && (523)\\
&= D(2h), && (524)\\
\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} &= \left\langle\left(\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathrm{e}^{\frac{1}{2}\left(\frac{v^*_{i\mathbf{k}}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v^*_{j\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}\right)}\right)^2\right\rangle_{\overline{H_{\bar{B}}}} && (525)\\
&= \left\langle\prod_{\mathbf{k}}D\left(2\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\mathrm{e}^{\frac{v^*_{i\mathbf{k}}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v^*_{j\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}}\right\rangle_{\overline{H_{\bar{B}}}} \ \text{(by (524))} && (526)\\
&= \prod_{\mathbf{k}}\mathrm{e}^{\frac{v^*_{i\mathbf{k}}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v^*_{j\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}}\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \ \text{(by (412))} && (527)\\
&= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v^*_{i\mathbf{k}}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v^*_{j\mathbf{k}}(t)}{2\omega^2_{\mathbf{k}}}}\mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^2\left(\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^2 && (528)\\
&= B^2_{ij}(t)\left|B_{ij}(t)\right|^2 \ \text{(by (214))}, && (529)\\
\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \ \text{(by (510))} && (530)\\
&= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2\left\langle\mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \ \text{(by expected value)} && (531)\\
&= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) && (532)\\
&= \frac{1}{4}\left(B^2_{10}(t)\left|B^2_{10}(t)\right|+2+B^2_{01}(t)\left|B^2_{01}(t)\right|-\left(B^2_{10}(t)+2B_{10}(t)B_{01}(t)+B^2_{01}(t)\right)\right) \ \text{(by (529))} && (533)\\
&= \frac{1}{4}\left(B^2_{10}(t)\left|B^2_{10}(t)\right|+2+B^2_{01}(t)\left|B^2_{10}(t)\right|-\left(B^2_{10}(t)+2\left|B^2_{10}(t)\right|+B^2_{01}(t)\right)\right) && (534)\\
&= \frac{1}{4}\left(B^2_{10}(t)+B^2_{01}(t)-2\right)\left(\left|B^2_{10}(t)\right|-1\right), && (535)\\
\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_y(t)\right)+\left\langle B_y(t)\right\rangle^2_{\overline{H_{\bar{B}}}} && (536)\\
&= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right) \ \left(\text{by } \left\langle B_y(t)\right\rangle_{\overline{H_{\bar{B}}}}=0 \text{ and (214)}\right) && (537)
\end{aligned}
$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t) + B_{10}(t) - B_{01}(t)\right)\ \text{(by } \langle B_y(t)\rangle_{\overline{H_{\bar{B}}}} = 0 \text{ and (214))} \tag{538}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right) \tag{539}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{540}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - 2\mathbb{I} + \left(B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{541}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - 2\,\langle\mathbb{I}\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)^2\right) \tag{542}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{01}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right)\ \text{(by (529))} \tag{543}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{01}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + 2B_{01}(t)\,B_{10}(t) - B_{10}^2(t)\right) \tag{544}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{10}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + 2\,|B_{10}(t)|^2 - B_{10}^2(t)\right) \tag{545}$$

$$= -\frac{1}{4}\left(B_{10}^2(t) + B_{01}^2(t) + 2\right)\left(|B_{10}(t)|^2 - 1\right), \tag{546}$$

$$\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_x(t)\,\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{547}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\,\langle B_x(t)\rangle_{\overline{H_{\bar{B}}}} \tag{548}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\cdot 0\ \text{(by } \langle B_x(t)\rangle_{\overline{H_{\bar{B}}}} = 0\text{)} \tag{549}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{550}$$

$$= \left\langle \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}}\ \text{(by (214))} \tag{551}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle\left(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)\right)\left(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\right)\big\rangle_{\overline{H_{\bar{B}}}} - \big\langle\left(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\right)\big\rangle_{\overline{H_{\bar{B}}}} \tag{552}$$

$$\times\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{553}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle B_1^+(t)\,B_0^-(t)\,B_0^+(t)B_1^-(t) - B_1^+(t)\,B_0^-(t)\,B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)\,B_0^+(t)\,B_1^-(t) - B_0^+(t)\,B_1^-(t) \tag{554}$$

$$\times B_1^+(t)B_0^-(t)\big\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{555}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\mathbb{I} - \left(B_1^+(t)\,B_0^-(t)\right)^2 + \left(B_0^+(t)\,B_1^-(t)\right)^2 - \mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{556}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{557}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{01}(t) + B_{10}(t)\right)\right) \tag{558}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}^2(t) - B_{10}^2(t)\right)\right) \tag{559}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\,|B_{10}(t)|^2 - B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + B_{10}^2(t)\right)\ \text{(by (529))} \tag{560}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t) - B_{10}^2(t)\right)\left(|B_{10}(t)|^2 - 1\right), \tag{561}$$

$$\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}}\ \text{(by (214))} \tag{562}$$

$$= \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_y(t)\,\frac{B_{10}(t) + B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{563}$$

$$= \frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t) + B_{01}(t)}{2}\,\langle B_y(t)\rangle_{\overline{H_{\bar{B}}}} \tag{564}$$

$$
\begin{aligned}
&= \frac{1}{2} \left\langle B_y(t) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t) + B_{01}(t)}{2} \cdot 0 \text{ (by } \langle B_y(t) \rangle_{\overline{H_{\bar{B}}}} = 0) && (565)\\
&= \frac{1}{2} \left\langle B_y(t) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} && (566)\\
&= \frac{1}{2} \left\langle \frac{B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (214))} && (567)\\
&= \frac{1}{4\mathrm{i}} \langle \left(B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) \right) \left(B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) \right) \rangle_{\overline{H_{\bar{B}}}} + \frac{(B_{10}(t) - B_{01}(t))}{4\mathrm{i}} \langle \left(B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) \right) \rangle_{\overline{H_{\bar{B}}}} && (568)\\
&= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t) \right) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} + \frac{(B_{10}(t) - B_{01}(t))\,(B_{10}(t) + B_{01}(t))}{4\mathrm{i}} && (569)\\
&= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t) \right) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} && (570)\\
&= \frac{1}{4\mathrm{i}} \left\langle B_0^+(t)\, B_1^-(t)\, B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t)\, B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t)\, B_1^+(t)\, B_0^-(t) - B_1^+(t)\, B_0^-(t)\, B_0^+(t)\, B_1^-(t) \right\rangle_{\overline{H_{\bar{B}}}} && (571)\\
&\quad + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} && (572)\\
&= \frac{1}{4\mathrm{i}} \left\langle \mathbb{I} + \left(B_0^+(t)\, B_1^-(t) \right)^2 - \left(B_1^+(t)\, B_0^-(t) \right)^2 - \mathbb{I} \right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} && (573)\\
&= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+(t)\, B_1^-(t) \right)^2 - \left(B_1^+(t)\, B_0^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} && (574)\\
&= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t)\, |B_{10}(t)|^2 - B_{10}^2(t)\, |B_{10}(t)|^2 \right) + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} \text{ (by (529))} && (575)\\
&= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t)\, |B_{10}(t)|^2 - B_{10}^2(t)\, |B_{10}(t)|^2 + B_{10}^2(t) - B_{01}^2(t) \right) && (576)\\
&= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t) - B_{10}^2(t) \right) \left(|B_{10}(t)|^2 - 1 \right). && (577)
\end{aligned}
$$

Summarizing the expected values obtained in the precedent lines we have:

$$
\begin{aligned}
\langle B_{iz}^2(t) \rangle_{\overline{H_{\bar{B}}}} &= \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), && (578)\\
\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_{\bar{B}}}} &= \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right), && (579)\\
\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_{\bar{B}}}} &= \frac{B_{10}(t) + B_{01}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right), && (580)\\
\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} &= \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) \right), && (581)\\
\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} &= \frac{B_{01}(t) + B_{10}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} \right), && (582)\\
\langle B_x^2(t) \rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{4} \left(B_{10}^2(t) + B_{01}^2(t) - 2 \right) \left(|B_{10}^2(t)| - 1 \right), && (583)\\
\langle B_y^2(t) \rangle_{\overline{H_{\bar{B}}}} &= -\frac{1}{4} \left(B_{10}^2(t) + B_{01}^2(t) + 2 \right) \left(|B_{10}(t)|^2 - 1 \right), && (584)\\
\langle B_x(t) B_y(t) \rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t) - B_{10}^2(t) \right) \left(|B_{10}(t)|^2 - 1 \right), && (585)\\
\langle B_y(t) B_x(t) \rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t) - B_{10}^2(t) \right) \left(|B_{10}(t)|^2 - 1 \right). && (586)
\end{aligned}
$$

The density matrix associated to $\rho_{\overline{S}} = \frac{\mathrm{e}^{-\beta \overline{H_{\bar{S}}}(t)}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \overline{H_{\bar{S}}}(t)}\right)}$ follows the form:

$$\rho_{\overline{S},00}=\frac{1}{2}-\frac{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{587}$$

$$\rho_{\overline{S},01}=-\frac{B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{588}$$

$$\rho_{\overline{S},10}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{589}$$

$$\rho_{\overline{S},11}=\frac{1}{2}+\frac{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}. \tag{590}$$

The expected values respect to the system $\overline{S}$ of relevance for calculating $\left\langle\overline{H_I}^2\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)}}$ are $\langle|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle|i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle|i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$ and $\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}$, we took account that $\sigma_x\sigma_y=\mathrm{i}\sigma_z$ and $\sigma_y\sigma_x=-\mathrm{i}\sigma_z$. The values needed for our calculation are:

$$\langle|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}-\frac{\left(\sum_i(-1)^i(\varepsilon_i\left(t\right)+R_i\left(t\right))\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i\left(t\right)+R_i\left(t\right))\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{591}$$

$$\langle|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}+\frac{\left(\sum_i(-1)^i(\varepsilon_i\left(t\right)+R_i\left(t\right))\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i\left(t\right)+R_i\left(t\right))\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{592}$$

$$\langle|0\rangle\langle 0|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{593}$$

$$\langle|1\rangle\langle 1|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{594}$$

$$\langle|0\rangle\langle 0|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i(-1)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{595}$$

$$\langle |1\rangle\langle 1|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}}, \tag{596}$$

$$\langle \sigma_x|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{597}$$

$$\langle \sigma_x|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{598}$$

$$\langle \sigma_y|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{599}$$

$$\langle \sigma_y|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{600}$$

$$\langle \sigma_z\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}. \tag{601}$$

Our next step is to find $v_3\left(t\right)$, the commutator $\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]$ is a central point for our calculations and it is equal to:

$$\left[\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right]=\left[\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\right. \tag{602}$$

$$\left.+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_xB_x(t)+\sigma_yB_y(t)\right)+V_{10}^{\Im}(t)\left(\sigma_xB_y(t)-\sigma_yB_x(t)\right)\right] \tag{603}$$

$$=\left[\sum_{i}\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\right. \tag{604}$$

$$\left.\sum_{i}B_{iz}\left(t\right)|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right)+\sigma_yB_y\left(t\right)\right)+V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y\left(t\right)-\sigma_yB_x\left(t\right)\right)\right] \tag{605}$$

$$=\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right)+\sigma_yB_y\left(t\right)\right)+\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y\left(t\right)-\sigma_yB_x\left(t\right)\right)+\sigma_x \tag{606}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_xB_x(t)+\sigma_yB_y(t)\right) \tag{607}$$

$$+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(\sigma_xB_y(t)-\sigma_yB_x(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i| \tag{608}$$

$$-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(\sigma_xB_y(t) \tag{609}$$

$$-\sigma_y B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t)\,|i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{610}$$

$$-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t)\,|i\rangle\langle i| \tag{611}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t))\,\sigma_x \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) \tag{612}$$

$$+V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))\sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{613}$$

$$\times \sum_i (\varepsilon_i(t) + R_i(t))\,|i\rangle\langle i| - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\,\sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\,\sigma_y \tag{614}$$

$$\times \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \tag{615}$$

$$= \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{616}$$

$$\times \sum_i B_{iz}(t)\,|i\rangle\langle i| + \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t)\,(B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) V_{10}^{\Im}(t) \tag{617}$$

$$\times (B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)\,|i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Re}(t)\,(-\mathrm{i}\sigma_z B_x(t) + B_y(t)) \tag{618}$$

$$-\left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t)|i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{619}$$

$$+\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{620}$$

$$\left.+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t)\,|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))\,|i\rangle\langle i| - V_{10}^{\Re}(t)\,(B_x(t) - \mathrm{i}\sigma_z B_y(t)) \tag{621}$$

$$\times \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{622}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))\,|i\rangle\langle i| - V_{10}^{\Im}(t)\,(B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) \tag{623}$$

$$+V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{624}$$

The term $\overline{H_{\bar{I}}}(t)\left[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)\right]$ is given by:

$$\overline{H_{\bar{I}}}(t)[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)] = \left(\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\right)\left(\sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| V_{10}^{\Re}(t)\right. \tag{625}$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{626}$$

$$\times \sum_i B_{iz}(t)|i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t)(B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{627}$$

$$\times V_{10}^{\Im}(t)(B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)|i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \tag{628}$$

$$\times V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t) + B_y(t)) - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) \tag{629}$$

$$\times |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \tag{630}$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t)|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{631}$$

$$-V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Re}(t)\,(B_x(t) - \mathrm{i}\sigma_z B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{632}$$

$$+V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{633}$$

$$-V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Im}(t)(B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{634}$$

$$\left.+V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) \tag{635}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{636}$$

$$+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\,(B_x(t) \tag{637}$$

$$+\mathrm{i}\sigma_z B_y(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{638}$$

$$\left.+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\,(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i| \tag{639}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{640}$$

$$\times V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_i B_{iz}(t)\,|i\rangle\langle i|\,\sigma_x \tag{641}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i| \tag{642}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{643}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t) \tag{644}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t) \tag{645}$$

$$\times(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) \tag{646}$$

$$-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{647}$$

$$+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x \tag{648}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t)) \tag{649}$$

$$+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_y \tag{650}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{651}$$

$$-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{652}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{653}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{654}$$

$$+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t) \tag{655}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{656}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t) \tag{657}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t) \tag{658}$$

$$+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{659}$$

$$\times V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{660}$$

$$+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i| \tag{661}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{662}$$

$$-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{663}$$

$$\times V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|-V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{664}$$

$$-\sigma_y B_x(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{665}$$

$$\times V^{\Im}_{10}(t)\left(-\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)+V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}} \tag{666}$$

$$\times V^{\Re}_{10}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{667}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{668}$$

$$-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t)) \tag{669}$$

$$\times|i\rangle\langle i|-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V^{\Re}_{10}(t)\left(B_x(t)-\mathrm{i}\sigma_z B_y(t)\right)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Re}_{10}(t) \tag{670}$$

$$\times\left(\mathrm{i}\sigma_z B_x(t)+B_y(t)\right)\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Re}_{10}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}} \tag{671}$$

$$-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Im}_{10}(t)\left(B_y(t)\right. \tag{672}$$

$$\left.+\mathrm{i}\sigma_z B_x(t)\right)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Im}_{10}(t)\left(\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{673}$$

$$-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}} \tag{674}$$

$$=V^{\Re}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\right)+V^{\Im}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y\right. \tag{675}$$

$$\left.\times B_{iz}(t)\,B_x(t)\right)+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_{i\neq i'}B_{iz}(t)\,B_{i'z}(t)\,|i\rangle\langle i'|+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Re}_{10}(t)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\right. \tag{676}$$

$$\left.\times B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)\right)+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\sum_i\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_x(t)\right)-\left(B^{\Re}_{10}(t)\right. \tag{677}$$

$$\left.\times V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_{i\neq i'}B_{iz}(t)B_{i'z}(t)\,|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|-\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)\sum_i\left(-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+B_{iz}(t)\right. \tag{678}$$

$$\left.\times B_y(t)\,|i\rangle\langle i|\right)+\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)+|i\rangle\langle i|B_{iz}(t)\,B_x(t)\right)+\sum_{i,\mathbf{k}}|i\rangle\langle i|B_{iz}(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}B_{iz}(t) \tag{679}$$

$$+V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}B_y(t)\right)+V^{\Im}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\right. \tag{680}$$

$$\left.\times\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}B_x(t)\right)-\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i B^2_{iz}(t)\,|i\rangle\langle i|\sigma_x+\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i B^2_{iz}(t)\,|i\rangle\langle i|\sigma_y-\sum_{i,\mathbf{k}}|i\rangle\langle i| \tag{681}$$

$$\times B^2_{iz}(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t)\sum_{i,i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|B_{iz}(t)B_x(t)+|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_y(t)\right)-V^{\Im}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\right. \tag{682}$$

$$\left.\times V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\,B_x(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)\right)+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+|i\rangle\langle i|\right. \tag{683}$$

$$\left.\times B_{iz}(t)\,B_y(t)\right)-V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right)-V^{\Im}_{10}(t)\sum_{i\neq i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|\right. \tag{684}$$

$$\left.\times B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)\,B_x(t)\right)-V^{\Im}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\,B_y(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_x(t)\right)+V^{\Im}_{10}(t) \tag{685}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)-|i\rangle\langle i|B_{iz}(t)\,B_x(t)\right)-V^{\Im}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}-|i\rangle\langle i|\sigma_y\right. \tag{686}$$

$$\left.\times B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right)+\left(V^{\Re}_{10}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B^2_x(t)+\sigma_x|i\rangle\langle i|\sigma_y B_x(t)\,B_y(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)\,B_x(t)+\sigma_y|i\rangle\langle i|\right. \tag{687}$$

$$\left.\times\sigma_y B^2_y(t)\right)+V^{\Re}_{10}(t)\,V^{\Im}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B^2_x(t)+\sigma_y|i\rangle\langle i|\sigma_x B^2_y(t)-\sigma_y|i\rangle\langle i|\sigma_y B_y(t)\,B_x(t)\right) \tag{688}$$

$$+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_x(t)\,B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_{iz}(t)\right)+\left(V^{\Re}_{10}(t)\right)^2\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right) \tag{689}$$

$$\times\left(\sigma_x B^2_x(t)+\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)-\sigma_x B^2_y(t)\right)+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\left(B_x(t)\,B_y(t)-\mathrm{i}\sigma_z B^2_y(t)\right. \tag{690}$$

$$\left.-\mathrm{i}\sigma_z B^2_x(t)-B_y(t)\,B_x(t)\right)-V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_{iz}(t)+|i\rangle\langle i|B_y(t)\,B_{iz}(t)\right)-V^{\Re}_{10}(t) \tag{691}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)\left(-\sigma_y B^2_x(t)+\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B^2_y(t)\right)-V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\right. \tag{692}$$

$$\left.\times V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)\left(-\sigma_y B_x(t)B_y(t)+\sigma_x B^2_y(t)-\sigma_x B^2_x(t)-\sigma_y B_y(t)B_x(t)\right)+V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)+\sigma_y|i\rangle\langle i|B_y(t)\right)\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}B_{iz}(t) \tag{693}$$

$$+\left(V_{10}^{\Re}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}} \tag{694}$$

$$\times\left(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{695}$$

$$\times\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)\,B_{iz}(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_y B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i|\right. \tag{696}$$

$$\left.\times\sigma_y B_y(t)\,B_{iz}(t)\right)-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\sigma_y|i\rangle\langle i|B_y(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-\left(V_{10}^{\Re}(t)\right)^{2}\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_x^2(t)\right. \tag{697}$$

$$\left.-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)+|i\rangle\langle i|B_y^2(t)\right)-\left(V_{10}^{\Re}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_x B_x^2(t)+\sigma_y B_y(t)\,B_x(t)\right. \tag{698}$$

$$\left.-\sigma_y B_x(t)\,B_y(t)+\sigma_x B_y^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_y B_x^2(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B_y^2(t)\right) \tag{699}$$

$$-\left(V_{10}^{\Re}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right) \tag{700}$$

$$\times\left(|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)-|i\rangle\langle i|B_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_x B_x(t)\right. \tag{701}$$

$$\left.\times B_y(t)+\sigma_y B_y^2(t)+\sigma_y B_x^2(t)-\sigma_x B_y(t)B_x(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_y B_x(t)B_y(t)-\sigma_x B_y^2(t)-\sigma_x B_x^2(t)\right. \tag{702}$$

$$\left.-\sigma_y B_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_x^2(t)-B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_i\left(\varepsilon_i(t)\right. \tag{703}$$

$$\left.+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B_y(t)\,B_x(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x^2(t)+\sigma_x|i\rangle\langle i|\sigma_y B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_y B_x(t)\,B_y(t)\right)+\left(V_{10}^{\Im}(t)\right)^{2}\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\right. \tag{704}$$

$$\left.\times\sigma_x B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x(t)\,B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B_y(t)\,B_x(t)+\sigma_y|i\rangle\langle i|\sigma_y B_x^2(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i\left(|i\rangle\langle i|B_y(t)\right. \tag{705}$$

$$\left.\times B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)+\sigma_y B_y^2(t)+\sigma_x B_x(t)B_y(t)\right) \tag{706}$$

$$+\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(B_y^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_x^2(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{707}$$

$$\times\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_{iz}(t)-|i\rangle\langle i|B_x(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(-\sigma_y B_y(t)\,B_x(t)-\sigma_x B_x^2(t)+\sigma_x B_y^2(t)\right. \tag{708}$$

$$\left.-\sigma_y B_x(t)\,B_y(t)\right)-\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(-\sigma_y B_y^2(t)-\sigma_x B_x(t)\,B_y(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_y B_x^2(t)\right)+V_{10}^{\Im}(t) \tag{709}$$

$$\times\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\right)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z\right. \tag{710}$$

$$\left.\times B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+\left(V_{10}^{\Im}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+B_x(t)\right. \tag{711}$$

$$\left.\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_x B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{712}$$

$$\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_y B_x(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{713}$$

$$-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_y(t)B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-|i\rangle\langle i|B_x(t)B_y(t)\right)-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{714}$$

$$\left.-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)-\sigma_y B_y^2(t)-\sigma_x B_x(t)B_y(t)\right)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_y B_y(t)B_x(t)\right. \tag{715}$$

$$\left.+\sigma_x B_x^2(t)+\sigma_x B_y^2(t)-\sigma_y B_x(t)B_y(t)\right)-\sum_{\mathbf{k}}V_{10}^{\Im}(t)V_{10}^{\Re}(t)\,\omega_{\mathbf{k}}\left(B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{716}$$

$$-\left(V_{10}^{\Im}(t)\right)^{2}\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_y^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+|i\rangle\langle i|B_x^2(t)\right)-\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{717}$$

$$\left.-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_x B_y^2(t)-\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)+\sigma_x B_x^2(t)\right)+\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_y B_y^2(t)\right. \tag{718}$$

$$\left.+\sigma_x B_x(t)\,B_y(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_y B_x^2(t)\right)-\left(V_{10}^{\Im}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_x^2(t)\right. \tag{719}$$

$$\left.\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right), \tag{720}$$

Now let's obtain the form of $\overline{H_{\bar{I}}}^{3}(t)$:

$$\overline{H_{\bar{I}}}^{3}(t)=\left(\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\right)\left(\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i\left(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x\right.\right. \tag{721}$$

$$+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) \quad (722)$$

$$+\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2 \quad (723)$$

$$\times\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right)\big) \quad (724)$$

$$=\sum_i B_{iz}\,(t)\,|i\rangle\langle i|\sum_i B_{iz}^2\,(t)\,|i\rangle\langle i|+\sum_i B_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+\sum_i B_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Im}\,(t) \quad (725)$$

$$\times\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\sum_i B_{iz}(t)|i\rangle\langle i| \quad (726)$$

$$\times\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) \quad (727)$$

$$+\sum_i B_{iz}(t)\,|i\rangle\langle i|\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t) \quad (728)$$

$$\times(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}\,(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}\,(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i\,(B_{iz}(t)B_y(t) \quad (729)$$

$$\times|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t) \quad (730)$$

$$+\sigma_yB_y(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t) \quad (731)$$

$$-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t) \quad (732)$$

$$-\sigma_yB_x(t))\sum_i B_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_y \quad (733)$$

$$\times B_x(t))V_{10}^{\Im}(t)\sum_i\,(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i| \quad (734)$$

$$\times B_y(t)B_{iz}(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t)) \quad (735)$$

$$\times\,V_{10}^{\Im}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\,\left(V_{10}^{\Im}(t)\right)^2\,\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)\right. \quad (736)$$

$$\left.+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right) \quad (737)$$

$$=\sum_i B_{iz}^3\,(t)\,|i\rangle\langle i|+V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}^2\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}^2\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}\,(t)\sum_i\left(B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_y\right) \quad (738)$$

$$+\,V_{10}^{\Re}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_x(t)B_{iz}(t)+|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_x^2(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)\right. \quad (739)$$

$$\left.\times B_x(t)B_y(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+|i\rangle\langle i|B_{iz}(t)B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)-|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)\right. \quad (740)$$

$$\left.B_x(t)B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_y^2(t)+|i\rangle\langle i|B_{iz}(t)B_x^2(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_x(t)B_y(t)\right)+V_{10}^{\Re}(t) \quad (741)$$

$$\times\sum_i\left(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}^2(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i(B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_x+B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t) \quad (742)$$

$$\times B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_x+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_x-B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_y \quad (743)$$

$$+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_x-B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_x^2(t)B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|\right. \quad (744)$$

$$\left.\times B_y(t)B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y^2(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^3\left(\sigma_xB_x^3(t)+\sigma_yB_x^2(t)B_y(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_xB_x(t)B_y^2(t)+\sigma_yB_y(t)B_x^2(t)\right. \quad (745)$$

$$\left.-\sigma_xB_y(t)B_x(t)B_y(t)+\sigma_xB_y^2(t)B_x(t)+\sigma_yB_y^3(t)\right)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i\left(|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_y^2(t)\right. \quad (746)$$

$$\left.\times B_{iz}(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_y(t)B_x(t)B_{iz}(t)\right)+V_{10}^{\Re}(t)\left(V_{10}^{\Im}(t)\right)^2\left(\sigma_xB_x(t)B_y^2(t)+\sigma_xB_x^3(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_yB_x^2(t)B_y(t)+\sigma_yB_y^3(t)\right. \quad (747)$$

$$\left.+\sigma_yB_y(t)\,B_x^2(t)+\sigma_xB_y^2(t)\,B_x(t)-\sigma_xB_y(t)\,B_x(t)\,B_y(t)\right)+V_{10}^{\Im}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}^2(t)-\sigma_y|i\rangle\langle i|B_x(t)\,B_{iz}^2(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t) \quad (748)$$

$$\times\,(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_x(t)+\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t)-\sigma_y|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t))+\left(V_{10}^{\Im}(t)\right)^2 \quad (749)$$

$$\times\,(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_x(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x(t)B_{iz}(t)B_y(t)+\sigma_y|i\rangle\langle i|\sigma_yB_x(t)B_{iz}(t)B_x(t))+V_{10}^{\Re}(t) \quad (750)$$

$$\times V_{10}^{\Im}(t)\sum_{i}\left(|i\rangle\langle i|B_{y}(t)\,B_{x}(t)\,B_{iz}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{y}^{2}(t)\,B_{iz}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{x}^{2}(t)\,B_{iz}(t)-|i\rangle\langle i|B_{x}(t)\,B_{y}(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(V_{10}^{\Re}(t)\right)^{2} \tag{751}$$

$$\times\left(\sigma_{x}B_{y}(t)B_{x}^{2}(t)+\sigma_{y}B_{y}(t)B_{x}(t)B_{y}(t)-\sigma_{y}B_{y}^{2}(t)B_{x}(t)+\sigma_{x}B_{y}^{3}(t)-\sigma_{y}B_{x}^{3}(t)+\sigma_{x}B_{x}^{2}(t)B_{y}(t)-\sigma_{x}B_{x}(t)B_{y}(t)B_{x}(t)-\sigma_{y}B_{x}(t)B_{y}^{2}(t)\right) \tag{752}$$

$$+\left(V_{10}^{\Im}(t)\right)^{2}\sum_{i}\left(|i\rangle\langle i|B_{y}^{2}(t)B_{iz}(t)-\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{y}(t)\,B_{x}(t)\,B_{iz}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{x}(t)\,B_{y}(t)\,B_{iz}(t)+|i\rangle\langle i|B_{x}^{2}(t)\,B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^{3}\left(\sigma_{x}B_{y}^{3}(t)\right. \tag{753}$$

$$\left.+\sigma_{x}B_{y}(t)\,B_{x}^{2}(t)-\sigma_{y}B_{y}^{2}(t)\,B_{x}(t)+\sigma_{y}B_{y}(t)\,B_{x}(t)\,B_{y}(t)-\sigma_{y}B_{x}(t)\,B_{y}^{2}(t)-\sigma_{y}B_{x}^{3}(t)-\sigma_{x}B_{x}(t)\,B_{y}(t)\,B_{x}(t)+\sigma_{x}B_{x}^{2}(t)\,B_{y}(t)\right). \tag{754}$$

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