

# Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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## I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality. Let's define the partition functions of  $\overline{H}(t)$  and  $\overline{H}_0(t)$  given by  $Z(t)$  and  $Z_0(t)$  respectively as:

$$Z(t) \equiv \text{Tr} \left( e^{-\beta \overline{H}(t)} \right), \quad (1)$$

$$Z_0(t) \equiv \text{Tr} \left( e^{-\beta \overline{H}_0(t)} \right). \quad (2)$$

where the transformed hamiltonians  $\overline{H}(t)$  and  $\overline{H}_0(t)$  are defined as:

$$\overline{H}(t) \equiv \overline{H}_I(t) + \overline{H}_0(t), \quad (3)$$

$$\overline{H}_0(t) \equiv \overline{H}_S(t) + \overline{H}_B. \quad (4)$$

For any operator  $A(t)$  we define the expected value respect to  $\overline{H}_0(t)$  as:

$$\langle A(t) \rangle_{\overline{H}_0(t)} \equiv \frac{\text{Tr} \left( A(t) e^{-\beta \overline{H}_0(t)} \right)}{\text{Tr} \left( e^{-\beta \overline{H}_0(t)} \right)}. \quad (5)$$

The terms  $\overline{H}_S(t)$ ,  $\overline{H}_B$  and  $\overline{H}_I(t)$  are related to the variational transformation performed in [1,2], this transformation allowed us to construct  $\overline{H}_I(t)$  such that  $\langle \overline{H}_I(t) \rangle_{\overline{H}_0(t)} = 0$ . The diagonalization of  $\overline{H}_0(t)$  in terms of it's eigenstates and eigenvalues such that  $\overline{H}_0(t) |n\rangle = E_{0,n}(t) |n\rangle$ , being  $|n\rangle$  an eigenstate of  $\overline{H}_0(t)$  with eigenvalue  $E_{0,n}(t)$  is  $\overline{H}_0(t) = \sum_n E_{0,n}(t) |n\rangle \langle n|$ , with  $\langle n|n'\rangle = \delta_{nn'}$ , so a simple form of  $e^{-\beta \overline{H}_0(t)}$  can be found as follows:

$$e^{r(X+Y)} = e^{rX} e^{rY} e^{-\frac{r^2}{2}[X,Y]} e^{\frac{r^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} \dots \text{ (Zassenhaus formula),} \quad (6)$$

$$e^{X+Y} = e^X e^Y e^{-\frac{r^2}{2}0} e^{\frac{r^3}{6}(2[Y,0] + [X,0])} \dots \text{ (setting } r = 1 \text{ and } [X,Y] = 0 \text{ in (6))} \quad (7)$$

$$= e^X e^Y \mathbb{I} \quad (8)$$

$$= e^X e^Y, \quad (9)$$

$$e^{-\beta \overline{H}_0(t)} = e^{-\sum_n \beta E_{0,n}(t) |n\rangle \langle n|} \text{ (by the diagonalization of } \overline{H}_0(t)) \quad (10)$$

$$e^{-\sum_n \beta E_{0,n}(t) |n\rangle\langle n|} = \prod_n e^{-\beta E_{0,n}(t) |n\rangle\langle n|} \text{ (by (9) and } [|n\rangle\langle n|, |n'\rangle\langle n'|] = 0) \quad (11)$$

$$= \prod_n \sum_{j=0}^{\infty} \frac{(-\beta E_{0,n}(t) |n\rangle\langle n|)^j}{j!} \text{ (by the exponential formula)} \quad (12)$$

$$= \prod_n \left( \mathbb{I} + \sum_{j=1}^{\infty} \frac{(-\beta E_{0,n}(t))^j |n\rangle\langle n|}{j!} \right) \text{ (using } (aA)^j = a^j A^j \text{ and } (|n\rangle\langle n|)^2 = |n\rangle\langle n|) \quad (13)$$

$$= \prod_n \left( \mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{(-\beta E_{0,n}(t))^j |n\rangle\langle n|}{j!} \right) \quad (14)$$

$$= \prod_n \left( \mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left( \sum_{j=0}^{\infty} \frac{(-\beta E_{0,n}(t))^j}{j!} \right) \right) \quad (15)$$

$$= \prod_n \left( \mathbb{I} - |n\rangle\langle n| + e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right) \text{ (by the exponential formula)} \quad (16)$$

$$= \prod_n \left( \mathbb{I} + \left( e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right). \quad (17)$$

We will prove by induction a neat form for (17), we will show that:

$$\prod_{j=1}^n \left( \mathbb{I} + \left( e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \mathbb{I} + \sum_{j=1}^n \left( e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \quad (18)$$

For  $n = 1$  the formula is trivial, in the case  $n = 2$  we obtain that:

$$\prod_{j=1}^2 \left( \mathbb{I} + \left( e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \left( \mathbb{I} + \left( e^{-\beta E_{0,1}(t)} - 1 \right) |1\rangle\langle 1| \right) \left( \mathbb{I} + \left( e^{-\beta E_{0,2}(t)} - 1 \right) |2\rangle\langle 2| \right) \quad (19)$$

$$= \mathbb{I} + \left( e^{-\beta E_{0,1}(t)} - 1 \right) |1\rangle\langle 1| + \left( e^{-\beta E_{0,2}(t)} - 1 \right) |2\rangle\langle 2| \text{ (by } \langle i|j \rangle = \delta_{ij}) \quad (20)$$

$$= \mathbb{I} + \sum_{j=1}^2 \left( e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \quad (21)$$

It is our case base, our induction step is (18), in the case  $n + 1$  we will have:

$$\prod_{j=1}^{n+1} \left( \mathbb{I} + \left( e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \left( \prod_{j=1}^n \left( \mathbb{I} + \left( e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) \right) \left( \mathbb{I} + \left( e^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| \right) \quad (22)$$

$$= \left( \mathbb{I} + \sum_n \left( e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right) \left( \mathbb{I} + \left( e^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| \right) \text{ (by induction step)} \quad (23)$$

$$= \mathbb{I} + \left( e^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| + \sum_n \left( e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \text{ (by } \langle i|j \rangle = \delta_{ij}) \quad (24)$$

$$= \mathbb{I} + \sum_{j=1}^{n+1} \left( e^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \quad (25)$$

By mathematical induction we proved that (18) is true for all  $n \in \mathbb{N}$ . Given that the resolution of the identity is  $\mathbb{I} = \sum_n |n\rangle\langle n|$  so we find that:

$$e^{-\beta \overline{H}_0(t)} = \prod_n \left( \mathbb{I} + \left( e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right) \quad (26)$$

$$= \mathbb{I} + \sum_n \left( e^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \quad (27)$$

$$= \mathbb{I} + \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| - \sum_n |n\rangle\langle n| \quad (28)$$

$$= \mathbb{I} + \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| - \mathbb{I} \text{ (by the resolution of the identity)} \quad (29)$$

$$= \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n|. \quad (30)$$

The partition function  $Z_0(t)$  is equal to:

$$Z_0(t) = \text{Tr} \left( \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right) \quad (31)$$

$$= \sum_n e^{-\beta E_{0,n}(t)} \text{Tr}(|n\rangle\langle n|) \quad (32)$$

$$= \sum_n e^{-\beta E_{0,n}(t)}. \quad (33)$$

The explicit form of the average value  $\langle A(t) \rangle_{\overline{H}_0(t)}$  can be found from the partition function  $Z_0(t)$ :

$$\langle A(t) \rangle_{\overline{H}_0(t)} = \frac{\text{Tr} \left( A(t) e^{-\beta \overline{H}_0(t)} \right)}{Z_0(t)} \quad (34)$$

$$= \frac{\text{Tr} \left( \sum_n A(t) e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right)}{\text{Tr} \left( \sum_n e^{-\beta \overline{H}_0(t)} \right)} \quad (35)$$

$$= \frac{\text{Tr} \left( \sum_n e^{-\beta E_{0,n}(t)} A(t) |n\rangle\langle n| \right)}{\text{Tr} \left( \sum_n e^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right)} \quad (36)$$

$$= \frac{\text{Tr} \left( \sum_n e^{-\beta E_{0,n}(t)} A(t) |n\rangle\langle n| \right)}{\sum_n e^{-\beta E_{0,n}(t)}} \quad (37)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \text{Tr} \left( A(t) |n\rangle\langle n| \right)}{\sum_n e^{-\beta E_{0,n}(t)}}. \quad (38)$$

At first we show a double sequence of inequalities of order  $M, N$  which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$Z(t) \geq Z_0(t) e^{-\langle \overline{H}_T(t) \rangle_{\overline{H}_0(t)}} (1 + F_M(\vec{u}(t); \alpha) + F_N(\vec{v}(t) - \vec{u}(t); \alpha)). \quad (39)$$

where the function  $F_N(\vec{v}(t); \alpha)$  is defined as:

$$F_N(\vec{w}(t); \alpha) \equiv e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{w_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!}. \quad (40)$$

In this case  $\alpha$  is a parameter that can be optimized,  $\beta \equiv \frac{1}{k_B T}$ ,  $\vec{w}(t)$  is a vector such that  $\vec{w}(t) = (w_1, w_2, \dots)$  and  $\vec{u}(t)$  and  $\vec{v}(t)$  are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian  $\overline{H}_{TD}(t)$  respect to the basis of eigenstates of  $\overline{H}_0(t)$  as:

$$\overline{H_{ID}}(t) \equiv \sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n|. \quad (41)$$

We will prove an important property related to  $\overline{H_{ID}}(t)$ , which is a Hamiltonian written as a linear combination of a set of orthonormal operators. Let's consider a vector space  $R$  with two operations  $+$  and  $\cdot$ , if there exist  $a, b \in R$  such that  $a \cdot b = 0$  and  $b \cdot a = 0$  then for any  $k \in \mathbb{N}$  we have  $(a + b)^k = a^k + b^k$  where  $a^k = a^{k-1} \cdot a$  is a recursive definition of the power of an element written in terms of  $\cdot$ . At first we prove that this result yields for any  $k \in \mathbb{N}$  by induction, the case  $k = 1$  is trivial so we will focus on the case  $k = 2$ , we have that:

$$(a + b)^2 = (a + b) \cdot (a + b) \quad (42)$$

$$= a \cdot a + a \cdot b + b \cdot a + b \cdot b \quad (43)$$

$$= a^2 + a \cdot b + b \cdot a + b^2 \quad (44)$$

$$= a^2 + 0 + 0 + b^2 \text{ (because } a \cdot b = b \cdot a = 0) \quad (45)$$

$$= a^2 + b^2. \quad (46)$$

This is the base case. By induction step we will consider that  $(a + b)^k = a^k + b^k$  with  $k \geq 2$ , now for  $k + 1$  we will have that:

$$(a + b)^{k+1} = (a + b)^k \cdot (a + b) \quad (47)$$

$$= (a^k + b^k) \cdot (a + b) \text{ (by induction step)} \quad (48)$$

$$= a^k \cdot a + a^k \cdot b + b^k \cdot a + b^k \cdot b \quad (49)$$

$$= a^{k+1} + a^{k-1} \cdot a \cdot b + b^{k-1} \cdot b \cdot a + b^{k+1} \text{ (by recursive definition of } a^k) \quad (50)$$

$$= a^{k+1} + a^{k-1} \cdot (a \cdot b) + b^{k-1} \cdot (b \cdot a) + b^{k+1} \text{ (by associativity on } R) \quad (51)$$

$$= a^{k+1} + a^{k-1} \cdot (0) + b^{k-1} \cdot (0) + b^{k+1} \text{ (because } a \cdot b = b \cdot a = 0) \quad (52)$$

$$= a^{k+1} + b^{k+1}. \quad (53)$$

By the principle of mathematical induction we can conclude that the proposition is true for all  $k \in \mathbb{N}$ . Now we will extend the result, let  $a_1, \dots, a_n \in R$  such that  $a_i \cdot a_j = 0$  for all  $i \neq j$  then  $(a_1 + \dots + a_n)^k = a_1^k + \dots + a_n^k$ . The case  $n = 1$  is trivial as well so we will focus on  $n = 2$ , this case was proved in the precedent lines so it will be our base case. By induction step we will consider that  $(a_1 + \dots + a_n)^k = a_1^k + \dots + a_n^k$  with  $n \geq 2$ , now for  $n + 1$  we will have that:

$$a_{n+1} \cdot (a_1 + \dots + a_n) = a_{n+1} \cdot a_1 + \dots + a_{n+1} \cdot a_n \quad (54)$$

$$= 0 \text{ (because } a_i \cdot a_j = 0 \text{ for all } i \neq j), \quad (55)$$

$$(a_1 + \dots + a_n + a_{n+1})^k = ((a_1 + \dots + a_n) + a_{n+1})^k \quad (56)$$

$$= (a_1 + \dots + a_n)^k + a_{n+1}^k \text{ (by (47) and (55))} \quad (57)$$

$$= a_1^k + \dots + a_n^k + a_{n+1}^k \text{ (by inductive step).} \quad (58)$$

So we can conclude by mathematical induction that the proposition is true for all  $n \in \mathbb{N}$ . We can prove the following property for  $(\overline{H_{ID}}(t))^k$ :

$$\langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \langle n' | \overline{H_I}(t) | n' \rangle |n'\rangle\langle n'| = \langle n | \overline{H_I}(t) | n \rangle \langle n' | \overline{H_I}(t) | n' \rangle |n\rangle\langle n|n'\rangle\langle n'| \quad (59)$$

$$= \langle n | \overline{H_I}(t) | n \rangle \langle n' | \overline{H_I}(t) | n' \rangle |n\rangle\langle n'| \delta_{nn'}, \quad (60)$$

$$(\overline{H_{ID}}(t))^k = \left( \sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \right)^k \quad (\text{by (41)}) \quad (61)$$

$$= \sum_n \left( \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \right)^k \quad (\text{by (58) and (60)}), \quad (62)$$

$$(aA)^k = a^k A^k \quad (\text{by the property of the power of a matrix}), \quad (63)$$

$$(|n\rangle\langle n|)^k = |n\rangle\langle n| \quad (\text{because } |n\rangle\langle n| \text{ is a projector and } k \in \mathbb{N}^*), \quad (64)$$

$$(\overline{H_{ID}}(t))^k = \sum_n \left( \langle n | \overline{H_I}(t) | n \rangle \right)^k |n\rangle\langle n| \quad (\text{by (63) and (64)}). \quad (65)$$

The vectors  $\vec{u}(t)$  and  $\vec{v}(t)$  are defined as  $\vec{u}(t) \equiv (u_1, u_2, \dots)$  and  $\vec{v}(t) \equiv (v_1, v_2, \dots)$ . We can define the elements of  $\vec{u}(t)$  and  $\vec{v}(t)$  in terms of the matrix  $\overline{H_{ID}}(t)$ :

$$u_k(t) \equiv \left\langle \left( \overline{H_{ID}}(t) - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k \right\rangle_{\overline{H_0}(t)} \quad (66)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \text{Tr} \left( \left( \sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n\rangle\langle n| \right)}{Z_0(t)} \quad (\text{by (38)}), \quad (67)$$

$$\left( \sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k = \sum_{j=0}^k (-1)^j \binom{k}{j} \left( \sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle\langle n| \right)^j \left( \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^{k-j} \quad (\text{by binomial theorem}) \quad (68)$$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} \left( \sum_n \langle n | \overline{H_I}(t) | n \rangle^j |n\rangle\langle n| \right) \left( \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^{k-j} \quad (\text{by (65)}) \quad (69)$$

$$= \sum_n \left( \sum_{j=0}^k (-1)^j \binom{k}{j} \langle n | \overline{H_I}(t) | n \rangle^j \left( \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^{k-j} \right) |n\rangle\langle n| \quad (70)$$

$$= \sum_n \left( \langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n\rangle\langle n|, \quad (71)$$

$$= \sum_n \left( \langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n\rangle\langle n|, \quad (72)$$

$$u_k(t) = \frac{\sum_n e^{-\beta E_{0,n}(t)} \text{Tr} \left( \sum_{n'} \left( \langle n' | \overline{H_I}(t) | n' \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n'\rangle\langle n'|n\rangle\langle n| \right)}{Z_0(t)} \quad (73)$$

$$= \frac{\sum_{nn'} e^{-\beta E_{0,n}(t)} \text{Tr} \left( \left( \langle n' | \overline{H_I}(t) | n' \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k |n'\rangle\langle n| \delta_{nn'} \right)}{Z_0(t)} \quad (74)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left( \langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k \text{Tr}(|n\rangle\langle n|)}{Z_0(t)} \quad (75)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left( \langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k 1}{Z_0(t)} \quad (76)$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left( \langle n | \overline{H_I}(t) | n \rangle - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k}{Z_0(t)}, \quad (77)$$

$$v_k(t) \equiv \frac{\sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \left( \overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t) - \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \right)^k \right| n \right\rangle}{Z_0(t)}. \quad (78)$$

By construction  $\langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} = 0$ , so we summarize the double inequality that generalizes the Bogoliubov inequality and its coefficients as:

$$Z(t) \geq Z_0(t) (1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))), \quad (79)$$

$$Z(t) = \text{Tr} \left( e^{-\beta \overline{H}(t)} \right), \quad (80)$$

$$Z_0(t) = \sum_n e^{-\beta E_{0,n}(t)}, \quad (81)$$

$$F_N(\vec{u}(t)) = e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!}, \quad (82)$$

$$u_k(t) = \frac{\sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) | n \rangle^k}{Z_0(t)}, \quad (83)$$

$$v_k(t) = \frac{\sum_n e^{-\beta E_{0,n}(t)} \langle n | (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^k | n \rangle}{Z_0(t)}. \quad (84)$$

As we can see the expression (83) was written in shorter terms, we want to do the same for (84) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for  $v_k(t)$ :

$$(\overline{H_0}(t) - E_{0,n}(t)) | n \rangle = \overline{H_0}(t) | n \rangle - E_{0,n}(t) | n \rangle \quad (85)$$

$$= E_{0,n}(t) | n \rangle - E_{0,n}(t) | n \rangle \quad (86)$$

$$= 0, \quad (87)$$

$$\langle n | (\overline{H_0}(t) - E_{0,n}(t)) = \langle n | \overline{H_0}(t) - \langle n | E_{0,n}(t) \quad (88)$$

$$= \langle n | E_{0,n}(t) - \langle n | E_{0,n}(t) \quad (89)$$

$$= 0. \quad (90)$$

At first we calculated  $v_1(t)$  using the definition (84):

$$v_1(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t) | n \rangle \quad (91)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_0}(t) - E_{0,n}(t) | n \rangle + \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) | n \rangle \quad (92)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} (\langle n | \overline{H_0}(t) | n \rangle - \langle n | E_{0,n}(t) | n \rangle) + \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \quad (93)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} (\langle n | E_{0,n}(t) | n \rangle - \langle n | E_{0,n}(t) | n \rangle) + \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \quad (94)$$

$$= 0 + \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \quad (\text{by construction } \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} = 0) \quad (95)$$

$$= 0. \quad (96)$$

For  $k \geq 2$  and  $k \in \mathbb{N}$  we calculated:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^k \right| n \right\rangle \quad (97)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (98)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (99)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (E_{0,n}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (E_{0,n}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (100)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle. \quad (101)$$

In general we can write a formula for  $v_k(t)$  that implies an expected value of a dependent expression of  $\overline{H_I}(t)$  and  $\overline{H_0}(t)$ :

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle \quad (102)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) + \overline{H_I}(t) - E_{0,n}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle \quad (103)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H}(t) - E_{0,n}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle \quad (104)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) \left( \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j}(t) E_{0,n}^j(t) \right) \overline{H_I}(t) \right| n \right\rangle \quad (105)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) E_{0,n}^j(t) \right| n \right\rangle \quad (106)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right| n \right\rangle \quad (107)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right| n \right\rangle \quad (108)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (109)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{k-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \quad (110)$$

The formula (110) is well defined taking as example  $k = 2, 3$ .

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{2-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (111)$$

$$= (-1)^0 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^0 \overline{H_I}(t) \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \quad (112)$$

$$= \left\langle \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)}, \quad (113)$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{3-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (114)$$

$$= \left\langle \sum_{j=0}^1 (-1)^j \binom{1}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{1-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (115)$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^1 \overline{H_I}(t) \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^0 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \quad (116)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \mathbb{I} - \overline{H_I}(t) \mathbb{I} \overline{H_I}(t) \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \quad (117)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \quad (118)$$

$$= \langle \overline{H_I}(t)^3 + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \quad (119)$$

$$= \langle \overline{H_I}(t)^3 + \overline{H_I}(t) (\overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t)) \rangle_{\overline{H_0}(t)} \quad (120)$$

$$= \langle \overline{H_I}(t)^3 + \overline{H_I}(t) [\overline{H_0}(t), \overline{H_I}(t)] \rangle_{\overline{H_0}(t)} \quad (\text{because } [\overline{H_0}(t), \overline{H_I}(t)] = \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t)). \quad (121)$$

So we summarize:

$$\overline{H_{ID}}(t) = \sum_n \langle n | \overline{H_I}(t) | n \rangle | n \rangle \langle n |, \quad (122)$$

$$u_k(t) = \left\langle (\overline{H_{ID}}(t))^k \right\rangle_{\overline{H_0}(t)}, \quad (123)$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{k-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \quad (124)$$

The free energy is defined as:

$$E_{\text{free}}(t) \equiv -\frac{1}{\beta} \ln(Z(t)). \quad (125)$$

We define the free energy at first order as:

$$E_{\text{free},1}(t) \equiv -\frac{1}{\beta} \ln(Z_0(t)). \quad (126)$$

It is well-known that the function  $f(x) = -\ln(x)$  is a decreasing function so we can transform (39):

$$E_{\text{free}}(t) \leq -\frac{1}{\beta} \ln(Z_0(t) (1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t)))) \quad (127)$$

$$= -\frac{1}{\beta} \ln(Z_0(t)) - \frac{1}{\beta} \ln(1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (128)$$

$$= E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (129)$$

$$\equiv E_{\text{free,MN}}(t). \quad (130)$$

here  $E_{\text{free,MN}}(t)$  is the free energy associate to the strong version of the Quantum Bogoliubov inequality of  $M, N$  order. In our approach we will set  $N = M$ , so the inequality (130) of  $N, N$  order is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (131)$$

$$= E_{\text{free,NN}}(t). \quad (132)$$



A weaker form of the inequality (132) is obtained making  $\vec{u}(t) = 0$  as suggest [3]:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{v}(t))) \quad (133)$$

$$\equiv E_{\text{free},N}(t). \quad (134)$$

The algebraic equation associated with  $\alpha_{\text{opt}}(t)$  such that  $E_{\text{free},N}(t)$  is closer to  $E_{\text{free}}(t)$  follows from the fact that in the optimal parameter  $\frac{\partial E_{\text{free},N}(t)}{\partial \alpha}|_{\alpha_{\text{opt}}(t)} = 0$ , calculating this derivate we have:

$$\frac{\partial E_{\text{free},N}(t)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left( E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{v}(t))) \right) \quad (135)$$

$$= -\frac{1}{\beta} \frac{\frac{\partial}{\partial \alpha} (F_N(\vec{v}(t)))}{1 + F_N(\vec{v}(t))} \quad (136)$$

$$= 0. \quad (137)$$

The precedent equation is equivalent to make the numerator equal to 0:

$$\frac{\partial F_N(\vec{v}(t))}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left( e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} \right) \quad (138)$$

$$= -e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} + e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\partial}{\partial \alpha} \frac{\alpha^i}{i!} \quad (\text{by product rule}) \quad (139)$$

$$= -e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} + e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=1}^{2N-1-k} \frac{\alpha^{i-1}}{(i-1)!} \quad (140)$$

$$= e^{-\alpha} \left( \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=1}^{2N-1-k} \frac{\alpha^{i-1}}{(i-1)!} - \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} \right) \quad (141)$$

$$= e^{-\alpha} \left( \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{j=0}^{2N-2-k} \frac{\alpha^j}{j!} - \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!} \right) \quad (\text{setting } j = i - 1) \quad (142)$$

$$= e^{-\alpha} \left( - \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \frac{\alpha^{2N-1-k}}{(2N-1-k)!} \right) \quad (\text{performing the difference}) \quad (143)$$

$$= 0. \quad (144)$$

Then the optimal value  $\alpha_{\text{opt}}(t)$  will satisfy the following equation:

$$G(\alpha_{\text{opt}}(t)) \equiv \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \quad (145)$$

$$= 0. \quad (146)$$

The elements presented are the required to find variational parameters of the system using the inequality (134) and the self consistent equation (SCE) (145) to a particular order required.

## II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify  $v_2(t)$ ,  $v_3(t)$ ,  $v_4(t)$ ,  $v_5(t)$  using the (124), we have already  $v_2(t)$ ,  $v_3(t)$  and the form of  $v_4(t)$  and  $v_5(t)$  is given by:



$$= \left\langle \overline{H_I}(t) \left( \overline{H_I}^3(t) + \overline{H_I}^2(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_0}^2(t) \overline{H_I}(t) + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}(t) \right. \right. \quad (177)$$

$$\left. \times \overline{H_0}^2(t) + \overline{H_0}^3(t) \right) \overline{H_I}(t) - 3\overline{H_I}(t) \left( \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^2(t) \right) \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) \left( \overline{H_I}(t) \right. \quad (178)$$

$$\left. + \overline{H_0}(t) \right) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}^3(t) \Big\rangle_{\overline{H_0}(t)} \quad (179)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}^3(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^3(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \right. \quad (180)$$

$$\left. \times \overline{H_0}^2(t) \overline{H_I}^2(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}^3(t) \overline{H_I}(t) - 3\overline{H_I}^4(t) \overline{H_0}(t) - 3\overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) - 3\overline{H_I}(t) \right. \quad (181)$$

$$\left. \times \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) - 3\overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}^3(t) \overline{H_0}^2(t) + 3\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}^2(t) \overline{H_0}^3(t) \right\rangle_{\overline{H_0}(t)} \quad (182)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_0}(t) \overline{H_I}^3(t) + \overline{H_0}^2(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) \right. \right. \quad (183)$$

$$\left. + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^3(t) \overline{H_I}(t) - 3\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) - 3\overline{H_I}^3(t) \overline{H_0}(t) - 3\overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}(t) \right. \quad (184)$$

$$\left. \times \overline{H_0}^3(t) + 3\overline{H_I}^2(t) \overline{H_0}^2(t) + 3\overline{H_0}(t) \overline{H_I}(t) \overline{H_0}^2(t) - 3\overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \quad (185)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}^3(t) - \overline{H_0}(t) \overline{H_I}^2(t) \right. \right. \quad (186)$$

$$\left. \times \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) + \overline{H_0}^3(t) \overline{H_I}(t) \right. \quad (187)$$

$$\left. - \overline{H_I}(t) \overline{H_0}^3(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) - \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) + 2\overline{H_I}^2(t) \overline{H_0}^2(t) - 2\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_0}(t) \overline{H_I}(t) \right. \quad (188)$$

$$\left. \times \overline{H_0}^2(t) - 3\overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_0}^2(t) \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (189)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \left( \overline{H_I}^2(t) \overline{H_0}(t) \right) \overline{H_I}(t) - \overline{H_I}(t) \left( \overline{H_I}^2(t) \overline{H_0}(t) \right) \right) + \left( \overline{H_I}(t) \overline{H_0}(t) \right) \overline{H_I}^2(t) - \overline{H_I}^2(t) \left( \overline{H_I}(t) \overline{H_0}(t) \right) \right. \quad (190)$$

$$\left. + \left( \overline{H_0}(t) \overline{H_I}^3(t) - \overline{H_I}^3(t) \overline{H_0}(t) \right) + \left( \overline{H_0}(t) \left( \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right) - \left( \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right) \overline{H_0}(t) \right) + \left( \overline{H_0}(t) \left( \overline{H_0}(t) \overline{H_I}^2(t) \right) \right. \quad (191)$$

$$\left. - \left( \overline{H_0}(t) \overline{H_I}^2(t) \right) \overline{H_0}(t) \right) + \left( \overline{H_0}^3(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}^3(t) \right) + \left( \left( \overline{H_I}(t) \overline{H_0}(t) \right) \left( \overline{H_0}(t) \overline{H_I}(t) \right) - \left( \overline{H_0}(t) \overline{H_I}(t) \right) \left( \overline{H_I}(t) \overline{H_0}(t) \right) \right) \quad (192)$$

$$\left. + 3\overline{H_0}(t) \left( \overline{H_I}(t) \overline{H_0}(t) - \overline{H_0}(t) \overline{H_I}(t) \right) \overline{H_0}(t) + 2\overline{H_I}(t) \left( \overline{H_I}(t) \overline{H_0}(t) - \overline{H_0}(t) \overline{H_I}(t) \right) \overline{H_0}(t) + \left( \left( \overline{H_I}^2(t) \overline{H_0}(t) \right) \left( \overline{H_0}(t) \right) \right. \quad (193)$$

$$\left. - \left( \overline{H_0}(t) \right) \left( \overline{H_I}^2(t) \overline{H_0}(t) \right) \right\rangle_{\overline{H_0}(t)} \quad (194)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[ \overline{H_0}(t), \overline{H_I}^3(t) \right] + \left[ \overline{H_0}(t), \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right] \right. \quad (195)$$

$$\left. + \left[ \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}^2(t) \right] + \left[ \overline{H_0}^3(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + 3\overline{H_0}(t) \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + 2\overline{H_I}(t) \right. \quad (196)$$

$$\left. \times \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right\rangle_{\overline{H_0}(t)}. \quad (197)$$

Summarizing we have that:

$$v_2(t) = \left\langle \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)}, \quad (198)$$

$$v_3(t) = \left\langle \overline{H_I}^3(t) + \overline{H_I}(t) \left[ \overline{H_0}(t), \overline{H_I}(t) \right] \right\rangle_{\overline{H_0}(t)}, \quad (199)$$

$$v_4(t) = \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \left( \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[ \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[ \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right) \right\rangle_{\overline{H_0}(t)}, \quad (200)$$

$$v_5(t) = \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[ \overline{H_0}(t), \overline{H_I}^3(t) \right] + \left[ \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}^2(t) \right] \right. \quad (201)$$

$$\left. + \left[ \overline{H_0}(t), \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right] + \left[ \overline{H_0}^3(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + 3\overline{H_0}(t) \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) \right. \quad (202)$$

$$\left. + 2\overline{H_I}(t) \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right\rangle_{\overline{H_0}(t)}. \quad (203)$$

Now we will obtain the expected values related to  $v_2(t)$ ,  $v_3(t)$ ,  $v_4(t)$  and  $v_5(t)$ . Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_S}(t) \equiv (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right), \quad (204)$$

$$\overline{H_I}(t) \equiv \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)), \quad (205)$$

$$\overline{H_B} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (206)$$

$$= H_B. \quad (207)$$

In this case  $\varepsilon_j(t)$ ,  $R_j(t)$  for  $j \in \{0, 1\}$ ,  $B_{10}^{\Re}(t)$ ,  $B_{10}^{\Im}(t)$ ,  $V_{10}^{\Re}(t)$  and  $V_{10}^{\Im}(t)$  are scalars and the other operators are:

$$\sigma_x \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \quad (208)$$

$$\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (209)$$

$$\sigma_y \equiv -i|1\rangle\langle 0| + i|0\rangle\langle 1| \quad (210)$$

$$\equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (211)$$

$$\sigma_z \equiv |1\rangle\langle 1| - |0\rangle\langle 0| \quad (212)$$

$$\equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (213)$$

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\chi_{ij}(t)} \end{pmatrix}, \quad (214)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right), \quad (215)$$

$$D(\pm v_{\mathbf{k}}(t)) \equiv e^{\pm \left( \frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}. \quad (216)$$

As we can see they verify the relationship  $\sigma_x \sigma_y = i \sigma_z$ . The explicit form of  $\overline{H_I}^2(t)$  is:

$$\overline{H_I}^2(t) = \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i B_{iz}(t) |i\rangle\langle i| (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \sum_i B_{iz}(t) |i\rangle\langle i| (\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Re}(t) \quad (217)$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i B_{iz}(t) |i\rangle\langle i| + \left( V_{10}^{\Re}(t) \right)^2 (\sigma_x B_x(t) + \sigma_y B_y(t))^2 + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (218)$$

$$\times (\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) (\sigma_x B_x(t) \quad (219)$$

$$+ \sigma_y B_y(t)) + \left( V_{10}^{\Im}(t) \right)^2 (\sigma_x B_y(t) - \sigma_y B_x(t))^2 \quad (220)$$

$$= \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}(t) \quad (221)$$

$$\times B_x(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t)) + \left( V_{10}^{\Re}(t) \right)^2 (\sigma_x^2 B_x^2(t) + \sigma_x \sigma_y B_x(t) B_y(t) + \sigma_y \quad (222)$$

$$\times \sigma_x B_y(t) B_x(t) + \sigma_y^2 B_y^2(t)) + V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) + \left( V_{10}^{\Im}(t) \right)^2 (\sigma_x^2 B_y^2(t) + \sigma_y^2 B_x^2(t) \quad (223)$$

$$- \sigma_x \sigma_y B_y(t) B_x(t) - \sigma_y \sigma_x B_x(t) B_y(t)) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x^2 B_y(t) B_x(t) + \sigma_x \sigma_y B_y^2(t) - \sigma_y \sigma_x B_x^2(t) - \sigma_y^2 B_x(t) B_y(t) \quad (224)$$

$$+ \sigma_x^2 B_x(t) B_y(t) - \sigma_x \sigma_y B_x^2(t) + \sigma_y \sigma_x B_y^2(t) - \sigma_y^2 B_y(t) B_x(t)), \quad (225)$$

$$\sigma_x \sigma_y = i\sigma_z \text{ (by Pauli matrices properties),} \quad (226)$$

$$\overline{H_T}^2(t) = \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}(t) \quad (227)$$

$$\times B_x(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_x^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z \quad (228)$$

$$\times B_y(t) B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_y^2(t) + B_x^2(t) - i\sigma_z \quad (229)$$

$$\times B_y(t) B_x(t) + i\sigma_z B_x(t) B_y(t)). \quad (230)$$

To introduce the direct calculation of the expected values recall that the hamiltonian  $\overline{H_0}(t)$  is a direct sum of the hamiltonians of two Hilbert spaces given by  $\overline{H_{\bar{S}}}(t)$  and  $\overline{H_{\bar{B}}}$ , so we can write in general the hamiltonian  $\overline{H_0}(t)$  as:

$$\overline{H_0}(t) = \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} + \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}. \quad (231)$$

where  $\mathbb{I}_{\bar{B}}$  and  $\mathbb{I}_{\bar{S}}$  are the identity of the systems  $\bar{B}$  and  $\bar{S}$  respectively.

We can show that:

$$[\overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}}, \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}] = \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} \cdot \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} \cdot \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} \quad (232)$$

$$= \overline{H_{\bar{S}}}(t) \mathbb{I}_{\bar{S}} \otimes \mathbb{I}_{\bar{B}} \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} \mathbb{I}_{\bar{B}} \quad (233)$$

$$= \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} - \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} \text{ (by definition of identity operator)} \quad (234)$$

$$= 0. \quad (235)$$

Let's introduce the following partition functions  $Z_{\bar{S}}(t)$  and  $Z_{\bar{B}}$  related to the systems  $\bar{S}$  and  $\bar{B}$  respectively.:

$$Z_{\bar{S}}(t) \equiv \text{Tr} \left( e^{-\beta \overline{H_{\bar{S}}}(t)} \right), \quad (236)$$

$$Z_{\bar{B}} \equiv \text{Tr} \left( e^{-\beta \overline{H_{\bar{B}}}} \right) \quad (237)$$

Using (9), (232) and  $\text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B)$  we can infer that the partition function  $Z_0(t)$  can be factorized as:

$$Z_0(t) = \text{Tr} \left( e^{-\beta \overline{H_0}(t)} \right). \quad (238)$$

$$= \text{Tr} \left( e^{-\beta (\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}})} \right) \text{ (by (4))}, \quad (239)$$

$$= \text{Tr} \left( e^{-\beta \overline{H_{\bar{S}}}(t)} e^{-\beta \overline{H_{\bar{B}}}} \right) \text{ (by (9))} \quad (240)$$

$$= \text{Tr} \left( e^{-\beta \overline{H_{\bar{S}}}(t)} \otimes e^{-\beta \overline{H_{\bar{B}}}} \right) \text{ (because } \bar{S} \text{ and } \bar{B} \text{ are disjoint Hilbert spaces)} \quad (241)$$

$$= \text{Tr} \left( e^{-\beta \overline{H_{\bar{S}}}(t)} \right) \text{Tr} \left( e^{-\beta \overline{H_{\bar{B}}}} \right) \text{ (by } \text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B)), \quad (242)$$

$$= Z_{\bar{S}}(t) Z_{\bar{B}} \text{ (by (236) and (237))}. \quad (243)$$

For an operator  $J(t)$  that can be factorized as  $J(t) = S(t) \otimes B(t)$  with  $S(t) \in \text{gen}(\overline{H_{\bar{S}}}(t))$  and  $B(t) \in \text{gen}(\overline{H_{\bar{B}}})$ , being  $\text{gen}(A)$  the vectorial space generated by the eigenvectors of the operator  $A$ , we calculate it's expected value respect to  $\overline{H_0}(t)$  using a simple way as follows:

$$\langle J(t) \rangle_{\overline{H_0(t)}} = \frac{\text{Tr} \left( J(t) e^{-\beta \overline{H_0(t)}} \right)}{\text{Tr} \left( e^{-\beta \overline{H_0(t)}} \right)} \text{ (by (5))} \quad (244)$$

$$= \frac{\text{Tr} \left( (S(t) \otimes B(t)) \left( e^{-\beta \overline{H_S(t)}} \otimes e^{-\beta \overline{H_B}} \right) \right)}{\text{Tr} \left( e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left( e^{-\beta \overline{H_B}} \right)} \text{ (by } J(t) = S(t) \otimes B(t) \text{ and } e^{-\beta \overline{H_0(t)}} = e^{-\beta \overline{H_S(t)}} \otimes e^{-\beta \overline{H_B}} \text{)} \quad (245)$$

$$= \frac{\text{Tr} \left( \left( S(t) e^{-\beta \overline{H_S(t)}} \right) \otimes \left( B(t) e^{-\beta \overline{H_B}} \right) \right)}{\text{Tr} \left( e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left( e^{-\beta \overline{H_B}} \right)} \text{ (rearranging and factorizing)} \quad (246)$$

$$= \frac{\text{Tr} \left( S(t) e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left( B(t) e^{-\beta \overline{H_B}} \right)}{\text{Tr} \left( e^{-\beta \overline{H_S(t)}} \right) \text{Tr} \left( e^{-\beta \overline{H_B}} \right)} \text{ (by } \text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B) \text{)} \quad (247)$$

$$= \frac{\text{Tr} \left( S(t) e^{-\beta \overline{H_S(t)}} \right)}{\text{Tr} \left( e^{-\beta \overline{H_S(t)}} \right)} \frac{\text{Tr} \left( B(t) e^{-\beta \overline{H_B}} \right)}{\text{Tr} \left( e^{-\beta \overline{H_B}} \right)} \quad (248)$$

$$= \langle S(t) \rangle_{\overline{H_S(t)}} \langle B(t) \rangle_{\overline{H_B}} \text{ (by (5))}. \quad (249)$$

The factorization of  $\langle \overline{H_I}^2(t) \rangle_{\overline{H_0(t)}}$  in terms of expected values of elements from  $\text{gen}(\overline{H_S}(t))$  and  $\text{gen}(\overline{H_B})$  is:

$$\langle \overline{H_I}^2(t) \rangle_{\overline{H_0(t)}} = \sum_i \langle |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_{iz}^2(t) \rangle_{\overline{H_B}} + V_{10}^{\Re}(t) \sum_i \left( \langle |i\rangle\langle i| \sigma_x \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} + \langle |i\rangle\langle i| \sigma_y \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} \right) \quad (250)$$

$$+ V_{10}^{\Im}(t) \sum_i \left( \langle |i\rangle\langle i| \sigma_x \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} - \langle |i\rangle\langle i| \sigma_y \rangle_{\overline{H_S(t)}} \langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} \right) + V_{10}^{\Re}(t) \sum_i \left( \langle \sigma_x |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \right) \quad (251)$$

$$\times \langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} + \langle \sigma_y |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} + \left( V_{10}^{\Re}(t) \right)^2 \left( \langle B_x^2(t) \rangle_{\overline{H_B}} + i \langle \sigma_z \rangle_{\overline{H_S(t)}} \langle B_x(t) B_y(t) \rangle_{\overline{H_B}} \right) \quad (252)$$

$$- i \langle \sigma_z \rangle_{\overline{H_S(t)}} \langle B_y(t) B_x(t) \rangle_{\overline{H_B}} + \langle B_y^2(t) \rangle_{\overline{H_B}} + V_{10}^{\Im}(t) \sum_i \left( \langle \sigma_x |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} - \langle \sigma_y |i\rangle\langle i| \rangle_{\overline{H_S(t)}} \right) \quad (253)$$

$$\times \langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} + \left( V_{10}^{\Im}(t) \right)^2 \left( \langle B_y^2(t) \rangle_{\overline{H_B}} + \langle B_x^2(t) \rangle_{\overline{H_B}} - i \langle \sigma_z \rangle_{\overline{H_S(t)}} \langle B_y(t) B_x(t) \rangle_{\overline{H_B}} + i \langle \sigma_z \rangle_{\overline{H_S(t)}} \right) \quad (254)$$

$$\times \langle B_x(t) B_y(t) \rangle_{\overline{H_B}}. \quad (255)$$

In order to obtain the expected values of  $\langle \overline{H_I}^2(t) \rangle_{\overline{H_0(t)}}$  respect to the part related to the bath we need to calculate the following expected values that appear in the equation (230) and can be obtained using the factorization of (250). The expected values relevant for calculations are  $\langle B_{iz}^2(t) \rangle_{\overline{H_B}}$ ,  $\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}}$ ,  $\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}}$ ,  $\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}}$ ,  $\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}}$ ,  $\langle B_x^2(t) \rangle_{\overline{H_B}}$ ,  $\langle B_y^2(t) \rangle_{\overline{H_B}}$ ,  $\langle B_x(t) B_y(t) \rangle_{\overline{H_B}}$  and  $\langle B_y(t) B_x(t) \rangle_{\overline{H_B}}$ . Recalling the form of the hamiltonian  $\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$  we can extend the result (243), introducing the notation:

$$A_1 \otimes \cdots \otimes A_n \equiv \bigotimes_k A_k, \quad (256)$$

$$Z_{\mathbf{k}} \equiv \text{Tr} \left( e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right) \quad (257)$$

$$= \left( 1 - e^{-\beta \omega_{\mathbf{k}}} \right)^{-1} \quad (258)$$

$$= f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}}). \quad (259)$$

with the creation  $b_{\mathbf{k}}$  and annihilation  $b_{\mathbf{k}}^{\dagger}$  operators defined in terms of their actions as:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle, \quad (260)$$

$$b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle. \quad (261)$$

being  $|j_{\mathbf{k}}\rangle$  an eigenstate of  $H_{\mathbf{k}} \equiv \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$ . With this notation we can write the partition function as:

$$Z_{\bar{B}} = \text{Tr} \left( e^{-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right), \quad (262)$$

$$e^{-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}, \quad (263)$$

$$Z_{\bar{B}} = \text{Tr} \left( \bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right) \text{ (by (263))} \quad (264)$$

$$= \prod_{\mathbf{k}} \text{Tr} \left( e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right) \text{ (by } \text{Tr} (A \otimes B) = \text{Tr} (A) \text{Tr} (B) \text{)} \quad (265)$$

$$= \prod_{\mathbf{k}} Z_{\mathbf{k}} \text{ (by (263))}. \quad (266)$$

For a function  $f(t)$  which can be factorized as:

$$f(t) \equiv \prod_{\mathbf{k}} f_{\mathbf{k}}(t). \quad (267)$$

with  $f_{\mathbf{k}}(t) \in \text{gen}(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}})$ , it's expected value is given by:

$$\langle f(t) \rangle_{\overline{H_B}} = \frac{\text{Tr} \left( f(t) e^{-\beta \overline{H_B}} \right)}{\text{Tr} \left( e^{-\beta \overline{H_B}} \right)} \quad (268)$$

$$= \frac{\text{Tr} \left( \prod_{\mathbf{k}} f_{\mathbf{k}}(t) \bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\text{Tr} \left( \bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \text{ (by (263) and (267))} \quad (269)$$

$$= \frac{\text{Tr} \left( \bigotimes_{\mathbf{k}} f_{\mathbf{k}}(t) e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\text{Tr} \left( \bigotimes_{\mathbf{k}} e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \quad (270)$$

$$= \frac{\prod_{\mathbf{k}} \text{Tr} \left( f_{\mathbf{k}}(t) e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\prod_{\mathbf{k}} \text{Tr} \left( e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \quad (271)$$

$$= \prod_{\mathbf{k}} \frac{\text{Tr} \left( f_{\mathbf{k}}(t) e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)}{\text{Tr} \left( e^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \right)} \quad (272)$$

$$= \prod_{\mathbf{k}} \langle f_{\mathbf{k}}(t) \rangle_{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}. \quad (273)$$

It means that for an operator that can be factorized in terms of functions generated by  $\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$  for each  $\mathbf{k}$  we only require to calculate the expected value respect to the Hilbert space where the operator belongs. This process lead us to the following explicit forms of the expected values relevant for our calculations:

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \left\langle \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right)^2 \right\rangle_{\overline{H_B}} \quad (274)$$

$$= \sum_{\mathbf{k}} \left\langle \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 + \sum_{\mathbf{k} \neq \mathbf{k}'} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_B}} \quad (275)$$

$$- v_{i\mathbf{k}'}(t)) b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} \rangle_{\overline{H_B}} \quad (276)$$

$$= \sum_{\mathbf{k}} \left\langle \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left\langle \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (277)$$

$$\times \left\langle \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_B}} \text{ (by (273))}, \quad (278)$$

$$\langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} = \frac{\text{Tr} \left( b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (279)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (280)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 1)} |j_{\mathbf{k}} + 1\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (281)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 1)} \text{Tr}(|j_{\mathbf{k}} + 1\rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (282)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 1)} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (283)$$

$$= 0, \quad (284)$$

$$\langle b_{\mathbf{k}} \rangle_{\overline{H_B}} = \frac{\text{Tr} \left( b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (285)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (286)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} |j_{\mathbf{k}} - 1\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (287)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \text{Tr}(|j_{\mathbf{k}} - 1\rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (288)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (289)$$

$$= 0, \quad (290)$$

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} \left\langle \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) \quad (291)$$

$$\times \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \langle b_{\mathbf{k}'}^\dagger \rangle_{\overline{H_B}} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \langle b_{\mathbf{k}'} \rangle_{\overline{H_B}} \right) \quad (292)$$

$$= \sum_{\mathbf{k}} \left\langle \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k} \neq \mathbf{k}'} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \cdot 0 + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \cdot 0) ((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \cdot 0 \quad (293)$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \cdot 0) \text{ (by (279) and (285))} \quad (294)$$

$$= \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right)^2 \right\rangle_{\overline{H_B}} \quad (295)$$

$$= \sum_{\mathbf{k}} \left\langle (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 (b_{\mathbf{k}}^\dagger)^2 + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*)^2 b_{\mathbf{k}}^2 \right\rangle_{\overline{H_B}} \quad (296)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 \left\langle (b_{\mathbf{k}}^\dagger)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \left\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*)^2 \langle b_{\mathbf{k}}^2 \rangle_{\overline{H_B}}, \quad (297)$$

$$\left\langle (b_{\mathbf{k}}^\dagger)^2 \right\rangle_{\overline{H_B}} = \frac{\text{Tr} \left( (b_{\mathbf{k}}^\dagger)^2 \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (299)$$



$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \left( b_{\mathbf{k}}^{\dagger} \right)^2 |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (300)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 2)(j_{\mathbf{k}} + 1)} |j_{\mathbf{k}} + 2 \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (301)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 2)(j_{\mathbf{k}} + 1)} \text{Tr}(|j_{\mathbf{k}} + 2 \rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (302)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}} + 2)(j_{\mathbf{k}} + 1)} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (303)$$

$$= 0, \quad (304)$$

$$\langle b_{\mathbf{k}}^2 \rangle_{\overline{H_B}} = \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^2 |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (305)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}(j_{\mathbf{k}} - 1)} |j_{\mathbf{k}} - 2 \rangle \langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (306)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}(j_{\mathbf{k}} - 1)} \text{Tr}(|j_{\mathbf{k}} - 2 \rangle \langle j_{\mathbf{k}}|)}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (307)$$

$$= \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}(j_{\mathbf{k}} - 1)} \cdot 0}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (308)$$

$$= 0, \quad (309)$$

$$\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \rangle_{\overline{H_B}} = (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (310)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (311)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (312)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1 \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (313)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}} + 1} \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (314)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} j_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (j_{\mathbf{k}} + 1) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (315)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (316)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) \text{Tr}(|j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|) \quad (317)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) \quad (318)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \sum_{j_{\mathbf{k}}=0}^{\infty} (e^{-\beta \omega_{\mathbf{k}}})^{j_{\mathbf{k}}} (2j_{\mathbf{k}} + 1), \quad (319)$$

$$\sum_{j_{\mathbf{k}}=0}^{\infty} x^{j_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) = \frac{1+x}{(1-x)^2}, \quad (320)$$

$$\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \rangle_{\overline{H_B}} = \left(1 - e^{-\beta\omega_{\mathbf{k}}}\right) \frac{e^{-\beta\omega_{\mathbf{k}}} + 1}{(1 - e^{-\beta\omega_{\mathbf{k}}})^2} \text{ (setting } x = e^{-\beta\omega_{\mathbf{k}}} \text{ in (320) and by (310))}, \quad (321)$$

$$= \frac{1 + e^{-\beta\omega_{\mathbf{k}}}}{1 - e^{-\beta\omega_{\mathbf{k}}}} \quad (322)$$

$$= \frac{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} e^{\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} e^{\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}} \quad (323)$$

$$= \frac{\cosh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\sinh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (324)$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (325)$$

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \text{ (by (299), (305) and (325))}, \quad (326)$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} = \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_B}} \quad (327)$$

$$= \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left( e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + e^{\chi_{01}(t)} \right. \right. \quad (328)$$

$$\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \right) \right\rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (329)$$

$$= \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left( e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + e^{\chi_{01}(t)} \right. \right. \quad (330)$$

$$\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \right) \right\rangle_{\overline{H_B}} \text{ (by (279) and (285))}, \quad (331)$$

$$\langle F(h) \rangle_{\overline{H_B}} \equiv \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | F(h) | \alpha \rangle d^2 \alpha \text{ (using the coherent representation with } N = (e^{\beta\omega} - 1)^{-1} \text{)}, \quad (332)$$

$$D(\alpha_{\mathbf{k}}) \equiv e^{\left(\frac{\alpha_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{\alpha_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)}, \quad (333)$$

$$|\alpha\rangle \equiv D(\alpha) |0\rangle, \quad (334)$$

$$\langle \alpha | \equiv \langle 0 | D(-\alpha), \quad (335)$$

$$D(-\alpha) D(h) D(\alpha) \equiv D(h) e^{h\alpha^* - h^* \alpha}, \quad (336)$$

$$D(0) \equiv \mathbb{I}, \quad (337)$$

$$D(-\alpha) D(0) D(\alpha) = D(0) e^{0 \cdot \alpha^* - 0^* \cdot \alpha} \quad (338)$$

$$= D(0) \quad (339)$$

$$= \mathbb{I}, \quad (340)$$

$$D(-\alpha) b^{\dagger} D(\alpha) = b^{\dagger} + \alpha^*, \quad (341)$$

$$D(-\alpha) b D(\alpha) = b + \alpha, \quad (342)$$

$$\langle D(h) \rangle_{\overline{H_B}} = e^{-\frac{|h|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}, \quad (343)$$

$$\langle b^{\dagger} D(h) \rangle_{\overline{H_B}} = -h^* \langle D(h) \rangle_{\overline{H_B}} N \quad (344)$$

$$\langle b D(h) \rangle_B = h \langle D(h) \rangle_B (N+1), \quad (345)$$

$$\langle D(h) b \rangle_B = h N \langle D(h) \rangle_B, \quad (346)$$

$$\langle D(h) b^{\dagger} \rangle_B = -h^* \langle D(h) \rangle_B (N+1) \quad (347)$$

$$\langle B_{iz}(t)B_x(t) \rangle_{\overline{H_B}} = \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left( e^{\chi_{10}(t)} \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right) \right. \quad (348)$$

$$\left. + e^{\chi_{01}(t)} \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \quad (349)$$

$$= \frac{1}{2} \left\langle e^{\chi_{10}(t)} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) + e^{\chi_{01}(t)} \right. \quad (350)$$

$$\left. \times \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \quad (351)$$

$$= \frac{e^{\chi_{10}(t)}}{2} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \right. \quad (352)$$

$$\times \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \rangle_{\overline{H_B}} + \frac{e^{\chi_{01}(t)}}{2} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right. \right. \quad (353)$$

$$\left. - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \right), \quad (354)$$

$$\langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = e^{-\frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (355)$$

$$N_{\mathbf{k}} = (e^{\beta\omega_{\mathbf{k}}} - 1)^{-1}, \quad (356)$$

$$\langle b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (357)$$

$$\langle b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (358)$$

$$\prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} = e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (359)$$

$$\left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} = \left\langle b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) \right\rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (360)$$

$$= -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (361)$$

$$= -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (362)$$

$$= -\alpha_{\mathbf{k}}^* N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (363)$$

$$\left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} = \langle b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (364)$$

$$= \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (365)$$

$$= \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (366)$$

$$= \alpha_{\mathbf{k}} (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (367)$$

$$\langle B_{iz}(t)B_x(t) \rangle_{\overline{H_B}} = \frac{e^{\chi_{10}(t)}}{2} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \langle b_{\mathbf{k}} \right. \quad (368)$$



$$\langle B_{iz}(t)B_y(t) \rangle_{\overline{H_B}} = \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right\rangle_{\overline{H_B}} \quad (388)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}}}{2i} + \frac{(B_{10}(t) - B_{01}(t))}{2i} \quad (389)$$

$$\times \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad (390)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}}}{2i} + \frac{(B_{10}(t) - B_{01}(t))}{2i} \cdot 0 \quad (391)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}}}{2i} \quad (392)$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left( e^{\chi_{01}(t)} \Pi_{\mathbf{k}} D \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - e^{\chi_{10}(t)} \Pi_{\mathbf{k}} D \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle}{2i} \overline{H_B} \quad (393)$$

$$= \frac{1}{2i} \left\langle \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} e^{\chi_{01}(t)} \prod_{\mathbf{k}} D \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} e^{\chi_{10}(t)} \prod_{\mathbf{k}} D \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \quad (394)$$

$$+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}e^{\chi_{01}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}e^{\chi_{10}(t)}\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\Bigg\rangle_{\overline{H_B}} \quad (395)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{01}(t)} \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} D \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_B}} - e^{\chi_{10}(t)} \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} D \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_B}} \right) \quad (396)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* e^{i\chi_{01}(t)} \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\overline{B}}}} - \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\overline{B}}}} \quad (397)$$

$$\times e^{\chi_{10}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \Big) \quad (398)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{01}(t)} \left( - \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \right) - e^{\chi_{10}(t)} \right) \quad (399)$$

$$\times \left( - \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* e^{\chi_{01}(t)} \quad (400)$$

$$\times \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \quad (401)$$

$$\times e^{\chi_{10}(t)} e^{-\sum_{\mathbf{k}} \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (402)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( -\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} B_{01}(t) \right) - \left( -\left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} B_{10}(t) \right) \right) \quad (403)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \quad (404)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \quad (405)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}B_{01}}(t) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}B_{10}}(t) \right) \quad (406)$$

$$-(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \quad (407)$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \quad (408)$$

$$\langle B_x(t)B_{iz}(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (409)$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \quad (410)$$

$$\times \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (411)$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \cdot 0 \quad (412)$$

$$= \frac{1}{2} \left\langle \left( B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) \right) \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_B}} \quad (413)$$

$$= \frac{1}{2} \sum_{\mathbf{k}} \left\langle \left( e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) + e^{\chi_{01}(t)} \prod_{\mathbf{k}'} D\left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \right. \right. \quad (414)$$

$$\left. + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (415)$$

$$\langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}} \rangle_{\overline{H_B}} = \alpha_{\mathbf{k}} N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (416)$$

$$\langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} = -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}}, \quad (417)$$

$$\left\langle \left( \prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} = \langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (418)$$

$$= -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (419)$$

$$= -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (420)$$

$$= -\alpha_{\mathbf{k}}^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (421)$$

$$\left\langle \left( \prod_{\mathbf{k}'} D(\alpha_{\mathbf{k}'}) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} = \langle D(\alpha_{\mathbf{k}}) b_{\mathbf{k}} \rangle_{\overline{H_B}} \left\langle \prod_{\mathbf{k}' \neq \mathbf{k}} D(\alpha_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (422)$$

$$= \alpha_{\mathbf{k}} N_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \prod_{\mathbf{k}' \neq \mathbf{k}} \langle D(\alpha_{\mathbf{k}'}) \rangle_{\overline{H_B}} \quad (423)$$

$$= \alpha_{\mathbf{k}} N_{\mathbf{k}} \prod_{\mathbf{k}} \langle D(\alpha_{\mathbf{k}}) \rangle_{\overline{H_B}} \quad (424)$$

$$= \alpha_{\mathbf{k}} N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \quad (425)$$

$$\langle B_x(t)B_{iz}(t) \rangle_{\overline{H_B}} = \frac{1}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{10}(t)} \left\langle \left( \prod_{\mathbf{k}'} D\left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + e^{\chi_{01}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \right. \quad (426)$$

$$\left. \times \left\langle \left( \prod_{\mathbf{k}'} D\left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \left( \prod_{\mathbf{k}'} D\left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right. \quad (427)$$

$$\left. \times e^{\chi_{10}(t)} + e^{\chi_{01}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \left( \prod_{\mathbf{k}'} D\left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right) \quad (428)$$

$$= \frac{1}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{10}(t)} \left( - \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \right. \quad (429)$$

$$\left. + e^{\chi_{01}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( - \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + e^{\chi_{10}(t)} \right) \quad (430)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + e^{x_{01}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \quad (431)$$

$$\times \left( \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} e^{-\Sigma_{\mathbf{k}}} \frac{\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (432)$$

$$= \frac{1}{2} \sum_{\mathbf{k}} \left( -(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) B_{10}(t) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right) \quad (433)$$

$$\times (N_{\mathbf{k}} + 1)B_{01}(t) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} B_{10}(t) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \quad (434)$$

$$\times N_{\mathbf{k}} B_{01}(t)) \quad (435)$$

$$= \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) \right) \quad (436)$$

$$\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (437)$$

$$= \left\langle \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} + \frac{B_{10}(t) - B_{01}(t)}{2i} \quad (438)$$

$$\times \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (439)$$

$$= \left\langle \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (440)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left\langle \left( e^{\chi_{01}(t)} \prod_{\mathbf{k}'} D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) - e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \right) \right. \quad (441)$$

$$+ (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}}) \rangle_{H_{\bar{B}}} \quad (442)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{01}(t)} \left\langle \left( \prod_{\mathbf{k}'} D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\overline{B}}}} - e^{\chi_{10}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \right) \quad (443)$$

$$\times \left\langle \left( \prod_{\mathbf{k}'} D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\overline{B}}}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \left( \prod_{\mathbf{k}'} D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_{\overline{B}}}} \quad (444)$$

$$\times e^{\chi_{01}(t)} - e^{\chi_{10}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \left( \prod_{\mathbf{k}'} D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_{\overline{B}}}} \right) \quad (445)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) e^{\chi_{01}(t)} \left( - \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right) \right) \right) \quad (446)$$

$$-e^{\chi_{10}(t)}(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( - \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) e^{-\sum_{\mathbf{k}} \left[ \frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right]^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + e^{\chi_{01}(t)} \quad (447)$$

$$\times (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} - e^{\chi_{10}(t)} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right) \quad (448)$$

$$\times \left( \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} e^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \right) \quad (449)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) B_{01}(t) + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right) \quad (450)$$

$$\times (N_{\mathbf{k}} + 1) B_{10}(t) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} B_{01}(t) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \quad (451)$$

$$\times N_{\mathbf{k}} B_{10}(t) \quad (452)$$

$$= \frac{B_{01}(t) + B_{10}(t)}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} \right) \quad (453)$$

$$\text{Var}_{\overline{H_B}}(A) \equiv \langle A^2 \rangle_{\overline{H_B}} - \langle A \rangle_{\overline{H_B}}^2, \quad (454)$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X), \quad (455)$$

$$\langle B_x(t) \rangle_{\overline{H_B}} = 0, \quad (456)$$

$$\langle B_y(t) \rangle_{\overline{H_B}} = 0, \quad (457)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \text{Var}_{\overline{H_B}}(B_x(t)) + \langle B_x(t) \rangle_{\overline{H_B}}^2 \quad (458)$$

$$= \text{Var}_{\overline{H_B}} \left( \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (459)$$

$$= \frac{1}{4} \text{Var}_{\overline{H_B}} (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)) \quad (460)$$

$$= \frac{1}{4} \text{Var}_{\overline{H_B}} (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \quad (461)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (462)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 + B_1^+(t) B_0^-(t) B_0^+(t) B_1^-(t) + B_0^+(t) B_1^-(t) B_1^+(t) B_0^-(t) + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} \right. \quad (463)$$

$$\left. - (B_{10}(t) + B_{01}(t))^2 \right) \quad (464)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 + 2\mathbb{I} + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right), \quad (465)$$

$$(D(h))^2 = D(h) D(h) \quad (466)$$

$$= D(h+h) e^{\frac{1}{2} \left( \frac{h^* h - h h^*}{\omega^2} \right)} \quad (467)$$

$$= D(2h), \quad (468)$$

$$\left\langle (B_i^+(t) B_j^-(t))^2 \right\rangle_{\overline{H_B}} = \left\langle \left( \prod_{\mathbf{k}} D \left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right)^2 \right\rangle_{\overline{H_B}} \quad (469)$$

$$= \left\langle \prod_{\mathbf{k}} D \left( 2 \left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} \right\rangle_{\overline{H_B}} \quad (470)$$

$$= \prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} e^{-2 \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (471)$$

$$= \left( \prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{2 \omega_{\mathbf{k}}^2}} e^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right)^2 \left( \prod_{\mathbf{k}} e^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right)^2 \quad (472)$$

$$= B_{ij}^2(t) |B_{ij}(t)|^2, \quad (473)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 + 2\mathbb{I} + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (474)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} + 2 \langle \mathbb{I} \rangle_{\overline{H_B}} + \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (475)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} + 2 + \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (476)$$

$$= \frac{1}{4} (B_{10}^2(t) |B_{10}(t)| + 2 + B_{01}^2(t) |B_{01}(t)| - (B_{10}^2(t) + 2 B_{10}(t) B_{01}(t) + B_{01}^2(t))) \quad (477)$$

$$= \frac{1}{4} (B_{10}^2(t) |B_{10}(t)| + 2 + B_{01}^2(t) |B_{10}(t)| - (B_{10}^2(t) + 2 |B_{10}(t)| + B_{01}^2(t))) \quad (478)$$

$$= \frac{1}{4} (B_{10}^2(t) + B_{01}^2(t) - 2) (|B_{10}(t)| - 1) \quad (479)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = \text{Var}_{\overline{H_B}}(B_y(t)) + \langle B_y(t) \rangle_{\overline{H_B}}^2 \quad (480)$$



$$= \text{Var}_{\overline{H_B}} \left( \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right) \quad (481)$$

$$= -\frac{1}{4} \text{Var}_{\overline{H_B}} (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)) \quad (482)$$

$$= -\frac{1}{4} \text{Var}_{\overline{H_B}} (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \quad (483)$$

$$= -\frac{1}{4} \left( \left\langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t))^2 - (B_{01}(t) - B_{10}(t))^2 \right\rangle_{\overline{H_B}} \right) \quad (484)$$

$$= -\frac{1}{4} \left( \left\langle (B_0^+(t) B_1^-(t))^2 - 2\mathbb{I} + (B_1^+(t) B_0^-(t))^2 - (B_{01}(t) - B_{10}(t))^2 \right\rangle_{\overline{H_B}} \right) \quad (485)$$

$$= -\frac{1}{4} \left( \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} + \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} - 2\langle \mathbb{I} \rangle_{\overline{H_B}} - (B_{01}(t) - B_{10}(t))^2 \right) \quad (486)$$

$$= -\frac{1}{4} (B_{01}^2(t) |B_{01}(t)|^2 - 2 + B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}(t) - B_{10}(t))^2) \quad (487)$$

$$= -\frac{1}{4} (B_{01}^2(t) |B_{01}(t)|^2 - 2 + B_{10}^2(t) |B_{10}(t)|^2 - B_{01}^2(t) + 2B_{01}(t) B_{10}(t) - B_{10}^2(t)) \quad (488)$$

$$= -\frac{1}{4} (B_{01}^2(t) |B_{10}(t)|^2 - 2 + B_{10}^2(t) |B_{10}(t)|^2 - B_{01}^2(t) + 2|B_{10}(t)|^2 - B_{10}^2(t)) \quad (489)$$

$$= -\frac{1}{4} (B_{10}^2(t) + B_{01}^2(t) + 2) (|B_{10}(t)|^2 - 1), \quad (490)$$

$$\langle B_x(t) B_y(t) \rangle_{\overline{H_B}} = \left\langle B_x(t) \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right\rangle_{\overline{H_B}} \quad (491)$$

$$= \left\langle B_x(t) \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)}{2i} \right\rangle_{\overline{H_B}} + \frac{B_{10}(t) - B_{01}(t)}{2i} \langle B_x(t) \rangle_{\overline{H_B}} \quad (492)$$

$$= \left\langle B_x(t) \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)}{2i} \right\rangle_{\overline{H_B}} + \frac{B_{10}(t) - B_{01}(t)}{2i} \cdot 0 \quad (493)$$

$$= \left\langle B_x(t) \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)}{2i} \right\rangle_{\overline{H_B}} \quad (494)$$

$$= \left\langle \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)}{2i} \right\rangle_{\overline{H_B}} \quad (495)$$

$$= \frac{1}{4i} \left( \left\langle (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}} - \left\langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}} \right) \quad (496)$$

$$\times (B_{10}(t) + B_{01}(t)) \quad (497)$$

$$= \frac{1}{4i} \left( \left\langle (B_1^+(t) B_0^-(t) B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) B_0^+(t) B_1^-(t) - B_0^+(t) B_1^-(t) \right\rangle_{\overline{H_B}} \right) \quad (498)$$

$$\times B_1^+(t) B_0^-(t) \rangle_{\overline{H_B}} - (B_{01}(t) - B_{10}(t)) (B_{10}(t) + B_{01}(t)) \quad (499)$$

$$= \frac{1}{4i} \left( \left\langle \mathbb{I} - (B_1^+(t) B_0^-(t))^2 + (B_0^+(t) B_1^-(t))^2 - \mathbb{I} \right\rangle_{\overline{H_B}} - (B_{01}(t) - B_{10}(t)) (B_{10}(t) + B_{01}(t)) \right) \quad (500)$$

$$= \frac{1}{4i} \left( \left\langle (B_0^+(t) B_1^-(t))^2 - (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{01}(t) - B_{10}(t)) (B_{10}(t) + B_{01}(t)) \right) \quad (501)$$

$$= \frac{1}{4i} \left( \left\langle (B_0^+(t) B_1^-(t))^2 - (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{01}(t) - B_{10}(t)) (B_{01}(t) + B_{10}(t)) \right) \quad (502)$$

$$= \frac{1}{4i} \left( \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{01}^2(t) - B_{10}^2(t)) \right) \quad (503)$$

$$= \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2 - B_{01}^2(t) + B_{10}^2(t)) \quad (504)$$

$$= \frac{1}{4i} (B_{01}^2(t) - B_{10}^2(t)) (|B_{10}(t)|^2 - 1), \quad (505)$$

$$\langle B_y(t) B_x(t) \rangle_{\overline{H_B}} = \left\langle B_y(t) \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_B}} \quad (506)$$

$$= \left\langle B_y(t) \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)}{2} \right\rangle_{\overline{H_B}} - \left\langle B_y(t) \frac{B_{10}(t) + B_{01}(t)}{2} \right\rangle_{\overline{H_B}} \quad (507)$$

$$= \frac{1}{2} \langle B_y(t) (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \langle B_y(t) \rangle_{\overline{H_B}} \quad (508)$$

$$= \frac{1}{2} \langle B_y(t) (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \rangle_{\overline{H_B}} - \frac{B_{10}(t) + B_{01}(t)}{2} \cdot 0 \quad (509)$$

$$= \frac{1}{2} \langle B_y(t) (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \rangle_{\overline{H_B}} \quad (510)$$

$$= \frac{1}{2} \left\langle \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \right\rangle_{\overline{H_B}} \quad (511)$$

$$= \frac{1}{4i} \langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \rangle_{\overline{H_B}} + \frac{(B_{10}(t) - B_{01}(t))}{4i} \langle (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \rangle_{\overline{H_B}} \quad (512)$$

$$= \frac{1}{4i} \langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \rangle_{\overline{H_B}} + \frac{(B_{10}(t) - B_{01}(t)) (B_{10}(t) + B_{01}(t))}{4i} \quad (513)$$

$$= \frac{1}{4i} \langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \rangle_{\overline{H_B}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4i} \quad (514)$$

$$= \frac{1}{4i} \langle B_0^+(t) B_1^-(t) B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) B_1^+(t) B_0^-(t) - B_1^+(t) B_0^-(t) B_0^+(t) B_1^-(t) \rangle_{\overline{H_B}} \quad (515)$$

$$+ \frac{B_{10}^2(t) - B_{01}^2(t)}{4i} \quad (516)$$

$$= \frac{1}{4i} \langle \mathbb{I} + (B_0^+(t) B_1^-(t))^2 - (B_1^+(t) B_0^-(t))^2 - \mathbb{I} \rangle_{\overline{H_B}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4i} \quad (517)$$

$$= \frac{1}{4i} \langle (B_0^+(t) B_1^-(t))^2 - (B_1^+(t) B_0^-(t))^2 \rangle_{\overline{H_B}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4i} \quad (518)$$

$$= \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2) + \frac{B_{10}^2(t) - B_{01}^2(t)}{4i} \quad (519)$$

$$= \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2 + B_{10}^2(t) - B_{01}^2(t)) \quad (520)$$

$$= \frac{1}{4i} (B_{01}^2(t) - B_{10}^2(t)) (|B_{10}(t)|^2 - 1). \quad (521)$$

Summarizing the expected values obtained in the precedent lines we have:

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right), \quad (522)$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right), \quad (523)$$

$$\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right), \quad (524)$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) \right), \quad (525)$$

$$\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} = \frac{B_{01}(t) + B_{10}(t)}{2i} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} \right), \quad (526)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \frac{1}{4} (B_{10}^2(t) + B_{01}^2(t) - 2) (|B_{10}^2(t)| - 1), \quad (527)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = -\frac{1}{4} (B_{10}^2(t) + B_{01}^2(t) + 2) (|B_{10}(t)|^2 - 1), \quad (528)$$

$$\langle B_x(t) B_y(t) \rangle_{\overline{H_B}} = \frac{1}{4i} (B_{01}^2(t) - B_{10}^2(t)) (|B_{10}(t)|^2 - 1), \quad (529)$$

$$\langle B_y(t) B_x(t) \rangle_{\overline{H_B}} = \frac{1}{4i} (B_{01}^2(t) - B_{10}^2(t)) (|B_{10}(t)|^2 - 1). \quad (530)$$

The density matrix associated to  $\rho_{\overline{S}} = \frac{e^{-\beta \overline{H}_{\overline{S}}(t)}}{\text{Tr}(e^{-\beta \overline{H}_{\overline{S}}(t)})}$  follows the form:

$$\rho_{\bar{S},00} = \frac{1}{2} - \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (531)$$

$$\rho_{\bar{S},01} = -\frac{B_{10}^*(t) V_{10}^*(t) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (532)$$

$$\rho_{\bar{S},10} = -\frac{B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (533)$$

$$\rho_{\bar{S},11} = \frac{1}{2} + \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t)|^2 |V_{10}(t)|^2}}. \quad (534)$$

The expected values respect to the system  $\bar{S}$  of relevance for calculating  $\langle \bar{H}_I^{-2}(t) \rangle_{\bar{H}_{\bar{S}}(t)}$  are  $\langle |i\rangle\langle i| \rangle_{\bar{H}_{\bar{S}}(t)}$ ,  $\langle |i\rangle\langle i| \sigma_x \rangle_{\bar{H}_{\bar{S}}(t)}$ ,  $\langle |i\rangle\langle i| \sigma_y \rangle_{\bar{H}_{\bar{S}}(t)}$ ,  $\langle \sigma_x |i\rangle\langle i| \rangle_{\bar{H}_{\bar{S}}(t)}$ ,  $\langle \sigma_y |i\rangle\langle i| \rangle_{\bar{H}_{\bar{S}}(t)}$  and  $\langle \sigma_z \rangle_{\bar{H}_{\bar{S}}(t)}$ , we took account that  $\sigma_x \sigma_y = i\sigma_z$  and  $\sigma_y \sigma_x = -i\sigma_z$ . The values needed for our calculation are:

$$\langle |0\rangle\langle 0| \rangle_{\bar{H}_{\bar{S}}(t)} = \frac{1}{2} - \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}}, \quad (535)$$

$$\langle |1\rangle\langle 1| \rangle_{\bar{H}_{\bar{S}}(t)} = \frac{1}{2} + \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}}, \quad (536)$$

$$\langle |0\rangle\langle 0| \sigma_x \rangle_{\bar{H}_{\bar{S}}(t)} = -\frac{B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}}, \quad (537)$$

$$\langle |1\rangle\langle 1| \sigma_x \rangle_{\bar{H}_{\bar{S}}(t)} = -\frac{B_{10}^*(t) V_{10}^*(t) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}}, \quad (538)$$

$$\langle |0\rangle\langle 0| \sigma_y \rangle_{\bar{H}_{\bar{S}}(t)} = -\frac{iB_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4|B_{10}(t) V_{10}(t)|^2}}, \quad (539)$$

$$\langle 1|1|\sigma_y\rangle_{\overline{H_S}(t)} = -\frac{iB_{10}^*(t)V_{10}^*(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)|^2|V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)|^2|V_{10}(t)|^2}}, \quad (540)$$

$$\langle \sigma_x|0|0\rangle_{\overline{H_S}(t)} = -\frac{B_{10}^*(t)V_{10}^*(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}}, \quad (541)$$

$$\langle \sigma_x|1|1\rangle_{\overline{H_S}(t)} = -\frac{B_{10}(t)V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}}, \quad (542)$$

$$\langle \sigma_y|0|0\rangle_{\overline{H_S}(t)} = -\frac{iB_{10}^*(t)V_{10}^*(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}}, \quad (543)$$

$$\langle \sigma_y|1|1\rangle_{\overline{H_S}(t)} = -\frac{iB_{10}(t)V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}}, \quad (544)$$

$$\langle \sigma_z\rangle_{\overline{H_S}(t)} = -\frac{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}\right)}{\sqrt{\left(\sum_i(-1)^i(\varepsilon_i(t)+R_i(t))\right)^2+4|B_{10}(t)V_{10}(t)|^2}}. \quad (545)$$

Our next step is to find  $v_3(t)$ , the commutator  $[\overline{H_0}(t), \overline{H_I}(t)]$  is a central point for our calculations and it is equal to:

$$[\overline{H_0}(t), \overline{H_I}(t)] = \left[ (\varepsilon_0(t) + R_0(t))|0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t))|1\rangle\langle 1| + \sigma_x \left( B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t) \right) - \sigma_y \left( B_{10}^{\Re}(t)V_{10}^{\Im}(t) \right. \right. \quad (546)$$

$$\left. + B_{10}^{\Im}(t)V_{10}^{\Re}(t) \right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \Big] \quad (547)$$

$$= \left[ \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| + \sigma_x \left( B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t) \right) - \sigma_y \left( B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t) \right) \right. \quad (548)$$

$$\left. + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \right] \quad (549)$$

$$= \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \quad (550)$$

$$\times \left( B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| + \sigma_x \left( B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (551)$$

$$+ \sigma_x \left( B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sigma_y \left( B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| \quad (552)$$

$$- \sigma_y \left( B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) - \sigma_y \left( B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (553)$$



$$= \sum_i B_{iz}(t) |i\rangle\langle i| \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\mathfrak{I}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (580)$$

$$+ \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_i B_{iz}(t) |i\rangle\langle i| \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (B_x(t) \quad (581)$$

$$+i\sigma_z B_y(t) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) (B_y(t) - i\sigma_z B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) \quad (582)$$

$$+B_{10}^{\mathfrak{S}}(t)V_{10}^{\mathfrak{R}}(t))\sum_i B_{iz}(t)|i\rangle\langle i|-\sum_i B_{iz}(t)|i\rangle\langle i|\left(B_{10}^{\mathfrak{R}}(t)V_{10}^{\mathfrak{S}}(t)+B_{10}^{\mathfrak{S}}(t)V_{10}^{\mathfrak{R}}(t)\right)V_{10}^{\mathfrak{R}}(t)(-i\sigma_z B_x(t)+B_y(t))-\sum_i B_{iz}(t)|i\rangle\langle i| \quad (583)$$

$$\times \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (-i\sigma_z B_y(t) - B_x(t)) + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (584)$$

$$\times V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \quad (585)$$

$$\times \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |\dot{x}| \dot{x} \left| \sum_i B_{iz}(t) |\dot{x}| \dot{x} \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - \sum_i B_{iz}(t) |\dot{x}| \dot{x} \left| \sum_i B_{iz}(t) |\dot{x}| \dot{x} \right| \right. \quad (586)$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathcal{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathcal{R}}(t) (B_x(t) - i\sigma_z B_y(t)) \quad (587)$$

$$\times \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Re}(t) (i\sigma_z B_x(t) + B_y(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Re}(t) \quad (588)$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{Z}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| - \sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{Z}}(t) \quad (589)$$

$$\times (B_y(t) + i\sigma_z B_x(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Im}(t) (i\sigma_z B_y(t) - B_x(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (590)$$

$$-\sum_i B_{iz}(t) |i\rangle \langle i| V_{10}^{\mathfrak{Z}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (591)$$

$$+\sigma_y B_y(t) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle \langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_x \quad (592)$$

$$\times \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}(t) |i\rangle \langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (B_x(t) + i\sigma_z B_y(t)) \quad (593)$$

$$+ V_{10}^{\mathcal{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_x \left( B_{10}^{\mathcal{R}}(t) V_{10}^{\mathcal{R}}(t) - B_{10}^{\mathcal{I}}(t) V_{10}^{\mathcal{I}}(t) \right) V_{10}^{\mathcal{I}}(t) (B_y(t) - i\sigma_z B_x(t)) - V_{10}^{\mathcal{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_y \quad (594)$$

$$X \left( D_{10}^{\text{L}}(t) \vee_{10}(t) + D_{10}^{\text{L}}(t) \vee_{10}(t) \right) \sum_i D_{iz}(t) |t| \wedge |t| (-\vee_{10}(t) (x X(t) + y Y(t))) \left( D_{10}^{\text{L}}(t) \vee_{10}(t) + D_{10}^{\text{L}}(t) \vee_{10}(t) \right) \vee_{10}(t) (-10z X(t) + D_y(t)) \quad (555)$$

$$= \sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger}(\mathbf{t}) \left[ V_{10}(\mathbf{t}) + V_{10}^{\mathfrak{R}}(\mathbf{t}) \right] + B_{\mathbf{k}}(\mathbf{t}) + B_{\mathbf{k}}(\mathbf{t}) \sum_{\mathbf{k}'} V_{10}^{\dagger}(\mathbf{t}) V_{10}^{\mathfrak{R}}(\mathbf{t}) + B_{\mathbf{k}}^{\dagger}(\mathbf{t}) + B_{\mathbf{k}}^{\dagger}(\mathbf{t}) + V_{10}^{\mathfrak{R}}(\mathbf{t}) + B_{\mathbf{k}}(\mathbf{t}) + B_{\mathbf{k}}(\mathbf{t}) \sum_{\mathbf{k}'} V_{10}^{\dagger}(\mathbf{t}) \quad (507)$$

[illegible]

[illegible]

$$+\partial_y D_y(y) \sum_i D_{iz}(z) |\lambda| \lambda | \partial_y \left( D_{10}^{(i)}(y) v_{10}^{(i)} + D_{10}^{(i)}(z) v_{10}^{(i)} \right) - v_{10}^{(i)} (\partial_x D_x(x) + \partial_y D_y(y)) \sum_i D_{iz}(z) |\lambda| \lambda | \sum_{\mathbf{k}} \omega_{\mathbf{k}} k_{\perp}^2 \mathbf{k}_{\parallel} - v_{10}^{(i)}(z) \rangle \rangle,$$

[illegible]

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i \omega_k \mathbf{k}_k^\dagger \mathbf{k}_k - V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i t) \quad (602)$$

$$+R_i(t)|i\rangle\langle i| - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))V_{10}^{\Im}(t)(B_y(t) + \mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (603)$$

$$\times V_{10}^{\mathfrak{Z}}(t)(i\sigma_z B_y(t) - B_x(t)) \left( B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{Z}}(t) + B_{10}^{\mathfrak{Z}}(t) V_{10}^{\mathfrak{R}}(t) \right) - V_{10}^{\mathfrak{R}}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\mathfrak{Z}}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (604)$$

$$+ V_{10}^{\mathfrak{S}}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\mathfrak{R}}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\mathfrak{S}}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| \quad (605)$$

$$\times V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left( B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{S}}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (606)$$

$$-\sigma_y B_x(t) \left( B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{I}}(t) V_{10}^{\mathfrak{I}}(t) \right) V_{10}^{\mathfrak{R}}(t) (B_x(t) + i\sigma_z B_y(t)) + V_{10}^{\mathfrak{I}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left( B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{R}}(t) - B_{10}^{\mathfrak{I}}(t) V_{10}^{\mathfrak{I}}(t) \right) \quad (607)$$

$$\times V_{10}^{\mathfrak{S}}(t) (B_y(t) - \mathrm{i}\sigma_z B_x(t)) - V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \left( B_{10}^{\mathfrak{R}}(t) V_{10}^{\mathfrak{S}}(t) + B_{10}^{\mathfrak{S}}(t) V_{10}^{\mathfrak{R}}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| - V_{10}^{\mathfrak{S}}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \quad (608)$$

$$\begin{aligned}
& -\sigma_y B_x(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) (-i\sigma_z B_x(t) + B_y(t)) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (609) \\
& \times V_{10}^{\Im}(t) (-i\sigma_z B_y(t) - B_x(t)) + V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (610) \\
& \times V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (611) \\
& \times \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (612) \\
& - V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) \quad (613) \\
& \times |i\rangle\langle i| - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (B_x(t) - i\sigma_z B_y(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) \quad (614) \\
& \times (i\sigma_z B_x(t) + B_y(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (615) \\
& - V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (B_y(t) \quad (616) \\
& + i\sigma_z B_x(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) (i\sigma_z B_y(t) - B_x(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (617) \\
& - V_{10}^{\Re}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (618) \\
& = V_{10}^{\Re}(t) \sum_i (\varepsilon_i(t) + R_i(t)) (|i\rangle\langle i| \sigma_x B_{iz}(t) B_x(t) + |i\rangle\langle i| \sigma_y B_{iz}(t) B_y(t)) + V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t)) (|i\rangle\langle i| \sigma_x B_{iz}(t) B_y(t) - |i\rangle\langle i| \sigma_y \quad (619) \\
& \times B_{iz}(t) B_x(t)) + \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_{i \neq i'} B_{iz}(t) B_{i'z}(t) |i\rangle\langle i'| + \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) \sum_i (|i\rangle\langle i| B_{iz}(t) \quad (620) \\
& \times B_x(t) + |i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t)) + \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) \sum_i (|i\rangle\langle i| \sigma_x B_{iz}(t) B_y(t) - |i\rangle\langle i| \sigma_y B_{iz}(t) B_x(t)) - \left( B_{10}^{\Re}(t) \quad (621) \\
& \times V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_{i \neq i'} B_{iz}(t) B_{i'z}(t) |i\rangle\langle i'| \sigma_y |i'\rangle\langle i'| - \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) \sum_i (-|i\rangle\langle i| \sigma_z B_{iz}(t) B_x(t) + B_{iz}(t) \quad (622) \\
& \times B_y(t) |i\rangle\langle i|) + \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) \sum_i (|i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t) + |i\rangle\langle i| B_{iz}(t) B_x(t)) + \sum_{i, \mathbf{k}} |i\rangle\langle i| B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) \quad (623) \\
& + V_{10}^{\Re}(t) \sum_{i, \mathbf{k}} \left( |i\rangle\langle i| \sigma_x B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + |i\rangle\langle i| \sigma_y B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) \right) + V_{10}^{\Im}(t) \sum_{i, \mathbf{k}} \left( |i\rangle\langle i| \sigma_x B_{iz}(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - |i\rangle\langle i| \sigma_y B_{iz}(t) \quad (624) \\
& \times \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) \right) - \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}^2(t) |i\rangle\langle i| \sigma_x + \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i B_{iz}^2(t) |i\rangle\langle i| \sigma_y - \sum_{i, \mathbf{k}} |i\rangle\langle i| \quad (625) \\
& \times B_{iz}^2(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t) \sum_{i, i'} (\varepsilon_{i'}(t) + R_{i'}(t)) (|i\rangle\langle i| \sigma_x |i'\rangle\langle i'| B_{iz}(t) B_x(t) + |i\rangle\langle i| \sigma_y |i'\rangle\langle i'| B_{iz}(t) B_y(t)) - V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) \quad (626) \\
& \times V_{10}^{\Im}(t) \right) \sum_i (|i\rangle\langle i| B_{iz}(t) B_x(t) - |i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t)) + V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i (|i\rangle\langle i| \sigma_z B_{iz}(t) B_x(t) + |i\rangle\langle i| \quad (627) \\
& \times B_{iz}(t) B_y(t)) - V_{10}^{\Im}(t) \sum_{i, \mathbf{k}} \left( |i\rangle\langle i| \sigma_x B_{iz}(t) B_x(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |i\rangle\langle i| \sigma_y B_{iz}(t) B_y(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) - V_{10}^{\Re}(t) \sum_{i \neq i'} (\varepsilon_{i'}(t) + R_{i'}(t)) (|i\rangle\langle i| \sigma_x |i'\rangle\langle i'| \quad (628) \\
& \times B_{iz}(t) B_y(t) - |i\rangle\langle i| \sigma_y |i'\rangle\langle i'| B_{iz}(t) B_x(t)) - V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i (|i\rangle\langle i| B_{iz}(t) B_y(t) + |i\rangle\langle i| \sigma_z B_x(t)) + V_{10}^{\Re}(t) \quad (629) \\
& \times \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i (|i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t) - |i\rangle\langle i| B_{iz}(t) B_x(t)) - V_{10}^{\Im}(t) \sum_{i, \mathbf{k}} \left( |i\rangle\langle i| \sigma_x B_{iz}(t) B_y(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - |i\rangle\langle i| \sigma_y \quad (630) \\
& \times B_{iz}(t) B_x(t) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) + \left( V_{10}^{\Re}(t) \right)^2 \sum_i (\varepsilon_i(t) + R_i(t)) (\sigma_x |i\rangle\langle i| \sigma_x B_x^2(t) + \sigma_x |i\rangle\langle i| \sigma_y B_x(t) B_y(t) + \sigma_y |i\rangle\langle i| \sigma_x B_y(t) B_x(t) + \sigma_y |i\rangle\langle i| \quad (631) \\
& \times \sigma_y B_y^2(t)) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t)) (\sigma_x |i\rangle\langle i| \sigma_x B_x(t) B_y(t) - \sigma_x |i\rangle\langle i| \sigma_y B_x^2(t) + \sigma_y |i\rangle\langle i| \sigma_x B_y^2(t) - \sigma_y |i\rangle\langle i| \sigma_y B_y(t) B_x(t)) \quad (632) \\
& + V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i (|i\rangle\langle i| B_x(t) B_{iz}(t) - i\sigma_z |i\rangle\langle i| B_y(t) B_{iz}(t)) + \left( V_{10}^{\Re}(t) \right)^2 \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (633) \\
& \times (\sigma_x B_x^2(t) + \sigma_y B_x(t) B_y(t) + \sigma_y B_y(t) B_x(t) - \sigma_x B_y^2(t)) + V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) (B_x(t) B_y(t) - i\sigma_z B_y^2(t) \quad (634) \\
& - i\sigma_z B_x^2(t) - B_y(t) B_x(t)) - V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i (i\sigma_z |i\rangle\langle i| B_x(t) B_{iz}(t) + |i\rangle\langle i| B_y(t) B_{iz}(t)) - V_{10}^{\Re}(t) \quad (635) \\
& \times \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) (-\sigma_y B_x^2(t) + \sigma_x B_y(t) B_x(t) + \sigma_x B_x(t) B_y(t) + \sigma_y B_y^2(t)) - V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) \quad (636) \\
& \times V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (-\sigma_y B_x(t) B_y(t) + \sigma_x B_y^2(t) - \sigma_x B_x^2(t) - \sigma_y B_y(t) B_x(t)) + V_{10}^{\Re}(t) \sum_{i, \mathbf{k}} (\sigma_x |i\rangle\langle i| B_x(t) + \sigma_y |i\rangle\langle i| B_y(t)) \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) \quad (637)
\end{aligned}$$

$$+ \left( V_{10}^{\Re}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) - i\sigma_z B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) + B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) \right) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \quad (638)$$

$$\times \left( B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - i\sigma_z B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) - B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) \right) - V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (639)$$

$$\times \sum_i \left( \sigma_x |i\rangle\langle i| \sigma_x B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| \sigma_x B_y(t) B_{iz}(t) \right) + V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i \left( \sigma_x |i\rangle\langle i| \sigma_y B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| \quad (640)$$

$$\times \sigma_y B_y(t) B_{iz}(t) \right) - V_{10}^{\Re}(t) \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left( \sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) - \left( V_{10}^{\Re}(t) \right)^2 \sum_i \left( \varepsilon_i(t) + R_i(t) \right) \left( |i\rangle\langle i| B_x^2(t) \quad (641)$$

$$- i\sigma_z |i\rangle\langle i| B_y(t) B_x(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_y(t) + |i\rangle\langle i| B_y^2(t) \right) - \left( V_{10}^{\Re}(t) \right)^2 \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left( \sigma_x B_x^2(t) + \sigma_y B_y(t) B_x(t) \quad (642)$$

$$- \sigma_y B_x(t) B_y(t) + \sigma_x B_y^2(t) \right) + \left( V_{10}^{\Re}(t) \right)^2 \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left( \sigma_y B_x^2(t) - \sigma_x B_y(t) B_x(t) + \sigma_x B_x(t) B_y(t) + \sigma_y B_y^2(t) \right) \quad (643)$$

$$- \left( V_{10}^{\Re}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B_x^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) - V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_i \left( \varepsilon_i(t) + R_i(t) \right) \quad (644)$$

$$\times \left( |i\rangle\langle i| B_x(t) B_y(t) - i\sigma_z |i\rangle\langle i| B_y^2(t) - i\sigma_z |i\rangle\langle i| B_x^2(t) - |i\rangle\langle i| B_y(t) B_x(t) \right) - V_{10}^{\Re}(t) V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left( \sigma_x B_x(t) \quad (645)$$

$$\times B_y(t) + \sigma_y B_y^2(t) + \sigma_y B_x^2(t) - \sigma_x B_y(t) B_x(t) \right) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left( \sigma_y B_x(t) B_y(t) - \sigma_x B_y^2(t) - \sigma_x B_x^2(t) \quad (646)$$

$$- \sigma_y B_y(t) B_x(t) \right) - V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_x^2(t) - B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) + V_{10}^{\Im}(t) V_{10}^{\Re}(t) \sum_i \left( \varepsilon_i(t) \quad (647)$$

$$+ R_i(t) \right) \left( \sigma_x |i\rangle\langle i| \sigma_x B_y(t) B_x(t) - \sigma_y |i\rangle\langle i| \sigma_x B_x^2(t) + \sigma_x |i\rangle\langle i| \sigma_y B_y^2(t) - \sigma_y |i\rangle\langle i| \sigma_y B_x(t) B_y(t) \right) + \left( V_{10}^{\Im}(t) \right)^2 \sum_i \left( \varepsilon_i(t) + R_i(t) \right) \left( \sigma_x |i\rangle\langle i| \quad (648)$$

$$\times \sigma_x B_y^2(t) - \sigma_y |i\rangle\langle i| \sigma_x B_x(t) B_y(t) - \sigma_x |i\rangle\langle i| \sigma_y B_y(t) B_x(t) + \sigma_y |i\rangle\langle i| \sigma_y B_x^2(t) \right) + V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i \left( |i\rangle\langle i| B_y(t) \quad (649)$$

$$\times B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_{iz}(t) \right) + V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) \left( \sigma_x B_y(t) B_x(t) - \sigma_y B_x^2(t) + \sigma_y B_y^2(t) + \sigma_x B_x(t) B_y(t) \right) \quad (650)$$

$$+ \left( V_{10}^{\Im}(t) \right)^2 \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left( B_y^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z B_y(t) B_x(t) + B_x^2(t) \right) - V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (651)$$

$$\times \sum_i \left( i\sigma_z |i\rangle\langle i| B_y(t) B_{iz}(t) - |i\rangle\langle i| B_x(t) B_{iz}(t) \right) - V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) \left( -\sigma_y B_y(t) B_x(t) - \sigma_x B_x^2(t) + \sigma_x B_y^2(t) \quad (652)$$

$$- \sigma_y B_x(t) B_y(t) \right) - \left( V_{10}^{\Im}(t) \right)^2 \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left( -\sigma_y B_y^2(t) - \sigma_x B_x(t) B_y(t) - \sigma_x B_y(t) B_x(t) + \sigma_y B_x^2(t) \right) + V_{10}^{\Im}(t) \quad (653)$$

$$\times \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left( \sigma_x |i\rangle\langle i| B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t) \right) + V_{10}^{\Im}(t) V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + i\sigma_z \quad (654)$$

$$\times B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) \right) + \left( V_{10}^{\Im}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) + i\sigma_z B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_y(t) - i\sigma_z B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) + B_x(t) \quad (655)$$

$$\times b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_x(t) \right) - V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i \left( \sigma_x |i\rangle\langle i| \sigma_x B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| \sigma_x B_x(t) B_{iz}(t) \right) + V_{10}^{\Im}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) \quad (656)$$

$$+ B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i \left( \sigma_x |i\rangle\langle i| \sigma_y B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| \sigma_y B_x(t) B_{iz}(t) \right) - V_{10}^{\Im}(t) \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left( \sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) \quad (657)$$

$$- V_{10}^{\Im}(t) V_{10}^{\Re}(t) \sum_i \left( \varepsilon_i(t) + R_i(t) \right) \left( |i\rangle\langle i| B_y(t) B_x(t) + i\sigma_z |i\rangle\langle i| B_x^2(t) + i\sigma_z |i\rangle\langle i| B_y^2(t) - |i\rangle\langle i| B_x(t) B_y(t) \right) - V_{10}^{\Im}(t) V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) \quad (658)$$

$$- B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left( \sigma_x B_y(t) B_x(t) - \sigma_y B_x^2(t) - \sigma_y B_y^2(t) - \sigma_x B_x(t) B_y(t) \right) + V_{10}^{\Im}(t) V_{10}^{\Re}(t) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left( \sigma_y B_y(t) B_x(t) \quad (659)$$

$$+ \sigma_x B_x^2(t) + \sigma_x B_y^2(t) - \sigma_y B_x(t) B_y(t) \right) - \sum_{\mathbf{k}} V_{10}^{\Im}(t) V_{10}^{\Re}(t) \omega_{\mathbf{k}} \left( B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_x^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) \quad (660)$$

$$- \left( V_{10}^{\Im}(t) \right)^2 \sum_i \left( \varepsilon_i(t) + R_i(t) \right) \left( |i\rangle\langle i| B_y^2(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_y(t) - i\sigma_z |i\rangle\langle i| B_y(t) B_x(t) + |i\rangle\langle i| B_x^2(t) \right) - \left( V_{10}^{\Im}(t) \right)^2 \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) \quad (661)$$

$$- B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \left( \sigma_x B_y^2(t) - \sigma_y B_x(t) B_y(t) + \sigma_y B_y(t) B_x(t) + \sigma_x B_x^2(t) \right) + \left( V_{10}^{\Im}(t) \right)^2 \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \left( \sigma_y B_y^2(t) \quad (662)$$

$$+ \sigma_x B_x(t) B_y(t) - \sigma_x B_y(t) B_x(t) + \sigma_y B_x^2(t) \right) - \left( V_{10}^{\Im}(t) \right)^2 \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B_y^2(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + i\sigma_z B_x(t) B_y(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - i\sigma_z B_y(t) B_x(t) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + B_x^2(t) \quad (663)$$

$$\times b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right), \quad (664)$$

Now let's obtain the form of  $\overline{H}_T^3(t)$ :

$$\overline{H}_T^3(t) = \left( \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \right) \left( \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x \quad (665)$$



$$\begin{aligned}
& + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y + V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t)) \quad (666) \\
& + \left( V_{10}^{\Re}(t) \right)^2 (B_x^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z B_y(t) B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) + \left( V_{10}^{\Im}(t) \right)^2 \quad (667) \\
& \times (B_y^2(t) + B_x^2(t) - i\sigma_z B_y(t) B_x(t) + i\sigma_z B_x(t) B_y(t)) \quad (668) \\
= & \sum_i B_{iz}(t) |i\rangle\langle i| \sum_i B_{iz}^2(t) |i\rangle\langle i| + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Im}(t) \quad (669) \\
& \times \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_y) + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t)) + \sum_i B_{iz}(t) |i\rangle\langle i| \quad (670) \\
& \times \left( V_{10}^{\Re}(t) \right)^2 (B_x^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z B_y(t) B_x(t) + B_y^2(t)) + \sum_i B_{iz}(t) |i\rangle\langle i| V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) \quad (671) \\
& + \sum_i B_{iz}(t) |i\rangle\langle i| \left( V_{10}^{\Im}(t) \right)^2 (B_y^2(t) + B_x^2(t) - i\sigma_z B_y(t) B_x(t) + i\sigma_z B_x(t) B_y(t)) + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Im}(t) \quad (672) \\
& \times (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) \quad (673) \\
& \times |i\rangle\langle i| \sigma_x - B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}(t)) + V_{10}^{\Re}(t) (\sigma_x B_x(t) \quad (674) \\
& + \sigma_y B_y(t)) \left( V_{10}^{\Re}(t) \right)^2 (B_x^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z B_y(t) B_x(t) + B_y^2(t)) + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) \quad (675) \\
& - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \left( V_{10}^{\Im}(t) \right)^2 (B_y^2(t) + B_x^2(t) - i\sigma_z B_y(t) B_x(t) + i\sigma_z B_x(t) B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) \quad (676) \\
& - \sigma_y B_x(t)) \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y \quad (677) \\
& \times B_x(t)) V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) V_{10}^{\Re}(t) \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}(t) + \sigma_y |i\rangle\langle i| \quad (678) \\
& \times B_y(t) B_{iz}(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \left( V_{10}^{\Re}(t) \right)^2 (B_x^2(t) + i\sigma_z B_x(t) B_y(t) - i\sigma_z B_y(t) B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (679) \\
& \times V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \left( V_{10}^{\Im}(t) \right)^2 (B_y^2(t) + B_x^2(t) - i\sigma_z B_y(t) B_x(t) \quad (680) \\
& + i\sigma_z B_x(t) B_y(t)) \quad (681) \\
= & \sum_i B_{iz}^3(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}^2(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}^2(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) \sum_i (B_{iz}^2(t) B_y(t) |i\rangle\langle i| \sigma_x - B_{iz}^2(t) B_x(t) |i\rangle\langle i| \sigma_y) \quad (682) \\
& + V_{10}^{\Re}(t) \sum_{i \neq i'} (|i'\rangle\langle i'| \sigma_x |i\rangle\langle i| B_{i'z}(t) B_x(t) B_{iz}(t) + |i'\rangle\langle i'| \sigma_y |i\rangle\langle i| B_{i'z}(t) B_y(t) B_{iz}(t)) + \left( V_{10}^{\Re}(t) \right)^2 \sum_i (|i\rangle\langle i| B_{iz}(t) B_x^2(t) + i|i\rangle\langle i| \sigma_z B_{iz}(t) \quad (683) \\
& \times B_x(t) B_y(t) - i|i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t) B_x(t) + |i\rangle\langle i| B_{iz}(t) B_y^2(t)) + V_{10}^{\Im}(t) \sum_{i \neq i'} (|i'\rangle\langle i'| \sigma_x |i\rangle\langle i| B_{i'z}(t) B_y(t) B_{iz}(t) - |i'\rangle\langle i'| \sigma_y |i\rangle\langle i| B_{i'z}(t) \quad (684) \\
& B_x(t) B_{iz}(t)) + \left( V_{10}^{\Im}(t) \right)^2 \sum_i (|i\rangle\langle i| B_{iz}(t) B_y^2(t) + |i\rangle\langle i| B_{iz}(t) B_x^2(t) - i|i\rangle\langle i| \sigma_z B_{iz}(t) B_y(t) B_x(t) + i|i\rangle\langle i| \sigma_z B_{iz}(t) B_x(t) B_y(t)) + V_{10}^{\Re}(t) \quad (685) \\
& \times \sum_i (\sigma_x |i\rangle\langle i| B_x(t) B_{iz}^2(t) + \sigma_y |i\rangle\langle i| B_y(t) B_{iz}^2(t)) + \left( V_{10}^{\Re}(t) \right)^2 \sum_i (B_x(t) B_{iz}(t) B_x(t) \sigma_x |i\rangle\langle i| \sigma_x + B_x(t) B_{iz}(t) B_y(t) \sigma_x |i\rangle\langle i| \sigma_y + B_y(t) \quad (686) \\
& \times B_{iz}(t) B_x(t) \sigma_y |i\rangle\langle i| \sigma_x + B_y(t) B_{iz}(t) B_y(t) \sigma_y |i\rangle\langle i| \sigma_y) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_i (B_x(t) B_{iz}(t) B_y(t) \sigma_x |i\rangle\langle i| \sigma_x - B_x(t) B_{iz}(t) B_x(t) \sigma_x |i\rangle\langle i| \sigma_y \quad (687) \\
& + B_y(t) B_{iz}(t) B_y(t) \sigma_y |i\rangle\langle i| \sigma_x - B_y(t) B_{iz}(t) B_x(t) \sigma_y |i\rangle\langle i| \sigma_y) + \left( V_{10}^{\Re}(t) \right)^2 \sum_i (|i\rangle\langle i| B_x^2(t) B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_y(t) B_{iz}(t) - i\sigma_z |i\rangle\langle i| \quad (688) \\
& \times B_y(t) B_x(t) B_{iz}(t) + |i\rangle\langle i| B_y^2(t) B_{iz}(t)) + \left( V_{10}^{\Re}(t) \right)^3 (\sigma_x B_x^3(t) + \sigma_y B_x^2(t) B_y(t) - \sigma_y B_x(t) B_y(t) B_x(t) + \sigma_x B_x(t) B_y^2(t) + \sigma_y B_y(t) B_x^2(t) \quad (689) \\
& - \sigma_x B_y(t) B_x(t) B_y(t) + \sigma_x B_y^2(t) B_x(t) + \sigma_y B_y^3(t)) + V_{10}^{\Re}(t) V_{10}^{\Im}(t) \sum_i (|i\rangle\langle i| B_x(t) B_y(t) B_{iz}(t) - i\sigma_z |i\rangle\langle i| B_x^2(t) B_{iz}(t) - i|i\rangle\langle i| \sigma_z B_y^2(t) \quad (690) \\
& \times B_{iz}(t) + i|i\rangle\langle i| \sigma_z B_y(t) B_x(t) B_{iz}(t)) + V_{10}^{\Re}(t) \left( V_{10}^{\Im}(t) \right)^2 (\sigma_x B_x(t) B_y^2(t) + \sigma_x B_x^3(t) - \sigma_y B_x(t) B_y(t) B_x(t) + \sigma_y B_x^2(t) B_y(t) + \sigma_y B_y^3(t) \quad (691) \\
& + \sigma_y B_y(t) B_x^2(t) + \sigma_x B_y^2(t) B_x(t) - \sigma_x B_y(t) B_x(t) B_y(t)) + V_{10}^{\Im}(t) \sum_i (\sigma_x |i\rangle\langle i| B_y(t) B_{iz}^2(t) - \sigma_y |i\rangle\langle i| B_x(t) B_{iz}^2(t)) + V_{10}^{\Im}(t) V_{10}^{\Im}(t) \quad (692) \\
& \times (\sigma_x |i\rangle\langle i| \sigma_x B_y(t) B_{iz}(t) B_x(t) + \sigma_x |i\rangle\langle i| \sigma_y B_y(t) B_{iz}(t) B_y(t) - \sigma_y |i\rangle\langle i| \sigma_x B_y(t) B_{iz}(t) B_y(t) - \sigma_x |i\rangle\langle i| \sigma_y B_y(t) B_{iz}(t) B_y(t)) + \left( V_{10}^{\Im}(t) \right)^2 \quad (693) \\
& \times (\sigma_x |i\rangle\langle i| \sigma_x B_y(t) B_{iz}(t) B_y(t) - \sigma_x |i\rangle\langle i| \sigma_y B_y(t) B_{iz}(t) B_x(t) - \sigma_y |i\rangle\langle i| \sigma_x B_x(t) B_{iz}(t) B_y(t) + \sigma_y |i\rangle\langle i| \sigma_y B_x(t) B_{iz}(t) B_x(t)) + V_{10}^{\Re}(t) \quad (694)
\end{aligned}$$

$$\times V_{10}^{\mathfrak{S}}(t) \sum_i (|i\rangle\langle i| B_y(t) B_x(t) B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_y^2(t) B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_x^2(t) B_{iz}(t) - |i\rangle\langle i| B_x(t) B_y(t) B_{iz}(t)) + V_{10}^{\mathfrak{S}}(t) \left(V_{10}^{\mathfrak{R}}(t)\right)^2 \quad (695)$$

$$\times (\sigma_x B_y(t) B_x^2(t) + \sigma_y B_y(t) B_x(t) B_y(t) - \sigma_y B_y^2(t) B_x(t) + \sigma_x B_y^3(t) - \sigma_y B_x^3(t) + \sigma_x B_x^2(t) B_y(t) - \sigma_x B_x(t) B_y(t) B_x(t) - \sigma_y B_x(t) B_y^2(t)) \quad (696)$$

$$+ \left(V_{10}^{\mathfrak{S}}(t)\right)^2 \sum_i (|i\rangle\langle i| B_y^2(t) B_{iz}(t) - i\sigma_z |i\rangle\langle i| B_y(t) B_x(t) B_{iz}(t) + i\sigma_z |i\rangle\langle i| B_x(t) B_y(t) B_{iz}(t) + |i\rangle\langle i| B_x^2(t) B_{iz}(t)) + \left(V_{10}^{\mathfrak{S}}(t)\right)^3 (\sigma_x B_y^3(t) \quad (697)$$

$$+ \sigma_x B_y(t) B_x^2(t) - \sigma_y B_y^2(t) B_x(t) + \sigma_y B_y(t) B_x(t) B_y(t) - \sigma_y B_x(t) B_y^2(t) - \sigma_y B_x^3(t) - \sigma_x B_x(t) B_y(t) B_x(t) + \sigma_x B_x^2(t) B_y(t)) . \quad (698)$$

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