

A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1)$$

$$H_S(t) = \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|, \quad (2)$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (3)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (4)$$

For the states $|0\rangle, |1\rangle$ we have the orthonormal condition:

$$\langle i|j\rangle = \delta_{ij} \quad (5)$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where V is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) \quad (6)$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}} \quad (7)$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \quad (8)$$

$$V = |0\rangle\langle 0| \hat{\varphi}_0 + |1\rangle\langle 1| \hat{\varphi}_1. \quad (9)$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6) will give us the following result:

$$e^{\pm V} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1)} \quad (10)$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + \dots \quad (11)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + \dots \quad (12)$$

$$= |0\rangle\langle 0| \left(1 \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm \dots\right) + |1\rangle\langle 1| \left(1 \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm \dots\right) \quad (13)$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} \quad (14)$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*b_{\mathbf{k}})} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*b_{\mathbf{k}})} \quad (15)$$

$$= |0\rangle\langle 0|B_{0\pm} + |1\rangle\langle 1|B_{1\pm}, \quad (16)$$

$$B_{i\pm} \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (17)$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} e^{\frac{t^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} e^{-\frac{t^4}{24}([[[X,Y],X],X] + 3[[[X,Y],X],Y] + 3[[[X,Y],Y],Y])} \dots \quad (18)$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all \mathbf{k}', \mathbf{k} and i, j we can show making $t = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right)} = e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right]} \dots \quad (19)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}0} \dots \quad (20)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} \quad (21)$$

By induction of this result we can write expression of $B_{i\pm}$ as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}})$:

$$B_{i\pm} = \prod_{\mathbf{k}} D\left(\pm \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \quad (22)$$

$$D(\pm v_{i\mathbf{k}}) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (23)$$

this will help us to write operators O in the variational frame :

$$\overline{O} \equiv e^V O e^{-V}. \quad (24)$$

We use the following identities:

$$\overline{|0\rangle\langle 0|} = e^V |0\rangle\langle 0| e^{-V} \quad (25)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |0\rangle\langle 0| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (26)$$

$$= (|0\rangle\langle 0|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (27)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (28)$$

$$= |0\rangle\langle 0|B_{0+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (29)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}B_{1-} \quad (30)$$

$$= |0\rangle\langle 0|, \quad (31)$$

$$\overline{|1\rangle\langle 1|} = (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |1\rangle\langle 1| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (32)$$

$$= (|0\rangle\langle 0|1\rangle\langle 1|B_{0+} + |1\rangle\langle 1|1\rangle\langle 1|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (33)$$

$$= |1\rangle\langle 1|B_{1+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (34)$$

$$= |1\rangle\langle 1|0\rangle\langle 0|B_{1+}B_{0-} + B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} \quad (35)$$

$$= B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} \quad (36)$$

$$= |1\rangle\langle 1|B_{1+}B_{1-} \quad (37)$$

$$= |1\rangle\langle 1|, \quad (38)$$

$$\overline{|0\rangle\langle 1|} = e^V |0\rangle\langle 1| e^{-V} \quad (39)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |0\rangle\langle 1| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (40)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_{0+} + |1\rangle\langle 1|B_{1+}|0\rangle\langle 1|) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (41)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_{0+} + |1\rangle\langle 1|0\rangle\langle 1|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (42)$$

$$= |0\rangle\langle 1|B_{0+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (43)$$

$$= |0\rangle\langle 1|0\rangle\langle 0|B_{0+}B_{0-} + |0\rangle\langle 1|1\rangle\langle 1|B_{0+}B_{1-} \quad (44)$$

$$= |0\rangle\langle 1|B_{0+}B_{1-}, \quad (45)$$

$$\overline{|1\rangle\langle 0|} = e^V |1\rangle\langle 0| e^{-V} \quad (46)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |1\rangle\langle 0| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (47)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (48)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_{0+} + |1\rangle\langle 1|1\rangle\langle 0|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (49)$$

$$= |1\rangle\langle 0|B_{1+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (50)$$

$$= |1\rangle\langle 0|B_{1+}|0\rangle\langle 0|B_{0-} + |1\rangle\langle 0|B_{1+}|1\rangle\langle 1|B_{1-} \quad (51)$$

$$= |1\rangle\langle 0|B_{1+}B_{0-} + |1\rangle\langle 0|1\rangle\langle 1|B_{1+}B_{1-} \quad (52)$$

$$= |1\rangle\langle 0|B_{1+}B_{0-}, \quad (53)$$

$$\overline{b_{\mathbf{k}}} = e^V b_{\mathbf{k}} e^{-V} \quad (54)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) b_{\mathbf{k}} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (55)$$

$$= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-}|0\rangle\langle 0| + |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}|1\rangle\langle 1|B_{1-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-}|1\rangle\langle 1| \quad (56)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}B_{1-} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-} \quad (57)$$

$$= |0\rangle\langle 0| \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (58)$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - |1\rangle\langle 1| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (59)$$

$$= b_{\mathbf{k}} - |1\rangle\langle 1| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (60)$$

$$\overline{b_{\mathbf{k}}^\dagger} = e^V b_{\mathbf{k}}^\dagger e^{-V} \quad (61)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) b_{\mathbf{k}}^\dagger (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (62)$$

$$= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^\dagger B_{0-}|0\rangle\langle 0| + |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^\dagger|1\rangle\langle 1|B_{1-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^\dagger|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^\dagger B_{1-}|1\rangle\langle 1| \quad (63)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^\dagger B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}^\dagger B_{1-} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}^\dagger B_{0-} + |1\rangle\langle 1|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^\dagger B_{1-} \quad (64)$$

$$= |0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \quad (65)$$

$$= b_{\mathbf{k}}^\dagger - |1\rangle\langle 1| \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} - |0\rangle\langle 0| \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (66)$$

We have used the following:

$$B_{i+} b_{\mathbf{k}} B_{i-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (67)$$

$$B_{i+} b_{\mathbf{k}}^{\dagger} B_{i-} = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (68)$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t) |0\rangle\langle 0|} = \varepsilon_0(t) |0\rangle\langle 0|, \quad (69)$$

$$\overline{\varepsilon_1(t) |1\rangle\langle 1|} = \varepsilon_1(t) |1\rangle\langle 1|, \quad (70)$$

$$\overline{V_{10}(t) |1\rangle\langle 0|} = V_{10}(t) |1\rangle\langle 0| B_{1+} B_{0-}, \quad (71)$$

$$\overline{V_{01}(t) |0\rangle\langle 1|} = V_{01}(t) |0\rangle\langle 1| B_{0+} B_{1-}, \quad (72)$$

$$\overline{g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}}} = g_{i\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) + g_{i\mathbf{k}}^* \left(|0\rangle\langle 0| \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (73)$$

$$= g_{i\mathbf{k}} \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (74)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| \quad (75)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |0\rangle\langle 0| - \left(g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |1\rangle\langle 1|, \quad (76)$$

$$\overline{|0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}})} = (|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}) |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}) \quad (77)$$

$$= |0\rangle\langle 0| B_{0+} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) |0\rangle\langle 0| B_{0-} \quad (78)$$

$$= |0\rangle\langle 0| B_{0+} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) B_{0-} \quad (79)$$

$$= |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (80)$$

$$\overline{|1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}})} = (|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}) |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}) \quad (81)$$

$$= |1\rangle\langle 1| B_{1+} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) |1\rangle\langle 1| B_{1-} \quad (82)$$

$$= |1\rangle\langle 1| B_{1+} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) B_{1-} \quad (83)$$

$$= |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (84)$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \omega_{\mathbf{k}} (|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} (|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}) \quad (85)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| B_{0+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{0-} + |1\rangle\langle 1| B_{1+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{1-} \right) \quad (86)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \Pi_{\mathbf{k}'} D \left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D \left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \left(|0\rangle\langle 0| \Pi_{\mathbf{k}'} D \left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D \left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right) \quad (87)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| D \left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \Pi_{\mathbf{k}' \neq \mathbf{k}} D \left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) D \left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) + |1\rangle\langle 1| D \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \Pi_{\mathbf{k}' \neq \mathbf{k}} D \left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) D \left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right) \quad (88)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| D \left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \mathbb{I} + |1\rangle\langle 1| D \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \mathbb{I} \right) \quad (89)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (92)$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (93)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (94)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \quad (95)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger) \right). \quad (96)$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)} |0\rangle\langle 0| + \overline{\varepsilon_1(t)} |1\rangle\langle 1| + \overline{V_{10}(t)} |1\rangle\langle 0| + \overline{V_{01}(t)} |0\rangle\langle 1| \quad (97)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_{1+} B_{0-} + V_{01}(t) |0\rangle\langle 1| B_{0+} B_{1-}, \quad (98)$$

$$\overline{H_I} = \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (99)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (100)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (101)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) + \sum_{\mathbf{k}} |1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (102)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (103)$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) \right) \quad (104)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) \right). \quad (105)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{j+} B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}) \right) \quad (106)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_S} + \overline{H_I} + \overline{H_B} \quad (107)$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (108)$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \quad (109)$$

We assume that the bath is at equilibrium with inverse temperature $\beta = 1/k_B T$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})} \quad (110)$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (111)$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (112)$$

So the product of the matrix representation of b^\dagger and b is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (113)$$

$$= \sum_{j=0}^{\infty} -j\beta\omega |j\rangle\langle j|, \quad (114)$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j .

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \quad (115)$$

$$\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \quad (116)$$

The value of $\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\text{Tr} \left(\exp \left(-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) \right) = \text{Tr} \left(\sum_{j_{\mathbf{k}}} \exp \left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (117)$$

$$= \sum_{j_{\mathbf{k}}} \exp \left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) \quad (118)$$

$$= \sum_{j_{\mathbf{k}}} \exp \left(-\beta \omega_{\mathbf{k}} \right)^{j_{\mathbf{k}}} \quad (119)$$

$$= \frac{1}{1 - \exp \left(-\beta \omega_{\mathbf{k}} \right)} \quad (\text{by geometric series}) \quad (120)$$

$$\equiv f_{\text{Bose-Einstein}} \left(-\beta \omega_{\mathbf{k}} \right). \quad (121)$$

$$\text{Tr} \left(\exp \left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right) \right) = \text{Tr} \left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp \left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (122)$$

$$= \prod_{\mathbf{k}} \text{Tr} \left(\sum_{j_{\mathbf{k}}} \exp \left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (123)$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}} \left(-\beta \omega_{\mathbf{k}} \right). \quad (124)$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr} (e^{-\beta H_B})} \quad (125)$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp \left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}} \left(-\beta \omega_{\mathbf{k}} \right)} \quad (126)$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp \left(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}} \right) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}} \left(-\beta \omega_{\mathbf{k}} \right)}. \quad (127)$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle, \quad (128)$$

$$b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle. \quad (129)$$

Then we can prove that $\langle B_{iz} \rangle_{\overline{H_B}} = 0$ using the following property based on (128)-(129):

$$\langle B_{iz} \rangle_{\overline{H_B}} = \text{Tr} (\rho_B B_{iz}) = \text{Tr} (B_{iz} \rho_B) \quad (130)$$

$$= \text{Tr} \left(\left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \right) \rho_B \right) \quad (131)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \rho_B \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \rho_B \right) \quad (132)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left(b_{\mathbf{k}}^\dagger \rho_B \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left(b_{\mathbf{k}} \rho_B \right) \quad (133)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) \quad (134)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left(b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right), \quad (135)$$

$$\text{Tr} \left(b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}}^\dagger |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}^\dagger) \quad (136)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (137)$$

$$= 0, \quad (138)$$

$$\text{Tr} \left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}) \quad (139)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (140)$$

$$= 0. \quad (141)$$

we therefore find that:

$$\langle B_{iz} \rangle_{\overline{H_B}} = 0 \quad (142)$$

Another important expected value is $B = \langle B_{\pm} \rangle_{\overline{H_B}}$, where $B_{\pm} = e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}$ is given by:

$$\langle B_{\pm} \rangle_{H_B} = \text{Tr} (\rho_B B_{\pm}) = \text{Tr} (B_{\pm} \rho_B) \quad (143)$$

$$= \text{Tr} \left(e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \rho_B \right) \quad (144)$$

$$= \prod_{\mathbf{k}} \text{Tr} (D(\pm \alpha_{\mathbf{k}}) \rho_B) \quad (145)$$

$$= \prod_{\mathbf{k}} \langle D(\pm \alpha_{\mathbf{k}}) \rangle. \quad (146)$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha) |\alpha \rangle \langle \alpha| d^2 \alpha \quad (147)$$

where $P(\alpha)$ satisfies $\int P(\alpha) d^2 \alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P(\alpha)$ is given by:

$$\langle A \rangle = \text{Tr} (A \rho) \quad (148)$$

$$= \int P(\alpha) \langle \alpha | A | \alpha \rangle d^2 \alpha \quad (149)$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N} \exp \left(-\frac{|\alpha|^2}{N} \right)$ where $N = (e^{\beta \omega} - 1)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | D(h) | \alpha \rangle d^2 \alpha \quad (150)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle d^2 \alpha \quad (151)$$

$$D(h) D(\alpha) = D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (152)$$

$$D(-\alpha) (D(h) D(\alpha)) = D(-\alpha) D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (153)$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (154)$$

$$= D(\alpha) e^{\frac{1}{2}(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2)} e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (155)$$

$$= D(\alpha) e^{(h\alpha^* - h^* \alpha)}, \quad (156)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(h) \exp(h\alpha^* - h^* \alpha) | 0 \rangle d^2 \alpha \quad (157)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle d^2 \alpha \quad (158)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \langle 0 | h \rangle d^2 \alpha \quad (159)$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (160)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \langle 0 | \exp\left(-\frac{|h|^2}{2}\right) \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle d^2 \alpha \quad (161)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \exp\left(-\frac{|h|^2}{2}\right) d^2 \alpha \quad (162)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N} + h\alpha^* - h^* \alpha\right) d^2 \alpha, \quad (163)$$

$$\alpha = x + iy, \quad (164)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 + y^2}{N} + h(x - iy) - h^*(x + iy)\right) dx dy \quad (165)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{N} + hx - h^* x\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N} - ihy - ih^* y\right) dy, \quad (166)$$

$$-\frac{x^2}{N} + hx - h^* x = -\frac{1}{N}(x^2 - Nhx + Nh^* x) \quad (167)$$

$$= -\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 + \frac{N(h^* - h)^2}{4}, \quad (168)$$

$$-\frac{y^2}{N} - ihy - ih^* y = -\frac{1}{N}(y^2 + iNhy + iNh^* y) \quad (169)$$

$$= -\frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right) - \frac{N(h + h^*)^2}{4}, \quad (170)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 - \frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right)\right) dx dy, \quad (171)$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx, \quad (172)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x + \frac{(Nh^* - Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{\left(y^2 + \frac{iN(h + h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dy \quad (173)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \left(\sqrt{2\pi} \sqrt{\frac{N}{2}} \right)^2 \quad (174)$$

$$= \exp \left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4} \right) \quad (175)$$

$$= \exp \left(-\frac{|h|^2}{2} + \frac{N(h^{*2} - 2hh^* + h^2) - N(h^2 + 2hh^* + h^{*2})}{4} \right) \quad (176)$$

$$= \exp \left(-|h|^2 \left(N + \frac{1}{2} \right) \right) \quad (177)$$

$$= \exp \left(-|h|^2 \left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2} \right) \right) \quad (178)$$

$$= \exp \left(-\frac{|h|^2}{2} \left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} \right) \right) \quad (179)$$

$$= \exp \left(-\frac{|h|^2}{2} \coth \left(\frac{\beta\omega}{2} \right) \right). \quad (180)$$

In the last line we used $\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} = \coth \left(\frac{\beta\omega}{2} \right)$. So the value of (145) using (180) is given by:

$$B = \exp \left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (181)$$

We will now force $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_+ \sigma_+$ and $B_- \sigma_-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$.

A final form of the terms of the Hamiltonian \overline{H} is:

$$\overline{H}(t) = \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| B_{jj'} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle \langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} + v_{j\mathbf{k}}) b_{\mathbf{k}} \right) \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (182)$$

$$= \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| B_{jj'} + \sum_{j\mathbf{k}} |j\rangle \langle j| B_{j\mathbf{k}} + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_{jj'} + B_{jj'} - B_{jj'}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (183)$$

$$\equiv \overline{H_S}(t) + \overline{H_I} + \overline{H_B}. \quad (184)$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_{1+} B_{0-} \rangle = B_{10} \quad (185)$$

$$= \left\langle \prod_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D \left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle \quad (186)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) D \left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \quad (187)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \right\rangle \quad (188)$$

$$= \prod_{\mathbf{k}} \left\langle D \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (189)$$

$$= \prod_{\mathbf{k}} \exp \left(-\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (190)$$

$$= \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)}. \quad (191)$$

From the definition $B_{01} = \langle B_{0+} B_{1-} \rangle$ using the displacement operator we have:

$$\langle B_{0+} B_{1-} \rangle = B_{01} \quad (192)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (193)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (194)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \right\rangle \quad (195)$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (196)$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (197)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (198)$$

This can be checked in the following way:

$$\langle B_{0+} B_{1-} \rangle = B_{01} \quad (199)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (200)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)^*} \quad (201)$$

$$= \langle B_{1+} B_{0-} \rangle^* \quad (202)$$

$$= B_{10}^*. \quad (203)$$

The parts of the Hamiltonian splitted are:

$$\overline{H_{\overline{S}}}(t) \equiv (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma_+ + V_{01}(t) B_{01} \sigma_-, \quad (204)$$

$$\overline{H_{\overline{I}}} \equiv V_{10}(t) (B_{1+} B_{0-} - B_{10}) \sigma_+ + V_{01}(t) (B_{0+} B_{1-} - B_{01}) \sigma_- + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z}, \quad (205)$$

$$\overline{H_{\overline{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (206)$$

$$= H_B. \quad (207)$$

Note that $\overline{H_{\overline{B}}}$, which is the bath acting on the effective “system” \overline{S} in the variational frame, is just the original bath, H_B , before transforming to the variational frame.

For the Hamiltonian (205) we can verify the condition $\langle \overline{H_{\overline{I}}} \rangle_{\overline{H_{\overline{B}}}} = 0$ in the following way:

$$\langle \overline{H_T} \rangle_{\overline{H_B}} = \langle \sum_{n\mathbf{k}} ((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}) |n\rangle \langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_j + B_{j'} - B_{jj'}) \rangle_{\overline{H_B}} \quad (208)$$

$$= \langle \sum_{n\mathbf{k}} ((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}) |n\rangle \langle n| \rangle_{\overline{H_B}} + \langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_j + B_{j'} - B_{jj'}) \rangle_{\overline{H_B}} \quad (209)$$

$$= \sum_{n\mathbf{k}} \left(\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + \langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle \langle n| + \sum_{j \neq j'} |j\rangle \langle j'| \left(\langle V_{jj'}(t) B_j + B_{j'} - \rangle_{\overline{H_B}} - \langle V_{jj'}(t) B_{jj'} \rangle_{\overline{H_B}} \right) \quad (210)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle \langle n| + \sum_{j \neq j'} |j\rangle \langle j'| V_{jj'}(t) \left(\langle B_j + B_{j'} - \rangle_{\overline{H_B}} - \langle B_{jj'} \rangle_{\overline{H_B}} \right) \quad (211)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle \langle n| + \sum_{j \neq j'} |j\rangle \langle j'| V_{jj'}(t) (B_{jj'} - B_{jj'}). \quad (212)$$

$$= 0. \quad (213)$$

We used (142) and (191) to evaluate the expected values.

Let's consider the following Hermitian combinations:

$$B_x = B_x^\dagger \quad (214)$$

$$= \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}, \quad (215)$$

$$B_y = B_y^\dagger \quad (216)$$

$$= \frac{B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10} - B_{01}}{2i}, \quad (217)$$

$$B_{iz} = B_{iz}^\dagger \quad (218)$$

$$= \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \quad (219)$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$\overline{H_S}(t) = (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + V_{10}(t) B_{10} \sigma_+ + V_{01}(t) B_{01} \sigma_- \quad (220)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + V_{10}(t) B_{10} \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01} \frac{\sigma_x - i\sigma_y}{2} \quad (221)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + V_{10}(t) (\Re(B_{10}(t)) + i\Im(B_{10}(t))) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) (\Re(B_{10}(t)) - i\Im(B_{10}(t))) \frac{\sigma_x - i\sigma_y}{2} \quad (222)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \Re(B_{10}(t)) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) + i\Im(B_{10}(t)) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) \quad (223)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \Re(B_{10}(t)) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + i\Im(B_{10}(t)) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (224)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \Re(B_{10}(t)) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + i\Im(B_{10}(t)) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (225)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \Re(B_{10}(t)) (\sigma_x \Re(V_{10}(t)) - \sigma_y \Im(V_{10}(t))) + i\Im(B_{10}(t)) (i\sigma_x \Im(V_{10}(t)) + i\sigma_y \Re(V_{10}(t))) \quad (226)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + (\sigma_x \Re(B_{10}(t)) \Re(V_{10}(t)) - \sigma_y \Re(B_{10}(t)) \Im(V_{10}(t))) - (\sigma_x \Im(B_{10}(t)) \Im(V_{10}(t)) + \sigma_y \Im(B_{10}(t)) \Re(V_{10}(t))) \quad (227)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \sigma_x (\Re(B_{10}(t)) \Re(V_{10}(t)) - \Im(B_{10}(t)) \Im(V_{10}(t))) - \sigma_y (\Re(B_{10}(t)) \Im(V_{10}(t)) + \Im(B_{10}(t)) \Re(V_{10}(t))) \quad (228)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \sigma_x (B_{10}^\Re(t) V_{10}^\Re(t) - B_{10}^\Im(t) V_{10}^\Im(t)) - \sigma_y (B_{10}^\Re(t) V_{10}^\Im(t) + B_{10}^\Im(t) V_{10}^\Re(t)). \quad (229)$$

$$\overline{H_I} V_{10}(t)(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}) + V_{01}(t)(\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} \quad (230)$$

$$= |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} + (\Re V_{10}(t) + i\Im V_{10}(t))(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}) + (\Re(V_{10}(t)) - i\Im(V_{10}(t)))(\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) \quad (231)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + \Re(V_{10}(t))(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} + \sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) + i\Im(V_{10}(t))(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} - \sigma_- B_{0+} B_{1-} + \sigma_- B_{01}) \quad (232)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + \Re(V_{10}(t))\left(\frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - i\sigma_y}{2} B_{01}\right) \quad (233)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + \Re V_{10}(t)\left(\frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - i\sigma_y}{2} B_{01}\right) + i\Im(V_{10}(t))\left(\frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} - \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - i\sigma_y}{2} B_{01}\right) \quad (234)$$

$$+ \frac{\sigma_x - i\sigma_y}{2} B_{01}) \quad (235)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} + i\sigma_y \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2}\right) + V_{10}^{\Im}(t)\left(\sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} \quad (236)$$

$$+ i\sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \quad (237)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t)\left(i\sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \quad (238)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t)\left(i^2 \sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \quad (239)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t)\left(i^2 \sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}\right) \quad (240)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t)(i^2 \sigma_x (-B_y) - \sigma_y B_x) \quad (241)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t)(\sigma_x B_y - \sigma_y B_x). \quad (242)$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right) + \langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} + O \left(\langle \overline{H_I}^2 \rangle_{\overline{H_S}(t) + \overline{H_B}} \right). \quad (243)$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_B (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = 0. \quad (244)$$

Using this condition and given that $[\overline{H_S}(t), \overline{H_B}] = 0$, we have:

$$e^{-\beta(\overline{H_S}(t) + \overline{H_B})} = e^{-\beta \overline{H_S}(t)} e^{-\beta \overline{H_B}}. \quad (245)$$

Then using the fact that $\overline{H_S}(t)$ and $\overline{H_B}$ relate to different Hilbert spaces, we obtain:

$$\text{Tr} \left(e^{-\beta \overline{H_S}(t)} e^{-\beta \overline{H_B}} \right) = \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right). \quad (246)$$

So Eq. (244) becomes:

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (247)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (248)$$

$$= -\frac{1}{\beta} \frac{\partial \left(\ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right) + \ln \left(\text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right) \right)}{\partial v_{i\mathbf{k}}} \quad (249)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (250)$$

$$= 0 \quad (\text{by Eq. (244)}). \quad (251)$$

But since $\bar{H}_B = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right) \right)}{\partial v_{i\mathbf{k}}} = \frac{1}{e^{-\beta \bar{H}_B(t)}} \frac{\partial \text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right)}{\partial v_{i\mathbf{k}}} \quad (252)$$

$$= 0. \quad (253)$$

This means we need to impose:

$$\frac{\partial \text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right)}{\partial v_{i\mathbf{k}}} = 0. \quad (254)$$

First we look at:

$$-\beta \bar{H}_B(t) = -\beta \left((\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma_+ + V_{01}(t) B_{01} \sigma_- \right). \quad (255)$$

Then the eigenvalues of $-\beta \bar{H}_B(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr} \left(-\beta \bar{H}_B(t) \right) + \text{Det} \left(-\beta \bar{H}_B(t) \right) = 0. \quad (256)$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr} \left(\bar{H}_B(t) \right), \quad (257)$$

$$\eta \equiv \sqrt{\left(\text{Tr} \left(\bar{H}_B(t) \right) \right)^2 - 4 \text{Det} \left(\bar{H}_B(t) \right)}. \quad (258)$$

The solutions of the equation (256) are:

$$\lambda = \beta \frac{-\text{Tr} \left(\bar{H}_B(t) \right) \pm \sqrt{\left(\text{Tr} \left(\bar{H}_B(t) \right) \right)^2 - 4 \text{Det} \left(\bar{H}_B(t) \right)}}{2} \quad (259)$$

$$= \beta \frac{-\varepsilon(t) \pm \eta(t)}{2} \quad (260)$$

$$= -\beta \frac{\varepsilon(t) \mp \eta(t)}{2}. \quad (261)$$

The value of $\text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right) = \exp \left(-\frac{\varepsilon(t) \beta}{2} \right) \exp \left(\frac{\eta(t) \beta}{2} \right) + \exp \left(-\frac{\varepsilon(t) \beta}{2} \right) \exp \left(-\frac{\eta(t) \beta}{2} \right) \quad (262)$$

$$= 2 \exp \left(-\frac{\varepsilon(t) \beta}{2} \right) \cosh \left(\frac{\eta(t) \beta}{2} \right). \quad (263)$$

Given that $v_{i\mathbf{k}}$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right) = A(\varepsilon(t), \eta(t))$ to calculate $\frac{\partial \text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right)}{\partial \Re(v_{i\mathbf{k}})}$ can lead to:

$$\frac{\partial \text{Tr} \left(e^{-\beta \bar{H}_B(t)} \right)}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial \left(2 \exp \left(-\frac{\varepsilon(t) \beta}{2} \right) \cosh \left(\frac{\eta(t) \beta}{2} \right) \right)}{\partial \Re(v_{i\mathbf{k}})} \quad (264)$$

$$= 2 \left(-\frac{\beta}{2} \frac{\partial \varepsilon(t)}{\partial \Re(v_{i\mathbf{k}})} \right) \exp \left(-\frac{\varepsilon(t) \beta}{2} \right) \cosh \left(\frac{\eta(t) \beta}{2} \right) + 2 \left(\frac{\beta}{2} \frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})} \right) \exp \left(-\frac{\varepsilon(t) \beta}{2} \right) \sinh \left(\frac{\eta(t) \beta}{2} \right) \quad (265)$$

$$= -\beta \exp \left(-\frac{\varepsilon(t) \beta}{2} \right) \left(\frac{\partial \varepsilon(t)}{\partial \Re(v_{i\mathbf{k}})} \cosh \left(\frac{\eta(t) \beta}{2} \right) - \frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})} \sinh \left(\frac{\eta(t) \beta}{2} \right) \right). \quad (266)$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial \varepsilon(t)}{\partial \Re(v_{i\mathbf{k}})} \cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})} \sinh\left(\frac{\eta(t)\beta}{2}\right) = 0. \quad (267)$$

The derivates included in the expression given are related to:

$$\langle B_{0+} B_{1-} \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (268)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right)^* \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (269)$$

$$= \langle B_{1+} B_{0-} \rangle^*, \quad (270)$$

$$R_i = \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (271)$$

$$= \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (272)$$

$$\langle B_{0+} B_{1-} \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (273)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{1}{2\omega_{\mathbf{k}}^2} (v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*) \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (274)$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = (\Re(v_{0\mathbf{k}}) - i\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}}) + i\Im(v_{1\mathbf{k}})) - (\Re(v_{0\mathbf{k}}) + i\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}}) - i\Im(v_{1\mathbf{k}})) \quad (275)$$

$$= \Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}) + i\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}) - i\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}) + \Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}) - (\Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}) - i\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}) + i\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}) + \Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})) \quad (276)$$

$$= 2i(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})), \quad (277)$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}} - v_{0\mathbf{k}})(v_{1\mathbf{k}} - v_{0\mathbf{k}})^* \quad (278)$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - (v_{1\mathbf{k}} v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}}) \quad (279)$$

$$= (\Re(v_{1\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}))^2 + (\Re(v_{0\mathbf{k}}))^2 + (\Im(v_{0\mathbf{k}}))^2 - ((\Re(v_{1\mathbf{k}}) + i\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}}) - i\Im(v_{0\mathbf{k}})) + (\Re(v_{1\mathbf{k}}) - i\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}}) + i\Im(v_{0\mathbf{k}}))) \quad (280)$$

$$= (\Re(v_{1\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}))^2 + (\Re(v_{0\mathbf{k}}))^2 + (\Im(v_{0\mathbf{k}}))^2 - 2(\Re(v_{1\mathbf{k}})\Re(v_{0\mathbf{k}}) + \Im(v_{1\mathbf{k}})\Im(v_{0\mathbf{k}})) \quad (281)$$

$$= (\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))^2. \quad (282)$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left(\frac{\Re(v_{i\mathbf{k}})^2 + \Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{\Re(v_{i\mathbf{k}}) - i\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{\Re(v_{i\mathbf{k}}) + i\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} \right) \right) \quad (283)$$

$$= \sum_{\mathbf{k}} \left(\frac{\Re(v_{i\mathbf{k}})^2 + \Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}} - \Re(v_{i\mathbf{k}}) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i\Im(v_{i\mathbf{k}}) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (284)$$

$$\langle B_{0+} B_{1-} \rangle = \left(\prod_{\mathbf{k}} \exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (285)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{2i(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (286)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{i(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (287)$$

Calculating the derivates respect to $\Re(\alpha_{i\mathbf{k}})$ and $\Im(\alpha_{i\mathbf{k}})$ we have:

$$\frac{\partial \varepsilon(t)}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial(\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial \Re(v_{i\mathbf{k}})} \quad (288)$$

$$= \frac{\partial \left(\left(\frac{\Re(v_{i\mathbf{k}})^2 + \Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}} - \Re(v_{i\mathbf{k}}) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i \Im(v_{i\mathbf{k}}) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial \Re(v_{i\mathbf{k}})} \quad (289)$$

$$= \frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \quad (290)$$

$$\frac{\partial |B_{10}|^2}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial \left(\exp \left(- \sum_{\mathbf{k}} \frac{(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right)}{\partial \Re(v_{i\mathbf{k}})} \quad (291)$$

$$= - \frac{2(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2} \frac{\partial(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))}{\partial \Re(v_{i\mathbf{k}})} \exp \left(- \sum_{\mathbf{k}} \frac{(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (292)$$

$$= - \frac{2(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2} \frac{\partial(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))}{\partial \Re(v_{i\mathbf{k}})} |B_{10}|^2, \quad (293)$$

$$\frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial \sqrt{\left(\text{Tr}(\overline{H_S}(t)) \right)^2 - 4 \text{Det}(\overline{H_S}(t))}}{\partial \Re(v_{i\mathbf{k}})} \quad (294)$$

$$= \frac{2 \text{Tr}(\overline{H_S}(t)) \frac{\partial \text{Tr}(\overline{H_S}(t))}{\partial \Re(v_{i\mathbf{k}})} - 4 \frac{\partial \text{Det}(\overline{H_S}(t))}{\partial \Re(v_{i\mathbf{k}})}}{2 \sqrt{\left(\text{Tr}(\overline{H_S}(t)) \right)^2 - 4 \text{Det}(\overline{H_S}(t))}} \quad (295)$$

$$= \frac{\varepsilon(t) \left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial \left((\varepsilon_1(t) + R_1)(\varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2 \right)}{\partial \Re(v_{i\mathbf{k}})}}{\eta(t)} \quad (296)$$

$$= \frac{\varepsilon(t) \left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i) \left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2} \frac{\partial(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))}{\partial \Re(v_{i\mathbf{k}})} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (297)$$

$$= \frac{\varepsilon(t) \left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i) \left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(\Re(v_{i\mathbf{k}}) - \Re(v_{i'\mathbf{k}}))}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (298)$$

$$= \frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(- \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right. \quad (299)$$

$$\left. + 4 \frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (300)$$

From the equation (267) and replacing the derivates obtained we have:

$$\tanh \left(\frac{\beta \eta(t)}{2} \right) = \frac{\frac{\partial \varepsilon(t)}{\partial \Re(v_{i\mathbf{k}})}}{\frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})}} \quad (301)$$

$$= \frac{\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{2\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}}{\frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} \left(\frac{\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + 2 \frac{(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}} + 2 \frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) - \frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}} \varepsilon(t)}{\eta(t)}}} \quad (302)$$

Rearrannging this equation will lead to:

$$\tanh \left(\frac{\beta \eta(t)}{2} \right) = \frac{(2v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}} - g_{i\mathbf{k}}^*) \eta(t)}{v_{i\mathbf{k}}^{\Re} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} + g_{i\mathbf{k}}^*) (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4 \frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (303)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re}) \eta(t)}{v_{i\mathbf{k}}^{\Re} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re} (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4 \frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (304)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re}) \eta(t)}{v_{i\mathbf{k}}^{\Re} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 4 \frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (305)$$

$$= \frac{(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re}) \eta(t)}{v_{i\mathbf{k}}^{\Re} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (306)$$

Separating (305) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(\Re(v_{i\mathbf{k}}) - \Re(g_{i\mathbf{k}}))\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = \Re(v_{i\mathbf{k}}) \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth(\frac{\beta\omega_{\mathbf{k}}}{2})}{\omega_{\mathbf{k}}} \right) - \Re(g_{i\mathbf{k}}) (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (307)$$

$$\Re(v_{i\mathbf{k}}) - \Re(g_{i\mathbf{k}}) = \Re(v_{i\mathbf{k}}) \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \Re(g_{i\mathbf{k}}) (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \quad (308)$$

$$+ 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (309)$$

$$\Re(g_{i\mathbf{k}}) \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{\Re(v_{i'\mathbf{k}})}{\Re(g_{i\mathbf{k}})} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (310)$$

$$\Re(v_{i\mathbf{k}}) = \frac{\Re(g_{i\mathbf{k}}) \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{\Re(v_{i'\mathbf{k}})}{\Re(g_{i\mathbf{k}})} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (311)$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon(t)}{\partial \Im(v_{i\mathbf{k}})} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial \Im(v_{i\mathbf{k}})} \quad (312)$$

$$= \frac{\partial \left(\left(\frac{\Re(v_{i\mathbf{k}})^2 + \Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}} - \Re(v_{i\mathbf{k}}) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i \Im(v_{i\mathbf{k}}) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial \Im(v_{i\mathbf{k}})} \quad (313)$$

$$= 2 \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (314)$$

$$\frac{\partial |B_{10}|^2}{\partial \Im(v_{i\mathbf{k}})} = \frac{\partial \left(\exp \left(- \sum_{\mathbf{k}} \frac{(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)}{\partial \Im(v_{i\mathbf{k}})} \quad (315)$$

$$= - \frac{2(\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2} \frac{\partial (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))}{\partial \Im(v_{i\mathbf{k}})} \exp \left(- \sum_{\mathbf{k}} \frac{(\Re(v_{1\mathbf{k}}) - \Re(v_{0\mathbf{k}}))^2 + (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (316)$$

$$= - \frac{2(\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2} \frac{\partial (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))}{\partial \Im(v_{i\mathbf{k}})} |B_{10}|^2 \quad (317)$$

$$\frac{\partial \eta(t)}{\partial \Re(v_{i\mathbf{k}})} = \frac{\partial \sqrt{(\text{Tr}(\overline{H_S}(t)))^2 - 4 \text{Det}(\overline{H_S}(t))}}{\partial \Re(v_{i\mathbf{k}})} \quad (318)$$

$$= \frac{2 \text{Tr}(\overline{H_S}(t)) \frac{\partial \text{Tr}(\overline{H_S}(t))}{\partial \Re(v_{i\mathbf{k}})} - 4 \frac{\partial \text{Det}(\overline{H_S}(t))}{\partial \Re(v_{i\mathbf{k}})}}{2 \sqrt{(\text{Tr}(\overline{H_S}(t)))^2 - 4 \text{Det}(\overline{H_S}(t))}} \quad (319)$$

$$= \frac{\varepsilon(t) \left(2 \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1)(\varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial \Re(v_{i\mathbf{k}})}}{\eta(t)} \quad (320)$$

$$= \frac{\varepsilon(t) \left(2 \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i) \left(2 \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \frac{2(\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2} \frac{\partial (\Im(v_{1\mathbf{k}}) - \Im(v_{0\mathbf{k}}))}{\partial \Re(v_{i\mathbf{k}})} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (321)$$

$$= \frac{\varepsilon(t) \left(2 \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i) \left(2 \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \frac{2(\Im(v_{i\mathbf{k}}) - \Im(v_{i'\mathbf{k}}))}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (322)$$

$$= \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(-i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i) i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (323)$$

$$+ 4 \frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (324)$$

From the equation (267) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial \varepsilon(t)}{\partial \Im(v_{i\mathbf{k}})}}{\frac{\partial \eta(t)}{\partial \Im(v_{i\mathbf{k}})}} \quad (325)$$

$$= \frac{2 \frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{2}{\eta(t)} \left(\frac{\Im(g_{i\mathbf{k}}^*)}{\omega_{\mathbf{k}}} \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{\Im(g_{i\mathbf{k}}^*)}{\omega_{\mathbf{k}}} + 2 \frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)} \quad (326)$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{(2\Im(v_{i\mathbf{k}}) - i(g_{i\mathbf{k}}^* - g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - i(g_{i\mathbf{k}}^* - g_{i\mathbf{k}})(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (327)$$

$$= \frac{2(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2\Im(g_{i\mathbf{k}})(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (328)$$

$$= \frac{2(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2\Im(g_{i\mathbf{k}})(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 4\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (329)$$

$$= \frac{(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}}))\eta(t)}{\Im(v_{i\mathbf{k}})\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \Im(g_{i\mathbf{k}})(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (330)$$

Separating (330) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}}))\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = \Im(v_{i\mathbf{k}})\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \Im(g_{i\mathbf{k}})(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (331)$$

$$\Im(v_{i\mathbf{k}}) - \Im(g_{i\mathbf{k}}) = \Im(v_{i\mathbf{k}})\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \quad (332)$$

$$- \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\Im(g_{i\mathbf{k}})(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (333)$$

$$\Im(v_{i\mathbf{k}}) = \frac{\Im(g_{i\mathbf{k}})\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (334)$$

$$\Im(v_{i\mathbf{k}}) = \frac{\Im(g_{i\mathbf{k}})\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (335)$$

The variational parameters are:

$$v_{i\mathbf{k}} = \Re(v_{i\mathbf{k}}) + i\Im(v_{i\mathbf{k}}) \quad (336)$$

$$= \frac{\Re(g_{i\mathbf{k}})\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (337)$$

$$+ i \frac{\Im(g_{i\mathbf{k}})\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (338)$$

$$= \frac{g_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (339)$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$, the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T(0) e^{-V} \quad (340)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) (\rho_S(0) \otimes \rho_B^{\text{Thermal}}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (341)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: |0\rangle\langle 0|B_{0+}|0\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \quad (342)$$

$$= |0\rangle\langle 0|B_{0+}|0\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \quad (343)$$

$$= |0\rangle\langle 0| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{0-} \quad (344)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \quad (345)$$

$$= |1\rangle\langle 1|B_{1+}\rho_B^{\text{Thermal}}B_{1-} \quad (346)$$

$$= |1\rangle\langle 1| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{1-} \quad (347)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1|: |0\rangle\langle 0|B_{0+}|0\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \quad (348)$$

$$= |0\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \quad (349)$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}B_{1-} \quad (350)$$

$$= |0\rangle\langle 1| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{1-} \quad (351)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0|: |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \quad (352)$$

$$= |1\rangle\langle 0| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{0-} \quad (353)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O}(t) \equiv U^\dagger(t)O(t)U(t) \quad (354)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (355)$$

Here \mathcal{T} denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\overline{\rho_S}(t)U(t), \text{ where} \quad (356)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\overline{\rho_T}(t)) \quad (357)$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ \Re(V_{10}(t)) & \Re(V_{10}(t)) & 1 & \Im(V_{10}(t)) & -\Im(V_{10}(t)) & 1 \end{pmatrix} \quad (358)$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_I}(t)$ as pointed in the equation (??):

$$\overline{H_I}(t) = \sum_i B_{iz} |i\rangle\langle i| + \Re(V_{10}(t)) (\sigma_x B_x + \sigma_y B_y) + \Im(V_{10}(t)) (\sigma_x B_y - \sigma_y B_x) \quad (359)$$

$$= B_{0z} |0\rangle\langle 0| + B_{1z} |1\rangle\langle 1| + \Re(V_{10}(t)) \sigma_x B_x + \Re(V_{10}(t)) \sigma_y B_y + \Im(V_{10}(t)) \sigma_x B_y - \Im(V_{10}(t)) \sigma_y B_x \quad (360)$$

$$= \sum_i C_i(t) (A_i \otimes B_i(t)) \quad (361)$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{d\widetilde{\rho}_T(t)}{dt} = -i[\widetilde{H}_I(t), \widetilde{\rho}_T(t)]. \quad (362)$$

This equation has the formal solution

$$\widetilde{\rho}_T(t) = \rho(0) - i \int_0^t [\widetilde{H}_I(s), \widetilde{\rho}_T(s)] ds. \quad (363)$$

Replacing the equation (363) in the equation (362) give us:

$$\frac{d\widetilde{\rho}_T(t)}{dt} = -i[\widetilde{H}_I(t), \rho_T(0)] - \int_0^t [\widetilde{H}_I(t), [\widetilde{H}_I(s), \widetilde{\rho}_T(s)]] ds. \quad (364)$$

This equation allow us to iterate and write in terms of a series expansion with $\rho_T(0)$ the solution as:

$$\widetilde{\rho}_T(t) = \rho_T(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \dots [\widetilde{H}_I(t_n), \rho_T(0)]] \dots] \quad (365)$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho}_{\overline{S}}(t) = \rho_{\overline{S}}(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \dots [\widetilde{H}_I(t_n), \rho_{\overline{S}}(0) \rho_B^{\text{Thermal}}]] \dots] \quad (366)$$

here we have assumed that $\rho_T(0) = \rho_{\overline{S}}(0) \otimes \rho_B^{\text{Thermal}}$. Consider the following notation:

$$\widetilde{\rho}_{\overline{S}}(t) = (1 + W_1(t) + W_2(t) + \dots) \rho_{\overline{S}}(0) \quad (367)$$

$$= W(t) \rho_{\overline{S}}(0) \quad (368)$$

in this case

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \dots [\widetilde{H}_I(t_n), (\cdot) \rho_B^{\text{Thermal}}]] \dots] \quad (369)$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{d\widetilde{\rho}_{\overline{S}}(t)}{dt} = (\dot{W}_1(t) + \dot{W}_2(t) + \dots) \rho_{\overline{S}}(0) \quad (370)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} W(t) \rho_{\overline{S}}(0) \quad (371)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} \widetilde{\rho}_{\overline{S}}(t) \quad (372)$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\text{Tr}_B [\widetilde{H}_I(t) \rho_B(0)] = 0$ so $W_1(t) = 0$. Thus, to second order and taking $W(t) \approx \mathbb{I}$ then the equation (370) becomes:

$$\frac{d\widetilde{\rho}_{\overline{S}}(t)}{dt} = -i[H_{\overline{S}}, \rho_{\overline{S}}(t)] - \int_0^t d\tau [H_I, [\widetilde{H}_I(-\tau), \rho_{\overline{S}}(t) \rho_B^{\text{Thermal}}]] \quad (373)$$

Replacing $t_1 \rightarrow t - \tau$

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\tilde{H}_I(t_1), [\tilde{H}_I(t_2), \dots [\tilde{H}_I(t_n), (\cdot) \rho_B^{\text{Thermal}}]] \dots] \quad (374)$$

Taking as reference state ρ_B^{Thermal} and truncating at second order in $\overline{H_I}(t)$, we obtain our master equation in the interaction picture in the transformed frame:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}}] \right] ds \quad (375)$$

From the interaction picture applied on $\overline{H_I}(t)$ we find:

$$\widetilde{H_I}(t) = U^\dagger(t) e^{iH_B t} \overline{H_I}(t) e^{-iH_B t} U(t) \quad (376)$$

we use the time-ordering operator \mathcal{T} because in general $\overline{H_S}(t)$ doesn't commute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{H_I}(t) = \sum_i C_i(t) \left(\widetilde{A_i}(t) \otimes \widetilde{B_i}(t) \right) \quad (377)$$

$$\widetilde{A_i}(t) = U^\dagger(t) e^{iH_B t} A_i e^{-iH_B t} U(t) \quad (378)$$

$$= U^\dagger(t) A_i U(t) e^{iH_B t} e^{-iH_B t} \quad (379)$$

$$= U^\dagger(t) A_i U(t) \mathbb{I} \quad (380)$$

$$= U^\dagger(t) A_i U(t) \quad (381)$$

$$\widetilde{B_i}(t) = U^\dagger(t) e^{iH_B t} B_i(t) e^{-iH_B t} U(t) \quad (382)$$

$$= U^\dagger(t) U(t) e^{iH_B t} B_i(t) e^{-iH_B t} \quad (383)$$

$$= \mathbb{I} e^{iH_B t} B_i(t) e^{-iH_B t} \quad (384)$$

$$= e^{iH_B t} B_i(t) e^{-iH_B t} \quad (385)$$

Here we have used the fact that $[\overline{H_S}(t), H_B] = 0$ because these operators belong to different Hilbert spaces, so $[U(t), e^{iH_B t}] = 0$.

Using the expression (377) to replace it in the equation (375)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}}] \right] ds \quad (386)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)), [\sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)), \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}}] \right] ds \quad (387)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)), \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} - \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \right] ds \quad (388)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} - \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \right) \quad (389)$$

$$- \sum_{i \in J} C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \quad (390)$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \left\langle \widetilde{B_j}(t) \widetilde{B_i}(s) \right\rangle_B \quad (391)$$

$$= \left\langle \widetilde{B_j}(\tau) \widetilde{B_i}(0) \right\rangle_B \quad (392)$$

The correlation functions relevant that appear in the equation (??) are:

$$\text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B^{\text{Thermal}} \right) = \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (393)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (394)$$

$$= \Lambda_{ji}(\tau) \quad (395)$$

$$\text{Tr}_B \left(\widetilde{B}_j(t) \rho_B^{\text{Thermal}} \widetilde{B}_i(s) \right) = \text{Tr}_B \left(\widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B^{\text{Thermal}} \right) \quad (396)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (397)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (398)$$

$$= \Lambda_{ij}(-\tau) \quad (399)$$

$$\text{Tr}_B \left(\widetilde{B}_i(s) \rho_B^{\text{Thermal}} \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B^{\text{Thermal}} \right) \quad (400)$$

$$= \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (401)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (402)$$

$$= \Lambda_{ji}(\tau) \quad (403)$$

$$\text{Tr}_B \left(\rho_B^{\text{Thermal}} \widetilde{B}_i(s) \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B^{\text{Thermal}} \right) \quad (404)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (405)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (406)$$

$$= \Lambda_{ij}(-\tau) \quad (407)$$

The cyclic property of the trace was use widely in the development of equations (393) and (407). Replacing in (??)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{ij} (C_i(t) C_j(s) (\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s)) + C_i(t) C_j(s) (\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t))) ds \quad (408)$$

$$= - \int_0^t \sum_{ij} (C_i(t) C_j(s) (\Lambda_{ij}(\tau) [\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t)] + \Lambda_{ji}(-\tau) [\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t)])) ds \quad (409)$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) = \Lambda_{ij}(\tau) [\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t)] \quad (410)$$

$$\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s) = \Lambda_{ji}(-\tau) [\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t)] \quad (411)$$

Returning to the Schroedinger picture we have:

$$U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) = U(t) \widetilde{A}_i(t) U^\dagger(t) U(t) \widetilde{A}_j(s) U^\dagger(t) U(t) \widetilde{\rho_S}(t) U^\dagger(t) \quad (412)$$

$$= \left(U(t) \widetilde{A}_i(t) U^\dagger(t) \right) \left(U(t) \widetilde{A}_j(s) U^\dagger(t) \right) \left(U(t) \widetilde{\rho_S}(t) U^\dagger(t) \right) \quad (413)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) \quad (414)$$

This procedure applying to the relevant commutators give us:

$$U(t) [\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t)] U^\dagger(t) = \left(U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) - U(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) U^\dagger(t) \right) \quad (415)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) - \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) A_i \quad (416)$$

$$= [A_i, \widetilde{A}_j(t - \tau, t) \widetilde{\rho_S}(t)] \quad (417)$$

Introducing this transformed commutators in the equation (??) allow us to obtain the master equation of the system

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau \left(C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[A_i, \widetilde{A_j}(t-\tau, t) \overline{\rho_S}(t) \right] \right. \quad (418)$$

$$\left. + C_j(t) C_i(t-\tau) \Lambda_{ji}(-\tau) \left[\overline{\rho_S}(t) \widetilde{A_j}(t-\tau, t), A_i \right] \right) \quad (419)$$

where $i, j \in \{1, 2, 3, 4, 5, 6\}$.

Here $\widetilde{A_j}(s, t) = U(t) U^\dagger(s) A_j U(s) U^\dagger(t)$ where $U(t)$ is given by (355). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. The environmental correlation functions are given by:

$$\Lambda_{ij}(\tau) = \text{Tr}_B \left(\widetilde{B_i}(t) \widetilde{B_j}(s) \rho_B^{\text{Thermal}} \right) \quad (420)$$

$$= \text{Tr}_B \left(\widetilde{B_i}(\tau) \widetilde{B_j}(0) \rho_B^{\text{Thermal}} \right) \quad (421)$$

Calculating the correlation functions allow us to obtain:

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \text{Tr}_B \left(\widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rho_B^{\text{Thermal}} \right) \quad (422)$$

$$= \int d^2\alpha P(\alpha) \langle \alpha | \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) | \alpha \rangle \quad (423)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) | \alpha \rangle d^2\alpha \quad (424)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) | \alpha \rangle d^2\alpha \quad (425)$$

$$\widetilde{B_{jz}}(\tau) = \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \quad (426)$$

$$\widetilde{B_{jz}}(0) = \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \quad (427)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \text{Tr}_B \left(\widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rho_B \right) \quad (428)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (429)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (430)$$

$$+ \text{Tr}_B \left(\sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \rho_B \right) \quad (431)$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}} = p_{j\mathbf{k}} \quad (432)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(p_{j\mathbf{k}'} b_{\mathbf{k}'}^\dagger + p_{j\mathbf{k}'}^* b_{\mathbf{k}'} \right) \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} \left(p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger + p_{j\mathbf{k}}^* b_{\mathbf{k}} \right) \rho_B \right) \quad (433)$$

$$= 0 + \text{Tr}_B \left(\sum_{\mathbf{k}} \left(p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger + p_{j\mathbf{k}}^* b_{\mathbf{k}} \right) \rho_B \right) \quad (434)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} \left(p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* p_{j\mathbf{k}} b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \rho_B \right) \quad (435)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^* p_{j\mathbf{k}} b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) \quad (436)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) \quad (437)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \text{Tr}_B \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rho_B \right) + e^{-i\omega_{\mathbf{k}}\tau} \text{Tr}_B \left(b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rho_B \right) \right) \quad (438)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}} \right) \quad (439)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) \sum_{\mathbf{k}'} ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (471)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k} \neq \mathbf{k}'} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (472)$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (473)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (474)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}}^\dagger b_{\mathbf{k}'} e^{i\omega_{\mathbf{k}}\tau} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}} b_{\mathbf{k}'}^\dagger e^{-i\omega_{\mathbf{k}}\tau} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (475)$$

$$= \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (476)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \mathbb{D}(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \mathbb{D}(\alpha_{\mathbf{k}}) \mathbb{D}(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} \mathbb{D}(\alpha_{\mathbf{k}}) | \mathbb{D} \rangle d^2 \alpha_{\mathbf{k}} \quad (477)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | \mathbb{D}(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \mathbb{D}(\alpha_{\mathbf{k}}) \mathbb{D}(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} \mathbb{D}(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (478)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (479)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}|^2 d^2 \alpha_{\mathbf{k}} \quad (480)$$

$$= N \quad (481)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \mathbb{D}(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} \mathbb{D}(\alpha_{\mathbf{k}}) \mathbb{D}(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \mathbb{D}(\alpha_{\mathbf{k}}) | \mathbb{D} \rangle d^2 \alpha_{\mathbf{k}} \quad (482)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (483)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (484)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (485)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \mathbb{D}(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger \mathbb{D}(\alpha_{\mathbf{k}}) | \mathbb{D} \rangle d^2 \alpha_{\mathbf{k}} \quad (486)$$

$$= N + 1 \quad (487)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \rangle_B = \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* e^{i\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) e^{-i\omega_{\mathbf{k}}\tau} (N+1) \quad (488)$$

$$= \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) e^{-i\omega_{\mathbf{k}}\tau} + N ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) e^{-i\omega_{\mathbf{k}}\tau})) \quad (489)$$

$$D(h') D(h) = \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) D(h' + h) \quad (490)$$

$$\langle D(h') D(h) \rangle_B = \text{Tr}_B \left(\exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) D(h' + h) \rho_B^{\text{Thermal}} \right) \quad (491)$$

$$= \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) \text{Tr}_B \left(D(h' + h) \rho_B^{\text{Thermal}} \right) \quad (492)$$

$$= \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) \frac{1}{\pi N} \int d^2 \alpha P(\alpha) \langle \alpha | D(h' + h) | \alpha \rangle \quad (493)$$

$$= \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (494)$$

$$h' = h \exp(i\omega\tau) \quad (495)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp\left(\frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau))\right) \exp\left(-\frac{|h + h \exp(i\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (496)$$

$$\frac{1}{2} |h|^2 (\exp(i\omega\tau) - \exp(-i\omega\tau)) = \frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau)) \quad (497)$$

$$= \frac{1}{2} |h|^2 (\cos(\omega\tau) + i \sin(\omega\tau) - \cos(\omega\tau) + i \sin(\omega\tau)) \quad (498)$$

$$= \frac{1}{2} |h|^2 (2i \sin(\omega\tau)) \quad (499)$$

$$= i |h|^2 \sin(\omega\tau) \quad (500)$$

$$-\frac{|h + h \exp(i\omega\tau)|^2}{2} = -|h|^2 \frac{|1 + \exp(i\omega\tau)|^2}{2} \quad (501)$$

$$= -|h|^2 \frac{|1 + \cos(\omega\tau) + i \sin(\omega\tau)|^2}{2} \quad (502)$$

$$= -|h|^2 \frac{(1 + \cos(\omega\tau))^2 + \sin^2(\omega\tau)}{2} \quad (503)$$

$$= -|h|^2 \frac{(1 + 2 \cos(\omega\tau) + \cos^2(\omega\tau)) + \sin^2(\omega\tau)}{2} \quad (504)$$

$$= -|h|^2 \frac{2 + 2 \cos(\omega\tau)}{2} \quad (505)$$

$$= -|h|^2 (1 + \cos(\omega\tau)) \quad (506)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp(i|h|^2 \sin(\omega\tau)) \exp\left(-|h|^2 (1 + \cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (507)$$

$$= \exp \left(i |h|^2 \sin(\omega\tau) - |h|^2 (1 + \cos(\omega\tau)) \coth \left(\frac{\beta\omega}{2} \right) \right) \quad (508)$$

$$= \exp \left(-|h|^2 \left(-i \sin(\omega\tau) + \cos(\omega\tau) \coth \left(\frac{\beta\omega}{2} \right) \right) \right) \exp \left(-|h|^2 \coth \left(\frac{\beta\omega}{2} \right) \right) \quad (509)$$

$$= \langle D(h) \rangle_B \exp(-\phi(\tau)) \quad (510)$$

$$\exp(-\phi(\tau)) = \exp\left(-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - i\sin(\omega\tau)\right)\right) \quad (511)$$

$$\phi(\tau) = |h|^2 \left(\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau) \right) \quad (512)$$

$$\langle D(h') D(h) \rangle_B = \exp \left(\frac{1}{2} (h' h^* - h'^* h) \right) \exp \left(-\frac{|h + h'|^2}{2} \coth \left(\frac{\beta \omega}{2} \right) \right) \quad (513)$$

$$h' = v \exp(i\omega\tau) \quad (514)$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau) \widetilde{B_{1+}B_{0-}}(0) \right\rangle_B = \text{Tr}_B \left(\widetilde{B_{1+}B_{0-}}(\tau) \widetilde{B_{1+}B_{0-}}(0) \rho_B^{\text{Thermal}} \right) \quad (515)$$

$$= \text{Tr}_B \left(\widetilde{B_{i+} B_{0-}}(\tau) \widetilde{B_{i+} B_{0-}}(0) \rho_B^{\text{Thermal}} \right) \quad (516)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (517)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (518)$$

$$= \text{Pr}_k \left(\exp \left(\frac{v_{1k}^* v_{0k} - v_{1k} v_{0k}^*}{\omega_k^2} \right) \right) \text{Tr}_B \left(\text{Pr}_k \left(D \left(\frac{v_{1k} - v_{0k}}{\omega_k} e^{i\omega_k \tau} \right) D \left(\frac{v_{1k} - v_{0k}}{\omega_k} \right) \right) \rho_B^{\text{Thermal}} \right) \quad (519)$$

$$= \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \Pi_{\mathbf{k}} \left(\exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \right) \quad (520)$$

$$= \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (521)$$

$$\langle \widetilde{B_0 + \bar{B}_1} - (\tau) \widetilde{B_0 + \bar{B}_1} - (0) \rangle_B = \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_0 \mathbf{k} v_1 \mathbf{k} - v_0 \mathbf{k} v_1^* \mathbf{k}}{\omega_{\mathbf{k}}^2} \right) \exp \left(- \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (522)$$

$$\langle \widetilde{B_{1+} B_{0-}}(\tau) \widetilde{B_{0+} B_{1-}}(0) \rangle_B = \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}} \omega_{\mathbf{k}}} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (523)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}} v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (524)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \Pi_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) \Pi_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \right) \quad (525)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} D \left(\frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) \Pi_{\mathbf{k}} D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \quad (526)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left(\left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \rho_B^{\text{Thermal}} \right) \quad (527)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left(\left(D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i(\omega \tau + \pi)} \right) D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \right) \rho_B^{\text{Thermal}} \right) \quad (528)$$

$$= \prod_{\mathbf{k}} \exp \left(- \left| \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega \tau + \pi) + \cos(\omega \tau + \pi) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (529)$$

$$= \Pi_{\mathbf{k}} \exp\left(-\left|\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}}\right|^2 \left(\sin(\omega \tau) - \cos(\omega \tau) \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)\right)\right) \exp\left(-\left|\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)\right) \quad (530)$$

$$\langle \widetilde{B_{0+}B_{1-}}(\tau) \widetilde{B_{1+}B_{0-}}(0) \rangle_B = \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (531)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \Pi_{\mathbf{k}} D \left(\frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \quad (532)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left(D \left(\frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i(\omega_{\mathbf{k}} \tau + \pi)} \right) D \left(\frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \quad (533)$$

$$= \Pi_{\mathbf{k}} \exp \left(- \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau + \pi) + \cos(\omega_{\mathbf{k}} \tau + \pi) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (534)$$

$$= \langle |B_1 + \widehat{B_0 - (\tau) B_0 + \widehat{B_1 - (0)}} \rangle_B \quad (535)$$

$$\langle \widehat{B_0 + \widehat{B_1 - (\tau) B_0 + \widehat{B_1 - (0)}}} \rangle_B = \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left(\frac{v_0^* \mathbf{k} v_1 \mathbf{k} - v_0 \mathbf{k} v_1^* \mathbf{k}}{\omega_{\mathbf{k}}^2} \right)} \right) \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B^{\text{Thermal}} \right) \quad (536)$$

$$\langle D(h)b \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h)b | \alpha \rangle \quad (537)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | D(-\alpha) D(h)b D(\alpha) | \alpha \rangle \quad (538)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h)b D(\alpha) | 0 \rangle \quad (539)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \quad (540)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \quad (541)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b + \alpha) | 0 \rangle \quad (542)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h)b | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h)\alpha | 0 \rangle \quad (543)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h)\alpha | 0 \rangle \quad (544)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(- \frac{|h|^2}{2} \right) d^2 \alpha \quad (545)$$

$$= hN \langle D(h) \rangle_B \quad (546)$$

$$\langle D(h)b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h)b^\dagger | \alpha \rangle \quad (547)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h)b^\dagger D(\alpha) | 0 \rangle \quad (548)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h)b^\dagger D(\alpha) | 0 \rangle \quad (549)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) | 0 \rangle \quad (550)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b^\dagger + \alpha^*) | 0 \rangle \quad (551)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b^\dagger + \alpha^*) | 0 \rangle \quad (552)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h)b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h)\alpha^* | 0 \rangle \quad (553)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h)\alpha^* | 0 \rangle \quad (554)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle -h | 1 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle \quad (555)$$

$$\langle -h | = \exp \left(- \frac{|-h^*|^2}{2} \right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n | \quad (556)$$

$$\langle -h | 1 \rangle = \exp \left(- \frac{|-h^*|^2}{2} \right) (-h^*) \quad (557)$$

$$\langle D(h)b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(- \frac{|-h^*|^2}{2} \right) (-h^*) + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \exp \left(- \frac{|-h^*|^2}{2} \right) \quad (558)$$

$$= -h^* \langle D(h) \rangle_B (N+1) \quad (559)$$

$$\langle bD(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | bD(h) | \alpha \rangle \quad (560)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(- \frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left(- \frac{|h|^2}{2} \right) \quad (561)$$

$$= h \langle D(h) \rangle_B (N+1) \quad (562)$$

$$\langle b^\dagger D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | b^\dagger D(h) | \alpha \rangle \quad (563)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(-\frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left(-\frac{|h|^2}{2} \right) \quad (564)$$

$$= -h^* \langle D(h) \rangle_B \quad (565)$$

$$\langle \widetilde{B_{1+} B_{0-}}(\tau) \rangle_B = \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \right\rangle_B \quad (566)$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \prod_{\mathbf{k}} \langle D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \rangle_B \quad (567)$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \prod_{\mathbf{k}} \langle D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \rangle_B \quad (568)$$

$$= \prod_{\mathbf{k}} \left(\exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \prod_{\mathbf{k}} \exp \left(-\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (569)$$

$$= B_{10} \quad (570)$$

The correlation functions can be found readily as:

$$\widetilde{B_{1+} B_{0-}}(\tau) = \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \quad (571)$$

$$\widetilde{B_{0+} B_{1-}}(\tau) = \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \quad (572)$$

$$\widetilde{B_x}(0) = \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{10}^*}{2} \quad (573)$$

$$\widetilde{B_y}(0) = \frac{B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10} - B_{10}^*}{2i} \quad (574)$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (575)$$

$$B_{iz} = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \quad (576)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} \tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} \tau} \right) \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \right\rangle_B \quad (577)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}} \tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}} \tau} (N_{\mathbf{k}} + 1) \quad (578)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \left\langle \frac{B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{10}^*}{2} \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{10}^*}{2} \right\rangle_B \quad (579)$$

$$= \frac{1}{4} \langle (B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{10}^*) (B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{10}^*) \rangle_B \quad (580)$$

$$= \frac{1}{4} \langle B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10} - B_{1+} B_{0-}(\tau) B_{10}^* + B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - \quad (581)$$

$$- B_{0+} B_{1-}(\tau) B_{10} - B_{0+} B_{1-}(\tau) B_{10}^* + B_{10} B_{10} - B_{10} B_{10}^* - B_{10}^* B_{10} + B_{10}^* B_{10} + B_{10}^* B_{10}^* \rangle \quad (582)$$

$$= \frac{1}{4} \langle B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10} - B_{1+} B_{0-}(\tau) B_{10}^* + B_{0+} B_{1-}(\tau) B_{1+} B_{0-} - \quad (583)$$

$$+ B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10} - B_{0+} B_{1-}(\tau) B_{10}^* \rangle \quad (584)$$

$$\langle \widetilde{B_{0+} B_{1-}(\tau) B_{0+} B_{1-}(0)} \rangle_B = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (585)$$

$$U = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (586)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (587)$$

$$S = \prod_{\mathbf{k}} \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (588)$$

$$\langle \widetilde{B_{0+} B_{1-}(\tau) B_{0+} B_{1-}(0)} \rangle_B = U \exp(-\phi(\tau)) S \quad (589)$$

$$\langle \widetilde{B_{1+} B_{0-}(\tau) B_{1+} B_{0-}(0)} \rangle_B = U^* \exp(-\phi(\tau)) S \quad (590)$$

$$\langle \widetilde{B_{1+} B_{0-}(\tau) B_{0+} B_{1-}(0)} \rangle_B = \exp(\phi(\tau)) S \quad (591)$$

$$\langle \widetilde{B_{0+} B_{1-}(\tau) B_{1+} B_{0-}(0)} \rangle_B = \langle \widetilde{B_{1+} B_{0-}(\tau) B_{0+} B_{1-}(0)} \rangle_B \quad (592)$$

$$\langle \widetilde{B_{1+} B_{0-}(\tau)} \rangle_B = \prod_{\mathbf{k}} \left(\exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \prod_{\mathbf{k}} \exp \left(- \frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (593)$$

$$= U^{*1/2} S^{1/2} \quad (594)$$

$$\langle \widetilde{B_x(\tau)} \widetilde{B_x(0)} \rangle_B = \frac{1}{4} \langle B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10-} B_{1+} B_{0-}(\tau) B_{10+} B_{0+} B_{1-}(\tau) B_{1+} B_{0-} - \quad (595)$$

$$+ B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10-} B_{0+} B_{1-}(\tau) B_{10+} \rangle \quad (596)$$

$$\langle \widetilde{B_x(\tau)} \widetilde{B_x(0)} \rangle_B = \frac{1}{4} \langle B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10-} B_{1+} B_{0-}(\tau) B_{10+} B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - \quad (597)$$

$$- B_{0+} B_{1-}(\tau) B_{10-} B_{0+} B_{1-}(\tau) B_{10+} \rangle \quad (598)$$

$$= \frac{1}{4} \left(U^* \exp(-\phi(\tau)) S + \exp(\phi(\tau)) S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau)) S + U \exp(-\phi(\tau)) S - B_{10}^{*2} - |B_{10}|^2 \right) \quad (599)$$

$$= \frac{1}{4} \left(2\Re(U) \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2\Re(B_{10}^2) - 2|B_{10}|^2 \right) \quad (600)$$

$$= \frac{1}{4} \left(2\Re(U) \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2\Re(U^*) S - 2S \right) \quad (601)$$

$$= \frac{S}{2} \left(\Re(U) \exp(-\phi(\tau)) + \exp(\phi(\tau)) - \Re(U^*) - 1 \right) \quad (602)$$

$$\langle \widetilde{B_y(\tau)} \widetilde{B_y(0)} \rangle_B = \left\langle \frac{B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10-} B_{10+} - B_{10-} B_{10+}}{2i} \frac{B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10-} B_{10+} - B_{10-} B_{10+}}{2i} \right\rangle_B \quad (603)$$

$$= -\frac{1}{4} \langle (B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10-} B_{10+} - B_{10-} B_{10+}) (B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10-} B_{10+} - B_{10-} B_{10+}) \rangle_B \quad (604)$$

$$= -\frac{1}{4} \langle B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}(\tau) B_{1+} B_{0-} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} \rangle \quad (605)$$

$$= -\frac{1}{4} \langle B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}(\tau) B_{1+} B_{0-} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} \rangle \quad (606)$$

$$= -\frac{1}{4} \langle B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}(\tau) B_{1+} B_{0-} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} \rangle \quad (607)$$

$$= -\frac{1}{4} \langle B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}(\tau) B_{1+} B_{0-} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} \rangle \quad (608)$$

$$= -\frac{1}{4} \langle B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}(\tau) B_{1+} B_{0-} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} \rangle \quad (609)$$

$$= -\frac{1}{4} \left(U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S - \exp(\phi(\tau)) S + U^* \exp(-\phi(\tau)) S + 2S - 2\Re(U^*) S \right) \quad (610)$$

$$= -\frac{S}{4} \left(2\Re(U) \exp(-\phi(\tau)) - 2\exp(\phi(\tau)) + 2 - 2\Re(U) \right) \quad (611)$$

$$= \frac{S}{2} \left(\exp(\phi(\tau)) - \Re(U) \exp(-\phi(\tau)) - 1 + \Re(U) \right) \quad (612)$$

$$\langle \widetilde{B_x(\tau)} \widetilde{B_y(0)} \rangle_B = \left\langle \frac{B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10-} B_{10+}}{2} \frac{B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10-} B_{10+} - B_{10-} B_{10+}}{2i} \right\rangle_B \quad (613)$$

$$= \frac{1}{4i} \langle (B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10-} B_{10+}) (B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10-} B_{10+} - B_{10-} B_{10+}) \rangle_B \quad (614)$$

$$= \frac{1}{4i} \langle B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{10-} B_{10+} - B_{1+} B_{0-}(\tau) B_{10-} B_{10+} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} \rangle \quad (615)$$

$$+ B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}(\tau) B_{1+} B_{0-} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} - B_{10-} B_{10+}(\tau) B_{10-} B_{10+} \rangle \quad (616)$$

$$= \frac{1}{4i} \langle B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{10-} B_{1+} B_{0-}(\tau) B_{10+}^* B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{0+} B_{1-}(\tau) B_{10+}^* B_{10-}^* B_{10+}^* \rangle \quad (617)$$

$$= \frac{1}{4i} \langle B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{10-} B_{10+}^* B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{10-} B_{10+}^* B_{10-}^* B_{10+}^* \rangle \quad (618)$$

$$= \frac{1}{4i} \langle B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{10-} B_{10+} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} - B_{10+}^* B_{10-}^* \rangle \quad (619)$$

$$= \frac{1}{4i} (\exp(\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S + U^* S - US) \quad (620)$$

$$= \frac{1}{4i} (-U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S + U^* S - US) \quad (621)$$

$$= \frac{S}{4i} (-U^* \exp(-\phi(\tau)) + U \exp(-\phi(\tau)) + U^* - U) \quad (622)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (623)$$

$$= \frac{2i\Im(U)S}{4i} (\exp(-\phi(\tau)) - 1) \quad (624)$$

$$= \frac{\Im(U)S}{2} (\exp(-\phi(\tau)) - 1) \quad (625)$$

$$\langle \widetilde{B}_y(\tau) \widetilde{B}_x(0) \rangle_B = \left\langle \frac{B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10-} B_{10+}^*}{2i} \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10-} B_{10+}^*}{2} \right\rangle_B \quad (626)$$

$$= \frac{1}{4i} \langle (B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10-} B_{10+}^*) (B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10-} B_{10+}^*) \rangle_B \quad (627)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+}^* - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} \quad (628)$$

$$+ B_{1+} B_{0-}(\tau) B_{10-} B_{10+}^* + B_{10-} B_{10+}^* B_{10-} B_{10+}^* - B_{10-} B_{10+}^* B_{10-} B_{10+}^* - B_{10-} B_{10+}^* B_{10-} B_{10+}^* + B_{10-} B_{10+}^* B_{10-} B_{10+}^* \rangle \quad (629)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10-} B_{10+}^* - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}^* B_{10-} B_{10+}^* \rangle \quad (630)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{10-} B_{10+}^* B_{10-} B_{10+}^* - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}^* B_{10-} B_{10+}^* \rangle \quad (631)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{10-} B_{10+}^* B_{10-} B_{10+}^* - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} - B_{1+} B_{0-}(\tau) B_{0+} B_{1-} + B_{10-} B_{10+}^* B_{10-} B_{10+}^* \rangle \quad (632)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + B_{10-}^2 - B_{10+}^{*2}) \quad (633)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U^* S - US) \quad (634)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (635)$$

$$= \frac{2i\Im(U)S}{4i} (\exp(-\phi(\tau)) - 1) \quad (636)$$

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$$\langle B_{0+} B_{1-} (\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right) N_{\mathbf{k}'} B_{10}^* \quad (708)$$

$$\langle B_{1+B_{10}-(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger} \rangle_B = -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^\tau (N_{\mathbf{k}'} + 1) B_{10} \quad (709)$$

$$\langle B_{1+B_0-(\tau(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*b_{\mathbf{k}'})_B} = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} \quad (710)$$

$$\langle \bar{\mathcal{B}}_{y'}(\tau) \bar{\mathcal{B}}_{iz}(0) \rangle_B = \frac{1}{2!} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}^* + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} \right) \quad (711)$$

$$+(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}^* - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} \quad (712)$$

$$= \frac{1}{2!} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}' - v_{1\mathbf{k}'}}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}^* + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}' - v_{0\mathbf{k}'}}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} \right) \quad (713)$$

$$+(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}^* - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} \quad (714)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) (N_{\mathbf{k}'} + 1) e^{-i\omega_{\mathbf{k}'} \tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + B_{10}^*) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'} \tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (-B_{10} - B_{10}^*) \right) \quad (715)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) (N_{\mathbf{k}'} + 1) e^{-i\omega_{\mathbf{k}'} \tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + B_{10}^*) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'} \tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (B_{10} + B_{10}^*) \right) \quad (716)$$

$$= \frac{1}{i} \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) (N_{\mathbf{k}'} + 1) e^{-i\omega_{\mathbf{k}'} \tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* \Re(B_{10}) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'} \tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \Re(B_{10}) \right) \quad (717)$$

$$= i \sum_{\mathbf{k}'} \Re(B_{10}) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) (N_{\mathbf{k}'} + 1) e^{-i\omega_{\mathbf{k}'}\tau} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* \right) \quad (718)$$

$$\left\langle \widetilde{B_{\pm 2}}(\tau) \widetilde{B_Y(0)} \right\rangle_B = \left\langle \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left(\frac{B_0 + B_{1-} - B_{1+} B_{0-} + B_{10} - B_{10}^*}{2i} \right) \right\rangle_B \quad (719)$$

$$= \frac{1}{2i} \left\langle \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) (B_{0+} + B_{1-} - B_{1+} + B_{0-} + B_{10} - B_{10}^*) \right\rangle_B \quad (720)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left\langle \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) (B_{0+} B_{1-} - B_{1+} B_{0-}) \right\rangle_B \quad (721)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}} \left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_{0+B_1} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} B_{1+B_0} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{-i\omega_{\mathbf{k}'}\tau} B_{0+B_1} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{-i\omega_{\mathbf{k}'}\tau} B_{1+B_0} \right\rangle \quad (722)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle \chi_{\mathbf{k}', B_0+B_1-}^\dagger - e^{i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle \chi_{\mathbf{k}', B_1+B_0-}^\dagger + e^{-i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle^* \langle \mathbf{b}_{\mathbf{k}', B_0+B_1-} \rangle - e^{-i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle^* \langle \mathbf{b}_{\mathbf{k}', B_1+B_0-} \rangle \right) \quad (723)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \chi_{\mathbf{k}'}^\dagger B_0 + B_1 -) e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \chi_{\mathbf{k}'}^\dagger B_1 + B_0 -) + e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \chi_{\mathbf{k}'}^\dagger B_0 + B_1 -)^* (b_{\mathbf{k}'} B_0 + B_1 -) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \chi_{\mathbf{k}'}^\dagger)^* (b_{\mathbf{k}'} B_1 + B_0 -) \quad (724)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle \langle b_{\mathbf{k}'}^\dagger, B_0 + B_1 - \rangle - e^{i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle \langle b_{\mathbf{k}'}^\dagger, B_1 + B_0 - \rangle + e^{-i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle^* \langle b_{\mathbf{k}'}', B_0 + B_1 - \rangle - e^{-i\omega_{\mathbf{k}'}\tau} \langle g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \rangle^* \langle b_{\mathbf{k}'}', B_1 + B_0 - \rangle \right) \quad (725)$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_{1+} B_{0-} \right\rangle_B = - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \quad (726)$$

$$\langle b_{\mathbf{k}'}^\dagger B_{0+} B_{1-} \rangle_B = - \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^* N_{\mathbf{k}'} \quad (727)$$

$$\langle b_{\mathbf{k}'} B_{1+B_0-} \rangle_B = \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \quad (728)$$

$$\langle b_{\mathbf{k}'} B_{0+} B_{1-} \rangle_B = \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10}^* \quad (729)$$

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$$\begin{aligned}
\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B &= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(- \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^* N_{\mathbf{k}'} \right) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(- \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) + e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10}^* \right) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \\
&= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} \left(- (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^* N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) + e^{-i\omega_{\mathbf{k}'}\tau} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10}^* \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \\
&= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} \left((-g_{i\mathbf{k}'} + v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^* N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) + e^{-i\omega_{\mathbf{k}'}\tau} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10}^* \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \\
&= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + B_{10}^*) N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (B_{10} + B_{10}^*) (N_{\mathbf{k}'} + 1) \right) \\
&= \frac{1}{i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* \Re(B_{10}) N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \Re(B_{10}) (N_{\mathbf{k}'} + 1) \right) \\
&= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \Re(B_{10}) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* \Re(B_{10}) N_{\mathbf{k}'} \right) \\
&= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \Re(B_{10}) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* \Re(B_{10}) N_{\mathbf{k}'} \right) \\
&= i \Re(B_{10}) \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* N_{\mathbf{k}'} \right) \\
\end{aligned}
\tag{737}$$

The correlation functions are equal to:

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right)
\tag{749}$$

$$U = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right)
\tag{750}$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)
\tag{751}$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{|B_{10}|^2}{2} (\Re(U) \exp(-\phi(\tau)) + \exp(\phi(\tau)) - \Re(U) - 1)
\tag{752}$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{|B_{10}|^2}{2} (\exp(\phi(\tau)) - \Re(U) \exp(-\phi(\tau)) - 1 + \Re(U))
\tag{753}$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{\Im(U) |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1)
\tag{754}$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{\Im(U) |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1)
\tag{755}$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B = i \Im(B_{10}) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right)
\tag{756}$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i \Im(B_{10}) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right)
\tag{757}$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = i \Re(B_{10}) \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right)
\tag{758}$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i \Re(B_{10}) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right)
\tag{759}$$

The spectral density is defined in the usual way:

$$J_n(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}})
\tag{760}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_n = \int_0^\infty \frac{J_n(\omega)}{\omega} d\omega \quad (761)$$

The integral version of the correlation functions are given by:

$$(762)$$

$$(763)$$

$$(764)$$

$$(765)$$

$$(766)$$

$$(767)$$

$$(768)$$

$$(769)$$

$$(770)$$

$$(771)$$

$$(772)$$

The eigenvalues of the Hamiltonian $\overline{H_S}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}(\overline{H_S}) \lambda + \text{Det}(\overline{H_S}) = 0 \quad (773)$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_\pm = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S |\pm\rangle = \lambda_\pm |\pm\rangle$. Using this notation is possible to write $H_S = \lambda_+ |+\rangle \langle +| + \lambda_- |-\rangle \langle -|$.

The time-dependence of the system operators $\widetilde{A}_i(t)$ may be made explicit using the Fourier decomposition:

$$\widetilde{A}_i(\tau) = e^{i\overline{H_S}\tau} A_i e^{-i\overline{H_S}\tau} \quad (774)$$

$$= \sum_w e^{-i w \tau} A_i(w) \quad (775)$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.

In order to use the equation (775) to descompose the equation (355) we need to consider the time ordering operator \mathcal{T} , it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right)$ like:

$$U(t) \equiv \mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right) \quad (776)$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) \dots H(t_n) \quad (777)$$

Here $0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right) \equiv \exp(-it H_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots$ so we can write:

$$U(t) = \exp\left(-it \left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots\right)\right) \quad (778)$$

The terms can be found expanding $\mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_S}(t') dt' \quad (779)$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' [\overline{H_S}(t'), \overline{H_S}(t'')] \quad (780)$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' ([\overline{H_S}(t'), \overline{H_S}(t'')], \overline{H_S}(t''')) + [[\overline{H_S}(t'''), \overline{H_S}(t'')], \overline{H_S}(t')] \quad (781)$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A}_i(t) = e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \quad (782)$$

$$= \sum_{w(t)} e^{-iw(t)t} A_i(w(t)) \quad (783)$$

$w(t)$ belongs to the set of differences of eigenvalues that depends of the time. As we can see the eigenvectors are time dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A}_j(t - \tau, t)$ using the Magnus expansion generates:

$$\widetilde{A}_j(t - \tau, t) = U(t - \tau) U^\dagger(t) A_j(t) U(t) U^\dagger(t - \tau) \quad (784)$$

$$= e^{-i(t-\tau)H_E(t-\tau)} e^{iH_E(t)t} A_j(t) e^{-iH_E(t)t} e^{i(t-\tau)H_E(t-\tau)} \quad (785)$$

$$= e^{-i(t-\tau)H_E(t-\tau)} \sum_{w(t)} e^{-iw(t)t} A_j(w(t)) e^{i(t-\tau)H_E(t-\tau)} \quad (786)$$

$$= \sum_{w(t), w'(t-\tau)} e^{-iw(t)t} e^{iw'(t-\tau)} A'_j(w(t), w'(t-\tau)) \quad (787)$$

where $w'(t - \tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $H_S(t - \tau)$ and $H_S(t)$ respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (775) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $V = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle | \alpha \rangle \langle \beta | \quad (788)$$

Given that $[|+\rangle \langle +|, |-\rangle \langle -|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{i\overline{H_S}\tau} = e^{i(\lambda_+|+\rangle \langle +| + \lambda_-|-\rangle \langle -|)\tau} \quad (789)$$

$$= e^{i\lambda_+|+\rangle \langle +|\tau} e^{i\lambda_-|-\rangle \langle -|\tau} \quad (790)$$

$$= (|-\rangle \langle -| + e^{i\lambda_+\tau} |+\rangle \langle +|) (|+\rangle \langle +| + e^{i\lambda_-\tau} |-\rangle \langle -|) \quad (791)$$

$$= e^{i\lambda_+\tau} |+\rangle \langle +| + e^{i\lambda_-\tau} |-\rangle \langle -| \quad (792)$$

Calculating the transformation (775) directly using the previous relationship we find that:

$$\widetilde{A}_i(\tau) = (e^{i\lambda_+\tau} |+\rangle \langle +| + e^{i\lambda_-\tau} |-\rangle \langle -|) \left(\sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle | \alpha \rangle \langle \beta | \right) (e^{-i\lambda_+\tau} |+\rangle \langle +| + e^{-i\lambda_-\tau} |-\rangle \langle -|) \quad (793)$$

$$= \langle + | A_i | + \rangle | + \rangle \langle + | + e^{i\eta\tau} \langle + | A_i | - \rangle | + \rangle \langle - | + e^{-i\eta\tau} \langle - | A_i | + \rangle | - \rangle \langle + | + \langle - | A_i | - \rangle | - \rangle \langle - | \quad (794)$$

Here $\eta = \lambda_+ - \lambda_-$. Comparing the RHS of the equations (775) and the explicit expression for $\widetilde{A}_i(\tau)$ and we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$A_i(0) = \langle + | A_i | + \rangle | + \rangle \langle + | + \langle - | A_i | - \rangle | - \rangle \langle - | \quad (795)$$

$$A_i(w) = \langle + | A_i | - \rangle | + \rangle \langle - | \quad (796)$$

$$A_i(-w) = \langle - | A_i | + \rangle | - \rangle \langle + | \quad (797)$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A}_i(\tau) = \widetilde{A}_i^\dagger(\tau)$ and $\widetilde{B}_i(\tau) = \widetilde{B}_i^\dagger(\tau)$ we can use the equation (775) to write the master equation in the following neater form:

$$\frac{d\bar{\rho}_S}{dt} = -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \frac{1}{2} \sum_{ij} \sum_{w, w'} \gamma_{ij}(w, w', t) [A_i, A_j(w, w') \bar{\rho}_S(t) - \bar{\rho}_S(t) A_j^\dagger(w, w')] - i \sum_{ij} \sum_w S_{ij}(w, w', t) [A_i, A_j(w, w') \bar{\rho}_S(t) + \bar{\rho}_S(t) A_j^\dagger(w, w')] \quad (798)$$

where $A_j^\dagger(w) = A(-w)$ as expected from the equations (796) and (797). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}(w, w', t) = 2\Re(K_{ij}(w, w', t))$ and $S_{ij}(w, w', t) = \Im(K_{ij}(w, w', t))$, respectively, defined in terms of the response functions

$$K_{ij}(w, w', t) = \int_0^t C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) e^{i w \tau} e^{-i t(w - w')} d\tau \quad (799)$$

$$= K_{ijww'}(t) \quad (800)$$

If we extend the upper limit of integration to ∞ in the equation (799) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{d\bar{\rho}_S}{dt} = -i [\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijww'} K_{ijww'}(t) [A_i, A_{jww'} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jww'}^\dagger] \quad (801)$$

Applying the inverse transformation we will obtain that:

$$e^{-V} \frac{d\bar{\rho}_S}{dt} e^V = \frac{d(e^{-V} \bar{\rho}_S e^V)}{dt} \quad (802)$$

$$= \frac{d\rho_S}{dt} \quad (803)$$

$$= -ie^{-V} [\bar{H}_S(t), \bar{\rho}_S(t)] e^V - \sum_{ijww'} K_{ijww'}(t) e^{-V} [A_i, A_{jww'} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jww'}^\dagger] e^V \quad (804)$$

For a product we have the following:

$$e^{-V} \overline{AB} e^V = e^{-V} \overline{A} \overline{B} e^V \quad (805)$$

$$= e^{-V} \overline{A} e^V e^{-V} \overline{B} e^V \quad (806)$$

$$= (e^{-V} \overline{A} e^V) (e^{-V} \overline{B} e^V) \quad (807)$$

$$= \overline{AB} \quad (808)$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V} \overline{[A, B]} e^V = e^{-V} \overline{(AB - BA)} e^V \quad (809)$$

$$= e^{-V} \overline{AB} e^V - e^{-V} \overline{BA} e^V \quad (810)$$

$$= \overline{AB} - \overline{BA} \quad (811)$$

$$= [A, B] \quad (812)$$

So we will obtain that

$$\frac{d\rho_{\overline{S}}}{dt} = -i e^{-V} [\overline{H}_{\overline{S}}(t), \overline{\rho}_{\overline{S}}(t)] e^V - \sum_{ijww'} K_{ijww'}(t) e^{-V} [A_i, A_{jww'} \overline{\rho}_{\overline{S}}(t) - \overline{\rho}_{\overline{S}}(t) A_{jww'}^\dagger] e^V \quad (813)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \sum_{ijww'} K_{ijww'}(t) [e^{-V} A_i e^V, e^{-V} A_{jww'} \overline{\rho}_{\overline{S}}(t) e^V - e^{-V} \overline{\rho}_{\overline{S}}(t) A_{jww'}^\dagger e^V] \quad (814)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \sum_{ijww'} K_{ijww'}(t) [e^{-V} A_i e^V, e^{-V} A_{jww'} e^V e^{-V} \overline{\rho}_{\overline{S}}(t) e^V - e^{-V} \overline{\rho}_{\overline{S}}(t) e^V e^{-V} A_{jww'}^\dagger e^V] \quad (815)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \sum_{ijww'} K_{ijww'}(t) [e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_{\overline{S}}(t) - \rho_{\overline{S}}(t) e^{-V} A_{jww'}^\dagger e^V] \quad (816)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \left(\sum_{ijww'} K_{ijww'}(t) \left([e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_{\overline{S}}(t)] - [e^{-V} A_i e^V, \rho_{\overline{S}}(t) e^{-V} A_{jww'}^\dagger e^V] \right) \right) \quad (817)$$

V. LIMIT CASES

In order to show the plausibility of the master equation (798) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (798) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1, 2, 3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_j g_k (b_k^\dagger + b_k) \right) |1\rangle\langle 1| + \frac{\Omega}{2} \sigma_x + \sum_k \omega_k b_k^\dagger b_k \quad (818)$$

After performing the transformation (24) on the Hamiltonian (818) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H}_S = (\delta + R) |1\rangle\langle 1| + \frac{\Omega_r}{2} \sigma_x \quad (819)$$

$$\overline{H}_I = B_z |1\rangle\langle 1| + \frac{\Omega}{2} (B_x \sigma_x + B_y \sigma_y) \quad (820)$$

$$H_B = \sum_k \omega_k b_k^\dagger b_k \quad (821)$$

The Hamiltonian (819) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z + \frac{\Omega_r}{2} \sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R) |1\rangle\langle 1| - \frac{\delta}{2} \mathbb{I} = \left(\frac{\delta}{2} + R \right) |1\rangle\langle 1| - \frac{\delta}{2} |0\rangle\langle 0| \quad (822)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\delta + R}{2} \sigma_z \quad (823)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z \quad (824)$$

In this Hamiltonian we can write $A_1 = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2}$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_S}$.

$$\lambda_+ = \frac{\epsilon + \eta}{2} \quad (825)$$

$$\lambda_- = \frac{\epsilon - \eta}{2} \quad (826)$$

$$|+\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} \begin{pmatrix} \epsilon + \eta \\ \Omega_r \end{pmatrix} \quad (827)$$

$$|-\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} \begin{pmatrix} -\Omega_r \\ \epsilon + \eta \end{pmatrix} \quad (828)$$

Using this basis we can find the decomposition matrices using the equations (796)-(797) and the fact that $|+\rangle = \cos(\theta) |1\rangle + \sin(\theta) |0\rangle$ and $|-\rangle = -\sin(\theta) |1\rangle + \cos(\theta) |0\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$:

$$\langle + | \sigma_x | + \rangle = (\cos(\theta) \ \sin(\theta)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (829)$$

$$= 2 \sin(\theta) \cos(\theta) \quad (830)$$

$$= \sin(2\theta) \quad (831)$$

$$\langle - | \sigma_x | - \rangle = (-\sin(\theta) \ \cos(\theta)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (832)$$

$$= -2 \sin(\theta) \cos(\theta) \quad (833)$$

$$= -\sin(2\theta) \quad (834)$$

$$\langle - | \sigma_x | + \rangle = (-\sin(\theta) \ \cos(\theta)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (835)$$

$$= \cos^2(\theta) - \sin^2(\theta) \quad (836)$$

$$= \cos(2\theta) \quad (837)$$

$$\langle + | \sigma_y | + \rangle = (\cos(\theta) \ \sin(\theta)) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (838)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (839)$$

$$= 0 \quad (840)$$

$$\langle - | \sigma_y | - \rangle = (-\sin(\theta) \ \cos(\theta)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (841)$$

$$= -i \sin(\theta) \cos(\theta) + i \sin(\theta) \cos(\theta) \quad (842)$$

$$= 0 \quad (843)$$

$$\langle - | \sigma_y | + \rangle = (-\sin(\theta) \ \cos(\theta)) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (844)$$

$$= i \cos^2(\theta) + i \sin^2(\theta) \quad (845)$$

$$= i \quad (846)$$

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = (\cos(\theta) \ \sin(\theta)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (847)$$

$$= \cos(\theta) \cos(\theta) \quad (848)$$

$$= \cos^2(\theta) \quad (849)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = (-\sin(\theta) \ \cos(\theta)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (850)$$

$$= \sin(\theta) \sin(\theta) \quad (851)$$

$$= \sin^2(\theta) \quad (852)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = (-\sin(\theta) \ \cos(\theta)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (853)$$

$$= -\sin(\theta) \cos(\theta) \quad (854)$$

$$= -\sin(\theta) \cos(\theta) \quad (855)$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta) (|+\rangle \langle +| - |-\rangle \langle -|) \quad (856)$$

$$A_1(\eta) = \cos(2\theta) |-\rangle \langle +| \quad (857)$$

$$A_2(0) = 0 \quad (858)$$

$$A_2(\eta) = i |-\rangle \langle +| \quad (859)$$

$$A_3(0) = \cos^2(\theta) |+\rangle \langle +| + \sin^2(\theta) |-\rangle \langle -| \quad (860)$$

$$A_3(\eta) = -\sin(\theta) \cos(\theta) |-\rangle \langle +| \quad (861)$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (421) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) \quad (862)$$

Using the notation of the master equation (798), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(??)-(??) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (421) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left(\widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (863)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right) \quad (864)$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left(\widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (865)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right) \quad (866)$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega J(\omega) (1 - F(\omega))^2 G_+(\tau) \quad (867)$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2} \int_0^\infty d\omega \frac{J(\omega)}{\omega} F(\omega) (1 - F(\omega)) iG_-(\tau) \quad (868)$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau) \quad (869)$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 \quad (870)$$

Finally taking the Hamiltonian (818) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\text{Det}(\overline{H_S}) = -\frac{\Omega_r^2}{4}$, $\text{Tr}(\overline{H_S}) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (336) we have that:

$$f_k = \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k} \right)} \quad (871)$$

$$= \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k} \right)} \quad (872)$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (419) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t) U^\dagger(t - \tau) = U(\tau)$. From the equation (798) and using the fact that

$$\widetilde{A}_j(t - \tau, t) = U(\tau) A_j U(-\tau) \quad (873)$$

$$= \sum_w e^{i\omega\tau} A_i(-w) \quad (874)$$

$$= \sum_w e^{-i\omega\tau} A_i(w) \quad (875)$$

because the matrices $U(t)$ and $U(t - \tau)$ commute from the fact that $H_S(t)$ and $H_S(t - \tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{d\bar{\rho}_S(t)}{dt} = -i[H_S(t), \bar{\rho}_S(t)] - \frac{1}{2} \sum_{ij} \sum_w \gamma_{ij}(w, t) [A_i, A_j(w) \bar{\rho}_S(t) - \bar{\rho}_S(t) A_j^\dagger(w)] \quad (876)$$

$$- \sum_{ij} \sum_w S_{ij}(w, t) [A_i, A_j(w) \bar{\rho}_S(t) + \bar{\rho}_S(t) A_j^\dagger(w)] \quad (877)$$

where $A_j^\dagger(w) = A(-w)$, as we can see the equation (877) contains the rates and energy shifts $\gamma_{ij}(w, t) = 2\Re(K_{ij}(w, t))$ and $S_{ij}(w, t) = \Im(K_{ij}(w, t))$, respectively, defined in terms of the response functions

$$K_{ij}(w, t) = \int_0^t \Lambda'_{ij}(\tau) e^{i\omega\tau} d\tau \quad (878)$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (109) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right) \right) |1\rangle\langle 1| + \frac{\Omega(t)}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (879)$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (879) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (223) and (??) we obtain the following transformed Hamiltonians:

$$\overline{H}_S = (\delta + R_1) |1\rangle\langle 1| + \frac{B\sigma_x}{2} \Omega(t) \quad (880)$$

$$\overline{H}_I = \frac{\Omega(t)}{2} (B_x \sigma_x + B_y \sigma_y) \quad (881)$$

In this case $R_1 = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} g_{\mathbf{k}} \right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}}^{-1} |g_{\mathbf{k}}|^2) = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}} | \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} |^2)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (??)-(??) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} G_+(\tau) d\omega \quad (882)$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau)$ so (882) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left(\coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau) \right) d\omega \quad (883)$$

Writing the interaction Hamiltonian (881) in the similar way to the equation (??) allow us to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (223) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2} \sigma_z + \frac{\delta'}{2} \mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then \overline{H}_S is equal to:

$$\overline{H}_S = \frac{\delta'}{2} \sigma_z + \frac{B\sigma_x}{2} \Omega(t) \quad (884)$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \text{Tr}_B \left(\widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (885)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right) \quad (886)$$

$$\Lambda_{22}(\tau) = \text{Tr}_B \left(\widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (887)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right) \quad (888)$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(419) is:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i \left[\frac{\delta'}{2} \sigma_z + \frac{\Omega_r(t) \sigma_x}{2}, \rho_S(t) \right] - \sum_{i=1}^2 \int_0^t d\tau \left(C_i(t) C_i(t-\tau) \Lambda_{ii}(\tau) \left[A_i, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right] \right. \quad (889)$$

$$\left. + C_i(t) C_i(t-\tau) \Lambda_{ii}(-\tau) \left[\rho_S(t) \widetilde{A}_i(t-\tau, t), A_i \right] \right) \quad (890)$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A}_i(t-\tau, t) = \widetilde{\sigma}_i(t-\tau, t)$, also using the equations (885) and (888) on the equation (890) we obtain that:

$$\frac{d\overline{\rho_S}(t)}{dt} = -\frac{i}{2} [\delta' \sigma_z + \Omega_r(t) \sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4} \int_0^t d\tau \Omega(t-\tau) ([\sigma_x, \widetilde{\sigma}_x(t-\tau, t) \rho_S(t)] \Lambda_x(\tau) \quad (891)$$

$$+ [\sigma_y, \widetilde{\sigma}_y(t-\tau, t) \rho_S(t)] \Lambda_y(\tau) + [\rho_S(t) \widetilde{\sigma}_x(t-\tau, t), \sigma_x] \Lambda_x(\tau) + [\rho_S(t) \widetilde{\sigma}_y(t-\tau, t), \sigma_y] \Lambda_y(\tau)) \quad (892)$$

As we can see $[A_j, \widetilde{A}_i(t-\tau, t) \rho_S(t)]^\dagger = [\rho_S(t) \widetilde{A}_i(t-\tau, t), A_j]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation (S17) of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (421) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\hbar = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta |1\rangle\langle 1| + \frac{\Omega(t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (893)$$

Using the equation (798), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{d\rho_S}{dt} = -i [H_S(t), \rho_S(t)] - \frac{1}{2} \sum_w \gamma_{33}(w, t) \left[A_3, A_3(w) \rho_S(t) - \rho_S(t) A_3^\dagger(w) \right] \quad (894)$$

$$- \sum_w S_{33}(w, t) \left[A_3, A_3(w) \rho_S(t) + \rho_S(t) A_3^\dagger(w) \right] \quad (895)$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty d\omega J(\omega) G_+(\tau) \quad (896)$$

$$\Lambda_{33}(-\tau) = \int_0^\infty d\omega J(\omega) G_+(-\tau) \quad (897)$$

In our case $A_3 = \frac{\mathbb{I} + \sigma_z}{2}$, the equation (895) can be transformed in

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \sum_w (K_{33}(w, t)[A_3, A_3(w)\rho_S(t)] + K_{33}^*(w, t)[\rho_S(t)A_3(w), A_3]) \quad (898)$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthermore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\tilde{\Omega}$. To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\tilde{\Omega}(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\tilde{\Omega}}{2} \quad (899)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \quad (900)$$

$$\lambda_- = -\frac{\tilde{\Omega}}{2} \quad (901)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |0\rangle) \quad (902)$$

The elements of the decomposition matrices are:

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (903)$$

$$= \frac{1}{2} \quad (904)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = \frac{1}{2} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (905)$$

$$= \frac{1}{2} \quad (906)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (907)$$

$$= -\frac{1}{2} \quad (908)$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \quad (909)$$

$$= \frac{\mathbb{I}}{2} \quad (910)$$

$$A_3(\eta) = -\frac{1}{2} |-\rangle \langle +| \quad (911)$$

$$= \frac{1}{4} (\sigma_z + i\sigma_y) \quad (912)$$

$$A_3(-\eta) = -\frac{1}{2} |+\rangle \langle -| \quad (913)$$

$$= \frac{1}{4} (\sigma_z - i\sigma_y) \quad (914)$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - K_{33}(\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z + i\sigma_y) \rho_S(t) \right] - K_{33}(-\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z - i\sigma_y) \rho_S(t) \right] \quad (915)$$

$$- K_{33}^*(\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4} (\sigma_z + i\sigma_y), \frac{\sigma_z}{2} \right] - K_{33}^*(-\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4} (\sigma_z - i\sigma_y), \frac{\sigma_z}{2} \right] \quad (916)$$

Calculating the response functions extending the upper limit of τ to ∞ , we obtain:

$$K_{33}(\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{i\tilde{\Omega}\tau} d\tau d\omega \quad (917)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (918)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} (n(\omega) + 1) e^{-i\tau\omega} d\tau d\omega \quad (919)$$

$$= \int_0^\infty \int_0^\infty J(\omega) (n(\omega) + 1) e^{i\tilde{\Omega}\tau - i\tau\omega} d\tau d\omega \quad (920)$$

$$= \int_0^\infty J(\omega) (n(\omega) + 1) \pi \delta(\tilde{\Omega} - \omega) d\omega \quad (921)$$

$$= \pi J(\tilde{\Omega}) (n(\tilde{\Omega}) + 1) \quad (922)$$

$$K_{33}(-\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{-i\tilde{\Omega}\tau} d\tau d\omega \quad (923)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (924)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} n(\omega) e^{i\tau\omega} d\tau d\omega \quad (925)$$

$$= \int_0^\infty \int_0^\infty J(\omega) n(\omega) e^{-i\tilde{\Omega}\tau + i\tau\omega} d\tau d\omega \quad (926)$$

$$= \int_0^\infty J(\omega) n(\omega) \pi \delta(-\tilde{\Omega} + \omega) d\omega \quad (927)$$

$$= \pi J(\tilde{\Omega}) n(\tilde{\Omega}) \quad (928)$$

Here we have used $\int_0^\infty ds e^{\pm i\epsilon s} = \pi \delta(\epsilon) \pm i \frac{\text{V.P.}}{\epsilon}$, where V.P. denotes the Cauchy's principal value. These principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (915) lead us to obtain:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\tilde{\Omega}) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (929)$$

$$- \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\tilde{\Omega}) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right) \quad (930)$$

This is the same result than the equation (S17), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{d\rho_S(t)}{dt} = -i\frac{\Omega(t)}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\Omega(t)) ((n(\Omega(t)) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\Omega(t)) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)]) \quad (931)$$

$$- \frac{\pi}{8} J(\Omega(t)) ((n(\Omega(t)) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\Omega(t)) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z]) \quad (932)$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (933)$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (934)$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right), \quad (935)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (936)$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}) \right) \quad (937)$$

At first let's obtain e^V under the transformation (937), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \quad (938)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{(\sum_{n=1} |n\rangle\langle n| \hat{\varphi})^2}{2!} + \dots \quad (939)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \quad (940)$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \quad (941)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \quad (942)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \quad (943)$$

Given that $[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \quad (944)$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (945)$$

$$= B_{\pm} \quad (946)$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-$. because this form imposes that $e^{-V} e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (947)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (933):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \quad (948)$$

$$= |0\rangle\langle 0|, \quad (949)$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \quad (950)$$

$$= |m\rangle\langle m| B_+ |m\rangle\langle n| n\rangle\langle n| B_-, \quad (951)$$

$$= |m\rangle\langle n|, \quad m \neq 0, \quad n \neq 0, \quad (952)$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \quad (953)$$

$$= |0\rangle\langle m| B_- \quad m \neq 0, \quad (954)$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (955)$$

$$= |0\rangle\langle m| B_+ \quad m \neq 0, \quad (956)$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (957)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B_+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_- \quad (958)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_+ b_{\mathbf{k}}^\dagger B_- \right) (B_+ b_{\mathbf{k}} B_-) \quad (959)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (960)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (961)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \quad (962)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (963)$$

The transformed Hamiltonians of the equations (934) to (936) written in terms of (948) to (963) are:

$$\overline{H_S(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (964)$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (965)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (966)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|}) + \sum_{m,n \neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \quad (967)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) B_- |0\rangle\langle n| + V_{n0}(t) B_+ |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (968)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (969)$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) \left(\left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (970)$$

$$= \left(\mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B_+ \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (971)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B_+ (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) B_- \quad (972)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (973)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (974)$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (975)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (976)$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (977)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (978)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (979)$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (980)$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (981)$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \right) \quad (982)$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (983)$$

Using the previous functions we have that (980) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (984)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) \quad (985)$$

Now in order to separate the elements of the hamiltonian (985) let's follow the references of the equations (??) and (223) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \quad (986)$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B_- - B) + V_{n0}(t) |n\rangle\langle 0| (B_+ - B)), \quad (987)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (988)$$

Here B is given by (??) The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (989)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (990)$$

$$B_x = \frac{B_+ + B_- - 2B}{2} \quad (991)$$

$$B_y = \frac{B_- - B_+}{2i} \quad (992)$$

Using this set of hermitian operators to write the Hamiltonians (934)-(936)

$$\overline{H_S(t)} = \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (993)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{0 < m < n} (V_{mn}(t) |m\rangle\langle n| + V_{nm}(t) |n\rangle\langle m|) \quad (994)$$

$$+ \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (995)$$

$$= \sum_{0 < m < n} ((\Re(V_{mn}(t)) + i\Im(V_{mn}(t))) |m\rangle\langle n| + (\Re(V_{mn}(t)) - i\Im(V_{mn}(t))) |n\rangle\langle m|) + \varepsilon_0(t) |0\rangle\langle 0| \quad (996)$$

$$+ B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (997)$$

$$= \sum_{0 < m < n} \left((\Re(V_{nm}(t)) + i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (998)$$

$$+ B \sum_{n=1} \left(V_{0n}(t) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + V_{n0}(t) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) + \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (999)$$

$$= \sum_{0 < m < n} (\Re(V_{nm}(t)) \sigma_{nm,x} + \Im(V_{nm}(t)) \sigma_{nm,y}) + B \sum_{n=1} (\Re(V_{0n}(t)) \sigma_{0n,x} + \Im(V_{mn}(t)) \sigma_{0n,y}) \quad (1000)$$

$$+ \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1001)$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B_- - B) + V_{n0}(t) |n\rangle\langle 0| (B_+ - B)) \quad (1002)$$

$$= \sum_{n=1} \left((\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) (B_- - B) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) (B_+ - B) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) \quad (1003)$$

$$+ \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1004)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} ((B_- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) + (B_+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))) \right) \quad (1005)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B_+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) - (B_- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))) \quad (1006)$$

$$+ \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1007)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} (B_+ + B_- - 2B) \Re(V_{0n}(t)) + i(B_- - B - B_+ + B) \Im(V_{0n}(t)) \right) \quad (1008)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B_+ - B - B_- + B) \Re(V_{0n}(t)) + i(B - B_- + B - B_+) \Im(V_{0n}(t))) + \sum_{n=1} B_{z,n} |n\rangle\langle n| \quad (1009)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (\sigma_{0n,x} (B_x \Re(V_{0n}(t)) - B_y \Im(V_{0n}(t))) \quad (1010)$$

$$+ \sigma_{0n,y} (B_y \Re(V_{0n}(t)) + B_x \Im(V_{0n}(t)))) \quad (1011)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H}_S + \overline{H}_B)} \right) \right) + \langle \overline{H}_I \rangle_{\overline{H}_S + \overline{H}_B} + O \left(\langle \overline{H}_I^2 \rangle_{\overline{H}_S + \overline{H}_B} \right). \quad (1012)$$

Taking the equations (244)-(252) and given that $\text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right) = C(R_1, R_2, \dots, R_{d-1}, B)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (252) and using the chain rule we obtain that:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \frac{\partial R_n}{\partial v_{\mathbf{k}}}, \quad (1013)$$

$$= 0 \quad (1014)$$

Let's recall the equations (981) and (983), we can write them in terms of the variational parameters

$$B = \exp \left(- (1/2) \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) \right) \quad (1015)$$

$$R_n = \sum_{\mathbf{k}} \omega_{\mathbf{k}}^{-1} (v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}} v_{\mathbf{k}}) \quad (1016)$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \quad (1017)$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1018)$$

Introducing this derivates in the equation (1013) give us:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \right) + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1019)$$

$$= v_{\mathbf{k}} \left(\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t) \quad (1020)$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1021)$$

$$= \frac{2g_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1022)$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}} v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}}. \quad (1023)$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) (|0\rangle\langle 0| \otimes \rho_B) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (1024)$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \quad (1025)$$

$$0 = \rho(0) \quad (1026)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1027)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1028)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1029)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}(t)) \quad (1030)$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I}(t) = B_{z,0} |0\rangle\langle 0| + \sum_{n=1} (\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n} |n\rangle\langle n|) \quad (1031)$$

$$+ \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x} \quad (1032)$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1033)$$

$$B_{z,n} = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \text{ if } n \neq 0 \quad (1034)$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \quad (1035)$$

$$A_{2n}(t) = \sigma_{0n,y} \quad (1036)$$

$$A_{3n}(t) = |n\rangle\langle n| \quad (1037)$$

$$A_{4n}(t) = A_{2n}(t) \quad (1038)$$

$$A_{5n}(t) = A_{1n}(t) \quad (1039)$$

$$B_{1n}(t) = B_x \quad (1040)$$

$$B_{2n}(t) = B_y \quad (1041)$$

$$B_{3n}(t) = B_{z,n} \quad (1042)$$

$$B_{4n}(t) = B_{1n}(t) \quad (1043)$$

$$B_{5n}(t) = B_{2n}(t) \quad (1044)$$

$$C_{10}(t) = 0 \quad (1045)$$

$$C_{20}(t) = 0 \quad (1046)$$

$$C_{40}(t) = 0 \quad (1047)$$

$$C_{50}(t) = 0 \quad (1048)$$

$$C_{30}(t) = 1 \quad (1049)$$

$$C_{1n}(t) = \Re(V_{0n}(t)) \quad (1050)$$

$$C_{2n}(t) = C_{1n}(t) \quad (1051)$$

$$C_{3n}(t) = 1 \quad (1052)$$

$$C_{4n}(t) = \Im(V_{0n}(t)) \quad (1053)$$

$$C_{5n}(t) = -\Im(V_{0n}(t)) \quad (1054)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J} \sum_{n=1} C_{jn}(t) (A_{jn} \otimes B_{jn}(t)) \quad (1055)$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1027) as:

$$\widetilde{H_I}(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t) \right) \quad (1056)$$

$$\widetilde{A_i}(t) = U^\dagger(t) A_i U(t) \quad (1057)$$

$$\widetilde{B_i}(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \quad (1058)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(s) \rho_B \right] \right] ds \quad (1059)$$

Replacing the equation (1056) in (1059) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1060)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \left[\sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1061)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] \quad (1062)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \Big] ds \quad (1063)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right) \quad (1064)$$

$$- \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \quad (1065)$$

$$- \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1066)$$

$$+ \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \Big) ds \quad (1067)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jn j'n'}(\tau) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(t)(s) \right\rangle_B \quad (1068)$$

$$= \left\langle \widetilde{B}_{jn}(\tau) \widetilde{B}_{j'n'}(0) \right\rangle_B \quad (1069)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \rho_B \right) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jn}(t)$ and elements related to the system given by $\widetilde{A}_{jn}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jn}(t) \widetilde{B}_{j'n'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jn}(t) \right) \text{Tr} \left(\widetilde{B}_{j'n'}(t) \right)$. The correlation functions relevant of the master equation (1067) are:

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1070)$$

$$= \left\langle \widetilde{B_{jn}}(0) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1071)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1072)$$

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \rho_B \widetilde{B_{j'n'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1073)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1074)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1075)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1076)$$

$$\text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \rho_B \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) \quad (1077)$$

$$= \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1078)$$

$$= \left\langle \widetilde{B_{jn}}(\tau) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1079)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1080)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1081)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1082)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1083)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1084)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1060) and (1067) and using the equations (1070)-(1084) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jn j'n'}(\tau) \widetilde{A_{jn}}(t) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'n' jn}(-\tau) \widetilde{A_{jn}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \right) \right. \quad (1085)$$

$$\left. + C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{j'n' jn}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \widetilde{A_{jn}}(t) - \Lambda_{jn j'n'}(\tau) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jn}}(t) \right) \right) ds \quad (1086)$$

$$= - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jn j'n'}(\tau) \left[\widetilde{A_{jn}}(t), \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'n' jn}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s), \widetilde{A_{jn}}(t) \right] \right) \right) \quad (1087)$$

$$\frac{d\overline{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(t-\tau) \left(\Lambda_{jn j'n'}(\tau) \left[A_{jn}(t), A_{j'n'}(t-\tau, t) \overline{\rho_S}(t) \right] + \Lambda_{j'n' jn}(-\tau) \left[\overline{\rho_S}(t) A_{j'n'}(t-\tau, t), A_{jn}(t) \right] \right) \right) d\tau - i [H_S(t), \overline{\rho_S}(t)] \quad (1088)$$

For this case we used that $A_{jn}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jn}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation and if we take $n = 2$ (two sites), $\mu_0(t) = 0$, $\mu_1(t) = 1$ then we can reproduce a similar expression to (419) as expected.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1089)$$

$$H_S(t) = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1090)$$

$$H_I = \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left(g_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} \right), \quad (1091)$$

$$H_B = \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger b_{\mathbf{u}\mathbf{k}}. \quad (1092)$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \omega_{\mathbf{u}\mathbf{k}}^{-1} \left(f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} \right) \quad (1093)$$

At first let's obtain $e^{\pm V}$ under the transformation (1093), consider $\hat{\varphi}_n = \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}}^{-1} \left(f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} \right)$, so the equation (1093) can be written as $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \quad (1094)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{(\sum_n |n\rangle\langle n| \hat{\varphi}_n)^2}{2!} + \dots \quad (1095)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1096)$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1097)$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots \right) \quad (1098)$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \quad (1099)$$

Given that $\left[f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}, f_{n\mathbf{u}'\mathbf{k}'} b_{\mathbf{u}'\mathbf{k}'}^\dagger - f_{n\mathbf{u}'\mathbf{k}'}^* b_{\mathbf{u}'\mathbf{k}'} \right] = 0$ for all \mathbf{k}', \mathbf{k} and u, u' then we can proof using the Zassenhaus formula and defining $D(\pm \alpha_{n\mathbf{u}\mathbf{k}}) = e^{\pm (\alpha_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - \alpha_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}})}$ in the same way than (23) with $\alpha_{n\mathbf{u}\mathbf{k}} = \frac{f_{n\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}}$:

$$e^{\pm \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}}^{-1} (f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}})} = \prod_{\mathbf{u}} e^{\pm \sum_{\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}}^{-1} (f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}})} \quad (1100)$$

$$= \prod_{\mathbf{u}} \left(\prod_{\mathbf{k}} e^{\pm \omega_{\mathbf{u}\mathbf{k}}^{-1} (f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}})} \right) \quad (1101)$$

$$= \prod_{\mathbf{u}} \left(\prod_{\mathbf{k}} D(\pm \alpha_{n\mathbf{u}\mathbf{k}}) \right) \quad (1102)$$

$$= \prod_{\mathbf{u}\mathbf{k}} D(\pm \alpha_{n\mathbf{u}\mathbf{k}}) \quad (1103)$$

$$= \prod_{\mathbf{u}} B_{n\mathbf{u}\pm} \quad (1104)$$

$$B_{n\mathbf{u}\pm} \equiv \prod_{\mathbf{k}} D(\pm \alpha_{n\mathbf{u}\mathbf{k}}) \quad (1105)$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1106)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1089):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1107)$$

$$= \prod_u B_{0u+} |0\rangle\langle 0| \prod_u B_{0u-}, \quad (1108)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \quad (1109)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \quad (1110)$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \quad (1111)$$

$$= |0\rangle\langle 0|. \quad (1112)$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1113)$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n| \prod_u B_{nu-}, \quad (1114)$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \quad (1115)$$

$$= |m\rangle\langle n| \prod_u (B_{mu+} B_{nu-}), \quad m \neq n, \quad (1116)$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D(\alpha_{m\mathbf{u}\mathbf{k}}) \prod_{\mathbf{k}} D(-\alpha_{n\mathbf{u}\mathbf{k}}) \right), \quad (1117)$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} (D(\alpha_{m\mathbf{u}\mathbf{k}}) D(-\alpha_{n\mathbf{u}\mathbf{k}})), \quad (1118)$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D(\alpha_{m\mathbf{u}\mathbf{k}} - \alpha_{n\mathbf{u}\mathbf{k}}) \exp \left(\frac{1}{2} (-\alpha_{m\mathbf{u}\mathbf{k}} \alpha_{n\mathbf{u}\mathbf{k}}^* + \alpha_{m\mathbf{u}\mathbf{k}}^* \alpha_{n\mathbf{u}\mathbf{k}}) \right) \right). \quad (1119)$$

$$\prod_u (B_{mu+} B_{nu-}) = \prod_{u\mathbf{k}} \left(D(\alpha_{m\mathbf{u}\mathbf{k}} - \alpha_{n\mathbf{u}\mathbf{k}}) \exp \left(\frac{1}{2} (-\alpha_{m\mathbf{u}\mathbf{k}} \alpha_{n\mathbf{u}\mathbf{k}}^* + \alpha_{m\mathbf{u}\mathbf{k}}^* \alpha_{n\mathbf{u}\mathbf{k}}) \right) \right). \quad (1120)$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1121)$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \dots \right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \right) \left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \dots \right), \quad (1122)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + \dots, \quad (1123)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots \right) \prod_u B_{0u-} \quad (1124)$$

$$+ |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots \right) \prod_u B_{1u-} + \dots \quad (1125)$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \dots \right) \quad (1126)$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \dots \right) + \dots \quad (1127)$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) \quad (1128)$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) + \dots \quad (1129)$$

The transformed Hamiltonians of the equations (1090) to (1092) written in terms of (1107) to (1132) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1135)$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1136)$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) \quad (1137)$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| (g_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}) \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1138)$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{\mathbf{u}\mathbf{k}} |0\rangle\langle 0| (g_{0\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{0\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}) + \dots \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1139)$$

$$= \prod_u B_{0u+} \sum_{\mathbf{u}\mathbf{k}} |0\rangle\langle 0| (g_{0\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{0\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}) \prod_u B_{0u-} \quad (1140)$$

$$+ \prod_u B_{1u+} \sum_{\mathbf{u}\mathbf{k}} |1\rangle\langle 1| (g_{1\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{1\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}) \prod_u B_{1u-} + \dots \quad (1141)$$

$$= \sum_{\mathbf{u}\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{u}\mathbf{k}} \prod_u B_{0u+} b_{\mathbf{u}\mathbf{k}}^\dagger \prod_u B_{0u-} + g_{0\mathbf{u}\mathbf{k}}^* \prod_u B_{0u+} b_{\mathbf{u}\mathbf{k}} \prod_u B_{0u-} \right) \quad (1142)$$

$$+ \sum_{\mathbf{u}\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{u}\mathbf{k}} \prod_u B_{1u+} b_{\mathbf{u}\mathbf{k}}^\dagger \prod_u B_{1u-} + g_{1\mathbf{u}\mathbf{k}}^* \prod_u B_{1u+} b_{\mathbf{u}\mathbf{k}} \prod_u B_{1u-} \right) + \dots \quad (1143)$$

$$= \sum_{\mathbf{u}\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{u}\mathbf{k}} \left(b_{\mathbf{u}\mathbf{k}}^\dagger - \frac{v_{0\mathbf{u}\mathbf{k}}^*}{\omega_{\mathbf{u}\mathbf{k}}} \right) + g_{0\mathbf{u}\mathbf{k}}^* \left(b_{\mathbf{u}\mathbf{k}} - \frac{v_{0\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}} \right) \right) \quad (1144)$$

$$+ \sum_{\mathbf{u}\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{u}\mathbf{k}} \left(b_{\mathbf{u}\mathbf{k}}^\dagger - \frac{v_{1\mathbf{u}\mathbf{k}}^*}{\omega_{\mathbf{u}\mathbf{k}}} \right) + g_{1\mathbf{u}\mathbf{k}}^* \left(b_{\mathbf{u}\mathbf{k}} - \frac{v_{1\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}} \right) \right) + \dots \quad (1145)$$

$$= \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left(g_{n\mathbf{u}\mathbf{k}} \left(b_{\mathbf{u}\mathbf{k}}^\dagger - \frac{v_{n\mathbf{u}\mathbf{k}}^*}{\omega_{\mathbf{u}\mathbf{k}}} \right) + g_{n\mathbf{u}\mathbf{k}}^* \left(b_{\mathbf{u}\mathbf{k}} - \frac{v_{n\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}} \right) \right) \quad (1146)$$

$$= \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left(g_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} - \left(g_{n\mathbf{u}\mathbf{k}} \frac{v_{n\mathbf{u}\mathbf{k}}^*}{\omega_{\mathbf{u}\mathbf{k}}} + g_{n\mathbf{u}\mathbf{k}}^* \frac{v_{n\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}} \right) \right) \quad (1147)$$

$$\overline{H_B} = \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger b_{\mathbf{u}\mathbf{k}} + \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{n\mathbf{u}\mathbf{k}}|^2}{\omega_{\mathbf{u}\mathbf{k}}} - (v_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + v_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}) \right) \quad (1148)$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger b_{\mathbf{u}\mathbf{k}} + \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{n\mathbf{u}\mathbf{k}}|^2}{\omega_{\mathbf{u}\mathbf{k}}} \right) \quad (1149)$$

$$- (v_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + v_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}) + \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left(g_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} - \left(g_{n\mathbf{u}\mathbf{k}} \frac{v_{n\mathbf{u}\mathbf{k}}^*}{\omega_{\mathbf{u}\mathbf{k}}} + g_{n\mathbf{u}\mathbf{k}}^* \frac{v_{n\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}} \right) \right) \quad (1150)$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{u}\mathbf{k}} \left(\frac{|v_{n\mathbf{u}\mathbf{k}}|^2}{\omega_{\mathbf{u}\mathbf{k}}} - \left(g_{n\mathbf{u}\mathbf{k}} \frac{v_{n\mathbf{u}\mathbf{k}}^*}{\omega_{\mathbf{u}\mathbf{k}}} + g_{n\mathbf{u}\mathbf{k}}^* \frac{v_{n\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}} \right) \right) \quad (1151)$$

$$B_{z,n}(t) = \sum_{\mathbf{u}\mathbf{k}} \left((g_{n\mathbf{u}\mathbf{k}} - v_{n\mathbf{u}\mathbf{k}}) b_{\mathbf{u}\mathbf{k}}^\dagger + (g_{n\mathbf{u}\mathbf{k}} - v_{n\mathbf{u}\mathbf{k}})^* b_{\mathbf{u}\mathbf{k}} \right) \quad (1152)$$

Using the previous functions we have that (1149) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \quad (1153)$$

$$+ \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \quad (1154)$$

Now in order to separate the elements of the hamiltonian (1154) let's follow the references of the equations (223) and (??) to separate the hamiltonian, before proceeding to do this we need to consider the term of the form:

$$\left\langle \prod_u (B_{mu+} B_{nu-}) \right\rangle_{\overline{H_0}} = \left\langle \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp \left(\frac{1}{2} (-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}}) \right) \right) \right\rangle_{\overline{H_0}} \quad (1155)$$

$$= \left(\prod_{u\mathbf{k}} \exp \left(\frac{1}{2} (-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}}) \right) \right) \left\langle \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \right\rangle_{\overline{H_0}} \quad (1156)$$

$$= \left(\prod_{u\mathbf{k}} \exp \left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{u\mathbf{k}}}{2} \right) \right) \quad (1157)$$

$$\equiv B_{nm} \quad (1158)$$

$$\left\langle \prod_u (B_{nu+} B_{mu-}) \right\rangle_{\overline{H_0}} = \left(\prod_{u\mathbf{k}} \exp \left(\frac{(v_{nu\mathbf{k}}^* v_{mu\mathbf{k}} - v_{nu\mathbf{k}} v_{mu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{u\mathbf{k}}}{2} \right) \right) \quad (1159)$$

$$= B_{nm}^* \quad (1160)$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+} B_{nu-}) - B_{nm} \quad (1161)$$

As we can see:

$$J_{nm}^\dagger = \left(\prod_u (B_{mu+} B_{nu-}) - B_{nm} \right)^\dagger \quad (1162)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{nm}^* \quad (1163)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{mn} \quad (1164)$$

$$= J_{mn} \quad (1165)$$

We can separate the Hamiltonian (1154) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_S}(t) = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \quad (1166)$$

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1167)$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \quad (1168)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + H_B)} \right) \right) + \langle \overline{H_I} \rangle_{\overline{H_S}(t) + H_B} + O \left(\left\langle \overline{H_I^2} \right\rangle_{\overline{H_S}(t) + H_B} \right). \quad (1169)$$

Taking the equations (244)-(252) and given that $\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) = C(R_0, R_1, R_2, \dots, R_{d-1}, B_{01}, B_{02}, \dots, B_{0(d-1)}, \dots, B_{(d-2)(d-1)})$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = \Re(v_{nu\mathbf{k}}) + i\Im(v_{nu\mathbf{k}})$. So our approach will be based on the derivation respect to $\Re(v_{nu\mathbf{k}})$ and $\Im(v_{nu\mathbf{k}})$. The Hamiltonian $\overline{H_S}(t)$ can be written like:

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \right) |n\rangle\langle n| \quad (1170)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp \left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \quad (1171)$$

$$\prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \quad (1172)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}} v_{nu\mathbf{k}}^* + g_{nu\mathbf{k}}^* v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \quad (1173)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp \left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \quad (1174)$$

$$\prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \quad (1175)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) \Re(v_{nu\mathbf{k}}) + i \Im(v_{nu\mathbf{k}}) (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \quad (1176)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp \left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \quad (1177)$$

$$\prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \quad (1178)$$

$$v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^* = (\Re(v_{mu\mathbf{k}}) - i \Im(v_{mu\mathbf{k}})) (\Re(v_{nu\mathbf{k}}) + i \Im(v_{nu\mathbf{k}})) - (\Re(v_{mu\mathbf{k}}) + i \Im(v_{mu\mathbf{k}})) (\Re(v_{nu\mathbf{k}}) - i \Im(v_{nu\mathbf{k}})) \quad (1179)$$

$$= (\Re(v_{mu\mathbf{k}}) \Re(v_{nu\mathbf{k}}) + i \Im(v_{nu\mathbf{k}}) \Re(v_{mu\mathbf{k}}) - i \Im(v_{mu\mathbf{k}}) \Re(v_{nu\mathbf{k}}) + \Im(v_{mu\mathbf{k}}) \Im(v_{nu\mathbf{k}})) \quad (1180)$$

$$- (\Re(v_{mu\mathbf{k}}) \Re(v_{nu\mathbf{k}}) - i \Im(v_{nu\mathbf{k}}) \Re(v_{mu\mathbf{k}}) + i \Im(v_{mu\mathbf{k}}) \Re(v_{nu\mathbf{k}}) + \Im(v_{mu\mathbf{k}}) \Im(v_{nu\mathbf{k}})) \quad (1181)$$

$$= 2i (\Im(v_{nu\mathbf{k}}) \Re(v_{mu\mathbf{k}}) - \Im(v_{mu\mathbf{k}}) \Re(v_{nu\mathbf{k}})) \quad (1182)$$

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) \Re(v_{nu\mathbf{k}}) + i \Im(v_{nu\mathbf{k}}) (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \quad (1183)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp \left(\frac{i (\Im(v_{nu\mathbf{k}}) \Re(v_{mu\mathbf{k}}) - \Im(v_{mu\mathbf{k}}) \Re(v_{nu\mathbf{k}}))}{\omega_{u\mathbf{k}}^2} \right) \right) \quad (1184)$$

$$\prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \quad (1185)$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}}) (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \quad (1186)$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}} v_{mu\mathbf{k}}^* + v_{nu\mathbf{k}}^* v_{mu\mathbf{k}}) \quad (1187)$$

$$= (\Re(v_{mu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}))^2 + (\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2 \quad (1188)$$

$$- ((\Re(v_{nu\mathbf{k}}) + i \Im(v_{nu\mathbf{k}})) (\Re(v_{mu\mathbf{k}}) - i \Im(v_{mu\mathbf{k}})) + (\Re(v_{nu\mathbf{k}}) - i \Im(v_{nu\mathbf{k}})) (\Re(v_{mu\mathbf{k}}) + i \Im(v_{mu\mathbf{k}}))) \quad (1189)$$

$$= (\Re(v_{mu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}))^2 + (\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2 - 2 (\Re(v_{nu\mathbf{k}}) \Re(v_{mu\mathbf{k}}) + \Im(v_{nu\mathbf{k}}) \Im(v_{mu\mathbf{k}})) \quad (1190)$$

$$= (\Re(v_{mu\mathbf{k}}) - \Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{mu\mathbf{k}}) - \Im(v_{nu\mathbf{k}}))^2 \quad (1191)$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1192)$$

$$= \sum_{u\mathbf{k}} \left(\frac{(\Re(v_{nu\mathbf{k}}))^2 + (\Im(v_{nu\mathbf{k}}))^2 - (g_{nu\mathbf{k}} + g_{nu\mathbf{k}}^*) \Re(v_{nu\mathbf{k}}) - i \Im(v_{nu\mathbf{k}}) (g_{nu\mathbf{k}}^* - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \quad (1193)$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial \Re(v_{nuk})} = \frac{\partial}{\partial \Re(v_{nuk})} \sum_{uk} \left(\frac{(\Re(v_{nuk}))^2 + (\Im(v_{nuk}))^2 - 2\Re(g_{nuk})\Re(v_{nuk}) - 2\Im(g_{nuk})\Im(v_{nuk})}{\omega_{uk}} \right) \quad (1198)$$

$$= \frac{2\Re(v_{nuk}) - 2\Re(g_{nuk})}{\omega_{uk}} \delta_{nn'} \quad (1199)$$

$$= 2 \frac{\Re(v_{nuk}) - \Re(g_{nuk})}{\omega_{uk}} \delta_{nn'} \quad (1200)$$

$$\frac{\partial R_{n'}}{\partial \Im(v_{nuk})} = \frac{\partial}{\partial \Im(v_{nuk})} \sum_{uk} \left(\frac{(\Re(v_{nuk}))^2 + (\Im(v_{nuk}))^2 - 2\Re(g_{nuk})\Re(v_{nuk}) - 2\Im(g_{nuk})\Im(v_{nuk})}{\omega_{uk}} \right) \quad (1201)$$

$$= \frac{2\Im(v_{nuk}) - 2\Im(g_{nuk})}{\omega_{uk}} \delta_{nn'} \quad (1202)$$

$$= 2 \frac{\Im(v_{nuk}) - \Im(g_{nuk})}{\omega_{uk}} \delta_{nn'} \quad (1203)$$

Given that:

$$\ln B_{mn} = \ln \left(\left(\prod_{uk} \exp \left(\frac{i(\Im(v_{nuk})\Re(v_{muk}) - \Im(v_{muk})\Re(v_{nuk}))}{\omega_{uk}^2} \right) \right) \right) \quad (1204)$$

$$\prod_u \exp \left(-\frac{1}{2} \sum_k \frac{(\Re(v_{muk}) - \Re(v_{nuk}))^2 + (\Im(v_{muk}) - \Im(v_{nuk}))^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1205)$$

$$= \sum_{uk} \ln \exp \left(\frac{i(\Im(v_{nuk})\Re(v_{muk}) - \Im(v_{muk})\Re(v_{nuk}))}{\omega_{uk}^2} \right) \quad (1206)$$

$$+ \sum_u \ln \exp \left(-\frac{1}{2} \sum_k \frac{(\Re(v_{muk}) - \Re(v_{nuk}))^2 + (\Im(v_{muk}) - \Im(v_{nuk}))^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1207)$$

$$= \sum_{uk} \left(\frac{i(\Im(v_{nuk})\Re(v_{muk}) - \Im(v_{muk})\Re(v_{nuk}))}{\omega_{uk}^2} \right) \quad (1208)$$

$$+ \sum_{uk} \left(-\frac{1}{2} \frac{(\Re(v_{muk}) - \Re(v_{nuk}))^2 + (\Im(v_{muk}) - \Im(v_{nuk}))^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1209)$$

$$\frac{\partial \ln B_{mn}}{\partial \Re(v_{nuk})} = \frac{-i\Im(v_{muk}) - (\Re(v_{nuk}) - \Re(v_{muk})) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1210)$$

$$\frac{\partial \ln B_{mn}}{\partial \Im(v_{nuk})} = \frac{i\Re(v_{muk}) - (\Im(v_{nuk}) - \Im(v_{muk})) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1211)$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \quad (1212)$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \quad (1213)$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial (B_{nm})^\dagger}{\partial a} \quad (1214)$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial \Re(v_{nuk})} = B_{mn} \frac{\partial \ln B_{mn}}{\partial \Re(v_{nuk})} \quad (1215)$$

$$= B_{mn} \left(\frac{-i\Im(v_{muk}) - (\Re(v_{nuk}) - \Re(v_{muk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1216)$$

$$= B_{mn} \left(\frac{-i\Im(v_{muk}) + (\Re(v_{muk}) - \Re(v_{nuk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1217)$$

$$\frac{\partial B_{nm}}{\partial \Re(v_{nuk})} = \left(\frac{\partial B_{mn}}{\partial \Re(v_{nuk})} \right)^\dagger \quad (1218)$$

$$= \left(B_{mn} \left(\frac{-i\Im(v_{muk}) + (\Re(v_{muk}) - \Re(v_{nuk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right)^\dagger \quad (1219)$$

$$= B_{nm} \left(\frac{i\Im(v_{muk}) + (\Re(v_{muk}) - \Re(v_{nuk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1220)$$

$$\frac{\partial B_{mn}}{\partial \Im(v_{nuk})} = B_{mn} \frac{\partial \ln B_{mn}}{\partial \Im(v_{nuk})} \quad (1221)$$

$$= B_{mn} \left(\frac{i\Re(v_{muk}) - (\Im(v_{nuk}) - \Im(v_{muk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1222)$$

$$= B_{mn} \left(\frac{i\Re(v_{muk}) + (\Im(v_{muk}) - \Im(v_{nuk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1223)$$

$$\frac{\partial B_{nm}}{\partial \Im(v_{nuk})} = \left(\frac{\partial B_{mn}}{\partial \Im(v_{nuk})} \right)^\dagger \quad (1224)$$

$$= (B_{mn})^\dagger \quad (1225)$$

$$= B_{nm} \left(\frac{-i\Re(v_{muk}) + (\Im(v_{muk}) - \Im(v_{nuk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1226)$$

Introducing this derivatives in the equation (1198) give us:

$$\frac{\partial A_B}{\partial \Re(v_{nuk})} = \frac{\partial A_B}{\partial R_n} \left(2 \frac{\Re(v_{nuk}) - \Re(g_{nuk})}{\omega_{uk}} \right) \quad (1227)$$

$$+ \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{i\Im(v_{muk}) + (\Re(v_{muk}) - \Re(v_{nuk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1228)$$

$$+ \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-i\Im(v_{muk}) + (\Re(v_{muk}) - \Re(v_{nuk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1229)$$

$$= 0 \quad (1230)$$

We can obtain the variational parameters:

$$-2 \frac{\partial A_B}{\partial R_n} \frac{\Re(v_{nuk})}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{\Re(v_{nuk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{\Re(v_{nuk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1231)$$

$$= -\frac{\partial A_B}{\partial R_n} \frac{2\Re(g_{nuk})}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{i\Im(v_{muk}) + \Re(v_{muk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-i\Im(v_{muk}) + \Re(v_{muk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1232)$$

$$\Re(v_{nuk}) = \frac{\frac{\partial A_B}{\partial R_n} \frac{2\Re(g_{nuk})}{\omega_{uk}} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{i\Im(v_{muk}) + \Re(v_{muk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-i\Im(v_{muk}) + \Re(v_{muk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right)}{2 \frac{\partial A_B}{\partial R_n} \frac{1}{\omega_{uk}} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right)} \quad (1233)$$

$$= \frac{2\Re(g_{nuk}) \omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(i\Im(v_{muk}) + \Re(v_{muk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(-i\Im(v_{muk}) + \Re(v_{muk}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right) \right)}{2\omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right)} \quad (1234)$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_B}{\partial \Im(v_{nuk})} = \frac{\partial A_B}{\partial R_n} \left(2 \frac{\Im(v_{nuk}) - \Im(g_{nuk})}{\omega_{uk}} \right) \quad (1235)$$

$$+ \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{-i\Re(v_{muk}) - (\Im(v_{nuk}) - \Im(v_{muk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1236)$$

$$+ \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{i\Re(v_{muk}) - (\Im(v_{nuk}) - \Im(v_{muk})) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1237)$$

$$= 0 \quad (1238)$$

Rearranging we obtain

We transform any operator O into the interaction picture in the following way:

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1255)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1256)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1257)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}(t)) \quad (1258)$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1259)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1260)$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \quad (1261)$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2i} \quad (1262)$$

We can proof that $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+} B_{n-} - B_m B_n)^\dagger \quad (1263)$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \quad (1264)$$

$$= B_{n+} B_{m-} - B_n B_m \quad (1265)$$

$$= B_{nm} \quad (1266)$$

So we can say that the set of matrices (1259) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1259) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1267)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}) \quad (1268)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) + i\Im(V_{nm}(t)) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) \right) \quad (1269)$$

$$+ \Re(V_{nm}(t)) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) - i\Im(V_{nm}(t)) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1270)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} + B_{mn}}{2} \right) + \Re(V_{nm}(t)) \sigma_{nm,y} \frac{i(B_{mn} - B_{nm})}{2} \right) \quad (1271)$$

$$+ i\Im(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} - B_{mn}}{2} \right) + \Im(V_{nm}(t)) \sigma_{nm,y} \left(\frac{B_{nm} + B_{mn}}{2} \right) \quad (1272)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (\Re(V_{nm}(t)) \sigma_{nm,x} B_{nm,x} - \Im(V_{nm}(t)) \sigma_{nm,x} B_{nm,y} + \Re(V_{nm}(t)) \sigma_{nm,y} B_{nm,y} \quad (1273)$$

$$+ \Im(V_{nm}(t)) \sigma_{nm,y} B_{nm,x}) \quad (1274)$$

Let's define the set

$$P = \{(n, m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\} \quad (1275)$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \quad (1276)$$

$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \quad (1277)$$

$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \quad (1278)$$

$$A_{4,nm}(t) = A_{2,mn}(t) \quad (1279)$$

$$A_{5,nm}(t) = A_{1,nm}(t) \quad (1280)$$

$$B_{1,nm}(t) = B_{nm,x} \quad (1281)$$

$$B_{2,nm}(t) = B_{nm,y} \quad (1282)$$

$$B_{3,nm}(t) = B_{z,n}(t) \quad (1283)$$

$$B_{4,nm}(t) = B_{1,nm}(t) \quad (1284)$$

$$B_{5,nm}(t) = B_{2,nm}(t) \quad (1285)$$

$$C_{1,nm}(t) = \Re(V_{nm}(t)) \quad (1286)$$

$$C_{2,nm}(t) = C_{1,nm}(t) \quad (1287)$$

$$C_{3,nm}(t) = 1 \quad (1288)$$

$$C_{4,nm}(t) = \Im(V_{nm}(t)) \quad (1289)$$

$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \quad (1290)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t) (A_{jp} \otimes B_{jp}(t)) \quad (1291)$$

Here $J = \{1, 2, 3, 4, 5\}$ and P the set defined in (1275).

We write the interaction Hamiltonian transformed under (1255) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t) (\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)) \quad (1292)$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \quad (1293)$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t) e^{-iH_B t} \quad (1294)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(s)\rho_B]] ds \quad (1295)$$

Replacing the equation (1292) in (1295) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1296)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \left[\sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1297)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1298)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \Big] ds \quad (1299)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right. \quad (1300)$$

$$\left. - \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \right. \quad (1301)$$

$$\left. - \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \right. \quad (1302)$$

$$\left. + \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \right) ds \quad (1303)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B \quad (1304)$$

$$= \left\langle \widetilde{B}_{jp}(\tau) \widetilde{B}_{j'p'}(0) \right\rangle_B \quad (1305)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jp}(t)$ and elements related to the system given by $\widetilde{A}_{jp}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jp}(t) \widetilde{B}_{j'p'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jp}(t) \right) \text{Tr} \left(\widetilde{B}_{j'p'}(t) \right)$. The correlation functions relevant of the master equation (1303) are:

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1306)$$

$$= \left\langle \widetilde{B_{jp}}(0) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1307)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1308)$$

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \rho_B \widetilde{B_{j'p'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1309)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1310)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1311)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1312)$$

$$\text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \rho_B \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) \quad (1313)$$

$$= \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1314)$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1315)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1316)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1317)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1318)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1319)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1320)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1296) and (1303) and using the equations (1306)-(1320) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \widetilde{A_{jp}}(t) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau) \widetilde{A_{jp}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \right) \right. \quad (1321)$$

$$\left. + C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{j'p'jp}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jp}}(t) \right) \right) ds \quad (1322)$$

$$= - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \left[\widetilde{A_{jp}}(t), \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s), \widetilde{A_{jp}}(t) \right] \right) \right) \quad (1323)$$

Rearranging and identifying the commutators allow us to write a more simplified version

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(t-\tau) \left(\Lambda_{jpj'p'}(\tau) \left[A_{jp}(t), A_{j'p'}(t-\tau, t) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) A_{j'p'}(t-\tau, t), A_{jp}(t) \right] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1324)$$

For this case we used that $A_{jp}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jp}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation.

VIII. BIBLIOGRAPHY

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