Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality. Let´s define the partition functions of $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ given by $Z\left(t\right)$ and $Z_0\left(t\right)$ respectively as:

$$Z\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}\left(t\right)}\right), \tag{1}$$

$$Z_0\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}_0\left(t\right)}\right). \tag{2}$$

where the transformed hamiltonians $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ are defined as:

$$\overline{H}\left(t\right) \equiv \overline{H_{\overline{I}}}\left(t\right) + \overline{H_0}\left(t\right), \tag{3}$$

$$\overline{H_0}\left(t\right) \equiv \overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}. \tag{4}$$

For any operator $A\left(t\right)$ we define the expected value respect to $\overline{H_0}\left(t\right)$ as:

$$\left\langle A\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} \equiv \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}\left(t\right)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}\left(t\right)}\right)}. \tag{5}$$

The terms $\overline{H_{\bar{S}}}\left(t\right)$, $\overline{H_{\bar{B}}}$ and $\overline{H_{\overline{I}}}\left(t\right)$ are related to the variational transformation performed in [1,2], this transformation allowed us to construct $\overline{H_{\overline{I}}}\left(t\right)$ such that $\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} = 0$. The diagonalization of $\overline{H_0}\left(t\right)$ in terms of it's eigenstates and eigenvalues such that $\overline{H_0}\left(t\right)|n\rangle = E_{0,n}\left(t\right)|n\rangle$, being $|n\rangle$ an eigenstate of $\overline{H_0}\left(t\right)$ with eigenvalue $E_{0,n}\left(t\right)$ is $\overline{H_0}\left(t\right) = \sum_n E_{0,n}\left(t\right)|n\rangle\langle n|$, with $\langle n|n'\rangle = \delta_{nn'}$, so a simple form of $\mathrm{e}^{-\beta\overline{H_0}\left(t\right)}$ can be found as follows:

$$\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\mathrm{e}^{rY}\mathrm{e}^{-\frac{r^2}{2}[X,Y]}\mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\cdots \text{ (Zassenhaus formula)}, \tag{6}$$

$$\mathrm{e}^{X+Y} = \mathrm{e}^{X}\mathrm{e}^{Y}\mathrm{e}^{-\frac{r^2}{2}0}\mathrm{e}^{\frac{r^3}{6}(2[Y,0]+[X,0])}\cdots \text{ (setting } r=1 \text{ and } [X,Y]=0 \text{ in (6))} \tag{7}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}\mathbb{I} \tag{8}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}, \tag{9}$$

$$\mathrm{e}^{-\beta\overline{H_0}\left(t\right)} = \mathrm{e}^{-\sum_n \beta E_{0,n}\left(t\right)|n\rangle\langle n|} \text{ (by the diagonalization of } \overline{H_0}\left(t\right)\text{)} \tag{10}$$

$$
\begin{aligned}
\mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\langle n|} &= \prod_n \mathrm{e}^{-\beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by (9) and } [|n\rangle\langle n|, |n'\rangle\langle n'|] = 0) && (11)\\
&= \prod_n \sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)|n\rangle\langle n|\right)^j}{j!} \text{ (by the exponential formula)} && (12)\\
&= \prod_n \left(\mathbb{I} + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)\right)^j |n\rangle\langle n|}{j!}\right) \text{ (using } (aA)^j = a^j A^j \text{ and } (|n\rangle\langle n|)^2 = |n\rangle\langle n|) && (13)\\
&= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)\right)^j |n\rangle\langle n|}{j!}\right) && (14)\\
&= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left(\sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)\right)^j}{j!}\right)\right) && (15)\\
&= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \text{ (by the exponential formula)} && (16)\\
&= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right). && (17)
\end{aligned}
$$

We will prove by induction a neat form for (17), we will show that:

$$
\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \mathbb{I} + \sum_{j=1}^{n} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \quad (18)
$$

For $n = 1$ the formula is trivial, in the case $n = 2$ we obtain that:

$$
\begin{aligned}
\prod_{j=1}^{2} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) &= \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2|\right) && (19)\\
&= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1| + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2| \text{ (by } \langle i|j\rangle = \delta_{ij}) && (20)\\
&= \mathbb{I} + \sum_{j=1}^{2} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|n\rangle\langle n|. && (21)
\end{aligned}
$$

It is our case base, our induction step is (18), in the case $n + 1$ we will have:

$$
\begin{aligned}
\prod_{j=1}^{n+1} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) &= \left(\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right)\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) && (22)\\
&= \left(\mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) \text{ (by induction step)} && (23)\\
&= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1| + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| \text{ (by } \langle i|j\rangle = \delta_{ij}) && (24)\\
&= \mathbb{I} + \sum_{j=1}^{n+1} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. && (25)
\end{aligned}
$$

By mathematical induction we proved that (18) is true for all $n \in \mathbb{N}$. Given that the resolution of the identity is $\mathbb{I} = \sum_n |n\rangle\langle n|$ so we find that:

$$
\begin{aligned}
\mathrm{e}^{-\beta\overline{H_0}(t)} &= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right) && (26)\\
&= \mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| && (27)\\
&= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \sum_n |n\rangle\langle n| && (28)\\
&= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \mathbb{I} \text{ (by the resolution of the identity)} && (29)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|. && (30)
\end{aligned}
$$

The partition function $Z_0\left(t\right)$ is equal to:

$$
\begin{aligned}
Z_0\left(t\right) &= \mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) && (31)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(|n\rangle\langle n|\right) && (32)\\
&= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}. && (33)
\end{aligned}
$$

The explicit form of the average value $\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)}$ can be found from the partition function $Z_0\left(t\right)$:

$$
\begin{aligned}
\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)} &= \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{Z_0\left(t\right)} && (34)\\
&= \frac{\mathrm{Tr}\left(\sum_n A\left(t\right)\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta\overline{H_0}(t)}\right)} && (35)\\
&= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)} && (36)\\
&= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} && (37)\\
&= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}}. && (38)
\end{aligned}
$$

At first we show a double sequence of inequalities of order M, N which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$
Z\left(t\right) \geq Z_0\left(t\right)\mathrm{e}^{-\left\langle \overline{H_I}(t)\right\rangle_{\overline{H_0}(t)}}\left(1 + F_M\left(\overrightarrow{u}\left(t\right);\alpha\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right);\alpha\right)\right). \tag{39}
$$

where the funcion $F_N\left(\overrightarrow{u}\left(t\right);\alpha\right)$ is defined as:

$$
F_N\left(\overrightarrow{w}\left(t\right);\alpha\right) \equiv \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{w_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. \tag{40}
$$

In this case α is a parameter that can be optimized, $\beta \equiv \frac{1}{\mathrm{k_B}T}$, $\overrightarrow{w}\left(t\right)$ is a vector such that $\overrightarrow{w}\left(t\right) = \left(w_1, w_2, ...\right)$ and $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian $\overline{H_{\overline{I}D}}\left(t\right)$ respect to the basis of eigenstates of $\overline{H_0}\left(t\right)$ as:

$$\overline{H_{\overline{I}D}}\,(t) \equiv \sum_{n} \left\langle n \left| \overline{H_{\overline{I}}}\,(t) \right| n \right\rangle |n\rangle\langle n|. \tag{41}$$

We will prove an important property related to $\overline{H_{\overline{I}D}}\,(t)$, which is a Hamiltonian written as a linear combination of a set of ortonormal operators. Let's consider a vector space R with two operations $+$ and $\cdot$, if there exist $a, b \in R$ such that $a \cdot b = 0$ and $b \cdot a = 0$ then for any $k \in \mathbb{N}$ we have $(a+b)^k = a^k + b^k$ where $a^k = a^{k-1} \cdot a$ is a recursive definition of the power of an element written in terms of $\cdot$. At first we prove that this result yields for any $k \in \mathbb{N}$ by induction, the case $k = 1$ is trivial so we will focus on the case $k = 2$, we have that:

$$\begin{aligned}
(a+b)^2 &= (a+b) \cdot (a+b) && (42)\\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b && (43)\\
&= a^2 + a \cdot b + b \cdot a + b^2 && (44)\\
&= a^2 + 0 + 0 + b^2 \text{ (because } a \cdot b = b \cdot a = 0) && (45)\\
&= a^2 + b^2. && (46)
\end{aligned}$$

This is the base case. By induction step we will consider that $(a+b)^k = a^k + b^k$ with $k \geq 2$, now for $k+1$ we will have that:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)^k \cdot (a+b) && (47)\\
&= \left(a^k + b^k\right) \cdot (a+b) \text{ (by induction step)} && (48)\\
&= a^k \cdot a + a^k \cdot b + b^k \cdot a + b^k \cdot b && (49)\\
&= a^{k+1} + a^{k-1} \cdot a \cdot b + b^{k-1} \cdot b \cdot a + b^{k+1} \text{ (by recursive definition of } a^k) && (50)\\
&= a^{k+1} + a^{k-1} \cdot (a \cdot b) + b^{k-1} \cdot (b \cdot a) + b^{k+1} \text{ (by associativity on } R) && (51)\\
&= a^{k+1} + a^{k-1} \cdot (0) + b^{k-1} \cdot (0) + b^{k+1} \text{ (because } a \cdot b = b \cdot a = 0) && (52)\\
&= a^{k+1} + b^{k+1}. && (53)
\end{aligned}$$

By the principle of mathematical induction we can conclude that the proposition is true for all $k \in \mathbb{N}$. Now we will extend the result, let $a_1, ..., a_n \in R$ such that $a_i \cdot a_j = 0$ for all $i \neq j$ then $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$. The case $n = 1$ is trivial as well so we will focus on $n = 2$, this case was proved in the precedent lines so it will be our base case. By induction step we will consider that $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$ with $n \geq 2$, now for $n+1$ we will have that:

$$\begin{aligned}
a_{n+1} \cdot (a_1 + ... + a_n) &= a_{n+1} \cdot a_1 + ... + a_{n+1} \cdot a_n && (54)\\
&= 0 \text{ (because } a_i \cdot a_j = 0 \text{ for all } i \neq j \text{),} && (55)\\
(a_1 + ... + a_n + a_{n+1})^k &= ((a_1 + ... + a_n) + a_{n+1})^k && (56)\\
&= (a_1 + ... + a_n)^k + a_{n+1}^k \text{ (by (47) and (55))} && (57)\\
&= a_1^k + ... + a_n^k + a_{n+1}^k \text{ (by inductive step).} && (58)
\end{aligned}$$

So we can conclude by mathematical induction that the proposition is true for all $n \in \mathbb{N}$. We can prove the following property for $\left(\overline{H_{\overline{I}D}}\,(t)\right)^k$:

$$\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle|n\rangle\langle n|\left\langle n'\left|\overline{H_{\overline{I}}}\left(t\right)\right|n'\right\rangle|n'\rangle\langle n'| = \left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle\left\langle n'\left|\overline{H_{\overline{I}}}\left(t\right)\right|n'\right\rangle|n\rangle\langle n|n'\rangle\langle n'| \tag{59}$$

$$= \left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle\left\langle n'\left|\overline{H_{\overline{I}}}\left(t\right)\right|n'\right\rangle|n\rangle\langle n'|\delta_{nn'}, \tag{60}$$

$$\left(\overline{H_{\overline{I}D}}\left(t\right)\right)^k = \left(\sum_n\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle|n\rangle\langle n|\right)^k \text{ (by (41))} \tag{61}$$

$$= \sum_n\left(\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle|n\rangle\langle n|\right)^k \text{ (by (58) and (60)) ,} \tag{62}$$

$$\left(aA\right)^k = a^k A^k \text{ (by the property of the power of a matrix) ,} \tag{63}$$

$$\left(|n\rangle\langle n|\right)^k = |n\rangle\langle n| \text{ (because } |n\rangle\langle n| \text{ is a projector and } k\in\mathbb{N}^*\text{) ,} \tag{64}$$

$$\left(\overline{H_{\overline{I}D}}\left(t\right)\right)^k = \sum_n\left(\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle\right)^k|n\rangle\langle n| \text{ (by (63) and (64)) .} \tag{65}$$

The vectors $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ are defined as $\overrightarrow{u}\left(t\right)\equiv\left(u_1,u_2,...\right)$ and $\overrightarrow{v}\left(t\right)\equiv\left(v_1,v_2,...\right)$. We can define the elements of $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ in terms of the matrix $\overline{H_{\overline{I}D}}\left(t\right)$:

$$u_k\left(t\right)\equiv\left\langle\left(\overline{H_{\overline{I}D}}\left(t\right)-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k\right\rangle_{\overline{H_0}(t)} \tag{66}$$

$$=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\sum_n\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle|n\rangle\langle n|-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k|n\rangle\langle n|\right)}{Z_0\left(t\right)} \text{ (by (38)) ,} \tag{67}$$

$$\left(\sum_n\langle n|\overline{H_{\overline{I}}}\left(t\right)|n\rangle|n\rangle\langle n|-\langle\overline{H_{\overline{I}}}(t)\rangle_{\overline{H_0}(t)}\right)^k=\sum_{j=0}^k(-1)^j\binom{k}{j}\left(\sum_n\langle n|\overline{H_{\overline{I}}}\left(t\right)|n\rangle|n\rangle\langle n|\right)^j\left(\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^{k-j} \text{ (by binomial theorem)} \tag{68}$$

$$=\sum_{j=0}^k\left(-1\right)^j\binom{k}{j}\left(\sum_n\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle^j|n\rangle\langle n|\right)\left(\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^{k-j} \text{ (by (65))} \tag{69}$$

$$=\sum_n\left(\sum_{j=0}^k\left(-1\right)^j\binom{k}{j}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle^j\left(\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^{k-j}\right)|n\rangle\langle n| \tag{70}$$

$$=\sum_n\left(\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k|n\rangle\langle n|, \tag{71}$$

$$=\sum_n\left(\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k|n\rangle\langle n|, \tag{72}$$

$$u_k\left(t\right)=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\sum_{n'}\left(\left\langle n'\left|\overline{H_{\overline{I}}}\left(t\right)\right|n'\right\rangle-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k|n'\rangle\langle n'|n\rangle\langle n|\right)}{Z_0\left(t\right)} \tag{73}$$

$$=\frac{\sum_{nn'} \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{\overline{I}}}\left(t\right)\right|n'\right\rangle-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k|n'\rangle\langle n|\delta_{nn'}\right)}{Z_0\left(t\right)} \tag{74}$$

$$=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k\mathrm{Tr}\left(|n\rangle\langle n|\right)}{Z_0\left(t\right)} \tag{75}$$

$$=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k 1}{Z_0\left(t\right)} \tag{76}$$

$$=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k}{Z_0\left(t\right)}, \tag{77}$$

$$v_k\left(t\right)\equiv\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)-\left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)}\right)^k\right|n\right\rangle}{Z_0\left(t\right)}. \tag{78}$$

By construction $\left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} = 0$, so we summarize the double inequality that generalizes the Bogoliubov inequality and it's coefficients as:

$$\begin{aligned}
Z\left(t\right) &\geq Z_0\left(t\right)\left(1 + F_M\left(\overrightarrow{u}\left(t\right)\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right)\right)\right), && (79)\\
Z\left(t\right) &= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}\left(t\right)}\right), && (80)\\
Z_0\left(t\right) &= \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}, && (81)\\
F_N\left(\overrightarrow{u}\left(t\right)\right) &= \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k \frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}, && (82)\\
u_k\left(t\right) &= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle^k}{Z_0\left(t\right)}, && (83)\\
v_k\left(t\right) &= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right)^k\right|n\right\rangle}{Z_0\left(t\right)}. && (84)
\end{aligned}$$

As we can see the expression (83) was written in shorter terms, we want to do the same for (84) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for $v_k\left(t\right)$:

$$\begin{aligned}
\left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right)\left|n\right\rangle &= \overline{H_0}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (85)\\
&= E_{0,n}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (86)\\
&= 0, && (87)\\
\left\langle n\right|\left(\overline{H_0}\left(t\right) - E_{0,n}\right) &= \left\langle n\right|\overline{H_0}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) && (88)\\
&= \left\langle n\right|E_{0,n}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) && (89)\\
&= 0. && (90)
\end{aligned}$$

At first we calculated $v_1\left(t\right)$ using the definition (84) :

$$\begin{aligned}
v_1\left(t\right) &= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right|n\right\rangle && (91)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right|n\right\rangle + \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle && (92)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|\overline{H_0}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} && (93)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} && (94)\\
&= 0 + \left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} && (95)\\
&= 0. && (96)
\end{aligned}$$

For $k \geq 2$ and $k \in \mathbb{N}$ we calculated:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^k \right| n \right\rangle \tag{97}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{98}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{99}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{100}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle . \tag{101}$$

In general we can write a formula for $v_k(t)$ that implies an expected value of a dependent expression of $\overline{H_{\overline{I}}}(t)$ and $\overline{H_0}(t)$:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{102}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) + \overline{H_{\overline{I}}}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{103}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{104}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j}(t)\, E_{0,n}^j(t) \right) \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{105}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, E_{0,n}^j(t) \right| n \right\rangle \tag{106}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{107}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{108}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t)\; \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{109}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} . \tag{110}$$

The formula (110) is well defined taking as example $k = 2, 3$.

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{2-2-j} \overline{H_{\overline{I}}}(t)\; \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \tag{111}$$

$$= (-1)^0 \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\; \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \tag{112}$$

$$= \left\langle \overline{H_{\overline{I}}}^2(t) \right\rangle_{\overline{H_0}(t)} , \tag{113}$$

$$v_3\left(t\right)=\left\langle\sum_{j=0}^{3-2}\left(-1\right)^j\binom{3-2}{j}\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{3-2-j}\overline{H_I}\left(t\right)\,\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{114}$$

$$=\left\langle\sum_{j=0}^{1}\left(-1\right)^j\binom{1}{j}\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{1-j}\overline{H_I}\left(t\right)\,\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{115}$$

$$=\left\langle(-1)^0\binom{1}{0}\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^1\overline{H_I}(t)\,\overline{H_0}^0(t)+(-1)^1\binom{1}{1}\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^0\overline{H_I}(t)\,\overline{H_0}^1(t)\right\rangle_{\overline{H_0}(t)} \tag{116}$$

$$=\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_I}\left(t\right)\,\mathbb{I}-\overline{H_I}\left(t\right)\mathbb{I}\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{117}$$

$$=\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_I}\left(t\right)-\overline{H_I}\left(t\right)\,\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{118}$$

$$=\left\langle\overline{H_I}\left(t\right)^3+\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)\,\overline{H_I}\left(t\right)-\overline{H_I}\left(t\right)\,\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{119}$$

$$=\left\langle\overline{H_I}\left(t\right)^3+\overline{H_I}\left(t\right)\left(\overline{H_0}\left(t\right)\,\overline{H_I}\left(t\right)-\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)\right)\right\rangle_{\overline{H_0}(t)} \tag{120}$$

$$=\left\langle\overline{H_I}\left(t\right)^3+\overline{H_I}\left(t\right)\left[\overline{H_0}\left(t\right),\overline{H_I}\left(t\right)\right]\right\rangle_{\overline{H_0}(t)}\ \text{(because }\left[\overline{H_0}\left(t\right),\overline{H_I}\left(t\right)\right]=\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)-\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\text{).} \tag{121}$$

So we summarize:

$$\overline{H_I}_D\left(t\right)=\sum_n\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle|n\rangle\langle n|, \tag{122}$$

$$u_k\left(t\right)=\left\langle\left(\overline{H_I}_D\left(t\right)\right)^k\right\rangle_{\overline{H_0}(t)}, \tag{123}$$

$$v_k\left(t\right)=\sum_{j=0}^{k-2}\left(-1\right)^j\binom{k-2}{j}\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{k-2-j}\overline{H_I}\left(t\right)\,\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0}(t)}. \tag{124}$$

Then we obtained finally:

$$Z\left(t\right)\geq Z_0\left(t\right)\left(1+F_M\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right), \tag{125}$$

The free energy is defined as:

$$E_{\text{free}}\left(t\right)\equiv-\frac{1}{\beta}\ln\left(Z\left(t\right)\right). \tag{126}$$

It is well-known that the function $f\left(x\right)=\ln\left(x\right)$ is monotonic and increasing so we can transform (125):

$$E_{\text{free},1}\left(t\right)=-\frac{1}{\beta}\ln\left(Z_0\left(t\right)\right), \tag{127}$$

$$E_{\text{free}}\left(t\right)\leq-\frac{1}{\beta}\ln\left(Z_0\left(t\right)\left(1+F_M\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right)\right) \tag{128}$$

$$E_{\text{free}}\left(t\right)\leq-\frac{1}{\beta}\ln\left(Z_0\left(t\right)\right)-\frac{1}{\beta}\ln\left(1+F_M\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right) \tag{129}$$

$$E_{\text{free}}\left(t\right)\leq E_{\text{free},1}\left(t\right)-\frac{1}{\beta}\ln\left(1+F_M\left(\overrightarrow{u}\left(t\right)\right)+F_N\left(\overrightarrow{v}\left(t\right)-\overrightarrow{u}\left(t\right)\right)\right) \tag{130}$$

$$\equiv E_{\text{free,MN}}\left(t\right). \tag{131}$$

here $E_{\text{free,MN}}\left(t\right)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of M,N order. In our approach we will set $N=M$, so our quantum Bogoliubov inequality of N order is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) \tag{132}$$

$$= E_{\text{free,NN}}(t). \tag{133}$$

A weaker form of the inequality (133) is obtained making $\overrightarrow{u}(t) = 0$ as suggest [3]:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{v}(t)\right)\right) \tag{134}$$

$$\equiv E_{\text{free,N}}(t). \tag{135}$$

The algebraic equation associated with $\alpha_{\text{opt}}(t)$ such that $E_{\text{free,N}}(t)$ is closer to $E_{\text{free}}(t)$ follows from the fact that in the optimal parameter $\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha}|_{\alpha_{\text{opt}(t)}} = 0$, calculating this derivate we have:

$$\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{v}(t)\right)\right)\right) \tag{136}$$

$$= -\frac{1}{\beta}\frac{\frac{\partial}{\partial\alpha}\left(F_N\left(\overrightarrow{v}(t)\right)\right)}{1 + F_N\left(\overrightarrow{v}(t)\right)} \tag{137}$$

$$= 0. \tag{138}$$

The precedent equation is equivalent to:

$$\frac{\partial F_N\left(\overrightarrow{v}(t)\right)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{139}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\partial}{\partial\alpha}\frac{\alpha^i}{i!} \text{ (by product rule)} \tag{140}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} \tag{141}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{142}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{j=0}^{2N-2-k}\frac{\alpha^j}{j!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \text{ (setting } j = i-1\text{)} \tag{143}$$

$$= \mathrm{e}^{-\alpha}\left(-\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha^{2N-1-k}}{(2N-1-k)!}\right) \text{ (performing the difference)} \tag{144}$$

$$= 0. \tag{145}$$

Then the optimal value $\alpha_{\text{opt}}(t)$ will sastisfy the following equation:

$$G\left(\alpha_{\text{opt}}(t)\right) \equiv \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \tag{146}$$

$$= 0. \tag{147}$$

The elements presented are the required to find variational parameters of the system using the inequality (135) and the self consistent equation (146) to a particular order expected.

II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2\left(t\right), v_3\left(t\right), v_4\left(t\right), v_5\left(t\right)$ using the (124), we have already $v_2\left(t\right), v_3\left(t\right)$ and the form of $v_4\left(t\right)$ and $v_5\left(t\right)$ is given by:

$$v_4\left(t\right)=\sum_{j=0}^{4-2}\left(-1\right)^j\binom{4-2}{j}\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{4-2-j}\overline{H_I}\left(t\right)\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{148}$$

$$=\sum_{j=0}^{2}\left(-1\right)^j\binom{2}{j}\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{2-j}\overline{H_I}\left(t\right)\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{149}$$

$$=\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}\left(t\right)\overline{H_0}^0(t)\right\rangle_{\overline{H_0(t)}}-2\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^1\overline{H_I}\left(t\right)\overline{H_0}^1(t)\right\rangle_{\overline{H_0(t)}}+\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)\right.\right. \tag{150}$$

$$\left.\left.+\overline{H_0}\left(t\right)\right)^0\overline{H_I}\left(t\right)\overline{H_0}^2\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{151}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)^2\overline{H_I}\left(t\right)\mathbb{I}\right\rangle_{\overline{H_0(t)}}-2\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^1\overline{H_I}\left(t\right)\overline{H_0}^1(t)\right\rangle_{\overline{H_0(t)}}+\left\langle\overline{H_I}^2(t)\,\overline{H_0}^2(t)\right\rangle_{\overline{H_0(t)}} \tag{152}$$

$$=\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^2\overline{H_I}\left(t\right)\right\rangle_{\overline{H_0(t)}}-2\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0(t)}}+\left\langle\overline{H_I}^2\left(t\right)\overline{H_0}^2\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{153}$$

$$=\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^2\overline{H_I}\left(t\right)-2\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)+\overline{H_I}^2\left(t\right)\overline{H_0}^2\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{154}$$

$$=\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}^2\left(t\right)+\overline{H_I}\left(t\right)\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}\left(t\right)+\overline{H_0}^2\left(t\right)\right)\overline{H_I}\left(t\right)-2\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)\overline{H_I}\left(t\right)\overline{H_0}(t)+\overline{H_I}^2\left(t\right)\right. \tag{155}$$

$$\left.\times\overline{H_0}^2\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{156}$$

$$=\left\langle\overline{H_I}^4\left(t\right)+\overline{H_I}^2\left(t\right)\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)+\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\overline{H_I}^2(t)+\overline{H_I}\left(t\right)\overline{H_0}^2\left(t\right)\overline{H_I}\left(t\right)-2\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\right. \tag{157}$$

$$\left.+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\right\rangle_{\overline{H_0(t)}} \tag{158}$$

$$=\left\langle\overline{H_I}^4(t)+\overline{H_I}^2(t)\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t)+\overline{H_I}(t)\overline{H_0}^2(t)\,\overline{H_I}(t)-2\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}^2(t)\overline{H_0}^2(t)-2\overline{H_I}\left(t\right)\right. \tag{159}$$

$$\left.\times\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{160}$$

$$=\left\langle\overline{H_I}^4(t)+\overline{H_I}^2(t)\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}\left(t\right)\overline{H_0}(t)\overline{H_I}^2(t)+\overline{H_I}\left(t\right)\overline{H_0}^2(t)\,\overline{H_I}\left(t\right)-\overline{H_I}^3(t)\overline{H_0}(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}^2(t)\right. \tag{161}$$

$$\left.\times\overline{H_0}^2\left(t\right)-\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)-\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{162}$$

$$=\left\langle\overline{H_I}^4(t)+\overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}\left(t\right)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}\left(t\right)\overline{H_0}(t)\,\overline{H_I}^2(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}\left(t\right)\overline{H_0}^2(t)\,\overline{H_I}\left(t\right)-\overline{H_I}\left(t\right)\right. \tag{163}$$

$$\left.\times\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)+\overline{H_I}^2\left(t\right)\overline{H_0}^2\left(t\right)-\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{164}$$

$$=\left\langle\overline{H_I}^4(t)+\overline{H_I}\left(t\right)\left(\left(\left(\overline{H_I}\left(t\right)\overline{H_0}(t)\right)\overline{H_I}\left(t\right)-\overline{H_I}\left(t\right)\left(\overline{H_I}(t)\overline{H_0}(t)\right)\right)+\left(\overline{H_0}(t)\overline{H_I}^2(t)-\overline{H_I}^2(t)\overline{H_0}(t)\right)+\left(\overline{H_0}(t)\left(\overline{H_0}(t)\right.\right.\right.\right. \tag{165}$$

$$\left.\left.\left.\left.\times\overline{H_I}\left(t\right)\right)-\left(\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)\right)\overline{H_0}\left(t\right)\right)+\left(\left(\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\right)\overline{H_0}\left(t\right)-\overline{H_0}\left(t\right)\left(\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\right)\right)\right)\right\rangle_{\overline{H_0(t)}} \tag{166}$$

$$=\left\langle\overline{H_I}^4(t)+\overline{H_I}\left(t\right)\left(\left[\overline{H_I}\left(t\right)\overline{H_0}\left(t\right),\overline{H_I}\left(t\right)\right]+\left[\overline{H_0}\left(t\right),\overline{H_I}^2(t)\right]+\left[\overline{H_0}\left(t\right),\overline{H_0}\left(t\right)\overline{H_I}\left(t\right)\right]+\left[\overline{H_I}\left(t\right)\overline{H_0}\left(t\right),\overline{H_0}\left(t\right)\right]\right)\right\rangle_{\overline{H_0(t)}}, \tag{167}$$

$$v_5\left(t\right)=\sum_{j=0}^{5-2}\left(-1\right)^j\binom{5-2}{j}\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{5-2-j}\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)^j\right\rangle_{\overline{H_0(t)}} \tag{168}$$

$$=\sum_{j=0}^{3}\left(-1\right)^j\binom{3}{j}\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{3-j}\overline{H_I}\left(t\right)\,\overline{H_0}\left(t\right)^j\right\rangle_{\overline{H_0(t)}} \tag{169}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)^3\overline{H_I}\left(t\right)\overline{H_0}^0(t)-3\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)^2\overline{H_I}\left(t\right)\overline{H_0}(t)-\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)^0\overline{H_I}\left(t\right)\overline{H_0}^3(t)\right. \tag{170}$$

$$\left.+3\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_I}\left(t\right)\overline{H_0}^2\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{171}$$

$$=\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)^3\overline{H_I}\left(t\right)-3\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)^2\overline{H_I}\left(t\right)\overline{H_0}(t)+3\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)\overline{H_I}\left(t\right)\overline{H_0}^2(t)-\overline{H_I}\left(t\right)\right. \tag{172}$$

$$\left.\times\overline{H_I}\left(t\right)\overline{H_0}^3\left(t\right)\right\rangle_{\overline{H_0(t)}} \tag{173}$$

$$=\Big\langle \overline{H_I}(t)\big(\overline{H_I}(t)+\overline{H_0}(t)\big)^3\overline{H_I}(t)-3\overline{H_I}(t)\big(\overline{H_I}(t)+\overline{H_0}(t)\big)^2\overline{H_I}(t)\overline{H_0}(t)+3\overline{H_I}(t)\big(\overline{H_I}(t)+\overline{H_0}(t)\big)\overline{H_I}(t)\overline{H_0}^2(t)-\overline{H_I}(t) \tag{174}$$

$$\times\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0(t)}} \tag{175}$$

$$=\Big\langle \overline{H_I}(t)\big(\overline{H_I}(t)+\overline{H_0}(t)\big)^3\,\overline{H_I}(t)-3\overline{H_I}(t)\Big(\overline{H_I}^2(t)+\overline{H_I}(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)+\overline{H_0}^2(t)\Big)\overline{H_I}(t)\overline{H_0}(t)+3\overline{H_I}(t)\big(\overline{H_I}(t) \tag{176}$$

$$+\overline{H_0}(t)\big)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0(t)}} \tag{177}$$

$$=\Big\langle \overline{H_I}(t)\,\Big(\overline{H_I}^3(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}(t) \tag{178}$$

$$\times\overline{H_0}^2(t)+\overline{H_0}^3(t)\Big)\,\overline{H_I}(t)-3\overline{H_I}(t)\,\Big(\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^2(t)\Big)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}(t)\,\big(\overline{H_I}(t) \tag{179}$$

$$+\overline{H_0}(t)\big)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-\overline{H_I}(t)\,\overline{H_I}(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0(t)}} \tag{180}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}^3(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^3(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t) \tag{181}$$

$$\times\,\overline{H_0}^2(t)\,\overline{H_I}^2(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}^4(t)\,\overline{H_0}(t)-3\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-3\overline{H_I}(t) \tag{182}$$

$$\times\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)-3\overline{H_I}(t)\overline{H_0}^2(t)\,\overline{H_I}(t)\overline{H_0}(t)+3\overline{H_I}^3(t)\overline{H_0}^2(t)+3\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}^2(t)-\overline{H_I}^2(t)\,\overline{H_0}^3(t)\Big\rangle_{\overline{H_0(t)}} \tag{183}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^2(t)+\overline{H_0}(t)\,\overline{H_I}^3(t)+\overline{H_0}^2(t)\,\overline{H_I}^2(t)+\overline{H_I}(t)\,\overline{H_0}^2(t)\,\overline{H_I}(t) \tag{184}$$

$$+\,\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}(t)-3\overline{H_I}^3(t)\overline{H_0}(t)-3\overline{H_0}^2(t)\,\overline{H_I}(t)\overline{H_0}(t)-\overline{H_I}(t) \tag{185}$$

$$\times\overline{H_0}^3(t)+3\overline{H_I}^2(t)\,\overline{H_0}^2(t)+3\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-3\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big\rangle_{\overline{H_0(t)}} \tag{186}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}^3(t)-\overline{H_0}(t)\overline{H_I}^2(t) \tag{187}$$

$$\times\,\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}(t)-\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)+\overline{H_0}^3(t)\,\overline{H_I}(t) \tag{188}$$

$$-\,\overline{H_I}(t)\,\overline{H_0}^3(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)-\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)+2\overline{H_I}^2(t)\,\overline{H_0}^2(t)-2\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_0}(t)\,\overline{H_I}(t) \tag{189}$$

$$\times\overline{H_0}^2(t)-3\overline{H_0}^2(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)+\overline{H_0}^2(t)\,\overline{H_I}^2(t)\Big)\Big\rangle_{\overline{H_0(t)}} \tag{190}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big(\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\overline{H_I}(t)-\overline{H_I}(t)\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big)+\Big(\big(\overline{H_I}(t)\,\overline{H_0}(t)\big)\overline{H_I}^2(t)-\overline{H_I}^2(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)\big)\Big) \tag{191}$$

$$+\,\Big(\overline{H_0}(t)\overline{H_I}^3(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)\Big)+\Big(\overline{H_0}(t)\big(\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\big)-\big(\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\big)\overline{H_0}(t)\Big)+\Big(\overline{H_0}(t)\Big(\overline{H_0}(t)\overline{H_I}^2(t)\Big) \tag{192}$$

$$-\,\Big(\overline{H_0}(t)\overline{H_I}^2(t)\Big)\overline{H_0}(t)\Big)+\Big(\overline{H_0}^3(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}^3(t)\Big)+\Big(\big(\overline{H_I}(t)\overline{H_0}(t)\big)\big(\overline{H_0}(t)\overline{H_I}(t)\big)-\big(\overline{H_0}(t)\overline{H_I}(t)\big)\big(\overline{H_I}(t)\overline{H_0}(t)\big)\Big) \tag{193}$$

$$+\,3\overline{H_0}(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_I}(t)\big)\overline{H_0}(t)+2\overline{H_I}(t)\big(\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_I}(t)\big)\overline{H_0}(t)+\Big(\Big(\overline{H_I}^2(t)\overline{H_0}(t)\Big)\big(\overline{H_0}(t)\big) \tag{194}$$

$$-\,\big(\overline{H_0}(t)\big)\,\Big(\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big)\Big)\Big\rangle_{\overline{H_0(t)}} \tag{195}$$

$$=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\,\Big(\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_I}^3(t)\Big]+\big[\overline{H_0}(t)\,,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\big] \tag{196}$$

$$+\,\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}^3(t)\,,\overline{H_I}(t)\Big]+\big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+2\overline{H_I}(t) \tag{197}$$

$$\times\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0(t)}}\,. \tag{198}$$

Summarizing we have that:

$$v_2(t)=\Big\langle \overline{H_I}^2(t)\Big\rangle_{\overline{H_0(t)}}\,, \tag{199}$$

$$v_3(t)=\Big\langle \overline{H_I}^3(t)+\overline{H_I}(t)\,\big[\overline{H_0}(t)\,,\overline{H_I}(t)\big]\Big\rangle_{\overline{H_0(t)}}\,, \tag{200}$$

$$v_4(t)=\Big\langle \overline{H_I}^4(t)+\overline{H_I}(t)\Big(\big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}(t)\big]+\Big[\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\big[\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_I}(t)\big]+\big[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\big]\Big)\Big\rangle_{\overline{H_0(t)}}, \tag{201}$$

$$v_5(t)=\Big\langle \overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_I}^3(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}^2(t)\Big] \tag{202}$$

$$+\,\big[\overline{H_0}(t)\,,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\big]+\Big[\overline{H_0}^3(t)\,,\overline{H_I}(t)\Big]+\big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t) \tag{203}$$

$$+2\overline{H_I}(t)\,\big[\overline{H_I}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0(t)}}\,. \tag{204}$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_{\bar{S}}}(t) \equiv (\varepsilon_0(t)+R_0(t))\,|0\rangle\langle 0| + (\varepsilon_1(t)+R_1(t))\,|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \tag{205}$$

$$\overline{H_{\bar{I}}}(t) \equiv \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)), \tag{206}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{207}$$

$$= H_B. \tag{208}$$

In this case $\varepsilon_j(t)$, $R_j(t)$ for $j \in \{0,1\}$, $B_{10}^{\Re}(t)$, $B_{10}^{\Im}(t)$, $V_{10}^{\Re}(t)$ and $V_{10}^{\Im}(t)$ are scalars and the other operators are:

$$\sigma_x \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \tag{209}$$

$$\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{210}$$

$$\sigma_y \equiv -\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1| \tag{211}$$

$$\equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{212}$$

$$\sigma_z \equiv |1\rangle\langle 1| - |0\rangle\langle 0| \tag{213}$$

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{214}$$

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right) & \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \\ \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \tag{215}$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)\,v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right), \tag{216}$$

$$D(\pm v_{\mathbf{k}}(t)) \equiv \mathrm{e}^{\pm\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \tag{217}$$

As we can see they verify the relationship $\sigma_x\sigma_y = \mathrm{i}\sigma_z$. The explicit form of $\overline{H_{\bar{I}}}^2(t)$ is:

$$\overline{H_{\bar{I}}}^2(t) = \sum_i B_{iz}^2(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i B_{iz}(t)|i\rangle\langle i|(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\sum_i B_{iz}(t)\,|i\rangle\langle i|\,(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Re}(t) \tag{218}$$

$$\times(\sigma_x B_x(t) + \sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i| + \left(V_{10}^{\Re}(t)\right)^2(\sigma_x B_x(t) + \sigma_y B_y(t))^2 + V_{10}^{\Re}(t)V_{10}^{\Im}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{219}$$

$$\times(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))(\sigma_x B_x(t) \tag{220}$$

$$+\sigma_y B_y(t)) + \left(V_{10}^{\Im}(t)\right)^2(\sigma_x B_y(t) - \sigma_y B_x(t))^2 \tag{221}$$

$$= \sum_i B_{iz}^2(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i(B_{iz}(t)\,B_x(t)\,|i\rangle\langle i|\sigma_x + B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i(B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_x - B_{iz}(t) \tag{222}$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2\left(\sigma_x^2 B_x^2(t) + \sigma_x\sigma_y B_x(t)B_y(t) + \sigma_y\right. \tag{223}$$

$$\left.\times\sigma_x B_y(t)B_x(t) + \sigma_y^2 B_y^2(t)\right) + V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2\left(\sigma_x^2 B_y^2(t) + \sigma_y^2 B_x^2(t)\right. \tag{224}$$

$$\left.-\sigma_x\sigma_y B_y(t)B_x(t) - \sigma_y\sigma_x B_x(t)B_y(t)\right) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)\left(\sigma_x^2 B_y(t)B_x(t) + \sigma_x\sigma_y B_y^2(t) - \sigma_y\sigma_x B_x^2(t) - \sigma_y^2 B_x(t)B_y(t)\right. \tag{225}$$

$$\left.+\sigma_x^2 B_x(t)B_y(t) - \sigma_x\sigma_y B_x^2(t) + \sigma_y\sigma_x B_y^2(t) - \sigma_y^2 B_y(t)B_x(t)\right), \tag{226}$$

$$
\begin{aligned}
&\sigma_x\sigma_y = \mathrm{i}\sigma_z \text{ (by Pauli matrices properties)}, &(227)\\
&\overline{H_I}^2(t) = \sum_i B_{iz}^2(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i (B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x + B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i (B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x - B_{iz}(t) &(228)\\
&\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i (\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2 (B_x^2(t) + \mathrm{i}\sigma_z B_x(t)B_y(t) - \mathrm{i}\sigma_z &(229)\\
&\times B_y(t)B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t)\sum_i (\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_y^2(t) + B_x^2(t) - \mathrm{i}\sigma_z &(230)\\
&\times B_y(t)B_x(t) + \mathrm{i}\sigma_z B_x(t)B_y(t)). &(231)
\end{aligned}
$$

To introduce the direct calculation of the expected values recall that the hamiltonian $\overline{H_0}(t)$ is a direct sum of the hamiltonians of two Hilbert spaces given by $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$, so we can write in general the hamiltonian $\overline{H_0}(t)$ as:

$$\overline{H_0}(t) = \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} + \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}. \tag{232}$$

where $\mathbb{I}_{\bar{B}}$ and $\mathbb{I}_{\bar{S}}$ are the identity of the systems $\bar{B}$ and $\bar{S}$ respectively.
We can show that:

$$
\begin{aligned}
\left[\overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}}, \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}\right] &= \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} \cdot \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} \cdot \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} &(233)\\
&= \overline{H_{\bar{S}}}(t)\,\mathbb{I}_{\bar{S}} \otimes \mathbb{I}_{\bar{B}}\overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}}\overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}}\mathbb{I}_{\bar{B}} &(234)\\
&= \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} - \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} \text{ (by definition of identity operator)} &(235)\\
&= 0. &(236)
\end{aligned}
$$

Let's introduce the following partition functions $Z_{\bar{S}}(t)$ and $Z_{\bar{B}}$ related to the systems $\bar{S}$ and $\bar{B}$ respectively.:

$$
\begin{aligned}
Z_{\bar{S}}(t) &\equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right), &(237)\\
Z_{\bar{B}} &\equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) &(238)
\end{aligned}
$$

Using (9), (233) and $\mathrm{Tr}\,(\mathrm{A} \otimes \mathrm{B}) = \mathrm{Tr}\,(\mathrm{A})\,\mathrm{Tr}\,(\mathrm{B})$ we can infer that the partition function $Z_0(t)$ can be factorized as:

$$
\begin{aligned}
Z_0(t) &= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0(t)}}\right). &(239)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right) \text{ (by (4))}, &(240)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by (9))} &(241)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)} \otimes \mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (because } \bar{S} \text{ and } \bar{B} \text{ are disjoint Hilbert spaces)} &(242)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by Tr}(A \otimes B)\text{=Tr}(A)\text{Tr}(B)), &(243)\\
&= Z_{\bar{S}}(t)\,Z_{\bar{B}} \text{ (by (237) and (238))}. &(244)
\end{aligned}
$$

For an operator $J(t)$ that can be factorized as $J(t) = S(t) \otimes B(t)$ with $S(t) \in \mathrm{gen}\left(\overline{H_{\bar{S}}}(t)\right)$ and $B(t) \in \mathrm{gen}\left(\overline{H_{\bar{B}}}\right)$, being $\mathrm{gen}\,(\mathrm{A})$ the vectorial space generated by the eigenvectors of the operator A, we calculate it's expected value respect to $\overline{H_0}(t)$ using a simple way as follows:

$$\langle J(t)\rangle_{\overline{H_0}(t)} = \frac{\mathrm{Tr}\left(J(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \text{ (by (5))} \tag{245}$$

$$= \frac{\mathrm{Tr}\left((S(t)\otimes B(t))\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\otimes \mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by } J(t)=S(t)\otimes B(t) \text{ and } \mathrm{e}^{-\beta\overline{H_0}(t)}=\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\otimes \mathrm{e}^{-\beta\overline{H_{\bar B}}}) \tag{246}$$

$$= \frac{\mathrm{Tr}\left(\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\otimes\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (rearranging and factorizing)} \tag{247}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by Tr}(A\otimes B)\text{=Tr}(A)\text{Tr}(B)) \tag{248}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)}\frac{\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \tag{249}$$

$$= \langle S(t)\rangle_{\overline{H_{\bar S}}(t)}\langle B(t)\rangle_{\overline{H_{\bar B}}} \text{ (by (5))}. \tag{250}$$

The factorization of $\left\langle \overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)}$ in terms of expected values of elements from gen $\left(\overline{H_{\bar S}}(t)\right)$ and gen $\left(\overline{H_{\bar B}}\right)$ is:

$$\left\langle\overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)} = \sum_i\langle|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar B}}} + V_{10}^{\Re}(t)\sum_i\left(\langle|i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \langle|i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}}\right) \tag{251}$$

$$+V_{10}^{\Im}(t)\sum_i\left(\langle|i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}} - \langle|i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}}(t)}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}}\right) + V_{10}^{\Re}(t)\sum_i\left(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\right. \tag{252}$$

$$\left.\times\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} + \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\right) + \left(V_{10}^{\Re}(t)\right)^2\left(\langle B_x^2(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar B}}}\right. \tag{253}$$

$$\left.-\mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \langle B_y^2(t)\rangle_{\overline{H_{\bar B}}}\right) + V_{10}^{\Im}(t)\sum_i\left(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} - \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}}(t)}\right. \tag{254}$$

$$\left.\times\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\right) + \left(V_{10}^{\Im}(t)\right)^2\left(\langle B_y^2(t)\rangle_{\overline{H_{\bar B}}} + \langle B_x^2(t)\rangle_{\overline{H_{\bar B}}} - \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}}(t)}\right. \tag{255}$$

$$\left.\times\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}\right). \tag{256}$$

In order to obtain the expected values of $\left\langle\overline{H_I}^2(t)\right\rangle_{\overline{H_0}(t)}$ respect to the part related to the bath we need to calculate the following expected values that appear in the equation (231) and can be obtained using the factorization of (251). The expected values relevant for calculations are $\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_y(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x^2(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_y^2(t)\rangle_{\overline{H_{\bar B}}}$. Recalling the form of the hamiltonian $\overline{H_{\bar B}}=\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}$ we can extend the result (244), also we introduce the notation:

$$A_1\otimes\cdots\otimes A_n \equiv \bigotimes_k A_k, \tag{257}$$

$$Z_{\mathbf{k}} \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right) \tag{258}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{-1} \tag{259}$$

$$= f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{260}$$

with the creation $b_{\mathbf{k}}$ and annihilation $b_{\mathbf{k}}^{\dagger}$ operators satisfying:

$$b_{\mathbf{k}}\mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{261}$$

$$b_{\mathbf{k}}^{\dagger}\mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1}\mid j_{\mathbf{k}}+1\rangle. \tag{262}$$

being $|j_{\mathbf{k}}\rangle$ an eigenstate of $H_{\mathbf{k}}\equiv\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}$. With this notation we can write the partition function as:

$$Z_{\bar{B}} = \mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right), \tag{263}$$

$$\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}, \tag{264}$$

$$Z_{\bar{B}} = \mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right) \text{ (by (264))} \tag{265}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right) \text{ (by } \mathrm{Tr}\left(A\otimes B\right) = \mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right)\text{)} \tag{266}$$

$$= \prod_{\mathbf{k}} Z_{\mathbf{k}} \text{ (by (264))}. \tag{267}$$

For a function $f\left(t\right)$ which can be factorized as:

$$f\left(t\right) \equiv \prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right). \tag{268}$$

with $f_{\mathbf{k}}\left(t\right) \in \mathrm{gen}\left(\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)$, it's expected value is given by:

$$\left\langle f\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(f\left(t\right)\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \tag{269}$$

$$= \frac{\mathrm{Tr}\left(\prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \text{ (by (264) and (268))} \tag{270}$$

$$= \frac{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{271}$$

$$= \frac{\prod_{\mathbf{k}}\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\prod_{\mathbf{k}}\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{272}$$

$$= \prod_{\mathbf{k}}\frac{\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{273}$$

$$= \prod_{\mathbf{k}}\left\langle f_{\mathbf{k}}\left(t\right)\right\rangle_{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}. \tag{274}$$

It means that for an operator that can be factorized in terms of functions generated by $\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}$ for each $\mathbf{k}$ we only require to calculate the expected value respect to the Hilbert space where the operator belongs. This process lead us to the following explicit forms of the expected values relevant for our calculations:

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{275}$$

$$= \sum_{\mathbf{k}}\left\langle\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)^{2}+\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(\left(g_{i\mathbf{k}'}\right.\right.\right. \tag{276}$$

$$\left.\left.\left.-v_{i\mathbf{k}'}(t)\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^{*}b_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{277}$$

$$= \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{278}$$

$$\times \left\langle \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^{*} b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (274))} \tag{279}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{280}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{281}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{282}$$

$$= 0, \tag{283}$$

$$\left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{284}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{285}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{286}$$

$$= 0, \tag{287}$$

$$\left\langle B_{iz}^{2}(t) \right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{288}$$

$$= \sum_{\mathbf{k}} \left\langle \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{2} \left(b_{\mathbf{k}}^{\dagger}\right)^{2} + \left|g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right|^{2} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) + \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right)^{2} b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{289}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{2} \left\langle \left(b_{\mathbf{k}}^{\dagger}\right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left|g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right|^{2} \left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right)^{2} b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{290}$$

$$\left\langle \left(b_{\mathbf{k}}^{\dagger}\right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger}\right)^{2} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{291}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger}\right)^{2} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{292}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+2)(j_{\mathbf{k}}+1)} |j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{293}$$

$$= 0, \tag{294}$$

$$\left\langle b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{2} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{295}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}(j_{\mathbf{k}}-1)} |j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{296}$$

$$= 0, \tag{297}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \left(1 - \mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \tag{298}$$

$$= \left(1 - \mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \tag{299}$$

$$= \left(1 - \mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \left(2 j_{\mathbf{k}} + 1\right) \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right) \tag{300}$$

$$=\left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\sum_{j_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right)\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \tag{301}$$

$$=\frac{1+\mathrm{e}^{-\beta\omega_{\mathbf{k}}}}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \tag{302}$$

$$=\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{303}$$

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\sum_{\mathbf{k}}\left|g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{304}$$

$$\left\langle B_{iz}\left(t\right)B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\frac{B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{305}$$

$$=\frac{1}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\right.\right. \tag{306}$$

$$\left.\left.\times\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{307}$$

$$\left\langle b^{\dagger}D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\langle\alpha|b^{\dagger}D\left(h\right)|\alpha\rangle\mathrm{d}^{2}\alpha \tag{308}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\langle0|D\left(-\alpha\right)b^{\dagger}D\left(\alpha\right)D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)|0\rangle\mathrm{d}^{2}\alpha \tag{309}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\langle0|D\left(-\alpha\right)b^{\dagger}D\left(\alpha\right)D\left(h\right)\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}|0\rangle\mathrm{d}^{2}\alpha \tag{310}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\langle0|\left(b^{\dagger}+\alpha^{*}\right)D\left(h\right)\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}|0\rangle\mathrm{d}^{2}\alpha \tag{311}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\langle0|\left(b^{\dagger}+\alpha^{*}\right)|h\rangle\mathrm{d}^{2}\alpha \tag{312}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\langle0|\left(b^{\dagger}+\alpha^{*}\right)|h\rangle\mathrm{d}^{2}\alpha, \tag{313}$$

$$|\alpha\rangle\equiv\mathrm{e}^{-\frac{|\alpha|^{2}}{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle, \tag{314}$$

$$\left\langle b^{\dagger}D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\left(\langle0|b^{\dagger}|h\rangle+\alpha^{*}\langle0|h\rangle\right)\mathrm{d}^{2}\alpha \tag{315}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\left(\langle0|b^{\dagger}\mathrm{e}^{-\frac{|\alpha|^{2}}{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle+\alpha^{*}\langle0|h\rangle\right)\mathrm{d}^{2}\alpha \tag{316}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\left(\langle0|\mathrm{e}^{-\frac{|\alpha|^{2}}{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\sqrt{n+1}|n+1\rangle+\alpha^{*}\langle0|h\rangle\right)\mathrm{d}^{2}\alpha \tag{317}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\alpha^{*}\langle0|h\rangle\mathrm{d}^{2}\alpha \tag{318}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\alpha^{*}\mathrm{e}^{-\frac{|h|^{2}}{2}}\mathrm{d}^{2}\alpha \tag{319}$$

$$=\frac{\mathrm{e}^{-\frac{|h|^{2}}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^{2}+y^{2}}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^{*}(x+\mathrm{i}y)}\left(x-\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \tag{320}$$

$$=-h^{*}N\left(\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}\right)^{2}, \tag{321}$$

$$\left\langle B_{iz}(t)B_{x}(t)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)+\mathrm{e}^{\chi_{01}(t)}\right.\right. \tag{322}$$

$$\left.\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{323}$$

$$=\frac{1}{2}\left\langle\mathrm{e}^{\chi_{10}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)+\mathrm{e}^{\chi_{01}(t)}\right. \tag{324}$$

$$\times \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \prod_{\mathbf{k}'} \left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \bigg\rangle_{\overline{H_{\bar{B}}}} \tag{325}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2} \left(\sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \langle b_{\mathbf{k}} \right. \tag{326}$$

$$\left. \times \prod_{\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}} \Bigg) \tag{327}$$

$$+ \frac{\mathrm{e}^{\chi_{01}(t)}}{2} \left(\sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}'} \left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right. \tag{328}$$

$$\left. \times \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} \left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{329}$$

$$= \frac{B_{10}(t)}{2} \left(-\sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} \right. \tag{330}$$

$$\left. + -v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}}+1)\, \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + \frac{B_{01}(t)}{2} \left(-\sum_{\mathbf{k}} (g_{i\mathbf{k}} \right. \tag{331}$$

$$-v_{i\mathbf{k}}(t)) \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right. \tag{332}$$

$$\left. \left. -\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}}+1)\, \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{333}$$

$$= \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left(-\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right. \tag{334}$$

$$\left. + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}}+1)\, \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right), \tag{335}$$

$$\langle B_{iz}(t) B_{y}(t) \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \frac{B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \right\rangle_{\overline{H_{\bar{B}}}} \tag{336}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{337}$$

$$= \frac{B_{10}(t)}{2\mathrm{i}} \left(\sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}}\, \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} - \sum_{\mathbf{k}} (\, g_{i\mathbf{k}} \right. \tag{338}$$

$$\left. -v_{i\mathbf{k}}(t))^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}}+1)\, \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) + \frac{B_{01}(t)}{2\mathrm{i}} \left(-\sum_{\mathbf{k}} (g_{i\mathbf{k}} \right. \tag{339}$$

$$-v_{i\mathbf{k}}(t)) \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right. \tag{340}$$

$$\left. \left. -\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}}+1)\, \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right) \tag{341}$$

$$= \frac{B_{10}(t)+B_{01}(t)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{342}$$

$$\left.-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{343}$$

$$\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{344}$$

$$= \frac{1}{2}\left\langle\left(B_1^{+}(t)\,B_0^{-}(t)+B_0^{+}(t)\,B_1^{-}(t)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{345}$$

$$= \frac{1}{2}\left\langle\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{346}$$

$$+\frac{1}{2}\left\langle\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{347}$$

$$\langle D(h)\,b\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle\alpha|D(h)\,b|\alpha\rangle\mathrm{d}^2\alpha \tag{348}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha)\,|0\rangle\mathrm{d}^2\alpha \tag{349}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\,(b+\alpha)\,|0\rangle\mathrm{d}^2\alpha \tag{350}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha|0\rangle\mathrm{d}^2\alpha \tag{351}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\langle 0|D(h)\,|0\rangle\mathrm{d}^2\alpha \tag{352}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle\mathrm{d}^2\alpha \tag{353}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int\alpha\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha \tag{354}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\,(x+\mathrm{i}y)\,\mathrm{d}x\mathrm{d}y \tag{355}$$

$$= Nh\mathrm{e}^{-|h|^2\coth\left(\frac{\beta\omega}{2}\right)} \tag{356}$$

$$= Nh\,\langle D(h)\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{357}$$

$$\left\langle D(h)\,b^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle\alpha|D(h)\,b^{\dagger}|\alpha\rangle\mathrm{d}^2\alpha \tag{358}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,b^{\dagger}D(\alpha)\,|0\rangle\mathrm{d}^2\alpha \tag{359}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\left(b^{\dagger}+\alpha^*\right)|0\rangle\mathrm{d}^2\alpha \tag{360}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0|D(h)\,b^{\dagger}|0\rangle\mathrm{d}^2\alpha+\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0|D(h)\,|0\rangle\mathrm{d}^2\alpha \tag{361}$$

$$= \frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle -h|1\rangle\mathrm{d}^2\alpha+\frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha, \tag{362}$$

$$\langle\alpha| = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\langle n|, \tag{363}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\mathrm{e}^{-\frac{|h|^{2}}{2}}\left(-h^{*}\right)\mathrm{d}^{2}\alpha+\frac{\mathrm{e}^{-\frac{|h|^{2}}{2}}}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\alpha^{*}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\mathrm{d}^{2}\alpha \tag{364}$$

$$=\frac{\mathrm{e}^{-\frac{|h|^{2}}{2}}}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^{2}}{N}}\mathrm{e}^{h\alpha^{*}-h^{*}\alpha}\left(-h^{*}+\alpha^{*}\right)\mathrm{d}^{2}\alpha \tag{365}$$

$$=\frac{\mathrm{e}^{-\frac{|h|^{2}}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^{2}+y^{2}}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^{*}(x+\mathrm{i}y)}\left(-h^{*}+x-\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \tag{366}$$

$$=-\left(N+1\right)h^{*}\mathrm{e}^{-|h|^{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{367}$$

$$=-\left(N+1\right)h^{*}\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{368}$$

$$\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}=\mathrm{e}^{-\frac{|h|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{369}$$

$$\left\langle B_{x}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{\mathrm{e}^{\chi_{01}(t)}}{2} \tag{370}$$

$$\times\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{371}$$

$$=\frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right.\right.\right.\right. \tag{372}$$

$$\left.\left.\left.\left.-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right.\right. \tag{373}$$

$$\left.\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{374}$$

$$=\frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right.\right.\right.\right. \tag{375}$$

$$\left.\left.\left.\left.-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}(g_{i\mathbf{k}}\right. \tag{376}$$

$$\left.-v_{i\mathbf{k}}\left(t\right))^{*}\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{377}$$

$$=\frac{B_{10}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+(g_{i\mathbf{k}}\right.\right. \tag{378}$$

$$\left.\left.-v_{i\mathbf{k}}\left(t\right))^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right)+\frac{B_{01}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\right.\right. \tag{379}$$

$$\left.\left.\times\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{380}$$

$$=\frac{B_{10}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+(g_{i\mathbf{k}}\right.\right. \tag{381}$$

$$\left.\left.-v_{i\mathbf{k}}\left(t\right))^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right)+\frac{B_{01}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\right.\right. \tag{382}$$

$$\left.\left.\times\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{383}$$

$$=\frac{B_{01}(t)-B_{10}(t)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))(N_{\mathbf{k}}+1)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right. \tag{384}$$

$$- \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{385}$$

$$\langle D(h)\, b\rangle_{\overline{H_{\bar{B}}}} = Nh \langle D(h)\rangle^{2}_{\overline{H_{\bar{B}}}}, \tag{386}$$

$$\left\langle D(h)\, b^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = -(N+1)\, h^{*} \langle D(h)\rangle^{2}_{\overline{H_{\bar{B}}}}, \tag{387}$$

$$\langle B_y(t) B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = \langle B_y(t)\, B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} \tag{388}$$

$$= \left\langle \frac{B_0^{+}(t)\, B_1^{-}(t) - B_1^{+}(t)\, B_0^{-}(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{389}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \left(B_0^{+}(t) B_1^{-}(t) - B_1^{+}(t) B_0^{-}(t) + B_{10}(t) - B_{01}(t)\right) \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{390}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \left(B_0^{+}(t)\, B_1^{-}(t) - B_1^{+}(t)\, B_0^{-}(t)\right) \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{391}$$

$$= \frac{B_{10}(t)}{2\mathrm{i}} \left(\sum_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right.\right. \tag{392}$$

$$\left.\left. \times N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right) + \frac{B_{01}(t)}{2\mathrm{i}} \left(\sum_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \left(-\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1) \right.\right. \tag{393}$$

$$\left.\left. \times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} N_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{394}$$

$$= \frac{B_{10}(t)}{2\mathrm{i}} \left(\sum_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right.\right. \tag{395}$$

$$\left.\left. \times N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right) + \frac{B_{01}(t)}{2\mathrm{i}} \left(\sum_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1) \right.\right. \tag{396}$$

$$\left.\left. \times \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\right) \tag{397}$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} \right. \tag{398}$$

$$\left. - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} N_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{399}$$

$$\mathrm{Var}_{\overline{H_{\bar{B}}}}(A) \equiv \left\langle A^{2}\right\rangle_{\overline{H_{\bar{B}}}} - \langle A\rangle^{2}_{\overline{H_{\bar{B}}}}, \tag{400}$$

$$\left\langle B_x^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_x(t)\right) + \langle B_x(t)\rangle^{2}_{\overline{H_{\bar{B}}}} \tag{401}$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_1^{+}(t)\, B_0^{-}(t) + B_0^{+}(t)\, B_1^{-}(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{402}$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^{+}(t)\, B_0^{-}(t) + B_0^{+}(t)\, B_1^{-}(t) - B_{10}(t) - B_{01}(t)\right) \tag{403}$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^{+}(t)\, B_0^{-}(t) + B_0^{+}(t)\, B_1^{-}(t)\right) \tag{404}$$

$$= \frac{1}{4}\left(\left\langle \left(B_1^{+}(t)\, B_0^{-}(t) + B_0^{+}(t)\, B_1^{-}(t)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{10}(t) + B_{01}(t)\right)^{2}\right) \tag{405}$$

$$= \frac{1}{4}\left(\left\langle \left(B_1^{+}(t)B_0^{-}(t)\right)^{2} + B_1^{+}(t)B_0^{-}(t)B_0^{+}(t)B_1^{-}(t) + B_0^{+}(t)B_1^{-}(t)B_1^{+}(t)B_0^{-}(t) + \left(B_0^{+}(t)B_1^{-}(t)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}}\right. \tag{406}$$

$$\left. - \left(B_{10}(t) + B_{01}(t)\right)^{2}\right) \tag{407}$$

$$= \frac{1}{4}\left(\left\langle \left(B_1^{+}(t)\, B_0^{-}(t)\right)^{2} + 2\mathbb{I} + \left(B_0^{+}(t)\, B_1^{-}(t)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{10}(t) + B_{01}(t)\right)^{2}\right), \tag{408}$$

$$\left(D(h)\right)^{2} = D(h)\, D(h) \tag{409}$$

$$= D\left(h+h\right)\mathrm{e}^{\frac{1}{2}\left(\frac{h^{*}h-hh^{*}}{\omega^{2}}\right)} \tag{410}$$

$$= D\left(2h\right), \tag{411}$$

$$\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{412}$$

$$= \left\langle\prod_{\mathbf{k}} D\left(2\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\mathrm{e}^{\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{413}$$

$$= \prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}}\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{414}$$

$$\left\langle B_x^2\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4}\left(\left\langle\left(B_1^+\left(t\right)B_0^-\left(t\right)\right)^2+2\mathbb{I}+\left(B_0^+\left(t\right)B_1^-\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right)^2\right) \tag{415}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+\left(t\right)B_0^-\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+\left(t\right)B_1^-\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right)^2\right) \tag{416}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+\left(t\right)B_0^-\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+\left(t\right)B_1^-\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right)^2\right) \tag{417}$$

$$= \frac{1}{4}\left(\mathrm{e}^{2\chi_{10}(t)}\prod_{\mathbf{k}}\mathrm{e}^{-2\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}+2+\mathrm{e}^{2\chi_{01}(t)}\prod_{\mathbf{k}}\mathrm{e}^{-2\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{418}$$

$$\left.-\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right)^2\right) \tag{419}$$

$$= \frac{1}{4}\left(B_{10}^2\left(t\right)\left|B_{10}^2\left(t\right)\right|+2+B_{01}^2\left(t\right)\left|B_{01}^2\left(t\right)\right|-\left(B_{10}^2\left(t\right)+2B_{10}\left(t\right)B_{01}\left(t\right)+B_{01}^2\left(t\right)\right)\right) \tag{420}$$

$$\left\langle B_y^2\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_y\left(t\right)\right)+\left\langle B_y\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}^2 \tag{421}$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_0^+\left(t\right)B_1^-\left(t\right)-B_1^+\left(t\right)B_0^-\left(t\right)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}\right) \tag{422}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+\left(t\right)B_1^-\left(t\right)-B_1^+\left(t\right)B_0^-\left(t\right)+B_{10}\left(t\right)-B_{01}\left(t\right)\right) \tag{423}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+\left(t\right)B_1^-\left(t\right)-B_1^+\left(t\right)B_0^-\left(t\right)\right) \tag{424}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+\left(t\right)B_1^-\left(t\right)-B_1^+\left(t\right)B_0^-\left(t\right)\right)^2-\left(B_{01}\left(t\right)-B_{10}\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{425}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+\left(t\right)B_1^-\left(t\right)\right)^2-2\mathbb{I}+\left(B_1^+\left(t\right)B_0^-\left(t\right)\right)^2-\left(B_{01}\left(t\right)-B_{10}\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{426}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+\left(t\right)B_1^-\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\left(B_1^+\left(t\right)B_0^-\left(t\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-2-\left(B_{01}\left(t\right)-B_{10}\left(t\right)\right)^2\right), \tag{427}$$

$$\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} = \prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}}\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{428}$$

$$= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{2\omega_{\mathbf{k}}^{2}}}\mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^2\left(\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^2 \tag{429}$$

$$= B_{ij}^2\left(t\right)\left|B_{ij}\left(t\right)\right|^2, \tag{430}$$

$$\left\langle B_y^2\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = -\frac{1}{4}\left(B_{01}^2\left(t\right)\left|B_{10}\left(t\right)\right|^2-2+B_{10}^2\left(t\right)\left|B_{10}\left(t\right)\right|^2-\left(B_{01}\left(t\right)-B_{10}\left(t\right)\right)^2\right), \tag{431}$$

$$\left\langle B_x\left(t\right)B_y\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)}{2}\frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{432}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)\right)\left(B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{433}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\mathbb{I}-\left(B_1^+(t)B_0^-(t)\right)^2+B_{10}^2(t)-B_{10}(t)B_{01}(t)+\left(B_0^+(t)B_1^-(t)\right)^2-\mathbb{I}+B_{10}(t)B_{01}(t)-B_{01}^2(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{434}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_0^+\left(t\right)B_1^-\left(t\right)\right)^2-\left(B_1^+\left(t\right)B_0^-\left(t\right)\right)^2-\left(B_{01}^2\left(t\right)-B_{10}^2\left(t\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{435}$$

$$= \frac{1}{4\mathrm{i}} \left(B_{01}^{2}\left(t\right) \left|B_{10}\left(t\right)\right|^{2} - B_{10}^{2}\left(t\right) \left|B_{10}\left(t\right)\right|^{2} - \left(B_{01}^{2}\left(t\right) - B_{10}^{2}\left(t\right)\right)\right), \tag{436}$$

$$\left\langle B_{y}(t) B_{x}(t)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \frac{B_{0}^{+}(t)\, B_{1}^{-}(t) - B_{1}^{+}(t)\, B_{0}^{-}(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \frac{B_{1}^{+}(t)\, B_{0}^{-}(t) + B_{0}^{+}(t)\, B_{1}^{-}(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{437}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t) + B_{10}(t) - B_{01}(t)\right) \left(B_{1}^{+}(t) B_{0}^{-}(t) + B_{0}^{+}(t) B_{1}^{-}(t) - B_{10}(t) - B_{01}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{438}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \mathbb{I} + \left(B_{0}^{+}(t) B_{1}^{-}(t)\right)^{2} - B_{10}(t) B_{01}(t) - B_{01}^{2}(t) - \left(B_{1}^{+}(t) B_{0}^{-}(t)\right)^{2} - \mathbb{I} + B_{10}^{2}(t) + B_{10}(t) B_{01}(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{439}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \left(B_{0}^{+}\left(t\right) B_{1}^{-}\left(t\right)\right)^{2} - B_{01}^{2}\left(t\right) - \left(B_{1}^{+}\left(t\right) B_{0}^{-}\left(t\right)\right)^{2} + B_{10}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{440}$$

$$= \frac{1}{4\mathrm{i}} \left(B_{01}^{2}\left(t\right) \left|B_{10}\left(t\right)\right|^{2} - B_{01}^{2}\left(t\right) - \left(B_{10}^{2}\left(t\right) \left|B_{10}\left(t\right)\right|^{2} - B_{10}^{2}\left(t\right)\right)\right). \tag{441}$$

The density matrix associated to $\rho_{\overline{S}} = \frac{\mathrm{e}^{-\beta \overline{H_{0}}(t)}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \overline{H_{0}}(t)}\right)}$ follows the form:

$$\rho_{\overline{S},00} = \frac{1}{2} + \frac{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}}, \tag{442}$$

$$\rho_{\overline{S},01} = -\frac{B_{10}^{*}\left(t\right) V_{10}^{*}\left(t\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}}, \tag{443}$$

$$\rho_{\overline{S},10} = -\frac{B_{10}\left(t\right) V_{10}\left(t\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}}, \tag{444}$$

$$\rho_{\overline{S},11} = \frac{1}{2} - \frac{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right) \tanh\left(\frac{\beta}{2} \sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i} \left(-1\right)^{i} \left(\varepsilon_{i}\left(t\right) + R_{i}\left(t\right)\right)\right)^{2} + 4\left|B_{10}\left(t\right)\right|^{2} \left|V_{10}\left(t\right)\right|^{2}}}. \tag{445}$$

The expected values respect to the system $\overline{S}$ of relevance for calculating $\left\langle \overline{H_{I}}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)}}$ are $\langle |i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle |i\rangle\langle i|\sigma_{x}\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle |i\rangle\langle i|\sigma_{y}\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle \sigma_{x}|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle \sigma_{y}|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$ and $\langle \sigma_{z}\rangle_{\overline{H_{\bar{S}}(t)}}$, we took account that $\sigma_{x}\sigma_{y} = \mathrm{i}\sigma_{z}$ and $\sigma_{y}\sigma_{x} = -\mathrm{i}\sigma_{z}$. The values needed for our calculation are:

$$\langle |0\rangle\langle 0| \rangle_{\overline{H_{\bar{S}}(t)}} = \frac{1}{2} - \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{446}$$

$$\langle |1\rangle\langle 1| \rangle_{\overline{H_{\bar{S}}(t)}} = \frac{1}{2} + \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{2\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{447}$$

$$\langle |0\rangle\langle 0|\sigma_x \rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{448}$$

$$\langle |1\rangle\langle 1|\sigma_x \rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}^{*}(t) V_{10}^{*}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{449}$$

$$\langle |0\rangle\langle 0|\sigma_y \rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\mathrm{i}B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{450}$$

$$\langle |1\rangle\langle 1|\sigma_y \rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^{*}(t) V_{10}^{*}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2}}, \tag{451}$$

$$\langle \sigma_x|0\rangle\langle 0| \rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}^{*}(t) V_{10}^{*}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{452}$$

$$\langle \sigma_x|1\rangle\langle 1| \rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{453}$$

$$\langle \sigma_y|0\rangle\langle 0| \rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B_{10}^{*}(t) V_{10}^{*}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{454}$$

$$\langle \sigma_y|1\rangle\langle 1| \rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\mathrm{i}B_{10}(t) V_{10}(t) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}, \tag{455}$$

$$\langle \sigma_z \rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right) \tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}\right)}{\sqrt{\left(\sum_i (-1)^i (\varepsilon_i(t) + R_i(t))\right)^2 + 4\left|B_{10}(t) V_{10}(t)\right|^2}}. \tag{456}$$

Summarizing the expected values of the bath we have:

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}}\left|g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{457}$$

$$\left\langle B_{iz}\left(t\right)B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right.\right. \tag{458}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\right)\right), \tag{459}$$

$$\left\langle B_{iz}\left(t\right)B_{y}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right.\right. \tag{460}$$

$$\left.\left.-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\right)\right), \tag{461}$$

$$\left\langle B_{x}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{B_{01}\left(t\right)-B_{10}\left(t\right)}{2}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right. \tag{462}$$

$$\left.\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right), \tag{463}$$

$$\left\langle B_{y}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right. \tag{464}$$

$$\left.\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right), \tag{465}$$

$$\left\langle B_{x}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4}\left(B_{10}^{2}\left(t\right)\left|B_{10}^{2}\left(t\right)\right|+2+B_{01}^{2}\left(t\right)\left|B_{01}^{2}\left(t\right)\right|-\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right)^{2}\right), \tag{466}$$

$$\left\langle B_{y}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = -\frac{1}{4}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-2+B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}\left(t\right)-B_{10}\left(t\right)\right)^{2}\right), \tag{467}$$

$$\left\langle B_{x}\left(t\right)B_{y}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4\mathrm{i}}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}^{2}\left(t\right)-B_{10}^{2}\left(t\right)\right)\right), \tag{468}$$

$$\left\langle B_{y}\left(t\right)B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4\mathrm{i}}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}^{2}\left(t\right)-B_{10}^{2}\left(t\right)\right)\right). \tag{469}$$

Our next step is to find $v_3\left(t\right)$, the commutator $\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]$ is a central point for our calculations and it is equal to:

$$\left[\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right]=\Big[\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)\right. \tag{470}$$

$$\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\Big] \tag{471}$$

$$=\left[\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right)-\sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right)\right. \tag{472}$$

$$\left.+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_{i}B_{iz}\left(t\right)|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right)+V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y\left(t\right)-\sigma_y B_x\left(t\right)\right)\right] \tag{473}$$

$$=\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right)+\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y\left(t\right)-\sigma_y B_x\left(t\right)\right)+\sigma_x \tag{474}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right) \tag{475}$$

$$+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i| \tag{476}$$

$$-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(\sigma_x B_y(t) \tag{477}$$

$$-\sigma_y B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t)\,|i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{478}$$

$$-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t)\,|i\rangle\langle i| \tag{479}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t))\,\sigma_x \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) \tag{480}$$

$$+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sigma_y \left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{481}$$

$$\times \sum_i (\varepsilon_i(t) + R_i(t))\,|i\rangle\langle i| - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\,\sigma_x \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\,\sigma_y \tag{482}$$

$$\times \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \tag{483}$$

$$= \sum_i (\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) + \sum_i (\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{484}$$

$$\times \sum_i B_{iz}(t)\,|i\rangle\langle i| + \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t)\,(B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) V_{10}^{\Im}(t) \tag{485}$$

$$\times (B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)\,|i\rangle\langle i| - \left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) V_{10}^{\Re}(t)\,(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{486}$$

$$-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t)|i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{487}$$

$$+\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{488}$$

$$\left.+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t)\,|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))\,|i\rangle\langle i| - V_{10}^{\Re}(t)\,(B_x(t) - \mathrm{i}\sigma_z B_y(t)) \tag{489}$$

$$\times \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{490}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))\,|i\rangle\langle i| - V_{10}^{\Im}(t)\,(B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) \tag{491}$$

$$+V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{492}$$

The term $\overline{H_{\bar{I}}}(t)\left[\,\overline{H_0}(t), \overline{H_{\bar{I}}}(t)\right]$ is given by:

$$\overline{H_{\bar{I}}}(t)[\,\overline{H_0}(t), \overline{H_{\bar{I}}}(t)] = \left(\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\right)\left(\sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\right. \tag{493}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t)) + \sum_i (\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{494}$$

$$\times \sum_i B_{iz}(t)|i\rangle\langle i| + \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t)(B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{495}$$

$$\times V_{10}^{\Im}(t)(B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)|i\rangle\langle i| - \left(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{496}$$

$$\times V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) - \left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) \tag{497}$$

$$\times |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \tag{498}$$

$$\times \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y \left(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t)|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{499}$$

$$-V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Re}(t)\,(B_x(t) - \mathrm{i}\sigma_z B_y(t)) \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{500}$$

$$+V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{501}$$

$$-V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Im}(t)(B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{502}$$

$$\left.+V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) \tag{503}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{504}$$

$$+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\,(B_x(t) \tag{505}$$

$$+\mathrm{i}\sigma_z B_y(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{506}$$

$$\left.+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-\sum_i B_{iz}(t)|i\rangle\langle i|\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\,(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-\sum_i B_{iz}(t)|i\rangle\langle i| \tag{507}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{508}$$

$$\times V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x \tag{509}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i| \tag{510}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{511}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t) \tag{512}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t) \tag{513}$$

$$\times(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) \tag{514}$$

$$-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{515}$$

$$+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x \tag{516}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t)) \tag{517}$$

$$+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_y \tag{518}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{519}$$

$$-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{520}$$

$$\times\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{521}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{522}$$

$$+\sigma_y B_y(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t) \tag{523}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{524}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t) \tag{525}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t) \tag{526}$$

$$+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{527}$$

$$\times V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{528}$$

$$+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i| \tag{529}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{530}$$

$$-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{531}$$

$$\times V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{532}$$

$$-\sigma_y B_x(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{533}$$

$$\times V^{\Im}_{10}(t)\left(-\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)+V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{534}$$

$$\times V^{\Re}_{10}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{535}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{536}$$

$$-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t)) \tag{537}$$

$$\times|i\rangle\langle i|-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V^{\Re}_{10}(t)\left(B_x(t)-\mathrm{i}\sigma_z B_y(t)\right)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Re}_{10}(t) \tag{538}$$

$$\times\left(\mathrm{i}\sigma_z B_x(t)+B_y(t)\right)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Re}_{10}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{539}$$

$$-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Im}_{10}(t)\left(B_y(t)\right. \tag{540}$$

$$\left.+\mathrm{i}\sigma_z B_x(t)\right)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Im}_{10}(t)\left(\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{541}$$

$$-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{542}$$

$$=V^{\Re}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\right)+V^{\Im}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y\right. \tag{543}$$

$$\left.\times B_{iz}(t)\,B_x(t)\right)+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_{i\neq i'}B_{iz}(t)\,B_{i'z}(t)\,|i\rangle\langle i'|+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Re}_{10}(t)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\right. \tag{544}$$

$$\left.\times B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)\right)+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\sum_i\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_x(t)\right)-\left(B^{\Re}_{10}(t)\right. \tag{545}$$

$$\left.\times V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_{i\neq i'}B_{iz}(t)B_{i'z}(t)\,|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|-\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)\sum_i\left(-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+B_{iz}(t)\right. \tag{546}$$

$$\left.\times B_y(t)\,|i\rangle\langle i|\right)+\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)+|i\rangle\langle i|B_{iz}(t)\,B_x(t)\right)+\sum_{i,\mathbf{k}}|i\rangle\langle i|B_{iz}(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_{iz}(t) \tag{547}$$

$$+V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_y(t)\right)+V^{\Im}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\right. \tag{548}$$

$$\left.\times\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_x(t)\right)-\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i B^2_{iz}(t)\,|i\rangle\langle i|\sigma_x+\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i B^2_{iz}(t)\,|i\rangle\langle i|\sigma_y-\sum_{i,\mathbf{k}}|i\rangle\langle i| \tag{549}$$

$$\times B^2_{iz}(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t)\sum_{i,i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|B_{iz}(t)B_x(t)+|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_y(t)\right)-V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\right. \tag{550}$$

$$\left.\times V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\,B_x(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)\right)+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+|i\rangle\langle i|\right. \tag{551}$$

$$\left.\times B_{iz}(t)\,B_y(t)\right)-V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\right)-V^{\Im}_{10}(t)\sum_{i\neq i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|\right. \tag{552}$$

$$\left.\times B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)\,B_x(t)\right)-V^{\Im}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\,B_y(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_x(t)\right)+V^{\Im}_{10}(t) \tag{553}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)-|i\rangle\langle i|B_{iz}(t)\,B_x(t)\right)-V^{\Im}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-|i\rangle\langle i|\sigma_y\right. \tag{554}$$

$$\left.\times B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\right)+\left(V^{\Re}_{10}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B^2_x(t)+\sigma_x|i\rangle\langle i|\sigma_y B_x(t)\,B_y(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)\,B_x(t)+\sigma_y|i\rangle\langle i|\right. \tag{555}$$

$$\left.\times\sigma_y B^2_y(t)\right)+V^{\Re}_{10}(t)\,V^{\Im}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B^2_x(t)+\sigma_y|i\rangle\langle i|\sigma_x B^2_y(t)-\sigma_y|i\rangle\langle i|\sigma_y B_y(t)\,B_x(t)\right) \tag{556}$$

$$+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_x(t)\,B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_{iz}(t)\right)+\left(V^{\Re}_{10}(t)\right)^2\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right) \tag{557}$$

$$\times\left(\sigma_x B^2_x(t)+\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)-\sigma_x B^2_y(t)\right)+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\left(B_x(t)\,B_y(t)-\mathrm{i}\sigma_z B^2_y(t)\right. \tag{558}$$

$$\left.-\mathrm{i}\sigma_z B^2_x(t)-B_y(t)\,B_x(t)\right)-V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_{iz}(t)+|i\rangle\langle i|B_y(t)\,B_{iz}(t)\right)-V^{\Re}_{10}(t) \tag{559}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)\left(-\sigma_y B^2_x(t)+\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B^2_y(t)\right)-V^{\Im}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\right. \tag{560}$$

$$\left.\times V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)\left(-\sigma_y B_x(t)B_y(t)+\sigma_x B^2_y(t)-\sigma_x B^2_x(t)-\sigma_y B_y(t)B_x(t)\right)+V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)+\sigma_y|i\rangle\langle i|B_y(t)\right)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_{iz}(t) \tag{561}$$

$$+\left(V_{10}^{\Re}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)-\mathrm{i}\sigma_{z}B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)+\mathrm{i}\sigma_{z}B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)+B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}} \tag{562}$$

$$\times\left(B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)-\mathrm{i}\sigma_{z}B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)-\mathrm{i}\sigma_{z}B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)-B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{563}$$

$$\times\sum_{i}\left(\sigma_{x}|i\rangle\langle i|\sigma_{x}B_{x}(t)\,B_{iz}(t)+\sigma_{y}|i\rangle\langle i|\sigma_{x}B_{y}(t)\,B_{iz}(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i}\left(\sigma_{x}|i\rangle\langle i|\sigma_{y}B_{x}(t)\,B_{iz}(t)+\sigma_{y}|i\rangle\langle i|\right. \tag{564}$$

$$\left.\times\sigma_{y}B_{y}(t)\,B_{iz}(t)\right)-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_{x}|i\rangle\langle i|B_{x}(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\sigma_{y}|i\rangle\langle i|B_{y}(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-\left(V_{10}^{\Re}(t)\right)^{2}\sum_{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\left(|i\rangle\langle i|B_{x}^{2}(t)\right. \tag{565}$$

$$\left.-\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{y}(t)\,B_{x}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{x}(t)\,B_{y}(t)+|i\rangle\langle i|B_{y}^{2}(t)\right)-\left(V_{10}^{\Re}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_{x}B_{x}^{2}(t)+\sigma_{y}B_{y}(t)\,B_{x}(t)\right. \tag{566}$$

$$\left.-\sigma_{y}B_{x}(t)\,B_{y}(t)+\sigma_{x}B_{y}^{2}(t)\right)+\left(V_{10}^{\Re}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_{y}B_{x}^{2}(t)-\sigma_{x}B_{y}(t)\,B_{x}(t)+\sigma_{x}B_{x}(t)\,B_{y}(t)+\sigma_{y}B_{y}^{2}(t)\right) \tag{567}$$

$$-\left(V_{10}^{\Re}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_{x}^{2}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_{z}B_{x}(t)\,B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_{z}B_{y}(t)\,B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_{y}^{2}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right) \tag{568}$$

$$\times\left(|i\rangle\langle i|B_{x}(t)\,B_{y}(t)-\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{y}^{2}(t)-\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{x}^{2}(t)-|i\rangle\langle i|B_{y}(t)\,B_{x}(t)\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_{x}B_{x}(t)\right. \tag{569}$$

$$\left.\times B_{y}(t)+\sigma_{y}B_{y}^{2}(t)+\sigma_{y}B_{x}^{2}(t)-\sigma_{x}B_{y}(t)B_{x}(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_{y}B_{x}(t)B_{y}(t)-\sigma_{x}B_{y}^{2}(t)-\sigma_{x}B_{x}^{2}(t)\right. \tag{570}$$

$$\left.-\sigma_{y}B_{y}(t)\,B_{x}(t)\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_{x}(t)\,B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_{z}B_{y}^{2}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_{z}B_{x}^{2}(t)-B_{y}(t)\,B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_{i}\left(\varepsilon_{i}(t)\right. \tag{571}$$

$$\left.+R_{i}(t)\right)\left(\sigma_{x}|i\rangle\langle i|\sigma_{x}B_{y}(t)\,B_{x}(t)-\sigma_{y}|i\rangle\langle i|\sigma_{x}B_{x}^{2}(t)+\sigma_{x}|i\rangle\langle i|\sigma_{y}B_{y}^{2}(t)-\sigma_{y}|i\rangle\langle i|\sigma_{y}B_{x}(t)\,B_{y}(t)\right)+\left(V_{10}^{\Im}(t)\right)^{2}\sum_{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\left(\sigma_{x}|i\rangle\langle i|\right. \tag{572}$$

$$\left.\times\sigma_{x}B_{y}^{2}(t)-\sigma_{y}|i\rangle\langle i|\sigma_{x}B_{x}(t)\,B_{y}(t)-\sigma_{x}|i\rangle\langle i|\sigma_{y}B_{y}(t)\,B_{x}(t)+\sigma_{y}|i\rangle\langle i|\sigma_{y}B_{x}^{2}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_{i}\left(|i\rangle\langle i|B_{y}(t)\right. \tag{573}$$

$$\left.\times B_{iz}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{x}(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_{x}B_{y}(t)B_{x}(t)-\sigma_{y}B_{x}^{2}(t)+\sigma_{y}B_{y}^{2}(t)+\sigma_{x}B_{x}(t)B_{y}(t)\right) \tag{574}$$

$$+\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(B_{y}^{2}(t)+\mathrm{i}\sigma_{z}B_{x}(t)B_{y}(t)-\mathrm{i}\sigma_{z}B_{y}(t)B_{x}(t)+B_{x}^{2}(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{575}$$

$$\times\sum_{i}\left(\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{y}(t)B_{iz}(t)-|i\rangle\langle i|B_{x}(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(-\sigma_{y}B_{y}(t)\,B_{x}(t)-\sigma_{x}B_{x}^{2}(t)+\sigma_{x}B_{y}^{2}(t)\right. \tag{576}$$

$$\left.-\sigma_{y}B_{x}(t)\,B_{y}(t)\right)-\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(-\sigma_{y}B_{y}^{2}(t)-\sigma_{x}B_{x}(t)\,B_{y}(t)-\sigma_{x}B_{y}(t)\,B_{x}(t)+\sigma_{y}B_{x}^{2}(t)\right)+V_{10}^{\Im}(t) \tag{577}$$

$$\times\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_{x}|i\rangle\langle i|B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)-\sigma_{y}|i\rangle\langle i|B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\right)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)+\mathrm{i}\sigma_{z}B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)+\mathrm{i}\sigma_{z}\right. \tag{578}$$

$$\left.\times B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)-B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)\right)+\left(V_{10}^{\Im}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)+\mathrm{i}\sigma_{z}B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{y}(t)-\mathrm{i}\sigma_{z}B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)+B_{x}(t)\right. \tag{579}$$

$$\left.\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{x}(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\sum_{i}\left(\sigma_{x}|i\rangle\langle i|\sigma_{x}B_{y}(t)\,B_{iz}(t)-\sigma_{y}|i\rangle\langle i|\sigma_{x}B_{x}(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{580}$$

$$\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i}\left(\sigma_{x}|i\rangle\langle i|\sigma_{y}B_{y}(t)B_{iz}(t)-\sigma_{y}|i\rangle\langle i|\sigma_{y}B_{x}(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_{x}|i\rangle\langle i|B_{y}(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sigma_{y}|i\rangle\langle i|B_{x}(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{581}$$

$$-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\left(|i\rangle\langle i|B_{y}(t)B_{x}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{x}^{2}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{y}^{2}(t)-|i\rangle\langle i|B_{x}(t)B_{y}(t)\right)-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{582}$$

$$\left.-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_{x}B_{y}(t)B_{x}(t)-\sigma_{y}B_{x}^{2}(t)-\sigma_{y}B_{y}^{2}(t)-\sigma_{x}B_{x}(t)B_{y}(t)\right)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_{y}B_{y}(t)B_{x}(t)\right. \tag{583}$$

$$\left.+\sigma_{x}B_{x}^{2}(t)+\sigma_{x}B_{y}^{2}(t)-\sigma_{y}B_{x}(t)B_{y}(t)\right)-\sum_{\mathbf{k}}V_{10}^{\Im}(t)V_{10}^{\Re}(t)\,\omega_{\mathbf{k}}\left(B_{y}(t)B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_{z}B_{y}^{2}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_{z}B_{x}^{2}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-B_{x}(t)B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{584}$$

$$-\left(V_{10}^{\Im}(t)\right)^{2}\sum_{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\left(|i\rangle\langle i|B_{y}^{2}(t)+\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{x}(t)\,B_{y}(t)-\mathrm{i}\sigma_{z}|i\rangle\langle i|B_{y}(t)\,B_{x}(t)+|i\rangle\langle i|B_{x}^{2}(t)\right)-\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{585}$$

$$\left.-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_{x}B_{y}^{2}(t)-\sigma_{y}B_{x}(t)\,B_{y}(t)+\sigma_{y}B_{y}(t)\,B_{x}(t)+\sigma_{x}B_{x}^{2}(t)\right)+\left(V_{10}^{\Im}(t)\right)^{2}\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_{y}B_{y}^{2}(t)\right. \tag{586}$$

$$\left.+\sigma_{x}B_{x}(t)\,B_{y}(t)-\sigma_{x}B_{y}(t)\,B_{x}(t)+\sigma_{y}B_{x}^{2}(t)\right)-\left(V_{10}^{\Im}(t)\right)^{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_{y}^{2}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_{z}B_{x}(t)B_{y}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_{z}B_{y}(t)B_{x}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_{x}^{2}(t)\right. \tag{587}$$

$$\left.\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right), \tag{588}$$

Now let's obtain the form of $\overline{H_{\overline{I}}}^{3}(t)$:

$$\overline{H_{\overline{I}}}^{3}(t)=\left(\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_{x}B_{x}(t)+\sigma_{y}B_{y}(t))+V_{10}^{\Im}(t)(\sigma_{x}B_{y}(t)-\sigma_{y}B_{x}(t))\right)\left(\sum_{i}B_{iz}^{2}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_{i}(B_{iz}(t)B_{x}(t)\,|i\rangle\langle i|\sigma_{x}\right. \tag{589}$$

$$+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) \tag{590}$$

$$+\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2 \tag{591}$$

$$\times\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right)) \tag{592}$$

$$=\sum_iB_{iz}\,(t)\,|i\rangle\langle i|\sum_iB_{iz}^2\,(t)\,|i\rangle\langle i|+\sum_iB_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+\sum_iB_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Im}\,(t) \tag{593}$$

$$\times\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+\sum_iB_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\sum_iB_{iz}(t)|i\rangle\langle i| \tag{594}$$

$$\times\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+\sum_iB_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) \tag{595}$$

$$+\sum_iB_{iz}(t)\,|i\rangle\langle i|\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\sum_iB_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t) \tag{596}$$

$$\times(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}\,(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}\,(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t) \tag{597}$$

$$\times|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t) \tag{598}$$

$$+\sigma_yB_y(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t) \tag{599}$$

$$-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t) \tag{600}$$

$$-\sigma_yB_x(t))\sum_iB_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_y \tag{601}$$

$$\times B_x(t))V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i| \tag{602}$$

$$\times B_y(t)B_{iz}(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t)) \tag{603}$$

$$\times V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)\right. \tag{604}$$

$$\left.+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right) \tag{605}$$

$$=\sum_iB_{iz}^3\,(t)\,|i\rangle\langle i|+V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}^2\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}^2\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}\,(t)\sum_i\left(B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_y\right) \tag{606}$$

$$+V_{10}^{\Re}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_x(t)B_{iz}(t)+|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_x^2(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)\right. \tag{607}$$

$$\times B_x(t)B_y(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+|i\rangle\langle i|B_{iz}(t)B_y^2(t)\Big)+V_{10}^{\Im}(t)\sum_{i\neq i'}\Big(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)-|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t) \tag{608}$$

$$B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_y^2(t)+|i\rangle\langle i|B_{iz}(t)B_x^2(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_x(t)B_y(t)\right)+V_{10}^{\Re}(t) \tag{609}$$

$$\times\sum_i\left(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}^2(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i(B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_x+B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t) \tag{610}$$

$$\times B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_x+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_x-B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_y \tag{611}$$

$$+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_x-B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\Big(|i\rangle\langle i|B_x^2(t)B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i| \tag{612}$$

$$\times B_y(t)B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y^2(t)B_{iz}(t)\Big)+\left(V_{10}^{\Re}(t)\right)^3\Big(\sigma_xB_x^3(t)+\sigma_yB_x^2(t)B_y(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_xB_x(t)B_y^2(t)+\sigma_yB_y(t)B_x^2(t) \tag{613}$$

$$-\sigma_xB_y(t)B_x(t)B_y(t)+\sigma_xB_y^2(t)B_x(t)+\sigma_yB_y^3(t)\Big)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i\Big(|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_y^2(t) \tag{614}$$

$$\times B_{iz}(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_y(t)B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)\left(V_{10}^{\Im}(t)\right)^2\Big(\sigma_xB_x(t)B_y^2(t)+\sigma_xB_x^3(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_yB_x^2(t)B_y(t)+\sigma_yB_y^3(t) \tag{615}$$

$$+\sigma_yB_y(t)\,B_x^2(t)+\sigma_xB_y^2(t)\,B_x(t)-\sigma_xB_y(t)\,B_x(t)\,B_y(t)\Big)+V_{10}^{\Im}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}^2(t)-\sigma_y|i\rangle\langle i|B_x(t)\,B_{iz}^2(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t) \tag{616}$$

$$\times(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_x(t)+\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t)-\sigma_y|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t))+\left(V_{10}^{\Im}(t)\right)^2 \tag{617}$$

$$\times(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_x(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x(t)B_{iz}(t)B_y(t)+\sigma_y|i\rangle\langle i|\sigma_yB_x(t)B_{iz}(t)B_x(t))+V_{10}^{\Re}(t) \tag{618}$$

$$\times V_{10}^{\Im}(t)\sum_{i}\left(|i\rangle\langle i|B_y(t)B_x(t)B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(V_{10}^{\Re}(t)\right)^2 \tag{619}$$

$$\times\left(\sigma_x B_y(t)B_x^2(t)+\sigma_y B_y(t)B_x(t)B_y(t)-\sigma_y B_y^2(t)B_x(t)+\sigma_x B_y^3(t)-\sigma_y B_x^3(t)+\sigma_x B_x^2(t)B_y(t)-\sigma_x B_x(t)B_y(t)B_x(t)-\sigma_y B_x(t)B_y^2(t)\right) \tag{620}$$

$$+\left(V_{10}^{\Im}(t)\right)^2\sum_{i}\left(|i\rangle\langle i|B_y^2(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_x(t)B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)+|i\rangle\langle i|B_x^2(t)B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^3\left(\sigma_x B_y^3(t)\right. \tag{621}$$

$$\left.+\sigma_x B_y(t)B_x^2(t)-\sigma_y B_y^2(t)B_x(t)+\sigma_y B_y(t)B_x(t)B_y(t)-\sigma_y B_x(t)B_y^2(t)-\sigma_y B_x^3(t)-\sigma_x B_x(t)B_y(t)B_x(t)+\sigma_x B_x^2(t)B_y(t)\right). \tag{622}$$

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