Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality. Let´s define the partition functions of $\overline{H}(t)$ and $\overline{H_0}(t)$ given by $Z(t)$ and $Z_0(t)$ respectively as:

$$Z(t) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}(t)}\right), \tag{1}$$

$$Z_0(t) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}_0(t)}\right). \tag{2}$$

where the transformed hamiltonians $\overline{H}(t)$ and $\overline{H_0}(t)$ are defined as:

$$\overline{H}(t) \equiv \overline{H_{\bar{I}}}(t) + \overline{H_0}(t), \tag{3}$$

$$\overline{H_0}(t) \equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{B}}}. \tag{4}$$

For any operator $A(t)$ we define the expected value respect to $\overline{H_0}(t)$ as:

$$\langle A(t)\rangle_{\overline{H_0}(t)} \equiv \frac{\mathrm{Tr}\left(A(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}. \tag{5}$$

The terms $\overline{H_{\bar{S}}(t)}$, $\overline{H_{\bar{B}}}$ and $\overline{H_{\bar{I}}}(t)$ are related to the variational transformation performed in [1,2], this transformation allowed us to construct $\overline{H_{\bar{I}}}(t)$ such that $\left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_0}(t)} = 0$. The diagonalization of $\overline{H_0}(t)$ in terms of it's eigenstates and eigenvalues such that $\overline{H_0}(t)|n\rangle = E_{0,n}(t)|n\rangle$, being $|n\rangle$ an eigenstate of $\overline{H_0}(t)$ with eigenvalue $E_{0,n}(t)$ is $\overline{H_0}(t) = \sum_n E_{0,n}(t)|n\rangle\langle n|$, with $\langle n|n'\rangle = \delta_{nn'}$, so a simple form of $\mathrm{e}^{-\beta\overline{H_0}(t)}$ can be found as follows:

$$\begin{aligned}
\mathrm{e}^{-\beta\overline{H_0}(t)} &= \mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by the diagonalization of } \overline{H_0}(t)) && (6)\\
&= \prod_n \mathrm{e}^{-\beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by the Zassenhaus formula)} && (7)\\
&= \prod_n \sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}(t)|n\rangle\langle n|\right)^j}{j!} \text{ (by the exponential formula)} && (8)\\
&= \prod_n \left(\mathbb{I} + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j |n\rangle\langle n|}{j!}\right) \text{ (using } (aA)^j = a^j A^j \text{ and } \left(|n\rangle\langle n|\right)^2 = |n\rangle\langle n|) && (9)\\
&= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j |n\rangle\langle n|}{j!}\right) && (10)
\end{aligned}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left(\sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}\left(t\right)\right)^j}{j!} \right) \right) \tag{11}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + \mathrm{e}^{-\beta E_{0,n}(t)} |n\rangle\langle n| \right) \text{ (by the exponential formula)} \tag{12}$$

$$= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right). \tag{13}$$

We will prove by induction a neat form for (13), we will show that:

$$\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \mathbb{I} + \sum_{j=1}^{n} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \tag{14}$$

For $n = 1$ the formula is trivial, in the case $n = 2$ we obtain that:

$$\prod_{j=1}^{2} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1 \right) |1\rangle\langle 1| \right) \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1 \right) |2\rangle\langle 2| \right) \tag{15}$$

$$= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1 \right) |1\rangle\langle 1| + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1 \right) |2\rangle\langle 2| \text{ (by } \langle i|j\rangle = \delta_{ij}) \tag{16}$$

$$= \mathbb{I} + \sum_{j=1}^{2} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1 \right) |n\rangle\langle n|. \tag{17}$$

It is our case base, our induction step is (14), in the case $n + 1$ we will have:

$$\prod_{j=1}^{n+1} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) = \left(\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j| \right) \right) \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| \right) \tag{18}$$

$$= \left(\mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right) \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| \right) \text{ (by induction step)} \tag{19}$$

$$= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1 \right) |n+1\rangle\langle n+1| + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \text{ (by } \langle i|j\rangle = \delta_{ij}) \tag{20}$$

$$= \mathbb{I} + \sum_{j=1}^{n+1} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1 \right) |j\rangle\langle j|. \tag{21}$$

By mathematical induction we proved that (14) is true for all $n \in \mathbb{N}$. Given that the resolution of the identity is $\mathbb{I} = \sum_n |n\rangle\langle n|$ so we find that:

$$\mathrm{e}^{-\beta \overline{H_0}(t)} = \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \right) \tag{22}$$

$$= \mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1 \right) |n\rangle\langle n| \tag{23}$$

$$= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} |n\rangle\langle n| - \sum_n |n\rangle\langle n| \tag{24}$$

$$= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} |n\rangle\langle n| - \mathbb{I} \text{ (by the resolution of the identity)} \tag{25}$$

$$= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} |n\rangle\langle n|. \tag{26}$$

The partition function $Z_0\left(t\right)$ is equal to:

$$Z_0(t) = \mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)} |n\rangle\langle n|\right) \tag{27}$$
$$= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \mathrm{Tr}\left(|n\rangle\langle n|\right) \tag{28}$$
$$= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}. \tag{29}$$

The explicit form of the average value $\langle A(t)\rangle_{\overline{H_0}(t)}$ can be found from the partition function $Z_0(t)$:

$$\langle A(t)\rangle_{\overline{H_0}(t)} = \frac{\mathrm{Tr}\left(A(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{Z_0(t)} \tag{30}$$
$$= \frac{\mathrm{Tr}\left(\sum_n A(t)\,\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \tag{31}$$
$$= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)} A(t)\,|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)} \tag{32}$$
$$= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)} A(t)\,|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} \tag{33}$$
$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(A(t)\,|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}}. \tag{34}$$

At first we show a double sequence of inequalities of order M, N which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$Z(t) \geq Z_0(t)\,\mathrm{e}^{-\langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)}}\left(1 + F_M\left(\overrightarrow{u}(t);\alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t);\alpha\right)\right). \tag{35}$$

where the funcion $F_N\left(\overrightarrow{u}(t);\alpha\right)$ is defined as:

$$F_N\left(\overrightarrow{w}(t);\alpha\right) \equiv \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\,\frac{w_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. \tag{36}$$

In this case α is a parameter that can be optimized, $\beta \equiv \frac{1}{\mathrm{k_B}T}$, $\overrightarrow{w}(t)$ is a vector such that $\overrightarrow{w}(t) = (w_1, w_2, ...)$ and $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian $\overline{H_{\overline{I}D}}(t)$ respect to the basis of eigenstates of $\overline{H_0}(t)$ as:

$$\overline{H_{\overline{I}D}}(t) \equiv \sum_n \left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle |n\rangle\langle n|. \tag{37}$$

We will prove an important property related to $\overline{H_{\overline{I}D}}(t)$, which is a Hamiltonian written as a linear combination of a set of ortonormal operators. Let's consider a vector space R with two operations $+$ and $\cdot$, if there exist $a, b \in R$ such that $a \cdot b = 0$ and $b \cdot a = 0$ then for any $k \in \mathbb{N}$ we have $(a+b)^k = a^k + b^k$ where $a^k = a^{k-1}\cdot a$ is a recursive definition of the power of an element written in terms of $\cdot$. At first we prove that this result yields for any $k \in \mathbb{N}$ by induction, the case $k = 1$ is trivial so we will focus on the case $k = 2$, we have that:

$$(a+b)^2 = (a+b)\cdot(a+b) \tag{38}$$
$$= a\cdot a + a\cdot b + b\cdot a + b\cdot b \tag{39}$$
$$= a^2 + a\cdot b + b\cdot a + b^2 \tag{40}$$
$$= a^2 + 0 + 0 + b^2 \text{ (because } a\cdot b = b\cdot a = 0) \tag{41}$$
$$= a^2 + b^2. \tag{42}$$

This is the base case. By induction step we will consider that $(a+b)^k = a^k + b^k$ with $k \geq 2$, now for $k+1$ we will have that:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)^k \cdot (a+b) && (43)\\
&= \left(a^k + b^k\right) \cdot (a+b) \text{ (by induction step)} && (44)\\
&= a^k \cdot a + a^k \cdot b + b^k \cdot a + b^k \cdot b && (45)\\
&= a^{k+1} + a^{k-1} \cdot a \cdot b + b^{k-1} \cdot b \cdot a + b^{k+1} \text{ (by recursive definition of } a^k\text{)} && (46)\\
&= a^{k+1} + a^{k-1} \cdot (a \cdot b) + b^{k-1} \cdot (b \cdot a) + b^{k+1} \text{ (by associativity on } R\text{)} && (47)\\
&= a^{k+1} + a^{k-1} \cdot (0) + b^{k-1} \cdot (0) + b^{k+1} \text{ (because } a \cdot b = b \cdot a = 0\text{)} && (48)\\
&= a^{k+1} + b^{k+1}. && (49)
\end{aligned}$$

By the principle of mathematical induction we can conclude that the proposition is true for all $k \in \mathbb{N}$. Now we will extend the result, let $a_1, ..., a_n \in R$ such that $a_i \cdot a_j = 0$ for all $i \neq j$ then $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$. The case $n = 1$ is trivial as well so we will focus on $n = 2$, this case was proved in the precedent lines so it will be our base case. By induction step we will consider that $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$ with $n \geq 2$, now for $n+1$ we will have that:

$$\begin{aligned}
a_{n+1} \cdot (a_1 + ... + a_n) &= a_{n+1} \cdot a_1 + ... + a_{n+1} \cdot a_n && (50)\\
&= 0 \text{ (because } a_i \cdot a_j = 0 \text{ for all } i \neq j \text{),} && (51)\\
(a_1 + ... + a_n + a_{n+1})^k &= ((a_1 + ... + a_n) + a_{n+1})^k && (52)\\
&= (a_1 + ... + a_n)^k + a_{n+1}^k \text{ (by (43) and (51))} && (53)\\
&= a_1^k + ... + a_n^k + a_{n+1}^k \text{ (by inductive step).} && (54)
\end{aligned}$$

So we can conclude by mathematical induction that the proposition is true for all $n \in \mathbb{N}$. We can prove the following property for $\left(\overline{H_{\overline{I}D}}(t)\right)^k$:

$$\begin{aligned}
\left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n| \left\langle n' \left| \overline{H_{\overline{I}}}(t) \right| n' \right\rangle |n'\rangle\langle n'| &= \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle \left\langle n' \left| \overline{H_{\overline{I}}}(t) \right| n' \right\rangle |n\rangle\langle n|n'\rangle\langle n'| && (55)\\
&= \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle \left\langle n' \left| \overline{H_{\overline{I}}}(t) \right| n' \right\rangle |n\rangle\langle n'| \delta_{nn'}, && (56)\\
\left(\overline{H_{\overline{I}D}}(t)\right)^k &= \left(\sum_n \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n| \right)^k \quad \text{(by (37))} && (57)\\
&= \sum_n \left(\left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n| \right)^k \text{ (by (54) and (56)) ,} && (58)\\
(aA)^k &= a^k A^k \text{ (by the property of the power of a matrix) ,} && (59)\\
(|n\rangle\langle n|)^k &= |n\rangle\langle n| \text{ (because } |n\rangle\langle n| \text{ is a projector and } k \in \mathbb{N}^*\text{) ,} && (60)\\
\left(\overline{H_{\overline{I}D}}(t)\right)^k &= \sum_n \left(\left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle \right)^k |n\rangle\langle n| \text{ (by (59) and (60)) .} && (61)
\end{aligned}$$

The vectors $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ are defined as $\overrightarrow{u}(t) \equiv (u_1, u_2, ...)$ and $\overrightarrow{v}(t) \equiv (v_1, v_2, ...)$. We can define the elements of $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ in terms of the matrix $\overline{H_{\overline{I}D}}(t)$:

$$\begin{aligned}
u_k(t) &\equiv \left\langle \left(\overline{H_{\overline{I}D}}(t) - \langle \overline{H_{\overline{I}}}(t) \rangle_{\overline{H_0(t)}} \right)^k \right\rangle_{\overline{H_0(t)}} && (62)\\
&= \frac{\sum_n e^{-\beta E_{0,n}(t)} \mathrm{Tr}\left(\left(\sum_n \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n| - \langle \overline{H_{\overline{I}}}(t) \rangle_{\overline{H_0(t)}} \right)^k |n\rangle\langle n| \right)}{Z_0(t)} \text{ (by (34)) ,} && (63)
\end{aligned}$$

$$\left(\sum_n \langle n | \overline{H_{\overline{I}}}(t) | n \rangle |n\rangle\langle n| - \langle \overline{H_{\overline{I}}}(t) \rangle_{\overline{H_0(t)}} \right)^k = \sum_{j=0}^k (-1)^j \binom{k}{j} \left(\sum_n \langle n | \overline{H_{\overline{I}}}(t) | n \rangle |n\rangle\langle n| \right)^j \left(\langle \overline{H_{\overline{I}}}(t) \rangle_{\overline{H_0(t)}} \right)^{k-j} \text{ (by binomial theorem)} \quad (64)$$

$$=\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left(\sum_{n}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle^{j}|n\rangle\langle n|\right)\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k-j}\quad\text{(by (61))}\tag{65}$$

$$=\sum_{n}\left(\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle^{j}\left(\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k-j}\right)|n\rangle\langle n|\tag{66}$$

$$=\sum_{n}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}|n\rangle\langle n|,\tag{67}$$

$$=\sum_{n}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}|n\rangle\langle n|,\tag{68}$$

$$u_{k}(t)=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\sum_{n'}\left(\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}|n'\rangle\langle n'|n\rangle\langle n|\right)}{Z_{0}(t)}\tag{69}$$

$$=\frac{\sum_{nn'}\mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{I}}(t)\right|n'\right\rangle-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}|n'\rangle\langle n|\delta_{nn'}\right)}{Z_{0}(t)}\tag{70}$$

$$=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}\mathrm{Tr}\left(|n\rangle\langle n|\right)}{Z_{0}(t)}\tag{71}$$

$$=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}1}{Z_{0}(t)}\tag{72}$$

$$=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}}{Z_{0}(t)},\tag{73}$$

$$v_{k}(t)\equiv\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_{0}}(t)-E_{0,n}(t)+\overline{H_{I}}(t)-\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}\right)^{k}\right|n\right\rangle}{Z_{0}(t)}.\tag{74}$$

By construction $\left\langle\overline{H_{I}}(t)\right\rangle_{\overline{H_{0}(t)}}=0$, so we summarize the double inequality that generalizes the Bogoliubov inequality and it's coefficients as:

$$Z(t)\geq Z_{0}(t)\left(1+F_{M}\left(\overrightarrow{u}(t)\right)+F_{N}\left(\overrightarrow{v}(t)-\overrightarrow{u}(t)\right)\right),\tag{75}$$

$$Z(t)=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}(t)}\right),\tag{76}$$

$$Z_{0}(t)=\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)},\tag{77}$$

$$F_{N}\left(\overrightarrow{u}(t)\right)=\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^{k}\frac{u_{k}(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^{i}}{i!},\tag{78}$$

$$u_{k}(t)=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{I}}(t)\right|n\right\rangle^{k}}{Z_{0}(t)},\tag{79}$$

$$v_{k}(t)=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_{0}}(t)-E_{0,n}(t)+\overline{H_{I}}(t)\right)^{k}\right|n\right\rangle}{Z_{0}(t)}.\tag{80}$$

As we can see the expression (79) was written in shorter terms, we want to do the same for (80) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for $v_{k}(t)$:

$$
\begin{aligned}
\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)\right)\left|n\right\rangle &= \overline{H_0}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (81)\\
&= E_{0,n}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (82)\\
&= 0, && (83)\\
\left\langle n\right|\left(\overline{H_0}\left(t\right)-E_{0,n}\right) &= \left\langle n\right|\overline{H_0}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) && (84)\\
&= \left\langle n\right|E_{0,n}\left(t\right) - \left\langle n\right|E_{0,n}\left(t\right) && (85)\\
&= 0. && (86)
\end{aligned}
$$

At first we calculated $v_1\left(t\right)$ using the definition (80) :

$$
\begin{aligned}
v_1\left(t\right) &= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle && (87)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)\right|n\right\rangle + \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle && (88)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|\overline{H_0}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0(t)}} && (89)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left(\left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0(t)}} && (90)\\
&= 0 + \left\langle\overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0(t)}} && (91)\\
&= 0. && (92)
\end{aligned}
$$

For $k\geq 2$ and $k\in\mathbb{N}$ we calculated:

$$
\begin{aligned}
v_k(t) &= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^k\right|n\right\rangle && (93)\\
&= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^{k-2}\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\right|n\right\rangle && (94)\\
&= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^{k-2}\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\right|n\right\rangle && (95)\\
&= \frac{1}{Z_0(t)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\left(E_{0,n}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\left(\overline{H_0}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)^{k-2}\left(E_{0,n}(t)-E_{0,n}(t)+\overline{H_{\overline{I}}}(t)\right)\right|n\right\rangle && (96)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle. && (97)
\end{aligned}
$$

In general we can write a formula for $v_k\left(t\right)$ that implies an expected value of a dependent expression of $\overline{H_{\overline{I}}}\left(t\right)$ and $\overline{H_0}\left(t\right)$:

$$
\begin{aligned}
v_k\left(t\right) &= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_0}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle && (98)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H_0}\left(t\right)+\overline{H_{\overline{I}}}\left(t\right)-E_{0,n}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle && (99)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\overline{H}\left(t\right)-E_{0,n}\left(t\right)\right)^{k-2}\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle && (100)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\left(\sum_{j=0}^{k-2}\left(-1\right)^j\binom{k-2}{j}\overline{H}^{k-2-j}\left(t\right)E_{0,n}^j\left(t\right)\right)\overline{H_{\overline{I}}}\left(t\right)\right|n\right\rangle && (101)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\sum_{j=0}^{k-2}\left(-1\right)^j\binom{k-2}{j}\left\langle n\left|\overline{H_{\overline{I}}}\left(t\right)\overline{H}^{k-2-j}\left(t\right)\overline{H_{\overline{I}}}\left(t\right)E_{0,n}^j\left(t\right)\right|n\right\rangle && (102)
\end{aligned}
$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{103}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{104}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t)\ \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{105}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{106}$$

The formula (106) is well defined taking as example $k = 2, 3$.

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{2-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \tag{107}$$

$$= (-1)^0 \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\ \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \tag{108}$$

$$= \left\langle \overline{H_{\overline{I}}}^2(t) \right\rangle_{\overline{H_0}(t)}, \tag{109}$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{3-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{110}$$

$$= \left\langle \sum_{j=0}^{1} (-1)^j \binom{1}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{1-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{111}$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^1 \overline{H_{\overline{I}}}(t)\, \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\, \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \tag{112}$$

$$= \langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right) \overline{H_{\overline{I}}}(t)\ \mathbb{I} - \overline{H_{\overline{I}}}(t)\, \mathbb{I}\overline{H_{\overline{I}}}(t)\ \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \tag{113}$$

$$= \langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right) \overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\ \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \tag{114}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t)\ \overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\ \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{115}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t)\ \overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\ \overline{H_0}(t)\right) \right\rangle_{\overline{H_0}(t)} \tag{116}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t) \left[\overline{H_0}(t), \overline{H_{\overline{I}}}(t)\right] \right\rangle_{\overline{H_0}(t)}. \tag{117}$$

So we summarize:

$$\overline{H_{\overline{I}D}}(t) = \sum_n \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n|, \tag{118}$$

$$u_k(t) = \left\langle \left(\overline{H_{\overline{I}D}}(t)\right)^k \right\rangle_{\overline{H_0}(t)}, \tag{119}$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{120}$$

Then we obtained finally:

$$Z(t) \geq Z_0(t) \left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right), \tag{121}$$

The free energy is defined as:

$$E_{\mathrm{free}}(t) \equiv -\frac{1}{\beta}\ln(Z(t)). \tag{122}$$

It is well-known that the function $f(x) = \ln(x)$ is monotonic and increasing so we can transform (121):

$$\begin{aligned}
E_{\mathrm{free},1}(t) &= -\frac{1}{\beta}\ln(Z_0(t)), &(123)\\
E_{\mathrm{free}}(t) &\leq -\frac{1}{\beta}\ln\left(Z_0(t)\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right)\right) &(124)\\
E_{\mathrm{free}}(t) &\leq -\frac{1}{\beta}\ln(Z_0(t)) - \frac{1}{\beta}\ln\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) &(125)\\
E_{\mathrm{free}}(t) &\leq E_{\mathrm{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) &(126)\\
&\equiv E_{\mathrm{free,MN}}(t). &(127)
\end{aligned}$$

here $E_{\mathrm{free,MN}}(t)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of M, N order. In our approach we will set $N = M$, so our quantum Bogoliubov inequality of N order is:

$$\begin{aligned}
E_{\mathrm{free}}(t) &\leq E_{\mathrm{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) &(128)\\
&= E_{\mathrm{free,NN}}(t). &(129)
\end{aligned}$$

A weaker form of the inequality (129) is obtained making $\overrightarrow{u}(t) = 0$ as suggest [3]:

$$\begin{aligned}
E_{\mathrm{free}}(t) &\leq E_{\mathrm{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{v}(t)\right)\right) &(130)\\
&\equiv E_{\mathrm{free,N}}(t). &(131)
\end{aligned}$$

The algebraic equation associated with $\alpha_{\mathrm{opt}}(t)$ such that $E_{\mathrm{free,N}}(t)$ is closer to $E_{\mathrm{free}}(t)$ follows from the fact that in the optimal parameter $\frac{\partial E_{\mathrm{free,N}}(t)}{\partial \alpha}|_{\alpha_{\mathrm{opt}(t)}} = 0$, calculating this derivate we have:

$$\begin{aligned}
\frac{\partial E_{\mathrm{free,N}}(t)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left(E_{\mathrm{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{v}(t)\right)\right)\right) &(132)\\
&= -\frac{1}{\beta}\frac{\frac{\partial}{\partial \alpha}\left(F_N\left(\overrightarrow{v}(t)\right)\right)}{1 + F_N\left(\overrightarrow{v}(t)\right)} &(133)\\
&= 0. &(134)
\end{aligned}$$

The precedent equation is equivalent to:

$$\frac{\partial F_N\left(\overrightarrow{v}\left(t\right)\right)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{135}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}+\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\partial}{\partial\alpha}\frac{\alpha^i}{i!}\ \text{(by product rule)} \tag{136}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}+\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{\left(i-1\right)!} \tag{137}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{\left(i-1\right)!}-\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{138}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{j=0}^{2N-2-k}\frac{\alpha^j}{j!}-\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right)\ \text{(setting } j=i-1\text{)} \tag{139}$$

$$= \mathrm{e}^{-\alpha}\left(-\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\frac{\alpha^{2N-1-k}}{\left(2N-1-k\right)!}\right)\ \text{(performing the difference)} \tag{140}$$

$$= 0. \tag{141}$$

Then the optimal value $\alpha_{\mathrm{opt}}\left(t\right)$ will sastisfy the following equation:

$$G\left(\alpha_{\mathrm{opt}}\left(t\right)\right) \equiv \sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{u_k\left(t\right)}{k!}\frac{\alpha_{\mathrm{opt}}^{2N-1-k}}{\left(2N-1-k\right)!} \tag{142}$$

$$= 0. \tag{143}$$

The elements presented are the required to find variational parameters of the system using the inequality (131) and the self consistent equation (142) to a particular order expected.

II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2\left(t\right), v_3\left(t\right), v_4\left(t\right), v_5\left(t\right)$ using the (120), we have already $v_2\left(t\right), v_3\left(t\right)$ and the form of $v_4\left(t\right)$ and $v_5\left(t\right)$ is given by:

$$v_4\left(t\right) = \sum_{j=0}^{4-2}\left(-1\right)^j\binom{4-2}{j}\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{4-2-j}\overline{H_I}\left(t\right)\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{144}$$

$$= \sum_{j=0}^{2}\left(-1\right)^j\binom{2}{j}\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^{2-j}\overline{H_I}\left(t\right)\overline{H_0}^j\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{145}$$

$$= \left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}\left(t\right)\overline{H_0}^0(t)\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^1\overline{H_I}\left(t\right)\overline{H_0}^1(t)\right\rangle_{\overline{H_0}(t)}+\langle\overline{H_I}\left(t\right)(\overline{H_I}\left(t\right) \tag{146}$$

$$+\overline{H_0}\left(t\right))^0\,\overline{H_I}\left(t\right)\overline{H_0}^2\left(t\right)\rangle_{\overline{H_0}(t)} \tag{147}$$

$$= \left\langle\overline{H_I}(t)\left(\overline{H_I}\left(t\right)+\overline{H_0}(t)\right)^2\overline{H_I}\left(t\right)\mathbb{I}\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^1\overline{H_I}\left(t\right)\overline{H_0}^1(t)\right\rangle_{\overline{H_0}(t)}+\left\langle\overline{H_I}^2(t)\,\overline{H_0}^2(t)\right\rangle_{\overline{H_0}(t)} \tag{148}$$

$$= \left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)^2\overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_I}\left(t\right)\left(\overline{H_I}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_I}\left(t\right)\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)}+\left\langle\overline{H_I}^2\left(t\right)\overline{H_0}^2\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{149}$$

$$=\left\langle\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^2\overline{H_{\bar{I}}}\left(t\right)-2\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)+\overline{H_{\bar{I}}}^{2}\left(t\right)\overline{H_0}^{2}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{150}$$

$$=\left\langle\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}^{2}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}^{2}\left(t\right)\right)\overline{H_{\bar{I}}}\left(t\right)-2\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}(t)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)+\overline{H_{\bar{I}}}^{2}\left(t\right)\right. \tag{151}$$

$$\left.\times\overline{H_0}^{2}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{152}$$

$$=\left\langle\overline{H_{\bar{I}}}^{4}\left(t\right)+\overline{H_{\bar{I}}}^{2}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)-2\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\right. \tag{153}$$

$$\left.+\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}^{2}(t)\right\rangle_{\overline{H_0}(t)} \tag{154}$$

$$=\left\langle\overline{H_{\bar{I}}}^{4}(t)+\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}(t)\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)-2\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}^{2}(t)-2\overline{H_{\bar{I}}}\left(t\right)\right. \tag{155}$$

$$\left.\times\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{156}$$

$$=\left\langle\overline{H_{\bar{I}}}^{4}(t)+\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}(t)\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)-\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_{\bar{I}}}^{2}(t)\right. \tag{157}$$

$$\left.\times\overline{H_0}^{2}\left(t\right)-\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)-\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{158}$$

$$=\left\langle\overline{H_{\bar{I}}}^{4}(t)+\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)-\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_{\bar{I}}}(t)\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}(t)\right. \tag{159}$$

$$\left.\times\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)+\overline{H_{\bar{I}}}^{2}\left(t\right)\overline{H_0}^{2}\left(t\right)-\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{160}$$

$$=\left\langle\overline{H_{\bar{I}}}^{4}(t)+\overline{H_{\bar{I}}}\left(t\right)\left(\left[\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]+\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}^{2}(t)\right]+\left[\overline{H_0}\left(t\right),\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\right]+\left[\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right),\overline{H_0}\left(t\right)\right]\right)\right\rangle_{\overline{H_0}(t)}, \tag{161}$$

$$v_5\left(t\right)=\sum_{j=0}^{5-2}\left(-1\right)^j\binom{5-2}{j}\left\langle\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^{5-2-j}\overline{H_{\bar{I}}}\left(t\right)\ \overline{H_0}\left(t\right)^j\right\rangle_{\overline{H_0}(t)} \tag{162}$$

$$=\sum_{j=0}^{3}\left(-1\right)^j\binom{3}{j}\left\langle\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^{3-j}\overline{H_{\bar{I}}}\left(t\right)\ \overline{H_0}\left(t\right)^j\right\rangle_{\overline{H_0}(t)} \tag{163}$$

$$=\left\langle\overline{H_{\bar{I}}}(t)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}(t)\right)^3\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{0}(t)-3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}(t)\right)^2\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)-\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}(t)\right)^0\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{3}(t)\right. \tag{164}$$

$$\left.+3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{165}$$

$$=\left\langle\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}(t)\right)^3\overline{H_{\bar{I}}}\left(t\right)-3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}(t)\right)^2\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)+3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}(t)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}(t)-\overline{H_{\bar{I}}}\left(t\right)\right. \tag{166}$$

$$\left.\times\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{3}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{167}$$

$$=\left\langle\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^3\overline{H_{\bar{I}}}\left(t\right)-3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^2\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)+3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}\left(t\right)\right. \tag{168}$$

$$\left.-\overline{H_{\bar{I}}}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{3}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{169}$$

$$=\left\langle\overline{H_{\bar{I}}}(t)\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)^3\overline{H_{\bar{I}}}(t)-3\overline{H_{\bar{I}}}(t)\left(\overline{H_{\bar{I}}}^{2}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)+\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}^{2}\left(t\right)\right)\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)+3\overline{H_{\bar{I}}}(t)\right. \tag{170}$$

$$\left.\times\left(\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}\left(t\right)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}\left(t\right)-\overline{H_{\bar{I}}}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{3}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{171}$$

$$=\left\langle\overline{H_{\bar{I}}}(t)\left(\overline{H_{\bar{I}}}^{3}(t)+\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}\left(t\right)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}(t)+\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)+\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}\left(t\right)+\overline{H_{\bar{I}}}(t)\right.\right. \tag{172}$$

$$\left.\times\overline{H_0}^{2}(t)+\overline{H_0}^{3}(t)\right)\overline{H_{\bar{I}}}\left(t\right)-3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_{\bar{I}}}\left(t\right)+\overline{H_0}^{2}(t)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)+3\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}\left(t\right)\right. \tag{173}$$

$$\left.\left.+\overline{H_0}\left(t\right)\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}\left(t\right)-\overline{H_{\bar{I}}}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{3}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{174}$$

$$=\left\langle\overline{H_{\bar{I}}}^{5}\left(t\right)+\overline{H_{\bar{I}}}^{3}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{I}}}^{2}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}^{2}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}^{3}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)\right. \tag{175}$$

$$+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}(t)\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{3}(t)\,\overline{H_{\bar{I}}}\left(t\right)-3\overline{H_{\bar{I}}}^{4}(t)\,\overline{H_0}(t)-3\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)\,\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t) \tag{176}$$

$$-3\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)-3\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)+3\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}^{2}(t)+3\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}(t)\,\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}(t)-\overline{H_{\bar{I}}}^{2}(t) \tag{177}$$

$$\left.\times\overline{H_0}^{3}\left(t\right)\right\rangle_{\overline{H_0}(t)} \tag{178}$$

$$=\left\langle\overline{H_{\bar{I}}}^{5}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\left(\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}^{2}(t)+\overline{H_0}\left(t\right)\overline{H_{\bar{I}}}^{3}(t)+\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}\left(t\right)\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}\left(t\right)\right.\right. \tag{179}$$

$$+\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^3(t)\,\overline{H_I}(t)-3\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}(t)\overline{H_0}(t)-3\overline{H_I}^3(t)\overline{H_0}(t)-3\overline{H_0}^2(t)\,\overline{H_I}(t)\overline{H_0}(t)-\overline{H_I}(t) \quad (180)$$
$$\times\overline{H_0}^3(t)+3\overline{H_I}^2(t)\,\overline{H_0}^2(t)+3\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}^2(t)-3\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)\Big)\Big\rangle_{\overline{H_0(t)}} \quad (181)$$
$$=\Big\langle\overline{H_I}^5(t)+\overline{H_I}(t)\Big(\overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t)-\overline{H_I}^3(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}^3(t)-\overline{H_0}(t)\overline{H_I}^2(t) \quad (182)$$
$$\times\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}(t)-\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t)+\overline{H_0}^3(t)\,\overline{H_I}(t) \quad (183)$$
$$-\overline{H_I}(t)\,\overline{H_0}^3(t)+\overline{H_I}^2(t)\,\overline{H_0}^2(t)-\overline{H_0}(t)\,\overline{H_I}^2(t)\,\overline{H_0}(t)+2\overline{H_I}^2(t)\,\overline{H_0}^2(t)-2\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_0}(t)\,\overline{H_I}(t) \quad (184)$$
$$\times\overline{H_0}^2(t)-3\overline{H_0}^2(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_I}^3(t)\,\overline{H_0}(t)+\overline{H_0}^2(t)\,\overline{H_I}^2(t)\Big)\Big\rangle_{\overline{H_0(t)}} \quad (185)$$
$$=\Big\langle\overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_I}^3(t)\Big]+\left[\overline{H_0}(t)\,,\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\right] \quad (186)$$
$$+\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}^3(t)\,,\overline{H_I}(t)\Big]+\left[\overline{H_I}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_I}(t)\right]+3\overline{H_0}(t)\left[\overline{H_I}(t)\,,\overline{H_0}(t)\right]\overline{H_0}(t)+2\overline{H_I}(t) \quad (187)$$
$$\times\left[\overline{H_I}(t)\,,\overline{H_0}(t)\right]\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0(t)}}. \quad (188)$$

Summarizing we have that:

$$v_2(t)=\Big\langle\overline{H_I}^2(t)\Big\rangle_{\overline{H_0(t)}}, \quad (189)$$
$$v_3(t)=\Big\langle\overline{H_I}^3(t)+\overline{H_I}(t)\left[\overline{H_0}(t)\,,\overline{H_I}(t)\right]\Big\rangle_{\overline{H_0(t)}}, \quad (190)$$
$$v_4(t)=\Big\langle\overline{H_I}^4(t)+\overline{H_I}(t)\Big(\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}(t)\right]+\Big[\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\left[\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_I}(t)\right]+\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\right]\Big)\Big\rangle_{\overline{H_0(t)}}, \quad (191)$$
$$v_5(t)=\Big\langle\overline{H_I}^5(t)+\overline{H_I}(t)\Big(\Big[\overline{H_I}^2(t)\,\overline{H_0}(t),\overline{H_I}(t)\Big]+\Big[\overline{H_I}(t)\,\overline{H_0}(t),\overline{H_I}^2(t)\Big]+\Big[\overline{H_0}(t),\overline{H_I}^3(t)\Big]+\Big[\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_I}^2(t)\Big] \quad (192)$$
$$+\left[\overline{H_0}(t),\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\right]+\Big[\overline{H_0}^3(t),\overline{H_I}(t)\Big]+\left[\overline{H_I}(t)\,\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_I}(t)\right]+3\overline{H_0}(t)\left[\overline{H_I}(t),\overline{H_0}(t)\right]\overline{H_0}(t) \quad (193)$$
$$+2\overline{H_I}(t)\left[\overline{H_I}(t)\,,\overline{H_0}(t)\right]\overline{H_0}(t)+\Big[\overline{H_I}^2(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0(t)}}. \quad (194)$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_{\bar{S}}(t)}\equiv(\varepsilon_0(t)+R_0(t))\,|0\rangle\langle 0|+(\varepsilon_1(t)+R_1(t))\,|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \quad (195)$$
$$\overline{H_{\bar{I}}}(t)\equiv\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t)), \quad (196)$$
$$\overline{H_{\bar{B}}}\equiv\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \quad (197)$$
$$=H_B. \quad (198)$$

Then the explicit form of $\overline{H_{\bar{I}}}^2(t)$ is:

$$\overline{H_{\bar{I}}}^2(t)=\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i B_{iz}(t)\,|i\rangle\langle i|\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)\sum_i B_{iz}(t)\,|i\rangle\langle i|\,(\sigma_x B_y(t)-\sigma_y B_x(t)) \quad (199)$$
$$+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|+\left(V_{10}^{\Re}(t)\right)^2(\sigma_x B_x(t)+\sigma_y B_y(t))^2+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\,(\sigma_x B_x(t) \quad (200)$$
$$+\sigma_y B_y(t))\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \quad (201)$$
$$-\sigma_y B_x(t))\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\left(V_{10}^{\Im}(t)\right)^2(\sigma_x B_y(t)-\sigma_y B_x(t))^2 \quad (202)$$
$$=\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i\left(B_{iz}(t)\,B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}(t)\sum_i\left(B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_x\right. \quad (203)$$

$$-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2(\sigma_x^2B_x^2(t)+\sigma_x\sigma_yB_x(t)B_y(t) \tag{204}$$

$$+\sigma_y\sigma_xB_y(t)B_x(t)+\sigma_y^2B_y^2(t))+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2(\sigma_x^2B_y^2(t)+\sigma_y^2B_x^2(t) \tag{205}$$

$$-\sigma_x\sigma_yB_y(t)B_x(t)-\sigma_y\sigma_xB_x(t)B_y(t))+V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x^2B_y(t)B_x(t)+\sigma_x\sigma_yB_y^2(t)-\sigma_y\sigma_xB_x^2(t)-\sigma_y^2B_x(t)B_y(t) \tag{206}$$

$$+\sigma_x^2B_x(t)B_y(t)-\sigma_x\sigma_yB_x^2(t)+\sigma_y\sigma_xB_y^2(t)-\sigma_y^2B_y(t)B_x(t))\,, \tag{207}$$

$$\sigma_x\sigma_y=\mathrm{i}\sigma_z\,, \tag{208}$$

$$\overline{H_{\bar{I}}}^2(t)=\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i(B_{iz}(t)\,B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_x \tag{209}$$

$$-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t) \tag{210}$$

$$-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t))+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2(B_y^2(t)+B_x^2(t) \tag{211}$$

$$-\mathrm{i}\sigma_zB_y(t)\,B_x(t)+\mathrm{i}\sigma_zB_x(t)\,B_y(t))\,. \tag{212}$$

In order to obtain the expected values of $\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_0(t)}}$ respect to the part related to the bath we need to calculate the following expected values $\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_{iz}(t)\,B_x(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_{iz}(t)\,B_y(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_y(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x^2(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_y^2(t)\rangle_{\overline{H_{\bar{B}}}}$:

$$\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\left(\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{213}$$

$$=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)^2+\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\sum_{\mathbf{k}'}((g_{i\mathbf{k}'}\right. \tag{214}$$

$$\left.-v_{i\mathbf{k}'}(t))\,b_{\mathbf{k}'}^\dagger+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*\,b_{\mathbf{k}'}\big)\right\rangle_{\overline{H_{\bar{B}}}} \tag{215}$$

$$=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{216}$$

$$\times\left\langle\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))\,b_{\mathbf{k}'}^\dagger+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*\,b_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{217}$$

$$=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^\dagger+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{218}$$

$$=\sum_{\mathbf{k}}\left\langle(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^2\left(b_{\mathbf{k}}^\dagger\right)^2+|g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)|^2\left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right)+((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*)^2\,b_{\mathbf{k}}^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{219}$$

$$=\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^2\left\langle\left(b_{\mathbf{k}}^\dagger\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}|g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)|^2\left\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+b_{\mathbf{k}}b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left\langle((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*)^2\,b_{\mathbf{k}}^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{220}$$

$$\left\langle\left(b_{\mathbf{k}}^\dagger\right)^2\right\rangle_{\overline{H_{\bar{B}}}}=\frac{\mathrm{Tr}\left(\left(b_{\mathbf{k}}^\dagger\right)^2\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\text{Bose}-\text{Einstein}}(-\beta\omega_{\mathbf{k}})} \tag{221}$$

$$=\frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger\right)^2|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\text{Bose}-\text{Einstein}}(-\beta\omega_{\mathbf{k}})} \tag{222}$$

$$=\frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\sqrt{(j_{\mathbf{k}}+2)\,(j_{\mathbf{k}}+1)}|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\text{Bose}-\text{Einstein}}(-\beta\omega_{\mathbf{k}})} \tag{223}$$

$$= 0, \tag{224}$$

$$\langle b_{\mathbf{k}}^2 \rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} |j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{225}$$

$$= 0, \tag{226}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{227}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{228}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{229}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \tag{230}$$

$$= \frac{1+\mathrm{e}^{-\beta\omega_{\mathbf{k}}}}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \tag{231}$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{232}$$

$$\langle B_{iz}^2 \left(t\right) \rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{233}$$

$$\langle B_{iz}\left(t\right) B_x\left(t\right) \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^* b_{\mathbf{k}}\right) \frac{B_1^+(t)\, B_0^-\left(t\right) + B_0^+\left(t\right) B_1^-\left(t\right) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{234}$$

$$= \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)\, b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^* b_{\mathbf{k}}\right) \left(\mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right) + \mathrm{e}^{\chi_{01}(t)} \right.\right. \tag{235}$$

$$\left.\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{236}$$

$$\left\langle b^{\dagger} D\left(h\right) \right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle \alpha | b^{\dagger} D\left(h\right) | \alpha \rangle \mathrm{d}^2\alpha \tag{237}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | D\left(-\alpha\right) b^{\dagger} D\left(\alpha\right) D\left(-\alpha\right) D\left(h\right) D\left(\alpha\right) | 0 \rangle \mathrm{d}^2\alpha \tag{238}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | D\left(-\alpha\right) b^{\dagger} D\left(\alpha\right) D\left(h\right) \mathrm{e}^{h\alpha^* - h^*\alpha} | 0 \rangle \mathrm{d}^2\alpha \tag{239}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle 0 | \left(b^{\dagger} + \alpha^*\right) D\left(h\right) \mathrm{e}^{h\alpha^* - h^*\alpha} | 0 \rangle \mathrm{d}^2\alpha \tag{240}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \langle 0 | \left(b^{\dagger} + \alpha^*\right) | h \rangle \mathrm{d}^2\alpha \tag{241}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \langle 0 | \left(b^{\dagger} + \alpha^*\right) | h \rangle \mathrm{d}^2\alpha, \tag{242}$$

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{243}$$

$$\left\langle b^{\dagger} D\left(h\right) \right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \left(\langle 0 | b^{\dagger} | h \rangle + \alpha^* \langle 0 | h \rangle\right) \mathrm{d}^2\alpha \tag{244}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \left(\langle 0 | b^{\dagger} \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle + \alpha^* \langle 0 | h \rangle\right) \mathrm{d}^2\alpha \tag{245}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \left(\langle 0 | \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1} |n+1\rangle + \alpha^* \langle 0 | h \rangle\right) \mathrm{d}^2\alpha \tag{246}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{e}^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 | h \rangle \mathrm{d}^2\alpha \tag{247}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\,\mathrm{d}^2\alpha \tag{248}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\,(x-\mathrm{i}y)\,\mathrm{d}x\mathrm{d}y \tag{249}$$

$$= -h^*N\left(\langle D\left(h\right)\rangle_{\overline{H_{\bar{B}}}}\right)^2 \tag{250}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right.\right. \tag{251}$$

$$\left.\left.+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{252}$$

$$= \frac{1}{2}\left\langle\mathrm{e}^{\chi_{10}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right. \tag{253}$$

$$\left.+\mathrm{e}^{\chi_{01}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{254}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\right. \tag{255}$$

$$\left.\times\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{256}$$

$$+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\right. \tag{257}$$

$$\left.\times\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{258}$$

$$= \frac{B_{10}\left(t\right)}{2}\left(-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{259}$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{260}$$

$$+\frac{B_{01}\left(t\right)}{2}\left(-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{261}$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{262}$$

$$= \frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{263}$$

$$\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{264}$$

$$\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*b_{\mathbf{k}}\right)\frac{B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{265}$$

$$=\frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\left(B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{266}$$

$$=\frac{B_{10}(t)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}-\sum_{\mathbf{k}}(g_{i\mathbf{k}}\right. \tag{267}$$

$$\left.-v_{i\mathbf{k}}\left(t\right))^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{268}$$

$$+\frac{B_{01}\left(t\right)}{2\mathrm{i}}\left(-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{269}$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{270}$$

$$=\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{271}$$

$$\left.-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{272}$$

$$\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle B_x\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{273}$$

$$=\left\langle\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{274}$$

$$=\frac{1}{2}\left\langle\left(B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{275}$$

$$=\frac{1}{2}\left\langle\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{276}$$

$$+\frac{1}{2}\left\langle\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{277}$$

$$\langle D\left(h\right)b\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha|D\left(h\right)b|\alpha\right\rangle\mathrm{d}^2\alpha \tag{278}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)D\left(-\alpha\right)bD\left(\alpha\right)|0\right\rangle\mathrm{d}^2\alpha \tag{279}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0|D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\left(b+\alpha\right)|0\right\rangle\mathrm{d}^2\alpha \tag{280}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0|D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha|0\right\rangle\mathrm{d}^2\alpha \tag{281}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\langle 0|D\left(h\right)|0\rangle\mathrm{d}^2\alpha \tag{282}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle\mathrm{d}^2\alpha \tag{283}$$

$$=\frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int\alpha\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha \tag{284}$$

$$=\frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(x+\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \tag{285}$$

$$= Nh\mathrm{e}^{-|h|^2\coth\left(\frac{\beta\omega}{2}\right)} \tag{286}$$

$$= Nh\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{287}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\right|D\left(h\right)b^{\dagger}\left|\alpha\right\rangle\mathrm{d}^2\alpha \tag{288}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\right|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)D\left(-\alpha\right)b^{\dagger}D\left(\alpha\right)\left|0\right\rangle\mathrm{d}^2\alpha \tag{289}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\right|D\left(h\right)\mathrm{e}^{h\alpha^*-h^*\alpha}\left(b^{\dagger}+\alpha^*\right)\left|0\right\rangle\mathrm{d}^2\alpha \tag{290}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|D\left(h\right)b^{\dagger}\left|0\right\rangle\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|D\left(h\right)\left|0\right\rangle\mathrm{d}^2\alpha \tag{291}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h|1\right\rangle\mathrm{d}^2\alpha + \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha, \tag{292}$$

$$\left\langle\alpha\right| = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\left(\alpha^*\right)^n}{\sqrt{n!}}\left\langle n\right|, \tag{293}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\left(-h^*\right)\mathrm{d}^2\alpha + \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\alpha^*\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha \tag{294}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left(-h^*+\alpha^*\right)\mathrm{d}^2\alpha \tag{295}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N}}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(-h^*+x-\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \tag{296}$$

$$= -\left(N+1\right)h^*\mathrm{e}^{-|h|^2\coth\left(\frac{\beta\omega}{2}\right)}, \tag{297}$$

$$= -\left(N+1\right)h^*\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{298}$$

$$\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{299}$$

$$\left\langle B_x\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{300}$$

$$+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{301}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right.\right.\right.\right. \tag{302}$$

$$\left.\left.\left.\left.-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\right.\right. \tag{303}$$

$$\left.\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{304}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right.\right.\right.\right. \tag{305}$$

$$\left.\left.\left.\left.-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}\right. \tag{306}$$

$$\left.+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left\langle\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{307}$$

$$= \frac{B_{10}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^*\right.\right. \tag{308}$$

$$+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg)\Bigg)+\frac{B_{01}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}\right.\right.\right. \tag{309}$$

$$-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg) \tag{310}$$

$$=\frac{B_{10}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+\left(g_{i\mathbf{k}}\right.\right.\right. \tag{311}$$

$$-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg)+\frac{B_{01}\left(t\right)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\right.\right. \tag{312}$$

$$\times\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg) \tag{313}$$

$$=\frac{B_{01}(t)-B_{10}(t)}{2}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right. \tag{314}$$

$$-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg) \tag{315}$$

$$\left\langle D\left(h\right)b\right\rangle_{\overline{H_{\bar{B}}}}=Nh\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{316}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}=-\left(N+1\right)h^{*}\left\langle D\left(h\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2}, \tag{317}$$

$$\langle B_{y}(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle B_{y}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{318}$$

$$=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{319}$$

$$=\frac{1}{2\mathrm{i}}\left\langle\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{320}$$

$$=\frac{1}{2\mathrm{i}}\left\langle\left(B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{321}$$

$$=\frac{B_{10}(t)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\right.\right. \tag{322}$$

$$\times N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg)+\frac{B_{01}\left(t\right)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\right.\right. \tag{323}$$

$$\times\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg) \tag{324}$$

$$=\frac{B_{10}\left(t\right)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)(N_{\mathbf{k}}+1)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\right.\right. \tag{325}$$

$$\times N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg)+\frac{B_{01}\left(t\right)}{2\mathrm{i}}\left(\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\right.\right. \tag{326}$$

$$\times\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg)\Bigg) \tag{327}$$

$$=\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right. \tag{328}$$

$$-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\Bigg) \tag{329}$$

$$\left\langle B_{x}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_{x}\left(t\right)\right)+\left\langle B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}^{2} \tag{330}$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)-B_{10}(t)-B_{01}(t)}{2}\right) \tag{331}$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)-B_{10}(t)-B_{01}(t)\right) \tag{332}$$

$$= \frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right) \tag{333}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{334}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)B_0^-(t)\right)^2+B_1^+(t)B_0^-(t)B_0^+(t)B_1^-(t)+B_0^+(t)B_1^-(t)B_1^+(t)B_0^-(t)+\left(B_0^+(t)B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right. \tag{335}$$

$$\left.-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{336}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right), \tag{337}$$

$$(D(h))^2 = D(h)\,D(h) \tag{338}$$

$$= D(h+h)\,\mathrm{e}^{\frac{1}{2}\left(\frac{h^*h-hh^*}{\omega^2}\right)} \tag{339}$$

$$= D(2h), \tag{340}$$

$$\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^2\right\rangle_{\overline{H_{\bar{B}}}} \tag{341}$$

$$= \left\langle\prod_{\mathbf{k}}D\left(2\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right)\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{342}$$

$$= \prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\,\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{343}$$

$$\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2+2\mathbb{I}+\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{344}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{345}$$

$$= \frac{1}{4}\left(\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+2+\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{346}$$

$$= \frac{1}{4}\left(\mathrm{e}^{2\chi_{10}(t)}\prod_{\mathbf{k}}\mathrm{e}^{-2\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}+2+\mathrm{e}^{2\chi_{01}(t)}\prod_{\mathbf{k}}\mathrm{e}^{-2\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{347}$$

$$\left.-\left(B_{10}(t)+B_{01}(t)\right)^2\right) \tag{348}$$

$$= \frac{1}{4}\left(B_{10}^2(t)\left|B_{10}^2(t)\right|+2+B_{01}^2(t)\left|B_{01}^2(t)\right|-\left(B_{10}^2(t)+2B_{10}(t)\,B_{01}(t)+B_{01}^2(t)\right)\right) \tag{349}$$

$$\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_y(t)\right)+\left\langle B_y(t)\right\rangle_{\overline{H_{\bar{B}}}}^2 \tag{350}$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}}\left(\frac{B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right) \tag{351}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)+B_{10}(t)-B_{01}(t)\right) \tag{352}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right) \tag{353}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right)^2-\left(B_{01}(t)-B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{354}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2-2\mathbb{I}+\left(B_1^+(t)\,B_0^-(t)\right)^2-\left(B_{01}(t)-B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{355}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}+\left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}-2-\left(B_{01}(t)-B_{10}(t)\right)^2\right), \tag{356}$$

$$\left\langle\left(B_i^+(t)B_j^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} = \prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}}\,\mathrm{e}^{-2\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{357}$$

$$= \left(\prod_{\mathbf{k}} \mathrm{e}^{\frac{v_{i\mathbf{k}}^{*}(t)v_{j\mathbf{k}}(t)-v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^{*}(t)}{2\omega_{\mathbf{k}}^{2}}} \mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{2} \left(\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{2} \tag{358}$$

$$= B_{ij}^{2}(t)\left|B_{ij}(t)\right|^{2}, \tag{359}$$

$$\left\langle B_{y}^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} = -\frac{1}{4}\left(B_{01}^{2}(t)\left|B_{10}(t)\right|^{2}-2+B_{10}^{2}(t)\left|B_{10}(t)\right|^{2}-\left(B_{01}(t)-B_{10}(t)\right)^{2}\right), \tag{360}$$

$$\left\langle B_{x}(t)\,B_{y}(t)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \frac{B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\,\frac{B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{361}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)\right)\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{362}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\mathbb{I}-\left(B_{1}^{+}(t)B_{0}^{-}(t)\right)^{2}+B_{10}^{2}(t)-B_{10}(t)B_{01}(t)+\left(B_{0}^{+}(t)B_{1}^{-}(t)\right)^{2}-\mathbb{I}+B_{10}(t)B_{01}(t)-B_{01}^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{363}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_{0}^{+}(t)\,B_{1}^{-}(t)\right)^{2}-\left(B_{1}^{+}(t)\,B_{0}^{-}(t)\right)^{2}-\left(B_{01}^{2}(t)-B_{10}^{2}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{364}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^{2}(t)\left|B_{10}(t)\right|^{2}-B_{10}^{2}(t)\left|B_{10}(t)\right|^{2}-\left(B_{01}^{2}(t)-B_{10}^{2}(t)\right)\right), \tag{365}$$

$$\left\langle B_{y}(t)B_{x}(t)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\,\frac{B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{366}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_{0}^{+}(t)B_{1}^{-}(t)-B_{1}^{+}(t)B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)\right)\left(B_{1}^{+}(t)B_{0}^{-}(t)+B_{0}^{+}(t)B_{1}^{-}(t)-B_{10}(t)-B_{01}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{367}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\mathbb{I}+\left(B_{0}^{+}(t)B_{1}^{-}(t)\right)^{2}-B_{10}(t)B_{01}(t)-B_{01}^{2}(t)-\left(B_{1}^{+}(t)B_{0}^{-}(t)\right)^{2}-\mathbb{I}+B_{10}^{2}(t)+B_{10}(t)B_{01}(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{368}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_{0}^{+}(t)\,B_{1}^{-}(t)\right)^{2}-B_{01}^{2}(t)-\left(B_{1}^{+}(t)\,B_{0}^{-}(t)\right)^{2}+B_{10}^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{369}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^{2}(t)\left|B_{10}(t)\right|^{2}-B_{01}^{2}(t)-\left(B_{10}^{2}(t)\left|B_{10}(t)\right|^{2}-B_{10}^{2}(t)\right)\right). \tag{370}$$

The density matrix associated to $\rho_{\overline{S}} = \frac{\mathrm{e}^{-\beta\overline{H_{0}}(t)}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{0}}(t)}\right)}$ follows the form:

$$\rho_{\overline{S},00} = \frac{1}{2}+\frac{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}}, \tag{371}$$

$$\rho_{\overline{S},01} = -\frac{B_{10}^{*}(t)\,V_{10}^{*}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}}, \tag{372}$$

$$\rho_{\overline{S},10} = -\frac{B_{10}(t)\,V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}}, \tag{373}$$

$$\rho_{\overline{S},11} = \frac{1}{2}-\frac{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}(-1)^{i}\left(\varepsilon_{i}(t)+R_{i}(t)\right)\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}}. \tag{374}$$

The expected values respect to the system $\overline{S}$ of relevance for calculating $\left\langle\overline{H_{\bar{I}}}^{2}(t)\right\rangle_{H_{\overline{S}}}$ are $\langle|i\rangle\langle i|\rangle_{H_{\overline{S}}}$, $\langle|i\rangle\langle i|\sigma_{x}\rangle_{H_{\overline{S}}}$, $\langle|i\rangle\langle i|\sigma_{y}\rangle_{H_{\overline{S}}}$, $\langle\sigma_{x}|i\rangle\langle i|\rangle_{H_{\overline{S}}}$, $\langle\sigma_{y}|i\rangle\langle i|\rangle_{H_{\overline{S}}}$ and $\langle\sigma_{z}\rangle_{H_{\overline{S}}}$, we took account that $\sigma_{x}\sigma_{y} = \mathrm{i}\sigma_{z}$ and $\sigma_{y}\sigma_{x} = -\mathrm{i}\sigma_{z}$. The values needed for our calculation are:

$$\langle|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}-\frac{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(375)$$

$$\langle|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}+\frac{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(376)$$

$$\langle|0\rangle\langle 0|\sigma_{x}\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}(t)V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(377)$$

$$\langle|1\rangle\langle 1|\sigma_{x}\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}^{*}(t)V_{10}^{*}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(378)$$

$$\langle|0\rangle\langle 0|\sigma_{y}\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{\mathrm{i}B_{10}(t)V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(379)$$

$$\langle|1\rangle\langle 1|\sigma_{y}\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{\mathrm{i}B_{10}^{*}(t)V_{10}^{*}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}}},\quad(380)$$

$$\langle\sigma_{x}|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}^{*}(t)V_{10}^{*}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(381)$$

$$\langle\sigma_{x}|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}(t)V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(382)$$

$$\langle\sigma_{y}|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{\mathrm{i}B_{10}^{*}(t)V_{10}^{*}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(383)$$

$$\langle\sigma_{y}|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{\mathrm{i}B_{10}(t)V_{10}(t)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}},\quad(384)$$

$$\langle\sigma_{z}\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}(-1)^{i}(\varepsilon_{i}(t)+R_{i}(t))\right)^{2}+4\left|B_{10}(t)V_{10}(t)\right|^{2}}}.\quad(385)$$

Summarizing the expected values of the bath we have:

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\sum_{\mathbf{k}}\left|g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{386}$$

$$\left\langle B_{iz}\left(t\right)B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right.\right. \tag{387}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\right)\right), \tag{388}$$

$$\left\langle B_{iz}\left(t\right)B_{y}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right.\right. \tag{389}$$

$$\left.\left.-\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)\right)\right), \tag{390}$$

$$\left\langle B_{x}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{01}\left(t\right)-B_{10}\left(t\right)}{2}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right. \tag{391}$$

$$\left.\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right), \tag{392}$$

$$\left\langle B_{y}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\mathrm{e}^{-\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(N_{\mathbf{k}}+1\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\right.\right. \tag{393}$$

$$\left.\left.-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}N_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right), \tag{394}$$

$$\left\langle B_{x}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4}\left(B_{10}^{2}\left(t\right)\left|B_{10}^{2}\left(t\right)\right|+2+B_{01}^{2}\left(t\right)\left|B_{01}^{2}\left(t\right)\right|-\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right)^{2}\right), \tag{395}$$

$$\left\langle B_{y}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=-\frac{1}{4}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-2+B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}\left(t\right)-B_{10}\left(t\right)\right)^{2}\right), \tag{396}$$

$$\left\langle B_{x}\left(t\right)B_{y}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4\mathrm{i}}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}^{2}\left(t\right)-B_{10}^{2}\left(t\right)\right)\right), \tag{397}$$

$$\left\langle B_{y}\left(t\right)B_{x}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{4\mathrm{i}}\left(B_{01}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-B_{10}^{2}\left(t\right)\left|B_{10}\left(t\right)\right|^{2}-\left(B_{01}^{2}\left(t\right)-B_{10}^{2}\left(t\right)\right)\right). \tag{398}$$

Our next step is to find $v_3\left(t\right)$, the commutator $\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]$ is a central point for our calculations and it is equal to:

$$\left[\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right]=\left[\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)\right.\right. \tag{399}$$

$$\left.\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_xB_x(t)+\sigma_yB_y(t)\right)+V_{10}^{\Im}(t)\left(\sigma_xB_y(t)-\sigma_yB_x(t)\right)\right] \tag{400}$$

$$=\left[\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right)-\sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right)\right. \tag{401}$$

$$\left.+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_{i}B_{iz}\left(t\right)|i\rangle\langle i|+V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right)+\sigma_yB_y\left(t\right)\right)+V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y\left(t\right)-\sigma_yB_x\left(t\right)\right)\right] \tag{402}$$

$$=\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right)+\sigma_yB_y\left(t\right)\right)+\sum_{i}\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y\left(t\right)-\sigma_yB_x\left(t\right)\right)+\sigma_x \tag{403}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_xB_x(t)+\sigma_yB_y(t)\right) \tag{404}$$

$$+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(\sigma_xB_y(t)-\sigma_yB_x(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i}B_{iz}(t)\,|i\rangle\langle i| \tag{405}$$

$$-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(\sigma_xB_y(t) \tag{406}$$

$$-\sigma_y B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{407}$$

$$-\sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t) |i\rangle\langle i| \tag{408}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{409}$$

$$+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \tag{410}$$

$$\times \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \tag{411}$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \tag{412}$$

$$= \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{413}$$

$$\times \sum_i B_{iz}(t) |i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t) (B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Im}(t) \tag{414}$$

$$\times (B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) |i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Re}(t) (-\mathrm{i}\sigma_z B_x(t) + B_y(t)) \tag{415}$$

$$-\left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t) (-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{416}$$

$$+\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t)\right. \tag{417}$$

$$\left.+B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t) (B_x(t) - \mathrm{i}\sigma_z B_y(t)) \tag{418}$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t) (\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \tag{419}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{420}$$

$$+V_{10}^{\Im}(t) (\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{421}$$

The term $\overline{H_{\bar{I}}}(t)\left[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)\right]$ is given by:

$$\overline{H_{\bar{I}}}(t)[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)] = \left(\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\right) \left(\sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| V_{10}^{\Re}(t)\right. \tag{422}$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{423}$$

$$\times \sum_i B_{iz}(t)|i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t)(B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{424}$$

$$\times V_{10}^{\Im}(t)(B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)|i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \tag{425}$$

$$\times V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t) + B_y(t)) - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_i B_{iz}(t) \tag{426}$$

$$\times |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \tag{427}$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t)|i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t)|i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{428}$$

$$-V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Re}(t)(B_x(t) - \mathrm{i}\sigma_z B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{429}$$

$$+V_{10}^{\Re}(t)(\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{430}$$

$$-V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| - V_{10}^{\Im}(t)(B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \tag{431}$$

$$\left.+V_{10}^{\Im}(t)(\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) \tag{432}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{433}$$

$$+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\,(B_x(t) \tag{434}$$

$$+\mathrm{i}\sigma_z B_y(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{435}$$

$$\left.+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\,(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i| \tag{436}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{437}$$

$$\times V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_i B_{iz}(t)\,|i\rangle\langle i|\,\sigma_x \tag{438}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i| \tag{439}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{440}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t) \tag{441}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t) \tag{442}$$

$$\times(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right) \tag{443}$$

$$-\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{444}$$

$$+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x \tag{445}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t)) \tag{446}$$

$$+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_y \tag{447}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{448}$$

$$-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{449}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{450}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{451}$$

$$+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t) \tag{452}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{453}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t) \tag{454}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t) \tag{455}$$

$$+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{456}$$

$$\times V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{457}$$

$$+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i| \tag{458}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{459}$$

$$-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{460}$$

$$\times V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|-V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{461}$$

$$-\sigma_y B_x(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{462}$$

$$\times V^{\Im}_{10}(t)\left(-\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)+V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{463}$$

$$\times V^{\Re}_{10}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{464}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{465}$$

$$-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t)) \tag{466}$$

$$\times|i\rangle\langle i|-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V^{\Re}_{10}(t)\left(B_x(t)-\mathrm{i}\sigma_z B_y(t)\right)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Re}_{10}(t) \tag{467}$$

$$\times\left(\mathrm{i}\sigma_z B_x(t)+B_y(t)\right)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Re}_{10}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{468}$$

$$-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|-V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V^{\Im}_{10}(t)\left(B_y(t)\right. \tag{469}$$

$$\left.+\mathrm{i}\sigma_z B_x(t)\right)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Im}_{10}(t)\left(\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right) \tag{470}$$

$$-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V^{\Im}_{10}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{471}$$

$$=V^{\Re}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)B_y(t)\right)+V^{\Im}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)B_y(t)-|i\rangle\langle i|\sigma_y\right. \tag{472}$$

$$\left.\times B_{iz}(t)B_x(t)\right)+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_{i\neq i'}B_{iz}(t)B_{i'z}(t)|i\rangle\langle i'|+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Re}_{10}(t)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\right. \tag{473}$$

$$\left.\times B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)\right)+\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\sum_i\left(|i\rangle\langle i|\sigma_x B_{iz}(t)B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)B_x(t)\right)-\left(B^{\Re}_{10}(t)\right. \tag{474}$$

$$\left.\times V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_{i\neq i'}B_{iz}(t)B_{i'z}(t)|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|-\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)\sum_i\left(-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_x(t)+B_{iz}(t)\right. \tag{475}$$

$$\left.\times B_y(t)|i\rangle\langle i|\right)+\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)+|i\rangle\langle i|B_{iz}(t)B_x(t)\right)+\sum_{i,\mathbf{k}}|i\rangle\langle i|B_{iz}(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_{iz}(t) \tag{476}$$

$$+V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_y(t)\right)+V^{\Im}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\right. \tag{477}$$

$$\left.\times\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_x(t)\right)-\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i B^2_{iz}(t)|i\rangle\langle i|\sigma_x+\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i B^2_{iz}(t)|i\rangle\langle i|\sigma_y-\sum_{i,\mathbf{k}}|i\rangle\langle i| \tag{478}$$

$$\times B^2_{iz}(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t)\sum_{i,i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|B_{iz}(t)B_x(t)+|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_y(t)\right)-V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\right. \tag{479}$$

$$\left.\times V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_x(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)\right)+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_x(t)+|i\rangle\langle i|\right. \tag{480}$$

$$\left.\times B_{iz}(t)B_y(t)\right)-V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)B_x(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}+|i\rangle\langle i|\sigma_y B_{iz}(t)B_y(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\right)-V^{\Im}_{10}(t)\sum_{i\neq i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|\right. \tag{481}$$

$$\left.\times B_{iz}(t)B_y(t)-|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_x(t)\right)-V^{\Im}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_y(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_x(t)\right)+V^{\Im}_{10}(t) \tag{482}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)-|i\rangle\langle i|B_{iz}(t)B_x(t)\right)-V^{\Im}_{10}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)B_y(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-|i\rangle\langle i|\sigma_y\right. \tag{483}$$

$$\left.\times B_{iz}(t)B_x(t)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\right)+\left(V^{\Re}_{10}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B^2_x(t)+\sigma_x|i\rangle\langle i|\sigma_y B_x(t)B_y(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)B_x(t)+\sigma_y|i\rangle\langle i|\right. \tag{484}$$

$$\left.\times\sigma_y B^2_y(t)\right)+V^{\Re}_{10}(t)V^{\Im}_{10}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B^2_x(t)+\sigma_y|i\rangle\langle i|\sigma_x B^2_y(t)-\sigma_y|i\rangle\langle i|\sigma_y B_y(t)B_x(t)\right) \tag{485}$$

$$+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i\left(|i\rangle\langle i|B_x(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_{iz}(t)\right)+\left(V^{\Re}_{10}(t)\right)^2\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right) \tag{486}$$

$$\times\left(\sigma_x B^2_x(t)+\sigma_y B_x(t)B_y(t)+\sigma_y B_y(t)B_x(t)-\sigma_x B^2_y(t)\right)+V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\left(B_x(t)B_y(t)-\mathrm{i}\sigma_z B^2_y(t)\right. \tag{487}$$

$$\left.-\mathrm{i}\sigma_z B^2_x(t)-B_y(t)B_x(t)\right)-V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y(t)B_{iz}(t)\right)-V^{\Re}_{10}(t) \tag{488}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)\left(-\sigma_y B^2_x(t)+\sigma_x B_y(t)B_x(t)+\sigma_x B_x(t)B_y(t)+\sigma_y B^2_y(t)\right)-V^{\Re}_{10}(t)\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\right. \tag{489}$$

$$\left.\times V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)\left(-\sigma_y B_x(t)B_y(t)+\sigma_x B^2_y(t)-\sigma_x B^2_x(t)-\sigma_y B_y(t)B_x(t)\right)+V^{\Re}_{10}(t)\sum_{i,\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)+\sigma_y|i\rangle\langle i|B_y(t)\right)\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}B_{iz}(t) \tag{490}$$

$$+\left(V_{10}^{\Re}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-\mathrm{i}\sigma_zB_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_zB_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}} \tag{491}$$

$$\times\left(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_zB_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_zB_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{492}$$

$$\times\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_xB_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i|\sigma_xB_y(t)\,B_{iz}(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_yB_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i|\right. \tag{493}$$

$$\left.\times\sigma_yB_y(t)\,B_{iz}(t)\right)-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\sigma_y|i\rangle\langle i|B_y(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_x^2(t)\right. \tag{494}$$

$$-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)+|i\rangle\langle i|B_y^2(t))-\left(V_{10}^{\Re}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_xB_x^2(t)+\sigma_yB_y(t)\,B_x(t)\right. \tag{495}$$

$$\left.-\sigma_yB_x(t)\,B_y(t)+\sigma_xB_y^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_yB_x^2(t)-\sigma_xB_y(t)\,B_x(t)+\sigma_xB_x(t)\,B_y(t)+\sigma_yB_y^2(t)\right) \tag{496}$$

$$-\left(V_{10}^{\Re}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_zB_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_zB_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right) \tag{497}$$

$$\times\left(|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)-|i\rangle\langle i|B_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_xB_x(t)\right. \tag{498}$$

$$\left.\times B_y(t)+\sigma_yB_y^2(t)+\sigma_yB_x^2(t)-\sigma_xB_y(t)B_x(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_yB_x(t)B_y(t)-\sigma_xB_y^2(t)-\sigma_xB_x^2(t)\right. \tag{499}$$

$$\left.-\sigma_yB_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_zB_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_zB_x^2(t)-B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_i\left(\varepsilon_i(t)\right. \tag{500}$$

$$\left.+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)\,B_x(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x^2(t)+\sigma_x|i\rangle\langle i|\sigma_yB_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_yB_x(t)\,B_y(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\right. \tag{501}$$

$$\left.\times\sigma_xB_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x(t)\,B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)\,B_x(t)+\sigma_y|i\rangle\langle i|\sigma_yB_x^2(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i\left(|i\rangle\langle i|B_y(t)\right. \tag{502}$$

$$\left.\times B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_xB_y(t)B_x(t)-\sigma_yB_x^2(t)+\sigma_yB_y^2(t)+\sigma_xB_x(t)B_y(t)\right) \tag{503}$$

$$+\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(B_y^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_x^2(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{504}$$

$$\times\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_{iz}(t)-|i\rangle\langle i|B_x(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(-\sigma_yB_y(t)\,B_x(t)-\sigma_xB_x^2(t)+\sigma_xB_y^2(t)\right. \tag{505}$$

$$\left.-\sigma_yB_x(t)\,B_y(t)\right)-\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(-\sigma_yB_y^2(t)-\sigma_xB_x(t)\,B_y(t)-\sigma_xB_y(t)\,B_x(t)+\sigma_yB_x^2(t)\right)+V_{10}^{\Im}(t) \tag{506}$$

$$\times\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\right)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_zB_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z\right. \tag{507}$$

$$\left.\times B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+\mathrm{i}\sigma_zB_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_zB_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+B_x(t)\right. \tag{508}$$

$$\left.\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{509}$$

$$\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_yB_x(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{510}$$

$$-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_y(t)B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-|i\rangle\langle i|B_x(t)B_y(t)\right)-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{511}$$

$$\left.-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_xB_y(t)B_x(t)-\sigma_yB_x^2(t)-\sigma_yB_y^2(t)-\sigma_xB_x(t)B_y(t)\right)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_yB_y(t)B_x(t)\right. \tag{512}$$

$$\left.+\sigma_xB_x^2(t)+\sigma_xB_y^2(t)-\sigma_yB_x(t)B_y(t)\right)-\sum_{\mathbf{k}}V_{10}^{\Im}(t)V_{10}^{\Re}(t)\,\omega_{\mathbf{k}}\left(B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_zB_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_zB_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{513}$$

$$-\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_y^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+|i\rangle\langle i|B_x^2(t)\right)-\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{514}$$

$$\left.-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_xB_y^2(t)-\sigma_yB_x(t)\,B_y(t)+\sigma_yB_y(t)\,B_x(t)+\sigma_xB_x^2(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_yB_y^2(t)\right. \tag{515}$$

$$\left.+\sigma_xB_x(t)\,B_y(t)-\sigma_xB_y(t)\,B_x(t)+\sigma_yB_x^2(t)\right)-\left(V_{10}^{\Im}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_zB_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_zB_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_x^2(t)\right. \tag{516}$$

$$\left.\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right), \tag{517}$$

Now let's obtain the form of $\overline{H_{\bar{I}}}^3(t)$:

$$\overline{H_{\bar{I}}}^3(t)=\left(\sum_iB_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\right)\left(\sum_iB_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x\right. \tag{518}$$

$$+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) \tag{519}$$

$$+\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2 \tag{520}$$

$$\times\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right)) \tag{521}$$

$$=\sum_iB_{iz}\,(t)\,|i\rangle\langle i|\sum_iB_{iz}^2\,(t)\,|i\rangle\langle i|+\sum_iB_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+\sum_iB_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Im}\,(t) \tag{522}$$

$$\times\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+\sum_iB_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\sum_iB_{iz}(t)|i\rangle\langle i| \tag{523}$$

$$\times\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+\sum_iB_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) \tag{524}$$

$$+\sum_iB_{iz}(t)\,|i\rangle\langle i|\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\sum_iB_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t) \tag{525}$$

$$\times(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}\,(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}\,(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i\,(B_{iz}(t)B_y(t) \tag{526}$$

$$\times|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t) \tag{527}$$

$$+\sigma_yB_y(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t) \tag{528}$$

$$-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t) \tag{529}$$

$$-\sigma_yB_x(t))\sum_iB_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_y \tag{530}$$

$$\times B_x(t))V_{10}^{\Im}(t)\sum_i\,(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i| \tag{531}$$

$$\times B_y(t)B_{iz}(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t)) \tag{532}$$

$$\times V_{10}^{\Im}(t)\sum_i\,(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Im}(t)\right)^2\,\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)\right. \tag{533}$$

$$\left.+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right) \tag{534}$$

$$=\sum_iB_{iz}^3\,(t)\,|i\rangle\langle i|+V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}^2\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}^2\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}\,(t)\sum_i\left(B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_y\right) \tag{535}$$

$$+V_{10}^{\Re}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_x(t)B_{iz}(t)+|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_x^2(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)\right. \tag{536}$$

$$\left.\times B_x(t)B_y(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+|i\rangle\langle i|B_{iz}(t)B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)-|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)\right. \tag{537}$$

$$\left.B_x(t)B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_y^2(t)+|i\rangle\langle i|B_{iz}(t)B_x^2(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_x(t)B_y(t)\right)+V_{10}^{\Re}(t) \tag{538}$$

$$\times\sum_i\left(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}^2(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i(B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_x+B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t) \tag{539}$$

$$\times B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_x+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_x-B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_y \tag{540}$$

$$+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_x-B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_x^2(t)B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|\right. \tag{541}$$

$$\left.\times B_y(t)B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y^2(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^3\left(\sigma_xB_x^3(t)+\sigma_yB_x^2(t)B_y(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_xB_x(t)B_y^2(t)+\sigma_yB_y(t)B_x^2(t)\right. \tag{542}$$

$$\left.-\sigma_xB_y(t)B_x(t)B_y(t)+\sigma_xB_y^2(t)B_x(t)+\sigma_yB_y^3(t)\right)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i\left(|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_y^2(t)\right. \tag{543}$$

$$\left.\times B_{iz}(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_y(t)B_x(t)B_{iz}(t)\right)+V_{10}^{\Re}(t)\left(V_{10}^{\Im}(t)\right)^2\,\left(\sigma_xB_x(t)B_y^2(t)+\sigma_xB_x^3(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_yB_x^2(t)B_y(t)+\sigma_yB_y^3(t)\right. \tag{544}$$

$$\left.+\sigma_yB_y(t)\,B_x^2(t)+\sigma_xB_y^2(t)\,B_x(t)-\sigma_xB_y(t)\,B_x(t)\,B_y(t)\right)+V_{10}^{\Im}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}^2(t)-\sigma_y|i\rangle\langle i|B_x(t)\,B_{iz}^2(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t) \tag{545}$$

$$\times\left(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_x(t)+\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t)-\sigma_y|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t)\right)+\left(V_{10}^{\Im}(t)\right)^2 \tag{546}$$

$$\times\left(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_x(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x(t)B_{iz}(t)B_y(t)+\sigma_y|i\rangle\langle i|\sigma_yB_x(t)B_{iz}(t)B_x(t)\right)+V_{10}^{\Re}(t) \tag{547}$$

$$\times V_{10}^{\Im}(t)\sum_{i}\left(|i\rangle\langle i|B_y(t)\,B_x(t)\,B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)\,B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)\,B_{iz}(t)-|i\rangle\langle i|B_x(t)\,B_y(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(V_{10}^{\Re}(t)\right)^2 \tag{548}$$

$$\times\left(\sigma_x B_y(t)B_x^2(t)+\sigma_y B_y(t)B_x(t)B_y(t)-\sigma_y B_y^2(t)B_x(t)+\sigma_x B_y^3(t)-\sigma_y B_x^3(t)+\sigma_x B_x^2(t)B_y(t)-\sigma_x B_x(t)B_y(t)B_x(t)-\sigma_y B_x(t)B_y^2(t)\right) \tag{549}$$

$$+\left(V_{10}^{\Im}(t)\right)^2\sum_{i}\left(|i\rangle\langle i|B_y^2(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)\,B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)\,B_{iz}(t)+|i\rangle\langle i|B_x^2(t)\,B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^3\left(\sigma_x B_y^3(t)\right. \tag{550}$$

$$\left.+\sigma_x B_y(t)\,B_x^2(t)-\sigma_y B_y^2(t)\,B_x(t)+\sigma_y B_y(t)\,B_x(t)\,B_y(t)-\sigma_y B_x(t)\,B_y^2(t)-\sigma_y B_x^3(t)-\sigma_x B_x(t)\,B_y(t)\,B_x(t)+\sigma_x B_x^2(t)\,B_y(t)\right). \tag{551}$$

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