A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$
$$H_S(t) = \varepsilon_0(t)\,|0\rangle\!\langle 0| + \varepsilon_1(t)\,|1\rangle\!\langle 1| + V_{10}(t)\,|1\rangle\!\langle 0| + V_{01}(t)\,|0\rangle\!\langle 1|, \tag{2}$$
$$H_I = |0\rangle\!\langle 0| \sum_{\mathbf{k}} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}} \right) + |1\rangle\!\langle 1| \sum_{\mathbf{k}} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}} \right), \tag{3}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

I. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $\mathrm{e}^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv |0\rangle\!\langle 0| \sum_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\!\langle 1| \sum_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}(t)$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}}(t). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \tag{8}$$
$$V(t) = |0\rangle\!\langle 0| \hat{\varphi}_0(t) + |1\rangle\!\langle 1| \hat{\varphi}_1(t). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $\mathrm{e}^{\pm V(t)}$ using the notation (6) will give us the following result:

$$
\begin{aligned}
\mathrm{e}^{\pm V(t)} &= \mathrm{e}^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t))} && (10)\\
&= \mathbb{I} \pm \left(|0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right)\right) + \frac{\left(\pm\left(|0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right)\right)\right)^2}{2!} + ... && (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm \left(|0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right)\right) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2\left(t\right)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2\left(t\right)}{2!} + ... && (12)\\
&= |0\rangle\langle 0|\left(\mathbb{I} \pm \hat{\varphi}_0\left(t\right) + \frac{\hat{\varphi}_0^2\left(t\right)}{2!} \pm ...\right) + |1\rangle\langle 1|\left(\mathbb{I} \pm \hat{\varphi}_1\left(t\right) + \frac{\hat{\varphi}_1^2\left(t\right)}{2!} \pm ...\right) && (13)\\
&= |0\rangle\langle 0|\mathrm{e}^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1|\mathrm{e}^{\pm\hat{\varphi}_1(t)} && (14)\\
&= |0\rangle\langle 0|\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{0\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1|\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger}-\alpha_{1\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_0^{\pm}\left(t\right) + |1\rangle\langle 1|B_1^{\pm}\left(t\right), && (16)\\
B_i^{\pm}\left(t\right) &\equiv \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\,\mathrm{e}^{rY}\,\mathrm{e}^{-\frac{r^2}{2}[X,Y]}\,\mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\,\mathrm{e}^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\cdots. \qquad (18)
$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}, i, j$ we can show plugging $r = 1$ in (18) the following result:

$$
\begin{aligned}
\mathrm{e}^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\cdots && (19)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{-\frac{1}{2}0}\cdots && (20)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathbb{I} && (21)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. && (22)
\end{aligned}
$$

By induction of this result we can write an expresion of $B_i^{\pm}\left(t\right)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D\left(\pm v_{i\mathbf{k}}\left(t\right)\right)$:

$$
\begin{aligned}
D\left(\pm v_{i\mathbf{k}}\left(t\right)\right) &\equiv \mathrm{e}^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, && (23)\\
B_i^{\pm}\left(t\right) &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right). && (24)
\end{aligned}
$$

this will help us to write operators $O\left(t\right)$ transformed in the variational frame as:

$$
\overline{O\left(t\right)} \equiv \mathrm{e}^{V(t)}O\left(t\right)\mathrm{e}^{-V(t)}. \qquad (25)
$$

We will use the following identities:

$$
\begin{aligned}
\overline{|0\rangle\langle 0|\,(t)} &= \mathrm{e}^{V(t)}|0\rangle\langle 0|\mathrm{e}^{-V(t)} \qquad (26)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (27)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (28)\\
&= |0\rangle\langle 0|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (29)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) \qquad (30)\\
&= |0\rangle\langle 0|, \qquad (31)\\
\overline{|1\rangle\langle 1|\,(t)} &= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (32)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (33)\\
&= |1\rangle\langle 1|B_1^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (34)\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) \qquad (35)\\
&= B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) \qquad (36)\\
&= |1\rangle\langle 1|, \qquad (37)\\
\overline{|0\rangle\langle 1|\,(t)} &= \mathrm{e}^{V(t)}|0\rangle\langle 1|\mathrm{e}^{-V(t)} \qquad (38)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (39)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (40)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (41)\\
&= |0\rangle\langle 1|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (42)\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) \qquad (43)\\
&= |0\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t)\,, \qquad (44)\\
\overline{|1\rangle\langle 0|\,(t)} &= \mathrm{e}^{V(t)}|1\rangle\langle 0|\mathrm{e}^{-V(t)} \qquad (45)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (46)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (47)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (48)\\
&= |1\rangle\langle 0|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+\,(t)\,B_1^-\,(t) \qquad (49)\\
&= |1\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t)\,, \qquad (50)\\
\overline{b_{\mathbf{k}}\,(t)} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}\mathrm{e}^{-V(t)} \qquad (51)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (52)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| \qquad (53)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,b_{\mathbf{k}}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,b_{\mathbf{k}}B_0^-(t) + |1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}B_1^-(t) \qquad (54)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \qquad (55)\\
&= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \qquad (56)\\
&= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \qquad (57)\\
\overline{b_{\mathbf{k}}\,(t)}^{\dagger} &= \mathrm{e}^{V(t)}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V(t)} \qquad (58)\\
&= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \qquad (59)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle 1| \qquad (60)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) \qquad (61)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}\right) \qquad (62)\\
&= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}}. \qquad (63)
\end{aligned}
$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+}\left(t\right)b_{\mathbf{k}}B_i^{-}\left(t\right)=b_{\mathbf{k}}-\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}, \tag{64}$$

$$B_i^{+}\left(t\right)b_{\mathbf{k}}^{\dagger}B_i^{-}\left(t\right)=b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}. \tag{65}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0\left(t\right)|0\rangle\langle 0|\left(t\right)}=\varepsilon_0\left(t\right)|0\rangle\langle 0|, \tag{66}$$

$$\overline{\varepsilon_1\left(t\right)|1\rangle\langle 1|\left(t\right)}=\varepsilon_1\left(t\right)|1\rangle\langle 1|, \tag{67}$$

$$\overline{V_{10}\left(t\right)|1\rangle\langle 0|\left(t\right)}=V_{10}\left(t\right)|1\rangle\langle 0|B_1^{+}\left(t\right)B_0^{-}\left(t\right), \tag{68}$$

$$\overline{V_{01}\left(t\right)|0\rangle\langle 1|\left(t\right)}=V_{01}\left(t\right)|0\rangle\langle 1|B_0^{+}\left(t\right)B_1^{-}\left(t\right), \tag{69}$$

$$\overline{\left(g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}\right)(t)}=g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\right)+g_{\mathbf{ik}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{70}$$

$$=g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$=g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{72}$$

$$=g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}-g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|-g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|-g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{73}$$

$$=g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}-\left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0|-\left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{74}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)(t)}=\left(|0\rangle\langle 0|B_0^{+}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{-}\left(t\right)\right) \tag{75}$$

$$=|0\rangle\langle 0|B_0^{+}\left(t\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^{-}\left(t\right) \tag{76}$$

$$=|0\rangle\langle 0|B_0^{+}\left(t\right)\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_0^{-}\left(t\right) \tag{77}$$

$$=|0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)+g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right), \tag{78}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)(t)}=\left(|0\rangle\langle 0|B_0^{+}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{-}\left(t\right)\right) \tag{79}$$

$$=|1\rangle\langle 1|B_1^{+}\left(t\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^{-}\left(t\right) \tag{80}$$

$$=|1\rangle\langle 1|B_1^{+}\left(t\right)\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_1^{-}\left(t\right) \tag{81}$$

$$=|1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)+g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right), \tag{82}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}(t)}=\omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{-}\left(t\right)\right) \tag{83}$$

$$=\omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{84}$$

$$=\omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}\left(t\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_0^{-}\left(t\right)+|1\rangle\langle 1|B_1^{+}\left(t\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_1^{-}\left(t\right)\right) \tag{85}$$

$$=\omega_{\mathbf{k}}\sum_{j}|j\rangle\langle j|D\left(\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{j\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{j\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right) \tag{86}$$

$$=\omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\mathbb{I}+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{87}$$

$$=\omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right) \tag{88}$$

$$=\omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{90}$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{91}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \tag{92}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \tag{93}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right). \tag{94}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)|0\rangle\langle 0|} + \overline{\varepsilon_1(t)|1\rangle\langle 1|} + \overline{V_{10}(t)|1\rangle\langle 0|} + \overline{V_{01}(t)|0\rangle\langle 1|} \tag{95}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| B_1^+(t) B_0^-(t) + V_{01}(t)|0\rangle\langle 1| B_0^+(t) B_1^-(t), \tag{96}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{97}$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{98}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \tag{99}$$

$$= \sum_{\mathbf{k},i} |i\rangle\langle i| \left(g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \tag{100}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} \tag{101}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right) \tag{102}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right). \tag{103}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} + \frac{\left| v_{j\mathbf{k}}(t) \right|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right). \tag{104}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}. \tag{105}$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \tag{106}$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^* b_{\mathbf{k}} \right), \tag{107}$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right). \tag{108}$$

$\chi_{ij}(t)$ is an imaginary number so $\mathrm{e}^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right) & \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \\ \frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \tag{109}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{110}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{111}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{\mathrm{e}^{-\beta H_B}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_B}\right)}. \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{113}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{114}$$

So the product of the matrix representation of $b^{\dagger}$ and b with $-\beta$ is:

$$-\beta\omega b^{\dagger}b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right|, \tag{116}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$\mathrm{e}^{-\beta\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}} = \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|,, \tag{117}$$
$$\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}} = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{118}$$

The value of $\mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}}\right)$ is:

$$\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}}\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{119}$$
$$= \sum_{j_{\mathbf{k}}}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} \tag{120}$$
$$= \frac{1}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \quad \text{(by geometric series)} \tag{121}$$
$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), \tag{122}$$
$$\mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}}\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{123}$$
$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{124}$$
$$= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{125}$$

So the density matrix of the bath is:

$$\rho_B = \frac{\mathrm{e}^{-\beta H_B}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_B}\right)} \tag{126}$$
$$= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{127}$$
$$= \prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{128}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{129}$$
$$b^\dagger_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. \tag{130}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (129)-(130):

$$\begin{aligned}
\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} =& \mathrm{Tr}\left(B_{iz}\left(t\right)\rho_B\right) && (131)\\
=& \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\rho_B\right) && (132)\\
=& \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\rho_B\right) && (133)\\
=& \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) && (134)\\
=& \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) && (135)\\
=& \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), && (136)\\
\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) =& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\ \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}\text{)} && (137)\\
=& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) && (138)\\
=& 0, && (139)\\
\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) =& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\ \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} && (140)\\
=& \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) && (141)\\
=& 0. && (142)
\end{aligned}$$

we therefore find that:

$$\langle B_{iz}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0. \quad (143)$$

Another important expected value is $B\left(t\right)=\langle B^{\pm}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm}\left(t\right)=\,^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\begin{aligned}
\left\langle B^{\pm}\left(t\right)\right\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B^{\pm}\left(t\right)\right)=\mathrm{Tr}\left(B^{\pm}\left(t\right)\rho_B\right) && (144)\\
&= \mathrm{Tr}\left(\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) && (145)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\rho_B\right) && (146)\\
&= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\right\rangle. && (147)
\end{aligned}$$

Given that we can write a density operator as:

$$\rho=\int P\left(\alpha\right)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \quad (148)$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right)\mathrm{d}^2\alpha=1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P\left(\alpha\right)$ is given by:

$$\begin{aligned}
\langle A\rangle =& \mathrm{Tr}\left(A\rho\right) && (149)\\
=& \int P\left(\alpha\right)\langle\alpha\left|A\right|\alpha\rangle\mathrm{d}^2\alpha. && (150)
\end{aligned}$$

We are typically interested in thermal state density operators, for which it can be shown that $P\left(\alpha\right) = \frac{1}{\pi N}\mathrm{e}^{-\frac{|\alpha|^2}{N}}$ where $N = \left(\mathrm{e}^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$.

Using the integral representation (150) we could obtain that the expected value for the displacement operator $D\left(h\right)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle\alpha|D\left(h\right)|\alpha\rangle \mathrm{d}^2\alpha \tag{151}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)|0\rangle \mathrm{d}^2\alpha, \tag{152}$$

$$D\left(h\right)D\left(\alpha\right) = D\left(h+\alpha\right)\mathrm{e}^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)}, \tag{153}$$

$$D\left(-\alpha\right)\left(D\left(h\right)D\left(\alpha\right)\right) = D\left(-\alpha\right)D\left(h+\alpha\right)\mathrm{e}^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{154}$$

$$= D\left(h\right)\mathrm{e}^{\frac{1}{2}\left(-\alpha\left(h+\alpha\right)^* + \alpha^*\left(h+\alpha\right)\right)}\mathrm{e}^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{155}$$

$$= D\left(\alpha\right)\mathrm{e}^{\frac{1}{2}\left(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2\right)}\mathrm{e}^{\frac{1}{2}\left(h\alpha^* - h^*\alpha\right)} \tag{156}$$

$$= D\left(\alpha\right)\mathrm{e}^{\left(h\alpha^* - h^*\alpha\right)}, \tag{157}$$

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D\left(h\right)\mathrm{e}^{\left(h\alpha^* - h^*\alpha\right)}|0\rangle \mathrm{d}^2\alpha \tag{158}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{\left(h\alpha^* - h^*\alpha\right)}\langle 0|D\left(h\right)|0\rangle \mathrm{d}^2\alpha \tag{159}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{\left(h\alpha^* - h^*\alpha\right)}\langle 0|h\rangle \mathrm{d}^2\alpha, \tag{160}$$

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{161}$$

$$\langle D\left(h\right)\rangle = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{\left(h\alpha^* - h^*\alpha\right)}\langle 0|\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha \tag{162}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{\left(h\alpha^* - h^*\alpha\right)}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{163}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha}\mathrm{d}^2\alpha, \tag{164}$$

$$\alpha = x + \mathrm{i}y, \tag{165}$$

$$\langle D\left(h\right)\rangle = \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N} + h\left(x - \mathrm{i}y\right) - h^*\left(x + \mathrm{i}y\right)}\mathrm{d}x\mathrm{d}y \tag{166}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{N} + hx - h^*x}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{N} + hx - h^*x}\mathrm{d}y, \tag{167}$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N}\left(x^2 - Nhx + Nh^*x\right) \tag{168}$$

$$= -\frac{1}{N}\left(x + \frac{\left(Nh^* - Nh\right)}{2}\right)^2 + \frac{N\left(h^* - h\right)^2}{4}, \tag{169}$$

$$-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^*y = -\frac{1}{N}\left(y^2 + \mathrm{i}Nhy + \mathrm{i}Nh^*y\right) \tag{170}$$

$$= -\frac{1}{N}\left(y^2 + \frac{\mathrm{i}N\left(h + h^*\right)}{2}\right) - \frac{N\left(h + h^*\right)^2}{4}, \tag{171}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\left(x-\mu\right)^2}{2\sigma^2}}\mathrm{d}x, \tag{172}$$

$$\langle D(h)\rangle = \frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}\mathrm{d}x\mathrm{d}y, \tag{173}$$

$$\begin{aligned}
\langle D\left(h\right)\rangle &= \frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}y && (174)\\
&= \frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 && (175)\\
&= \mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}} && (176)\\
&= \mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}} && (177)\\
&= \mathrm{e}^{-|h|^2\left(N+\frac{1}{2}\right)} && (178)\\
&= \mathrm{e}^{-|h|^2\left(\frac{1}{\mathrm{e}^{\beta\omega}-1}+\frac{1}{2}\right)} && (179)\\
&= \mathrm{e}^{-\frac{|h|^2}{2}\left(\frac{\mathrm{e}^{\beta\omega}+1}{\mathrm{e}^{\beta\omega}-1}\right)} && (180)\\
&= \mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}. && (181)
\end{aligned}$$

In the last line we used $\frac{\mathrm{e}^{\beta\omega}+1}{\mathrm{e}^{\beta\omega}-1}=\coth\left(\frac{\beta\omega}{2}\right)$. So the value of (146) using (181) is given by:

$$B=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{182}$$

We will now force $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$. We will also introduce the bath renormalizing driving in $\overline{H_S}\left(t\right)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+\left(t\right)\sigma^+$ and $B_i^-\left(t\right)\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}\left(t\right)$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$.

A final form of the terms of the Hamiltonian $\overline{H}\left(t\right)$ is:

$$\begin{aligned}
\overline{H(t)} &= \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+\sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^*b_{\mathbf{k}}+\frac{\left|v_{j\mathbf{k}}(t)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}+g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) && (183)\\
&= \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_{jj'}(t)+\sum_j|j\rangle\langle j|B_{jz}(t)+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+(t)\,B_{j'}^-(t)-B_{jj'}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (184)\\
&\equiv \overline{H_{\bar{S}}\left(t\right)}+\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{B}}}. && (185)
\end{aligned}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\begin{aligned}
\langle B_i^+\left(t\right)B_j^-\left(t\right)\rangle &= B_{ij}\left(t\right) && (186)\\
&= \left\langle\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right\rangle && (187)\\
&= \left\langle\prod_{\mathbf{k}}\left(D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle && (188)
\end{aligned}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \right\rangle \tag{189}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \tag{190}$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{-\frac{1}{2}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \tag{191}$$

$$= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)}. \tag{192}$$

From the definition $B_{01}(t) = \langle B_0^{+}(t) B_1^{-}(t) \rangle$ using the displacement operator we have:

$$\begin{aligned}
\langle B_0^{+}(t) B_1^{-}(t) \rangle &= B_{01}(t) && (193)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle && (194)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle && (195)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \right\rangle && (196)\\
&= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) && (197)\\
&= \prod_{\mathbf{k}} \left(\mathrm{e}^{-\frac{1}{2}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) && (198)\\
&= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)}. && (199)
\end{aligned}$$

We can check:

$$\begin{aligned}
\langle B_0^{+}(t) B_1^{-}(t) \rangle &= B_{01}(t) && (200)\\
&= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} && (201)\\
&= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)^{*}} && (202)\\
&= \langle B_1^{+}(t) B_0^{-}(t) \rangle^{*} && (203)\\
&= B_{10}^{*}(t). && (204)
\end{aligned}$$

The parts of the splitted Hamiltonian are:

$$\begin{aligned}
\overline{H_{\bar{S}}(t)} &\equiv (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + V_{10}(t) B_{10}\sigma^{+} + V_{01}(t) B_{01}\sigma^{-}, && (205)\\
\overline{H_{\bar{I}}} &\equiv V_{10}(t) \left(B_1^{+}(t) B_0^{-}(t) - B_{10}(t) \right) \sigma^{+} + V_{01}(t) \left(B_0^{+}(t) B_1^{-}(t) - B_{01}(t) \right) \sigma^{-} + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t), && (206)\\
\overline{H_{\bar{B}}} &\equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} && (207)\\
&= H_B. && (208)
\end{aligned}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.
For the Hamiltonian (206) we can verify the condition $\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$
\begin{aligned}
\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} =& \left\langle \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^{+}(t) B_{j'}^{-}(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} && (209) \\
=& \left\langle \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^{+}(t) B_{j'}^{-}(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} && (210) \\
=& \sum_{n\mathbf{k}} \left(\left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| \left(\left\langle V_{jj'}(t) B_j^{+}(t) B_{j'}^{-}(t) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle V_{jj'}(t) B_{jj'}(t) \right\rangle_{\overline{H_{\bar{B}}}} \right) && (211) \\
=& \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(\left\langle B_j^{+}(t) B_{j'}^{-}(t) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_{jj'}(t) \right\rangle_{\overline{H_{\bar{B}}}} \right) && (212) \\
=& \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t) \right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(B_{jj'}(t) - B_{jj'}(t) \right) && (213) \\
=& 0. && (214)
\end{aligned}
$$

We used (143) and (199) to evaluate the expression. Let's consider the following Hermitian combinations:

$$
\begin{aligned}
B_x(t) &= B_x^{\dagger}(t) && (215) \\
&= \frac{B_1^{+}(t) B_0^{-}(t) + B_0^{+}(t) B_1^{-}(t) - B_{10}(t) - B_{01}(t)}{2}, && (216) \\
B_y(t) &= B_y^{\dagger}(t) && (217) \\
&= \frac{B_0^{+}(t) B_1^{-}(t) - B_1^{+}(t) B_0^{-}(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, && (218) \\
B_{iz}(t) &= B_{iz}^{\dagger}(t) && (219) \\
&= \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right). && (220)
\end{aligned}
$$

Writing the equations (205) and (206) using the previous combinations we obtain that:

$$
\begin{aligned}
\overline{H_{\bar{S}}}(t) =& \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t) \right) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \sigma^{+} + V_{01}(t) B_{01}(t) \sigma^{-} && (221) \\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t) \right) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) B_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} && (222) \\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t) \right) |j\rangle\langle j| + V_{10}(t) \left(B_{10}^{\Re}(t) + \mathrm{i}B_{10}^{\Im}(t) \right) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) \left(B_{10}^{\Re}(t) - \mathrm{i}B_{10}^{\Im}(t) \right) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} && (223) \\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t) \right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \right) && (224) \\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t) \right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) && (225) \\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t) \right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\mathrm{i}\sigma_x V_{10}^{\Im}(t) + \mathrm{i}\sigma_y V_{10}^{\Re}(t) \right) && (226) \\
=& \left(\varepsilon_0(t) + R_0(t) \right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t) \right) |1\rangle\langle 1| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\mathrm{i}\sigma_x V_{10}^{\Im}(t) + \mathrm{i}\sigma_y V_{10}^{\Re}(t) \right) && (227) \\
=& \left(\varepsilon_0(t) + R_0(t) \right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t) \right) |1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) - \left(\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) && (228) \\
=& \left(\varepsilon_0(t) + R_0(t) \right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t) \right) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) && (229)
\end{aligned}
$$

$$= \left(\varepsilon_0(t) + R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \tag{230}$$

$$\overline{H_{\bar{I}}}(t) = V_{10}(t)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t)\right) + V_{01}(t)\left(\sigma^- B_0^+(t)\,B_1^-(t) - \sigma^- B_{01}(t)\right) + |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) \tag{231}$$

$$= |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) + \left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t)\right) + \left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^- B_0^+(t)\,B_1^-(t) - \sigma^- B_{01}(t)\right) \tag{232}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t) + \sigma^- B_0^+(t)B_1^-(t) - \sigma^- B_{01}(t)\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma^+ B_1^+(t)B_0^-(t) - \sigma^+ B_{10}(t) - \sigma^- B_0^+(t)B_1^-(t) + \sigma^- B_{01}(t)\right) \tag{233}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+(t)\,B_1^-(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) \tag{234}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+(t)\,B_1^-(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) \tag{235}$$

$$+ \mathrm{i}V_{10}^{\Im}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+(t)\,B_1^-(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) \tag{236}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2}\right) \tag{237}$$

$$+ \mathrm{i}V_{10}^{\Im}(t)\left(\sigma_x\frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{238}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x\frac{B_1^+(t)B_0^-(t) - B_0^+(t)B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} - \sigma_y\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{239}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+(t)B_0^-(t) - B_0^+(t)B_1^-(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{240}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+ B_0^-(t) - B_0^+ B_1^-(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{241}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\left(-B_y(t)\right) - \sigma_y B_x(t)\right) \tag{242}$$

$$= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right). \tag{243}$$

II. FREE-ENERGY MINIMIZATION

The true free energy $E_{\mathrm{Free}}(t)$ is bounded by the Bogoliubov inequality:

$$E_{\mathrm{Free}}(t) \leq E_{\mathrm{Free,B}}(t) \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right)\right) + \left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right). \tag{244}$$

We will optimize the set of variational parameters $\{v_{\mathbf{k}}(t)\}$ in order to minimize $E_{\mathrm{Free,B}}(t)$ (i.e. to make it as close to the true free energy $E_{\mathrm{Free}}(t)$ as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{\mathbf{k}}(t)\}$:

$$\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{245}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}(t), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = \mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}. \tag{246}$$

Then using the fact that $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right). \tag{247}$$

So Eq. (245) becomes:

$$\frac{\partial E_{\text{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{248}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{249}$$

$$= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{250}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{251}$$

$$= 0 \quad \text{(by Eq. (245))}. \tag{252}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial \ln\left(\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} \tag{253}$$

$$= 0. \tag{254}$$

This means we need to impose:

$$\frac{\partial \text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{255}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left((\varepsilon_0(t)+R_0(t))|0\rangle\langle 0| + (\varepsilon_1(t)+R_1(t))|1\rangle\langle 1| + V_{10}(t)B_{10}(t)\sigma^+ + V_{01}(t)B_{01}(t)\sigma^-\right). \tag{256}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \text{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{257}$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{258}$$

$$\eta \equiv \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{259}$$

The solutions of the equation (257) are:

$$\lambda = \beta\frac{-\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{260}$$

$$= \beta\frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{261}$$

$$= -\beta\frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{262}$$

The value of $\text{Tr}\left(\text{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right) = \mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{\frac{\eta(t)\beta}{2}} + \mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{-\frac{\eta(t)\beta}{2}} \tag{263}$$

$$= 2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right). \tag{264}$$

Given that $v_{i\mathbf{k}}(t)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right) = A(\varepsilon(t), \eta(t))$ to calculate $\frac{\partial\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial\left(2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{265}$$

$$= 2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\sinh\left(\frac{\eta(t)\beta}{2}\right) \tag{266}$$

$$= -\beta\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\left(\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right)\right). \tag{267}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right) = 0. \tag{268}$$

The derivates included in the expression given are related to:

$$\langle B_0^+(t)\,B_1^-(t)\rangle = \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{269}$$

$$= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^*\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{270}$$

$$= \langle B_1^+(t)\,B_0^-(t)\rangle^*, \tag{271}$$

$$R_i(t) = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{272}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{273}$$

$$\langle B_0^+(t)\,B_1^-(t)\rangle = \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{274}$$

$$= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{275}$$

$$v_{0\mathbf{k}}^*(t)\,v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\,v_{1\mathbf{k}}^*(t) = \left(v_{0\mathbf{k}}^{\Re}(t) - \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right) - \left(v_{0\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t) - \mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right) \tag{276}$$

$$= \left(v_{0\mathbf{k}}^{\Re}(t)\,v_{1\mathbf{k}}^{\Re}(t) + \mathrm{i}v_{0\mathbf{k}}^{\Re}(t)\,v_{1\mathbf{k}}^{\Im}(t) - \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\,v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t)\,v_{1\mathbf{k}}^{\Im}(t)\right) \tag{277}$$

$$- \left(v_{0\mathbf{k}}^{\Re}(t)\,v_{1\mathbf{k}}^{\Re}(t) - \mathrm{i}v_{0\mathbf{k}}^{\Re}(t)\,v_{1\mathbf{k}}^{\Im}(t) + \mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\,v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t)\,v_{1\mathbf{k}}^{\Im}(t)\right) \tag{278}$$

$$= 2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)\,v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t)\,v_{1\mathbf{k}}^{\Re}(t)\right), \tag{279}$$

$$\left|v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right|^2=\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)^* \tag{280}$$

$$=\left|v_{1\mathbf{k}}(t)\right|^2+\left|v_{0\mathbf{k}}(t)\right|^2-\left(v_{1\mathbf{k}}(t)\,v_{0\mathbf{k}}^*(t)+v_{1\mathbf{k}}^*(t)\,v_{0\mathbf{k}}(t)\right) \tag{281}$$

$$=\left(v_{1\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Im}(t)\right)^2-2\left(v_{1\mathbf{k}}^{\Re}(t)\,v_{0\mathbf{k}}^{\Re}(t)+v_{1\mathbf{k}}^{\Im}(t)\,v_{0\mathbf{k}}^{\Im}(t)\right) \tag{282}$$

$$=\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2. \tag{283}$$

Rewriting in terms of real and imaginary parts.

$$R_i(t)=\sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{284}$$

$$=\sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\,\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\,\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{285}$$

$$\left\langle B_0^+(t)\,B_1^-(t)\right\rangle=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{286}$$

$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{2\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{287}$$

$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{288}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}=\frac{\partial\left(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{289}$$

$$=\frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\,\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\,\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{290}$$

$$=\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{291}$$

$$\frac{\partial\left|B_{10}(t)\right|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)}=\frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{292}$$

$$=-\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{293}$$

$$=-\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\left|B_{10}(t)\right|^2, \tag{294}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}=\frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{295}$$

$$=\frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{296}$$

$$=\frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \tag{297}$$

$$= \frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon(t) - \varepsilon_i(t) - R_i(t))\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)}|B_{10}(t)\,V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{298}$$

$$= \frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon(t) - \varepsilon_i(t) - R_i(t))\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Re}_{i\mathbf{k}}(t)-v^{\Re}_{i'\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{299}$$

$$= \frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + \frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t))\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{300}$$

$$\left. +4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{301}$$

From the equation (268) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}}{\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}} \tag{302}$$

$$= \frac{\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{2g^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}}{\frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t)) - \frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + 2\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}} + 2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{303}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v^{\Re}_{i\mathbf{k}}(t) - g_{i\mathbf{k}} - g^{*}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4(\varepsilon(t)-\varepsilon_i(t)-R_i(t)) - \frac{4|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \left(g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}\right)(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))) + 4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{304}$$

$$= \frac{\left(2v^{\Re}_{i\mathbf{k}}(t) - 2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4(\varepsilon(t)-\varepsilon_i(t)-R_i(t)) - \frac{4|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g^{\Re}_{i\mathbf{k}}(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))) + 4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$= \frac{\left(2v^{\Re}_{i\mathbf{k}}(t) - 2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4(\varepsilon(t)-\varepsilon_i(t)-R_i(t)) - \frac{4|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)) + 4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$= \frac{\left(v^{\Re}_{i\mathbf{k}}(t) - g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t) - 2(\varepsilon(t)-\varepsilon_i(t)-R_i(t)) - \frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)) + 2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{307}$$

Separating (306) such that the terms with $v_{i\mathbf{k}}(t)$ are located at one side of the equation permit us to write:

$$\frac{\left(v^{\Re}_{i\mathbf{k}}(t) - g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g^{\Re}_{i\mathbf{k}}(2(\varepsilon_i(t)+R_i(t)) - \varepsilon(t)) + 2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{308}$$

$$v^{\Re}_{i\mathbf{k}}(t) - g^{\Re}_{i\mathbf{k}} = v^{\Re}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) \tag{309}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{310}$$

$$v^{\Re}_{i\mathbf{k}}(t) = \frac{g^{\Re}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{311}$$

$$v^{\Re}_{i\mathbf{k}}(t) = \frac{g^{\Re}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{312}$$

The imaginary part can be found in the following way:

$$
\begin{aligned}
\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)} &= \frac{\partial\left(\varepsilon_1(t)+R_1(t)+\varepsilon_0(t)+R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} && (313)\\
&= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} - \mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} && (314)\\
&= 2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, && (315)\\
\frac{\partial |B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Im}(t)} &= \frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} && (316)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (317)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}|B_{10}(t)|^2, && (318)\\
\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (319)\\
&= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} - 4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} && (320)\\
&= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|B_{10}(t)|^2|V_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\eta(t)} && (321)\\
&= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|B_{10}(t)V_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\eta(t)} && (322)\\
&= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} && (323)\\
\delta_{1i}-\delta_{0i} &= \frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}} && (324)\\
\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-v_{i'\mathbf{k}}^{\Im}(t)\right)|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^2}\right)}{\eta(t)} && (325)\\
&= \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\frac{4(\varepsilon_i(t)+R_i(t))-2\varepsilon(t)-\frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}}{\eta(t)}+\frac{1}{\eta(t)}\left(2\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Im}(t)|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^2}\right). && (326)
\end{aligned}
$$

From the equation (268) and replacing the derivatives obtained we have:

$$
\begin{aligned}
\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}} &= \tanh\left(\frac{\beta\eta(t)}{2}\right) && (327)\\
&= \frac{2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{v_{i\mathbf{k}}^{\Im}(t)\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\right)+\frac{2}{\eta(t)}\left(\frac{g_{i\mathbf{k}}^{*\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g_{i\mathbf{k}}^{*\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^2}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. && (328)
\end{aligned}
$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v_{i\mathbf{k}}^{\Im}(t)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$=\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{330}$$

$$=\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{331}$$

$$=\frac{\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{332}$$

Separating (332) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v_{i\mathbf{k}}^{\Im}(t)-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v_{i\mathbf{k}}^{\Im}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{333}$$

$$v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}=v_{i\mathbf{k}}^{\Im}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right) \tag{334}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{335}$$

$$v_{i\mathbf{k}}^{\Im}(t)=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{336}$$

$$v_{i\mathbf{k}}^{\Im}(t)=\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{337}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t)=v_{i\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{i\mathbf{k}}^{\Im}(t) \tag{338}$$

$$=\frac{g_{i\mathbf{k}}^{\Re}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{339}$$

$$+\mathrm{i}\frac{g_{i\mathbf{k}}^{\Im}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{340}$$

$$=\frac{g_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{341}$$

Let's obtain the explicit form of $v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)$ and $v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)$, at first we have:

$$a_i(\omega_{\mathbf{k}},t)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{342}$$

$$b_i(\omega_{\mathbf{k}},t)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}.\tag{343}$$

So the equation (338) written in explicit form is:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t),\tag{344}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t).\tag{345}$$

This system of equations has the following solutions:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)\tag{346}$$

$$=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+\left(g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)\right)b_0(\omega_{\mathbf{k}},t)\tag{347}$$

$$=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)\tag{348}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)(1-b_1(\omega_{\mathbf{k}},t)b_0(\omega_{\mathbf{k}},t))=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)\tag{349}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=\frac{g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)},\tag{350}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+\frac{g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}b_1(\omega_{\mathbf{k}},t)\tag{351}$$

$$=\frac{g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}.\tag{352}$$

For a shorter representation let's define:

$$r_i(\omega_{\mathbf{k}},t)=\frac{a_i(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)},\tag{353}$$

$$s_i(\omega_{\mathbf{k}},t)=\frac{a_{(i+1)\mathrm{mod}2}(\omega_{\mathbf{k}},t)\,b_{i\mathrm{mod}2}(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}.\tag{354}$$

So the variational parameters are given by:

$$\begin{pmatrix}v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\\ v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\end{pmatrix}\equiv\begin{pmatrix}r_0(\omega_{\mathbf{k}},t) & s_0(\omega_{\mathbf{k}},t)\\ r_1(\omega_{\mathbf{k}},t) & s_1(\omega_{\mathbf{k}},t)\end{pmatrix}\begin{pmatrix}g_0(\omega_{\mathbf{k}})\\ g_1(\omega_{\mathbf{k}})\end{pmatrix}.\tag{355}$$

Given that $v_{i\mathbf{k}}(\omega_{\mathbf{k}},t)\equiv g_i(\omega_{\mathbf{k}})\,F_i(\omega_{\mathbf{k}},t)$ then we can write:

$$F_0(\omega_{\mathbf{k}},t)=r_0(\omega_{\mathbf{k}},t)+\frac{g_1(\omega_{\mathbf{k}})}{g_0(\omega_{\mathbf{k}})}s_0(\omega_{\mathbf{k}},t)\tag{356}$$

$$F_1(\omega_{\mathbf{k}},t)=\frac{g_0(\omega_{\mathbf{k}})}{g_1(\omega_{\mathbf{k}})}r_1(\omega_{\mathbf{k}},t)+s_1(\omega_{\mathbf{k}},t)\tag{357}$$

III. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$, where $\rho_B^{\text{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$
\begin{aligned}
\overline{\rho_T(0)} &\equiv e^{V(0)}\rho_T(0)\,e^{-V(0)} && (358)\\
&= \left(|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)\right)(\rho_S(0)\otimes\rho_B)\left(|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)\right), && (359)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|: \quad & |0\rangle\langle 0|0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (360)\\
&= |0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (361)\\
&= |0\rangle\langle 0| \otimes B_0^+(0)\,\rho_B B_0^-(0)\,, && (362)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|: \quad & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (363)\\
&= |1\rangle\langle 1|B_1^+(0)\,\rho_B B_1^-(0) && (364)\\
&= |1\rangle\langle 1| \otimes B_1^+(0)\,\rho_B B_1^-(0)\,, && (365)\\
\text{for } \rho_S(0) = |0\rangle\langle 1|: \quad & |0\rangle\langle 0|B_0^+(0)\,|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (366)\\
&= |0\rangle\langle 1|B_0^+(0)\,\rho_B|1\rangle\langle 1|B_1^-(0) && (367)\\
&= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0)\,\rho_B B_1^-(0) && (368)\\
&= |0\rangle\langle 1| \otimes B_0^+(0)\,\rho_B B_1^-(0)\,, && (369)\\
\text{for } \rho_S(0) = |1\rangle\langle 0|: \quad & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (370)\\
&= |1\rangle\langle 0| \otimes B_1^+(0)\,\rho_B B_0^-(0)\,. && (371)
\end{aligned}
$$

We transform any operator O into the interaction picture in the following way:

$$
\widetilde{O}(t) \equiv U^\dagger(t)\,O(t)\,U(t)\,, \tag{372}
$$

$$
U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\overline{H_{\bar{S}}}(t')\right). \tag{373}
$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$
\widetilde{\overline{\rho_S}}(t) = U^\dagger(t)\,\overline{\rho_S}(t)\,U(t)\,, \text{ where} \tag{374}
$$

$$
\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}_T(t)\right). \tag{375}
$$

In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$
\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \tag{376}
$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (243):

$$
\begin{aligned}
\overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) && (377)\\
&= B_{0z}(t)\,|0\rangle\langle 0| + B_{1z}(t)\,|1\rangle\langle 1| + V_{10}^{\Re}(t)\,\sigma_x B_x(t) + V_{10}^{\Re}(t)\,\sigma_y B_y(t) + V_{10}^{\Im}(t)\,\sigma_x B_y(t) - V_{10}^{\Im}(t)\,\sigma_y B_x(t) && (378)\\
&= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right). && (379)
\end{aligned}
$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{380}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) \text{-i} \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]\mathrm{d}t'. \tag{381}$$

Replacing the equation (381) in the equation (380) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]]\mathrm{d}t'. \tag{382}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdot\cdot]. \tag{383}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]] \cdot\cdot]. \tag{384}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\overline{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\overline{\rho_S}(0) \tag{385}$$

$$= W(t)\overline{\rho_S}(0). \tag{386}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]] \cdot\cdot]. \tag{387}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{388}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} W(t)\overline{\rho_S}(0) \tag{389}$$

$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} \widetilde{\overline{\rho_S}}(t). \tag{390}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (388) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\widetilde{\overline{\rho_S}}(t) \tag{391}$$

$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B \left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]. \tag{392}$$

Replacing $t_1 \to t - \tau$

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right), \overline{\rho_S}\left(t\right)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B \left[\overline{H_{\bar{I}}}\left(t\right), \left[\widetilde{\overline{H_{\bar{I}}}}\left(-\tau\right), \overline{\rho_S}\left(t\right)\rho_B\right]\right]. \tag{393}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}\left(t\right)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right) = U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right). \tag{394}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\begin{aligned}
\widetilde{\overline{H_{\bar{I}}}}\left(t\right) &= \sum_i C_i\left(t\right)\left(\widetilde{A_i}\left(t\right)\otimes\widetilde{B_i}\left(t\right)\right), && (395)\\
\widetilde{A_i}\left(t\right) &= U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) && (396)\\
&= U^{\dagger}\left(t\right)A_iU\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} && (397)\\
&= U^{\dagger}\left(t\right)A_iU\left(t\right)\mathbb{I} && (398)\\
&= U^{\dagger}\left(t\right)A_iU\left(t\right), && (399)\\
\widetilde{B_i}\left(t\right) &= U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}U\left(t\right) && (400)\\
&= U^{\dagger}\left(t\right)U\left(t\right)\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} && (401)\\
&= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t} && (402)\\
&= \mathrm{e}^{\mathrm{i}H_B t}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_B t}. && (403)
\end{aligned}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}\left(t\right), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (395) to replace it in the equation (392)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}\left(t\right), \left[\widetilde{\overline{H_{\bar{I}}}}\left(t'\right), \widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' && (404)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j\left(t\right)\left(\widetilde{A_j}\left(t\right)\otimes\widetilde{B_j}\left(t\right)\right), \left[\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right), \widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' && (405)\\
&= -\int_0^t \mathrm{Tr}_B\Big[\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big), \sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\overline{\rho_S}}(t)\rho_B - \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\Big]\mathrm{d}t' && (406)\\
&= -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\overline{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big) && (407)\\
&\quad - \sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}t'. && (408)
\end{aligned}$$

In order to calculate the correlation functions we define:

$$\mathcal{B}_{ij}\left(t, t'\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(t'\right)\rho_B\right). \tag{409}$$

An useful property is

$$\begin{aligned}
\mathcal{B}^{*}_{ji}\left(t, t'\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right)^{\dagger} && (410)\\
&= \mathrm{Tr}_B\left(\rho_B^{\dagger}\widetilde{B_i}^{\dagger}\left(t'\right)\widetilde{B_j}^{\dagger}\left(t\right)\right) && (411)\\
&= \mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) && (412)\\
&= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) && (413)\\
&= \mathcal{B}_{ij}\left(t', t\right). && (414)
\end{aligned}$$

The correlation functions relevant that appear in the equation (408) are:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) &= \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\right\rangle_B \qquad &(415)\\
&= \mathcal{B}_{ji}\left(t,t'\right) &(416)\\
&= \mathcal{B}^*_{ij}\left(t',t\right) &(417)\\
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B\widetilde{B_i}\left(t'\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) &(418)\\
&= \mathcal{B}_{ij}\left(t',t\right) &(419)\\
\mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\rho_B\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) &(420)\\
&= \mathcal{B}^*_{ij}\left(t',t\right) &(421)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) &(422)\\
&= \mathcal{B}_{ij}\left(t',t\right) &(423)
\end{aligned}$$

The cyclic property of the trace was use widely in the development of equations (415) and (423). Replacing in (408)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} =& -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big) &(424)\\
&-\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}t'. &(425)\\
=& -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\Big) &(426)\\
&+\sum_{ij} C_i(t')C_j(t)\Big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' &(427)\\
=& -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\Big) &(428)\\
&+\sum_{ij} C_i(t')C_j(t)\Big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' &(429)\\
=& -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\Big)\ \text{(by permuting i and j because i,j}\in\text{J)} &(430)\\
&+\sum_{ij} C_i(t')C_j(t)\Big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' &(431)\\
=& -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t') &(432)\\
&+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' &(433)\\
=& -\int_0^t \Big(\sum_{ij} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\mathcal{B}_{ji}(t,t')-\widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\mathcal{B}_{ij}(t',t) &(434)\\
&+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\mathcal{B}_{ij}(t',t)-\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\mathcal{B}_{ji}(t,t')\Big)\Big)\mathrm{d}t' &(435)\\
=& -\int_0^t\left(\sum_{ij} C_j\left(t\right)C_i\left(t'\right)\left(\mathcal{B}_{ji}\left(t,t'\right)\left[\widetilde{A_j}\left(t\right),\widetilde{A_i}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]+\mathcal{B}_{ij}\left(t',t\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t'\right),\widetilde{A_j}\left(t\right)\right]\right)\right)\mathrm{d}t' &(436)\\
=& -\int_0^t \Big(\sum_{ij} C_i(t)C_j(t')\Big(\mathcal{B}_{ij}(t,t')\big[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)\big]+\mathcal{B}_{ji}(t',t)\big[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t'),\widetilde{A_i}(t)\big]\Big)\Big)\mathrm{d}t'\ \text{(exchanging i and j)} &(437)\\
=& -\int_0^t\left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]+\mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}t' &(438)\\
=& -\int_0^t\left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right]-\mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right)\right]\right)\right)\mathrm{d}t' &(439)
\end{aligned}$$

We could identify the following commutators in the equation deduced:

$$\mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)-\mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)=\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_s}}\left(t\right)\right], \tag{440}$$

$$\mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)=\mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]. \tag{441}$$

Returning to the Schroedinger picture we have:

$$U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(t')\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t) = U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\,U(t)\,\widetilde{A_j}(t')\,U^\dagger(t)\,U(t)\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t)\,, \tag{442}$$

$$= \left(U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{A_j}(t')\,U^\dagger(t)\right)\left(U(t)\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t)\right), \tag{443}$$

$$= A_i(t)\,\widetilde{A_j}(t',t)\,\overline{\rho_S}(t)\,. \tag{444}$$

This procedure applying to the relevant commutators give us:

$$U(t)\left[\widetilde{A_i}(t)\,,\widetilde{A_j}(t')\,\widetilde{\widetilde{\rho_S}}(t)\right]U^\dagger(t) = \left(U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(t')\,\widetilde{\widetilde{\rho_S}}(t)\,U^\dagger(t) - U(t)\,\widetilde{A_j}(t')\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right) \tag{445}$$

$$= A_i(t)\,\widetilde{A_j}(t',t)\,\overline{\rho_S}(t) - \widetilde{A_j}(t',t)\,\overline{\rho_S}(t)\,A_i \tag{446}$$

$$= \left[A_i(t)\,,\widetilde{A_j}(t',t)\,\overline{\rho_S}(t)\right]. \tag{447}$$

Introducing this transformed commutators in the equation (439) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathcal{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t)\,,\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}s C_i(t)\,C_j(t')\left(\mathcal{B}_{ij}(t,t')\left[A_i(t)\,,\widetilde{A_j}(t',t)\,\overline{\rho_S}(t)\right] + \mathcal{B}^*_{ij}(t,t')\left[\overline{\rho_S}(t)\widetilde{A_j}(t',t),A_i\right]\right), \tag{448}$$

$$t' = t-\tau \text{ (Change of variables in the integration process)}, \tag{449}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)\,C_j(t')\left(\mathcal{B}_{ij}(t,t')\left[A_i(t),\widetilde{A_j}(t',t)\overline{\rho_S}(t)\right] + \mathcal{B}^*_{ij}(t,t')\left[\overline{\rho_S}(t)\widetilde{A_j}(t',t),A_i(t)\right]\right). \tag{450}$$

where $i,j \in \{1,2,3,4,5.6\}$ and $t' = t-\tau$.
Here $\widetilde{A_j}(t-\tau,t) = U(t)\,U^\dagger(t-\tau)\,A_j(t)\,U(t-\tau)\,U^\dagger(t)$ where $U(t)$ is given by (373). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. In order to write in a simplified way we define the following notation:

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(t')\,\rho_B\right) \tag{451}$$

$$= \mathrm{Tr}_B\left(\mathrm{e}^{iH_Bt}B_i(t)\,\mathrm{e}^{-iH_Bt}\mathrm{e}^{iH_Bt'}B_j(t')\,\mathrm{e}^{-iH_Bt'}\rho_B\right) \tag{452}$$

$$\mathrm{e}^A = \sum_{k=0}^{\infty}\frac{A^k}{k!} \tag{453}$$

$$\mathrm{e}^{-iH_Bt'}\mathrm{e}^{-\beta H_B} = \sum_{m=0}^{\infty}\frac{(-iH_Bt')^m}{m!}\sum_{n=0}^{\infty}\frac{(-\beta H_B)^n}{n!} \tag{454}$$

$$= \sum_{m,n}\frac{(-iH_Bt')^m}{m!}\frac{(-\beta H_B)^n}{n!} \tag{455}$$

$$= \sum_{m,n}\frac{(-it')^m}{m!}\frac{(-\beta)^n}{n!}H_B^mH_B^n \tag{456}$$

$$= \sum_{m,n}\frac{(-it')^m}{m!}\frac{(-\beta)^n}{n!}H_B^nH_B^m \text{ (because the powers of a matrix commute)} \tag{457}$$

$$= \sum_{m,n}\frac{(-\beta)^n}{n!}H_B^n\frac{(-it')^m}{m!}H_B^m \tag{458}$$

$$= \sum_{m,n}\frac{(-\beta H_B)^n}{n!}\frac{(-it'H_B)^m}{m!} \tag{459}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_B t')^m}{m!} && (460)\\
&= \mathrm{e}^{-\beta H_B} \mathrm{e}^{-iH_B t'} && (461)\\
0 &= \mathrm{e}^{-iH_B t'} \mathrm{e}^{-\beta H_B} - \mathrm{e}^{-\beta H_B} \mathrm{e}^{-iH_B t'} \text{ (then } \mathrm{e}^{-iH_B t'} \text{ and } \rho_B \text{ commute)} && (462)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left(\mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \rho_B \mathrm{e}^{-iH_B t'} \right) \text{ (by permuting } \mathrm{e}^{-iH_B t'} \text{ and } \rho_B \text{)} && (463)\\
&= \mathrm{Tr}_B \left(\left(\mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \right) \rho_B \mathrm{e}^{-iH_B t'} \right) \text{ (by associative property)} && (464)\\
&= \mathrm{Tr}_B \left(\mathrm{e}^{-iH_B t'} \left(\mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \right) \rho_B \right) \text{ (by cyclic property of the trace)} && (465)\\
&= \mathrm{Tr}_B \left(\left(\mathrm{e}^{-iH_B t'} \mathrm{e}^{iH_B t} \right) B_i(t) \left(\mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} \right) B_j(t') \rho_B \right) \text{ (by associative property)} && (466)\\
[iH_B t, -iH_B t'] &= iH_B t (-iH_B t') - (-iH_B t') iH_B t && (467)\\
&= tt' H_B^2 - tt' H_B^2 && (468)\\
&= 0 \text{ (so } iH_B t \text{ and } -iH_B t' \text{ commute)} && (469)\\
\mathrm{e}^{-iH_B t'} \mathrm{e}^{iH_B t} &= \mathrm{e}^{iH_B t - iH_B t'} \text{ (by the Zassenhaus formula because } iH_B t \text{ and } -iH_B t' \text{ commute)} && (470)\\
&= \mathrm{e}^{iH_B (t-t')} && (471)\\
&= \mathrm{e}^{iH_B \tau} && (472)\\
\mathrm{e}^{iH_B t'} \mathrm{e}^{-iH_B t} &= \mathrm{e}^{-iH_B t + iH_B t'} \text{ (by the Zassenhaus formula because } -iH_B t \text{ and } iH_B t' \text{ commute)} && (473)\\
&= \mathrm{e}^{iH_B (-t+t')} && (474)\\
&= \mathrm{e}^{-iH_B \tau} && (475)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left(\mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} B_j(t') \rho_B \right) && (476)\\
B_i(t,\tau) &\equiv \mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} && (477)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left(\mathrm{e}^{iH_B (t-t')} B_i(t) \mathrm{e}^{-iH_B (t-t')} B_j(t') \rho_B \right) && (478)\\
t' &= t - \tau && (479)\\
\mathcal{B}_{ij}(t,t') &= \mathrm{Tr}_B \left(\mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} B_j(t') \rho_B \right) && (480)\\
&= \mathrm{Tr}_B \left(B_i(t,\tau) B_j(t',0) \rho_B \right) && (481)
\end{aligned}
$$

For the following results $i, j \in \{3, 6\}$, calculating the correlation functions allow us to obtain:

$$
\begin{aligned}
\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B &= \mathrm{Tr}_B \left(B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \rho_B \right) && (482)\\
&= \int \mathrm{d}^2\alpha P(\alpha) \left\langle \alpha \left| B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \right| \alpha \right\rangle && (483)\\
&= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle \alpha \left| B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \right| \alpha \right\rangle \mathrm{d}^2\alpha, && (484)\\
q_{j\mathbf{k}}(t) &= g_{j\mathrm{mod}2\mathbf{k}} - v_{j\mathrm{mod}2\mathbf{k}}(t) && (485)\\
B_{j\mathrm{mod}2z}(t,\tau) &= \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), && (486)\\
B_{j\mathrm{mod}2z}(t',0) &= \sum_{\mathbf{k}'} \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^{\dagger} + q_{j\mathbf{k}'}^{*}(t') b_{\mathbf{k}'} \right), && (487)\\
\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B &= \mathrm{Tr}_B \left(B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \rho_B \right) && (488)\\
&= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left(q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^{\dagger} + q_{j\mathbf{k}'}^{*}(t') b_{\mathbf{k}'} \right) \rho_B \right) && (489)
\end{aligned}
$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{490}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{491}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_B\right), \tag{492}$$

$$\left\langle\widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{493}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_B\right) \tag{494}$$

$$0=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_B\right) \tag{495}$$

$$\left\langle\widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B=0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_B\right) \tag{496}$$

$$=\mathrm{Tr}_B\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\left(b_{\mathbf{k}}^{\dagger}\right)^2\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{2}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B \tag{497}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{498}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right)+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right) \tag{499}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{500}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{501}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{502}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{503}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{504}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{505}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{506}$$

$$=\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right. \tag{507}$$

$$\left.+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{508}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right. \tag{509}$$

$$\left.+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{510}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\sum_{\mathbf{k}}\left(\left\langle0\left|q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right)+q_{j\mathbf{k}}^{*}(t)\,q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right)\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{511}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\left\langle0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle+\left\langle0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{512}$$

$$+\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\left\langle0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|0\right\rangle+\left\langle0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{513}$$

$$1=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{514}$$

$$b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left|0\right\rangle=0, \tag{515}$$

$$b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\left|0\right\rangle=\left|0\right\rangle, \tag{516}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}} \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left(\left(q_{j\mathbf{k}}(t)\, q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\langle 0\left||\alpha_{\mathbf{k}}|^2\right| 0\rangle \right. \tag{517}$$

$$\left. + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|0\right\rangle\right)\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{518}$$

$$= \frac{1}{\pi N}\int\sum_{\mathbf{k}}\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(\left(q_{j\mathbf{k}}(t)q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)|\alpha_{\mathbf{k}}|^2 + q^*_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{519}$$

$$\int_0^{2\pi}\int_0^{+\infty} r^2\mathrm{e}^{-\frac{r^2}{N}} r\mathrm{d}r\mathrm{d}\theta = \int |\alpha_{\mathbf{k}}|^2\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\mathrm{d}^2\alpha_{\mathbf{k}} \tag{520}$$

$$= \pi N^2 \tag{521}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}}\left(\left(q_{j\mathbf{k}}(t)\, q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)N + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{522}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j'\mathrm{mod}2z}}(t')\right\rangle_B = \mathrm{Tr}_B\left(B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0)\,\rho_B\right) \tag{523}$$

$$= \int \mathrm{d}^2\alpha P(\alpha)\,\langle\alpha\left|B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0)\right|\alpha\rangle \tag{524}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle\alpha\left|B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0)\right|\alpha\rangle\,\mathrm{d}^2\alpha \tag{525}$$

$$= \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}'} + q^*_{j'\mathbf{k}'}(t')\, b_{\mathbf{k}'}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{526}$$

$$= \langle\alpha|\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}'} + q^*_{j'\mathbf{k}'}(t')b_{\mathbf{k}'}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{527}$$

$$+ \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}} + q^*_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{528}$$

$$= \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}}(t')\, b^\dagger_{\mathbf{k}} + q^*_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{529}$$

$$= \left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\, b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{530}$$

$$+ \left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{531}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{532}$$

$$+ \sum_{\mathbf{k}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{533}$$

$$\left\langle b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right\rangle_B = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\, D(-\alpha_{\mathbf{k}})\, b^\dagger_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{534}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{535}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + \alpha_{\mathbf{k}}b^\dagger_{\mathbf{k}} + b_{\mathbf{k}}\alpha^*_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{536}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{537}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{538}$$

$$= N+1, \tag{539}$$

$$\left\langle b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right\rangle_B = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{540}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}|\alpha_{\mathbf{k}}|^2\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{541}$$

$$= N, \tag{542}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j'\mathrm{mod}2z}}\left(t'\right)\right\rangle_{B} = \sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)q^{*}_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N + q^{*}_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N+1\right)\right) \tag{543}$$

$$= \sum_{\mathbf{k}}2N\left(q_{j\mathbf{k}}\left(t\right)q^{*}_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Re} + \sum_{\mathbf{k}}q^{*}_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{544}$$

$$D\left(h'\right)D\left(h\right) = \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}D\left(h'+h\right), \tag{545}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_{B} = \mathrm{Tr}_{B}\left(\mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}D\left(h'+h\right)\rho_{B}\right) \tag{546}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\mathrm{Tr}_{B}\left(D\left(h'+h\right)\rho_{B}\right) \tag{547}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\frac{1}{\pi N}\int \mathrm{d}^{2}\alpha P\left(\alpha\right)\left\langle\alpha\left|D\left(h'+h\right)\right|\alpha\right\rangle \tag{548}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\mathrm{e}^{-\frac{\left|h+h'\right|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{549}$$

$$h' = h\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{550}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_{B} = \mathrm{e}^{\frac{1}{2}\left(hh^{*}\mathrm{e}^{\mathrm{i}\omega\tau}-h^{*}h\mathrm{e}^{-\mathrm{i}\omega\tau}\right)}\mathrm{e}^{-\frac{\left|h+h\mathrm{e}^{\mathrm{i}\omega\tau}\right|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{551}$$

$$\frac{1}{2}\left|h\right|^{2}\left(\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\right) = \frac{1}{2}\left(hh^{*}\mathrm{e}^{\mathrm{i}\omega\tau}-h^{*}h\mathrm{e}^{-\mathrm{i}\omega\tau}\right) \tag{552}$$

$$= \frac{1}{2}\left|h\right|^{2}\left(\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)-\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{553}$$

$$= \frac{1}{2}\left|h\right|^{2}\left(2\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{554}$$

$$= \mathrm{i}\left|h\right|^{2}\sin\left(\omega\tau\right), \tag{555}$$

$$-\frac{\left|h+h\mathrm{e}^{\mathrm{i}\omega\tau}\right|^{2}}{2} = -\left|h\right|^{2}\frac{\left|1+\mathrm{e}^{\mathrm{i}\omega\tau}\right|^{2}}{2} \tag{556}$$

$$= -\left|h\right|^{2}\frac{\left(1+2\cos\left(\omega\tau\right)+\cos^{2}\left(\omega\tau\right)\right)+\sin^{2}\left(\omega\tau\right)}{2} \tag{557}$$

$$= -\left|h\right|^{2}\frac{2+2\cos\left(\omega\tau\right)}{2} \tag{558}$$

$$= -\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right), \tag{559}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_{B} = \mathrm{e}^{\mathrm{i}\left|h\right|^{2}\sin\left(\omega\tau\right)}\mathrm{e}^{-\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)} \tag{560}$$

$$= \mathrm{e}^{\mathrm{i}\left|\mathrm{h}\right|^{2}\sin\left(\omega\tau\right)-\left|h\right|^{2}\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)} \tag{561}$$

$$= \mathrm{e}^{-\left|h\right|^{2}\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)}\mathrm{e}^{-\left|h\right|^{2}\coth\left(\frac{\beta\omega}{2}\right)} \tag{562}$$

$$= \left\langle D\left(h\right)\right\rangle_{B}\mathrm{e}^{-\phi\left(\tau\right)}, \tag{563}$$

$$\mathrm{e}^{-\phi\left(\tau\right)} = \mathrm{e}^{-\left|h\right|^{2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)}, \tag{564}$$

$$\phi\left(\tau\right) = \left|h\right|^{2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right), \tag{565}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_{B} = \mathrm{e}^{\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)}\mathrm{e}^{-\frac{\left|h+h'\right|^{2}}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{566}$$

$$h' = v\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{567}$$

$$m_{\mathbf{k}}\left(t\right) = \frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}, \tag{568}$$

$$\Gamma_{\mathbf{k}}\left(t\right) = \frac{v^{*}_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v^{*}_{0\mathbf{k}}\left(t\right)}{\omega^{2}_{\mathbf{k}}} \tag{569}$$

$$\left\langle \widetilde{B^{+}_{1}B^{-}_{0}}\left(t\right)\widetilde{B^{+}_{1}B^{-}_{0}}\left(t'\right)\right\rangle_{B} = \left\langle B^{+}_{1}B^{-}_{0}\left(t,\tau\right)B^{+}_{1}B^{-}_{0}\left(t',0\right)\right\rangle_{B} \tag{570}$$

$$= \left\langle B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\right\rangle_{B} \tag{571}$$

$$= \mathrm{Tr}_{B}\left(B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\rho_{B}\right) \tag{572}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t')\right)\right)\rho_B\right) \tag{573}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(m_{\mathbf{k}}(t')\right)\right)\rho_B\right) \tag{574}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')-\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}\right)^*m_{\mathbf{k}}(t')\right)-\frac{|m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}+m_{\mathbf{k}}(t')|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{575}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')\right)^{\Im}-\frac{|\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\mathrm{e}^{\mathrm{i}\omega\tau}+\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{576}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')\right)^{\Im}-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{577}$$

$$\left\langle\widetilde{B_0^+B_1^-}(t)\widetilde{B_0^+B_1^-}(t')\right\rangle_B=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\left(\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')\right)^{\Im}-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{578}$$

$$\left\langle D(h)\,b\right\rangle_B=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle\alpha\left|D(h)\,b\right|\alpha\right\rangle \tag{579}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle\alpha\left|D(-\alpha)\,D(h)\,bD(\alpha)\right|\alpha\right\rangle \tag{580}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D(-\alpha)\,D(h)\,bD(\alpha)\right|0\right\rangle \tag{581}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha)\right|0\right\rangle \tag{582}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D(-\alpha)\,D(h)\,D(\alpha)\,(b+\alpha)\right|0\right\rangle \tag{583}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,(b+\alpha)\right|0\right\rangle \tag{584}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b\right|0\right\rangle+\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,\alpha\right|0\right\rangle \tag{585}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,\alpha\right|0\right\rangle \tag{586}$$

$$=\frac{1}{\pi N}\int\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{587}$$

$$=hN\left\langle D(h)\right\rangle_B, \tag{588}$$

$$\left\langle D(h)\,b^\dagger\right\rangle_B=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle\alpha\left|D(h)\,b^\dagger\right|\alpha\right\rangle \tag{589}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\right|0\right\rangle \tag{590}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\right|0\right\rangle \tag{591}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,b^\dagger D(\alpha)\right|0\right\rangle \tag{592}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D(-\alpha)\,D(h)\,D(\alpha)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{593}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{594}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b^\dagger\right|0\right\rangle+\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,\alpha^*\right|0\right\rangle \tag{595}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\right|1\right\rangle+\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0\left|D(h)\right|0\right\rangle \tag{596}$$

$$
\begin{aligned}
&=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h|1\right\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0\left|D\left(h\right)\right|0\right\rangle, && (597)\\
\left\langle -h\right| &=\mathrm{e}^{-\frac{|-h^*|^2}{2}}\sum_n\frac{\left(-h^*\right)^n}{\sqrt{n!}}\left\langle n\right| && (598)\\
\left\langle -h|1\right\rangle &=\mathrm{e}^{-\frac{|-h^*|^2}{2}}\sum_n\frac{\left(-h^*\right)^n}{\sqrt{n!}}\left\langle n|1\right\rangle && (599)\\
\left\langle -h|1\right\rangle &=\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right), && (600)\\
\left\langle D\left(h\right)b^\dagger\right\rangle_B &=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right)+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|-h^*|^2}{2}} && (601)\\
\left\langle D\left(h\right)b^\dagger\right\rangle_B &=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right)+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|-h^*|^2}{2}} && (602)\\
&=-h^*\left\langle D\left(h\right)\right\rangle_B\left(N+1\right), && (603)\\
\left\langle bD\left(h\right)\right\rangle_B &=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle \alpha\left|bD\left(h\right)\right|\alpha\right\rangle && (604)\\
&=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}} && (605)\\
&=h\left\langle D\left(h\right)\right\rangle_B\left(N+1\right), && (606)\\
\left\langle b^\dagger D\left(h\right)\right\rangle_B &=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle \alpha\left|b^\dagger D\left(h\right)\right|\alpha\right\rangle && (607)\\
&=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}} && (608)\\
&=-h^*\left\langle D\left(h\right)\right\rangle_B N. && (609)
\end{aligned}
$$

The correlation functions can be found readily as:

$$
\begin{aligned}
B_1^+B_0^-\left(t,\tau\right) &=\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), && (610)\\
B_0^+B_1^-\left(t,\tau\right) &=\prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), && (611)\\
B_{10}\left(t\right) &=\mathrm{e}^{\chi_{10}(t)}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), && (612)\\
B_x\left(t,\tau\right) &=\frac{B_1^+B_0^-\left(t,\tau\right)+B_0^+B_1^-\left(t,\tau\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}, && (613)\\
B_y\left(t,\tau\right) &=\frac{B_0^+B_1^-\left(t,\tau\right)-B_1^+B_0^-\left(t,\tau\right)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}, && (614)\\
B_{i\mathrm{mod}2z}\left(t,\tau\right) &=\sum_{\mathbf{k}}\left(q_{i\mathbf{k}}\left(t\right)b_{\mathbf{k}}^\dagger\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{i\mathbf{k}}^*\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right), && (615)\\
\left\langle \widetilde{B_{i\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j\mathrm{mod}2z}}\left(t'\right)\right\rangle_B &=\left\langle B_{i\mathrm{mod}2z}\left(t,\tau\right)B_{j\mathrm{mod}2z}\left(t',0\right)\right\rangle_B && (616)\\
&=\left\langle \sum_{\mathbf{k}}\left(q_{i\mathbf{k}}\left(t\right)b_{\mathbf{k}}^\dagger\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{i\mathbf{k}}^*\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^\dagger+q_{j\mathbf{k}}^*\left(t'\right)b_{\mathbf{k}}\right)\right\rangle_B && (617)\\
&=\sum_{\mathbf{k}}q_{i\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^*\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}q_{i\mathbf{k}}^*\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), && (618)\\
\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_x}\left(t'\right)\right\rangle_B &=\left\langle B_x\left(t,\tau\right)B_x\left(t',0\right)\right\rangle_B && (619)
\end{aligned}
$$

$$=\left\langle\left(\frac{B_1^+B_0^-(t,\tau)+B_0^+B_1^-(t,\tau)-B_{10}(t)-B_{01}(t)}{2}\right)\left(\frac{B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_B \quad (620)$$

$$=\frac{1}{4}\left\langle\left(B_1^+B_0^-(t,\tau)+B_0^+B_1^-(t,\tau)-B_{10}(t)-B_{01}(t)\right)\left(B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')\right)\right\rangle_B \quad (621)$$

$$=\frac{1}{4}\left\langle B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)\,B_{10}(t')-B_1^+B_0^-(\tau)\,B_{01}(t')\right. \quad (622)$$

$$+B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)-B_0^+B_1^-(t,\tau)\,B_{10}(t')-B_0^+B_1^-(t,\tau)\,B_{01}(t') \quad (623)$$

$$-B_{10}(t)B_1^+B_0^-(t',0)-B_{10}(t)B_0^+B_1^-(t',0)+B_{10}(t)\,B_{10}(t')+B_{10}(t)\,B_{01}(t')-B_{01}(t)\,B_1^+B_0^-(t',0) \quad (624)$$

$$\left.-B_{01}(t)\,B_0^+B_1^-(t',0)+B_{01}(t)\,B_{10}(t')+B_{01}(t)\,B_{01}(t')\right\rangle \quad (625)$$

$$=\frac{1}{4}\left\langle B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0)+B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right. \quad (626)$$

$$\left.+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle-\frac{(B_{01}(t)+B_{10}(t))\,(B_{01}(t')+B_{10}(t'))}{4}, \quad (627)$$

$$U_{10}(t,t')=\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)m_{\mathbf{k}}^*(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Im}}, \quad (628)$$

$$\left\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle_B=\left\langle\Pi_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau})\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right)\Pi_{\mathbf{k}}\left(D(-m_{\mathbf{k}}(t'))\mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B \quad (629)$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}(t')\right)\right)\right\rangle_B \quad (630)$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\prod_{\mathbf{k}}\left\langle\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(-m_{\mathbf{k}}(t')\right)\right)\right\rangle_B \quad (631)$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (632)$$

$$\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\right\rangle_B=\left\langle\prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right)\prod_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B \quad (633)$$

$$=\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle D\left(-m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(m_{\mathbf{k}}(t')\right)\right\rangle_B \quad (634)$$

$$=\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\left\langle D\left(-m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(m_{\mathbf{k}}(t')\right)\right\rangle_B \quad (635)$$

$$=\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (636)$$

$$\left\langle B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)\right\rangle_B=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (637)$$

$$\left\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle_B=\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (638)$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{4}\left\langle B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0)+B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right. \quad (639)$$

$$\left.+B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right\rangle-\frac{(B_{01}(t)+B_{10}(t))\,(B_{01}(t')+B_{10}(t'))}{4}, \quad (640)$$

$$=\frac{1}{4}\left(2U_{10}(t,t')\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \quad (641)$$

$$\left.+2U_{10}^*(t,t')\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \quad (642)$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \quad (643)$$

$$=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \quad (644)$$

$$+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{645}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^{2}\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{646}$$

$$\left\langle\widetilde{B_{y}}(t)\widetilde{B_{y}}(t')\right\rangle_{B}=\left\langle B_{y}\left(t,\tau\right)B_{y}\left(t',0\right)\right\rangle_{B} \tag{647}$$

$$=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{648}$$

$$\left.+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{649}$$

$$=\left\langle\left(\frac{B_{0}^{+}B_{1}^{-}(t,\tau)-B_{1}^{+}B_{0}^{-}(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\left(\frac{B_{0}^{+}B_{1}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_{B} \tag{650}$$

$$=-\frac{1}{4}\left\langle\left(B_{0}^{+}B_{1}^{-}(t,\tau)-B_{1}^{+}B_{0}^{-}(t,\tau)+B_{10}(t)-B_{01}(t)\right)\left(B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{10}\left(t'\right)-B_{01}\left(t'\right)\right)\right\rangle_{B} \tag{651}$$

$$=-\frac{1}{4}\left\langle B_{0}^{+}B_{1}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{0}^{+}B_{1}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t,\tau)B_{10}(t')-B_{0}^{+}B_{1}^{-}(\tau)B_{01}(t')-B_{1}^{+}B_{0}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)\right. \tag{652}$$

$$+B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)-B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{10}\left(t'\right)+B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{01}\left(t'\right)+B_{10}(t)B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{10}(t)B_{1}^{+}B_{0}^{-}\left(t',0\right) \tag{653}$$

$$\left.+B_{10}(t)B_{10}(t')-B_{10}\left(t\right)B_{01}\left(t'\right)-B_{01}\left(t\right)B_{0}^{+}B_{1}^{-}\left(t',0\right)+B_{01}\left(t\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)-B_{01}\left(t\right)B_{10}\left(t'\right)+B_{01}\left(t\right)B_{01}\left(t'\right)\right\rangle \tag{654}$$

$$=-\frac{1}{4}\left\langle B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)-B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{0}^{+}B_{1}^{-}\left(t',0\right)+B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle \tag{655}$$

$$+\left(B_{01}\left(t\right)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Im} \tag{656}$$

$$=-\frac{1}{4}\left(2\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{657}$$

$$\left.-2\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{658}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{659}$$

$$=-\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{660}$$

$$\left.-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{661}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{662}$$

$$\left\langle\widetilde{B_{x}}(t)\widetilde{B_{y}}(t')\right\rangle_{B}=\left\langle B_{x}(t,\tau)B_{y}(t',0)\right\rangle_{B} \tag{663}$$

$$=\left\langle\left(\frac{B_{1}^{+}B_{0}^{-}\left(t,\tau\right)+B_{0}^{+}B_{1}^{-}\left(t,\tau\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\right)\left(\frac{B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{10}\left(t'\right)-B_{01}\left(t'\right)}{2\mathrm{i}}\right)\right\rangle_{B} \tag{664}$$

$$=\frac{1}{4i}\left\langle B_{1}^{+}B_{0}^{-}(t,\tau)B_{0}^{+}B_{1}^{-}(t',0)-B_{1}^{+}B_{0}^{-}(t,\tau)B_{1}^{+}B_{0}^{-}(t',0)+B_{1}^{+}B_{0}^{-}(t,\tau)B_{10}(t')-B_{1}^{+}B_{0}^{-}(t,\tau)B_{01}(t')+B_{01}(t)B_{1}^{+}B_{0}^{-}(t',0)\right. \tag{665}$$

$$+B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{10}\left(t'\right)-B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{01}\left(t'\right)+B_{01}(t)B_{01}\left(t'\right) \tag{666}$$

$$\left.-B_{10}(t)B_{0}^{+}B_{1}^{-}(t',0)+B_{10}(t)B_{1}^{+}B_{0}^{-}(t',0)-B_{10}(t)B_{10}(t')+B_{10}(t)B_{01}(t')-B_{01}(t)B_{0}^{+}B_{1}^{-}(t',0)-B_{01}(t)B_{10}(t')\right\rangle_{B} \tag{667}$$

$$=\frac{1}{4i}\left\langle B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{10}\left(t'\right)-B_{1}^{+}B_{0}^{-}\left(t,\tau\right)B_{01}\left(t'\right)\right. \tag{668}$$

$$+B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{10}\left(t'\right)-B_{0}^{+}B_{1}^{-}\left(t,\tau\right)B_{01}\left(t'\right) \tag{669}$$

$$-B_{10}\left(t\right)B_{0}^{+}B_{1}^{-}\left(t',0\right)+B_{10}\left(t\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)-B_{10}\left(t\right)B_{10}\left(t'\right)+B_{10}\left(t\right)B_{01}\left(t'\right)-B_{01}\left(t\right)B_{0}^{+}B_{1}^{-}\left(t',0\right) \tag{670}$$

$$\left.+B_{01}\left(t\right)B_{1}^{+}B_{0}^{-}\left(t',0\right)-B_{01}\left(t\right)B_{10}\left(t'\right)+B_{01}\left(t\right)B_{01}\left(t'\right)\right\rangle_{B} \tag{671}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\right\rangle \tag{672}$$

$$+\frac{1}{4i}\left(B_{10}(t)+B_{01}(t)\right)\left(B_{10}(t')-B_{01}(t')\right) \tag{673}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right. \tag{674}$$

$$\left. - B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right\rangle + \frac{1}{4i}\left(B_{10}(t)+B_{01}(t)\right)\left(B_{10}(t')-B_{01}(t')\right) \tag{675}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right. \tag{676}$$

$$\left. - B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right\rangle + \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{677}$$

$$= \frac{1}{4i}\left(\left(e^{\chi_{10}(t)+\chi_{01}(t')} - e^{\chi_{01}(t)+\chi_{10}(t')}\right)U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{678}$$

$$\left. + \left(e^{\chi_{01}(t)+\chi_{01}(t')} - e^{\chi_{10}(t)+\chi_{10}(t')}\right)U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{679}$$

$$+\left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{680}$$

$$= \frac{1}{2}\left(\left(e^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{681}$$

$$\left. + \left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{682}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B = \left\langle \left(\frac{B_0^+B_1^-(t,\tau) - B_1^+B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2i}\right)\left(\frac{B_1^+B_0^-(t',0) + B_0^+B_1^-(t',0) - B_{10}(t') - B_{01}(t')}{2}\right)\right\rangle_B \tag{683}$$

$$= \frac{1}{4i}\left\langle \left(B_0^+B_1^-(t,\tau) - B_1^+B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)\right)\left(B_1^+B_0^-(t',0) + B_0^+B_1^-(t',0) - B_{10}(t') - B_{01}(t')\right)\right\rangle_B \tag{684}$$

$$= \frac{1}{4i}\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_0^+B_1^-(t,\tau)B_{10}(t') - B_0^+B_1^-(t,\tau)B_{01}(t') + B_{01}(t)B_{01}(t')\right. \tag{685}$$

$$- B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) - B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) + B_1^+B_0^-(t,\tau)\,B_{10}(t') + B_1^+B_0^-(t,\tau)\,B_{01}(t') \tag{686}$$

$$+ B_{10}(t)\,B_0^+B_1^-(t',0) - B_{10}(t)\,B_{10}(t') - B_{10}(t)\,B_{01}(t') - B_{01}(t)\,B_1^+B_0^-(t',0) - B_{01}(t)\,B_0^+B_1^-(t',0) \tag{687}$$

$$\left. + B_{01}(t)\,B_{10}(t') + B_{10}(t)\,B_1^+B_0^-(t',0)\right\rangle \tag{688}$$

$$= \frac{1}{4i}\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle \tag{689}$$

$$+\left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{690}$$

$$= \frac{1}{4i}\left(e^{\chi_{01}(t)+\chi_{10}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{691}$$

$$+ e^{\chi_{01}(t)+\chi_{01}(t')}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{692}$$

$$- e^{\chi_{10}(t)+\chi_{10}(t')}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{693}$$

$$\left. - e^{\chi_{10}(t)+\chi_{01}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{694}$$

$$= \frac{1}{4i}\left(2i\left(e^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{695}$$

$$\left. + 2i\left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{696}$$

$$=(B_{10}(t))^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re}+\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}\left(t'\right)}\right)^{\Im}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{697}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}\left(t'\right)}\right)^{\Im}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{698}$$

$$\left\langle b^{\dagger}D\left(h\right)\right\rangle_{B}=-h^{*}\left\langle D\left(h\right)\right\rangle_{B}N \tag{699}$$

$$\left\langle bD\left(h\right)\right\rangle_{B}=h\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{700}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{B}=-h^{*}\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{701}$$

$$\left\langle D\left(h\right)b\right\rangle_{B}=h\left\langle D\left(h\right)\right\rangle_{B}N \tag{702}$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)=q_{j\mathbf{k}}\left(t\right) \tag{703}$$

$$\left\langle B_{1}^{+}B_{0}^{-}\left(t,\tau\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}=\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} \tag{704}$$

$$=\mathrm{e}^{\chi_{10}(t)}\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{705}$$

$$=\mathrm{e}^{\chi_{10}(t)}q_{i\mathbf{k}'}\left(t'\right)\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{706}$$

$$=\mathrm{e}^{\chi_{10}(t)}q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{707}$$

$$=q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{708}$$

$$=-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right) \tag{709}$$

$$\left\langle B_{1}^{+}B_{0}^{-}(t,\tau)q_{i\mathbf{k}'}^{*}(t')b_{\mathbf{k}'}\right\rangle_{B}=q_{i\mathbf{k}'}^{*}(t')\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}\left\langle\prod_{\mathbf{k}}D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle \tag{710}$$

$$=q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right), \tag{711}$$

$$\left\langle B_{0}^{+}B_{1}^{-}(t,\tau)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}=-q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right), \tag{712}$$

$$\left\langle B_{0}^{+}B_{1}^{-}(t,\tau)q_{i\mathbf{k}'}^{*}(t')b_{\mathbf{k}'}\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right), \tag{713}$$

$$q_{i\mathbf{k}'}\left(0\right)=g_{i\mathbf{k}'}-v_{i\mathbf{k}'} \tag{714}$$

$$\left\langle B_{x}\left(t,\tau\right)B_{i\bmod 2z}\left(t',0\right)\right\rangle_{B}=\left\langle\left(\frac{B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}}{2}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{715}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}\right)\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{716}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)\right)\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{717}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle B_{1+}B_{0-}\left(\tau\right)q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+B_{0+}B_{1-}\left(\tau\right)q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+B_{1+}B_{0-}\left(\tau\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right. \tag{718}$$

$$\left.+B_{0+}B_{1-}\left(\tau\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right\rangle_{B} \tag{719}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right)\right. \tag{720}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right)\right) \tag{721}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right)\right. \tag{722}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right)\right) \tag{723}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}\left(t\right)+\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{01}\left(t\right)\right)\right. \tag{724}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{01}\left(t\right)\right)\right) \tag{725}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}\left(t\right)-\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{01}\left(t\right)\right)\right. \tag{726}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{01}\left(t\right)\right)\right) \tag{727}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(B_{10}\left(t\right)-B_{01}\left(t\right)\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(B_{10}\left(t\right)\right.\right. \tag{728}$$

$$-B_{01}\left(t\right))) \tag{729}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}2\mathrm{i}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{730}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{731}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right), \tag{732}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) \tag{733}$$

$$\left\langle B_{i\mathrm{mod}2z}\left(t,\tau\right)B_{x}\left(t',0\right)\right\rangle_{B}=\left\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_{B} \tag{734}$$

$$=\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)\,b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_{B} \tag{735}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)\,b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{10}\left(t'\right)-B_{01}\left(t'\right)\right)\right\rangle_{B} \tag{736}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}\left(t\right)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)\right)\right\rangle_{B} \tag{737}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Big\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)+q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right) \tag{738}$$

$$+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)\Big\rangle, \tag{739}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B} \tag{740}$$

$$=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B} \tag{741}$$

$$=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(m_{\mathbf{k}'}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}'}\left(t'\right)}{2}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B} \tag{742}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{743}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{744}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{745}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left(-m_{\mathbf{k}'}^{*}\left(t'\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}N_{\mathbf{k}'}\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{746}$$

$$=-m_{\mathbf{k}'}^{*}\left(t'\right)q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{747}$$

$$=-m_{\mathbf{k}'}^{*}\left(t'\right)q_{i\mathbf{k}'}\left(t\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{748}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)\right\rangle_{B}=m_{\mathbf{k}'}^{*}\left(t'\right)q_{i\mathbf{k}'}\left(t\right)B_{01}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{749}$$

$$\left\langle q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B} \tag{750}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B} \tag{751}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle b_{\mathbf{k}'}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B} \tag{752}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B} \tag{753}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B} \tag{754}$$

$$= q^*_{i\mathbf{k}'}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)B_{10}(t'), \tag{755}$$

$$\left\langle (q_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_1^-(t',0)\right\rangle_B = q^*_{i\mathbf{k}'}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(-m_{\mathbf{k}'}(t'))(N_{\mathbf{k}'}+1)B_{01}(t'), \tag{756}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_x(t',0)\right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'}\Big(-m^*_{\mathbf{k}'}(t')q_{i\mathbf{k}'}(t)B_{10}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - (-m^*_{\mathbf{k}'}(t'))q_{i\mathbf{k}'}(t)B_{01}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{757}$$

$$+q^*_{i\mathbf{k}'}(t)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)B_{10}(t') + q^*_{i\mathbf{k}'}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(-m_{\mathbf{k}'}(t'))(N_{\mathbf{k}'}+1)B_{01}(t')\Big) \tag{758}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t')(B_{01}(t')-B_{10}(t')) + q^*_{i\mathbf{k}'}(t)m_{\mathbf{k}'}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{759}$$

$$\times(N_{\mathbf{k}'}+1)(B_{10}(t')-B_{01}(t'))) \tag{760}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t')(B_{01}(t')-B_{10}(t')) - q^*_{i\mathbf{k}'}(t)m_{\mathbf{k}'}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{761}$$

$$\times(N_{\mathbf{k}'}+1)(B_{01}(t')-B_{10}(t'))) \tag{762}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}B_{01}^{\Im}(t')\Big(q_{i\mathbf{k}'}(t)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t') - q^*_{i\mathbf{k}'}(t)m_{\mathbf{k}'}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)\Big), \tag{763}$$

$$\left\langle B_y(t,\tau)B_{i\mathrm{mod}2z}(t',0)\right\rangle_B = \left\langle\left(\frac{B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}+q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right)\right\rangle_B \tag{764}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)\right)\left(q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}+q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right)\right\rangle_B \tag{765}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)\right)\left(q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}+q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right)\right\rangle_B \tag{766}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle B_0^+B_1^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'} - B_1^+B_0^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'} + B_0^+B_1^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'} \tag{767}$$

$$-B_1^+B_0^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\Big\rangle, \tag{768}$$

$$\left\langle B_0^+B_1^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}\right\rangle_B = -q_{i\mathbf{k}'}(t')\left(-m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}(t), \tag{769}$$

$$\left\langle B_0^+B_1^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right\rangle_B = q^*_{i\mathbf{k}'}(t')\left(-m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t), \tag{770}$$

$$\left\langle B_1^+B_0^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}\right\rangle_B = -q_{i\mathbf{k}'}(t')\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}(t), \tag{771}$$

$$\left\langle B_1^+B_0^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right\rangle_B = q^*_{i\mathbf{k}'}(t')m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}N_{\mathbf{k}'}B_{10}(t), \tag{772}$$

$$\left\langle B_y(t,\tau)B_{i\mathrm{mod}2z}(t',0)\right\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}(B_{01}(t)+B_{10}(t))\Big(q_{i\mathbf{k}'}(t')\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1) - q^*_{i\mathbf{k}'}(t')m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}N_{\mathbf{k}'}\Big) \tag{773}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(q^*_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t) - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)\Big)(B_{10}(t)+B_{01}(t)), \tag{774}$$

$$= \frac{2}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t)(B_{10}(t))^{\Re} - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)(B_{10}(t))^{\Re}\Big), \tag{775}$$

$$= \frac{(B_{10}(t))^{\Re}}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t) - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)\Big), \tag{776}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_y(t',0)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_B \tag{777}$$

$$= \frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\left(B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t')\right.\right. \tag{778}$$

$$\left.\left.-B_{01}(t')\right)\right\rangle_B \tag{779}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)\right)\right\rangle_B \tag{780}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0) - q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0) \tag{781}$$

$$+q^*_{i\mathbf{k}'}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-\left(t',0\right)-q^*_{i\mathbf{k}'}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-\left(t',0\right)\Big\rangle \tag{782}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\Big\rangle \tag{783}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\right\rangle-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\right\rangle\Big\rangle \tag{784}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\Big\rangle+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\big\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\big\rangle \tag{785}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\right\rangle \tag{786}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\Big\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\Big\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\big\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\big\rangle \tag{787}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\right\rangle\Big) \tag{788}$$

$$\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\Big\rangle_B=-m^*_{\mathbf{k}'}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}, \tag{789}$$

$$\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\Big\rangle_B=m^*_{\mathbf{k}'}\left(t'\right)B_{01}\left(t'\right)N_{\mathbf{k}'}, \tag{790}$$

$$\big\langle b_{\mathbf{k}'}B_1^+B_0^-\left(t',0\right)\big\rangle_B=m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right), \tag{791}$$

$$\big\langle b_{\mathbf{k}'}B_0^+B_1^-\left(t',0\right)\big\rangle_B=-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right), \tag{792}$$

$$\big\langle B_{i\mathrm{mod}2z}(t,\tau)B_y\left(t',0\right)\big\rangle_B=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-\left(-m^*_{\mathbf{k}'}\left(t'\right)\right)B_{01}\left(t'\right)N_{\mathbf{k}'}\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-m^*_{\mathbf{k}'}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \tag{793}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)\right)-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\Big) \tag{794}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-q_{i\mathbf{k}'}\left(t\right)\left(-m^*_{\mathbf{k}'}\left(t'\right)\right)B_{01}\left(t'\right)N_{\mathbf{k}'}+q_{i\mathbf{k}'}\left(t\right)m^*_{\mathbf{k}'}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \tag{795}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(q^*_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)-q^*_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\right)\Big) \tag{796}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(B_{10}\left(t'\right)+B_{01}\left(t'\right)\right)\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m^*_{\mathbf{k}'}\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big) \tag{797}$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m^*_{\mathbf{k}'}\left(t'\right)B^{\Re}_{10}\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B^{\Re}_{10}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big) \tag{798}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B^{\Re}_{10}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m^*_{\mathbf{k}'}\left(t'\right)B^{\Re}_{10}\left(t'\right)N_{\mathbf{k}'}\Big) \tag{799}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B^{\Re}_{10}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m^*_{\mathbf{k}'}\left(t'\right)B^{\Re}_{10}\left(t'\right)N_{\mathbf{k}'}\Big) \tag{800}$$

$$=\mathrm{i}B^{\Re}_{10}\left(t'\right)\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m^*_{\mathbf{k}'}\left(t'\right)N_{\mathbf{k}'}\Big). \tag{801}$$

The correlation functions are equal to:

$$\Big\langle\widetilde{B_{i\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j\mathrm{mod}2z}}\left(t'\right)\Big\rangle_B=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)^*\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{802}$$

$$\Big\langle\widetilde{B_x}\left(t\right)\widetilde{B_x}\left(t'\right)\Big\rangle_B=\frac{1}{2}\Bigg(\Big(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\Big)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{803}$$

$$+\Big(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\Big)^{\Re}U^*_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{804}$$

$$-\Bigg(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)^{\Re}\Bigg(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)^{\Re} \tag{805}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B = -\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(\mathrm{v}_{1\mathbf{k}}(\mathrm{t})-\mathrm{v}_{0\mathbf{k}}(\mathrm{t}))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')-\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{806}$$

$$\left. - \left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{807}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}-\frac{\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im} \tag{808}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im} U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{809}$$

$$\left. + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im} U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{810}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im} U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{811}$$

$$\left. + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im} U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{812}$$

$$\left\langle\widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_x}(t')\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}} B_{01}^{\Im}(t')\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{813}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_{i\mathrm{mod}2z}}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t')\right)^{*}N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t')\right)\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{814}$$

$$\left\langle\widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_y}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Re}(t')\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{815}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_{i\mathrm{mod}2z}}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Re}(t)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t')\right)^{*}N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t')\right)(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\right). \tag{816}$$

Let’s consider the following expression related to the sum of coupling constants for a bath over all the frequences:

$$L_i(\omega) \equiv \sum_{\mathbf{k}} g_{i\mathbf{k}}\sqrt{\delta(\omega-\omega_{\mathbf{k}})}. \tag{817}$$

Under this definition we have the following expression for a function $f(\omega) \in L^2$:

$$\int_0^{\infty} f(\omega)\, L_i(\omega)\, L_j^{*}(\omega)\,\mathrm{d}\omega \approx \int_0^{\infty} f(\omega)\sum_{\mathbf{k}} g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sum_{\mathbf{k}'} g_j(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega \tag{818}$$

$$= \int_0^{\infty} f(\omega)\sum_{\mathbf{k},\mathbf{k}'} g_i(\omega_{\mathbf{k}})\, g_j(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega, \tag{819}$$

$$\int_0^{\infty} f(\omega)\sum_{\mathbf{k}} g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\mathrm{d}\omega = \sum_{\mathbf{k}}\int_0^{\infty} f(\omega)\, g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\mathrm{d}\omega. \tag{820}$$

Now we will approach to the function $\sqrt{\delta(\omega-\omega_{\mathbf{k}})}$ using the normal distribution, so:

$$\delta\left(\omega-\omega_{\mathbf{k}}\right) = \lim_{\sigma\to 0^+} \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(\omega-\omega_{\mathbf{k}})^2}{2\sigma^2}} \tag{821}$$

$$\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)} = \lim_{\sigma\to 0^+} \sqrt{\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(\omega-\omega_{\mathbf{k}})^2}{2\sigma^2}}} \tag{822}$$

$$= \lim_{\sigma\to 0^+} \sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{(\omega-\omega_{\mathbf{k}})^2}{4\sigma^2}} \tag{823}$$

$$= \lim_{\sigma\to 0^+} \sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{(\omega-\omega_{\mathbf{k}})^2}{2(\sqrt{2}\sigma)^2}} \tag{824}$$

$$= \lim_{\sigma\to 0^+} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right). \tag{825}$$

So we can obtain that:

$$\sum_{\mathbf{k}}\int_0^\infty f\left(\omega\right)g_i\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega = \sum_{\mathbf{k}}\int_0^\infty f\left(\omega\right)g_i\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega \tag{826}$$

$$= \sum_{\mathbf{k}}\int_0^\infty f\left(\omega\right)g_i\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to 0^+}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{827}$$

$$= \sum_{\mathbf{k}}\int_0^\infty f\left(\omega\right)g_i\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to 0^+}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{828}$$

$$= \lim_{\sigma\to 0^+}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}\int_0^\infty f\left(\omega\right)g_i\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega \tag{829}$$

$$= \sum_{\mathbf{k}}\left(\lim_{\sigma\to 0^+}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)\left(\lim_{\sigma\to 0^+}\int_0^\infty f\left(\omega\right)g_i\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega\right) \tag{830}$$

$$= \sum_{\mathbf{k}}\left(\lim_{\sigma\to 0^+}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)f\left(\omega_{\mathbf{k}}\right)g_i\left(\omega_{\mathbf{k}}\right)\text{ (with }f\left(\omega\right)\text{,}g_i\left(\omega\right)\in L^2\text{)} \tag{831}$$

$$= \lim_{\sigma\to 0^+}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_i\left(\omega_{\mathbf{k}}\right)\text{ (with }f\left(\omega\right)\text{,}g_i\left(\omega\right)\in L^2\text{)} \tag{832}$$

$$= 0\,\text{(because the sum }\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_i\left(\omega_{\mathbf{k}}\right)\text{ is finite).} \tag{833}$$

Then we can proof the following:

$$\int_0^\infty f(\omega)L_i(\omega)L_j^*\left(\omega\right)\mathrm{d}\omega \approx \int_0^\infty f\left(\omega\right)\sum_{\mathbf{k}}g_i\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sum_{\mathbf{k}'}g_j^*\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{834}$$

$$= \int_0^\infty f\left(\omega\right)\sum_{\mathbf{k},\mathbf{k}'}g_i\left(\omega_{\mathbf{k}}\right)g_j^*\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{835}$$

$$= \sum_{\mathbf{k}\neq\mathbf{k}'}\int_0^\infty f(\omega)g_i(\omega_{\mathbf{k}})g_j^*(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega + \sum_{\mathbf{k}}\int_0^\infty f(\omega)g_i(\omega_{\mathbf{k}})g_j^*(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{836}$$

$$= 0 + \sum_{\mathbf{k}}\int_0^\infty f(\omega)g_i(\omega_{\mathbf{k}})g_j^*(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{837}$$

$$= \sum_{\mathbf{k}}\int_0^\infty f(\omega)g_i(\omega_{\mathbf{k}})g_j^*(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{838}$$

$$= \sum_{\mathbf{k}}f(\omega_{\mathbf{k}})g_i(\omega_{\mathbf{k}})g_j^*(\omega_{\mathbf{k}}) \tag{839}$$

if $i=j$ we recover the spectral density defined in the usual way when we integrate for a function $f\left(\omega\right)$ that belongs to the set L^2:

$$\sum_{\mathbf{k}} f(\omega_{\mathbf{k}})g_i(\omega_{\mathbf{k}})g_i^*(\omega_{\mathbf{k}}) = \sum_{\mathbf{k}} \int_0^\infty f(\omega)g_i(\omega_{\mathbf{k}})g_i^*(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{840}$$

$$= \int_0^\infty f(\omega)J_{ii}(\omega)\mathrm{d}\omega \tag{841}$$

$$= \int_0^\infty f(\omega)|L_i(\omega)|^2\,\mathrm{d}\omega. \tag{842}$$

where

$$J_{ii}(\omega) = \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2\,\delta(\omega-\omega_{\mathbf{k}}), \tag{843}$$

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}},t) = g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}},t). \tag{844}$$

In this case $g_i(\omega)$ and $v_i(\omega,t)$ are the continuous version of $g_i(\omega_{\mathbf{k}})$ and $v_{i\mathbf{k}}(\omega_{\mathbf{k}},t)$ respectively.
The integral version of the correlation functions can be obtained as follows:

$$\left\langle \widetilde{B_{iz}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t')\right)^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t')\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1), \tag{845}$$

$$= \sum_{\mathbf{k}} \Big(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}},t))\,g_{j\mathbf{k}}^*\left(1-F_j(\omega_{\mathbf{k}},t')\right)^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}} + g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}},t))^*\,g_{j\mathbf{k}}\left(1-F_j(\omega_{\mathbf{k}},t')\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\Big) \tag{846}$$

$$\approx \int_0^\infty \Big(L_i(\omega)L_j^*(\omega)(1-F_i(\omega,t))\Big(1-F_j^*(\omega,t')\Big)\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + L_i^*(\omega)L_j(\omega)\big(1-F_i^*(\omega,t)\big)\left(1-F_j(\omega,t')\right)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\Big)\mathrm{d}\omega, \tag{847}$$

$$\chi_{10}(t) = \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)\,v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)\,v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right) \tag{848}$$

$$= \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{g_{1\mathbf{k}}^*F_1^*(\omega_{\mathbf{k}},t)\,g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t) - g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)\,g_{0\mathbf{k}}^*F_0^*(\omega_{\mathbf{k}},t)}{\omega_{\mathbf{k}}^2}\right) \tag{849}$$

$$= \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{g_{1\mathbf{k}}^*g_{0\mathbf{k}}F_1^*(\omega_{\mathbf{k}},t)\,F_0(\omega_{\mathbf{k}},t) - g_{1\mathbf{k}}g_{0\mathbf{k}}^*F_1(\omega_{\mathbf{k}},t)\,F_0^*(\omega_{\mathbf{k}},t)}{\omega_{\mathbf{k}}^2}\right) \tag{850}$$

$$\approx \int_0^\infty \frac{L_0(\omega)\,L_1^*(\omega)\,F_1^*(\omega,t)\,F_0(\omega,t) - L_1(\omega)\,L_0^*(\omega)\,F_1(\omega,t)\,F_0^*(\omega,t)}{2\omega^2}\mathrm{d}\omega, \tag{851}$$

$$U_{10}(t,t') = \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\left(\frac{(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))^*\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}} \tag{852}$$

$$= \mathrm{e}^{\mathrm{i}\sum_{\mathbf{k}}\left(\frac{(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))^*\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}} \tag{853}$$

$$= \mathrm{e}^{\mathrm{i}\left(\sum_{\mathbf{k}}\frac{(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))^*\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}} \tag{854}$$

$$= \mathrm{e}^{\mathrm{i}\left(\sum_{\mathbf{k}}\frac{(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t'))^*\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}} \tag{855}$$

$$\approx \mathrm{e}^{\mathrm{i}\left(\int_0^\infty\frac{(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))(L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t'))^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)^{\Im}}, \tag{856}$$

$$B_{10}(t) = \left(\prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{857}$$

$$= \mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{858}$$

$$\approx \mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\int_0^\infty\left|\frac{L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{859}$$

$$\xi^+(t,t') = \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{860}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{861}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{862}$$

$$\approx\mathrm{e}^{-\int_0^\infty\frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}+L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t')|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{863}$$

$$\xi^-(t,t')=\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{864}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{865}$$

$$=\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t'))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{866}$$

$$\approx\mathrm{e}^{-\int_0^\infty\frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}-(L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t'))|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{867}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}(t,t')\xi^+(t,t')+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*(t,t')\,\xi^-(t,t')\right)-(B_{10}(t))^{\Re}(B_{01}(t'))^{\Re} \tag{868}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B=-\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}(t,t')\,\xi^+(t,t')-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*(t,t')\,\xi^-(t,t')\right)+(B_{01}(t))^{\Im}(B_{10}(t'))^{\Im} \tag{869}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*(t,t')\,\xi^-(t,t')+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\,\xi^+(t,t')\right)+(B_{10}(t))^{\Re}(B_{10}(t'))^{\Im} \tag{870}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^*(t,t')\,\xi^-(t,t')+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\,\xi^+(t,t')\right)+(B_{10}(t))^{\Im}(B_{10}(t'))^{\Re} \tag{871}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B=\mathrm{i}B_{01}^{\Im}(t')\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{872}$$

$$=\mathrm{i}B_{01}^{\Im}(t')\sum_{\mathbf{k}}\left(-g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}},t))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right. \tag{873}$$

$$\left.+g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}},t))\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{874}$$

$$Q(\omega,t)=\frac{L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t)}{\omega} \tag{875}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B\approx\mathrm{i}B_{01}^{\Im}(t')\int_0^\infty\left(L_i(\omega)(1-F_i(\omega,t))Q^*(\omega,t')N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}-L_i^*(\omega)(1-F_i^*(\omega,t))Q(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega, \tag{876}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_{iz}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{877}$$

$$\approx\mathrm{i}B_{01}^{\Im}(t)\int_0^\infty\left(L_i^*(\omega)\left(1-F_i^*(\omega,t')\right)Q(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}-L_i(\omega)\left(1-F_i(\omega,t')\right)Q^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{878}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_y}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t')\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right), \tag{879}$$

$$\approx\mathrm{i}B_{10}^{\Re}(t')\int_0^\infty\left(L_i^*(\omega)\left(1-F_i^*(\omega,t')\right)Q(\omega,t')(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau}-L_i(\omega)\left(1-F_i(\omega,t')\right)Q^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{880}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_{iz}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))(N_{\mathbf{k}}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\right) \tag{881}$$

$$\approx i B_{10}^{\Re}(t)\int_0^\infty\left(L_i^*(\omega)\left(1-F_i^*(\omega,t')\right)Q(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}-L_i(\omega)\left(1-F_i(\omega,t')\right)Q^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega. \tag{882}$$

The integral version of $F_0(\omega,t)$ and $F_1(\omega,t)$ are:

$$a_i(\omega_{\mathbf{k}},t)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{883}$$

$$b_i(\omega_{\mathbf{k}},t)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{884}$$

$$r_i(\omega_{\mathbf{k}},t)=\frac{a_i(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)},\tag{885}$$

$$s_i(\omega_{\mathbf{k}},t)=\frac{a_{(i+1)\mathrm{mod}2}(\omega_{\mathbf{k}},t)\,b_{i\mathrm{mod}2}(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}.\tag{886}$$

$$\begin{aligned}F_0(\omega,t)&=r_0(\omega_{\mathbf{k}},t)+\frac{g_1(\omega_{\mathbf{k}})}{g_0(\omega_{\mathbf{k}})}s_0(\omega_{\mathbf{k}},t)&(887)\\&\approx r_0(\omega,t)+\frac{g_1(\omega)}{g_0(\omega)}s_0(\omega,t)&(888)\\&=r_0(\omega,t)+\frac{L_1(\omega)}{L_0(\omega)}s_0(\omega,t),&(889)\\F_1(\omega,t)&=\frac{g_0(\omega_{\mathbf{k}})}{g_1(\omega_{\mathbf{k}})}r_1(\omega_{\mathbf{k}},t)+s_1(\omega_{\mathbf{k}},t)&(890)\\&\approx\frac{g_0(\omega)}{g_1(\omega)}r_1(\omega,t)+s_1(\omega,t)&(891)\\&=\frac{L_0(\omega)}{L_1(\omega)}r_1(\omega,t)+s_1(\omega,t).&(892)\end{aligned}$$

The expressions showed are well defined because the relevant products present in the correlations functions are of the form:

$$\begin{aligned}\int_0^\infty f(\omega)L_j(\omega)F_j(\omega,t)L_i^*(\omega)F_i^*(\omega,t)\,\mathrm{d}\omega&=\int_0^\infty f(\omega)L_j(\omega)\left(r_j(\omega,t)+\frac{L_i(\omega)}{L_j(\omega)}s_j(\omega,t)\right)L_i^*(\omega)\left(r_i^*(\omega,t)+\frac{L_j^*(\omega)}{L_i^*(\omega)}s_i^*(\omega,t)\right)\mathrm{d}\omega&(893)\\&=\int_0^\infty f(\omega)\left(L_j(\omega)\,r_j(\omega,t)+L_i(\omega)\,s_j(\omega,t)\right)\left(L_i^*(\omega)\,r_i^*(\omega,t)+L_j^*(\omega)\,s_i^*(\omega,t)\right)\mathrm{d}\omega&(894)\\&=\int_0^\infty f(\omega)\Big(L_j(\omega)\,L_i^*(\omega)\,r_j(\omega,t)\,r_i^*(\omega,t)+|L_j(\omega)|^2\,r_j(\omega,t)\,s_i^*(\omega,t)&(895)\\&\quad+|L_i(\omega)|^2\,s_j(\omega,t)\,r_i^*(\omega,t)+L_i(\omega)\,L_j^*(\omega)\,s_j(\omega,t)\,s_i^*(\omega,t)\Big)\,\mathrm{d}\omega.&(896)\end{aligned}$$

here $f(\omega)\in L^2$. As we could proof these integral are convergent.
So the integral version of the correlation functions $\mathcal{B}_{ij}(t,t')$ is can be written in a neater form as:

$$\mathcal{B}(t,t')\equiv\begin{pmatrix}\mathcal{B}_{11}(t,t')&\mathcal{B}_{12}(t,t')&\mathcal{B}_{13}(t,t')&\mathcal{B}_{12}(t,t')&\mathcal{B}_{11}(t,t')&\mathcal{B}_{16}(t,t')\\\mathcal{B}_{21}(t,t')&\mathcal{B}_{22}(t,t')&\mathcal{B}_{23}(t,t')&\mathcal{B}_{22}(t,t')&\mathcal{B}_{21}(t,t')&\mathcal{B}_{26}(t,t')\\\mathcal{B}_{31}(t,t')&\mathcal{B}_{32}(t,t')&\mathcal{B}_{33}(t,t')&\mathcal{B}_{32}(t,t')&\mathcal{B}_{31}(t,t')&\mathcal{B}_{36}(t,t')\\\mathcal{B}_{21}(t,t')&\mathcal{B}_{22}(t,t')&\mathcal{B}_{23}(t,t')&\mathcal{B}_{22}(t,t')&\mathcal{B}_{21}(t,t')&\mathcal{B}_{26}(t,t')\\\mathcal{B}_{11}(t,t')&\mathcal{B}_{12}(t,t')&\mathcal{B}_{13}(t,t')&\mathcal{B}_{12}(t,t')&\mathcal{B}_{11}(t,t')&\mathcal{B}_{16}(t,t')\\\mathcal{B}_{61}(t,t')&\mathcal{B}_{62}(t,t')&\mathcal{B}_{63}(t,t')&\mathcal{B}_{62}(t,t')&\mathcal{B}_{61}(t,t')&\mathcal{B}_{66}(t,t')\end{pmatrix},\tag{897}$$

$$\mathcal{B}_{11}(t,t')=\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^+(t,t')+\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^*(t,t')\,\xi_{10}^-(t,t')\right)-B_{10}^{\Re}(t)\,B_{01}^{\Re}(t'),\tag{898}$$

$$\mathcal{B}_{22}(t,t')=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^+(t,t')-\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^*(t,t')\,\xi_{10}^-(t,t')\right)+B_{01}^{\Im}(t)B_{10}^{\Im}(t'),\tag{899}$$

$$\mathcal{B}_{12}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^*(t,t')\,\xi_{10}^-(t,t')+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^+(t,t')\right)+B_{10}^{\Re}(t)\,B_{10}^{\Im}(t'),\tag{900}$$

$$\mathcal{B}_{21}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^*(t,t')\,\xi_{10}^-(t,t')+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^+(t,t')\right)+B_{10}^{\Im}(t)\,B_{10}^{\Re}(t'),\tag{901}$$

$$\mathcal{B}_{ij}\left(t,t'\right)=\int_0^\infty \left(P_i\left(\omega,t\right)P_j^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_i^*\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\{3,6\}\,, \tag{902}$$

$$\mathcal{B}_{i1}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t'\right)\int_0^\infty \left(P_i\left(\omega,t\right)Q_{10}^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i^*\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}\,, \tag{903}$$

$$\mathcal{B}_{1i}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}\,, \tag{904}$$

$$\mathcal{B}_{i2}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t'\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\}\,, \tag{905}$$

$$\mathcal{B}_{2i}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}\,, \tag{906}$$

$$\zeta_{ij}\left(t,t'\right)=\mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i\left(\omega,t'\right)-L_j(\omega)F_j\left(\omega,t'\right)\right)^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)}, \tag{907}$$

$$\xi_{ij}^{\pm}\left(t,t'\right)=\mathrm{e}^{-\int_0^\infty \frac{\left|\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm\left(L_i(\omega)F_i\left(\omega,t'\right)-L_j(\omega)F_j\left(\omega,t'\right)\right)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{908}$$

$$P_i\left(\omega,t\right)=L_{i\mathrm{mod}2}\left(\omega\right)\left(1-F_{i\mathrm{mod}2}\left(\omega,t\right)\right), \tag{909}$$

$$Q_{ij}(\omega,t)=\frac{L_i\left(\omega\right)F_1\left(\omega,t\right)-L_j\left(\omega\right)F_j\left(\omega,t\right)}{\omega}, \tag{910}$$

$$a_i\left(\omega,t\right)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)}, \tag{911}$$

$$b_i\left(\omega,t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)}, \tag{912}$$

$$r_i\left(\omega,t\right)=\frac{a_i\left(\omega,t\right)}{1-b_0\left(\omega,t\right)b_1\left(\omega,t\right)}, \tag{913}$$

$$s_i\left(\omega,t\right)=\frac{a_{(i+1)\mathrm{mod}2}\left(\omega,t\right)b_{i\mathrm{mod}2}\left(\omega,t\right)}{1-b_0\left(\omega,t\right)b_1\left(\omega,t\right)}, \tag{914}$$

$$F_0\left(\omega,t\right)=r_0\left(\omega,t\right)+\frac{L_1\left(\omega\right)}{L_0\left(\omega\right)}s_0\left(\omega,t\right), \tag{915}$$

$$F_1\left(\omega,t\right)=\frac{L_0\left(\omega\right)}{L_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right). \tag{916}$$

The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_i}\left(\tau\right)=e^{\mathrm{i}\overline{H_{\bar{S}}}\tau}A_i\left(t\right)\mathrm{e}^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} \tag{917}$$

$$=\sum_w \mathrm{e}^{-\mathrm{i}w\tau}A_i\left(w\right). \tag{918}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w\in\{0,\pm\eta\}$.

In order to use the equation (918) to descompose the equation (373) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U\left(t\right)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_0^t\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \tag{919}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1) H(t_2) ... H(t_n). \tag{920}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_{\text{eff}}(t)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \equiv \exp(-\mathrm{i}tH_{\text{eff}}(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_{\text{eff}}(t) = H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + ...\right)\right). \tag{921}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \tag{922}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{923}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{924}$$

We can summarize that:

$$\widetilde{O}(t) \equiv U^{\dagger}(t)\, O(t)\, U(t), \tag{925}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_T}(t')\right) \tag{926}$$

$$= \exp\left(-\mathrm{i}\overline{H_{T,\text{eff}}}(t)\right), \text{ where} \tag{927}$$

$$H_X^{\text{eff}}(t) \equiv \frac{1}{t}\int_0^t H_X(t')\mathrm{d}t' - \frac{\mathrm{i}}{2t}\int_0^t\int_0^{t'}[H_X(t'), H_X(t'')]\mathrm{d}t'\mathrm{d}t'' + \frac{1}{6t}\int_0^t\int_0^{t'}\int_0^{t''}\left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t''' + \cdots. \tag{928}$$

In order to show the explicit form of the matrices present in the RHS of the equation (918) for a general 2×2 matrix in a given time let's write the matrix $A_i(t)$ in the base $W(t) = \left\{\left|\overline{H_{\bar{S},\text{eff},1}}(t)\right\rangle, \left|\overline{H_{\bar{S},\text{eff},0}}(t)\right\rangle\right\}$, formed by the time-dependent eigenvectors of $\overline{H_{\bar{S},\text{eff}}}(t)$ in the following way:

$$A_i(t) = \sum_{j,j'} \left\langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \left| A_i(t) \right| \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \right\rangle \left|\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\right\rangle\!\left\langle\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right|. \tag{929}$$

Let's obtain $U^{\dagger}(t')\, A_i(t)\, U(t')$ in explicit form:

$$U^{\dagger}(t')A_i(t)U(t') = U^{\dagger}(t')\left(\sum_{j,j'}\left\langle\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\left|A_i(t)\right|\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right\rangle\left|\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\right\rangle\!\left\langle\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right|\right)U(t') \tag{930}$$

$$= \sum_{j,j'}\left\langle\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\left|A_i(t)\right|\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right\rangle U^{\dagger}(t')\left|\overline{H_{\bar{S},\text{eff},j}}(t-\tau)\right\rangle\!\left\langle\overline{H_{\bar{S},\text{eff},j'}}(t-\tau)\right|U(t') \tag{931}$$

$$
\begin{aligned}
&= \sum_{j,j'} \left\langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \left| A_i(t) \right| \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\lambda_j(t-\tau)} \left| \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \middle\rangle\middle\langle \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \right| \mathrm{e}^{-\mathrm{i}(t-\tau)\lambda_{j'}(t-\tau)} && (932)\\
&= \sum_{j,j'} \left\langle \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \left| A_i(t) \right| \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\left(\lambda_j(t-\tau)-\lambda_{j'}(t-\tau)\right)} \left| \overline{H_{\bar{S},\text{eff},j}}(t-\tau) \middle\rangle\middle\langle \overline{H_{\bar{S},\text{eff},j'}}(t-\tau) \right|, && (933)\\
M_{jj'}(t-\tau) &= \left\langle \overline{H_{\bar{S},\text{eff},j}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\text{eff},j'}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\text{eff},j}}\,(t-\tau) \middle\rangle\middle\langle \overline{H_{\bar{S},\text{eff},j'}}\,(t-\tau) \right|, && (934)\\
U^\dagger(t')A_i(t)U(t') &= M_{00}(t-\tau) + M_{01}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)(\lambda_0(t-\tau)-\lambda_1(t-\tau))} + M_{10}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)(\lambda_1(t-\tau)-\lambda_0(t-\tau))} + M_{11}(t-\tau), && (935)\\
w(t-\tau) &= \lambda_1\,(t-\tau) - \lambda_0\,(t-\tau), && (936)\\
U^\dagger(t')A_i(t)U(t') &= M_{00}\,(t-\tau) + M_{01}\,(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} + M_{10}\mathrm{e}^{\mathrm{i}(t-\tau)w(t-\tau)} + M_{11} && (937)\\
&= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i\,(w\,(t-\tau)) && (938)\\
&= A_i\,(0) + A_i\,(w\,(t-\tau))\,\mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} + A_i\,(-w\,(t-\tau))\,\mathrm{e}^{\mathrm{i}(t-\tau)w(t-\tau)}. && (939)
\end{aligned}
$$

By direct comparison we obtain that:

$$
\begin{aligned}
A_i\,(w\,(t-\tau)) &= \left\langle \overline{H_{\bar{S},\text{eff},0}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\text{eff},1}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\text{eff},0}}\,(t-\tau) \middle\rangle\middle\langle \overline{H_{\bar{S},\text{eff},1}}\,(t-\tau) \right|, && (940)\\
A_i\,(-w\,(t-\tau)) &= \left\langle \overline{H_{\bar{S},\text{eff},1}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\text{eff},0}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\text{eff},1}}\,(t-\tau) \middle\rangle\middle\langle \overline{H_{\bar{S},\text{eff},0}}\,(t-\tau) \right|, && (941)\\
A_i\,(0) &= \sum_j \left\langle \overline{H_{\bar{S},\text{eff},j}}\,(t-\tau) \left| A_i\,(t) \right| \overline{H_{\bar{S},\text{eff},j}}\,(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\text{eff},j}}\,(t-\tau) \middle\rangle\middle\langle \overline{H_{\bar{S},\text{eff},j}}\,(t-\tau) \right|. && (942)
\end{aligned}
$$

These matrix have the following property $A_i\,(w\,(t-\tau)) = A_i^\dagger\,(-w\,(t-\tau))$. Now in order to perform the double Fourier decomposition let's recall:

$$
\widetilde{A_i}\,(t,t') \equiv U\,(t)\,U^\dagger\,(t')\,A_i\,(t)\,U\,(t')\,U^\dagger\,(t)\,. \tag{943}
$$

In this case the decomposition can be written as:

$$
\begin{aligned}
\widetilde{A_i}\,(t,t-\tau) &\equiv U\,(t)\,U^\dagger\,(t-\tau)\,A_i\,(t)\,U\,(t-\tau)\,U^\dagger\,(t) && (944)\\
&= U\,(t)\left(\sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i\,(w\,(t-\tau))\right) U^\dagger\,(t)\,. && (945)
\end{aligned}
$$

Now writting $A_i\,(w\,(t-\tau))$ in terms of the eigenstates of $\overline{H_{\bar{S},\text{eff}}}\,(t)$ we find:

$$
A_i\,(w\,(t-\tau)) = \sum_{j,j'} \left\langle \overline{H_{\bar{S},\text{eff},j}}\,(t) \left| A_i\,(w\,(t-\tau)) \right| \overline{H_{\bar{S},\text{eff},j'}}\,(t) \right\rangle \left| \overline{H_{\bar{S},\text{eff},j}}\,(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\text{eff},j'}}\,(t) \right|. \tag{946}
$$

Then the time evolution is given by:

$$\widetilde{A_i}(t,t-\tau) = U(t)\left(\sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i(w(t-\tau))\right) U^\dagger(t) \tag{947}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} U(t) A_i(w(t-\tau)) U^\dagger(t) \tag{948}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} U(t)\left(\sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right|\right) U^\dagger(t) \tag{949}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right\rangle U(t) \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right| U^\dagger(t) \tag{950}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{j,j'} \left\langle \overline{H_{\bar{S},\mathrm{eff},j}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right\rangle \mathrm{e}^{-\mathrm{i}t\lambda_j(t)} \left| \overline{H_{\bar{S},\mathrm{eff},j}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},j'}}(t) \right| \mathrm{e}^{\mathrm{i}t\lambda_{j'}(t)}, \tag{951}$$

$$w'(t) = \lambda_1(t) - \lambda_0(t), \tag{952}$$

$$\widetilde{A_i}(t,t-\tau) = \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{953}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{954}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{955}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big) \tag{956}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{w'(t)} \mathrm{e}^{\mathrm{i}tw'(t)} A_{iww'}(t-\tau,t) \tag{957}$$

$$= \sum_{w(t-\tau),w'(t)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \mathrm{e}^{\mathrm{i}tw'(t)} A_{iww'}(t-\tau,t) \tag{958}$$

$$= \sum_{w(t-\tau),w'(t)} \mathrm{e}^{\mathrm{i}\tau w(t-\tau)} \mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} A_{iww'}(t-\tau,t). \tag{959}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{960}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{961}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{962}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big) \tag{963}$$

$$= \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(0) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| + \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(0) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{964}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{965}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{966}$$

$$+ \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{967}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{968}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{969}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big) \tag{970}$$

$$+ e^{\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{971}$$

$$+ \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \tag{972}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right| \tag{973}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(-w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right| \Big). \tag{974}$$

Directly we can find that the decomposition matrices are:

$$A_{i0w'}(t-\tau,t) = \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(0) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right|, \tag{975}$$

$$A_{iww'}(t-\tau,t) = \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_i(w(t-\tau)) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right\rangle\!\left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right|, \tag{976}$$

$$A_{iw(-w')}\left(t-\tau,t\right)=\left\langle \overline{H_{\bar{S},\text{eff},1}}\left(t\right)\left|A_{i}\left(w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},0}}\left(t-\tau\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},1}}\left(t-\tau\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},0}}\left(t-\tau\right)\right|, \tag{977}$$

$$A_{iw0}\left(t-\tau,t\right)=\sum_{j}\left\langle \overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\left|A_{i}\left(w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right|, \tag{978}$$

$$A_{i00}\left(t-\tau,t\right)=\sum_{j}\left\langle \overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_{i}\left(0\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right|, \tag{979}$$

$$A_{i0(-w')}\left(t-\tau,t\right)=\left\langle \overline{H_{\bar{S},\text{eff},1}}\left(t\right)\left|A_{i}\left(0\right)\right|\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\right|, \tag{980}$$

$$A_{i(-w)0}\left(t-\tau,t\right)=\sum_{j}\left\langle \overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right|, \tag{981}$$

$$A_{i(-w)w'}\left(t-\tau,t\right)=\left\langle \overline{H_{\bar{S},\text{eff},0}}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\right|, \tag{982}$$

$$A_{i(-w)(-w')}\left(t-\tau,t\right)=\left\langle \overline{H_{\bar{S},\text{eff},1}}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},0}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},1}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},0}}\left(t-\tau\right)\right|. \tag{983}$$

Let's prove that $A_{jww'}\left(t-\tau,t\right)=A_{j(-w)(-w')}^{\dagger}\left(t-\tau,t\right)$:

$$\begin{aligned}
\left(\left\langle \overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j'}}\left(t-\tau\right)\right|\right)^{\dagger} &= \left\langle \overline{H_{\bar{S},\text{eff},j}}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\right\rangle^{*} \left|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right| && (984)\\
&= \left\langle \overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\left|A_{i}^{\dagger}\left(-w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right| && (985)\\
&= \left\langle \overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\left|A_{i}\left(w\left(t-\tau\right)\right)\right|\overline{H_{\bar{S},\text{eff},j}}\left(t\right)\right\rangle \left|\overline{H_{\bar{S},\text{eff},j'}}\left(t\right)\middle\rangle\middle\langle\overline{H_{\bar{S},\text{eff},j}}\left(t-\tau\right)\right|. && (986)
\end{aligned}$$

It can be seen that the index $-w$ and $-w'$ change to the functions w and w'.

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}\left(\tau\right)=\widetilde{A_i}^{\dagger}\left(\tau\right)$ and $\widetilde{B_i}\left(\tau\right)=\widetilde{B_i}^{\dagger}\left(\tau\right)$ we can use the equation (918) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ij}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big]+\mathcal{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \tag{987}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\Big[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\overline{\rho_S}(t)\Big] \tag{988}$$

$$-\mathcal{B}_{ji}\left(-\tau\right)\Big[A_i,\overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)\Big]\Big). \tag{989}$$

Now let's consider the following trace recalling that $\mathrm{Tr}\left(A\right)^{*}=\mathrm{Tr}\left(A^{\dagger}\right)$ so:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}\left(-\tau\right)\widetilde{B_i}\left(0\right)\rho_B\right) &= \mathrm{Tr}_B\left(\mathrm{e}^{-\mathrm{i}\tau H_B(\tau)}B_j\mathrm{e}^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\right) && (990)\\
&= \mathrm{Tr}_B\left(B_j\mathrm{e}^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\mathrm{e}^{-\mathrm{i}\tau H_B(\tau)}\right) \text{ (by cyclic permutivity of trace)} && (991)\\
&= \mathrm{Tr}_B\left(B_j\mathrm{e}^{\mathrm{i}\tau H_B(\tau)}B_i\mathrm{e}^{-\mathrm{i}\tau H_B(\tau)}\rho_B\right) \text{ (by commutativity of } \mathrm{e}^{-\mathrm{i}\tau H_B(\tau)} \text{ and } \rho_B\text{)} && (992)\\
&= \mathrm{Tr}_B\left(B_j\widetilde{B_i}\left(\tau\right)\rho_B\right) \text{ (by definition of time evolution)} && (993)\\
&= \mathrm{Tr}_B\left(B_j\widetilde{B_i}\left(\tau\right)\rho_B\right) && (994)\\
&= \mathrm{Tr}_B\left(\rho_B B_j\widetilde{B_i}\left(\tau\right)\right) && (995)\\
&= \mathrm{Tr}_B\left(\left(\widetilde{B_i}\left(\tau\right)B_j\rho_B\right)^{\dagger}\right) \text{ (by definition of adjoint)} && (996)\\
&= \mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)B_j\rho_B\right)^{*} && (997)\\
&= \mathcal{B}_{ij}^{*}\left(\tau\right) && (998)
\end{aligned}$$

Given that $A_{jww'}\left(t-\tau,t\right)=A_{j(-w)(-w')}^{\dagger}\left(t-\tau,t\right)$ and $w\left(t-\tau\right),w'\left(t\right)$ belong to the set of differences of eigenvalues of $\overline{H_{\bar{S}}^{\text{eff}}}\left(t-\tau\right)$ and $\overline{H_{\bar{S}}^{\text{eff}}}\left(t\right)$ denoted by J_t and $J_{t-\tau}$ respectively that depends of the time we can take an application where $w\left(t-\tau\right)\rightarrow-w\left(t-\tau\right)$ and $w'\left(t\right)\rightarrow-w'\left(t\right)$ such that the sum:

$$\sum_{ww'}\int_0^t \mathrm{d}\tau \mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)=\sum_{ww'}\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j(-w)(-w')}\left(t-\tau,t\right) \tag{999}$$

$$=\sum_{ww'}\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right). \tag{1000}$$

is invariant because if $\left(w\left(t-\tau\right),w'\left(t\right)\right)\in J_{t-\tau}\times J_t$ then $\left(-w\left(t-\tau\right),-w'\left(t\right)\right)\in J_{t-\tau}\times J_t$ where J_t denotes the set of differences of eigenvalues at time t. So the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t)\,,\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)\,C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\left[A_i(t)\,,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\,\overline{\rho_S}(t)\right] \tag{1001}$$

$$+\mathcal{B}^*_{ij}\left(\tau\right)\left[\overline{\rho_S}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right),A_i\left(t\right)\right]\Big) \tag{1002}$$

With the definition:

$$L_{ijww'}\left(t,t'\right)\equiv\int_0^t C_i\left(t\right)C_j\left(t'\right)\mathcal{B}_{ij}\left(\tau\right)\mathrm{e}^{\mathrm{i}\tau w\left(t'\right)}\mathrm{e}^{-\mathrm{i}t\left(w\left(t'\right)-w'\left(t\right)\right)}A_{jww'}\left(t,t'\right)\mathrm{d}\tau. \tag{1003}$$

We can show that:

$$L^{\dagger}_{ijww'}\left(t,t'\right)=\int_0^t\left(C_i\left(t\right)C_j\left(t'\right)\mathcal{B}_{ij}\left(t,t'\right)\mathrm{e}^{\mathrm{i}\tau w\left(t'\right)}\mathrm{e}^{-\mathrm{i}t\left(w\left(t'\right)-w'\left(t\right)\right)}A_{jww'}\left(t,t'\right)\mathrm{d}\tau\right)^{\dagger} \tag{1004}$$

$$=\int_0^t C^*_i\left(t\right)C^*_j\left(t'\right)\mathcal{B}^*_{ij}\left(t,t'\right)\mathrm{e}^{-\mathrm{i}\tau w^*\left(t'\right)}\mathrm{e}^{\mathrm{i}t\left(w\left(t'\right)-w'\left(t\right)\right)^*}A^{\dagger}_{jww'}\left(t,t'\right)\mathrm{d}\tau \tag{1005}$$

$$=\int_0^t C_i\left(t\right)C_j\left(t'\right)\mathcal{B}^*_{ij}\left(\tau\right)\mathrm{e}^{-\mathrm{i}\tau w\left(t'\right)}\mathrm{e}^{\mathrm{i}t\left(w\left(t'\right)-w'\left(t\right)\right)}A^{\dagger}_{jww'}\left(t,t'\right)\mathrm{d}\tau. \tag{1006}$$

So we can write the master equation as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\left[A_i,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{1007}$$

$$-\mathcal{B}^*_{ij}\left(\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right)\right]\Big) \tag{1008}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_i,L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left[\overline{\rho_{\bar{S}}}\left(t\right)L^{\dagger}_{ijww'}\left(t\right),A_i\right]\right). \tag{1009}$$

If we extend the upper limit of integration to ∞ in the equation (1006) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$\mathrm{e}^{-V}\overline{AB}\mathrm{e}^{V}=\mathrm{e}^{-V}\overline{A\mathbb{I}B}\mathrm{e}^{V} \tag{1010}$$

$$=\mathrm{e}^{-V}\overline{A}\mathrm{e}^{V}\mathrm{e}^{-V}\bar{B}\mathrm{e}^{V} \tag{1011}$$

$$=\left(\mathrm{e}^{-V}\overline{A}\mathrm{e}^{V}\right)\left(\mathrm{e}^{-V}\bar{B}\mathrm{e}^{V}\right) \tag{1012}$$

$$=AB. \tag{1013}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$\mathrm{e}^{-V}\overline{[A,B]}\mathrm{e}^{V}=\mathrm{e}^{-V}\overline{(AB-BA)}\mathrm{e}^{V} \tag{1014}$$
$$=\mathrm{e}^{-V}\overline{AB}\mathrm{e}^{V}-\mathrm{e}^{-V}\overline{BA}\mathrm{e}^{V} \tag{1015}$$
$$=AB-BA \tag{1016}$$
$$=[A,B]. \tag{1017}$$

So we will obtain that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=-\mathrm{i}\mathrm{e}^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{S}}\left(t\right)\right]\mathrm{e}^{V}-\mathrm{e}^{-V}\sum_{ijww'}\left(\left[A_{i},L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]-\left[A_{i},\overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right)\right]\right)\mathrm{e}^{V} \tag{1018}$$
$$=-\mathrm{i}\mathrm{e}^{-V}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{S}}\left(t\right)\right]\mathrm{e}^{V}-\sum_{ijww'}\left(\mathrm{e}^{-V}\left[A_{i},L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]\mathrm{e}^{V}-\mathrm{e}^{-V}\left[A_{i},\overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right)\right]\mathrm{e}^{V}\right) \tag{1019}$$
$$=-\mathrm{i}\left[H\left(t\right),\rho\left(t\right)\right]-\sum_{ijww'}\left(\left[A_{i},\mathrm{e}^{-V}L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\mathrm{e}^{V}\right]-\left[A_{i},\mathrm{e}^{-V}\overline{\rho_{S}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right)\mathrm{e}^{V}\right]\right) \tag{1020}$$
$$=-\mathrm{i}\left[H(t),\rho(t)\right]-\sum_{ijww'}\left(\left[A_{i},\mathrm{e}^{-V}L_{ijww'}\left(t\right)\mathrm{e}^{V}\mathrm{e}^{-V}\overline{\rho_{S}}(t)\mathrm{e}^{V}\right]-\left[\mathrm{e}^{-V}A_{i}\mathrm{e}^{V},\mathrm{e}^{-V}\overline{\rho_{S}}(t)\mathrm{e}^{V}\mathrm{e}^{-V}L_{ijww'}\left(t\right)\mathrm{e}^{V}\right]\right) \tag{1021}$$
$$=-\mathrm{i}\left[H\left(t\right),\rho\left(t\right)\right]-\sum_{ijww'}\left(\left[A_{i},L_{ijww'}\left(t\right)\rho\left(t\right)\right]-\left[A_{i},\rho\left(t\right)L_{ijww'}\left(t\right)\right]\right). \tag{1022}$$

Our master equation in the variationally optimized frame is:

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_{i},L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left[\overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right),A_{i}\right]\right), \tag{1023}$$
$$\dot{\rho}\left(t\right)=-\mathrm{i}\left[H\left(t\right),\rho\left(t\right)\right]-\sum_{ijww'}\left(\left[A_{i},L_{ijww'}\left(t\right)\rho\left(t\right)\right]+\left[\rho\left(t\right)L_{ijww'}^{\dagger}\left(t\right),A_{i}\right]\right). \tag{1024}$$

IV. LIMIT CASES

In order to show the plausibility of the master equation (1023) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-dependent VPQME for 2LS with real-valued system Hamiltonian and real-valued uniform coupling

This hamiltonian has as particular feature that the coupling constants are real, so we know that $g_{\mathbf{k}}=g_{\mathbf{k}}^{*}$ then:

$$H_{T}\left(t\right)=H_{S}\left(t\right)+H_{I}+H_{B}, \tag{1025}$$
$$H_{S}\left(t\right)=\sum_{i}\varepsilon_{i}\left(t\right)\left|i\right\rangle\!\left\langle i\right|+\sum_{i\neq j}V_{ij}\left(t\right)\left|i\right\rangle\!\left\langle j\right|, \tag{1026}$$
$$H_{I}=\sum_{i}\left|i\right\rangle\!\left\langle i\right|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right), \tag{1027}$$
$$H_{B}=\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{1028}$$

The transformed hamiltonian is:

$$\overline{H_{\bar{S}}}\left(t\right)\equiv\sum_{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\left|i\right\rangle\!\left\langle i\right|+\sigma_{x}\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right)-\sigma_{y}\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right), \tag{1029}$$

We can summarize the principal results of the elements of the variational parameters and the transformed hamiltonians as:

$$\overline{H_{\bar{S}}}(t) \equiv \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| + \sigma_x B_{10}(t) V_{10}(t) - \sigma_y B_{10}(t) V_{10}(t), \tag{1030}$$

$$R_i(t) = \int_0^\infty \frac{J(\omega)}{\omega} \left(F_i^2(\omega,t) - 2F_i(\omega,t)\right) \mathrm{d}\omega, \tag{1031}$$

$$\chi_{ij}(t) = 0, \tag{1032}$$

$$B_{ij}(t) = \mathrm{e}^{-\frac{1}{2}\int_0^\infty \frac{J(\omega)\left(F_i(\omega,t)-F_j(\omega,t)\right)^2}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{1033}$$

$$F_i(\omega,t) = \frac{\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{F_{i'}(\omega,t)g(\omega)}{\omega}B_{10}^2(t)V_{10}^2(t)\coth(\beta\omega/2)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2B_{10}^2(t)V_{10}^2(t)\coth(\beta\omega/2)}{\omega}\right)}, \tag{1034}$$

$$\eta(t) \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}, \tag{1035}$$

$$\varepsilon(t) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{1036}$$

$$J(\omega) \equiv \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}). \tag{1037}$$

The Fourier decomposition remains without change:

$$L_{ijww'}(t,t') \equiv \int_0^t C_i(t) C_j(t') \mathcal{B}_{ij}(t,t') \mathrm{e}^{\mathrm{i}\tau w(t')} \mathrm{e}^{-\mathrm{i}t\left(w(t')-w'(t)\right)} A_{jww'}(t,t')\,\mathrm{d}\tau, \tag{1038}$$

$$t' = t - \tau, \tag{1039}$$

$$A \equiv \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \end{pmatrix}, \tag{1040}$$

$$C(t) \equiv \begin{pmatrix} V_{10}(t) & V_{10}(t) & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{1041}$$

$$A_{j00}(t,t') = \sum_i \left\langle \overline{H_{\bar{S},\mathrm{eff},i}}(t) \left| A_{j0}(t') \right| \overline{H_{\bar{S},\mathrm{eff},i}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},i}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},i}}(t) \right|, \tag{1042}$$

$$A_{j0w'}(t,t') = \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_{j0}(t') \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right|, \tag{1043}$$

$$A_{jw0}(t,t') = \sum_i \left\langle \overline{H_{\bar{S},\mathrm{eff},i}}(t) \left| A_{jw}(t') \right| \overline{H_{\bar{S},\mathrm{eff},i}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},i}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},i}}(t) \right|, \tag{1044}$$

$$A_{jww'}(t,t') = \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \left| A_{jw}(t') \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \right|, \tag{1045}$$

$$A_{jw(-w')}(t,t') = \left\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t) \left| A_{jw}(t') \right| \overline{H_{\bar{S},\mathrm{eff},0}}(t-\tau) \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},1}}(t) \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t) \right|, \tag{1046}$$

$$A_{j(-w)(-w')}(t,t') = A_{jww'}^\dagger(t,t') \tag{1047}$$

$$A_{j0}(t') = \sum_i \left\langle \overline{H_{\bar{S},\mathrm{eff},i}}(t') \left| A_j(t) \right| \overline{H_{\bar{S},\mathrm{eff},i}}(t') \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},i}}(t') \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},i}}(t') \right|, \tag{1048}$$

$$A_{jw}(t') = \left\langle \overline{H_{\bar{S},\mathrm{eff},0}}(t') \left| A_j(t) \right| \overline{H_{\bar{S},\mathrm{eff},1}}(t') \right\rangle \left| \overline{H_{\bar{S},\mathrm{eff},0}}(t') \middle\rangle\middle\langle \overline{H_{\bar{S},\mathrm{eff},1}}(t') \right|, \tag{1049}$$

$$A_{j(-w)}(t') = A_{jw}^\dagger(t'). \tag{1050}$$

The effective hamiltonian is:

$$H_X^{\mathrm{eff}}(t) \equiv \frac{1}{t}\int_0^t H_X(t')\mathrm{d}t' - \frac{\mathrm{i}}{2t}\int_0^t\int_0^{t'}[H_X(t'), H_X(t'')]\mathrm{d}t'\mathrm{d}t'' + \frac{1}{6t}\int_0^t\int_0^{t'}\int_0^{t''}\left(\left[\left[\overline{H_{\bar{S}}}(t'),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}(t')\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t''' + \cdots, \tag{1051}$$

The correlation functions are:

$$\mathcal{B}\left(t,t'\right)\equiv\begin{pmatrix}\mathcal{B}_{11}\left(t,t'\right) & 0 & 0 & 0 & \mathcal{B}_{11}\left(t,t'\right) & 0\\ 0 & \mathcal{B}_{22}\left(t,t'\right) & \mathcal{B}_{23}\left(t,t'\right) & \mathcal{B}_{22}\left(t,t'\right) & 0 & \mathcal{B}_{26}\left(t,t'\right)\\ 0 & \mathcal{B}_{32}\left(t,t'\right) & \mathcal{B}_{33}\left(t,t'\right) & \mathcal{B}_{32}\left(t,t'\right) & 0 & \mathcal{B}_{36}\left(t,t'\right)\\ 0 & \mathcal{B}_{22}\left(t,t'\right) & \mathcal{B}_{23}\left(t,t'\right) & \mathcal{B}_{22}\left(t,t'\right) & 0 & \mathcal{B}_{26}\left(t,t'\right)\\ \mathcal{B}_{11}\left(t,t'\right) & 0 & 0 & 0 & \mathcal{B}_{11}\left(t,t'\right) & 0\\ 0 & \mathcal{B}_{62}\left(t,t'\right) & \mathcal{B}_{63}\left(t,t'\right) & \mathcal{B}_{62}\left(t,t'\right) & 0 & \mathcal{B}_{66}\left(t,t'\right)\end{pmatrix}, \tag{1052}$$

$$v_{i\mathbf{k}}^{*}\left(t\right)=v_{i\mathbf{k}}\left(t\right), \tag{1053}$$

$$\chi_{10}\left(t\right)=\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}^{*}\left(t\right)v_{j\mathbf{k}}\left(t\right)-v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}^{*}\left(t\right)}{2\omega_{\mathbf{k}}^{2}}\right) \tag{1054}$$

$$=\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}\left(t\right)-v_{i\mathbf{k}}\left(t\right)v_{j\mathbf{k}}\left(t\right)}{2\omega_{\mathbf{k}}^{2}}\right) \tag{1055}$$

$$=0, \tag{1056}$$

$$B_{10}\left(t\right)=B_{10}^{*}\left(t\right), \tag{1057}$$

$$Q_{ij}(\omega,t)=Q_{ij}^{*}(\omega,t), \tag{1058}$$

$$\zeta_{ij}\left(t,t'\right)=\mathrm{e}^{\mathrm{i}\Im\left(\int_{0}^{\infty}\frac{\left(L_{i}(\omega)F_{i}(\omega,t)-L_{j}(\omega)F_{j}(\omega,t)\right)\left(L_{i}(\omega)F_{i}(\omega,t')-L_{j}(\omega)F_{j}(\omega,t')\right)^{*}\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^{2}}\mathrm{d}\omega\right)} \tag{1059}$$

$$=\mathrm{e}^{\mathrm{i}\Im\left(\int_{0}^{\infty}\frac{\left(L_{i}(\omega)F_{i}(\omega,t)-L_{j}(\omega)F_{j}(\omega,t)\right)\left(L_{i}(\omega)F_{i}(\omega,t')-L_{j}(\omega)F_{j}(\omega,t')\right)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^{2}}\mathrm{d}\omega\right)} \tag{1060}$$

$$=\zeta_{ji}\left(t,t'\right), \tag{1061}$$

$$\mathcal{B}_{11}(t,t')=\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}^{\Re}\left(t\right)B_{01}^{\Re}\left(t'\right) \tag{1062}$$

$$=\frac{1}{2}\left(\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}\left(t\right)B_{01}\left(t'\right) \tag{1063}$$

$$=\frac{1}{2}\left(\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}\left(t\right)B_{01}\left(t'\right), \tag{1064}$$

$$\mathcal{B}_{22}(t,t')=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\xi_{10}^{+}(t,t')-\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}(t,t')\xi_{10}^{-}(t,t')\right)+B_{01}^{\Im}(t)B_{10}^{\Im}(t') \tag{1065}$$

$$=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right) \tag{1066}$$

$$=-\frac{1}{2}\left(\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right), \tag{1067}$$

$$\mathcal{B}_{12}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) \tag{1068}$$

$$=\frac{1}{2}\left(\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) \tag{1069}$$

$$=\frac{1}{2}\left(0\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+0\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)0 \tag{1070}$$

$$=0, \tag{1071}$$

$$\mathcal{B}_{21}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Im}\left(t\right)B_{10}^{\Re}\left(t'\right) \tag{1072}$$

$$=\frac{1}{2}\left(\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+0B_{10}^{\Re}\left(t'\right) \tag{1073}$$

$$=0, \tag{1074}$$

$$\mathcal{B}_{i2}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t'\right)\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega,i\in\{3,6\} \tag{1075}$$

$$=\mathrm{i}B_{10}\left(t'\right)\int_{0}^{\infty}\left(P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega,i\in\{3,6\} \tag{1076}$$

$$=\mathrm{i}B_{10}\left(t'\right)\int_{0}^{\infty}P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega,i\in\{3,6\}, \tag{1077}$$

$$
\begin{aligned}
\mathcal{B}_{2i}\left(t,t'\right) &= \mathrm{i}B_{10}^{\Re}\left(t\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\left\{3,6\right\}, && (1078)\\
&= \mathrm{i}B_{10}\left(t\right)\int_0^\infty \left(P_i\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\left\{3,6\right\}, && (1079)\\
&= \mathrm{i}B_{10}\left(t\right)\int_0^\infty P_i\left(\omega,t'\right)Q_{10}\left(\omega,t\right)\left(N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\left\{3,6\right\}, && (1080)\\
P_i\left(\omega,t\right) &= P_i^*\left(\omega,t\right), && (1081)\\
\mathcal{B}_{ij}\left(t,t'\right) &= \int_0^\infty \left(P_i\left(\omega,t\right)P_j^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_i^*\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\left\{3,6\right\}, && (1082)\\
&= \int_0^\infty \left(P_i\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_i\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\left\{3,6\right\}, && (1083)\\
&= \int_0^\infty P_i\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}\left(N\left(\omega\right)+\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\left\{3,6\right\}, && (1084)\\
\mathcal{B}_{i1}\left(t,t'\right) &= \mathrm{i}B_{01}^{\Im}\left(t'\right)\int_0^\infty \left(P_i\left(\omega,t\right)Q_{10}^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i^*\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega && (1085)\\
&= \mathrm{i}0\int_0^\infty \left(P_i\left(\omega,t\right)Q_{10}^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i^*\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega && (1086)\\
&= 0, i\in\left\{3,6\right\}, && (1087)\\
\mathcal{B}_{1i}\left(t,t'\right) &= \mathrm{i}B_{01}^{\Im}\left(t\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega && (1088)\\
&= \mathrm{i}0\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega && (1089)\\
&= 0, i\in\left\{3,6\right\}, && (1090)\\
\zeta_{ij}\left(t,t'\right) &= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i\left(\omega,t'\right)-L_j(\omega)F_j\left(\omega,t'\right)\right)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)}, && (1091)\\
\xi_{ij}^{\pm}\left(t,t'\right) &= \mathrm{e}^{-\int_0^\infty \frac{\left|\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm\left(L_i(\omega)F_i\left(\omega,t'\right)-L_j(\omega)F_j\left(\omega,t'\right)\right)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, && (1092)\\
P_i\left(\omega,t\right) &= L_{i\mathrm{mod}2}\left(\omega\right)\left(1-F_{i\mathrm{mod}2}\left(\omega,t\right)\right), && (1093)\\
Q_{ij}(\omega,t) &= \frac{L_i\left(\omega\right)F_1\left(\omega,t\right)-L_j\left(\omega\right)F_j\left(\omega,t\right)}{\omega}. && (1094)
\end{aligned}
$$

B. Time-independent variational quantum master equation

At first let's show that the master equation (1023) reproduces the results of the reference [1], for the latter case we have that $i,j\in\{1,2,3\}$ and $\omega\in(0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$
H=\left(\delta+\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \qquad (1095)
$$

After performing the transformation (25) on the Hamiltonian (1095) it's possible to split that result in the following set of Hamiltonians:

$$
\begin{aligned}
\overline{H_S} &= \left(\delta+R\right)|1\rangle\langle 1|+\frac{\Omega_r}{2}\sigma_x, && (1096)\\
\overline{H_I} &= B_z|1\rangle\langle 1|+\frac{\Omega}{2}\left(B_x\sigma_x+B_y\sigma_y\right), && (1097)\\
H_B &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. && (1098)
\end{aligned}
$$

The Hamiltonian (1096) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$, where $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ (this base for the Pauli matrix is different from the base assumed in PRB2011 which is $\sigma_z' = |1\rangle\langle 1| - |0\rangle\langle 0|$) by a term proportional to the identity given by $-\frac{\delta}{2}\mathbb{I}$ which is independent of the variational parameters, this can be seen in the following way, with $\epsilon = \delta + R$:

$$
\begin{aligned}
(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} &= (\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| - \frac{\delta}{2}|1\rangle\langle 1| && (1099)\\
&= \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1100)\\
&= \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1101)\\
&= \frac{R}{2}|1\rangle\langle 1| + \left(\frac{\delta}{2} + \frac{R}{2}\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1102)\\
&= \frac{R}{2}|1\rangle\langle 1| + \frac{R}{2}|0\rangle\langle 0| + \left(\frac{\delta}{2} + \frac{R}{2}\right)|1\rangle\langle 1| - \frac{R}{2}|0\rangle\langle 0| - \frac{\delta}{2}|0\rangle\langle 0| && (1103)\\
&= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) && (1104)\\
&= \frac{R}{2}\mathbb{I} - \frac{\delta + R}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) && (1105)\\
&= \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) && (1106)\\
&= \frac{R}{2}\mathbb{I} - \frac{\epsilon}{2}\sigma_z. && (1107)
\end{aligned}
$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$
\begin{aligned}
0 &= \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, && (1108)\\
\lambda_\pm &= \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} && (1109)\\
&= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} && (1110)\\
&= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} && (1111)\\
\eta &= \sqrt{\epsilon^2 + \Omega_r^2}, && (1112)\\
\lambda_\pm &= \frac{R \pm \eta}{2}. && (1113)
\end{aligned}
$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$
\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \quad (1114)
$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle$.

For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{1115}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle$. Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon+\eta}{2}, \tag{1116}$$
$$\lambda_- = \frac{\epsilon-\eta}{2}, \tag{1117}$$
$$|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle\,, \tag{1118}$$
$$|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle\,, \tag{1119}$$
$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}, \tag{1120}$$
$$\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}. \tag{1121}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{1122}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{1123}$$

So the value of $\tan(\theta)$ using (1123) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}+1} \tag{1124}$$
$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{1125}$$
$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{1126}$$

This proves our assertion.
Using this basis we can find the decomposition matrices using the equations (940)-(942) and the fact that $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1127}$$
$$= 2\sin(\theta)\cos(\theta) \tag{1128}$$
$$= \sin(2\theta)\,, \tag{1129}$$
$$\langle -|\sigma_x|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{1130}$$
$$= -2\sin(\theta)\cos(\theta) \tag{1131}$$
$$= -\sin(2\theta)\,, \tag{1132}$$
$$\langle -|\sigma_x|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1133}$$
$$= \cos^2(\theta) - \sin^2(\theta) \tag{1134}$$
$$= \cos(2\theta)\,, \tag{1135}$$
$$\langle +|\sigma_y|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1136}$$
$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{1137}$$
$$= 0, \tag{1138}$$
$$\langle -|\sigma_y|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{1139}$$
$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{1140}$$
$$= 0, \tag{1141}$$
$$\langle -|\sigma_y|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1142}$$
$$= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) \tag{1143}$$
$$= \mathrm{i}. \tag{1144}$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1145}$$
$$= \cos(\theta)\cos(\theta) \tag{1146}$$
$$= \cos^2(\theta)\,, \tag{1147}$$
$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{1148}$$
$$= \sin(\theta)\sin(\theta) \tag{1149}$$
$$= \sin^2(\theta)\,, \tag{1150}$$
$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1151}$$
$$= -\sin(\theta)\cos(\theta) \tag{1152}$$
$$= -\sin(\theta)\cos(\theta)\,. \tag{1153}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), \tag{1154}$$
$$A_1(\eta) = \cos(2\theta)\,|-\rangle\langle+|, \tag{1155}$$

$$A_2(0) = 0, \tag{1156}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{1157}$$
$$A_3(0) = \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-|, \tag{1158}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)\,|-\rangle\langle+|. \tag{1159}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (1023) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (1003) is equivalent to:

$$L_{ijww'}(t,t-\tau) \equiv \int_0^t C_i(t)\,C_j(t-\tau)\,\mathcal{B}_{ij}(t,t-\tau)\,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t,t-\tau)\,\mathrm{d}\tau, \tag{1160}$$
$$= \int_0^t C_i(t)\,C_j(t-\tau)\,\mathcal{B}_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}\tau w}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}A_j(w,w')\,\mathrm{d}\tau. \tag{1161}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda_{ij}(\tau)$ relate with the correlation functions defined in the equation (414) in the following way:

$$\Lambda_{ij}(\tau) = C_i(t)\,C_j(t-\tau)\,\mathcal{B}_{ij}(\tau). \tag{1162}$$

So the response matrix can be rewritten as:

$$L_{ijww'}(t,t-\tau) = \left(\int_0^t \mathrm{d}\tau\Lambda_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}\tau w}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}\right)A_j(w,w'). \tag{1163}$$

Let's define the response function like:

$$K_{ij}(w,w',t) = \int_0^t C_i(t)\,C_j(t-\tau)\,\mathcal{B}_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}w\tau}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{1164}$$
$$= \int_0^t \Lambda_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}w\tau}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{1165}$$
$$= K_{ijww'}(t). \tag{1166}$$

Then we have the following equivalence:

$$L_{ijww'}(t) = K_{ijww'}(t)\,A_j(w,w') \tag{1167}$$
$$= K_{ijww'}(t)\,A_{jww'}. \tag{1168}$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, L_{ijww'}(t)\,\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\,L^{\dagger}_{ijww'}(t)\right]\right) \tag{1169}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, K_{ijww'}(t)\,A_{jww'}\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\,K^{*}_{ijww'}(t)\,A^{\dagger}_{jww'}\right]\right) \tag{1170}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(K_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho_S}(t)\right] - K^{*}_{ijww'}(t)\left[A_i, \overline{\rho_S}(t)\,A^{\dagger}_{jww'}\right]\right) \tag{1171}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t)+\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,A_{jww'}\overline{\rho_S}(t)\right] - \left(K^{\Re}_{ijww'}(t)-\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\overline{\rho_S}(t)A^{\dagger}_{jww'}\right]\right) \tag{1172}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}K^{\Re}_{ijww'}(t)\left[A_i,A_{jww'}\overline{\rho_S}(t)-\overline{\rho_S}(t)A^{\dagger}_{jww'}\right] - \mathrm{i}\sum_{ijww'}K^{\Im}_{ijww'}(t)\left[A_i,A_{jww'}\overline{\rho_S}(t)+\overline{\rho_S}(t)A^{\dagger}_{jww'}\right] \tag{1173}$$

For the time-independent PRB2011 we have the following correlations obtained from the general model, we take account from the fact that $L_0(\omega) = 0$, $\Im(L_1(\omega)) = 0$, $\Im(F_1(\omega)) = 0$, $F_0(\omega) = 0$ and $\int_0^\infty |L_1(\omega)|^2 f(\omega)\,\mathrm{d}\omega = \int_0^\infty J(\omega) f(\omega)\,\mathrm{d}\omega$ for $f(\omega) \in L^2$. We can drop the time vector (t, t') and instead write the correlation functions as function of τ, we will drop t and t' from the expressions that contain them given the time independence of the hamiltonian:

$$\chi_{ij}(t) = \int_0^\infty \frac{L_1^*(\omega) L_0(\omega) F_1^*(\omega,t) F_0(\omega,t) - L_1(\omega) L_0^*(\omega) F_1(\omega,t) F_0^*(\omega,t)}{2\omega^2}\mathrm{d}\omega, \tag{1174}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^+(t,t') + \Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^*(t,t')\,\xi_{10}^-(t,t')\right) - B_{10}^{\Re}(t)\,B_{01}^{\Re}(t'), \tag{1175}$$

$$= \frac{1}{2}\left(\mathrm{e}^{\chi_{10}+\chi_{10}}\zeta_{10}\xi_{10}^+ + \mathrm{e}^{\chi_{10}+\chi_{01}}\zeta_{10}^*\xi_{10}^-\right) - B^2, \tag{1176}$$

$$\chi_{ij} = \int_0^\infty \frac{L_1(\omega) L_0(\omega) F_1(\omega) F_0(\omega) - L_1(\omega) L_0(\omega) F_1(\omega) F_0(\omega)}{2\omega^2}\mathrm{d}\omega, \tag{1177}$$

$$= 0 \tag{1178}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\mathrm{e}^0\zeta_{10}\xi_{10}^+ + \mathrm{e}^0\zeta_{10}^*\xi_{10}^-\right) - B^2, \tag{1179}$$

$$\zeta_{10} = \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{(L_1(\omega)F_1(\omega) - L_0(\omega)F_0(\omega))(L_1(\omega)F_1(\omega) - L_0(\omega)F_0(\omega))^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1180}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{(L_1(\omega)F_1(\omega))(L_1(\omega)F_1(\omega))^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1181}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{J(\omega)F^2(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1182}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}(\cos(\omega\tau)+\mathrm{i}\sin(\omega\tau))\mathrm{d}\omega\right)} \tag{1183}$$

$$= \mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega} \tag{1184}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\xi_{10}^+ + \mathrm{e}^{-\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\xi_{10}^-\right) - B^2, \tag{1185}$$

$$\xi_{10}^{\pm} = \mathrm{e}^{-\int_0^\infty \frac{\left|(L_1(\omega)F_1(\omega) - L_0(\omega)F_0(\omega))\mathrm{e}^{\mathrm{i}\omega\tau} \pm L_1(\omega)F_1(\omega) \mp L_0(\omega)F_0(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1186}$$

$$= \mathrm{e}^{-\int_0^\infty \frac{\left|L_1(\omega)F_1(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} \pm L_1(\omega)F_1(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1187}$$

$$= {}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left|\mathrm{e}^{\mathrm{i}\omega\tau}\pm 1\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1188}$$

$$\left|\mathrm{e}^{\mathrm{i}\omega\tau} \pm 1\right|^2 = 2\left(1 \pm \cos(\omega\tau)\right) \tag{1189}$$

$$\xi_{10}^{\pm} = \mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1\pm\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1190}$$

$$\mathcal{B}_{11}(\tau) = -B^2 + \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1+\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right. \tag{1191}$$

$$\left. + \mathrm{e}^{-\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1-\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right) \tag{1192}$$

$$= -B^2 + \frac{\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}}{2}\left(\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right. \tag{1193}$$

$$\left. + \mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(-\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) + \mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right) \tag{1194}$$

$$B = \mathrm{e}^{-\frac{1}{2}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{1195}$$

$$G_+(\omega) = \mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + \mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega) + 1\right) \tag{1196}$$

$$
\begin{aligned}
&= \left(\cos\left(\omega\tau\right) + \mathrm{i}\sin\left(\omega\tau\right)\right) N\left(\omega\right) + \left(\cos\left(\omega\tau\right) - \mathrm{i}\sin\left(\omega\tau\right)\right)\left(N\left(\omega\right)+1\right) && (1197)\\
&= \cos\left(\omega\tau\right)\left(2N\left(\omega\right)+1\right) - \mathrm{i}\sin\left(\omega\tau\right) && (1198)\\
&= \cos\left(\omega\tau\right)\left(\frac{2}{\mathrm{e}^{\beta\omega}-1}+1\right) - \mathrm{i}\sin\left(\omega\tau\right) && (1199)\\
&= \cos\left(\omega\tau\right)\left(\frac{1+\mathrm{e}^{\beta\omega}}{\mathrm{e}^{\beta\omega}-1}\right) - \mathrm{i}\sin\left(\omega\tau\right) && (1200)\\
&= \cos\left(\omega\tau\right)\left(\frac{\mathrm{e}^{-\beta\omega/2}+\mathrm{e}^{\beta\omega/2}}{-\mathrm{e}^{-\beta\omega/2}+\mathrm{e}^{\beta\omega/2}}\right) - \mathrm{i}\sin\left(\omega\tau\right) && (1201)\\
&= \cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right), && (1202)\\
\phi\left(\tau\right) &= \int_0^\infty \frac{J\left(\omega\right)F^2\left(\omega\right)}{\omega^2} G_+\left(\omega,\tau\right)\mathrm{d}\omega && (1203)\\
&= \int_0^\infty \frac{J\left(\omega\right)F^2\left(\omega\right)}{\omega^2}\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right) - \mathrm{i}\sin\left(\omega\tau\right)\right)\mathrm{d}\omega, && (1204)\\
\mathcal{B}_{11}(\tau) &= -B^2 + \frac{\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}}{2}\left(\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right. && (1205)\\
&\left.+\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(-\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)+\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right) && (1206)\\
&= \frac{B^2}{2}\left(\mathrm{e}^{-\phi(\tau)}+\mathrm{e}^{\phi(\tau)}-2\right) && (1207)\\
\mathcal{B}_{22}(\tau) &= -\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\xi_{10}^{+}(t,t') - \Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}(t,t')\xi_{10}^{-}(t,t')\right) + B_{01}^{\Im}(t)B_{10}^{\Im}(t'), && (1208)\\
&= -\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}+\chi_{01}}\right)\zeta_{10}\xi_{10}^{+} - \left(\mathrm{e}^{\chi_{10}+\chi_{01}}\right)\zeta_{10}^{*}\xi_{10}^{-}\right), && (1209)\\
&= \frac{1}{2}\left(\zeta_{10}^{*}\xi_{10}^{-} - \zeta_{10}\xi_{10}^{+}\right), && (1210)\\
&= \frac{1}{2}\left(\mathrm{e}^{-\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1-\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right. && (1211)\\
&\left.-\mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1+\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\right) && (1212)\\
&= \frac{1}{2}\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\left(\mathrm{e}^{\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right. && (1213)\\
&\left.-\mathrm{e}^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}{\omega^2}\mathrm{d}\omega}\right) && (1214)\\
&= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}-\mathrm{e}^{-\phi(\tau)}\right), && (1215)\\
\mathcal{B}_{12}(\tau) &= \frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right) + \Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right) + B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) && (1216)\\
&= 0, && (1217)\\
\mathcal{B}_{21}(\tau) &= \frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^{*}(t,t')\,\xi_{10}^{-}(t,t') + \Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^{+}(t,t')\right) + B_{10}^{\Im}(t)\,B_{10}^{\Re}(t'), && (1218)\\
&= 0, && (1219)\\
\mathcal{B}_{ij}\left(t,t'\right) &= \int_0^\infty \left(P_i\left(\omega,t\right)P_j^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right) + P_i^{*}\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\{3,6\}, && (1220)\\
\mathcal{B}_{66}\left(t,t'\right) &= \int_0^\infty \left(P_6\left(\omega,t\right)P_6^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right) + P_6^{*}\left(\omega,t\right)P_6\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega && (1221)
\end{aligned}
$$

$$P_6(\omega,t) = L_{6\mathrm{mod}2}(\omega)\left(1-F_{6\mathrm{mod}2}(\omega,t)\right), \tag{1222}$$

$$= L_0(\omega)\left(1-F_0(\omega,t)\right), \tag{1223}$$

$$= 0, \tag{1224}$$

$$\mathcal{B}_{66}(\tau) = 0, \tag{1225}$$

$$\mathcal{B}_{36}(t,t') = \int_0^\infty \left(P_3(\omega,t)P_6^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)P_6(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1226}$$

$$= 0, \tag{1227}$$

$$\mathcal{B}_{63}(t,t') = \int_0^\infty \left(P_6(\omega,t)P_3^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_6^*(\omega,t)P_3(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1228}$$

$$= 0, \tag{1229}$$

$$\mathcal{B}_{33}(t,t') = \int_0^\infty \left(P_3(\omega,t)P_3^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)P_3(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1230}$$

$$= \int_0^\infty \left(P_3(\omega,t)P_3^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)P_3(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1231}$$

$$P_3(\omega,t) = L_{3\mathrm{mod}2}(\omega)\left(1-F_{3\mathrm{mod}2}(\omega,t)\right), \tag{1232}$$

$$= L_1(\omega)\left(1-F_1(\omega,t)\right), \tag{1233}$$

$$P_3(\omega,t)P_3^*(\omega,t') = L_1(\omega)\left(1-F_1(\omega)\right)L_1^*(\omega)\left(1-F_1(\omega)\right), \tag{1234}$$

$$= |L_1(\omega)|^2\left(1-F_1(\omega)\right)^2 \tag{1235}$$

$$\mathcal{B}_{33}(t,t') = \int_0^\infty |L_1(\omega)|^2\left(1-F_1(\omega)\right)^2\left(\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + \mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1236}$$

$$= \int_0^\infty J(\omega)\left(1-F(\omega)\right)^2 G_+(\omega)\,\mathrm{d}\omega \tag{1237}$$

$$\mathcal{B}_{i1}(t,t') = \mathrm{i}B_{01}^{\Im}(t')\int_0^\infty \left(P_i(\omega,t)Q_{10}^*(\omega,t')N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_i^*(\omega,t)Q_{10}(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega,\ i\in\{3,6\} \tag{1238}$$

$$= 0, \tag{1239}$$

$$\mathcal{B}_{1i}(t,t') = \mathrm{i}B_{01}^{\Im}(t)\int_0^\infty \left(P_i^*(\omega,t')Q_{10}(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_i(\omega,t')Q_{10}^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega,\ i\in\{3,6\}, \tag{1240}$$

$$= 0, \tag{1241}$$

$$\mathcal{B}_{62}(t,t') = \mathrm{i}B_{10}^{\Re}(t')\int_0^\infty \left(P_6^*(\omega,t')Q_{10}(\omega,t')(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} - P_6(\omega,t')Q_{10}^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{1242}$$

$$= 0, \tag{1243}$$

$$\mathcal{B}_{26}(t,t') = \mathrm{i}B_{10}^{\Re}(t)\int_0^\infty \left(P_6^*(\omega,t')Q_{10}(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_6(\omega,t')Q_{10}^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1244}$$

$$= 0, \tag{1245}$$

$$\mathcal{B}_{32}(t,t') = \mathrm{i}B_{10}^{\Re}(t')\int_0^\infty \left(P_3^*(\omega,t')Q_{10}(\omega,t')(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} - P_3(\omega,t')Q_{10}^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{1246}$$

$$= \mathrm{i}B\int_0^\infty \left(P_3^*(\omega)Q_{10}(\omega)(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} - P_3(\omega)Q_{10}^*(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega, \tag{1247}$$

$$Q_{ij}(\omega,t) = \frac{L_i(\omega)F_j(\omega,t) - L_i(\omega)F_j(\omega,t)}{\omega}, \tag{1248}$$

$$Q_{10}(\omega,t) = \frac{L_1(\omega)F_1(\omega,t)}{\omega}, \tag{1249}$$

$$\mathcal{B}_{32}(t,t') = \mathrm{i}B\int_0^\infty \Bigg(L_1^*(\omega)\left(1-F_1(\omega,t)\right)\frac{L_1(\omega)F_1(\omega,t)}{\omega}(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} \tag{1250}$$

$$-L_1(\omega)\left(1-F_1(\omega,t)\right)\frac{L_1^*(\omega)F_1(\omega,t)}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\Bigg)\mathrm{d}\omega \tag{1251}$$

$$
\begin{aligned}
&= \mathrm{i}B \int_0^\infty \left|L_1(\omega)\right|^2 \left(\left(1 - F_1(\omega)\right) \frac{F_1(\omega)}{\omega} \left(N(\omega) + 1\right) \mathrm{e}^{-\mathrm{i}\omega\tau} - \left(1 - F_1(\omega)\right) \frac{F_1(\omega)}{\omega} \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) \right) \mathrm{d}\omega && (1252) \\
&= \mathrm{i}B \int_0^\infty J(\omega) \left(1 - F(\omega, t)\right) \frac{F(\omega)}{\omega} \left(\left(N(\omega) + 1\right) \mathrm{e}^{-\mathrm{i}\omega\tau} - \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) \right) \mathrm{d}\omega && (1253) \\
&= \mathrm{i}B \int_0^\infty J(\omega) \left(1 - F(\omega, t)\right) \frac{F(\omega)}{\omega} G_-(\omega) \, \mathrm{d}\omega && (1254)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_{23}(t, t') &= \mathrm{i}B_{10}^{\Re}(t) \int_0^\infty \left(P_3^*(\omega, t') Q_{10}(\omega, t) N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - P_3(\omega, t') Q_{10}^*(\omega, t) \mathrm{e}^{-\mathrm{i}\omega\tau} \left(N(\omega) + 1\right) \right) \mathrm{d}\omega && (1255) \\
&= \mathrm{i}B \int_0^\infty \left(P_3^*(\omega, t') Q_{10}(\omega, t) N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - P_3(\omega, t') Q_{10}^*(\omega, t) \mathrm{e}^{-\mathrm{i}\omega\tau} \left(N(\omega) + 1\right) \right) \mathrm{d}\omega && (1256) \\
&= \mathrm{i}B \textstyle\int_0^\infty \left(L_1^*(\omega)(1 - F_1(\omega, t)) \frac{L_1(\omega) F_1(\omega, t)}{\omega} N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - L_1(\omega)(1 - F_1(\omega, t)) \frac{L_1^*(\omega) F_1(\omega, t)}{\omega} \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega) + 1) \right) \mathrm{d}\omega && (1257) \\
&= \mathrm{i}B \int_0^\infty J(\omega) \left(1 - F_1(\omega, t)\right) \frac{F_1(\omega, t)}{\omega} \left(N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} - \mathrm{e}^{-\mathrm{i}\omega\tau} \left(N(\omega) + 1\right) \right) \mathrm{d}\omega && (1258) \\
&= -\mathrm{i}B \int_0^\infty J(\omega) \left(1 - F_1(\omega, t)\right) \frac{F_1(\omega, t)}{\omega} \left(-N(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} + \mathrm{e}^{-\mathrm{i}\omega\tau} \left(N(\omega) + 1\right) \right) \mathrm{d}\omega && (1259) \\
&= -\mathcal{B}_{32}(t, t') && (1260)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{ij}(t, t') &= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{\left(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t)\right)\left(L_i(\omega) F_i(\omega, t') - L_j(\omega) F_j(\omega, t')\right)^* \mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2} \mathrm{d}\omega \right)}, && (1261) \\
\xi_{ij}^{\pm}(t, t') &= \mathrm{e}^{-\int_0^\infty \frac{\left| \left(L_i(\omega) F_i(\omega, t) - L_j(\omega) F_j(\omega, t)\right) \mathrm{e}^{\mathrm{i}\omega\tau} \pm L_i(\omega) F_i(\omega, t') \mp L_j(\omega) F_j(\omega, t') \right|^2}{2\omega^2} \coth\left(\frac{\beta\omega}{2}\right) \mathrm{d}\omega}, && (1262) \\
P_i(\omega, t) &= L_{i\mathrm{mod}2}(\omega) \left(1 - F_{i\mathrm{mod}2}(\omega, t)\right), && (1263) \\
Q_{ij}(\omega, t) &= \frac{L_i(\omega) F_j(\omega, t) - L_i(\omega) F_j(\omega, t)}{\omega}, && (1264) \\
\mathcal{B}(\tau) &\equiv \begin{pmatrix}
\mathcal{B}_{11}(\tau) & 0 & 0 & 0 & \mathcal{B}_{11}(\tau) & 0 \\
0 & \mathcal{B}_{22}(\tau) & \mathcal{B}_{23}(\tau) & \mathcal{B}_{22}(\tau) & 0 & 0 \\
0 & -\mathcal{B}_{23}(\tau) & \mathcal{B}_{33}(\tau) & -\mathcal{B}_{23}(\tau) & 0 & 0 \\
0 & \mathcal{B}_{22}(\tau) & \mathcal{B}_{23}(\tau) & \mathcal{B}_{22}(\tau) & 0 & 0 \\
\mathcal{B}_{11}(\tau) & 0 & 0 & 0 & \mathcal{B}_{11}(\tau) & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. && (1265)
\end{aligned}
$$

The correlation functions as we can see in PRB2011 can be obtained using the following definition:

$$
\Lambda_{ij}(\tau) \equiv C_i C_j \mathcal{B}_{ij}(\tau) \tag{1266}
$$

Also the matrix C(t) can be decomposed as:

$$
\begin{aligned}
C_1 = C_2 &= \frac{\Omega}{2}, && (1267) \\
C_3 = C_6 &= 1, && (1268) \\
C_4 = C_4 &= 0 && (1269)
\end{aligned}
$$

Let's recall that $\Omega_r = B\Omega$. So the correlation functions Λ are:

$$
\begin{aligned}
\Lambda_{11}(\tau) &= C_1 C_1 \mathcal{B}_{11}(\tau) && (1270) \\
&= \left(\frac{\Omega}{2}\right)^2 \frac{B^2}{2} \left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2 \right) && (1271)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\Omega B)^2}{8}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}-2\right) && (1272)\\
&= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}-2\right), && (1273)\\
\Lambda_{22}(\tau) &= C_2C_2\mathcal{B}_{22}(\tau) && (1274)\\
&= \left(\frac{\Omega}{2}\right)^2\frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}-\mathrm{e}^{-\phi(\tau)}\right) && (1275)\\
&= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)}-\mathrm{e}^{-\phi(\tau)}\right), && (1276)\\
\Lambda_{33}(\tau) &= C_3C_3\mathcal{B}_{33}(\tau) && (1277)\\
&= (1)^2\int_0^\infty J(\omega)(1-F(\omega))^2 G_+(\omega)\,\mathrm{d}\omega && (1278)\\
&= \int_0^\infty J(\omega)(1-F(\omega))^2 G_+(\omega)\,\mathrm{d}\omega, && (1279)\\
\Lambda_{23}(\tau) &= C_2C_3\mathcal{B}_{23}(\tau) && (1280)\\
&= \frac{\Omega}{2}1\mathrm{i}B\int_0^\infty J(\omega)(1-F(\omega,t))\frac{F(\omega)}{\omega}G_-(\omega)\,\mathrm{d}\omega && (1281)\\
&= \mathrm{i}\frac{\Omega_r}{2}\int_0^\infty J(\omega)(1-F(\omega,t))\frac{F(\omega)}{\omega}G_-(\omega)\,\mathrm{d}\omega, && (1282)\\
\Lambda_{12}(\tau) &= \Lambda_{13}(\tau)=\Lambda_{16}(\tau) && (1283)\\
&= \Lambda_{21}(\tau)=\Lambda_{26}(\tau) && (1284)\\
&= \Lambda_{31}(\tau)=\Lambda_{36}(\tau)=0. && (1285)
\end{aligned}
$$

Now let's define:

$$K_{ijw}(t)=\int_0^t C_iC_j\mathcal{B}_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}w\tau}\mathrm{d}\tau. \tag{1286}$$

So

$$L_{ijw}(t)=A_{jw}K_{ijw}(t). \tag{1287}$$

Now for a time-independent hamiltonian is possible to show that for the decomposition matrix $A_j(w(t))=A_j(w)$:

$$U^\dagger(t)A_j(t)U(t)=\sum_w \mathrm{e}^{-\mathrm{i}wt}A_j(w). \tag{1288}$$

It means that a decomposition matrix of $\widetilde{A_j}(t)$ associated to the eigenvector under evolution for the same time-independent hamiltonian $U(t)A_j(w)U^\dagger(t)$ generates the same decomposition matrix multiplied by a phase $\mathrm{e}^{\mathrm{i}wt}$. It means that the decomposition matrix $A_{jww'}$ for a time-independent hamiltonian fulfill $A_{jww'}=A_j(w)\delta_{ww'}$ so only if $w=w'$ then the response function is relevant for taking account and it's equal to:

$$
\begin{aligned}
K_{ijww}(t) &= \int_0^t C_iC_j\mathcal{B}_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}w\tau}\mathrm{d}\tau && (1289)\\
&\equiv K_{ijw}(t). && (1290)
\end{aligned}
$$

Finally taking the Hamiltonian (1095) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t)=\frac{\Omega}{2}$, $\varepsilon_0(t)=0$ and $\varepsilon_1(t)=\delta$, then we obtain that Det $\left(\overline{H_S}\right)=-\frac{\Omega_r^2}{4}$, Tr $\left(\overline{H_S}\right)=\epsilon$. Now $\eta=\sqrt{\epsilon^2+\Omega_r^2}$ and using the equation (338) we have that:

$$f_k = \frac{g_k\left(1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1-\frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon-\frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{1291}$$

$$= \frac{g_k\left(1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1-\frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{1292}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (450) under a time-independent approach providing similar results.

The master equation for this special case is:

$$L_{ijww'}(t)=\delta_{ww'}A_{jw}K_{ijw}(t), \tag{1293}$$

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijww'}\left(\left[A_i,L_{ijww'}(t)\overline{\rho_{\bar{S}}}(t)\right]+\left[\overline{\rho_{\bar{S}}}(t)L_{ijww'}^{\dagger}(t),A_i\right]\right) \tag{1294}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(\left[A_i,L_{ijw}(t)\overline{\rho_{\bar{S}}}(t)\right]+\left[\overline{\rho_{\bar{S}}}(t)L_{ijw}^{\dagger}(t),A_i\right]\right) \tag{1295}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(\left[A_i,A_{jw}K_{ijw}(t)\overline{\rho_{\bar{S}}}(t)\right]+\left[\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}K_{ijw}^{*}(t),A_i\right]\right) \tag{1296}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(\left(K_{ijw}^{\Re}(t)+\mathrm{i}K_{ijw}^{\Im}(t)\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]+\left(K_{ijw}^{\Re}(t)-\mathrm{i}K_{ijw}^{\Im}(t)\right)\left[\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger},A_i\right]\right) \tag{1297}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(K_{ijw}^{\Re}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]+\left[\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger},A_i\right]\right)+\mathrm{i}K_{ijw}^{\Im}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]-\left[\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger},A_i\right]\right)\right) \tag{1298}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(K_{ijw}^{\Re}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]-\left[A_i,\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right]\right)+\mathrm{i}K_{ijw}^{\Im}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]+\left[A_i,\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right]\right)\right) \tag{1299}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(K_{ijw}^{\Re}(t)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)-\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right]+\mathrm{i}K_{ijw}^{\Im}(t)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)+\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right]\right) \tag{1300}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}K_{ijw}^{\Re}(t)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)-\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right]-\mathrm{i}\sum_{ijw}K_{ijw}^{\Im}(t)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)+\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right], \tag{1301}$$

$$\gamma_{ij}(w,t)\equiv 2K_{ijw}^{\Re}(t) \tag{1302}$$

$$S_{ij}(w,t)\equiv K_{ijw}^{\Im}(t) \tag{1303}$$

$$A_j(\omega)\equiv A_{jw} \tag{1304}$$

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\frac{1}{2}\sum_{ijw}\gamma_{ij}(w,t)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)-\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right]-\mathrm{i}\sum_{ijw}S_{ij}(w,t)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)+\overline{\rho_{\bar{S}}}(t)A_{jw}^{\dagger}\right] \tag{1305}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\frac{1}{2}\sum_{ijw}\gamma_{ij}(w,t)\left[A_i,A_j(w)\overline{\rho_{\bar{S}}}(t)-\overline{\rho_{\bar{S}}}(t)A_j^{\dagger}(w)\right]-\mathrm{i}\sum_{ijw}S_{ij}(w,t)\left[A_i,A_j(w)\overline{\rho_{\bar{S}}}(t)+\overline{\rho_{\bar{S}}}(t)A_j^{\dagger}(w)\right]. \tag{1306}$$

C. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z = 0$. The Hamiltonian studied is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega(t)}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{1307}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (1307) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (1096) and (1097) we obtain ther following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)\,|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega\,(t)\,, \tag{1308}$$

$$\overline{H_{\mathrm{I}}} = \frac{\Omega\,(t)}{2}\,(B_x\sigma_x + B_y\sigma_y)\,. \tag{1309}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F\,(\omega_{\mathbf{k}}) = 1$ and using the equations (1270)-(1285) we can deduce that the only terms that survive are $\Lambda_{11}\,(\tau)$ and $\Lambda_{22}\,(\tau)$. The phonon propagator for this case is:

$$\phi\,(\tau) = \int_0^\infty \frac{J\,(\omega)}{\omega^2}G_+\,(\tau)\,\mathrm{d}\omega. \tag{1310}$$

Writing $G_+\,(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos\,(\omega\tau) - i\sin\,(\omega\tau)$ so (1310) can be written as:

$$\phi\,(\tau) = \int_0^\infty \frac{J\,(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos\,(\omega\tau) - i\sin\,(\omega\tau)\right)\mathrm{d}\omega. \tag{1311}$$

Writing the interaction Hamiltonian (1309) in the similar way to the equation (1097) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1\,(t) = B_x$, $B_2\,(t) = B_y$ and $C_1\,(t) = \frac{\Omega(t)}{2} = C_2\,(t)$. Now taking the equation (1096) with $\delta'|1\rangle\langle 1|{=}\frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega\,(t)\,. \tag{1312}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}\,(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}\,(\tau)\,\widetilde{B_1}\,(0)\,\rho_B\right) \tag{1313}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1314}$$

$$\Lambda_{22}\,(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}\,(\tau)\,\widetilde{B_2}\,(0)\,\rho_B\right) \tag{1315}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right). \tag{1316}$$

These functions match with the equations $\Lambda_x\,(\tau)$ and $\Lambda_y\,(\tau)$ of the reference [2] and $\Lambda_i\,(\tau) = \Lambda_i\,(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(450) is:

$$\frac{\mathrm{d}\rho_S\,(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r\,(t)\,\sigma_x}{2}, \rho_S\,(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i\,(t)\,C_i\,(t-\tau)\,\Lambda_{ii}\,(\tau)\left[A_i, \widetilde{A_i}\,(t-\tau,t)\,\rho_S\,(t)\right]\right. \tag{1317}$$

$$\left.+C_i\,(t)\,C_i\,(t-\tau)\,\Lambda_{ii}\,(-\tau)\left[\rho_S\,(t)\,\widetilde{A_i}\,(t-\tau,t)\,, A_i\right]\right). \tag{1318}$$

Replacing $C_i\,(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}\,(t-\tau,t) = \widetilde{\sigma_i}\,(t-\tau,t)$, also using the equations (1313) and (1316) on the equation (1318) we obtain that:

$$\frac{\mathrm{d}\rho_S\,(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r\,(t)\,\sigma_x, \rho_S\,(t)\right] - \frac{\Omega\,(t)}{4}\int_0^t \mathrm{d}\tau\Omega\,(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}\,(t-\tau,t)\,\rho_S\,(t)\right]\Lambda_x\,(\tau)\right. \tag{1319}$$

$$\left.+\left[\sigma_y, \widetilde{\sigma_y}\,(t-\tau,t)\,\rho_S\,(t)\right]\Lambda_y\,(\tau) + \left[\rho_S\,(t)\,\widetilde{\sigma_x}\,(t-\tau,t)\,, \sigma_x\right]\Lambda_x\,(\tau) + \left[\rho_S\,(t)\,\widetilde{\sigma_y}\,(t-\tau,t)\,, \sigma_y\right]\Lambda_y\,(\tau)\right). \tag{1320}$$

As we can see $\left[A_j, \widetilde{A_i}\,(t-\tau,t)\,\rho_S\,(t)\right]^\dagger = \left[\rho_S\,(t)\,\widetilde{A_i}\,(t-\tau,t)\,, A_j\right]$, $\Lambda_x\,(\tau) = \Lambda_x\,(-\tau)$ and $\Lambda_y\,(\tau) = \Lambda_y\,(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

D. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega)=0$, so there is no transformation in this case. As we can see from the definition (414) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h}=1$ the Hamiltonian of the reference can be written in the form:

$$H=\Delta|1\rangle\langle 1|+\frac{\Omega(t)}{2}\left(|1\rangle\langle 0|+|0\rangle\langle 1|\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right). \tag{1321}$$

Using the equation (1023), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t}=-\mathrm{i}\left[H_S(t),\rho_S(t)),\rho_S(t)\right]-\frac{1}{2}\sum_{w}\gamma_{33}(w,t)\left[A_3,A_3(w)\rho_S(t)-\rho_S(t)A_3^{\dagger}(w)\right] \tag{1322}$$

$$-\sum_{w}S_{33}(w,t)\left[A_3,A_3(w)\rho_S(t)+\rho_S(t)A_3^{\dagger}(w)\right]\Bigg). \tag{1323}$$

The correlation functions are relevant if $F(\omega)=0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau)=\int_0^{\infty}\mathrm{d}\omega J(\omega)G_{+}(\tau), \tag{1324}$$

$$\Lambda_{33}(-\tau)=\int_0^{\infty}\mathrm{d}\omega J(\omega)G_{+}(-\tau). \tag{1325}$$

In our case $A_3=\frac{\mathbb{I}+\sigma_z}{2}$, the equation (1323) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t}=-\mathrm{i}\left[H_S(t),\rho_S(t))\right]-\sum_{w}\left(K_{33}(w,t)\left[A_3,A_3(w)\rho_S(t)\right]+K_{33}^{*}(w,t)\left[\rho_S(t)A_3(w),A_3\right]\right). \tag{1326}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta=0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S=\frac{\Omega(t)}{2}\left(|1\rangle\langle 0|+|0\rangle\langle 1|\right)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_{+}=\frac{\widetilde{\Omega}}{2}, \tag{1327}$$

$$|+\rangle=\frac{1}{\sqrt{2}}\left(|1\rangle+|0\rangle\right), \tag{1328}$$

$$\lambda_{-}=-\frac{\widetilde{\Omega}}{2}, \tag{1329}$$

$$|-\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right). \tag{1330}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1331}$$

$$= \frac{1}{2}, \tag{1332}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{1333}$$

$$= \frac{1}{2}, \tag{1334}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1335}$$

$$= -\frac{1}{2}. \tag{1336}$$

The decomposition matrices are

$$A_3\left(0\right) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{1337}$$

$$= \frac{\mathbb{I}}{2}, \tag{1338}$$

$$A_3\left(\eta\right) = -\frac{1}{2}|-\rangle\langle+| \tag{1339}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right), \tag{1340}$$

$$A_3\left(-\eta\right) = -\frac{1}{2}|+\rangle\langle-| \tag{1341}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right). \tag{1342}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{1343}$$

$$- K_{33}^{*}\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^{*}\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right]. \tag{1344}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)G_{+}\left(\tau\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1345}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1346}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1347}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)\left(n\left(\omega\right)+1\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1348}$$

$$= \int_0^{\infty} J\left(\omega\right)\left(n\left(\omega\right)+1\right)\pi\delta\left(\widetilde{\Omega} - \omega\right)\mathrm{d}\omega \tag{1349}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right), \tag{1350}$$

$$K_{33}\left(-\widetilde{\Omega}\right)=\int_0^{\infty}\int_0^{\infty}J\left(\omega\right)G_{+}\left(\tau\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1351}$$

$$=\int_0^{\infty}\int_0^{\infty}J\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}+n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1352}$$

$$=\int_0^{\infty}\int_0^{\infty}J\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1353}$$

$$=\int_0^{\infty}\int_0^{\infty}J\left(\omega\right)n\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau+\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1354}$$

$$=\int_0^{\infty}J\left(\omega\right)n\left(\omega\right)\pi\delta\left(-\widetilde{\Omega}+\omega\right)\mathrm{d}\omega \tag{1355}$$

$$=\pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right). \tag{1356}$$

Here we have used $\int_0^{\infty}\mathrm{d}s\,\mathrm{e}^{\pm i\varepsilon s}=\pi\delta\left(\varepsilon\right)\pm\mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3\left(0\right)$ because the spectral density $J\left(\omega\right)$ is equal to zero when $\omega=0$. Replacing in the equation (1343) lead us to obtain:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t}=-\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x,\rho_S\left(t\right)\right)]-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z,\left(\sigma_z+\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]+n\left(\widetilde{\Omega}\right)\left[\sigma_z,\left(\sigma_z-\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{1357}$$

$$-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z+\mathrm{i}\sigma_y\right),\sigma_z\right]+n\left(\widetilde{\Omega}\right)\left[\rho_S\left(t\right)\left(\sigma_z-\mathrm{i}\sigma_y\right),\sigma_z\right]\right). \tag{1358}$$

This is the same result than the equation ($S17$), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t}=-\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x,\rho_S(t))\right]-\frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right)+1\right)[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\rho_S(t)]+n\left(\Omega(t)\right)[\sigma_z,\left(\sigma_z-\mathrm{i}\sigma_y\right)\rho_S(t)]\right) \tag{1359}$$

$$-\frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z+\mathrm{i}\sigma_y\right),\sigma_z\right]+n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z-\mathrm{i}\sigma_y\right),\sigma_z\right]\right). \tag{1360}$$

V. FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving the free energy of the system using the variational parameters optimization we consider the generalization in [5] of the inequality (244), at first we consider the convex function of the form:

$$f_N\left(x\right)=\mathrm{e}^x-1-\frac{x}{1!}-\cdots-\frac{x^{2N-1}}{(2N-1)!}. \tag{1361}$$

with $N\in\mathbb{N}^*$. By the Jensen inequality we can proof that for an arbitrary constant α:

$$\left\langle\mathrm{e}^{x-\langle x\rangle}\right\rangle\geq 1+\mathrm{e}^{-\alpha}\left(\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^2\right\rangle-\alpha^2}{2!}+\cdots+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{2N-1}\right\rangle-\alpha^{2N-1}}{(2N-1)!}\right) \tag{1362}$$

$$=1+\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^k\right\rangle-\alpha^k}{k!}. \tag{1363}$$

For $N=3$ we can obtain the initial step to get the third Bogoliubov inequality, the RHS of (1363) is written explicitly as follows:

$$1+\mathrm{e}^{-\alpha}\sum_{k=2}^{5}\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{k}\right\rangle-\alpha^{k}}{k!}=1+\mathrm{e}^{-\alpha}\left(\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{2}\right\rangle-\alpha^{2}}{2!}+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{3}\right\rangle-\alpha^{3}}{3!}\right. \tag{1364}$$

$$\left.+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{4}\right\rangle-\alpha^{4}}{4!}+\frac{\left\langle\left(x-\langle x\rangle+\alpha\right)^{5}\right\rangle-\alpha^{5}}{5!}\right). \tag{1365}$$

We consider the partition functions of $\overline{H}$ and $\overline{H_0}$ respect to $\overline{H_0}$ as:

$$Z=\left\langle\mathrm{e}^{-\beta\overline{H}}\right\rangle_{\overline{H_0}} \tag{1366}$$
$$=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}}\right), \tag{1367}$$
$$\overline{H}=\overline{H_{\bar{I}}}+\overline{H_0}, \tag{1368}$$
$$Z_0=\mathrm{e}^{-\beta\left\langle\overline{H}\right\rangle_{\overline{H_0}}} \tag{1369}$$
$$=\mathrm{e}^{-\beta\left\langle\overline{H_{\bar{I}}}+\overline{H_0}\right\rangle_{\overline{H_0}}} \tag{1370}$$
$$=\mathrm{e}^{-\beta\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_0}}-\beta\left\langle\overline{H_0}\right\rangle_{\overline{H_0}}} \tag{1371}$$
$$=\mathrm{e}^{0}\mathrm{e}^{-\beta\left\langle\overline{H_0}\right\rangle_{\overline{H_0}}} \tag{1372}$$
$$=\mathrm{e}^{-\beta\left\langle\overline{H_0}\right\rangle_{\overline{H_0}}} \tag{1373}$$
$$=\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}}\right). \tag{1374}$$

Taking the Quantum Bogoliubov inequality from [4]:

$$Z\geq Z_0\mathrm{e}^{-\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_0}}}\left(1+F_M\left(\overrightarrow{u}\right)+F_N\left(\overrightarrow{v}-\overrightarrow{u}\right)\right), \tag{1375}$$
$$F_N\left(\overrightarrow{u}\right)=\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\frac{u_k}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. \tag{1376}$$

where

$$\overline{H_{\bar{I}D}}=\sum_n\left\langle n\left|\overline{H_{\bar{I}}}\right|n\right\rangle|n\rangle\langle n|\quad\text{(where $|n\rangle$ is an eigenstate of $\overline{H_0}$)}, \tag{1377}$$
$$\overline{H_0}\,|n\rangle=E_0^n\,|n\rangle\,, \tag{1378}$$
$$Z_0=\sum_n\mathrm{e}^{-\beta E_0^n}, \tag{1379}$$
$$u_k=\left\langle\left(\overline{H_{\bar{I}D}}-\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_0}}\right)^k\right\rangle_{\overline{H_0}} \tag{1380}$$
$$=\frac{1}{Z_0}\sum_n\mathrm{e}^{-\beta E_0^n}\left(\left\langle n\left|\overline{H_{\bar{I}}}\right|n\right\rangle-\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_0}}\right)^k, \tag{1381}$$
$$v_k=\frac{1}{Z_0}\sum_n\mathrm{e}^{-\beta E_0^n}\left\langle n\left|\left(\overline{H_0}-E_0^n+\overline{H_{\bar{I}}}-\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_0}}\right)^k\right|n\right\rangle. \tag{1382}$$

By construction $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_0}}=0$, so we arrive to:

$$Z \geq Z_0 \left(1 + F_M \left(\overrightarrow{u}\right) + F_N \left(\overrightarrow{v} - \overrightarrow{u}\right)\right), \tag{1383}$$

$$u_k = \left\langle \left(\overline{H}_{\overline{I}D}\right)^k \right\rangle_{\overline{H_0}}, \tag{1384}$$

$$v_k = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right)^k \right| n \right\rangle. \tag{1385}$$

As we can see the expression (1383) was written in terms of the expected value of an operator, we want to do the same for (1385) in order to write that expressions in a short form, following this we obtained:

$$\left(\overline{H_0} - E_0^n\right) |n\rangle = \overline{H_0} \, |n\rangle - E_0^n \, |n\rangle \tag{1386}$$

$$= E_0^n \, |n\rangle - E_0^n \, |n\rangle \tag{1387}$$

$$= 0, \tag{1388}$$

$$\langle n| \left(\overline{H_0} - E_0^n\right) = \langle n| \, \overline{H_0} - \langle n| \, E_0^n \tag{1389}$$

$$= \langle n| \, E_0^n - \langle n| \, E_0^n \tag{1390}$$

$$= 0. \tag{1391}$$

At first we calculated v_1 like:

$$v_1 = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \overline{H_0} - E_0^n + \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1392}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \overline{H_0} - E_0^n \right| n \right\rangle + \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1393}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| E_0^n - E_0^n \right| n \right\rangle + \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1394}$$

$$= 0 + \left\langle \overline{H_{\overline{I}}} \right\rangle_{\overline{H_0}} \tag{1395}$$

$$= 0. \tag{1396}$$

For $k \geq 2$ and $k \in N$ we calculated:

$$v_k = \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right)^k \right| n \right\rangle \tag{1397}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right) \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right)^{k-2} \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right) \right| n \right\rangle \tag{1398}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right) \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right)^{k-2} \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right) \right| n \right\rangle \tag{1399}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \left(E_0^n - E_0^n + \overline{H_{\overline{I}}}\right) \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right)^{k-2} \left(E_0^n - E_0^n + \overline{H_{\overline{I}}}\right) \right| n \right\rangle \tag{1400}$$

$$= \frac{1}{Z_0} \sum_n \mathrm{e}^{-\beta E_0^n} \left\langle n \left| \overline{H_{\overline{I}}} \left(\overline{H_0} - E_0^n + \overline{H_{\overline{I}}}\right)^{k-2} \overline{H_{\overline{I}}} \right| n \right\rangle \tag{1401}$$

Then we have the following inequality:

$$Z \geq Z_0 \mathrm{e}^{-\left\langle \overline{H_{\overline{I}}} \right\rangle_{\overline{H_0}}} \left(1 + \mathrm{e}^{-\alpha} \sum_{k=2}^{2N-1} \sum_{j=2}^{k} \frac{\binom{k}{j} \beta^j \alpha^{k-j} \left\langle \overline{H_{\overline{I}}}^j \right\rangle_{\overline{H_0}}}{k!}\right). \tag{1402}$$

We know that $\left\langle \overline{H_{\overline{I}}} \right\rangle_{\overline{H_0}} = 0$ by construction, then obtain finally:

$$Z \geq Z_0 \left(1 + \mathrm{e}^{-\alpha} \sum_{k=2}^{2N-1} \sum_{j=2}^{k} \frac{\binom{k}{j} \beta^j \alpha^{k-j} \left\langle \overline{H_{\overline{I}}}^j \right\rangle_{\overline{H_0}}}{k!} \right). \tag{1403}$$

The free energy is defined as:

$$E_{\text{free}} = -\frac{1}{\beta} \ln \left(Z \right). \tag{1404}$$

It is well known that the function $f(x) = \ln(x)$ is monotonic and increasing so we can transform (1403):

$$Z \geq Z_0 \left(1 + \mathrm{e}^{-\alpha} \sum_{k=2}^{2N-1} \sum_{j=2}^{k} \frac{\binom{k}{j} \beta^j \alpha^{k-j} \left\langle \overline{H_{\overline{I}}}^j \right\rangle_{\overline{H_0}}}{k!} \right), \tag{1405}$$

$$E_{\text{free},1} = -\frac{1}{\beta} \ln \left(Z_0 \right), \tag{1406}$$

$$E_{\text{free}} \leq E_{\text{free},1} - \frac{1}{\beta} \ln \left(1 + \mathrm{e}^{-\alpha} \sum_{k=2}^{2N-1} \sum_{j=2}^{k} \frac{\binom{k}{j} \beta^j \alpha^{k-j} \left\langle \overline{H_{\overline{I}}}^j \right\rangle_{\overline{H_0}}}{k!} \right) \tag{1407}$$

$$\equiv E_{\text{free,N}}. \tag{1408}$$

Now we consider the expected value of the form:

$$G(t) \equiv \left\langle \mathrm{e}^{-t\beta \overline{H_{\overline{I}}}} \right\rangle_{\overline{H_0}}, \tag{1409}$$

$$G^{(j)}(t) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} \left\langle \mathrm{e}^{-t\beta \overline{H_{\overline{I}}}} \right\rangle_{\overline{H_0}} \tag{1410}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j} \mathrm{e}^{-t\beta \overline{H_{\overline{I}}}} \right\rangle_{\overline{H_0}} \tag{1411}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j} \left(\mathbb{I} - \frac{t\beta \overline{H_{\overline{I}}}}{1!} + \frac{t^2 \beta^2 \overline{H_{\overline{I}}}^2}{2!} - \dots \right) \right\rangle_{\overline{H_0}} \tag{1412}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j} \left(\sum_{k=0}^{\infty} \frac{\left(-t\beta \overline{H_{\overline{I}}} \right)^k}{k!} \right) \right\rangle_{\overline{H_0}} \tag{1413}$$

$$= \left\langle \frac{\mathrm{d}^j}{\mathrm{d}t^j} \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^k \beta^k \overline{H_{\overline{I}}}^k}{k!} \right) \right\rangle_{\overline{H_0}}, \tag{1414}$$

$$G^{(j)}(0) = (-1)^j \beta^j \left\langle \overline{H_{\overline{I}}}^j \right\rangle_{\overline{H_0}}, \tag{1415}$$

$$\beta^j \left\langle \overline{H_{\overline{I}}}^j \right\rangle_{\overline{H_0}} = (-1)^j G^{(j)}(0). \tag{1416}$$

So we obtain that:

$$E_{\text{free,N}} = E_{\text{free},1} - \frac{1}{\beta} \ln \left(1 + \mathrm{e}^{-\alpha} \sum_{k=2}^{2N-1} \sum_{j=2}^{k} \frac{(-1)^j \binom{k}{j} G^{(j)}(0) \alpha^{k-j}}{k!} \right). \tag{1417}$$

From the fact that α is a free parameter then we can minimize the expression:

$$h(\alpha) = \ln\left(1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right). \tag{1418}$$

This process leads is to:

$$h'(\alpha) = \frac{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\alpha^{k-j}}{k!}\right)}{1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\alpha^{k-j}}{k!}} \tag{1419}$$

$$= 0, \tag{1420}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(1 + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right) \tag{1421}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,(k-j)\,\alpha^{k-j-1}}{k!} \tag{1422}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,(k-j)\,\alpha^{k-j-1}}{k!} - \sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right) \tag{1423}$$

$$= \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\left(\frac{(-1)^j\binom{k-1}{j}G^{(j)}(0)\,\alpha^{k-j-1}}{(k-1)!} - \frac{(-1)^j\binom{k}{j}G^{(j)}(0)\,\alpha^{k-j}}{k!}\right). \tag{1424}$$

Then α fullfils the following algebraic equation:

$$\sum_{k=2}^{2N-1}\sum_{j=2}^{k}\frac{(-1)^j G^{(j)}(0)}{\alpha^j}\left(\frac{\binom{k}{j}\alpha^k}{k!} - \frac{\binom{k-1}{j}\alpha^{k-1}}{(k-1)!}\right) = 0. \tag{1425}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1426}$$

$$H_S(t) = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{1427}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right), \tag{1428}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}}. \tag{1429}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{1430}$$

At first let's obtain $e^{\pm V}$ under the transformation (1430), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)$, so the equation (1430) can be written as $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$, then we have:

$$\begin{aligned}
e^{\pm V} &= e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} && (1431)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n| \hat{\varphi}_n\right)^2}{2!} + \ldots && (1432)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \ldots && (1433)\\
&= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \ldots && (1434)\\
&= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \ldots \right) && (1435)\\
&= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} && (1436)
\end{aligned}$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*} b_{u'\mathbf{k}'} \right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right) = e^{\pm\left(\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)}$ in the same way than (24) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$\begin{aligned}
e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} &= \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} && (1437)\\
&= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} \right)} \right) && (1438)\\
&= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \right) && (1439)\\
&= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) && (1440)\\
&= \prod_u B_{nu\pm} && (1441)\\
B_{nu\pm} &\equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) && (1442)
\end{aligned}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V} A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \tag{1443}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1426):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1444}$$
$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1445}$$
$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1446}$$
$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1447}$$
$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1448}$$
$$= |0\rangle\langle 0|. \tag{1449}$$
$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1450}$$
$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1451}$$
$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1452}$$
$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, \tag{1453}$$
$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1454}$$
$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1455}$$
$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1456}$$
$$\textstyle\prod_u (B_{mu+}B_{nu-}) = \displaystyle\prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1457}$$
$$\overline{\textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1458}$$
$$= \left(|0\rangle\langle 0| \textstyle\prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right)\left(\textstyle\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \textstyle\prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), \tag{1459}$$
$$= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, \tag{1460}$$
$$= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots \tag{1461}$$
$$= |0\rangle\langle 0| \left(\textstyle\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) \tag{1462}$$
$$+ |1\rangle\langle 1| \left(\textstyle\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots \tag{1463}$$
$$= |0\rangle\langle 0| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) \tag{1464}$$
$$+ |1\rangle\langle 1| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots \tag{1465}$$
$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1466}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1467}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) \tag{1468}$$
$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) \tag{1469}$$
$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{1470}$$

The transformed Hamiltonians of the equations (1427) to (1429) written in terms of (1444) to (1468) are:

$$\overline{H_S\left(t\right)} = \overline{\sum_n \varepsilon_n\left(t\right)\left|n\middle\rangle\!\middle\langle n\right| + \sum_{n\neq m} V_{nm}\left(t\right)\left|n\middle\rangle\!\middle\langle m\right|} \tag{1471}$$

$$= \overline{\sum_n \varepsilon_n\left(t\right)\left|n\middle\rangle\!\middle\langle n\right|} + \overline{\sum_{n\neq m} V_{nm}\left(t\right)\left|n\middle\rangle\!\middle\langle m\right|} \tag{1472}$$

$$= \sum_n \varepsilon_n\left(t\right)\left|n\middle\rangle\!\middle\langle n\right| + \sum_{n\neq m} V_{nm}\left(t\right)\left|n\middle\rangle\!\middle\langle m\right| \prod_u \left(B_{mu+}B_{nu-}\right) \tag{1473}$$

$$\overline{H_I} = \left(\sum_n \left|n\middle\rangle\!\middle\langle n\right| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} \left|n\middle\rangle\!\middle\langle n\right| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_n \left|n\middle\rangle\!\middle\langle n\right| \prod_u B_{nu-}\right) \tag{1474}$$

$$= \left(\sum_n \left|n\middle\rangle\!\middle\langle n\right| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} \left|0\middle\rangle\!\middle\langle 0\right| \left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right) + ...\right)\left(\sum_n \left|n\middle\rangle\!\middle\langle n\right| \prod_u B_{nu-}\right) \tag{1475}$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} \left|0\middle\rangle\!\middle\langle 0\right|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{0u-}+\prod_u B_{1u+}\sum_{u\mathbf{k}} \left|1\middle\rangle\!\middle\langle 1\right|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{1u-}+\ldots \tag{1476}$$

$$=\sum_{u\mathbf{k}} \left|0\middle\rangle\!\middle\langle 0\right|\left(g_{0u\mathbf{k}}\prod_u B_{0u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{0u-}+g_{0u\mathbf{k}}^{*}\prod_u B_{0u+}b_{u\mathbf{k}}\prod_u B_{0u-}\right)+\sum_{u\mathbf{k}} \left|1\middle\rangle\!\middle\langle 1\right|\left(g_{1u\mathbf{k}}\prod_u B_{1u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{1u-}+g_{1u\mathbf{k}}^{*}\prod_u B_{1u+}b_{u\mathbf{k}}\prod_u B_{1u-}\right)+\ldots \tag{1477}$$

$$=\sum_{u\mathbf{k}} \left|0\middle\rangle\!\middle\langle 0\right|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\sum_{u\mathbf{k}} \left|1\middle\rangle\!\middle\langle 1\right|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\ldots \tag{1478}$$

$$= \sum_{nu\mathbf{k}} \left|n\middle\rangle\!\middle\langle n\right| \left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger} - \frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right) + g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1479}$$

$$= \sum_{nu\mathbf{k}} \left|n\middle\rangle\!\middle\langle n\right| \left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1480}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} \left|n\middle\rangle\!\middle\langle n\right| \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1481}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H}=\sum_n \varepsilon_n(t)\left|n\middle\rangle\!\middle\langle n\right|+\sum_{n\neq m} V_{nm}(t)\left|n\middle\rangle\!\middle\langle m\right|\prod_u\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}} \left|n\middle\rangle\!\middle\langle n\right|\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1482}$$

$$+\sum_{nu\mathbf{k}} \left|n\middle\rangle\!\middle\langle n\right|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1483}$$

Let's define the following functions:

$$R_n\left(t\right) = \sum_{u\mathbf{k}} \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1484}$$

$$B_{z,n}\left(t\right) = \sum_{u\mathbf{k}} \left(\left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right)b_{u\mathbf{k}}^{\dagger} + \left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right)^{*}b_{u\mathbf{k}}\right) \tag{1485}$$

Using the previous functions we have that (1482) can be re-written in the following way:

$$\overline{H}=\sum_n \varepsilon_n(t)\left|n\middle\rangle\!\middle\langle n\right|+\sum_{n\neq m} V_{nm}(t)\left|n\middle\rangle\!\middle\langle m\right|\prod_u\left(B_{mu+}B_{nu-}\right)+\sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_n R_n(t)\left|n\middle\rangle\!\middle\langle n\right|+\sum_n B_{z,n}(t)\left|n\middle\rangle\!\middle\langle n\right| \tag{1486}$$

$$\tag{1487}$$

Now in order to separate the elements of the hamiltonian (1487) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \prod_u (B_{mu+}B_{nu-})\right\rangle_{\overline{H_0}} = \left\langle \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \tag{1488}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\tfrac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\left\langle \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \tag{1489}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1490}$$

$$\equiv B_{nm} \tag{1491}$$

$$\left\langle \prod_u (B_{nu+}B_{mu-})\right\rangle_{\overline{H_0}} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1492}$$

$$= B^*_{nm} \tag{1493}$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+}B_{nu-}) - B_{nm} \tag{1494}$$

As we can see:

$$J^\dagger_{nm} = \left(\prod_u (B_{mu+}B_{nu-}) - B_{nm}\right)^\dagger \tag{1495}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B^*_{nm} \tag{1496}$$

$$= \prod_u (B_{nu+}B_{mu-}) - B_{mn} \tag{1497}$$

$$= J_{mn} \tag{1498}$$

We can separate the Hamiltonian (1487) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)} = \sum_n (\varepsilon_n(t) + R_n)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|B_{nm} \tag{1499}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|J_{nm} + \sum_n B_{z,n}(t)|n\rangle\langle n|, \tag{1500}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \tag{1501}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right) + \langle \overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle \overline{H^2_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_{B}}}\right). \tag{1502}$$

Taking the equations (245)-(253) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right) = C\left(R_0, R_1, ..., R_{d-1}, B_{01}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^{\Re}_{nu\mathbf{k}}$ and $v^{\Im}_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1503}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}} - v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1504}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}}v^*_{nu\mathbf{k}} + g^*_{nu\mathbf{k}}v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1505}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}} - v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1506}$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}} - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1507}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}} - v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1508}$$

$$v^*_{mu\mathbf{k}}v_{nu\mathbf{k}} - v_{mu\mathbf{k}}v^*_{nu\mathbf{k}} = \left(v^{\Re}_{mu\mathbf{k}} - \mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}\right) - \left(v^{\Re}_{mu\mathbf{k}} + \mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}} - \mathrm{i}v^{\Im}_{nu\mathbf{k}}\right) \tag{1509}$$

$$= \left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}} - \mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}} + v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1510}$$

$$- \left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}} - \mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}} + \mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}} + v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1511}$$

$$= 2\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right) \tag{1512}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}} - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1513}$$

$$+ \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1514}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1515}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}}v^*_{mu\mathbf{k}} + v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}) \tag{1516}$$

$$= \left(v^{\Re}_{mu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}}\right)^2 + \left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - \left(v^{\Re}_{nu\mathbf{k}} + \mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}} - \mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1517}$$

$$- \left(v^{\Re}_{nu\mathbf{k}} - \mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}} + \mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1518}$$

$$= \left(v^{\Re}_{mu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}}\right)^2 + \left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2\left(v^{\Re}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}} + v^{\Im}_{nu\mathbf{k}}v^{\Im}_{mu\mathbf{k}}\right) \tag{1519}$$

$$= \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2 \tag{1520}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1521}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - (g_{nu\mathbf{k}} + g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}} - \mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}} - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right) \tag{1522}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}}v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \tag{1523}$$

$$B_{mn} = \left(\prod_{u\mathbf{k}} \exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}} - v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1524}$$

$$= \left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1525}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1526}$$

$$= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1527}$$

$$= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1528}$$

$$\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1529}$$

$$= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1530}$$

$$= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{1531}$$

Given that:

$$\ln B_{mn} = \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) \tag{1532}$$

$$= \sum_{u\mathbf{k}} \ln \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1533}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) \tag{1534}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1535}$$

$$\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} \tag{1536}$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \tag{1537}$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \tag{1538}$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} \tag{1539}$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1540}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1541}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1542}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1543}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1544}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1545}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1546}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1547}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1548}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1549}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1550}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1551}$$

Introducing this derivates in the equation (1526) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1552}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1553}$$

$$= 0 \tag{1554}$$

We can obtain the variational parameters:

$$
\begin{aligned}
&- 2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) && (1555)\\
&= -\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) && (1556)\\
v^{\Re}_{nu\mathbf{k}} &= \frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} && (1557)\\
&= \frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} && (1558)
\end{aligned}
$$

Let's consider the imaginary part of the variation parameters

$$
\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. && (1559)\\
&\quad\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) && (1560)\\
&= 0 && (1561)
\end{aligned}
$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Im}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1562}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1563}$$

$$v^{\Im}_{nu\mathbf{k}}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1564}$$

$$=\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1565}$$

$$v_{nu\mathbf{k}}=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}} \tag{1566}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1567}$$

$$\mathrm{i}\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1568}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1569}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1570}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1571}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1572}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1573}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1574}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^{V}\rho\left(0\right)e^{-V}=\left(\sum_{n}|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_{n}|n\rangle\langle n|B_{n+}\right) \tag{1575}$$

$$0=\left(B^{+}_{0}|0\rangle\langle 0|B^{-}_{0}\right)\otimes\rho_B \tag{1576}$$

$$0=\rho\left(0\right) \tag{1577}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t) O U(t) \tag{1578}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1579}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t) \overline{\rho_S}(t) U(t), \text{ where} \tag{1580}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{1581}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1582}$$
$$\sigma_{nm,y} = \mathrm{i}(|n\rangle\langle m| - |m\rangle\langle n|) \tag{1583}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1584}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1585}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = (B_{m+}B_{n-} - B_m B_n)^{\dagger} \tag{1586}$$
$$= B_{n-}^{\dagger} B_{m+}^{\dagger} - B_n B_m \tag{1587}$$
$$= B_{n+}B_{m-} - B_n B_m \tag{1588}$$
$$= B_{nm} \tag{1589}$$

So we can say that the set of matrices (1582) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1582) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \tag{1590}$$
$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}(V_{nm}(t)|n\rangle\langle m|B_{nm} + V_{mn}(t)|m\rangle\langle n|B_{mn}) \tag{1591}$$
$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}\left(\Re(V_{nm}(t)) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1592}$$
$$\left. + \Re(V_{nm}(t)) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1593}$$
$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}\left(\Re(V_{nm}(t))\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re(V_{nm}(t))\sigma_{nm,y}\frac{\mathrm{i}(B_{mn} - B_{nm})}{2}\right. \tag{1594}$$
$$\left. + \mathrm{i}\Im(V_{nm}(t))\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im(V_{nm}(t))\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1595}$$
$$= \sum_n B_{z,n}(t)|n\rangle\langle n| + \sum_{n<m}(\Re(V_{nm}(t))\sigma_{nm,x}B_{nm,x} - \Im(V_{nm}(t))\sigma_{nm,x}B_{nm,y} + \Re(V_{nm}(t))\sigma_{nm,y}B_{nm,y} \tag{1596}$$
$$+ \Im(V_{nm}(t))\sigma_{nm,y}B_{nm,x}) \tag{1597}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\right\} \tag{1598}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1-\delta_{mn}) \tag{1599}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1-\delta_{mn}) \tag{1600}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1601}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1602}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1603}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1604}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1605}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1606}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1607}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1608}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1609}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1610}$$
$$C_{3,nm}(t) = 1 \tag{1611}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1612}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1613}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J, p\in P} C_{jp}(t)\left(A_{jp}\otimes B_{jp}(t)\right) \tag{1614}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1598).

We write the interaction Hamiltonian transformed under (1578) as:

$$\widetilde{H_I}(t) = \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes \widetilde{B_{jp}}(t)\right) \tag{1615}$$
$$\widetilde{A_{jp}}(t) = U^{\dagger}(t)\,A_{jp}U(t) \tag{1616}$$
$$\widetilde{B_{jp}}(t) = \mathrm{e}^{\mathrm{i}H_B t}B_{jp}(t)\,(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{1617}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1618}$$

Replacing the equation (1615) in (1618) we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1619}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1620}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1621}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1622}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1623}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1624}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1625}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1626}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1627}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1628}$$

Here $s\to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1626) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1629}$$

$$= \left\langle \widetilde{B_{jp}}(0)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1630}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1631}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\rho_B\widetilde{B_{j'p'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1632}$$

$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1633}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1634}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1635}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\rho_B\widetilde{B_{jp}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\,\rho_B\right) \tag{1636}$$

$$= \left\langle \widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1637}$$

$$= \left\langle \widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1638}$$

$$= \Lambda_{jpj'p'}(\tau) \tag{1639}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\,\rho_B\right) \tag{1640}$$

$$= \left\langle \widetilde{B_{j'p'}}(s)\,\widetilde{B_{jp}}(t)\right\rangle_B \tag{1641}$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau)\,\widetilde{B_{jp}}(0)\right\rangle_B \tag{1642}$$

$$= \Lambda_{j'p'jp}(-\tau) \tag{1643}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1619)and (1626) and using the equations (1629)-(1643) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{jpj'p'}(\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'p'jp}(-\tau)\,\widetilde{A_{jp}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\right)\right. \tag{1644}$$

$$\left.+C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{j'p'jp}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\,\widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau)\,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jp}}(t)\right)\right)\mathrm{d}s \tag{1645}$$

$$= -\int_0^t \sum_{jj'pp'} \left(C_{jp}(t)\,C_{j'p'}(s)\left(\Lambda_{jpj'p'}(\tau)\left[\widetilde{A_{jp}}(t)\,,\widetilde{A_{j'p'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \Lambda_{j'p'jp}(-\tau)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}(s)\,,\widetilde{A_{jp}}(t)\right]\right)\right) \tag{1646}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'} \left(C_{jp}(t)\,C_{j'p'}(t-\tau)\left(\Lambda_{jpj'p'}(\tau)\left[A_{jp}(t)\,,A_{j'p'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'p'jp}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'p'}(t-\tau,t)\,,A_{jp}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1647}$$

For this case we used that $A_{jp}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jp}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1648}$$
$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1649}$$
$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1650}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1651}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1652}$$

At first let's obtain e^V under the transformation (1652), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \tag{1653}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n| \hat{\varphi}\right)^2}{2!} + ... \tag{1654}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + ... \tag{1655}$$
$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + ...\right) \tag{1656}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1657}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+} \tag{1658}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (24):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1659}$$
$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1660}$$
$$= B_{\pm} \tag{1661}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_{-}$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+}\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{-}\right) \tag{1662}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1648):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1663}$$

$$= |0\rangle\langle 0|, \tag{1664}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1665}$$

$$= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, \tag{1666}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1667}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1668}$$

$$= |0\rangle\langle m|B^{-}\ m \neq 0, \tag{1669}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1670}$$

$$= |0\rangle\langle m|B^{+}\ m \neq 0, \tag{1671}$$

$$\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1672}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B^{+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B^{-} \tag{1673}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B^{+}b_{\mathbf{k}}^{\dagger}B^{-}\right)\left(B^{+}b_{\mathbf{k}}B^{-}\right) \tag{1674}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1675}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1676}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right) \tag{1677}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1678}$$

The transformed Hamiltonians of the equations (1649) to (1651) written in terms of (1663) to (1678) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1679}$$

$$= \overline{\sum_{n=0}\varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|} \tag{1680}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \overline{\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n|} \tag{1681}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)\overline{|0\rangle\langle n|} + V_{n0}(t)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0}V_{mn}(t)\overline{|m\rangle\langle n|} \tag{1682}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)B^{-}|0\rangle\langle n| + V_{n0}(t)B^{+}|n\rangle\langle 0|\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1683}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1684}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(\left(\sum_{n=0}\mu_n(t)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1685}$$

$$= \left(\mu_0(t)|0\rangle\langle 0| + \sum_{n=1}\mu_n(t)|n\rangle\langle n|B^{+}\right)\left(\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1686}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}B^{+}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)B^{-} \tag{1687}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1688}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1689}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1690}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1691}$$

$$+\sum_{n=0}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}}2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1692}$$

$$= \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \tag{1693}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1694}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1695}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1696}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\right) \tag{1697}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1698}$$

Using the previous functions we have that (1695) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^- + V_{n0}(t)|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1699}$$
$$+\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n|n\rangle\langle n| + \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1700}$$

Now in order to separate the elements of the hamiltonian (1700) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| + \sum_{n=1} R_n|n\rangle\langle n| \tag{1701}$$
$$\overline{H_I} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^+ - B\right)\right), \tag{1702}$$
$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{1703}$$

Here B is given by:

$$B = \left\langle B^+ \right\rangle = \left\langle B^- \right\rangle$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1704}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1705}$$
$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1706}$$
$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1707}$$

Using this set of hermitian operators to write the Hamiltonians (1649)-(1651)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1708}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \tag{1709}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1710}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \tag{1711}$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1712}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1713}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1714}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1715}$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1716}$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^+ - B\right)\right) \tag{1717}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\left(B^- - B\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\left(B^+ - B\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1718}$$

$$+ \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1719}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + \left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\right)\right. \tag{1720}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - \left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\right)\right) \tag{1721}$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1722}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^+ + B^- - 2B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B^- - B - B^+ + B\right)\Im(V_{0n}(t))\right. \tag{1723}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B - B^- + B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B - B^- + B - B^+\right)\Im(V_{0n}(t))\right)\right) + \sum_{n=1} B_{z,n}|n\rangle\langle n| \tag{1724}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1725}$$

$$+ \sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1726}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\rm B} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right) + \langle \overline{H_I}\rangle_{\overline{H_S+H_B}} + O\left(\left\langle \overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1727}$$

Taking the equations (245)-(253) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (253) and using the chain rule we obtain that:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1728}$$

$$= 0 \tag{1729}$$

Let's recall the equations (1696) and (1698), we can write them in terms of the variational parameters

$$B = \exp\left(-\left(1/2\right)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1730}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1731}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1732}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1733}$$

Introducing this derivates in the equation (1728) give us:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1734}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1735}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1736}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1737}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1738}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho(0)e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right)(|0\rangle\langle 0| \otimes \rho_B)\left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right) \tag{1739}$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \tag{1740}$$

$$0 = \rho(0) \tag{1741}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1742}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1743}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\,,\ \text{where} \tag{1744}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{1745}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle 0| + \sum_{n=1}\left(\Re(V_{0n}(t))\, B_x\sigma_{0n,x} + \Re(V_{0n}(t))\, B_y\sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right. \tag{1746}$$

$$\left. + \Im(V_{0n}(t))\, B_x\sigma_{0n,y} - \Im(V_{0n}(t))\, B_y\sigma_{0n,x}\right) \tag{1747}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}}\mu_0(t)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1748}$$

$$B_{z,n} = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\ \text{if} n \neq 0 \tag{1749}$$

Now consider the following set of operators:

$$
\begin{aligned}
A_{1n}(t) &= \sigma_{0n,x} && (1750)\\
A_{2n}(t) &= \sigma_{0n,y} && (1751)\\
A_{3n}(t) &= |n\rangle\langle n| && (1752)\\
A_{4n}(t) &= A_{2n}(t) && (1753)\\
A_{5n}(t) &= A_{1n}(t) && (1754)\\
B_{1n}(t) &= B_x && (1755)\\
B_{2n}(t) &= B_y && (1756)\\
B_{3n}(t) &= B_{z,n} && (1757)\\
B_{4n}(t) &= B_{1n}(t) && (1758)\\
B_{5n}(t) &= B_{2n}(t) && (1759)\\
C_{10}(t) &= 0 && (1760)\\
C_{20}(t) &= 0 && (1761)\\
C_{40}(t) &= 0 && (1762)\\
C_{50}(t) &= 0 && (1763)\\
C_{30}(t) &= 1 && (1764)\\
C_{1n}(t) &= \Re\left(V_{0n}(t)\right) && (1765)\\
C_{2n}(t) &= C_{1n}(t) && (1766)\\
C_{3n}(t) &= 1 && (1767)\\
C_{4n}(t) &= \Im\left(V_{0n}(t)\right) && (1768)\\
C_{5n}(t) &= -\Im\left(V_{0n}(t)\right) && (1769)
\end{aligned}
$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$
\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn}\otimes B_{jn}(t)\right) \tag{1770}
$$

Here $J=\{1,2,3,4,5\}$.

We write the interaction Hamiltonian transformed under (1742) as:

$$
\begin{aligned}
\widetilde{H_I}(t) &= \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) && (1771)\\
\widetilde{A_i}(t) &= U^{\dagger}(t)\,A_i U(t) && (1772)\\
\widetilde{B_i}(t) &= \mathrm{e}^{\mathrm{i}H_B t}B_i(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} && (1773)
\end{aligned}
$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$
\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1774}
$$

Replacing the equation (1771)in (1774)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1775}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1776}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1777}$$

$$\left. -\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1778}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1779}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1780}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1781}$$

$$\left. +\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1782}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1783}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1784}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1782) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1785}$$

$$= \left\langle \widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1786}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1787}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1788}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1789}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1790}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1791}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1792}$$

$$= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1793}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1794}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1795}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1796}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1797}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1798}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1799}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1775)and (1782) and using the equations (1785)-(1799) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1800}$$

$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1801}$$

$$= -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t)\,,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,,\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1802}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t)\,,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t)\,,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1803}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (450) as expected.

VIII. BIBLIOGRAPHY

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