

A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

Nike Dattani*

Harvard-Smithsonian Center for Astrophysics

Camilo Chaparro Sogamoso[†]

National University of Colombia

I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1)$$

$$H_S(t) = \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|, \quad (2)$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (3)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (4)$$

For the states $|0\rangle, |1\rangle$ we have the orthonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \quad (5)$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right). \quad (6)$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}}(t). \quad (7)$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \quad (8)$$

$$V = |0\rangle\langle 0| \hat{\varphi}_0 + |1\rangle\langle 1| \hat{\varphi}_1. \quad (9)$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6) will give us the following result:

$$e^{\pm V} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1)} \quad (10)$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + \dots \quad (11)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + \dots \quad (12)$$

$$= |0\rangle\langle 0| \left(\mathbb{I} \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm \dots \right) + |1\rangle\langle 1| \left(\mathbb{I} \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm \dots \right) \quad (13)$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} \quad (14)$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*b_{\mathbf{k}})} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*b_{\mathbf{k}})} \quad (15)$$

$$= |0\rangle\langle 0|B_0^\pm + |1\rangle\langle 1|B_1^\pm, \quad (16)$$

$$B_i^\pm \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (17)$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} e^{\frac{t^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} e^{-\frac{t^4}{24}([[[X,Y],X],X] + 3[[[X,Y],X],Y] + 3[[[X,Y],Y],Y])} \dots \quad (18)$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}\right] = 0$ for all \mathbf{k}', \mathbf{k} and i, j we can show making $t = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}\right)} = e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}\right]} \dots \quad (19)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}} e^{-\frac{1}{2}0} \dots \quad (20)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}'b_{\mathbf{k}'}}. \quad (21)$$

By induction of this result we can write an expression of B_i^\pm (shown in equation (17)) as a product of exponentials, which we will call “displacement” operators $D(\pm v_{i\mathbf{k}})$:

$$D(\pm v_{i\mathbf{k}}) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, \quad (22)$$

$$B_i^\pm = \prod_{\mathbf{k}} D\left(\pm \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right). \quad (23)$$

this will help us to write operators O transformed in the variational frame as:

$$\overline{O} \equiv e^V O e^{-V}. \quad (24)$$

We will use the following identities:

$$\overline{|0\chi 0\rangle} = e^V |0\chi 0\rangle e^{-V} \quad (25)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |0\chi 0\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (26)$$

$$= (|0\chi 0\rangle |0\chi 0\rangle B_0^+ + |1\chi 1\rangle |0\chi 0\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (27)$$

$$= |0\chi 0\rangle B_0^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (28)$$

$$= |0\chi 0\rangle |0\chi 0\rangle B_0^+ B_0^- + |0\chi 0\rangle |1\chi 1\rangle B_0^+ B_1^- \quad (29)$$

$$= |0\chi 0\rangle, \quad (30)$$

$$\overline{|1\chi 1\rangle} = (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |1\chi 1\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (31)$$

$$= (|0\chi 0\rangle |1\chi 1\rangle B_0^+ + |1\chi 1\rangle |1\chi 1\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (32)$$

$$= |1\chi 1\rangle B_1^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (33)$$

$$= |1\chi 1\rangle |0\chi 0\rangle B_1^+ B_0^- + B_1^+ |1\chi 1\rangle |1\chi 1\rangle B_1^- \quad (34)$$

$$= B_1^+ |1\chi 1\rangle |1\chi 1\rangle B_1^- \quad (35)$$

$$= |1\chi 1\rangle, \quad (36)$$

$$\overline{|0\chi 1\rangle} = e^V |0\chi 1\rangle e^{-V} \quad (37)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |0\chi 1\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (38)$$

$$= (|0\chi 0\rangle |0\chi 1\rangle B_0^+ + |1\chi 1\rangle |0\chi 1\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (39)$$

$$= (|0\chi 0\rangle |0\chi 1\rangle B_0^+ + |1\chi 1\rangle |0\chi 1\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (40)$$

$$= |0\chi 1\rangle B_0^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (41)$$

$$= |0\chi 1\rangle |0\chi 0\rangle B_0^+ B_0^- + |0\chi 1\rangle |1\chi 1\rangle B_0^+ B_1^- \quad (42)$$

$$= |0\chi 1\rangle B_0^+ B_1^-, \quad (43)$$

$$\overline{|1\chi 0\rangle} = e^V |1\chi 0\rangle e^{-V} \quad (44)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |1\chi 0\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (45)$$

$$= (|0\chi 0\rangle |1\chi 0\rangle B_0^+ + |1\chi 1\rangle |1\chi 0\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (46)$$

$$= (|0\chi 0\rangle |1\chi 0\rangle B_0^+ + |1\chi 1\rangle |1\chi 0\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (47)$$

$$= |1\chi 0\rangle B_1^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (48)$$

$$= |1\chi 0\rangle B_1^+ |0\chi 0\rangle B_0^- + |1\chi 0\rangle B_1^+ |1\chi 1\rangle B_1^- \quad (49)$$

$$= |1\chi 0\rangle |0\chi 0\rangle B_1^+ B_0^- + |1\chi 0\rangle |1\chi 1\rangle B_1^+ B_1^- \quad (50)$$

$$= |1\chi 0\rangle B_1^+ B_0^-, \quad (51)$$

$$\overline{b_{\mathbf{k}}} = e^V b_{\mathbf{k}} e^{-V} \quad (52)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) b_{\mathbf{k}} (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (53)$$

$$= |0\chi 0\rangle B_0^+ b_{\mathbf{k}} B_0^- |0\chi 0\rangle + |0\chi 0\rangle B_0^+ b_{\mathbf{k}} |1\chi 1\rangle B_1^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}} |0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}} B_1^- |1\chi 1\rangle \quad (54)$$

$$= |0\chi 0\rangle |0\chi 0\rangle B_0^+ b_{\mathbf{k}} B_0^- + |0\chi 0\rangle |1\chi 1\rangle B_0^+ b_{\mathbf{k}} B_1^- + |1\chi 1\rangle |0\chi 0\rangle B_1^+ b_{\mathbf{k}} B_0^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}} B_1^- \quad (55)$$

$$= |0\chi 0\rangle \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\chi 1\rangle \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (56)$$

$$= (|0\chi 0\rangle + |1\chi 1\rangle) b_{\mathbf{k}} - |1\chi 1\rangle \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\chi 0\rangle \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (57)$$

$$= b_{\mathbf{k}} - |1\chi 1\rangle \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\chi 0\rangle \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (58)$$

$$\overline{b_{\mathbf{k}}^\dagger} = e^V b_{\mathbf{k}}^\dagger e^{-V} \quad (59)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) b_{\mathbf{k}}^\dagger (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (60)$$

$$= |0\chi 0\rangle B_0^+ b_{\mathbf{k}}^\dagger B_0^- |0\chi 0\rangle + |0\chi 0\rangle B_0^+ b_{\mathbf{k}}^\dagger |1\chi 1\rangle B_1^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}}^\dagger |0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}}^\dagger B_1^- |1\chi 1\rangle \quad (61)$$

$$= |0\chi 0\rangle |0\chi 0\rangle B_0^+ b_{\mathbf{k}}^\dagger B_0^- + |0\chi 0\rangle |1\chi 1\rangle B_0^+ b_{\mathbf{k}}^\dagger B_1^- + |1\chi 1\rangle |0\chi 0\rangle B_1^+ b_{\mathbf{k}}^\dagger B_0^- + |1\chi 1\rangle |1\chi 1\rangle B_1^+ b_{\mathbf{k}}^\dagger B_1^- \quad (62)$$

$$= |0\chi 0\rangle \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\chi 1\rangle \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \quad (63)$$

$$= b_{\mathbf{k}}^\dagger - |1\chi 1\rangle \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} - |0\chi 0\rangle \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (64)$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$:

$$B_i^+ b_{\mathbf{k}} B_i^- = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (65)$$

$$B_i^+ b_{\mathbf{k}}^\dagger B_i^- = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (66)$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t) |0\rangle\langle 0|} = \varepsilon_0(t) |0\rangle\langle 0|, \quad (67)$$

$$\overline{\varepsilon_1(t) |1\rangle\langle 1|} = \varepsilon_1(t) |1\rangle\langle 1|, \quad (68)$$

$$\overline{V_{10}(t) |1\rangle\langle 0|} = V_{10}(t) |1\rangle\langle 0| B_1^+ B_0^-, \quad (69)$$

$$\overline{V_{01}(t) |0\rangle\langle 1|} = V_{01}(t) |0\rangle\langle 1| B_0^+ B_1^-, \quad (70)$$

$$\overline{g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}}} = g_{i\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) + g_{i\mathbf{k}}^* \left(|0\rangle\langle 0| \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (71)$$

$$= g_{i\mathbf{k}} \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (72)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| \quad (73)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |0\rangle\langle 0| - \left(g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |1\rangle\langle 1|, \quad (74)$$

$$\overline{|0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} = (|0\rangle\langle 0| B_0^+ + |1\rangle\langle 1| B_1^+) |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_0^- + |1\rangle\langle 1| B_1^-) \quad (75)$$

$$= |0\rangle\langle 0| B_0^+ |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) |0\rangle\langle 0| B_0^- \quad (76)$$

$$= |0\rangle\langle 0| B_0^+ \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) B_0^- \quad (77)$$

$$= |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (78)$$

$$\overline{|1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} = (|0\rangle\langle 0| B_0^+ + |1\rangle\langle 1| B_1^+) |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_0^- + |1\rangle\langle 1| B_1^-) \quad (79)$$

$$= |1\rangle\langle 1| B_1^+ |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) |1\rangle\langle 1| B_1^- \quad (80)$$

$$= |1\rangle\langle 1| B_1^+ \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) B_1^- \quad (81)$$

$$= |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (82)$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \omega_{\mathbf{k}} (|0\rangle\langle 0| B_0^+ + |1\rangle\langle 1| B_1^+) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} (|0\rangle\langle 0| B_0^- + |1\rangle\langle 1| B_1^-) \quad (83)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) \quad (84)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| B_0^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_0^- + |1\rangle\langle 1| B_1^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_1^- \right) \quad (85)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \Pi_{\mathbf{k}' \neq \mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \Pi_{\mathbf{k}' \neq \mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) \right) \quad (86)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \mathbb{I} \right) \quad (87)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (88)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) \right) \quad (89)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (92)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \quad (93)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right). \quad (94)$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)} |0\rangle\langle 0| + \overline{\varepsilon_1(t)} |1\rangle\langle 1| + \overline{V_{10}(t)} |1\rangle\langle 0| + \overline{V_{01}(t)} |0\rangle\langle 1| \quad (95)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_1^+ B_0^- + V_{01}(t) |0\rangle\langle 1| B_0^+ B_1^-, \quad (96)$$

$$\overline{H_I} = \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (97)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (98)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (99)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (100)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (101)$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) \right) \quad (102)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) \right). \quad (103)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}) \right). \quad (104)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_S} + \overline{H_I} + \overline{H_B}. \quad (105)$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (106)$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right), \quad (107)$$

$$\chi_{ij} \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right). \quad (108)$$

χ_{ij} is an imaginary number so $e^{\chi_{ij}}$ is the phase associated to B_{ij} . We can summarize these definitions with other that we will proof and use from now in the following matrix:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_{ij} \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{\chi_{ij}} \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2i} & \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix}. \quad (109)$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \quad (110)$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \quad (111)$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_B T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})}. \quad (112)$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (113)$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (114)$$

So the product of the matrix representation of b^\dagger and b with $-\beta$ is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (115)$$

$$= \sum_{j=0}^{\infty} -j\beta\omega |j\rangle\langle j|, \quad (116)$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j .

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \quad (117)$$

$$\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \quad (118)$$

The value of $\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$ is:

$$\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (119)$$

$$= \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) \quad (120)$$

$$= \sum_{j_{\mathbf{k}}} \exp(-\beta\omega_{\mathbf{k}})^{j_{\mathbf{k}}} \quad (121)$$

$$= \frac{1}{1 - \exp(-\beta\omega_{\mathbf{k}})} \quad (\text{by geometric series}) \quad (122)$$

$$\equiv f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}), \quad (123)$$

$$\text{Tr}\left(\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (124)$$

$$= \prod_{\mathbf{k}} \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (125)$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}). \quad (126)$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})} \quad (127)$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (128)$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}. \quad (129)$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle, \quad (130)$$

$$b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle. \quad (131)$$

Then we can prove that $\langle B_{iz} \rangle_{\overline{H_B}} = 0$ using the following property based on (130)-(131):

$$\langle B_{iz} \rangle_{\overline{H_B}} = \text{Tr}(\rho_B B_{iz}) = \text{Tr}(B_{iz} \rho_B) \quad (132)$$

$$= \text{Tr} \left(\left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \right) \rho_B \right) \quad (133)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \rho_B \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \rho_B \right) \quad (134)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left(b_{\mathbf{k}}^{\dagger} \rho_B \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left(b_{\mathbf{k}} \rho_B \right) \quad (135)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) \quad (136)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left(b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right), \quad (137)$$

$$\text{Tr} \left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}^{\dagger}) \quad (138)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle \langle j_{\mathbf{k}}| \right) \quad (139)$$

$$= 0, \quad (140)$$

$$\text{Tr} \left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}) \quad (141)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle \langle j_{\mathbf{k}}| \right) \quad (142)$$

$$= 0. \quad (143)$$

we therefore find that:

$$\langle B_{iz} \rangle_{\overline{H_B}} = 0. \quad (144)$$

Another important expected value is $B = \langle B^{\pm} \rangle_{\overline{H_B}}$, where $B^{\pm} = e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}$ is given by:

$$\langle B^{\pm} \rangle_{H_B} = \text{Tr}(\rho_B B^{\pm}) = \text{Tr}(B^{\pm} \rho_B) \quad (145)$$

$$= \text{Tr} \left(e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \rho_B \right) \quad (146)$$

$$= \prod_{\mathbf{k}} \text{Tr}(D(\pm \alpha_{\mathbf{k}}) \rho_B) \quad (147)$$

$$= \prod_{\mathbf{k}} \langle D(\pm \alpha_{\mathbf{k}}) \rangle. \quad (148)$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha) |\alpha\rangle\langle\alpha| d^2\alpha. \quad (149)$$

where $P(\alpha)$ satisfies $\int P(\alpha) d^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P(\alpha)$ is given by:

$$\langle A \rangle = \text{Tr}(A\rho) \quad (150)$$

$$= \int P(\alpha) \langle \alpha | A | \alpha \rangle d^2\alpha. \quad (151)$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N} \exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = (e^{\beta\omega} - 1)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (151) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | D(h) | \alpha \rangle d^2\alpha \quad (152)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle d^2\alpha, \quad (153)$$

$$D(h) D(\alpha) = D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)}, \quad (154)$$

$$D(-\alpha) (D(h) D(\alpha)) = D(-\alpha) D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (155)$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (156)$$

$$= D(\alpha) e^{\frac{1}{2}(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2)} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (157)$$

$$= D(\alpha) e^{(h\alpha^* - h^*\alpha)}, \quad (158)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(h) \exp(h\alpha^* - h^*\alpha) | 0 \rangle d^2\alpha \quad (159)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) | 0 \rangle d^2\alpha \quad (160)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | h \rangle d^2\alpha, \quad (161)$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (162)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | \exp\left(-\frac{|h|^2}{2}\right) \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle d^2\alpha \quad (163)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|h|^2}{2}\right) d^2\alpha \quad (164)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha\right) d^2\alpha, \quad (165)$$

$$\alpha = x + iy, \quad (166)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 + y^2}{N} + h(x - iy) - h^*(x + iy)\right) dx dy \quad (167)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{N} + hx - h^*x\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N} - ihy - ih^*y\right) dy, \quad (168)$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N} (x^2 - Nhx + Nh^*x) \quad (169)$$

$$= -\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 + \frac{N(h^* - h)^2}{4}, \quad (170)$$

$$\frac{y^2}{N} - ihy - ih^*y = -\frac{1}{N} (y^2 + iNhy + iNh^*y) \quad (171)$$

$$= -\frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right) - \frac{N(h + h^*)^2}{4}, \quad (172)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 - \frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right)\right) dx dy, \quad (173)$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx, \quad (174)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x + \frac{(Nh^* - Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{\left(y^2 + \frac{iN(h + h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dy \quad (175)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \quad (176)$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right) \quad (177)$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N(h^{*2} - 2hh^* + h^2) - N(h^2 + 2hh^* + h^{*2})}{4}\right) \quad (178)$$

$$= \exp\left(-|h|^2 \left(N + \frac{1}{2}\right)\right) \quad (179)$$

$$= \exp\left(-|h|^2 \left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2}\right)\right) \quad (180)$$

$$= \exp\left(-\frac{|h|^2}{2} \left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1}\right)\right) \quad (181)$$

$$= \exp\left(-\frac{|h|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right). \quad (182)$$

In the last line we used $\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (147) using (182) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \quad (183)$$

We will now force $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B^+ \sigma^+$ and $B^- \sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$.

A final form of the terms of the Hamiltonian \overline{H} is:

$$\overline{H}(t) = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (184)$$

$$= \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{jj'} + \sum_j |j\rangle\langle j| B_{jj} + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| (B_j^+ B_{j'}^- - B_{jj'}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (185)$$

$$\equiv \overline{H_S}(t) + \overline{H_I} + \overline{H_B}. \quad (186)$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_i^+ B_j^- \rangle = B_{ij} \quad (187)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (188)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (189)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (190)$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (191)$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (192)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)}. \quad (193)$$

From the definition $B_{01} = \langle B_0^+ B_1^- \rangle$ using the displacement operator we have:

$$\langle B_0^+ B_1^- \rangle = B_{01} \quad (194)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (195)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (196)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (197)$$

$$= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (198)$$

$$= \prod_{\mathbf{k}} \left(\exp\left(-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (199)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)}. \quad (200)$$

We can check:

$$\langle B_0^+ B_1^- \rangle = B_{01} \quad (201)$$

$$= \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (202)$$

$$= \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)^*} \quad (203)$$

$$= \langle B_1^+ B_0^- \rangle^* \quad (204)$$

$$= B_{10}^*. \quad (205)$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}}(t) \equiv (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma^+ + V_{01}(t) B_{01} \sigma^-, \quad (206)$$

$$\overline{H_{\bar{I}}}(t) \equiv V_{10}(t) (B_1^+ B_0^- - B_{10}) \sigma^+ + V_{01}(t) (B_0^+ B_1^- - B_{01}) \sigma^- + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z}, \quad (207)$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (208)$$

$$= H_B. \quad (209)$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective “system” \bar{S} in the variational frame, is just the original bath, H_B , before transforming to the variational frame.

For the Hamiltonian (207) we can verify the condition $\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+ B_{j'}^- - B_{jj'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad (210)$$

$$= \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+ B_{j'}^- - B_{jj'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad (211)$$

$$= \sum_{n\mathbf{k}} \left(\left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| \left(\left\langle V_{jj'}(t) B_j^+ B_{j'}^- \right\rangle_{\overline{H_{\bar{B}}}} - \langle V_{jj'}(t) B_{jj'} \rangle_{\overline{H_{\bar{B}}}} \right) \quad (212)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}} \rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(\left\langle B_j^+ B_{j'}^- \right\rangle_{\overline{H_{\bar{B}}}} - \langle B_{jj'} \rangle_{\overline{H_{\bar{B}}}} \right) \quad (213)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}} \rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) (B_{jj'} - B_{jj'}) \quad (214)$$

$$= 0. \quad (215)$$

We used (144) and (193) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_x = B_x^\dagger \quad (216)$$

$$= \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}, \quad (217)$$

$$B_y = B_y^\dagger \quad (218)$$

$$= \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2i}, \quad (219)$$

$$B_{iz} = B_{iz}^\dagger \quad (220)$$

$$= \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \quad (221)$$

Writing the equations (206) and (207) using the previous combinations we obtain that:

$$\overline{H_S}(\hbar) = \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + V_{10}(t) B_{10} \sigma^+ + V_{01}(t) B_{01} \sigma^- \quad (222)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + V_{10}(t) B_{10} \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01} \frac{\sigma_x - i\sigma_y}{2} \quad (223)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + V_{10}(t) (B_{10}^{\Re}(t) + iB_{10}^{\Im}(t)) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) (B_{10}^{\Re}(t) - iB_{10}^{\Im}(t)) \frac{\sigma_x - i\sigma_y}{2} \quad (224)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) + iB_{10}^{\Im}(t) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) \quad (225)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + iB_{10}^{\Im}(t) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (226)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + B_{10}^{\Re}(t) (\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t)) + iB_{10}^{\Im}(t) (i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t)) \quad (227)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + B_{10}^{\Re}(t) (\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t)) + iB_{10}^{\Im}(t) (i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t)) \quad (228)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + (\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t)) - (\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \quad (229)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \quad (230)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \quad (231)$$

$$\overline{H_I} = V_{10}(t) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10}) + V_{01}(t) (\sigma^- B_0^+ B_1^- - \sigma^- B_{01}) + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} \quad (232)$$

$$= |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} + (V_{10}^{\Re}(t) + iV_{10}^{\Im}(t)) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10}) + (V_{10}^{\Re}(t) - iV_{10}^{\Im}(t)) (\sigma^- B_0^+ B_1^- - \sigma^- B_{01}) \quad (233)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10} + \sigma^- B_0^+ B_1^- - \sigma^- B_{01}) + iV_{10}^{\Im}(t) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10} - \sigma^- B_0^+ B_1^- + \sigma^- B_{01}) \quad (234)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+ B_0^- - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_0^+ B_1^- - \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (235)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+ B_0^- - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_0^+ B_1^- - \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (236)$$

$$+ iV_{10}^{\Im}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+ B_0^- - \frac{\sigma_x + i\sigma_y}{2} B_{10} - \frac{\sigma_x - i\sigma_y}{2} B_0^+ B_1^- + \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (237)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\sigma_x \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} + i\sigma_y \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2} \right) \quad (238)$$

$$+ iV_{10}^{\Im}(t) \left(\sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2} + i\sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (239)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) \left(i\sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2} - \sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (240)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) \left(i^2 \sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (241)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) \left(i^2 \sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (242)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) (i^2 \sigma_x (-B_y) - \sigma_y B_x) \quad (243)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) (\sigma_x B_y - \sigma_y B_x) \quad (244)$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right) + \langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} + O \left(\langle \overline{H_I}^2 \rangle_{\overline{H_S}(t) + \overline{H_B}} \right). \quad (245)$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_B (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = 0. \quad (246)$$

Using this condition and given that $[\overline{H_S}(t), \overline{H_B}] = 0$, we have:

$$e^{-\beta(\overline{H_S}(t) + \overline{H_B})} = e^{-\beta\overline{H_S}(t)} e^{-\beta\overline{H_B}}. \quad (247)$$

Then using the fact that $\overline{H_S}(t)$ and $\overline{H_B}$ relate to different Hilbert spaces, we obtain:

$$\text{Tr} \left(e^{-\beta\overline{H_S}(t)} e^{-\beta\overline{H_B}} \right) = \text{Tr} \left(e^{-\beta\overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta\overline{H_B}} \right). \quad (248)$$

So Eq. (246) becomes:

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (249)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta\overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta\overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (250)$$

$$= -\frac{1}{\beta} \frac{\partial \left(\ln \left(\text{Tr} \left(e^{-\beta\overline{H_S}(t)} \right) \right) + \ln \left(\text{Tr} \left(e^{-\beta\overline{H_B}} \right) \right) \right)}{\partial v_{i\mathbf{k}}} \quad (251)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta\overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta\overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (252)$$

$$= 0 \quad (\text{by Eq. (246)}). \quad (253)$$

But since $\overline{H_B} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\frac{\partial \ln \left(\text{Tr} \left(e^{-\beta\overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} = \frac{1}{e^{-\beta\overline{H_S}(t)}} \frac{\partial \text{Tr} \left(e^{-\beta\overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}} \quad (254)$$

$$= 0. \quad (255)$$

This means we need to impose:

$$\frac{\partial \text{Tr} \left(e^{-\beta\overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}} = 0. \quad (256)$$

First we look at:

$$-\beta \overline{H_{\bar{S}}}(t) = -\beta \left((\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma^+ + V_{01}(t) B_{01} \sigma^- \right). \quad (257)$$

Then the eigenvalues of $-\beta \overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}(-\beta \overline{H_{\bar{S}}}(t)) + \text{Det}(-\beta \overline{H_{\bar{S}}}(t)) = 0. \quad (258)$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr}(\overline{H_{\bar{S}}}(t)), \quad (259)$$

$$\eta \equiv \sqrt{(\text{Tr}(\overline{H_{\bar{S}}}(t)))^2 - 4\text{Det}(\overline{H_{\bar{S}}}(t))}. \quad (260)$$

The solutions of the equation (258) are:

$$\lambda = \beta \frac{-\text{Tr}(\overline{H_{\bar{S}}}(t)) \pm \sqrt{(\text{Tr}(\overline{H_{\bar{S}}}(t)))^2 - 4\text{Det}(\overline{H_{\bar{S}}}(t))}}{2} \quad (261)$$

$$= \beta \frac{-\varepsilon(t) \pm \eta(t)}{2} \quad (262)$$

$$= -\beta \frac{\varepsilon(t) \mp \eta(t)}{2}. \quad (263)$$

The value of $\text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)})$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)}) = \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \exp\left(\frac{\eta(t)\beta}{2}\right) + \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \exp\left(-\frac{\eta(t)\beta}{2}\right) \quad (264)$$

$$= 2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \cosh\left(\frac{\eta(t)\beta}{2}\right). \quad (265)$$

Given that v_{ik} is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)}) = A(\varepsilon(t), \eta(t))$ to calculate $\frac{\partial \text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)})}{\partial v_{ik}^{\Re}}$ can lead to:

$$\frac{\partial \text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)})}{\partial v_{ik}^{\Re}} = \frac{\partial \left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \cosh\left(\frac{\eta(t)\beta}{2}\right) \right)}{\partial v_{ik}^{\Re}} \quad (266)$$

$$= 2 \left(-\frac{\beta}{2} \frac{\partial \varepsilon(t)}{\partial v_{ik}^{\Re}} \right) \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \cosh\left(\frac{\eta(t)\beta}{2}\right) + 2 \left(\frac{\beta}{2} \frac{\partial \eta(t)}{\partial v_{ik}^{\Re}} \right) \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \sinh\left(\frac{\eta(t)\beta}{2}\right) \quad (267)$$

$$= -\beta \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \left(\frac{\partial \varepsilon(t)}{\partial v_{ik}^{\Re}} \cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial \eta(t)}{\partial v_{ik}^{\Re}} \sinh\left(\frac{\eta(t)\beta}{2}\right) \right). \quad (268)$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial \varepsilon(t)}{\partial v_{ik}^{\Re}} \cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial \eta(t)}{\partial v_{ik}^{\Re}} \sinh\left(\frac{\eta(t)\beta}{2}\right) = 0. \quad (269)$$

The derivates included in the expression given are related to:

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (270)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right)^* \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (271)$$

$$= \langle B_1^+ B_0^- \rangle^*, \quad (272)$$

$$R_i = \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (273)$$

$$= \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (274)$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (275)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{1}{2\omega_{\mathbf{k}}^2} (v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*) \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (276)$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = (v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im}) (v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im}) - (v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im}) (v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im}) \quad (277)$$

$$= (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}) - (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} + i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}) \quad (278)$$

$$= 2i (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re}), \quad (279)$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}} - v_{0\mathbf{k}}) (v_{1\mathbf{k}} - v_{0\mathbf{k}})^* \quad (280)$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - (v_{1\mathbf{k}} v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}}) \quad (281)$$

$$= (v_{1\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im})^2 + (v_{0\mathbf{k}}^{\Re})^2 + (v_{0\mathbf{k}}^{\Im})^2 - ((v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im})(v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im}) + (v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im})(v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im})) \quad (282)$$

$$= (v_{1\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im})^2 + (v_{0\mathbf{k}}^{\Re})^2 + (v_{0\mathbf{k}}^{\Im})^2 - 2 (v_{1\mathbf{k}}^{\Re} v_{0\mathbf{k}}^{\Re} + v_{1\mathbf{k}}^{\Im} v_{0\mathbf{k}}^{\Im}) \quad (283)$$

$$= (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2. \quad (284)$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left(\frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re} - i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re} + i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \right) \right) \quad (285)$$

$$= \sum_{\mathbf{k}} \left(\frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}} - g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right), \quad (286)$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} \exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (287)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{2i (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re})}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (288)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{i (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (289)$$

Calculating the derivatives respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (290)$$

$$= \frac{\partial \left(\left(\frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (291)$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \quad (292)$$

$$\frac{\partial |B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \left(\exp \left(- \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (293)$$

$$= - \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} \exp \left(- \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (294)$$

$$= - \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2, \quad (295)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \sqrt{\left(\text{Tr} \left(\overline{H_{\bar{S}}(t)} \right) \right)^2 - 4 \text{Det} \left(\overline{H_{\bar{S}}(t)} \right)}}{\partial v_{i\mathbf{k}}^{\Re}} \quad (296)$$

$$= \frac{2 \text{Tr} \left(\overline{H_{\bar{S}}(t)} \right) \frac{\partial \text{Tr} \left(\overline{H_{\bar{S}}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}} - 4 \frac{\partial \text{Det} \left(\overline{H_{\bar{S}}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2 \sqrt{\left(\text{Tr} \left(\overline{H_{\bar{S}}(t)} \right) \right)^2 - 4 \text{Det} \left(\overline{H_{\bar{S}}(t)} \right)}} \quad (297)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1)(\varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta(t)} \quad (298)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i) \left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (299)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i) \left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{i\mathbf{k}}^{\Re} - v_{i\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (300)$$

$$= \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(- \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \varepsilon(t) \right) \quad (301)$$

$$+ 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + 4 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right). \quad (302)$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh \left(\frac{\beta \eta(t)}{2} \right) = \frac{\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \quad (303)$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left(2 \frac{\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + 2 \frac{(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} + 2 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) - \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \varepsilon(t)}{\eta(t)}}}. \quad (304)$$

Rearrannng this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{(2v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}} - g_{i\mathbf{k}}^*)\eta(t)}{v_{i\mathbf{k}}^{\Re} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} + g_{i\mathbf{k}}^*)(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (305)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re}(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (306)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 4\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (307)$$

$$= \frac{(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (308)$$

Separating (307) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re})\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Re} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (309)$$

$$v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re} = v_{i\mathbf{k}}^{\Re} \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (310)$$

$$v_{i\mathbf{k}}^{\Re} = \frac{g_{i\mathbf{k}}^{\Re} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}, \quad (311)$$

$$v_{i\mathbf{k}}^{\Re} = \frac{g_{i\mathbf{k}}^{\Re} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}. \quad (312)$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (313)$$

$$= \frac{\partial \left(\left(\frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (314)$$

$$= 2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (315)$$

$$\frac{\partial |B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial \left(\exp \left(- \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (316)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Im}} \exp\left(-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (317)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Im}} |B_{10}|^2, \quad (318)$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \quad (319)$$

$$= \frac{2\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right) \frac{\partial\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} - 4\frac{\partial\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \quad (320)$$

$$= \frac{\varepsilon(t) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial((\varepsilon_1(t) + R_1)(\varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta(t)} \quad (321)$$

$$= \frac{\varepsilon(t) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \quad (322)$$

$$= \frac{\varepsilon(t) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2(v_{i\mathbf{k}}^{\Im} - v_{i'\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \quad (323)$$

$$= \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(-i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i) i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right). \quad (324)$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \quad (325)$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(\left(\frac{g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)^{\Im} \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \left(\frac{g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)^{\Im} + 2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}. \quad (326)$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - i\left(g_{i\mathbf{k}}^* - g_{i\mathbf{k}}\right) (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (327)$$

$$= \frac{2\left(\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^{\Im}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Im} (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (328)$$

$$= \frac{2\left(\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^{\Im}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (329)$$

$$= \frac{\left(\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^{\Im}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \quad (330)$$

Separating (330) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(v_{i\mathbf{k}}^{\mathfrak{S}} - g_{i\mathbf{k}}^{\mathfrak{S}})\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\mathfrak{S}} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\mathfrak{S}} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\mathfrak{S}}}{\omega_{\mathbf{k}}} |B_{10}V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (331)$$

$$v_{i\mathbf{k}}^{\mathfrak{S}} - g_{i\mathbf{k}}^{\mathfrak{S}} = v_{i\mathbf{k}}^{\mathfrak{S}} \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\mathfrak{S}} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \quad (332)$$

$$+ 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\mathfrak{S}}}{\omega_{\mathbf{k}}} |B_{10}V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (333)$$

$$v_{i\mathbf{k}}^{\mathfrak{S}} = \frac{g_{i\mathbf{k}}^{\mathfrak{S}} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\mathfrak{S}}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}, \quad (334)$$

$$v_{i\mathbf{k}}^{\mathfrak{S}} = \frac{g_{i\mathbf{k}}^{\mathfrak{S}} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\mathfrak{S}}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}. \quad (335)$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}) = v_{i\mathbf{k}}^{\mathfrak{R}}(\omega_{\mathbf{k}}) + i v_{i\mathbf{k}}^{\mathfrak{S}}(\omega_{\mathbf{k}}) \quad (336)$$

$$g_{i\mathbf{k}}^{\mathfrak{R}}(\omega_{\mathbf{k}}) \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\mathfrak{R}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \\ = \frac{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}{\quad} \quad (337)$$

$$+ i \frac{g_{i\mathbf{k}}^{\mathfrak{S}}(\omega_{\mathbf{k}}) \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\mathfrak{S}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (338)$$

$$= \frac{g_{i\mathbf{k}}(\omega_{\mathbf{k}}) \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}. \quad (339)$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$, where $\rho_B^{\text{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T(0) e^{-V} \quad (340)$$

$$= (|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+) (\rho_S(0) \otimes \rho_B) (|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-), \quad (341)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: |0\rangle\langle 0|B_0^+|0\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^- \quad (342)$$

$$= |0\rangle\langle 0|B_0^+|0\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^- \quad (343)$$

$$= |0\rangle\langle 0| \otimes B_0^+ \rho_B B_0^-, \quad (344)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: |1\rangle\langle 1|B_1^+|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^- \quad (345)$$

$$= |1\rangle\langle 1|B_1^+ \rho_B B_1^- \quad (346)$$

$$= |1\rangle\langle 1| \otimes B_1^+ \rho_B B_1^-, \quad (347)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1|: |0\rangle\langle 0|B_0^+|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^- \quad (348)$$

$$= |0\rangle\langle 1|B_0^+ \rho_B |1\rangle\langle 1|B_1^- \quad (349)$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_0^+ \rho_B B_1^- \quad (350)$$

$$= |0\rangle\langle 1| \otimes B_0^+ \rho_B B_1^-, \quad (351)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0|: |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^- \quad (352)$$

$$= |1\rangle\langle 0| \otimes B_1^+ \rho_B B_0^-. \quad (353)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O}(t) \equiv U^\dagger(t) O(t) U(t), \quad (354)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (355)$$

Here \mathcal{T} denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (356)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}_T(t)). \quad (357)$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \quad (358)$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_I}(t)$ as pointed in the equation (236):

$$\overline{H_I}(t) = \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) (\sigma_x B_y - \sigma_y B_x) \quad (359)$$

$$= B_{0z} |0\rangle\langle 0| + B_{1z} |1\rangle\langle 1| + V_{10}^{\Re}(t) \sigma_x B_x + V_{10}^{\Re}(t) \sigma_y B_y + V_{10}^{\Im}(t) \sigma_x B_y - V_{10}^{\Im}(t) \sigma_y B_x \quad (360)$$

$$= \sum_i C_i(t) (A_i \otimes B_i(t)). \quad (361)$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \widetilde{\rho_T}(t)]. \quad (362)$$

This equation has the formal solution

$$\widetilde{\rho_T}(t) = \overline{\rho_T}(0) - i \int_0^t [\widetilde{H_I}(s), \widetilde{\rho_T}(s)] ds. \quad (363)$$

Replacing the equation (363) in the equation (362) gives us:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_T}(s)]] ds. \quad (364)$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\rho_T}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \overline{\rho_T}(0)]] \dots]. \quad (365)$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho_S}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \overline{\rho_S}(0) \rho_B]] \dots]. \quad (366)$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\rho_S}(t) = (1 + W_1(t) + W_2(t) + \dots) \overline{\rho_S}(0) \quad (367)$$

$$= W(t) \overline{\rho_S}(0). \quad (368)$$

in this case

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), (\cdot) \rho_B]] \dots]. \quad (369)$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = (\dot{W}_1(t) + \dot{W}_2(t) + \dots) \overline{\rho_S}(0) \quad (370)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} W(t) \overline{\rho_S}(0) \quad (371)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} \widetilde{\rho_S}(t). \quad (372)$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\text{Tr}_B [\widetilde{H_I}(t) \rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (370) becomes:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = \dot{W}_2(t) \widetilde{\rho_S}(t) \quad (373)$$

$$= - \int_0^t dt_1 \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(t_1), \widetilde{\rho_S}(t) \rho_B]]. \quad (374)$$

Replacing $t_1 \rightarrow t - \tau$

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \int_0^t d\tau \text{Tr}_B [\overline{H_I}(t), [\widetilde{\overline{H_I}}(-\tau), \overline{\rho_S}(t) \rho_B]] . \quad (375)$$

From the interaction picture applied on $\overline{H_I}(t)$ we find:

$$\widetilde{\overline{H_I}}(t) = U^\dagger(t) e^{iH_B t} \overline{H_I}(t) e^{-iH_B t} U(t) . \quad (376)$$

we use the time-ordering operator \mathcal{T} because in general $\overline{H_S}(t)$ doesn't commute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_I}}(t) = \sum_i C_i(t) (\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)) , \quad (377)$$

$$\widetilde{A_i}(t) = U^\dagger(t) e^{iH_B t} A_i e^{-iH_B t} U(t) \quad (378)$$

$$= U^\dagger(t) A_i U(t) e^{iH_B t} e^{-iH_B t} \quad (379)$$

$$= U^\dagger(t) A_i U(t) \mathbb{I} \quad (380)$$

$$= U^\dagger(t) A_i U(t) , \quad (381)$$

$$\widetilde{B_i}(t) = U^\dagger(t) e^{iH_B t} B_i(t) e^{-iH_B t} U(t) \quad (382)$$

$$= U^\dagger(t) U(t) e^{iH_B t} B_i(t) e^{-iH_B t} \quad (383)$$

$$= \mathbb{I} e^{iH_B t} B_i(t) e^{-iH_B t} \quad (384)$$

$$= e^{iH_B t} B_i(t) e^{-iH_B t} . \quad (385)$$

Here we have used the fact that $[\overline{H_S}(t), H_B] = 0$ because these operators belong to different Hilbert spaces, so $[U(t), e^{iH_B t}] = 0$.

Using the expression (377) to replace it in the equation (374)

$$\frac{d\widetilde{\overline{\rho_S}}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{\overline{H_I}}(t), [\widetilde{\overline{H_I}}(s), \widetilde{\overline{\rho_S}}(t) \rho_B]] ds \quad (386)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \left[\sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) , \widetilde{\overline{\rho_S}}(t) \rho_B \right] \right] ds \quad (387)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B - \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \right] ds \quad (388)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B - \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \right) \quad (389)$$

$$- \sum_{i \in J} C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) + \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \Big) ds. \quad (390)$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \langle \widetilde{B_j}(t) \widetilde{B_i}(s) \rangle_B \quad (391)$$

$$= \langle \widetilde{B_j}(\tau) \widetilde{B_i}(0) \rangle_B . \quad (392)$$

The correlation functions relevant that appear in the equation (390) are:

$$\text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B \right) = \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (393)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (394)$$

$$= \Lambda_{ji}(\tau), \quad (395)$$

$$\text{Tr}_B \left(\widetilde{B}_j(t) \rho_B \widetilde{B}_i(s) \right) = \text{Tr}_B \left(\widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B \right) \quad (396)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (397)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (398)$$

$$= \Lambda_{ij}(-\tau), \quad (399)$$

$$\text{Tr}_B \left(\widetilde{B}_i(s) \rho_B \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B \right) \quad (400)$$

$$= \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (401)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (402)$$

$$= \Lambda_{ji}(\tau), \quad (403)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B \right) \quad (404)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (405)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (406)$$

$$= \Lambda_{ij}(-\tau). \quad (407)$$

The cyclic property of the trace was use widely in the development of equations (393) and (407). Replacing in (390)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = -\int_0^t \sum_{ij} (C_i(t) C_j(s) (\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s)) + C_i(t) C_j(s) (\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t))) ds \quad (408)$$

$$= -\int_0^t \sum_{ij} \left(C_i(t) C_j(s) \left(\Lambda_{ij}(\tau) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] + \Lambda_{ji}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t) \right] \right) \right) ds. \quad (409)$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) = \Lambda_{ij}(\tau) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right], \quad (410)$$

$$\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s) = \Lambda_{ji}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t) \right]. \quad (411)$$

Returning to the Schroedinger picture we have:

$$U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) = U(t) \widetilde{A}_i(t) U^\dagger(t) U(t) \widetilde{A}_j(s) U^\dagger(t) U(t) \widetilde{\rho_S}(t) U^\dagger(t), \quad (412)$$

$$= \left(U(t) \widetilde{A}_i(t) U^\dagger(t) \right) \left(U(t) \widetilde{A}_j(s) U^\dagger(t) \right) \left(U(t) \widetilde{\rho_S}(t) U^\dagger(t) \right), \quad (413)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t). \quad (414)$$

This procedure applying to the relevant commutators give us:

$$U(t) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] U^\dagger(t) = \left(U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) - U(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) U^\dagger(t) \right) \quad (415)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) - \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) A_i \quad (416)$$

$$= \left[A_i, \widetilde{A}_j(t - \tau, t) \widetilde{\rho_S}(t) \right]. \quad (417)$$

Introducing this transformed commutators in the equation (409) allow us to obtain the master equation of the system

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau \left(C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[A_i, \widetilde{A}_j(t-\tau, t) \overline{\rho_S}(t) \right] \right. \quad (418)$$

$$\left. + C_j(t) C_i(t-\tau) \Lambda_{ji}(-\tau) \left[\overline{\rho_S}(t) \widetilde{A}_j(t-\tau, t), A_i \right] \right). \quad (419)$$

where $i, j \in \{1, 2, 3, 4, 5, 6\}$.

Here $\widetilde{A}_j(s, t) = U(t) U^\dagger(s) A_j U(s) U^\dagger(t)$ where $U(t)$ is given by (355). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. The environmental correlation functions are given by:

$$\Lambda_{ij}(\tau) = \text{Tr}_B \left(\widetilde{B}_i(t) \widetilde{B}_j(s) \rho_B \right) \quad (420)$$

$$= \text{Tr}_B \left(\widetilde{B}_i(\tau) \widetilde{B}_j(0) \rho_B \right). \quad (421)$$

Calculating the correlation functions allow us to obtain:

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left(\widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rho_B \right) \quad (422)$$

$$= \int d^2\alpha P(\alpha) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle \quad (423)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle d^2\alpha \quad (424)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle d^2\alpha, \quad (425)$$

$$\widetilde{B}_{jz}(\tau) = \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right), \quad (426)$$

$$\widetilde{B}_{jz}(0) = \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right), \quad (427)$$

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left(\widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rho_B \right) \quad (428)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (429)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (430)$$

$$+ \text{Tr}_B \left(\sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \rho_B \right), \quad (431)$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}} = p_{j\mathbf{k}}, \quad (432)$$

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} (p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) (p_{j\mathbf{k}'} b_{\mathbf{k}'}^\dagger + p_{j\mathbf{k}'}^* b_{\mathbf{k}'}) \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} (p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) (p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger + p_{j\mathbf{k}}^* b_{\mathbf{k}}) \rho_B \right) \quad (433)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 (2 \cos(\omega_{\mathbf{k}}\tau) N + e^{-i\omega_{\mathbf{k}}\tau}) \quad (461)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2 \cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} + e^{-i\omega_{\mathbf{k}}\tau} \right) \quad (462)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{2 \cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} + \cos(\omega_{\mathbf{k}}\tau) - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (463)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{(2 + e^{\beta\omega_{\mathbf{k}}} - 1) \cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (464)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{(1 + e^{\beta\omega_{\mathbf{k}}}) \cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (465)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{\frac{\beta\omega_{\mathbf{k}}}{2}} \right) \cos(\omega_{\mathbf{k}}\tau)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{\frac{\beta\omega_{\mathbf{k}}}{2}}} - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (466)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \cos(\omega_{\mathbf{k}}\tau) - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (467)$$

$$= \sum_{\mathbf{k}} |g_{j\mathbf{k}} - v_{j\mathbf{k}}|^2 \left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \cos(\omega_{\mathbf{k}}\tau) - i \sin(\omega_{\mathbf{k}}\tau) \right), \quad (468)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \rangle_B = \int d^2 \alpha_{\mathbf{k}} P(\alpha_{\mathbf{k}}) \langle \alpha_{\mathbf{k}} | \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) | \alpha_{\mathbf{k}} \rangle \quad (469)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (470)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) \sum_{\mathbf{k}'} ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}'} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (471)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k} \neq \mathbf{k}'} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}^*) b_{\mathbf{k}'} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (472)$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (473)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (474)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} | \alpha_{\mathbf{k}} \rangle^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} | \alpha_{\mathbf{k}} \rangle^2 \alpha_{\mathbf{k}} \quad (475)$$

$$= \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}}, \quad (476)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (477)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (478)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (479)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}|^2 d^2 \alpha_{\mathbf{k}} \quad (480)$$

$$= N, \quad (481)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (482)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (483)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (484)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (485)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (486)$$

$$= N + 1, \quad (487)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \rangle_B = \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N+1) \quad (488)$$

$$= \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} + N ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau})), \quad (489)$$

$$D(h') D(h) = \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) D(h' + h), \quad (490)$$

$$\langle D(h') D(h) \rangle_B = \text{Tr}_B \left(\exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) D(h' + h) \rho_B \right) \quad (491)$$

$$= \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) \text{Tr}_B (D(h' + h) \rho_B) \quad (492)$$

$$= \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) \frac{1}{\pi N} \int d^2 \alpha P(\alpha) \langle \alpha | D(h' + h) | \alpha \rangle \quad (493)$$

$$= \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \quad (494)$$

$$h' = h \exp(i\omega\tau), \quad (495)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp\left(\frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau))\right) \exp\left(-\frac{|h + h \exp(i\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \quad (496)$$

$$\frac{1}{2} |h|^2 (\exp(i\omega\tau) - \exp(-i\omega\tau)) = \frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau)) \quad (497)$$

$$= \frac{1}{2} |h|^2 (\cos(\omega\tau) + i \sin(\omega\tau) - \cos(\omega\tau) + i \sin(\omega\tau)) \quad (498)$$

$$= \frac{1}{2} |h|^2 (2i \sin(\omega\tau)) \quad (499)$$

$$= i |h|^2 \sin(\omega\tau), \quad (500)$$

$$-\frac{|h + h \exp(i\omega\tau)|^2}{2} = -|h|^2 \frac{|1 + \exp(i\omega\tau)|^2}{2} \quad (501)$$

$$= -|h|^2 \frac{|1 + \cos(\omega\tau) + i \sin(\omega\tau)|^2}{2} \quad (502)$$

$$= -|h|^2 \frac{(1 + \cos(\omega\tau))^2 + \sin^2(\omega\tau)}{2} \quad (503)$$

$$= -|h|^2 \frac{(1 + 2 \cos(\omega\tau) + \cos^2(\omega\tau)) + \sin^2(\omega\tau)}{2} \quad (504)$$

$$= -|h|^2 \frac{2 + 2 \cos(\omega\tau)}{2} \quad (505)$$

$$= -|h|^2 (1 + \cos(\omega\tau)), \quad (506)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp(i|h|^2 \sin(\omega\tau)) \exp\left(-|h|^2(1 + \cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (507)$$

$$= \exp \left(i |h|^2 \sin(\omega\tau) - |h|^2 (1 + \cos(\omega\tau)) \coth \left(\frac{\beta\omega}{2} \right) \right) \quad (508)$$

$$= \exp \left(-|h|^2 \left(-i \sin(\omega\tau) + \cos(\omega\tau) \coth \left(\frac{\beta\omega}{2} \right) \right) \right) \exp \left(-|h|^2 \coth \left(\frac{\beta\omega}{2} \right) \right) \quad (509)$$

$$= \langle D(h) \rangle_B \exp(-\phi(\tau)), \quad (510)$$

$$\exp(-\phi(\tau)) = \exp\left(-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - i\sin(\omega\tau)\right)\right), \quad (511)$$

$$\phi(\tau) = |h|^2 \left(\cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau) \right), \quad (512)$$

$$\langle D(h') D(h) \rangle_B = \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \exp\left(-\frac{|h+h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \quad (513)$$

$$h' = v \exp(i\omega\tau), \quad (514)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \right\rangle_B = \text{Tr}_B \left(\widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \rho_B \right) \quad (515)$$

$$= \text{Tr}_B \left(\widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \rho_B \right) \quad (516)$$

$$= \text{Tr}_B \left(\prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right)^{\rho_B} \right) \quad (517)$$

$$= \text{Tr}_{\mathbf{B}} \left(\Pi_{\mathbf{K}} \left(D \left(\frac{v_{1\mathbf{K}} - v_{0\mathbf{K}}}{\omega_{\mathbf{K}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{K}}^* v_{0\mathbf{K}} - v_{1\mathbf{K}} v_{0\mathbf{K}}^*}{\omega_{\mathbf{K}}^2} \right)} D \left(\frac{v_{1\mathbf{K}} - v_{0\mathbf{K}}}{\omega_{\mathbf{K}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{K}}^* v_{0\mathbf{K}} - v_{1\mathbf{K}} v_{0\mathbf{K}}^*}{\omega_{\mathbf{K}}^2} \right)} \right) \rho_{\mathbf{B}} \right) \quad (518)$$

$$= \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)^{\rho_B} \right) \quad (519)$$

$$= \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \Pi_{\mathbf{k}} \left(\exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (520)$$

$$= \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (521)$$

$$\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_0^+ B_1^-(0)} \rangle_B = \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_{\mathbf{k}}^* v_{\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (522)$$

$$\left\langle \widetilde{B_1^+ B_0^-(\tau)} \widetilde{B_0^+ B_1^-(0)} \right\rangle_B = \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B \right) \quad (523)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B \right) \right) \quad (524)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \Pi_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \Pi_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B \right) \right) \quad (525)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} D \left(\frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) \Pi_{\mathbf{k}} D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \rho_B \right) \quad (526)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left(\left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \rho_B \right) \quad (527)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left(\left(D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i(\omega \tau + \pi)} \right) D \left(\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \right) \rho_B \right) \quad (528)$$

$$= \prod_{\mathbf{k}} \exp \left(- \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right)^2 \left(-i \sin(\omega \tau + \pi) + \cos(\omega \tau + \pi) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \exp \left(- \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right)^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \quad (529)$$

$$= \Pi_{\mathbf{k}} \exp \left(- \left| \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \left(i \sin(\omega \tau) - \cos(\omega \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (530)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \right\rangle_B = \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B \right) \quad (531)$$

$$= \text{Tr}_B \left(\Pi_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \Pi_{\mathbf{k}} D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B \right) \quad (532)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i(\omega_{\mathbf{k}}\tau + \pi)} \right) D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B \right) \quad (533)$$

$$= \Pi_{\mathbf{k}} \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}}\tau + \pi) + \cos(\omega_{\mathbf{k}}\tau + \pi) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (534)$$

$$= \left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B, \quad (535)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_{jz}^-}(0) \right\rangle_B = \text{Tr}_B \left(\Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right), \quad (536)$$

$$\langle D(h)b \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h)b | \alpha \rangle \quad (537)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | D(-\alpha) D(h) b D(\alpha) | \alpha \rangle \quad (538)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b D(\alpha) | 0 \rangle \quad (539)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \quad (540)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \quad (541)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b + \alpha) | 0 \rangle \quad (542)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (543)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (544)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(- \frac{|h|^2}{2} \right) d^2 \alpha \quad (545)$$

$$= hN \langle D(h) \rangle_B, \quad (546)$$

$$\langle D(h)b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h)b^\dagger | \alpha \rangle \quad (547)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (548)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (549)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) | 0 \rangle \quad (550)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b^\dagger + \alpha^*) | 0 \rangle \quad (551)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b^\dagger + \alpha^*) | 0 \rangle \quad (552)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \quad (553)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \quad (554)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle -h | 1 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \langle 0 | D(h) | 0 \rangle, \quad (555)$$

$$\langle -h | = \exp \left(- \frac{|-h^*|^2}{2} \right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n |, \quad (556)$$

$$\langle -h | 1 \rangle = \exp \left(- \frac{|-h^*|^2}{2} \right) (-h^*), \quad (557)$$

$$\langle D(h)b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(- \frac{|-h^*|^2}{2} \right) (-h^*) + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \exp \left(- \frac{|-h^*|^2}{2} \right) \quad (558)$$

$$= -h^* \langle D(h) \rangle_B (N+1), \quad (559)$$

$$\langle bD(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | bD(h) | \alpha \rangle \quad (560)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(- \frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left(- \frac{|h|^2}{2} \right) \quad (561)$$

$$= h \langle D(h) \rangle_B (N+1), \quad (562)$$

$$\langle b^\dagger D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | b^\dagger D(h) | \alpha \rangle \quad (563)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(-\frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left(-\frac{|h|^2}{2} \right) \quad (564)$$

$$= -h^* \langle D(h) \rangle_B N, \quad (565)$$

$$\langle \widetilde{B_1^+ B_0^-}(\tau) \rangle_B = \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle_B \quad (566)$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \langle D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \rangle_B \quad (567)$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \langle D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \rangle_B \quad (568)$$

$$= \prod_{\mathbf{k}} \left(\exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp \left(-\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (569)$$

$$= B_{10}. \quad (570)$$

The correlation functions can be found readily as:

$$\widetilde{B_1^+ B_0^-}(\tau) = \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right), \quad (571)$$

$$\widetilde{B_0^+ B_1^-}(\tau) = \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right), \quad (572)$$

$$\widetilde{B_x}(0) = \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}, \quad (573)$$

$$\widetilde{B_y}(0) = \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2i}, \quad (574)$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (575)$$

$$B_{iz} = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right), \quad (576)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} \tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} \tau} \right) \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \right\rangle_B \quad (577)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}} \tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}} \tau} (N_{\mathbf{k}} + 1), \quad (578)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \left\langle \frac{B_1^+ B_0^- (\tau) + B_0^+ B_1^- (\tau) - B_{10} - B_{01}}{2} \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right\rangle_B \quad (579)$$

$$= \frac{1}{4} \left\langle \left(B_1^+ B_0^- (\tau) + B_0^+ B_1^- (\tau) - B_{10} - B_{01} \right) \left(B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01} \right) \right\rangle_B \quad (580)$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^- (\tau) B_1^+ B_0^- + B_1^+ B_0^- (\tau) B_0^+ B_1^- - B_1^+ B_0^- (\tau) B_{10} - B_1^+ B_0^- (\tau) B_{01} + B_0^+ B_1^- (\tau) B_1^+ B_0^- + B_0^+ B_1^- (\tau) B_0^+ B_1^- \right. \quad (581)$$

$$\left. - B_0^+ B_1^- (\tau) B_{10} - B_0^+ B_1^- (\tau) B_{01} + B_1^+ B_0^- (\tau) B_{10} - B_1^+ B_0^- (\tau) B_{01} + B_0^+ B_1^- (\tau) B_{10} - B_0^+ B_1^- (\tau) B_{01} + B_{10} B_{01} \right\rangle \quad (582)$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^- (\tau) B_1^+ B_0^- + B_1^+ B_0^- (\tau) B_0^+ B_1^- - B_1^+ B_0^- (\tau) B_{10} - B_1^+ B_0^- (\tau) B_{01} + B_0^+ B_1^- (\tau) B_1^+ B_0^- + B_0^+ B_1^- (\tau) B_0^+ B_1^- \right. \quad (583)$$

$$\left. + B_0^+ B_1^- (\tau) B_{10} - B_0^+ B_1^- (\tau) B_{01} \right\rangle, \quad (584)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B = \Pi_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (585)$$

$$U = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right), \quad (586)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (587)$$

$$S = \prod_{\mathbf{k}} \exp \left(- \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (588)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B = U \exp(-\phi(\tau)) S, \quad (589)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \right\rangle_B = U^* \exp(-\phi(\tau)) S, \quad (590)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B = \exp(\phi(\tau)) S, \quad (591)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \right\rangle_B = \left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B, \quad (592)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \right\rangle_B = \prod_{\mathbf{k}} \left(\exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp \left(- \frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (593)$$

$$= U^{*1/2} S^{1/2}, \quad (594)$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \right\rangle_B = \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \quad (595)$$

$$\left. + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right\rangle, \quad (596)$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \right\rangle_B = \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \quad (597)$$

$$\left. - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right\rangle \quad (598)$$

$$= \frac{1}{4} \left(U^* \exp(-\phi(\tau)) S + \exp(\phi(\tau)) S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau)) S + U \exp(-\phi(\tau)) S - B_{10}^{*2} - |B_{10}|^2 \right) \quad (599)$$

$$= \frac{1}{4} \left(2U^{\Re} \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2(B_{10}^2)^{\Re} - 2|B_{10}|^2 \right) \quad (600)$$

$$= \frac{1}{4} \left(2U^{\Re} \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2(U^*)^{\Re} S - 2S \right) \quad (601)$$

$$= \frac{S}{2} \left(U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - (U^*)^{\Re} - 1 \right), \quad (602)$$

$$\left\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \right\rangle_B = \left\langle \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \right\rangle_B \quad (603)$$

$$= -\frac{1}{4} \left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01} \right) \left(B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01} \right) \right\rangle_B \quad (604)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_1^+ B_0^-(\tau) B_1^+ B_0^- \right. \quad (605)$$

$$\left. - B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} + B_{10} B_0^+ B_1^- - B_{10} B_1^+ B_0^- + B_{10} B_{10} - B_{10} B_{01} - B_{01} B_0^+ B_1^- + B_{01} B_1^+ B_0^- - B_{01} B_{10} + B_{01} B_{01} \right\rangle \quad (606)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right. \quad (607)$$

$$\left. - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} \right\rangle \quad (608)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_{01} B_{10} - B_{01} B_{01} - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_{10} B_{10} + B_{10} B_{01} \right\rangle \quad (609)$$

$$= -\frac{1}{4} \left(U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S - \exp(\phi(\tau)) S + U^* \exp(-\phi(\tau)) S + 2S - 2(U^*)^{\Re} S \right) \quad (610)$$

$$= -\frac{S}{4} \left(2U^{\Re} \exp(-\phi(\tau)) - 2\exp(\phi(\tau)) + 2 - 2U^{\Re} \right) \quad (611)$$

$$= \frac{S}{2} \left(\exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re} \right), \quad (612)$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \right\rangle_B = \left\langle \frac{B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01}}{2} \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \right\rangle_B \quad (613)$$

$$= \frac{1}{4i} \left\langle \left(B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01} \right) \left(B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01} \right) \right\rangle_B \quad (614)$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \quad (615)$$

$$\left. + B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_{10} B_0^+ B_1^- + B_{10} B_1^+ B_0^- - B_{10} B_{10} + B_{10} B_{01} - B_{01} B_0^+ B_1^- + B_{01} B_1^+ B_0^- - B_{01} B_{10} + B_{01} B_{01} \right\rangle \quad (616)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_{10}^- - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_{10} \rangle \quad (617)$$

$$- B_0^+ B_1^-(\tau) B_{01} \rangle \quad (618)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_{10} B_{10}^- - B_{10} B_{01} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_{01} B_{10} - B_{01} B_{01} \rangle \quad (619)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_{10} B_{10} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- - B_{01} B_{01} \rangle \quad (620)$$

$$= \frac{1}{4i} (\exp(\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S + U^* S - US) \quad (621)$$

$$= \frac{1}{4i} (-U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S + U^* S - US) \quad (622)$$

$$= \frac{S}{4i} (-U^* \exp(-\phi(\tau)) + U \exp(-\phi(\tau)) + U^* - U) \quad (623)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (624)$$

$$= \frac{2iU^3 S}{4i} (\exp(-\phi(\tau)) - 1) \quad (625)$$

$$= \frac{U^3 S}{2} (\exp(-\phi(\tau)) - 1), \quad (626)$$

$$\langle \widetilde{B} y(\tau) \widetilde{B} x(0) \rangle_B = \left\langle \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right\rangle_B \quad (627)$$

$$= \frac{1}{4i} \left\langle \left(B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01} \right) \left(B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01} \right) \right\rangle_B \quad (628)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- \rangle \quad (629)$$

$$+ B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} + B_{10} B_1^+ B_0^- + B_{10} B_0^+ B_1^- - B_{10} B_{10} - B_{10} B_{01} - B_{01} B_1^+ B_0^- - B_{01} B_0^+ B_1^- + B_{01} B_{10} + B_{01} B_{01} \rangle \quad (630)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- \rangle \quad (631)$$

$$+ B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} \rangle \quad (632)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_{01} B_{10} - B_{01} B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_{10} B_{10} + B_{10} B_{01} \rangle \quad (633)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_{01} B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_{10} B_{10} \rangle \quad (634)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + B_{10}^2 - B_{10}^{*2}) \quad (635)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U^* S - US) \quad (636)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (637)$$

$$= \frac{2iU^3 S}{4i} (\exp(-\phi(\tau)) - 1) \quad (638)$$

$$= -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}, \quad (639)$$

$$\langle B_1^+ B_0^-(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \Pi_{\mathbf{k}} \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}'}^* v_{0\mathbf{k}'} - v_{1\mathbf{k}'} v_{0\mathbf{k}'}}{\omega_{\mathbf{k}}^2} \right)\right) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} \langle \Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right) \rangle \quad (640)$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}, \quad (641)$$

$$\langle B_0^+ B_1^-(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger \rangle_B = -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01}. \quad (642)$$

$$\langle B_0^+ B_1^-(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}, \quad (643)$$

$$\langle \widetilde{B} x(\tau) \widetilde{B} iz(0) \rangle_B = \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01} \right. \quad (644)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \quad (645)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01} \right. \quad (646)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \quad (647)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) (N_{\mathbf{k}'} + 1) \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{01} \right) \right. \quad (648)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) B_{01} \right) \right) \quad (649)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(- \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} \right) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(- \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (708)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \quad (709)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} \left(- (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (710)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \quad (711)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} \left((-g_{i\mathbf{k}'} + v_{i\mathbf{k}'}) \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (712)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \quad (713)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + N_{\mathbf{k}'}) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (B_{10} + (N_{\mathbf{k}'} + 1)) \right) \quad (714)$$

$$= \frac{1}{i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) \right) \quad (715)$$

$$= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} \right) \quad (716)$$

$$= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} \right) \quad (717)$$

$$= i B_{10}^{\Re} \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* N_{\mathbf{k}'} \right). \quad (718)$$

The correlation functions are equal to:

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (719)$$

$$U = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right), \quad (720)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (721)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{|B_{10}|^2}{2} \left(U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^{\Re} - 1 \right), \quad (722)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{|B_{10}|^2}{2} \left(\exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re} \right), \quad (723)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (724)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (725)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (726)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (727)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = i B_{10}^{\Re} \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right), \quad (728)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Re} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right). \quad (729)$$

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \quad (730)$$

$$v_{i\mathbf{k}} = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}). \quad (731)$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i(\omega)}{\omega} d\omega. \quad (732)$$

The integral version of the correlation functions are given by:

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}})(g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^*(g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (733)$$

$$= \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))^*(g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}})) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (734)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) g_{j\mathbf{k}}^* (1 - F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}))^* g_{j\mathbf{k}} (1 - F_j(\omega_{\mathbf{k}})) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (735)$$

$$\approx \int_0^\infty \left(\sqrt{J_i(\omega) J_j^*(\omega)} (1 - F_i(\omega)) (1 - F_j^*(\omega)) e^{i\omega\tau} N(\omega) + \sqrt{J_i^*(\omega) J_j(\omega)} (1 - F_i^*(\omega)) (1 - F_j(\omega)) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega, \quad (736)$$

$$U = \prod_{\mathbf{k}} \left(\exp \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (737)$$

$$= \exp \left(\sum_{\mathbf{k}} \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \quad (738)$$

$$= \exp \left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}) g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (739)$$

$$= \exp \left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}) g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (740)$$

$$= \exp \left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* g_{1\mathbf{k}} F_0^*(\omega_{\mathbf{k}}) F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} g_{1\mathbf{k}}^* F_0(\omega_{\mathbf{k}}) F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (741)$$

$$\approx \exp \left(\int_0^\infty \frac{\sqrt{J_0^*(\omega) J_1(\omega)} F_0^*(\omega) F_1(\omega) - \sqrt{J_0(\omega) J_1^*(\omega)} F_0(\omega) F_1^*(\omega)}{\omega^2} d\omega \right), \quad (742)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (743)$$

$$= \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \left(-i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (744)$$

$$\approx \int_0^\infty \left| \frac{\sqrt{J_1(\omega) F_1(\omega)} - \sqrt{J_0(\omega) F_0(\omega)}}{\omega} \right|^2 \left(-i \sin(\omega\tau) + \cos(\omega\tau) \coth \left(\frac{\beta\omega}{2} \right) \right) d\omega, \quad (745)$$

$$B_{10} = \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left(\sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (746)$$

$$= \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left(\sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (747)$$

$$= \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left(\sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \right) \quad (748)$$

$$\approx \exp \left(-\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1(\omega) F_1(\omega)} - \sqrt{J_0(\omega) F_0(\omega)}}{\omega} \right|^2 \coth \left(\frac{\beta\omega}{2} \right) d\omega \right) \exp \left(\int_0^\infty \frac{1}{2} \left(\frac{\sqrt{J_0(\omega) J_1^*(\omega)} F_0(\omega) F_1^*(\omega) - \sqrt{J_0^*(\omega) J_1(\omega)} F_0^*(\omega) F_1(\omega)}{\omega^2} \right) d\omega \right), \quad (749)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{|B_{10}|^2}{2} (U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^{\Re} - 1), \quad (750)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{|B_{10}|^2}{2} (\exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re}), \quad (751)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (752)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (753)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (754)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))^* \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (755)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right)^* - g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}))^* \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (756)$$

$$Q(\omega) = \sqrt{J_i(\omega)} (1 - F_i(\omega)) \left(\frac{\sqrt{J_1(\omega)} F_1(\omega) - \sqrt{J_0(\omega)} F_0(\omega)}{\omega} \right)^*, \quad (757)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B \approx i B_{10}^{\Im} \int_0^\infty (Q(\omega) N(\omega) e^{i\omega\tau} - Q^*(\omega) (N(\omega) + 1) e^{-i\omega\tau}) d\omega, \quad (758)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (759)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left(g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} - g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (760)$$

$$\approx iB_{10}^{\mathfrak{S}} \int_0^\infty \left(Q^*(\omega) N(\omega) e^{i\omega\tau} - Q(\omega) (N(\omega) + 1) e^{-i\omega\tau} \right) d\omega, \quad (761)$$

$$\langle \widetilde{B}_{iz}(\tau) \widetilde{B}_y(0) \rangle_B = iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (762)$$

$$= iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (763)$$

$$\approx iB_{10}^{\mathfrak{R}} \int_0^\infty \left(e^{-i\omega\tau} Q^*(\omega) (N(\omega) + 1) - e^{i\omega\tau} Q(\omega) N(\omega) \right) d\omega, \quad (764)$$

$$\langle \widetilde{B}_y(\tau) \widetilde{B}_{iz}(0) \rangle_B = iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \quad (765)$$

$$= iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left(g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \quad (766)$$

$$= iB_{10}^{\mathfrak{R}} \int_0^\infty \left(e^{i\omega\tau} Q^*(\omega) N(\omega) - e^{-i\omega\tau} Q(\omega) (N(\omega) + 1) \right) d\omega. \quad (767)$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}(\overline{H_{\bar{S}}}) \lambda + \text{Det}(\overline{H_{\bar{S}}}) = 0. \quad (768)$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S |\pm\rangle = \lambda_{\pm} |\pm\rangle$. Using this notation is possible to write $H_{\bar{S}} = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$.

The time-dependence of the system operators $\widetilde{A}_i(t)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A}_i(\tau) = e^{i\overline{H_{\bar{S}}}\tau} A_i e^{-i\overline{H_{\bar{S}}}\tau} \quad (769)$$

$$= \sum_w e^{-i w \tau} A_i(w). \quad (770)$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.

In order to use the equation (770) to descompose the equation (355) we need to consider the time ordering operator \mathcal{T} , it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right)$ like:

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right) \quad (771)$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) \dots H(t_n). \quad (772)$$

Here $0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right) \equiv \exp(-it H_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots$ so we can write:

$$U(t) = \exp \left(-it \left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots \right) \right). \quad (773)$$

The terms can be found expanding $\mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_S}(t') dt', \quad (774)$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' [\overline{H_S}(t'), \overline{H_S}(t'')], \quad (775)$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' ([\overline{H_S}(t'), \overline{H_S}(t'')], \overline{H_S}(t''')) + [[\overline{H_S}(t'''), \overline{H_S}(t'')], \overline{H_S}(t')]. \quad (776)$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A}_i(t) = U^\dagger(t) A_i(t) U(t) \quad (777)$$

$$= e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \quad (778)$$

$$= \sum_{w(t)} e^{iw(t)t} A_i(w(t)). \quad (779)$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A}_j(t - \tau, t)$ using the Magnus expansion generates:

$$\widetilde{A}_j(t - \tau, t) = U(t - \tau) U^\dagger(t) A_j(t) U(t) U^\dagger(t - \tau) \quad (780)$$

$$= e^{-i(t-\tau)H_E(t-\tau)} e^{iH_E(t)t} A_j(t) e^{-iH_E(t)t} e^{i(t-\tau)H_E(t-\tau)} \quad (781)$$

$$= e^{-i(t-\tau)H_E(t-\tau)} \sum_{w(t)} e^{iw(t)t} A_j(w(t)) e^{i(t-\tau)H_E(t-\tau)} \quad (782)$$

$$= \sum_{w(t), w'(t-\tau)} e^{iw(t)t} e^{-iw'(t-\tau)} A'_j(w(t), -w'(t-\tau)). \quad (783)$$

$$\widetilde{A}_j(t - \tau, t) = U(t) U^\dagger(t - \tau) A_j(t) U(t - \tau) U^\dagger(t) \quad (784)$$

$$= e^{-itH_E(t)} e^{i(t-\tau)H_E(t-\tau)} A_j(t) e^{-i(t-\tau)H_E(t-\tau)} e^{itH_E(t)} \quad (785)$$

$$= e^{-itH_E(t)} \left(\sum_{w'(t-\tau)} e^{w'(t-\tau)} A_j(w'(t-\tau)) \right) e^{itH_E(t)} \quad (786)$$

$$= \sum_{w(t), w'(t-\tau)} e^{iw(t)t} e^{-iw'(t-\tau)} A'_j(w(t), -w'(t-\tau)). \quad (787)$$

where $w'(t - \tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}(t - \tau)$ and $\overline{H_E}(t)$ respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (770) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $V = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle |\alpha\rangle \langle \beta|. \quad (788)$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{i\overline{H_E}\tau} = e^{i(\lambda_+|+\rangle\langle+| + \lambda_-|-\rangle\langle-|)\tau} \quad (789)$$

$$= e^{i\lambda_+|+\rangle\langle+|\tau} e^{i\lambda_-|-\rangle\langle-|\tau} \quad (790)$$

$$= (|-\rangle\langle-| + e^{i\lambda_+\tau}|+\rangle\langle+|) (|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|) \quad (791)$$

$$= e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|. \quad (792)$$

Calculating the transformation (770) directly using the previous relationship we find that:

$$U^\dagger(\tau) A_i(\tau) U(\tau) = (e^{i\lambda_+\tau} |+\rangle\langle+| + e^{i\lambda_-\tau} |-\rangle\langle-|) \left(\sum_{\alpha, \beta \in V} \langle \alpha | A_i(\tau) | \beta \rangle | \alpha \rangle \langle \beta | \right) (e^{-i\lambda_+\tau} |+\rangle\langle+| + e^{-i\lambda_-\tau} |-\rangle\langle-|) \quad (793)$$

$$= \langle + | A_i(\tau) | + \rangle |+\rangle\langle+| + e^{i\eta\tau} \langle + | A_i(\tau) | - \rangle |+\rangle\langle-| + e^{-i\eta\tau} \langle - | A_i(\tau) | + \rangle |-\rangle\langle+| + \langle - | A_i(\tau) | - \rangle |-\rangle\langle-|. \quad (794)$$

$$= \mathcal{A}_i(0) + \mathcal{A}_i(-w) e^{iw\tau} + \mathcal{A}_i(w) e^{-iw\tau} \quad (795)$$

Here $w = \lambda_+ - \lambda_-$. Comparing the RHS of the equations (770) and the explicit expression for $\widetilde{A}_i(\tau)$ in (778), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$\mathcal{A}_i(0) = \langle + | A_i(\tau) | + \rangle |+\rangle\langle+| + \langle - | A_i(\tau) | - \rangle |-\rangle\langle-|, \quad (796)$$

$$\mathcal{A}_i(-w) = \langle + | A_i(\tau) | - \rangle |+\rangle\langle-|, \quad (797)$$

$$\mathcal{A}_i(w) = \langle - | A_i(\tau) | + \rangle |-\rangle\langle+|. \quad (798)$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A}_i(\tau) = \widetilde{A}_i^\dagger(\tau)$ and $\widetilde{B}_i(\tau) = \widetilde{B}_i^\dagger(\tau)$ we can use the equation (770) to write the master equation in the following neater form:

$$\frac{d\overline{\rho}_S(t)}{dt} = -i[\overline{H}_S(t), \overline{\rho}_S(t)] - \frac{1}{2} \sum_{ij} \sum_{w, w'} \gamma_{ij}(w, w', t) [A_i, A_j(w, w') \overline{\rho}_S(t) - \overline{\rho}_S(t) A_j^\dagger(w, w')] - i \sum_{ij} \sum_w S_{ij}(w, w', t) [A_i, A_j(w, w') \overline{\rho}_S(t) + \overline{\rho}_S(t) A_j^\dagger(w, w')]. \quad (799)$$

where $\mathcal{A}_j^\dagger(w) = \mathcal{A}_j(-w)$ as expected from the equations (797) and (798). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}(w, w', t) = 2K_{ij}^{\mathcal{R}}(w, w', t)$ and $S_{ij}(w, w', t) = K_{ij}^{\mathcal{S}}(w, w', t)$, respectively, defined in terms of the response functions

$$K_{ij}(w, w', t) = \int_0^t C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) e^{iw\tau} e^{-it(w-w')} d\tau \quad (800)$$

$$= K_{ijww'}(t). \quad (801)$$

If we extend the upper limit of integration to ∞ in the equation (800) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{d\overline{\rho}_S}{dt} = -i[\overline{H}_S(t), \overline{\rho}_S(t)] - \sum_{ijww'} \left(K_{ijww'}(t) [A_i, A_{jww'} \overline{\rho}_S(t)] - K_{ijww'}^*(t) [A_i, \overline{\rho}_S(t) A_{jww'}^\dagger] \right) \quad (802)$$

$$= -i[\overline{H}_S(t), \overline{\rho}_S(t)] - \sum_{ijww'} \left(K_{ijww'}^{\mathcal{R}}(t) [A_i, A_{jww'} \overline{\rho}_S(t) - \overline{\rho}_S(t) A_{jww'}^\dagger] + iK_{ijww'}^{\mathcal{S}}(t) [A_i, A_{jww'} \overline{\rho}_S(t) + \overline{\rho}_S(t) A_{jww'}^\dagger] \right) \quad (803)$$

Applying the inverse transformation we will obtain that:

$$e^{-V} \frac{d\overline{\rho}_S}{dt} e^V = \frac{d(e^{-V} \overline{\rho}_S e^V)}{dt} \quad (804)$$

$$= \frac{d\rho_S}{dt} \quad (805)$$

$$= -ie^{-V} [\overline{H}_S(t), \overline{\rho}_S(t)] e^V - \sum_{ijww'} K_{ijww'}(t) e^{-V} [A_i, A_{jww'} \overline{\rho}_S(t) - \overline{\rho}_S(t) A_{jww'}^\dagger] e^V. \quad (806)$$

For a product we have the following:

$$e^{-V} \overline{AB} e^V = e^{-V} \overline{A} \overline{B} e^V \quad (807)$$

$$= e^{-V} \overline{A} e^V e^{-V} \overline{B} e^V \quad (808)$$

$$= (e^{-V} \overline{A} e^V) (e^{-V} \overline{B} e^V) \quad (809)$$

$$= \overline{AB}. \quad (810)$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V} [A, B] e^V = e^{-V} (\overline{AB} - \overline{BA}) e^V \quad (811)$$

$$= e^{-V} \overline{AB} e^V - e^{-V} \overline{BA} e^V \quad (812)$$

$$= \overline{AB} - \overline{BA} \quad (813)$$

$$= [A, B]. \quad (814)$$

So we will obtain that

$$\frac{d\rho_S}{dt} = -i e^{-V} [H_{\bar{S}}(t), \bar{\rho}_S(t)] e^V - \sum_{ijww'} K_{ijww'}(t) e^{-V} [A_i, A_{jww'} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jww'}^\dagger] e^V \quad (815)$$

$$= -i [H_{\bar{S}}(t), \rho_S(t)] - \sum_{ijww'} K_{ijww'}(t) [e^{-V} A_i e^V, e^{-V} A_{jww'} \bar{\rho}_S(t) e^V - e^{-V} \bar{\rho}_S(t) A_{jww'}^\dagger e^V] \quad (816)$$

$$= -i [H_{\bar{S}}(t), \rho_S(t)] - \sum_{ijww'} K_{ijww'}(t) [e^{-V} A_i e^V, e^{-V} A_{jww'} e^V e^{-V} \bar{\rho}_S(t) e^V - e^{-V} \bar{\rho}_S(t) e^V e^{-V} A_{jww'}^\dagger e^V] \quad (817)$$

$$= -i [H_{\bar{S}}(t), \rho_S(t)] - \sum_{ijww'} K_{ijww'}(t) [e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_S(t) - \rho_S(t) e^{-V} A_{jww'}^\dagger e^V] \quad (818)$$

$$= -i [H_{\bar{S}}(t), \rho_S(t)] - \left(\sum_{ijww'} K_{ijww'}(t) ([e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_S(t)] - [e^{-V} A_i e^V, \rho_S(t) e^{-V} A_{jww'}^\dagger e^V]) \right). \quad (819)$$

V. LIMIT CASES

In order to show the plausibility of the master equation (799) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (799) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1, 2, 3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) |1\rangle\langle 1| + \frac{\Omega}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (820)$$

After performing the transformation (24) on the Hamiltonian (820) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R) |1\rangle\langle 1| + \frac{\Omega_r}{2} \sigma_x, \quad (821)$$

$$\overline{H_I} = B_z |1\rangle\langle 1| + \frac{\Omega}{2} (B_x \sigma_x + B_y \sigma_y), \quad (822)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (823)$$

The Hamiltonian (821) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z + \frac{\Omega_r}{2} \sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R) |1\rangle\langle 1| - \frac{\delta}{2} \mathbb{I} = \left(\frac{\delta}{2} + R \right) |1\rangle\langle 1| - \frac{\delta}{2} |0\rangle\langle 0| \quad (824)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\delta + R}{2} \sigma_z \quad (825)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z. \quad (826)$$

In this Hamiltonian we can write $A_1 = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I + \sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_S}$. Given that $\overline{H_S} = \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z + \frac{\Omega_r}{2} \sigma_x$ then $\text{Tr}(\overline{H_S}) = R$ and $\text{Det}(\overline{H_S}) = \frac{R^2 - \epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by:

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \quad (827)$$

$$\lambda_{\pm} = \frac{R \pm \sqrt{(-R)^2 - 4 \left(\frac{R^2 - \epsilon^2 - \Omega_r^2}{4} \right)}}{2} \quad (828)$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \quad (829)$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \quad (830)$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \quad (831)$$

$$\lambda_{\pm} = \frac{R \pm \eta}{2}. \quad (832)$$

For $\lambda_+ = \frac{R + \eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R + \eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R + \eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \quad (833)$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $-\frac{\epsilon + \eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon + \eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}|0\rangle + \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.

For $\lambda_- = \frac{R - \eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R - \eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R - \eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \quad (834)$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon + \eta}{2}b = 0$, so $a = -\frac{\epsilon + \eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$. Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon + \eta}{2}, \quad (835)$$

$$\lambda_- = \frac{\epsilon - \eta}{2}, \quad (836)$$

$$|+\rangle = \sin(\theta) |0\rangle + \cos(\theta) |1\rangle, \quad (837)$$

$$|-\rangle = \cos(\theta) |0\rangle - \sin(\theta) |1\rangle, \quad (838)$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}, \quad (839)$$

$$\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}. \quad (840)$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{\Omega_r}{\epsilon} \right). \quad (841)$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1} \left(\frac{\Omega_r}{\epsilon} \right)$.

$$\tan \left(\frac{x}{2} \right) = \frac{\sin(x)}{\cos(x) + 1}. \quad (842)$$

So the value of $\tan(\theta)$ is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} + 1} \quad (843)$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}}{\frac{\epsilon + \sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}} \quad (844)$$

$$= \frac{\Omega_r}{\epsilon + \eta}. \quad (845)$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (797)-(798) and the fact that $|+\rangle = \sin(\theta) |0\rangle + \cos(\theta) |1\rangle = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$ and $|-\rangle = \cos(\theta) |0\rangle - \sin(\theta) |1\rangle = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (846)$$

$$= 2 \sin(\theta) \cos(\theta) \quad (847)$$

$$= \sin(2\theta), \quad (848)$$

$$\langle -|\sigma_x|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (849)$$

$$= -2 \sin(\theta) \cos(\theta) \quad (850)$$

$$= -\sin(2\theta), \quad (851)$$

$$\langle -|\sigma_x|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (852)$$

$$= \cos^2(\theta) - \sin^2(\theta) \quad (853)$$

$$= \cos(2\theta), \quad (854)$$

$$\langle +|\sigma_y|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (855)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (856)$$

$$= 0, \quad (857)$$

$$\langle -|\sigma_y|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (858)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (859)$$

$$= 0, \quad (860)$$

$$\langle -|\sigma_y|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (861)$$

$$= i \cos^2(\theta) + i \sin^2(\theta) \quad (862)$$

$$= i. \quad (863)$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (864)$$

$$= \cos(\theta) \cos(\theta) \quad (865)$$

$$= \cos^2(\theta), \quad (866)$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (867)$$

$$= \sin(\theta) \sin(\theta) \quad (868)$$

$$= \sin^2(\theta), \quad (869)$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (870)$$

$$= -\sin(\theta) \cos(\theta) \quad (871)$$

$$= -\sin(\theta) \cos(\theta). \quad (872)$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta) (|+\rangle\langle+| - |-\rangle\langle-|), \quad (873)$$

$$A_1(\eta) = \cos(2\theta) |-\rangle\langle+|, \quad (874)$$

$$A_2(0) = 0, \quad (875)$$

$$A_2(\eta) = i|-\rangle\langle+|, \quad (876)$$

$$A_3(0) = \cos^2(\theta) |+\rangle\langle+| + \sin^2(\theta) |-\rangle\langle-|, \quad (877)$$

$$A_3(\eta) = -\sin(\theta) \cos(\theta) |-\rangle\langle+|. \quad (878)$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (421) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau). \quad (879)$$

Using the notation of the master equation (799), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations (719)-(729) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (421) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left(\widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (880)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \quad (881)$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left(\widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (882)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right), \quad (883)$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega J(\omega) (1 - F(\omega))^2 G_+(\tau), \quad (884)$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2} \int_0^\infty d\omega \frac{J(\omega)}{\omega} F(\omega) (1 - F(\omega)) iG_-(\tau), \quad (885)$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \quad (886)$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \quad (887)$$

$$= \Lambda'_{13}(\tau) \quad (888)$$

$$= \Lambda'_{31}(\tau) \quad (889)$$

$$= 0. \quad (890)$$

Finally taking the Hamiltonian (820) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\text{Det}(\overline{H_S}) = -\frac{\Omega_r^2}{4}$, $\text{Tr}(\overline{H_S}) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (336) we have that:

$$f_k = \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k} \right)} \quad (891)$$

$$= \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k} \right)}. \quad (892)$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (419) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (799) and using the fact that

$$\widetilde{A_j}(t-\tau, t) = U(\tau) A_j U(-\tau) \quad (893)$$

$$= \sum_w e^{i w \tau} A_i(-w) \quad (894)$$

$$= \sum_w e^{-i w \tau} A_i(w). \quad (895)$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \frac{1}{2} \sum_{ij} \sum_w \gamma_{ij}(w, t) [A_i, A_j(w) \overline{\rho_S}(t) - \overline{\rho_S}(t) A_j^\dagger(w)] \quad (896)$$

$$- \sum_{ij} \sum_w S_{ij}(w, t) [A_i, A_j(w) \overline{\rho_S}(t) + \overline{\rho_S}(t) A_j^\dagger(w)]. \quad (897)$$

where $A_j^\dagger(w) = A(-w)$, as we can see the equation (897) contains the rates and energy shifts $\gamma_{ij}(w, t) = 2K_{ij}^{\Re}(w, t)$ and $S_{ij}(w, t) = K_{ij}^{\Im}(w, t)$, respectively, defined in terms of the response functions

$$K_{ij}(w, t) = \int_0^t \Lambda'_{ij}(\tau) e^{i w \tau} d\tau. \quad (898)$$

This last proposition can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{i\overline{H_S}(t-\tau)} = e^{i(\lambda_+|+\rangle\langle+| + \lambda_-|-\rangle\langle-|)(t-\tau)} \quad (899)$$

$$= e^{i\lambda_+|+\rangle\langle+|(t-\tau)} e^{i\lambda_-|-\rangle\langle-|(t-\tau)} \quad (900)$$

$$= \left(|-\rangle\langle-| + e^{i\lambda_+(t-\tau)}|+\rangle\langle+|\right) \left(|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|\right) \quad (901)$$

$$= e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|. \quad (902)$$

Where λ_+, λ_- are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (770) of (796)-(798) directly using the previous relationship we find that:

$$\widetilde{A_i(0)}(t-\tau) = \left(e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|\right) (\langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|) \left(e^{-i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-i\lambda_-(t-\tau)}|-\rangle\langle-|\right) \quad (903)$$

$$= \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|, \quad (904)$$

$$\widetilde{A_i(w)}(t-\tau) = \left(e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|\right) (\langle+|A_i|-\rangle|+\rangle\langle-|) \left(e^{-i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-i\lambda_-(t-\tau)}|-\rangle\langle-|\right) \quad (905)$$

$$= \langle+|A_i|-\rangle|+\rangle\langle-| e^{i w (t-\tau)}, \quad (906)$$

$$\widetilde{A_i(-w)}(t-\tau) = \left(e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|\right) (\langle-|A_i|+\rangle|-\rangle\langle+|) \left(e^{-i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-i\lambda_-(t-\tau)}|-\rangle\langle-|\right) \quad (907)$$

$$= \langle-|A_i|+\rangle|-\rangle\langle+| e^{-i w (t-\tau)}. \quad (908)$$

Here $w = \lambda_+ - \lambda_-$. So we can see that for the equation (780) it's possible to deduce for this case of time-independent matrix \overline{H}_S if $w \neq w'$ then $A'_j(w, w') = 0$ so:

$$\begin{aligned}
\widetilde{A}_j(t - \tau, t) &= U(t) U^\dagger(t - \tau) A_j(t) U(t - \tau) U^\dagger(t) \\
&= U(t) \left(\sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} A_j(w(t-\tau)) \right) U^\dagger(t) \\
&= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} U(t) A_j(w(t-\tau)) U^\dagger(t) \\
&= \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_j(w(t-\tau), w'(t)) \\
&= \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_{jww'} \\
&= \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_j(w) \delta_{ww'} \\
&= \sum_w e^{-i(t-\tau)w} e^{itw} A_j(w) \\
&= \sum_w e^{i\tau w} A_j(w) \\
&= U^\dagger(-\tau) A_j U(-\tau)
\end{aligned}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right) \right) |1\rangle\langle 1| + \frac{\Omega(t)}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (909)$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (909) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (225) and (236) we obtain the following transformed Hamiltonians:

$$\overline{H}_S = (\delta + R_1) |1\rangle\langle 1| + \frac{B\sigma_x}{2} \Omega(t), \quad (910)$$

$$\overline{H}_I = \frac{\Omega(t)}{2} (B_x \sigma_x + B_y \sigma_y). \quad (911)$$

In this case $R_1 = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} g_{\mathbf{k}} \right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}}^{-1} |g_{\mathbf{k}}|^2) = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}} | \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} |^2)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (880)-(887) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} G_+(\tau) d\omega. \quad (912)$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau)$ so (912) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left(\coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau) \right) d\omega. \quad (913)$$

Writing the interaction Hamiltonian (911) in the similar way to the equation (236) allow us to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (225) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$ is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t). \quad (914)$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \text{Tr}_B \left(\widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (915)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \quad (916)$$

$$\Lambda_{22}(\tau) = \text{Tr}_B \left(\widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (917)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right). \quad (918)$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(419) is:

$$\frac{d\rho_S(t)}{dt} = -i \left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t) \right] - \sum_{i=1}^2 \int_0^t d\tau \left(C_i(t) C_i(t-\tau) \Lambda_{ii}(\tau) \left[A_i, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right] \right. \quad (919)$$

$$\left. + C_i(t) C_i(t-\tau) \Lambda_{ii}(-\tau) \left[\rho_S(t) \widetilde{A}_i(t-\tau, t), A_i \right] \right). \quad (920)$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A}_i(t-\tau, t) = \widetilde{\sigma}_i(t-\tau, t)$, also using the equations (915) and (918) on the equation (920) we obtain that:

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{2} [\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4} \int_0^t d\tau \Omega(t-\tau) ([\sigma_x, \widetilde{\sigma}_x(t-\tau, t) \rho_S(t)] \Lambda_x(\tau) \quad (921)$$

$$+ [\sigma_y, \widetilde{\sigma}_y(t-\tau, t) \rho_S(t)] \Lambda_y(\tau) + [\rho_S(t) \widetilde{\sigma}_x(t-\tau, t), \sigma_x] \Lambda_x(\tau) + [\rho_S(t) \widetilde{\sigma}_y(t-\tau, t), \sigma_y] \Lambda_y(\tau)). \quad (922)$$

As we can see $\left[A_j, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right]^\dagger = \left[\rho_S(t) \widetilde{A}_i(t-\tau, t), A_j \right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation (S17) of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (421) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\hbar = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right). \quad (923)$$

Using the equation (799), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{d\rho_S}{dt} = -i [H_S(t), \rho_S(t)] - \frac{1}{2} \sum_w \gamma_{33}(w, t) \left[A_3, A_3(w) \rho_S(t) - \rho_S(t) A_3^\dagger(w) \right] \quad (924)$$

$$- \sum_w S_{33}(w, t) \left[A_3, A_3(w) \rho_S(t) + \rho_S(t) A_3^\dagger(w) \right] \Bigg). \quad (925)$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty d\omega J(\omega) G_+(\tau), \quad (926)$$

$$\Lambda_{33}(-\tau) = \int_0^\infty d\omega J(\omega) G_+(-\tau). \quad (927)$$

In our case $A_3 = \frac{\mathbb{I} + \sigma_z}{2}$, the equation (925) can be transformed in

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \sum_w (K_{33}(w, t) [A_3, A_3(w) \rho_S(t)] + K_{33}^*(w, t) [\rho_S(t) A_3(w), A_3]). \quad (928)$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthermore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\tilde{\Omega}$. To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian using the approximation $\tilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\tilde{\Omega}}{2}, \quad (929)$$

$$|+\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |0\rangle), \quad (930)$$

$$\lambda_- = -\frac{\tilde{\Omega}}{2}, \quad (931)$$

$$|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \quad (932)$$

The elements of the decomposition matrices are:

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (933)$$

$$= \frac{1}{2}, \quad (934)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (935)$$

$$= \frac{1}{2}, \quad (936)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (937)$$

$$= -\frac{1}{2}. \quad (938)$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2} |+\rangle\langle +| + \frac{1}{2} |-\rangle\langle -| \quad (939)$$

$$= \frac{\mathbb{I}}{2}, \quad (940)$$

$$A_3(\eta) = -\frac{1}{2} |-\rangle\langle +| \quad (941)$$

$$= \frac{1}{4} (\sigma_z + i\sigma_y), \quad (942)$$

$$A_3(-\eta) = -\frac{1}{2} |+\rangle\langle -| \quad (943)$$

$$= \frac{1}{4} (\sigma_z - i\sigma_y). \quad (944)$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - K_{33}(\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + i\sigma_y) \rho_S(t) \right] - K_{33}(-\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - i\sigma_y) \rho_S(t) \right] \quad (945)$$

$$- K_{33}^*(\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4}(\sigma_z + i\sigma_y), \frac{\sigma_z}{2} \right] - K_{33}^*(-\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4}(\sigma_z - i\sigma_y), \frac{\sigma_z}{2} \right]. \quad (946)$$

Calculating the response functions extending the upper limit of τ to ∞ , we obtain:

$$K_{33}(\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{i\tilde{\Omega}\tau} d\tau d\omega \quad (947)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (948)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} (n(\omega) + 1) e^{-i\tau\omega} d\tau d\omega \quad (949)$$

$$= \int_0^\infty \int_0^\infty J(\omega) (n(\omega) + 1) e^{i\tilde{\Omega}\tau - i\tau\omega} d\tau d\omega \quad (950)$$

$$= \int_0^\infty J(\omega) (n(\omega) + 1) \pi \delta(\tilde{\Omega} - \omega) d\omega \quad (951)$$

$$= \pi J(\tilde{\Omega}) (n(\tilde{\Omega}) + 1), \quad (952)$$

$$K_{33}(-\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{-i\tilde{\Omega}\tau} d\tau d\omega \quad (953)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (954)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} n(\omega) e^{i\tau\omega} d\tau d\omega \quad (955)$$

$$= \int_0^\infty \int_0^\infty J(\omega) n(\omega) e^{-i\tilde{\Omega}\tau + i\tau\omega} d\tau d\omega \quad (956)$$

$$= \int_0^\infty J(\omega) n(\omega) \pi \delta(-\tilde{\Omega} + \omega) d\omega \quad (957)$$

$$= \pi J(\tilde{\Omega}) n(\tilde{\Omega}). \quad (958)$$

Here we have used $\int_0^\infty ds e^{\pm i\epsilon s} = \pi \delta(\epsilon) \pm i \frac{\text{V.P.}}{\epsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (945) lead us to obtain:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\tilde{\Omega}) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (959)$$

$$- \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\tilde{\Omega}) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right). \quad (960)$$

This is the same result than the equation (S17), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{d\rho_S(t)}{dt} = -i\frac{\Omega(t)}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\Omega(t)) ((n(\Omega(t)) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\Omega(t)) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)]) \quad (961)$$

$$- \frac{\pi}{8} J(\Omega(t)) ((n(\Omega(t)) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\Omega(t)) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z]). \quad (962)$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (963)$$

$$H_S(t) = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (964)$$

$$H_I = \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left(g_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} \right), \quad (965)$$

$$H_B = \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger b_{\mathbf{u}\mathbf{k}}. \quad (966)$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \omega_{\mathbf{u}\mathbf{k}}^{-1} \left(f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} \right) \quad (967)$$

At first let's obtain $e^{\pm V}$ under the transformation (967), consider $\hat{\varphi}_n = \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}}^{-1} \left(f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} \right)$, so the equation (967) can be written as $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \quad (968)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{(\sum_n |n\rangle\langle n| \hat{\varphi}_n)^2}{2!} + \dots \quad (969)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (970)$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (971)$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots \right) \quad (972)$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \quad (973)$$

Given that $\left[f_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - f_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}}, f_{n\mathbf{u}'\mathbf{k}'} b_{\mathbf{u}'\mathbf{k}'}^\dagger - f_{n\mathbf{u}'\mathbf{k}'}^* b_{\mathbf{u}'\mathbf{k}'} \right] = 0$ for all \mathbf{k}', \mathbf{k} and \mathbf{u}, \mathbf{u}' then we can proof using the Zassenhaus formula and defining $D(\pm \alpha_{n\mathbf{u}\mathbf{k}}) = e^{\pm (\alpha_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger - \alpha_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}})}$ in the same way than (23) with $\alpha_{n\mathbf{u}\mathbf{k}} = \frac{f_{n\mathbf{u}\mathbf{k}}}{\omega_{\mathbf{u}\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} (f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}})} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1} (f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}})} \quad (974)$$

$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1} (f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}})} \right) \quad (975)$$

$$= \prod_u \left(\prod_{\mathbf{k}} D(\pm \alpha_{nu\mathbf{k}}) \right) \quad (976)$$

$$= \prod_{u\mathbf{k}} D(\pm \alpha_{nu\mathbf{k}}) \quad (977)$$

$$= \prod_u B_{nu\pm} \quad (978)$$

$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D(\pm \alpha_{nu\mathbf{k}}) \quad (979)$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (980)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (963):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (981)$$

$$= \prod_u B_{0u+} |0\rangle\langle 0| |0\rangle\langle 0| \prod_u B_{0u-}, \quad (982)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \quad (983)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \quad (984)$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \quad (985)$$

$$= |0\rangle\langle 0|. \quad (986)$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (987)$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n| |n\rangle\langle n| \prod_u B_{nu-}, \quad (988)$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \quad (989)$$

$$= |m\rangle\langle n| \prod_u (B_{mu+} B_{nu-}), \quad m \neq n, \quad (990)$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D(\alpha_{muk}) \prod_{\mathbf{k}} D(-\alpha_{nuk}) \right), \quad (991)$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} (D(\alpha_{muk}) D(-\alpha_{nuk})), \quad (992)$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D(\alpha_{muk} - \alpha_{nuk}) \exp \left(\frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (993)$$

$$\Pi_u(B_{mu+} B_{nu-}) = \prod_{u\mathbf{k}} \left(D(\alpha_{muk} - \alpha_{nuk}) \exp \left(\frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (994)$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = (\sum_n |n\rangle\langle n| \prod_u B_{nu+}) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} (\sum_n |n\rangle\langle n| \prod_u B_{nu-}), \quad (995)$$

$$= (|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \dots) (\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}) (|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \dots), \quad (996)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + \dots, \quad (997)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{1u-} + \dots \quad (998)$$

$$= |0\rangle\langle 0| (\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \dots) \quad (999)$$

$$+ |1\rangle\langle 1| (\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \dots) + \dots \quad (1000)$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) \quad (1001)$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left(b_{1\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) + \dots \quad (1002)$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \dots \quad (1003)$$

$$= \sum_{nuk} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1004)$$

$$= \sum_{nuk} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left| \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right|^2 \right) \right) \quad (1005)$$

$$= \sum_{nuk} |n\rangle\langle n| \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nuk} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - v_{nu\mathbf{k}}^* b_{u\mathbf{k}} \right) \quad (1006)$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nuk} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - (v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}) \right) \quad (1007)$$

The transformed Hamiltonians of the equations (964) to (966) written in terms of (981) to (1005) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1008)$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1009)$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) \quad (1010)$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{nuk} |n\rangle\langle n| (g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk}) \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1011)$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}) + \dots \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1012)$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}) \prod_u B_{0u-} + \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| (g_{1u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{1u\mathbf{k}}^* b_{u\mathbf{k}}) \prod_u B_{1u-} + \dots \quad (1013)$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} \prod_u B_{0u+} b_{u\mathbf{k}}^\dagger \prod_u B_{0u-} + g_{0u\mathbf{k}}^* \prod_u B_{0u+} b_{u\mathbf{k}} \prod_u B_{0u-}) + \sum_{u\mathbf{k}} |1\rangle\langle 1| (g_{1u\mathbf{k}} \prod_u B_{1u+} b_{u\mathbf{k}}^\dagger \prod_u B_{1u-} + g_{1u\mathbf{k}}^* \prod_u B_{1u+} b_{u\mathbf{k}} \prod_u B_{1u-}) + \dots \quad (1014)$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{0u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{1u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \dots \quad (1015)$$

$$= \sum_{nuk} |n\rangle\langle n| \left(g_{nuk} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nuk}^*}{\omega_{u\mathbf{k}}} \right) + g_{nuk}^* \left(b_{u\mathbf{k}} - \frac{v_{nuk}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1016)$$

$$= \sum_{nuk} |n\rangle\langle n| \left(g_{nuk} b_{u\mathbf{k}}^\dagger + g_{nuk}^* b_{u\mathbf{k}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{u\mathbf{k}}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1017)$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nuk} |n\rangle\langle n| \left(\frac{|v_{nuk}|^2}{\omega_{u\mathbf{k}}} - (v_{nuk} b_{u\mathbf{k}}^\dagger + v_{nuk}^* b_{u\mathbf{k}}) \right) \quad (1018)$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{nuk} |n\rangle\langle n| \left(\frac{|v_{nuk}|^2}{\omega_{u\mathbf{k}}} - (v_{nuk} b_{u\mathbf{k}}^\dagger + v_{nuk}^* b_{u\mathbf{k}}) \right) \quad (1019)$$

$$+ \sum_{nuk} |n\rangle\langle n| \left(g_{nuk} b_{u\mathbf{k}}^\dagger + g_{nuk}^* b_{u\mathbf{k}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{u\mathbf{k}}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1020)$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nuk}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{u\mathbf{k}}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1021)$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nuk} - v_{nuk}) b_{u\mathbf{k}}^\dagger + (g_{nuk} - v_{nuk})^* b_{u\mathbf{k}} \right) \quad (1022)$$

Using the previous functions we have that (1019) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \quad (1023)$$

$$(1024)$$

Now in order to separate the elements of the hamiltonian (1024) let's follow the references of the equations (225) and (236) to separate the hamiltonian, before proceeding to do this we need to consider the term of the form:

$$\langle \Pi_u (B_{mu+} B_{nu-}) \rangle_{\overline{H_0}} = \langle \Pi_{u\mathbf{k}} (D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp(\frac{1}{2}(-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}})) \rangle_{\overline{H_0}} \quad (1025)$$

$$= (\Pi_{u\mathbf{k}} \exp(\frac{1}{2}(-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}}))) \langle \Pi_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \rangle_{\overline{H_0}} \quad (1026)$$

$$= \left(\Pi_{u\mathbf{k}} \exp\left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{nu\mathbf{k}} v_{mu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \quad (1027)$$

$$\equiv B_{nm} \quad (1028)$$

$$\langle \Pi_u (B_{nu+} B_{mu-}) \rangle_{\overline{H_0}} = \left(\Pi_{u\mathbf{k}} \exp\left(\frac{(v_{nu\mathbf{k}}^* v_{mu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \quad (1029)$$

$$= B_{nm}^* \quad (1030)$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+} B_{nu-}) - B_{nm} \quad (1031)$$

As we can see:

$$J_{nm}^\dagger = \left(\prod_u (B_{mu+} B_{nu-}) - B_{nm} \right)^\dagger \quad (1032)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{nm}^* \quad (1033)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{mn} \quad (1034)$$

$$= J_{mn} \quad (1035)$$

We can separate the Hamiltonian (1024) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}}(t) = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \quad (1036)$$

$$\overline{H_{\bar{I}}}(t) = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1037)$$

$$\overline{H_{\bar{B}}}(t) = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \quad (1038)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}})} \right) \right) + \langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} + O \left(\left\langle \overline{H_{\bar{I}}}^2 \right\rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} \right). \quad (1039)$$

Taking the equations (246)-(254) and given that $\text{Tr} \left(e^{-\beta \overline{H_{\bar{S}}}(t)} \right) = C(R_0, R_1, \dots, R_{d-1}, B_{01}, \dots, B_{0(d-1)}, \dots, B_{(d-2)(d-1)})$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v_{nu\mathbf{k}}^{\Re} + i v_{nu\mathbf{k}}^{\Im}$. So our approach will be based on the derivation respect to $v_{nu\mathbf{k}}^{\Re}$ and $v_{nu\mathbf{k}}^{\Im}$. The Hamiltonian $\overline{H_{\bar{S}}}(t)$ can be written like:

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \right) |n\rangle\langle n| \quad (1040)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1041)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \frac{g_{nuk} v_{nuk}^* + g_{nuk}^* v_{nuk}}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1042)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1043)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2}{\omega_{uk}} - \frac{(g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} + i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1044)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1045)$$

$$v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^* = (v_{muk}^{\Re} - i v_{muk}^{\Im})(v_{nuk}^{\Re} + i v_{nuk}^{\Im}) - (v_{muk}^{\Re} + i v_{muk}^{\Im})(v_{nuk}^{\Re} - i v_{nuk}^{\Im}) \quad (1046)$$

$$= (v_{muk}^{\Re} v_{nuk}^{\Re} + i v_{nuk}^{\Im} v_{muk}^{\Re} - i v_{muk}^{\Re} v_{nuk}^{\Im} + v_{muk}^{\Im} v_{nuk}^{\Im}) \quad (1047)$$

$$- (v_{muk}^{\Re} v_{nuk}^{\Re} - i v_{nuk}^{\Im} v_{muk}^{\Re} + i v_{muk}^{\Re} v_{nuk}^{\Im} + v_{muk}^{\Im} v_{nuk}^{\Im}) \quad (1048)$$

$$= 2i (v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re}) \quad (1049)$$

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2}{\omega_{uk}} - \frac{(g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} + i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1050)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1051)$$

$$|v_{muk} - v_{nuk}|^2 = (v_{muk} - v_{nuk})(v_{muk} - v_{nuk})^* \quad (1052)$$

$$= |v_{muk}|^2 + |v_{nuk}|^2 - (v_{nuk} v_{muk}^* + v_{muk} v_{nuk}^*) \quad (1053)$$

$$= (v_{muk}^{\Re})^2 + (v_{muk}^{\Im})^2 + (v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - (v_{nuk}^{\Re} + i v_{nuk}^{\Im})(v_{muk}^{\Re} - i v_{muk}^{\Im}) \quad (1054)$$

$$- (v_{muk}^{\Re} - i v_{muk}^{\Im})(v_{nuk}^{\Re} + i v_{nuk}^{\Im}) \quad (1055)$$

$$= (v_{muk}^{\Re})^2 + (v_{muk}^{\Im})^2 + (v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2(v_{nuk}^{\Re} v_{muk}^{\Re} + v_{nuk}^{\Im} v_{muk}^{\Im}) \quad (1056)$$

$$= (v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2 \quad (1057)$$

$$R_n(t) = \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1058)$$

$$= \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - (g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} - i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \quad (1059)$$

$$= \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1060)$$

$$B_{mn} = \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1061)$$

$$= \left(\Pi_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1062)$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Re}} = \frac{\partial}{\partial v_{nuk}^{\Re}} \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1063)$$

$$= \frac{2v_{nuk}^{\Re} - 2g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1064)$$

$$= 2 \frac{v_{nuk}^{\Re} - g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1065)$$

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Im}} = \frac{\partial}{\partial v_{nuk}^{\Im}} \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1066)$$

$$= \frac{2v_{nuk}^{\Im} - 2g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1067)$$

$$= 2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1068)$$

Given that:

$$\ln B_{mn} = \ln \left(\left(\prod_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \right) \quad (1069)$$

$$= \sum_{uk} \ln \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_u \ln \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1070)$$

$$= \sum_{uk} \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_{uk} \left(-\frac{1}{2} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1071)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Re}} = \frac{-i v_{muk}^{\Im} - (v_{nuk}^{\Re} - v_{muk}^{\Re}) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1072)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Im}} = \frac{i v_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1073)$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \quad (1074)$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \quad (1075)$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial (B_{nm})^{\dagger}}{\partial a} \quad (1076)$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \quad (1077)$$

$$= B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} - (v_{n\mathbf{k}}^{\Re} - v_{m\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1078)$$

$$= B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1079)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Re}} = \left(\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \right)^{\dagger} \quad (1080)$$

$$= \left(B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \quad (1081)$$

$$= B_{nm} \left(\frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1082)$$

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \quad (1083)$$

$$= B_{mn} \left(\frac{iv_{m\mathbf{k}}^{\Re} - (v_{n\mathbf{k}}^{\Im} - v_{m\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1084)$$

$$= B_{mn} \left(\frac{iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1085)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Im}} = \left(\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \right)^{\dagger} \quad (1086)$$

$$= (B_{mn})^{\dagger} \quad (1087)$$

$$= B_{nm} \left(\frac{-iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1088)$$

Introducing this derivatives in the equation (1063) give us:

$$\frac{\partial A_{\mathbf{B}}}{\partial v_{n\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathbf{B}}}{\partial R_n} \left(2 \frac{v_{n\mathbf{k}}^{\Re} - g_{u\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n < m} \left(\frac{\partial A_{\mathbf{B}}}{\partial B_{nm}} B_{nm} \left(\frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \quad (1089)$$

$$+ \frac{\partial A_{\mathbf{B}}}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1090)$$

$$= 0 \quad (1091)$$

We can obtain the variational parameters:

$$-2 \frac{\partial A_B}{\partial R_n} \frac{v_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1092)$$

$$= -\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1093)$$

$$v_{nuk}^{\Re} = \frac{\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right)}{2 \frac{\partial A_B}{\partial R_n} \frac{1}{\omega_{uk}} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right)} \quad (1094)$$

$$= \frac{2g_{nuk}^{\Re} \omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} (iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} (-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) \right)}{2\omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right)} \quad (1095)$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_B}{\partial v_{nuk}^{\Im}} = \frac{\partial A_B}{\partial R_n} \left(2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \right) + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{-iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1096)$$

$$+ \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1097)$$

$$= 0 \quad (1098)$$

Rearranging we obtain

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1115)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1116)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1117)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\bar{\rho}(t)) \quad (1118)$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1119)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1120)$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \quad (1121)$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2i} \quad (1122)$$

We can proof that $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+}B_{n-} - B_m B_n)^\dagger \quad (1123)$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \quad (1124)$$

$$= B_{n+} B_{m-} - B_n B_m \quad (1125)$$

$$= B_{nm} \quad (1126)$$

So we can say that the set of matrices (1119) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1119) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1127)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}) \quad (1128)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) + iV_{nm}^\Im(t) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) \right) \quad (1129)$$

$$+ \Re(V_{nm}(t)) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) - iV_{nm}^\Im(t) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1130)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} + B_{mn}}{2} \right) + \Re(V_{nm}(t)) \sigma_{nm,y} \frac{i(B_{mn} - B_{nm})}{2} \right) \quad (1131)$$

$$+ i\Im(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} - B_{mn}}{2} \right) + \Im(V_{nm}(t)) \sigma_{nm,y} \left(\frac{B_{nm} + B_{mn}}{2} \right) \quad (1132)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (\Re(V_{nm}(t)) \sigma_{nm,x} B_{nm,x} - \Im(V_{nm}(t)) \sigma_{nm,x} B_{nm,y} + \Re(V_{nm}(t)) \sigma_{nm,y} B_{nm,y} \quad (1133)$$

$$+ \Im(V_{nm}(t)) \sigma_{nm,y} B_{nm,x}) \quad (1134)$$

Let's define the set

$$P = \{(n, m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\} \quad (1135)$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \quad (1136)$$

$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \quad (1137)$$

$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \quad (1138)$$

$$A_{4,nm}(t) = A_{2,mn}(t) \quad (1139)$$

$$A_{5,nm}(t) = A_{1,nm}(t) \quad (1140)$$

$$B_{1,nm}(t) = B_{nm,x} \quad (1141)$$

$$B_{2,nm}(t) = B_{nm,y} \quad (1142)$$

$$B_{3,nm}(t) = B_{z,n}(t) \quad (1143)$$

$$B_{4,nm}(t) = B_{1,nm}(t) \quad (1144)$$

$$B_{5,nm}(t) = B_{2,nm}(t) \quad (1145)$$

$$C_{1,nm}(t) = \Re(V_{nm}(t)) \quad (1146)$$

$$C_{2,nm}(t) = C_{1,nm}(t) \quad (1147)$$

$$C_{3,nm}(t) = 1 \quad (1148)$$

$$C_{4,nm}(t) = \Im(V_{nm}(t)) \quad (1149)$$

$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \quad (1150)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t) (A_{jp} \otimes B_{jp}(t)) \quad (1151)$$

Here $J = \{1, 2, 3, 4, 5\}$ and P the set defined in (1135).

We write the interaction Hamiltonian transformed under (1115) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t) (\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)) \quad (1152)$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \quad (1153)$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t) e^{-iH_B t} \quad (1154)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B]] ds \quad (1155)$$

Replacing the equation (1152) in (1155) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1156)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \left[\sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1157)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] \quad (1158)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \Big] ds \quad (1159)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right. \quad (1160)$$

$$\left. - \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \right) \quad (1161)$$

$$- \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \quad (1162)$$

$$\left. + \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \right) ds \quad (1163)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B \quad (1164)$$

$$= \left\langle \widetilde{B}_{jp}(\tau) \widetilde{B}_{j'p'}(0) \right\rangle_B \quad (1165)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jp}(t)$ and elements related to the system given by $\widetilde{A}_{jp}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jp}(t) \widetilde{B}_{j'p'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jp}(t) \right) \text{Tr} \left(\widetilde{B}_{j'p'}(t) \right)$. The correlation functions relevant of the master equation (1163) are:

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1166)$$

$$= \left\langle \widetilde{B_{jp}}(0) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1167)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1168)$$

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \rho_B \widetilde{B_{j'p'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1169)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1170)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1171)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1172)$$

$$\text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \rho_B \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) \quad (1173)$$

$$= \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1174)$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1175)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1176)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1177)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1178)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1179)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1180)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1156) and (1163) and using the equations (1166)-(1180) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \widetilde{A_{jp}}(t) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau) \widetilde{A_{jp}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \right) \right. \quad (1181)$$

$$\left. + C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{j'p'jp}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jp}}(t) \right) \right) ds \quad (1182)$$

$$= - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \left[\widetilde{A_{jp}}(t), \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s), \widetilde{A_{jp}}(t) \right] \right) \right) \quad (1183)$$

Rearranging and identifying the commutators allow us to write a more simplified version

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(t-\tau) \left(\Lambda_{jpj'p'}(\tau) \left[A_{jp}(t), A_{j'p'}(t-\tau, t) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) A_{j'p'}(t-\tau, t), A_{jp}(t) \right] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1184)$$

For this case we used that $A_{jp}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jp}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1185)$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1186)$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right), \quad (1187)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1188)$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}) \right) \quad (1189)$$

At first let's obtain e^V under the transformation (1189), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \quad (1190)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{(\sum_{n=1} |n\rangle\langle n| \hat{\varphi})^2}{2!} + \dots \quad (1191)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \quad (1192)$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \quad (1193)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \quad (1194)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \quad (1195)$$

Given that $[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \quad (1196)$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (1197)$$

$$= B_{\pm} \quad (1198)$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B$. because this form imposes that $e^{-V} e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1199)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1185):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1200)$$

$$= |0\rangle\langle 0|, \quad (1201)$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1202)$$

$$= |m\rangle\langle m| B^+ |m\rangle\langle n| n\rangle\langle n| B^-, \quad (1203)$$

$$= |m\rangle\langle n|, \quad m \neq 0, \quad n \neq 0, \quad (1204)$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1205)$$

$$= |0\rangle\langle m| B^- \quad m \neq 0, \quad (1206)$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1207)$$

$$= |0\rangle\langle m| B^+ \quad m \neq 0, \quad (1208)$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1209)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B^- \quad (1210)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B^+ b_{\mathbf{k}}^\dagger B^- \right) \left(B^+ b_{\mathbf{k}} B^- \right) \quad (1211)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1212)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1213)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \quad (1214)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1215)$$

The transformed Hamiltonians of the equations (1186) to (1188) written in terms of (1200) to (1215) are:

$$\overline{H_S(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1216)$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1217)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1218)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|}) + \sum_{m,n \neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \quad (1219)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) B^- |0\rangle\langle n| + V_{n0}(t) B^+ |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1220)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1221)$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \left(\left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1222)$$

$$= \left(\mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B^+ \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1223)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B^+ (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) B^- \quad (1224)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1225)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1226)$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1227)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1228)$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (1229)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1230)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1231)$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1232)$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1233)$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \right) \quad (1234)$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1235)$$

Using the previous functions we have that (1232) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1236)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1237)$$

Now in order to separate the elements of the hamiltonian (1237) let's follow the references of the equations (236) and (225) to separate the hamiltonian like:

$$\overline{H_S}(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \quad (1238)$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)), \quad (1239)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (1240)$$

Here B is given by:

$$\begin{aligned} B &= \langle B^+ \rangle \\ &= \langle B^- \rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1241)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1242)$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \quad (1243)$$

$$B_y = \frac{B^- - B^+}{2i} \quad (1244)$$

Using this set of hermitian operators to write the Hamiltonians (1186)-(1188)

$$\overline{H_S(t)} = \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1245)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{0 < m < n} (V_{mn}(t) |m\rangle\langle n| + V_{nm}(t) |n\rangle\langle m|) \quad (1246)$$

$$+ \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1247)$$

$$= \sum_{0 < m < n} ((\Re(V_{mn}(t)) + i\Im(V_{mn}(t))) |m\rangle\langle n| + (\Re(V_{mn}(t)) - i\Im(V_{mn}(t))) |n\rangle\langle m|) + \varepsilon_0(t) |0\rangle\langle 0| \quad (1248)$$

$$+ B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1249)$$

$$= \sum_{0 < m < n} \left((\Re(V_{nm}(t)) + i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1250)$$

$$+ B \sum_{n=1} \left(V_{0n}(t) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + V_{n0}(t) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) + \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1251)$$

$$= \sum_{0 < m < n} (\Re(V_{nm}(t)) \sigma_{nm,x} + \Im(V_{nm}(t)) \sigma_{nm,y}) + B \sum_{n=1} (\Re(V_{0n}(t)) \sigma_{0n,x} + \Im(V_{mn}(t)) \sigma_{0n,y}) \quad (1252)$$

$$+ \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1253)$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)) \quad (1254)$$

$$= \sum_{n=1} \left((\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) (B^- - B) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) (B^+ - B) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) \quad (1255)$$

$$+ \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1256)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} ((B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) + (B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))) \right) \quad (1257)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) - (B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))) \quad (1258)$$

$$+ \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1259)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} (B^+ + B^- - 2B) \Re(V_{0n}(t)) + i(B^- - B - B^+ + B) \Im(V_{0n}(t)) \right) \quad (1260)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B - B^- + B) \Re(V_{0n}(t)) + i(B - B^- + B - B^+) \Im(V_{0n}(t))) + \sum_{n=1} B_{z,n} |n\rangle\langle n| \quad (1261)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (\sigma_{0n,x} (B_x \Re(V_{0n}(t)) - B_y \Im(V_{0n}(t))) \quad (1262)$$

$$+ \sigma_{0n,y} (B_y \Re(V_{0n}(t)) + B_x \Im(V_{0n}(t)))) \quad (1263)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H}_S + \overline{H}_B)} \right) \right) + \langle \overline{H}_I \rangle_{\overline{H}_S + \overline{H}_B} + O \left(\langle \overline{H}_I^2 \rangle_{\overline{H}_S + \overline{H}_B} \right). \quad (1264)$$

Taking the equations (246)-(254) and given that $\text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right) = C(R_1, R_2, \dots, R_{d-1}, B)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (254) and using the chain rule we obtain that:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \frac{\partial R_n}{\partial v_{\mathbf{k}}}, \quad (1265)$$

$$= 0 \quad (1266)$$

Let's recall the equations (1233) and (1235), we can write them in terms of the variational parameters

$$B = \exp \left(- (1/2) \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) \right) \quad (1267)$$

$$R_n = \sum_{\mathbf{k}} \omega_{\mathbf{k}}^{-1} (v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}} v_{\mathbf{k}}) \quad (1268)$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \quad (1269)$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1270)$$

Introducing this derivates in the equation (1265) give us:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \right) + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1271)$$

$$= v_{\mathbf{k}} \left(\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t) \quad (1272)$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1273)$$

$$= \frac{2g_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1274)$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}} v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}}. \quad (1275)$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) (|0\rangle\langle 0| \otimes \rho_B) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1276)$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \quad (1277)$$

$$0 = \rho(0) \quad (1278)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1279)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1280)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1281)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}(t)) \quad (1282)$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I}(t) = B_{z,0} |0\rangle\langle 0| + \sum_{n=1} (\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n} |n\rangle\langle n|) \quad (1283)$$

$$+ \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x} \quad (1284)$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1285)$$

$$B_{z,n} = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \text{ if } n \neq 0 \quad (1286)$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \quad (1287)$$

$$A_{2n}(t) = \sigma_{0n,y} \quad (1288)$$

$$A_{3n}(t) = |n\rangle\langle n| \quad (1289)$$

$$A_{4n}(t) = A_{2n}(t) \quad (1290)$$

$$A_{5n}(t) = A_{1n}(t) \quad (1291)$$

$$B_{1n}(t) = B_x \quad (1292)$$

$$B_{2n}(t) = B_y \quad (1293)$$

$$B_{3n}(t) = B_{z,n} \quad (1294)$$

$$B_{4n}(t) = B_{1n}(t) \quad (1295)$$

$$B_{5n}(t) = B_{2n}(t) \quad (1296)$$

$$C_{10}(t) = 0 \quad (1297)$$

$$C_{20}(t) = 0 \quad (1298)$$

$$C_{40}(t) = 0 \quad (1299)$$

$$C_{50}(t) = 0 \quad (1300)$$

$$C_{30}(t) = 1 \quad (1301)$$

$$C_{1n}(t) = \Re(V_{0n}(t)) \quad (1302)$$

$$C_{2n}(t) = C_{1n}(t) \quad (1303)$$

$$C_{3n}(t) = 1 \quad (1304)$$

$$C_{4n}(t) = \Im(V_{0n}(t)) \quad (1305)$$

$$C_{5n}(t) = -\Im(V_{0n}(t)) \quad (1306)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J} \sum_{n=1} C_{jn}(t) (A_{jn} \otimes B_{jn}(t)) \quad (1307)$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1279) as:

$$\widetilde{H_I}(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t) \right) \quad (1308)$$

$$\widetilde{A_i}(t) = U^\dagger(t) A_i U(t) \quad (1309)$$

$$\widetilde{B_i}(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \quad (1310)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1311)$$

Replacing the equation (1308) in (1311) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1312)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \left[\sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1313)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1314)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \Big] ds \quad (1315)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right) ds \quad (1316)$$

$$- \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \quad (1317)$$

$$- \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1318)$$

$$+ \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \Big) ds \quad (1319)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jn j'n'}(\tau) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B \quad (1320)$$

$$= \left\langle \widetilde{B}_{jn}(\tau) \widetilde{B}_{j'n'}(0) \right\rangle_B \quad (1321)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \rho_B \right) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jn}(t)$ and elements related to the system given by $\widetilde{A}_{jn}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jn}(t) \widetilde{B}_{j'n'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jn}(t) \right) \text{Tr} \left(\widetilde{B}_{j'n'}(t) \right)$. The correlation functions relevant of the master equation (1319) are:

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1322)$$

$$= \left\langle \widetilde{B_{jn}}(0) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1323)$$

$$= \Lambda_{jnj'n'}(\tau) \quad (1324)$$

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \rho_B \widetilde{B_{j'n'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1325)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1326)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1327)$$

$$= \Lambda_{j'n'jn}(-\tau) \quad (1328)$$

$$\text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \rho_B \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) \quad (1329)$$

$$= \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1330)$$

$$= \left\langle \widetilde{B_{jn}}(\tau) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1331)$$

$$= \Lambda_{jnj'n'}(\tau) \quad (1332)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1333)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1334)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1335)$$

$$= \Lambda_{j'n'jn}(-\tau) \quad (1336)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1312) and (1319) and using the equations (1322)-(1336) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jnj'n'}(\tau) \widetilde{A_{jn}}(t) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'n'jn}(-\tau) \widetilde{A_{jn}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \right) \right. \quad (1337)$$

$$\left. + C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{j'n'jn}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jn}}(t) \right) \right) ds \quad (1338)$$

$$= - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jnj'n'}(\tau) \left[\widetilde{A_{jn}}(t), \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'n'jn}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s), \widetilde{A_{jn}}(t) \right] \right) \right) \quad (1339)$$

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(t-\tau) \left(\Lambda_{jnj'n'}(\tau) [A_{jn}(t), A_{j'n'}(t-\tau, t) \widetilde{\rho_S}(t)] + \Lambda_{j'n'jn}(-\tau) [\widetilde{\rho_S}(t) A_{j'n'}(t-\tau, t), A_{jn}(t)] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1340)$$

For this case we used that $A_{jn}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jn}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation and if we take $n = 2$ (two sites), $\mu_0(t) = 0$, $\mu_1(t) = 1$ then we can reproduce a similar expression to (419) as expected.

VIII. BIBLIOGRAPHY

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* n.dattani@cfa.harvard.edu
† edcchaparroso@unal.edu.co