A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$
$$H_S(t) = \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| + V_{01}(t)|0\rangle\langle 1|, \tag{2}$$
$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij} \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}} \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$
$$V = |0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6)will give us the following result:

$$
\begin{aligned}
e^{\pm V} &= e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)} & (10)\\
&= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + ... & (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + ... & (12)\\
&= |0\rangle\langle 0| \left(1 \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm ...\right) + |1\rangle\langle 1| \left(1 \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm ...\right) & (13)\\
&= |0\rangle\langle 0| e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1| e^{\pm\hat{\varphi}_1} & (14)\\
&= |0\rangle\langle 0| e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^* b_{\mathbf{k}}\right)} + |1\rangle\langle 1| e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^* b_{\mathbf{k}}\right)} & (15)\\
&= |0\rangle\langle 0| B_{0\pm} + |1\rangle\langle 1| B_{1\pm}, & (16)\\
B_{i\pm} &\equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. & (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
e^{t(X+Y)} = e^{tX}\ e^{tY}\ e^{-\frac{t^2}{2}[X,Y]}\ e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\ e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \dots \tag{18}
$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}^{'}, \mathbf{k}$ and i, j we can show making $t = 1$ in (18) the following result:

$$
\begin{aligned}
e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]} \dots & (19)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}0} \dots & (20)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} & (21)
\end{aligned}
$$

By induction of this result we can write expresion of $B_{i\pm}$ as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}})$:

$$
\begin{aligned}
B_{i\pm} &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), & (22)\\
D(\pm v_{i\mathbf{k}}) &\equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. & (23)
\end{aligned}
$$

this will help us to write operators O in the variational frame :

$$
\overline{O} \equiv \mathrm{e}^{V} O \mathrm{e}^{-V}. \tag{24}
$$

We use the following identities:

$$
\begin{aligned}
\overline{|0\rangle\langle 0|} &= \mathrm{e}^{V}|0\rangle\langle 0|\mathrm{e}^{-V} && (25)\\
&= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (26)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|0\rangle\langle 0|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (27)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (28)\\
&= |0\rangle\langle 0|B_{0+}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (29)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}B_{0-}+|0\rangle\langle 0|1\rangle\langle 1|B_{0+}B_{1-} && (30)\\
&= |0\rangle\langle 0|, && (31)\\
\overline{|1\rangle\langle 1|} &= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (32)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_{0+}+|1\rangle\langle 1|1\rangle\langle 1|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (33)\\
&= |1\rangle\langle 1|B_{1+}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (34)\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_{1+}B_{0-}+B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} && (35)\\
&= B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} && (36)\\
&= |1\rangle\langle 1|B_{1+}B_{1-} && (37)\\
&= |1\rangle\langle 1|, && (38)\\
\overline{|0\rangle\langle 1|} &= \mathrm{e}^{V}|0\rangle\langle 1|\mathrm{e}^{-V} && (39)\\
&= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (40)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_{0+}+|1\rangle\langle 1|B_{1+}|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (41)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_{0+}+|1\rangle\langle 1|0\rangle\langle 1|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (42)\\
&= |0\rangle\langle 1|B_{0+}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (43)\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_{0+}B_{0-}+|0\rangle\langle 1|1\rangle\langle 1|B_{0+}B_{1-} && (44)\\
&= |0\rangle\langle 1|B_{0+}B_{1-}, && (45)\\
\overline{|1\rangle\langle 0|} &= \mathrm{e}^{V}|1\rangle\langle 0|\mathrm{e}^{-V} && (46)\\
&= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (47)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (48)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_{0+}+|1\rangle\langle 1|1\rangle\langle 0|B_{1+}\right)\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (49)\\
&= |1\rangle\langle 0|B_{1+}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (50)\\
&= |1\rangle\langle 0|B_{1+}|0\rangle\langle 0|B_{0-}+|1\rangle\langle 0|B_{1+}|1\rangle\langle 1|B_{1-} && (51)\\
&= |1\rangle\langle 0|B_{1+}B_{0-}+|1\rangle\langle 0|1\rangle\langle 1|B_{1+}B_{1-} && (52)\\
&= |1\rangle\langle 0|B_{1+}B_{0-}, && (53)\\
\overline{b_{\mathbf{k}}} &= \mathrm{e}^{V}b_{\mathbf{k}}\mathrm{e}^{-V} && (54)\\
&= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (55)\\
&= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-}|0\rangle\langle 0|+|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}|1\rangle\langle 1|B_{1-}+|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-}|1\rangle\langle 1| && (56)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-}+|0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}B_{1-}+|1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}B_{0-}+|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-} && (57)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (58)\\
&= \left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)b_{\mathbf{k}}-|1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-|0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} && (59)\\
&= b_{\mathbf{k}}-|1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-|0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, && (60)\\
\overline{b_{\mathbf{k}}}^{\dagger} &= \mathrm{e}^{V}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V} && (61)\\
&= \left(|0\rangle\langle 0|B_{0+}+|1\rangle\langle 1|B_{1+}\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1-}\right) && (62)\\
&= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-}|0\rangle\langle 0|+|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_{1-}+|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_{0-}+|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-}|1\rangle\langle 1| && (63)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^{\dagger}B_{0-}+|0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}^{\dagger}B_{1-}+|1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}^{\dagger}B_{0-}+|1\rangle\langle 1|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^{\dagger}B_{1-} && (64)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) && (65)\\
&= b_{\mathbf{k}}^{\dagger}-|1\rangle\langle 1|\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-|0\rangle\langle 0|\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. && (66)
\end{aligned}
$$

We have used the following:

$$B_{i+} b_{\mathbf{k}} B_{i-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{67}$$

$$B_{i+} b_{\mathbf{k}}^{\dagger} B_{i-} = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{68}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{69}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{70}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|} = V_{10}(t)\,|1\rangle\langle 0| B_{1+} B_{0-}, \tag{71}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|} = V_{01}(t)\,|0\rangle\langle 1| B_{0+} B_{1-}, \tag{72}$$

$$\overline{g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}}} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\right) + g_{\mathbf{ik}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{73}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{74}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{75}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{76}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}\right) \tag{77}$$

$$= |0\rangle\langle 0| B_{0+} |0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)|0\rangle\langle 0| B_{0-} \tag{78}$$

$$= |0\rangle\langle 0| B_{0+}\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_{0-} \tag{79}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{80}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)} = \left(|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}\right) \tag{81}$$

$$= |1\rangle\langle 1| B_{1+} |1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)|1\rangle\langle 1| B_{1-} \tag{82}$$

$$= |1\rangle\langle 1| B_{1+}\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_{1-} \tag{83}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), \tag{84}$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_{0+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{0-} + |1\rangle\langle 1| B_{1+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{1-}\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}} D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}} D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \mathbb{I}\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{90}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{91}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{92}$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{93}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \tag{94}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \tag{95}$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right). \tag{96}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} \tag{97}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0| B_{1+} B_{0-} + V_{01}(t)\,|0\rangle\langle 1| B_{0+} B_{1-}, \tag{98}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{99}$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \tag{100}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{101}$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{102}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \tag{103}$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right) \tag{104}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right). \tag{105}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{j+} B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{106}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_S} + \overline{H_I} + \overline{H_B} \tag{107}$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{108}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^*\, b_{\mathbf{k}} \right). \tag{109}$$

We assume that the bath is at equilibrium with inverse temperature $\beta = 1/k_\mathrm{B}T$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{110}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{111}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{112}$$

So the product of the matrix representation of $b^\dagger$ and b is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{113}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right|, \tag{114}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\!\langle j|, |i\rangle\!\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_\mathbf{k} b_\mathbf{k}^\dagger b_\mathbf{k}\right) = \sum_{j_\mathbf{k}} \exp\left(-j_\mathbf{k}\beta\omega_\mathbf{k}\right) |j_\mathbf{k}\rangle\!\langle j_\mathbf{k}|, \tag{115}$$

$$\exp\left(-\beta\sum_\mathbf{k} \omega_\mathbf{k} b_\mathbf{k}^\dagger b_\mathbf{k}\right) = \prod_\mathbf{k}\sum_{j_\mathbf{k}} \exp\left(-j_\mathbf{k}\beta\omega_\mathbf{k}\right) |j_\mathbf{k}\rangle\!\langle j_\mathbf{k}|. \tag{116}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_\mathbf{k} b_\mathbf{k}^\dagger b_\mathbf{k}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{117}$$

$$= \sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{118}$$

$$= \sum_{j_{\mathbf{k}}}\exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{119}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{120}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{121}$$

$$\mathrm{Tr}\left(\exp\left(-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{122}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{123}$$

$$= \prod_{\mathbf{k}}f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{124}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{125}$$

$$= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}}f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{126}$$

$$= \prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{127}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{128}$$

$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. \tag{129}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\overline{B}}}} = 0$ using the following property based on (128)-(129):

$$\langle B_{iz}\rangle_{\overline{H_{\overline{B}}}} = \mathrm{Tr}\left(\rho_B B_{iz}\right) = \mathrm{Tr}\left(B_{iz}\rho_B\right) \tag{130}$$

$$= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{131}$$

$$= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}\rho_B\right) \tag{132}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \,\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \,\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) \tag{133}$$

$$= \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} \mathrm{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) \tag{134}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \mathrm{Tr}\left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right), \tag{135}$$

$$\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right) b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}) \tag{136}$$

$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right) \sqrt{j_{\mathbf{k}}+1}\, |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{137}$$

$$= 0, \tag{138}$$

$$\mathrm{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right) b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}) \tag{139}$$

$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right) \sqrt{j_{\mathbf{k}}}\, |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) \tag{140}$$

$$= 0. \tag{141}$$

we therefore find that:

$$\langle B_{iz}\rangle_{\overline{H_B}} = 0 \tag{142}$$

Another important expected value is $B = \langle B_{\pm}\rangle_{\overline{H_B}}$, where $B_{\pm} = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)}$ is given by:

$$\langle B_{\pm}\rangle_{H_B} = \mathrm{Tr}\left(\rho_B B_{\pm}\right) = \mathrm{Tr}\left(B_{\pm}\rho_B\right) \tag{143}$$

$$= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)} \rho_B\right) \tag{144}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\right)\rho_B\right) \tag{145}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\pm\alpha_{\mathbf{k}}\right)\right\rangle. \tag{146}$$

Given that we can write a density operator as:

$$\rho = \int P\left(\alpha\right) |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha \tag{147}$$

where $P(\alpha)$ satisfies $\int P(\alpha)\,\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P(\alpha)$ is given by:

$$\langle A\rangle = \mathrm{Tr}\left(A\rho\right) \tag{148}$$

$$= \int P\left(\alpha\right) \langle\alpha\left|A\right|\alpha\rangle \mathrm{d}^2\alpha \tag{149}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle\alpha|D(h)|\alpha\rangle \mathrm{d}^2\alpha \tag{150}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(-\alpha)D(h)D(\alpha)|0\rangle \mathrm{d}^2\alpha \tag{151}$$

$$D(h)D(\alpha) = D(h+\alpha)e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} \tag{152}$$

$$D(-\alpha)(D(h)D(\alpha)) = D(-\alpha)D(h+\alpha)e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} \tag{153}$$

$$= D(h)e^{\frac{1}{2}(-\alpha(h+\alpha)^*+\alpha^*(h+\alpha))}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} \tag{154}$$

$$= D(\alpha)e^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^*h+|\alpha|^2\right)}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} \tag{155}$$

$$= D(\alpha)e^{(h\alpha^*-h^*\alpha)}, \tag{156}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(h)\exp(h\alpha^*-h^*\alpha)|0\rangle \mathrm{d}^2\alpha \tag{157}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|0\rangle \mathrm{d}^2\alpha \tag{158}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|h\rangle \mathrm{d}^2\alpha \tag{159}$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{160}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha \tag{161}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{162}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha\right)\mathrm{d}^2\alpha, \tag{163}$$

$$\alpha = x+\mathrm{i}y, \tag{164}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{N}+h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)\right)\mathrm{d}x\mathrm{d}y \tag{165}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{N}+hx-h^*x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y\right)\mathrm{d}y, \tag{166}$$

$$-\frac{x^2}{N}+hx-h^*x=-\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) \tag{167}$$

$$= -\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2+\frac{N(h^*-h)^2}{4}, \tag{168}$$

$$-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y=-\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) \tag{169}$$

$$= -\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)-\frac{N(h+h^*)^2}{4}, \tag{170}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, \tag{171}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x, \tag{172}$$

$$\langle D(h)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(x+\frac{(Nh^*-Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}y \tag{173}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{174}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^* - h\right)^2}{4} - \frac{N\left(h + h^*\right)^2}{4}\right) \tag{175}$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N\left(h^{*2} - 2hh^* + h^2\right) - N\left(h^2 + 2hh^* + h^{*2}\right)}{4}\right) \tag{176}$$

$$= \exp\left(-|h|^2\left(N + \frac{1}{2}\right)\right) \tag{177}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2}\right)\right) \tag{178}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1}\right)\right) \tag{179}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{180}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (145) using (**??**) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{181}$$

We will now force $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_+\sigma_+$and $B_-\sigma_-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{j+}B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j|\left((g_{j\mathbf{k}} - v_{j\mathbf{k}})b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{\left|v_{j\mathbf{k}}\right|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{182}$$

$$= \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|B_{jj'} + \sum_j |j\rangle\langle j|B_{jz} + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-} - B_{jj'}\right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{183}$$

$$\equiv \overline{H_{\overline{S}}(t)} + \overline{H_{\overline{I}}} + \overline{H_{\overline{B}}}. \tag{184}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_{1+}B_{0-}\rangle = B_{10} \tag{185}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle \tag{186}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle \tag{187}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\right\rangle \tag{188}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{189}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \tag{190}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}. \tag{191}$$

From the definition $B_{01} = \langle B_{0+}B_{1-}\rangle$ using the displacement operator we have:

$$\begin{aligned}
\langle B_{0+}B_{1-}\rangle &= B_{01} && (192)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle && (193)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \right\rangle && (194)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right) \right\rangle && (195)\\
&= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} && (196)\\
&= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} && (197)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} && (198)
\end{aligned}$$

This can be checked in the following way:

$$\begin{aligned}
\langle B_{0+}B_{1-}\rangle &= B_{01} && (199)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} && (200)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)^*} && (201)\\
&= \langle B_{1+}B_{0-}\rangle^* && (202)\\
&= B_{10}^*. && (203)
\end{aligned}$$

The parts of the Hamiltonian splitted are:

$$\begin{aligned}
\overline{H_{\overline{S}}(t)} &\equiv (\varepsilon_0(t) + R_0)\,|0\rangle\langle 0| + (\varepsilon_1(t) + R_1)\,|1\rangle\langle 1| + V_{10}(t)\,B_{10}\sigma_+ + V_{01}(t)\,B_{01}\sigma_-, && (204)\\
\overline{H_{\overline{I}}} &\equiv V_{10}(t)\,(B_{1+}B_{0-} - B_{10})\,\sigma_+ + V_{01}(t)\,(B_{0+}B_{1-} - B_{01})\,\sigma_- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, && (205)\\
\overline{H_{\overline{B}}} &\equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (206)\\
&= H_B. && (207)
\end{aligned}$$

Note that $\overline{H_{\overline{B}}}$, which is the bath acting on the effective "system" $\overline{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (205) we can verify the condition $\langle \overline{H_{\overline{I}}}\rangle_{\overline{H_{\overline{B}}}} = 0$ in the following way:

$$
\begin{aligned}
\left\langle\overline{H_{I}}\right\rangle_{\overline{H_{B}}} &= \left\langle\sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})b_{\mathbf{k}}^{\dagger}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^{*}b_{\mathbf{k}}\right)|n\rangle\langle n|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-}-B_{jj'}\right)\right\rangle_{\overline{H_{B}}} && (208)\\
&= \left\langle\sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})b_{\mathbf{k}}^{\dagger}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^{*}b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{B}}}+\left\langle\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_{j+}B_{j'-}-B_{jj'}\right)\right\rangle_{\overline{H_{B}}} && (209)\\
&= \sum_{n\mathbf{k}}\left(\left\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}})b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{B}}}+\langle(g_{n\mathbf{k}}-v_{n\mathbf{k}})^{*}b_{\mathbf{k}}\rangle_{\overline{H_{B}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}(t)B_{j+}B_{j'-}\right\rangle_{\overline{H_{B}}}-\left\langle V_{jj'}(t)B_{jj'}\right\rangle_{\overline{H_{B}}}\right) && (210)\\
&= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})\left\langle b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{B}}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^{*}\langle b_{\mathbf{k}}\rangle_{\overline{H_{B}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(\left\langle B_{j+}B_{j'-}\right\rangle_{\overline{H_{B}}}-\left\langle B_{jj'}\right\rangle_{\overline{H_{B}}}\right) && (211)\\
&= \sum_{n\mathbf{k}}\left((g_{n\mathbf{k}}-v_{n\mathbf{k}})\left\langle b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{B}}}+(g_{n\mathbf{k}}-v_{n\mathbf{k}})^{*}\langle b_{\mathbf{k}}\rangle_{\overline{H_{B}}}\right)|n\rangle\langle n|+\sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}(t)\left(B_{jj'}-B_{jj'}\right). && (212)\\
&= 0. && (213)
\end{aligned}
$$

We used (142) and (191) to evaluate the expected values.
Let's consider the following Hermitian combinations:

$$
\begin{aligned}
B_{x} &= B_{x}^{\dagger} && (214)\\
&= \frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{01}}{2}, && (215)\\
B_{y} &= B_{y}^{\dagger} && (216)\\
&= \frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{01}}{2\mathrm{i}}, && (217)\\
B_{iz} &= B_{iz}^{\dagger} && (218)\\
&= \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}\right). && (219)
\end{aligned}
$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$
\begin{aligned}
\overline{H_{\overline{S}}}(t) &= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+V_{10}(t)\,B_{10}\sigma_{+}+V_{01}(t)\,B_{01}\sigma_{-} && (220)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+V_{10}(t)\,B_{10}\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\,B_{01}\frac{\sigma_x-\mathrm{i}\sigma_y}{2} && (221)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+V_{10}(t)\left(\Re\left(B_{10}(t)\right)+\mathrm{i}\Im\left(B_{10}(t)\right)\right)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\left(\Re\left(B_{10}(t)\right)-\mathrm{i}\Im\left(B_{10}(t)\right)\right)\frac{\sigma_x-\mathrm{i}\sigma_y}{2} && (222)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\Re\left(B_{10}(t)\right)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right)+\mathrm{i}\Im\left(B_{10}(t)\right)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}-V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right) && (223)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\Re\left(B_{10}(t)\right)\left(\sigma_x\frac{V_{10}(t)+V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)-V_{01}(t)}{2}\right)+\mathrm{i}\Im\left(B_{10}(t)\right)\left(\sigma_x\frac{V_{10}(t)-V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)+V_{01}(t)}{2}\right) && (224)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\Re\left(B_{10}(t)\right)\left(\sigma_x\frac{V_{10}(t)+V_{10}^{*}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)-V_{10}^{*}(t)}{2}\right)+\mathrm{i}\Im\left(B_{10}(t)\right)\left(\sigma_x\frac{V_{10}(t)-V_{10}^{*}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)+V_{10}^{*}(t)}{2}\right) && (225)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\Re\left(B_{10}(t)\right)\left(\sigma_x\Re\left(V_{10}(t)\right)-\sigma_y\Im\left(V_{10}(t)\right)\right)+\mathrm{i}\Im\left(B_{10}(t)\right)\left(\mathrm{i}\sigma_x\Im\left(V_{10}(t)\right)+\mathrm{i}\sigma_y\Re\left(V_{10}(t)\right)\right) && (226)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\left(\sigma_x\Re\left(B_{10}(t)\right)\Re\left(V_{10}(t)\right)-\sigma_y\Re\left(B_{10}(t)\right)\Im\left(V_{10}(t)\right)\right)-\left(\sigma_x\Im\left(B_{10}(t)\right)\Im\left(V_{10}(t)\right)+\sigma_y\Im\left(B_{10}(t)\right)\Re\left(V_{10}(t)\right)\right) && (227)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\sigma_x\left(\Re\left(B_{10}(t)\right)\Re\left(V_{10}(t)\right)-\Im\left(B_{10}(t)\right)\Im\left(V_{10}(t)\right)\right)-\sigma_y\left(\Re\left(B_{10}(t)\right)\Im\left(V_{10}(t)\right)+\Im\left(B_{10}(t)\right)\Re\left(V_{10}(t)\right)\right) && (228)\\
&= (\varepsilon_0(t)+R_0)|0\rangle\langle 0|+(\varepsilon_1(t)+R_1)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right). && (229)
\end{aligned}
$$

$$
\begin{aligned}
\overline{H_{\overline{I}}} &= V_{10}(t)\left(\sigma_{+}B_{1+}B_{0-}-\sigma_{+}B_{10}\right)+V_{01}(t)\left(\sigma_{-}B_{0+}B_{1-}-\sigma_{-}B_{01}\right)+|0\rangle\langle 0|B_{0z}+|1\rangle\langle 1|B_{1z} && (230)\\
&= |0\rangle\langle 0|B_{0z}+|1\rangle\langle 1|B_{1z}+\left(\Re(V_{10}(t))+\mathrm{i}\Im(V_{10}(t))\right)\left(\sigma_{+}B_{1+}B_{0-}-\sigma_{+}B_{10}\right)+\left(\Re(V_{10}(t))-\mathrm{i}\Im(V_{10}(t))\right)\left(\sigma_{-}B_{0+}B_{1-}-\sigma_{-}B_{01}\right) && (231)\\
&= \sum_i B_{iz}|i\rangle\langle i|+\Re(V_{10}(t))\left(\sigma_{+}B_{1+}B_{0-}-\sigma_{+}B_{10}+\sigma_{-}B_{0+}B_{1-}-\sigma_{-}B_{01}\right)+\mathrm{i}\Im(V_{10}(t))\left(\sigma_{+}B_{1+}B_{0-}-\sigma_{+}B_{10}-\sigma_{-}B_{0+}B_{1-}+\sigma_{-}B_{01}\right) && (232)\\
&= \sum_i B_{iz}|i\rangle\langle i|+\Re(V_{10}(t))\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{1+}B_{0-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{0+}B_{1-}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) && (233)\\
&= \sum_i B_{iz}|i\rangle\langle i|+\Re(V_{10}(t))\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{1+}B_{0-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{0+}B_{1-}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right)+\mathrm{i}\Im(V_{10}(t))\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{1+}B_{0-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{0+}B_{1-}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) && (234)\\
&= \sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x\frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{01}}{2}+\mathrm{i}\sigma_y\frac{B_{1+}B_{0-}-B_{0+}B_{1-}-B_{10}+B_{01}}{2}\right)+\mathrm{i}V_{10}^{\Im}(t)\left(\sigma_x\frac{B_{1+}B_{0-}-B_{0+}B_{1-}-B_{10}+B_{01}}{2}+\sigma_y\frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{01}}{2}\right) && (235)\\
&= \sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right)+V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x\frac{B_{1+}B_{0-}-B_{0+}B_{1-}-B_{10}+B_{01}}{2}-\sigma_y\frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{01}}{2}\right) && (236)\\
&= \sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right)+V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_{1+}B_{0-}-B_{0+}B_{1-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_y\frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{01}}{2}\right) && (237)\\
&= \sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right)+V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_{1+}B_{0-}-B_{0+}B_{1-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_y\frac{B_{1+}B_{0-}+B_{0+}B_{1-}-B_{10}-B_{01}}{2}\right) && (238)\\
&= \sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right)+V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\left(-B_y\right)-\sigma_y B_x\right) && (239)\\
&= \sum_i B_{iz}|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right)+V_{10}^{\Im}(t)\left(\sigma_x B_y-\sigma_y B_x\right). && (240)
\end{aligned}
$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\overline{S}}}(t)+\overline{H_{\overline{B}}}\right)}\right)\right) + \langle\overline{H_{\overline{I}}}\rangle_{\overline{H_{\overline{S}}}(t)+\overline{H_{\overline{B}}}} + O\left(\left\langle\overline{H_{\overline{I}}}^{2}\right\rangle_{\overline{H_{\overline{S}}}(t)+\overline{H_{\overline{B}}}}\right). \tag{241}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\langle\overline{H_{\overline{I}}}\rangle_{\overline{H_{\overline{S}}}(t)+\overline{H_{\overline{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = 0. \tag{242}$$

Using this condition and given that $\left[\overline{H_{\overline{S}}}\left(t\right), \overline{H_{\overline{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\overline{S}}}(t)+\overline{H_{\overline{B}}}\right)} = e^{-\beta\overline{H_{\overline{S}}}(t)}e^{-\beta\overline{H_{\overline{B}}}}. \tag{243}$$

Then using the fact that $\overline{H_{\overline{S}}}\left(t\right)$ and $\overline{H_{\overline{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}e^{-\beta\overline{H_{\overline{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{B}}}}\right). \tag{244}$$

So Eq. (242) becomes:

$$\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} &= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\overline{S}}}(t)+\overline{H_{\overline{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}} && (245)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (246)\\
&= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)\right)+\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}} && (247)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (248)\\
&= 0 \quad (\text{by Eq. (242)}). && (249)
\end{aligned}$$

But since $\bar{H}_{\overline{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\begin{aligned}
\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} &= \frac{1}{e^{-\beta\overline{H_{\overline{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} && (250)\\
&= 0. && (251)
\end{aligned}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} = 0. \tag{252}$$

First we look at:

$$-\beta\overline{H_{\overline{S}}}(t)=-\beta\left(\left(\varepsilon_0(t)+R_0\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1\right)|1\rangle\langle 1|+V_{10}(t)B_{10}\sigma_{+}+V_{01}(t)B_{01}\sigma_{-}\right). \tag{253}$$

Then the eigenvalues of $-\beta\overline{H_{\overline{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2-\text{Tr}\left(-\beta\overline{H_{\overline{S}}}(t)\right)+\text{Det}\left(-\beta\overline{H_{\overline{S}}}(t)\right)=0. \tag{254}$$

Let's define:

$$\varepsilon(t)\equiv\text{Tr}\left(\overline{H_{\overline{S}}}(t)\right), \tag{255}$$

$$\eta\equiv\sqrt{\left(\text{Tr}\left(\overline{H_{\overline{S}}}(t)\right)\right)^2-4\text{Det}\left(\overline{H_{\overline{S}}}(t)\right)}. \tag{256}$$

The solutions of the equation (254) are:

$$\lambda=\beta\frac{-\text{Tr}\left(\overline{H_{\overline{S}}}(t)\right)\pm\sqrt{\left(\text{Tr}\left(\overline{H_{\overline{S}}}(t)\right)\right)^2-4\text{Det}\left(\overline{H_{\overline{S}}}(t)\right)}}{2} \tag{257}$$

$$=\beta\frac{-\varepsilon(t)\pm\eta(t)}{2} \tag{258}$$

$$=-\beta\frac{\varepsilon(t)\mp\eta(t)}{2}. \tag{259}$$

The value of $\text{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)=\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\exp\left(\frac{\eta(t)\beta}{2}\right)+\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\exp\left(-\frac{\eta(t)\beta}{2}\right) \tag{260}$$

$$=2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right). \tag{261}$$

Given that $v_{i\mathbf{k}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)=A\left(\varepsilon(t),\eta(t)\right)$ to calculate $\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)}{\partial\Re(v_{i\mathbf{k}})}$ can lead to:

$$\frac{\partial\text{Tr}\left(e^{-\beta\overline{H_{\overline{S}}}(t)}\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)}=\frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial\Re\left(v_{i\mathbf{k}}\right)} \tag{262}$$

$$=2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\right)\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\right)\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\sinh\left(\frac{\eta(t)\beta}{2}\right) \tag{263}$$

$$=-\beta\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\left(\frac{\partial\varepsilon(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\cosh\left(\frac{\eta(t)\beta}{2}\right)-\frac{\partial\eta(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\sinh\left(\frac{\eta(t)\beta}{2}\right)\right). \tag{264}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\cosh\left(\frac{\eta(t)\beta}{2}\right)-\frac{\partial\eta(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)}\sinh\left(\frac{\eta(t)\beta}{2}\right)=0. \tag{265}$$

The derivates included in the expression given are related to:

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{266}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)^*\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{267}$$

$$= \langle B_{1+}B_{0-}\rangle^*, \tag{268}$$

$$R_i = \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) \tag{269}$$

$$= \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{270}$$

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{271}$$

$$= \left(\prod_{\mathbf{k}}\exp\left(\frac{1}{2\omega_{\mathbf{k}}^2}(v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{272}$$

$$v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^* = (\Re(v_{0\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{1\mathbf{k}}))-(\Re(v_{0\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}}))(\Re(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{1\mathbf{k}})) \tag{273}$$

$$= (\Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\mathrm{i}\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}}))-(\Re(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})-\mathrm{i}\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})+\Im(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})) \tag{274}$$

$$= 2\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}})), \tag{275}$$

$$|v_{1\mathbf{k}}-v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}}-v_{0\mathbf{k}})(v_{1\mathbf{k}}-v_{0\mathbf{k}})^* \tag{276}$$

$$= |v_{1\mathbf{k}}|^2+|v_{0\mathbf{k}}|^2-(v_{1\mathbf{k}}v_{0\mathbf{k}}^*+v_{1\mathbf{k}}^*v_{0\mathbf{k}}) \tag{277}$$

$$= (\Re(v_{1\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}}))^2+(\Re(v_{0\mathbf{k}}))^2+(\Im(v_{0\mathbf{k}}))^2-((\Re(v_{1\mathbf{k}})+\mathrm{i}\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}})-\mathrm{i}\Im(v_{0\mathbf{k}}))+(\Re(v_{1\mathbf{k}})-\mathrm{i}\Im(v_{1\mathbf{k}}))(\Re(v_{0\mathbf{k}})+\mathrm{i}\Im(v_{0\mathbf{k}}))) \tag{278}$$

$$= (\Re(v_{1\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}}))^2+(\Re(v_{0\mathbf{k}}))^2+(\Im(v_{0\mathbf{k}}))^2-2(\Re(v_{1\mathbf{k}})\Re(v_{0\mathbf{k}})+\Im(v_{1\mathbf{k}})\Im(v_{0\mathbf{k}})) \tag{279}$$

$$= (\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2. \tag{280}$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}}\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{\Re(v_{i\mathbf{k}})-\mathrm{i}\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{\Re(v_{i\mathbf{k}})+\mathrm{i}\Im(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\right)\right) \tag{281}$$

$$= \sum_{\mathbf{k}}\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\Re(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}\Im(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{282}$$

$$\langle B_{0+}B_{1-}\rangle = \left(\prod_{\mathbf{k}}\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{283}$$

$$= \left(\prod_{\mathbf{k}}\exp\left(\frac{2\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{2\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{284}$$

$$= \left(\prod_{\mathbf{k}}\exp\left(\frac{\mathrm{i}(\Re(v_{0\mathbf{k}})\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}})\Re(v_{1\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{285}$$

Calculating the derivates respect to $\Re(\alpha_{i\mathbf{k}})$ and $\Im(\alpha_{i\mathbf{k}})$ we have:

$$\frac{\partial\varepsilon(t)}{\partial\Re(v_{i\mathbf{k}})} = \frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial\Re(v_{i\mathbf{k}})} \tag{286}$$

$$= \frac{\partial\left(\left(\frac{\Re(v_{i\mathbf{k}})^2+\Im(v_{i\mathbf{k}})^2}{\omega_{\mathbf{k}}}-\Re(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}\Im(v_{i\mathbf{k}})\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial\Re(v_{i\mathbf{k}})} \tag{287}$$

$$= \frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{288}$$

$$\frac{\partial|B_{10}|^2}{\partial\Re(v_{i\mathbf{k}})} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial\Re(v_{i\mathbf{k}})} \tag{289}$$

$$= -\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial\Re(v_{i\mathbf{k}})}\exp\left(-\sum_{\mathbf{k}}\frac{(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))^2+(\Im(v_{1\mathbf{k}})-\Im(v_{0\mathbf{k}}))^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{290}$$

$$= -\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial\Re(v_{i\mathbf{k}})}|B_{10}|^2, \tag{291}$$

$$\frac{\partial\eta(t)}{\partial\Re(v_{i\mathbf{k}})} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\overline{S}}(t)}\right)}}{\partial\Re(v_{i\mathbf{k}})} \tag{292}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)}{\partial\Re(v_{i\mathbf{k}})}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\overline{S}}(t)}\right)}{\partial\Re(v_{i\mathbf{k}})}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\overline{S}}(t)}\right)}} \tag{293}$$

$$= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial\Re(v_{i\mathbf{k}})}}{\eta(t)} \tag{294}$$

$$= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)+\frac{2(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\omega_{\mathbf{k}}^2}\frac{\partial(\Re(v_{1\mathbf{k}})-\Re(v_{0\mathbf{k}}))}{\partial\Re(v_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{295}$$

$$= \frac{\varepsilon(t)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\left((\varepsilon(t)-\varepsilon_i(t)-R_i)\left(\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)+\frac{2(\Re(v_{i\mathbf{k}})-\Re(v_{i'\mathbf{k}}))}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{296}$$

$$= \frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) \tag{297}$$

$$+\frac{1}{\eta(t)}\left(-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\varepsilon(t)+2(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{298}$$

From the equation (265) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial\Re(v_{i\mathbf{k}})}}{\frac{\partial\eta(t)}{\partial\Re(v_{i\mathbf{k}})}} \tag{299}$$

$$=\frac{\frac{2\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{2\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}}{\frac{\Re(v_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i)\frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}+2\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}^2}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{\Re(g_{i\mathbf{k}})}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}} \tag{300}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{(2\Re(v_{i\mathbf{k}})-g_{i\mathbf{k}}-g_{i\mathbf{k}}^*)\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}+g_{i\mathbf{k}}^*)(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{301}$$

$$=\frac{(2\Re(v_{i\mathbf{k}})-2\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2B_{10}^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Re(g_{i\mathbf{k}})(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{302}$$

$$=\frac{(2\Re(v_{i\mathbf{k}})-2\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+4\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{303}$$

$$=\frac{(\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}}))\eta(t)}{\Re(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Re(v_{i'\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{304}$$

Separating (303) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}}))\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=\Re(v_{i\mathbf{k}})\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)-\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{305}$$

$$\Re(v_{i\mathbf{k}})-\Re(g_{i\mathbf{k}})=\Re(v_{i\mathbf{k}})\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \tag{306}$$

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\Re(g_{i\mathbf{k}})(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{307}$$

$$\Re(v_{i\mathbf{k}})=\frac{\Re(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\Re(g_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{308}$$

$$\Re(v_{i\mathbf{k}})=\frac{\Re(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\Re(g_{i\mathbf{k}})}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{309}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon(t)}{\partial\Im\left(v_{i\mathbf{k}}\right)} = \frac{\partial(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0)}{\partial\Im\left(v_{i\mathbf{k}}\right)} \tag{310}$$

$$= \frac{\partial\left(\left(\frac{\Re\left(v_{i\mathbf{k}}\right)^2+\Im\left(v_{i\mathbf{k}}\right)^2}{\omega_{\mathbf{k}}}-\Re\left(v_{i\mathbf{k}}\right)\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\Im\left(v_{i\mathbf{k}}\right)\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)} \tag{311}$$

$$= 2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{312}$$

$$\frac{\partial|B_{10}|^2}{\partial\Im\left(v_{i\mathbf{k}}\right)} = \frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(\Re\left(v_{1\mathbf{k}}\right)-\Re\left(v_{0\mathbf{k}}\right)\right)^2+\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)} \tag{313}$$

$$= -\frac{2\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}\exp\left(-\sum_{\mathbf{k}}\frac{\left(\Re\left(v_{1\mathbf{k}}\right)-\Re\left(v_{0\mathbf{k}}\right)\right)^2+\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{314}$$

$$= -\frac{2\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}|B_{10}|^2 \tag{315}$$

$$\frac{\partial\eta(t)}{\partial\Re\left(v_{i\mathbf{k}}\right)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\overline{S}}(t)}\right)}}{\partial\Re\left(v_{i\mathbf{k}}\right)} \tag{316}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\overline{S}}(t)}\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\overline{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\overline{S}}(t)}\right)}} \tag{317}$$

$$= \frac{\varepsilon(t)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}}{\eta(t)} \tag{318}$$

$$= \frac{\varepsilon(t)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(\Im\left(v_{1\mathbf{k}}\right)-\Im\left(v_{0\mathbf{k}}\right)\right)}{\partial\Im\left(v_{i\mathbf{k}}\right)}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{319}$$

$$= \frac{\varepsilon(t)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\left(2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(\Im\left(v_{i\mathbf{k}}\right)-\Im\left(v_{i'\mathbf{k}}\right)\right)}{\omega^2_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{320}$$

$$= \frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\eta(t)}\right) \tag{321}$$

$$+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega^2_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{322}$$

From the equation (265) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial\Im(v_{i\mathbf{k}})}}{\frac{\partial\eta(t)}{\partial\Im(v_{i\mathbf{k}})}} \tag{323}$$

$$=\frac{2\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{\Im\left(v_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)-\frac{4}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\Im\left(g^*_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i\right)\frac{\Im\left(g^*_{i\mathbf{k}}\right)}{\omega_{\mathbf{k}}}+2\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega^2_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)} \tag{324}$$

Rearrannging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2\Im\left(v_{i\mathbf{k}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\right)\eta(t)}{\Im\left(v_{i\mathbf{k}}\right)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{325}$$

$$= \frac{2\left(\Im\left(v_{i\mathbf{k}}\right)-\Im\left(g_{i\mathbf{k}}\right)\right)\eta(t)}{\Im\left(v_{i\mathbf{k}}\right)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}B_{10}^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Im\left(g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{326}$$

$$= \frac{2\left(\Im\left(v_{i\mathbf{k}}\right)-\Im\left(g_{i\mathbf{k}}\right)\right)\eta(t)}{\Im\left(v_{i\mathbf{k}}\right)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2\Im\left(g_{i\mathbf{k}}\right)\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+4\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$= \frac{\left(\Im\left(v_{i\mathbf{k}}\right)-\Im\left(g_{i\mathbf{k}}\right)\right)\eta(t)}{\Im\left(v_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\Im\left(g_{i\mathbf{k}}\right)\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+2\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

Separating (**??**) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(\Im\left(v_{i\mathbf{k}}\right)-\Im\left(g_{i\mathbf{k}}\right)\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = \Im\left(v_{i\mathbf{k}}\right)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\Im\left(g_{i\mathbf{k}}\right)\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+2\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{329}$$

$$\Im(v_{i\mathbf{k}})-\Im(g_{i\mathbf{k}}) = \Im(v_{i\mathbf{k}})\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) \tag{330}$$

$$-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\Im(g_{i\mathbf{k}})\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{331}$$

$$\Im\left(v_{i\mathbf{k}}\right) = \frac{\Im(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{332}$$

$$\Im\left(v_{i\mathbf{k}}\right) = \frac{\Im(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{333}$$

The variational parameters are:

$$v_{i\mathbf{k}} = \Re\left(v_{i\mathbf{k}}\right)+\mathrm{i}\Im\left(v_{i\mathbf{k}}\right) \tag{334}$$

$$= \frac{\Re(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Re\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{335}$$

$$+\mathrm{i}\frac{\Im(g_{i\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{\Im\left(v_{i'\mathbf{k}}\right)}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{336}$$

$$= \frac{g_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}-\varepsilon\left(t\right)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_{T}\left(0\right)=\rho_{S}(0)\otimes\rho_{B}^{\mathrm{Thermal}}$, the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T\left(0\right) e^{-V} \tag{338}$$
$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) \left(\rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}\right) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \tag{339}$$
$$\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } |0\rangle\langle 0|0\rangle B_{0+}\langle 0|\rho_B^{\mathrm{Thermal}}|0\rangle\langle 0|B_{0-} \tag{340}$$
$$= |0\rangle B_{0+}\langle 0|\rho_B^{\mathrm{Thermal}}|0\rangle\langle 0|B_{0-} \tag{341}$$
$$= |0\rangle\langle 0| \otimes B_{0+}\rho_B^{\mathrm{Thermal}}B_{0-} \tag{342}$$
$$\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\rho_B^{\mathrm{Thermal}}|1\rangle\langle 1|B_{1-} \tag{343}$$
$$= |1\rangle\langle 1|B_{1+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{344}$$
$$= |1\rangle\langle 1| \otimes B_{1+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{345}$$
$$\text{for } \rho_S(0) = |0\rangle\langle 1| : |0\rangle\langle 0|B_{0+}|0\rangle\langle 1|\rho_B^{\mathrm{Thermal}}|1\rangle\langle 1|B_{1-} \tag{346}$$
$$= |0\rangle\langle 1|B_{0+}\rho_B^{\mathrm{Thermal}}|1\rangle\langle 1|B_{1-} \tag{347}$$
$$= |0\rangle\langle 1|1\rangle\langle 1|B_{0+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{348}$$
$$= |0\rangle\langle 1| \otimes B_{0+}\rho_B^{\mathrm{Thermal}}B_{1-} \tag{349}$$
$$\text{for } \rho_S(0) = |1\rangle\langle 0| : |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\rho_B^{\mathrm{Thermal}}|0\rangle\langle 0|B_{0-} \tag{350}$$
$$= |1\rangle\langle 0| \otimes B_{1+}\rho_B^{\mathrm{Thermal}}B_{0-} \tag{351}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}\left(t\right) \equiv U^\dagger(t) O\left(t\right) U(t) \tag{352}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\overline{S}}}\left(t'\right)\right). \tag{353}$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_{\overline{S}}}}(t) = U^\dagger(t)\overline{\rho_{\overline{S}}}(t)U(t), \text{ where} \tag{354}$$
$$\overline{\rho_{\overline{S}}}(t) = \mathrm{Tr}_B\left(\overline{\rho}_T(t)\right) \tag{355}$$

. In order to separate the Hamiltonian we define the matrix $\Lambda\left(t\right)$ such that $\Lambda_{1i}\left(t\right) = A_i$, $\Lambda_{2i}\left(t\right) = B_i$ and $\Lambda_{3i}\left(t\right) = C_i\left(t\right)$ written as:

$$\begin{pmatrix} A\left(t\right) \\ B\left(t\right) \\ C\left(t\right) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ \Re\left(V_{10}\left(t\right)\right) & \Re\left(V_{10}\left(t\right)\right) & 1 & \Im\left(V_{10}\left(t\right)\right) & -\Im\left(V_{10}\left(t\right)\right) & 1 \end{pmatrix} \tag{356}$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\overline{I}}}\left(t\right)$ as pointed in the equation (**??**):

$$\overline{H_{\overline{I}}}\left(t\right) = \sum_i B_{iz}|i\rangle\langle i| + \Re\left(V_{10}\left(t\right)\right)\left(\sigma_x B_x + \sigma_y B_y\right) + \Im\left(V_{10}\left(t\right)\right)\left(\sigma_x B_y - \sigma_y B_x\right) \tag{357}$$
$$= B_{0z}|0\rangle\langle 0| + B_{1z}|1\rangle\langle 1| + \Re\left(V_{10}\left(t\right)\right)\sigma_x B_x + \Re\left(V_{10}\left(t\right)\right)\sigma_y B_y + \Im\left(V_{10}\left(t\right)\right)\sigma_x B_y - \Im\left(V_{10}\left(t\right)\right)\sigma_y B_x \tag{358}$$
$$= \sum_i C_i\left(t\right)\left(A_i \otimes B_i\left(t\right)\right) \tag{359}$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\rho_T}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{H_{\overline{I}}}(t)\,,\widetilde{\rho_T}\,(t)]. \tag{360}$$

This equation has the formal solution

$$\widetilde{\rho_T}(t) = \rho\,(0)\,\text{-i}\int_0^t [\widetilde{H_{\overline{I}}}\,(s)\,,\widetilde{\rho_T}\,(s)]\mathrm{d}s. \tag{361}$$

Replacing the equation (361) in the equation (360)give us:

$$\frac{\mathrm{d}\widetilde{\rho_T}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{H_{\overline{I}}}\,(t)\,,\rho_T\,(0)]\text{-}\int_0^t [\widetilde{H_{\overline{I}}}\,(t)\,,[\widetilde{H_{\overline{I}}}\,(s)\,,\widetilde{\rho_T}\,(s)]]\mathrm{d}s. \tag{362}$$

This equation allow us to iterate and write in terms of a series expansion with $\rho_T\,(0)$ the solution as:

$$\widetilde{\rho_T}(t) = \rho_T\,(0) + \sum_{n=0}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_n[\widetilde{H_{\overline{I}}}\,(t_1)\,,[\widetilde{H_{\overline{I}}}\,(t_2)\,,\cdots[\widetilde{H_{\overline{I}}}\,(t_n)\,,\rho_T\,(0)]]\cdots] \tag{363}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho}_{\overline{S}}(t) = \rho_{\overline{S}}\,(0) + \sum_{n=1}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_n\mathrm{Tr}_B[\widetilde{H_{\overline{I}}}\,(t_1)\,,[\widetilde{H_{\overline{I}}}\,(t_2)\,,\cdots[\widetilde{H_{\overline{I}}}\,(t_n)\,,\rho_{\overline{S}}\,(0)\,\rho_B^{\mathrm{Thermal}}]]\cdots] \tag{364}$$

here we have assumed that $\rho_T\,(0) = \rho_{\overline{S}}(0)\otimes\rho_B^{\mathrm{Thermal}}$. Consider the following notation:

$$\widetilde{\rho}_{\overline{S}}(t) = (1 + W_1\,(t) + W_2\,(t) + ...)\,\rho_S\,(0) \tag{365}$$
$$= W\,(t)\,\rho_{\overline{S}}\,(0) \tag{366}$$

in this case

$$W_n\,(t) = (-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_n\mathrm{Tr}_B[\widetilde{H_{\overline{I}}}\,(t_1)\,,[\widetilde{H_{\overline{I}}}\,(t_2)\,,\cdots[\widetilde{H_{\overline{I}}}\,(t_n)\,,(\cdot)\,\rho_B^{\mathrm{Thermal}}]]\cdots] \tag{367}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\rho}_{\overline{S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1\,(t) + \dot{W}_2\,(t) + ...\right)\rho_{\overline{S}}\,(0) \tag{368}$$
$$= \left(\dot{W}_1\,(t) + \dot{W}_2\,(t) + ...\right)W\,(t)^{-1}\,W\,(t)\,\rho_{\overline{S}}\,(0) \tag{369}$$
$$= \left(\dot{W}_1\,(t) + \dot{W}_2\,(t) + ...\right)W\,(t)^{-1}\,\widetilde{\rho}_{\overline{S}}(t) \tag{370}$$

where we assumed that $W\,(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{H_{\overline{I}}}(t)\rho_B(0)] = 0$ so $W_1\,(t) = 0$. Thus, to second order and taking $W\,(t) \approx \mathbb{I}$ then the equation (368) becomes:

$$\frac{\mathrm{d}\rho_{\overline{S}}\,(t)}{\mathrm{d}t} = -\mathrm{i}\,[H_{\overline{S}}, \rho_{\overline{S}}\,(t)] - \int_0^t \mathrm{d}\tau\left[H_I, \left[\widetilde{H}_I(-\tau), \rho_{\overline{S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right] \tag{371}$$

Replacing $t_1 \to t - \tau$

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B[\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \cdots [\widetilde{H}_I(t_n), (\cdot)\, \rho_B^{\mathrm{Thermal}}]] \cdots] \tag{372}$$

Taking as reference state $\rho_B^{\mathrm{Thermal}}$ and truncating at second order in $\overline{H_I}(t)$, we obtain our master equation in the interaction picture in the transformed frame:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_I}}(t), \left[\widetilde{\overline{H_I}}(s), \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right] \mathrm{d}s \tag{373}$$

From the interaction picture applied on $\overline{H_I}(t)$ we find:

$$\widetilde{\overline{H_I}}(t) = U^\dagger(t)\, \mathrm{e}^{\mathrm{i}H_B t}\overline{H_I}(t)\, \mathrm{e}^{-\mathrm{i}H_B t}U(t) \tag{374}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_S}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\begin{aligned}
\widetilde{\overline{H_I}}(t) &= \sum_i C_i(t)\left(\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)\right) && (375)\\
\widetilde{A_i}(t) &= U^\dagger(t)\, \mathrm{e}^{\mathrm{i}H_B t}A_i \mathrm{e}^{-\mathrm{i}H_B t}U(t) && (376)\\
&= U^\dagger(t)\, A_i U(t)\, \mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} && (377)\\
&= U^\dagger(t)\, A_i U(t)\, \mathbb{I} && (378)\\
&= U^\dagger(t)\, A_i U(t) && (379)\\
\widetilde{B_i}(t) &= U^\dagger(t)\, \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t}U(t) && (380)\\
&= U^\dagger(t)\, U(t)\, \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t} && (381)\\
&= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t} && (382)\\
&= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\, \mathrm{e}^{-\mathrm{i}H_B t} && (383)
\end{aligned}$$

Here we have used the fact that $\left[\overline{H_S}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (375) to replace it in the equation (373)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_I}}(t), \left[\widetilde{\overline{H_I}}(s), \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\right]\right] \mathrm{d}s && (384)\\
&= -\int_0^t \mathrm{Tr}_B\Big[\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big), \Big[\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big), \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\Big]\Big]\mathrm{d}s && (385)\\
&= -\int_0^t \mathrm{Tr}_B\Big[\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big), \sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}} - \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\Big]\mathrm{d}s && (386)\\
&= -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}} - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) && (387)\\
&\quad - \sum_{i\in J} C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\overline{\rho_S}}(t)\rho_B^{\mathrm{Thermal}}\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}s && (388)
\end{aligned}$$

In order to calculate the correlation functions we define:

$$\begin{aligned}
\Lambda_{ji}(\tau) &= \left\langle \widetilde{B_j}(t)\, \widetilde{B_i}(s) \right\rangle_B && (389)\\
&= \left\langle \widetilde{B_j}(\tau)\, \widetilde{B_i}(0) \right\rangle_B && (390)
\end{aligned}$$

The correlation functions relevant that appear in the equation (**??**) are:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\rho_B^{\text{Thermal}}\right) &= \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\right\rangle_B && (391)\\
&= \left\langle \widetilde{B_j}\left(\tau\right)\widetilde{B_i}\left(0\right)\right\rangle_B && (392)\\
&= \Lambda_{ji}\left(\tau\right) && (393)\\
\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B^{\text{Thermal}}\widetilde{B_i}\left(s\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\rho_B^{\text{Thermal}}\right) && (394)\\
&= \left\langle \widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right\rangle_B && (395)\\
&= \left\langle \widetilde{B_i}\left(-\tau\right)\widetilde{B_j}\left(0\right)\right\rangle_B && (396)\\
&= \Lambda_{ij}\left(-\tau\right) && (397)\\
\mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\rho_B^{\text{Thermal}}\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\rho_B^{\text{Thermal}}\right) && (398)\\
&= \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(s\right)\right\rangle_B && (399)\\
&= \left\langle \widetilde{B_j}\left(\tau\right)\widetilde{B_i}\left(0\right)\right\rangle_B && (400)\\
&= \Lambda_{ji}\left(\tau\right) && (401)\\
\mathrm{Tr}_B\left(\rho_B^{\text{Thermal}}\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\rho_B^{\text{Thermal}}\right) && (402)\\
&= \left\langle \widetilde{B_i}\left(s\right)\widetilde{B_j}\left(t\right)\right\rangle_B && (403)\\
&= \left\langle \widetilde{B_i}\left(-\tau\right)\widetilde{B_j}\left(0\right)\right\rangle_B && (404)\\
&= \Lambda_{ij}\left(-\tau\right) && (405)
\end{aligned}$$

The cyclic property of the trace was use widely in the development of equations (391) and (405). Replacing in (**??**)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\big(\Lambda_{ij}(\tau)\widetilde{A_i}(t)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)-\Lambda_{ji}(-\tau)\widetilde{A_i}(t)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\big)+C_i(t)C_j(s)\big(\Lambda_{ji}(-\tau)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s)\widetilde{A_i}(t)-\Lambda_{ij}(\tau)\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}(t)\big)\Big)\mathrm{d}s && (406)\\
&= -\int_0^t \sum_{ij}\Big(C_i(t)C_j(s)\Big(\Lambda_{ij}(\tau)\Big[\widetilde{A_i}(t),\widetilde{A_j}(s)\widetilde{\overline{\rho_S}}(t)\Big]+\Lambda_{ji}(-\tau)\Big[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}(s),\widetilde{A_i}(t)\Big]\Big)\Big)\mathrm{d}s && (407)
\end{aligned}$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}\left(\tau\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)-\Lambda_{ij}\left(\tau\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}\left(t\right)=\Lambda_{ij}\left(\tau\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right] \qquad (408)$$

$$\Lambda_{ji}\left(-\tau\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\Lambda_{ji}\left(-\tau\right)\widetilde{A_i}\left(t\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_j}\left(s\right)=\Lambda_{ji}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_j},\widetilde{A_i}\left(t\right)\right] \qquad (409)$$

Returning to the Schroedinger picture we have:

$$\begin{aligned}
U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right) &= U\left(t\right)\widetilde{A_i}\left(t\right)U^{\dagger}\left(t\right)U\left(t\right)\widetilde{A_j}\left(s\right)U^{\dagger}\left(t\right)U\left(t\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right) && (410)\\
&= \left(U\left(t\right)\widetilde{A_i}\left(t\right)U^{\dagger}\left(t\right)\right)\left(U\left(t\right)\widetilde{A_j}\left(s\right)U^{\dagger}\left(t\right)\right)\left(U\left(t\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right)\right) && (411)\\
&= A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}(t) && (412)
\end{aligned}$$

This procedure applying to the relevant commutators give us:

$$\begin{aligned}
U\left(t\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right]U^{\dagger}\left(t\right) &= \left(U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)U^{\dagger}\left(t\right)-U\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_i}\left(t\right)U^{\dagger}\left(t\right)\right) && (413)\\
&= A_i\widetilde{A_j}\left(s,t\right)\overline{\rho_S}(t)-\widetilde{A_j}\left(s,t\right)\overline{\rho_S}(t)A_i && (414)\\
&= \left[A_i,\widetilde{A_j}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] && (415)
\end{aligned}$$

Introducing this transformed commutators in the equation (**??**) allow us to obtain the master equation of the system

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau \left(C_i\left(t\right)C_j\left(t-\tau\right)\Lambda_{ij}\left(\tau\right)\left[A_i, \widetilde{A_j}\left(t-\tau,t\right)\overline{\rho_S}(t)\right]\right. \tag{416}$$

$$\left. + C_j\left(t\right)C_i\left(t-\tau\right)\Lambda_{ji}\left(-\tau\right)\left[\overline{\rho_S}(t)\widetilde{A_j}\left(t-\tau,t\right), A_i\right]\right) \tag{417}$$

where $i,j \in \{1,2,3,4,5.6\}$.
Here $\widetilde{A_j}\left(s,t\right) = U\left(t\right)U^{\dagger}\left(s\right)A_jU\left(s\right)U^{\dagger}\left(t\right)$ where $U\left(t\right)$ is given by (353). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. The environmental correlation functions are given by:

$$\Lambda_{ij}\left(\tau\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B^{\mathrm{Thermal}}\right) \tag{418}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B^{\mathrm{Thermal}}\right) \tag{419}$$

Calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B^{\mathrm{Thermal}}\right) \tag{420}$$

$$= \int \mathrm{d}^2\alpha P\left(\alpha\right)\left\langle \alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle \tag{421}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle \alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{422}$$

$$= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\left\langle \alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle \mathrm{d}^2\alpha \tag{423}$$

$$\widetilde{B_{jz}}\left(\tau\right) = \sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{424}$$

$$\widetilde{B_{jz}}\left(0\right) = \sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger} + \left(g_{j\mathbf{k}'} - v_{j\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right) \tag{425}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\rho_B\right) \tag{426}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger} + \left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\rho_B\right) \tag{427}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger} + \left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\rho_B\right) \tag{428}$$

$$+ \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}}-v_{j\mathbf{k}})^* b_{\mathbf{k}}\right)\rho_B\right) \tag{429}$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}} = p_{j\mathbf{k}} \tag{430}$$

$$\left\langle \widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}'}b_{\mathbf{k}'}^{\dagger} + p_{j\mathbf{k}'}^* b_{\mathbf{k}'}\right)\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger} + p_{j\mathbf{k}}^* b_{\mathbf{k}}\right)\rho_B\right) \tag{431}$$

$$= 0 + \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger} + p_{j\mathbf{k}}^* b_{\mathbf{k}}\right)\rho_B\right) \tag{432}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B\right) \tag{433}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{434}$$

$$= \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) + \mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2 b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{435}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right) + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right)\right) \tag{436}$$

$$= \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{437}$$

$$= \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{438}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{439}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{440}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{441}$$

$$+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{442}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{443}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{444}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)\right. \tag{445}$$

$$\left.+\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)\right) \tag{446}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{447}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{448}$$

$$+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{449}$$

$$1 = \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{450}$$

$$b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left|0\right\rangle = 0 \tag{451}$$

$$b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\left|0\right\rangle = \left|0\right\rangle \tag{452}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{453}$$

$$+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{454}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{455}$$

$$= \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{456}$$

$$\frac{1}{\pi N}\int_0^{2\pi}\int_0^{\infty}r^2\exp\left(-\frac{r^2}{N}\right)r\,\mathrm{d}r\,\mathrm{d}\theta=\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{457}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2 e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{458}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{459}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{460}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}+\cos\left(\omega_{\mathbf{k}}\tau\right)-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{461}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(2+e^{\beta\omega_{\mathbf{k}}}-1\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{462}$$

$$= \sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(1+e^{\beta\omega_{\mathbf{k}}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{463}$$

$$= \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}+e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos(\omega_{\mathbf{k}}\tau)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}-e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)\right) \tag{464}$$

$$= \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau)-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)\right) \tag{465}$$

$$= \sum_{\mathbf{k}}|g_{j\mathbf{k}}-v_{j\mathbf{k}}|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos(\omega_{\mathbf{k}}\tau)-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)\right) \tag{466}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\right\rangle_B = \int\mathrm{d}^2\alpha_{\mathbf{k}}P(\alpha_{\mathbf{k}})\left\langle\alpha_{\mathbf{k}}\middle|\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\middle|\alpha_{\mathbf{k}}\right\rangle \tag{467}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{468}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left(\left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right)^{*} b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{469}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}\right)^{*} b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{470}$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} + \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^{*} b_{\mathbf{k}} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{471}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(\left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} + \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^{*} b_{\mathbf{k}} \right) | \alpha_{\mathbf{k}} \rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{472}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^{*} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{473}$$

$$= \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^{*} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} + \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{474}$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{475}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D\left(-\alpha_{\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} D\left(\alpha_{\mathbf{k}}\right) D\left(-\alpha_{\mathbf{k}}\right) b_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{476}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| \left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{477}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}|^2 \, \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{478}$$

$$= N \tag{479}$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{480}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| \left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right) \left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{481}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}} \alpha_{\mathbf{k}}^{*} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{482}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{483}$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle 0 \middle| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \middle| 0 \right\rangle \mathrm{d}^2 \alpha_{\mathbf{k}} \tag{484}$$

$$= N + 1 \tag{485}$$

$$\left\langle \widetilde{B_{jz}}\left(\tau\right) \widetilde{B_{j'z}}\left(0\right) \right\rangle_{B} = \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^{*} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N+1) \tag{486}$$

$$= \sum_{\mathbf{k}} \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} + N \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right)^{*} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^{*} \left(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right) \tag{487}$$

$$D\left(h'\right) D\left(h\right) = \exp\left(\frac{1}{2}\left(h' h^{*} - h'^{*} h\right)\right) D\left(h' + h\right) \tag{488}$$

$$\left\langle D\left(h'\right) D\left(h\right) \right\rangle_{B} = \mathrm{Tr}_{B}\left(\exp\left(\frac{1}{2}\left(h' h^{*} - h'^{*} h\right)\right) D\left(h' + h\right) \rho_{B}^{\mathrm{Thermal}}\right) \tag{489}$$

$$= \exp\left(\frac{1}{2}\left(h' h^{*} - h'^{*} h\right)\right) \mathrm{Tr}_{B}\left(D\left(h' + h\right) \rho_{B}^{\mathrm{Thermal}}\right) \tag{490}$$

$$= \exp\left(\frac{1}{2}\left(h' h^{*} - h'^{*} h\right)\right) \frac{1}{\pi N} \int \mathrm{d}^2 \alpha P\left(\alpha\right) \left\langle \alpha \middle| D\left(h' + h\right) \middle| \alpha \right\rangle \tag{491}$$

$$= \exp\left(\frac{1}{2}\left(h' h^{*} - h'^{*} h\right)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{492}$$

$$h' = h \exp\left(\mathrm{i}\omega\tau\right) \tag{493}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau)) D(h) \right\rangle_{B} = \exp\left(\frac{1}{2}\left(h h^{*} \exp(\mathrm{i}\omega\tau) - h^{*} h \exp(-\mathrm{i}\omega\tau)\right)\right) \exp\left(-\frac{|h + h\exp(\mathrm{i}\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \tag{494}$$

$$\frac{1}{2} |h|^2 \left(\exp\left(\mathrm{i}\omega\tau\right) - \exp\left(-\mathrm{i}\omega\tau\right)\right) = \frac{1}{2}\left(h h^{*} \exp\left(\mathrm{i}\omega\tau\right) - h^{*} h \exp\left(-\mathrm{i}\omega\tau\right)\right) \tag{495}$$

$$= \frac{1}{2} |h|^2 \left(\cos\left(\omega\tau\right) + \mathrm{i} \sin\left(\omega\tau\right) - \cos\left(\omega\tau\right) + \mathrm{i} \sin\left(\omega\tau\right)\right) \tag{496}$$

$$= \frac{1}{2} |h|^2 \left(2\mathrm{i} \sin\left(\omega\tau\right)\right) \tag{497}$$

$$= \mathrm{i} \, |h|^2 \sin\left(\omega\tau\right) \tag{498}$$

$$-\frac{|h + h\exp\left(\mathrm{i}\omega\tau\right)|^2}{2} = -|h|^2 \frac{|1 + \exp\left(\mathrm{i}\omega\tau\right)|^2}{2} \tag{499}$$

$$
\begin{aligned}
&= -|h|^2 \, \frac{|1+\cos(\omega\tau)+\mathrm{i}\sin(\omega\tau)|^2}{2} && (500)\\
&= -|h|^2 \, \frac{(1+\cos(\omega\tau))^2+\sin^2(\omega\tau)}{2} && (501)\\
&= -|h|^2 \, \frac{\left(1+2\cos(\omega\tau)+\cos^2(\omega\tau)\right)+\sin^2(\omega\tau)}{2} && (502)\\
&= -|h|^2 \, \frac{2+2\cos(\omega\tau)}{2} && (503)\\
&= -|h|^2 \left(1+\cos(\omega\tau)\right) && (504)\\
\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\rangle_B &= \exp\!\left(\mathrm{i}|h|^2\sin(\omega\tau)\right)\exp\!\left(-|h|^2(1+\cos(\omega\tau))\coth\left(\tfrac{\beta\omega}{2}\right)\right) && (505)\\
&= \exp\left(\mathrm{i}\,|h|^2\sin(\omega\tau)-|h|^2\left(1+\cos(\omega\tau)\right)\coth\left(\frac{\beta\omega}{2}\right)\right) && (506)\\
&= \exp\left(-|h|^2\left(-\mathrm{i}\sin(\omega\tau)+\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-|h|^2\coth\left(\frac{\beta\omega}{2}\right)\right) && (507)\\
&= \langle D(h)\rangle_B \exp\left(-\phi(\tau)\right) && (508)\\
\exp\left(-\phi(\tau)\right) &= \exp\left(-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)\right) && (509)\\
\phi(\tau) &= |h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right) && (510)\\
\left\langle D(h')\,D(h)\right\rangle_B &= \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right) && (511)\\
h' &= v\exp(i\omega\tau) && (512)\\
\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B &= \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}(\tau)\,\widetilde{B_{1+}B_{0-}}(0)\,\rho_B^{\mathrm{Thermal}}\right) && (513)\\
&= \mathrm{Tr}_B\left(\widetilde{B_{1+}B_{0-}}(\tau)\,\widetilde{B_{1+}B_{0-}}(0)\,\rho_B^{\mathrm{Thermal}}\right) && (514)\\
&= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) && (515)\\
&= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) && (516)\\
&= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) && (517)\\
&= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\prod_{\mathbf{k}}\left(\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (518)\\
&= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (519)\\
\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B &= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) && (520)\\
\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B &= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) && (521)\\
&= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) && (522)\\
&= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) && (523)\\
&= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) && (524)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) && (525)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega\tau+\pi)}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B^{\mathrm{Thermal}}\right) && (526)\\
&= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) && (527)\\
&= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(\mathrm{i}\sin(\omega\tau)-\cos(\omega\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) && (528)\\
\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B &= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B^{\mathrm{Thermal}}\right) && (529)
\end{aligned}
$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{530}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}_B\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{531}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega\tau+\pi)+\cos(\omega\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{532}$$

$$= \left\langle |B_{\widetilde{1+B}_{0-}}(\tau)B_{\widetilde{0+B}_{1-}}(0)\right\rangle_B \tag{533}$$

$$\left\langle B_{\widetilde{0+B}_{1-}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\rho_B^{\mathrm{Thermal}}\right) \tag{534}$$

$$\langle D(h)b\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|D(h)b|\alpha\rangle \tag{535}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\left|D(-\alpha)D(h)bD(\alpha)\right|\alpha\rangle \tag{536}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)bD(\alpha)\right|0\rangle \tag{537}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)D(\alpha)D(-\alpha)bD(\alpha)\right|0\rangle \tag{538}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\left|D(-\alpha)D(h)D(\alpha)(b+\alpha)\right|0\rangle \tag{539}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0\left|D(h)(b+\alpha)\right|0\rangle \tag{540}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)b|0\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \tag{541}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0\left|D(h)\alpha\right|0\rangle \tag{542}$$

$$= \frac{1}{\pi N}\int \alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \tag{543}$$

$$= hN\langle D(h)\rangle_B \tag{544}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle\alpha\left|D(h)b^\dagger\right|\alpha\right\rangle \tag{545}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)b^\dagger D(\alpha)\right|0\right\rangle \tag{546}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)b^\dagger D(\alpha)\right|0\right\rangle \tag{547}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)D(-\alpha)b^\dagger D(\alpha)\right|0\right\rangle \tag{548}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{549}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\left\langle 0\left|D(h)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{550}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\left\langle 0\left|D(h)b^\dagger\right|0\right\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha^*|0\rangle \tag{551}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|1\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle \tag{552}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle -h|1\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle \tag{553}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right)\sum_n\frac{(-h^*)^n}{\sqrt{n!}}\langle n| \tag{554}$$

$$\langle -h|1\rangle = \exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*) \tag{555}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*)+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\exp\left(-\frac{|-h^*|^2}{2}\right) \tag{556}$$

$$= -h^*\langle D(h)\rangle_B(N+1) \tag{557}$$

$$\langle bD(h)\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\left|bD(h)\right|\alpha\rangle \tag{558}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)h+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\frac{|h|^2}{2}\right) \tag{559}$$

$$= h \langle D(h) \rangle_B (N+1) \tag{560}$$

$$\left\langle b^\dagger D(h) \right\rangle_B = \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \left\langle \alpha \left| b^\dagger D(h) \right| \alpha \right\rangle \tag{561}$$

$$= \frac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \exp\left(-\frac{|h|^2}{2}\right) h + \tfrac{1}{\pi N} \int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right) \exp\left(h\alpha^* - h^*\alpha\right) \alpha \exp\left(-\tfrac{|h|^2}{2}\right) \tag{562}$$

$$= -h^* \langle D(h) \rangle_B N \tag{563}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \right) \right\rangle_B \tag{564}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{565}$$

$$= \prod_{\mathbf{k}} \left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)} \right) \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \right\rangle_B \tag{566}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(\tfrac{1}{2} \left(\frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) \right) \prod_{\mathbf{k}} \exp\left(-\tfrac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \tag{567}$$

$$= B_{10} \tag{568}$$

The correlation functions can be found readily as:

$$\widetilde{B_{1+}B_{0-}}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \tag{569}$$

$$\widetilde{B_{0+}B_{1-}}(\tau) = \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \exp\left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \tag{570}$$

$$\widetilde{B_x}(0) = \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2} \tag{571}$$

$$\widetilde{B_y}(0) = \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10} - B_{10}^*}{2\mathrm{i}} \tag{572}$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega}{2} \right) \right) \right) \tag{573}$$

$$B_{iz} = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \tag{574}$$

$$\left\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{jk})^* b_{\mathbf{k}} \right) \right\rangle_B \tag{575}$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \tag{576}$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \right\rangle_B = \left\langle \frac{B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*}{2} \frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2} \right\rangle_B \tag{577}$$

$$= \frac{1}{4} \langle (B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10} - B_{10}^*) (B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*) \rangle_B \tag{578}$$

$$= \frac{1}{4} \langle B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{10} - B_{1+}B_{0-}(\tau) B_{10}^* + B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{0+}B_{1-} \tag{579}$$

$$- B_{0+}B_{1-}(\tau) B_{10} - B_{0+}B_{1-}(\tau) B_{10}^* - B_{10} B_{1+}B_{0-} - B_{10} B_{0+}B_{1-} + B_{10} B_{10} + B_{10} B_{10}^* - B_{10}^* B_{1+}B_{0-} - B_{10}^* B_{0+}B_{1-} + B_{10}^* B_{10} + B_{10}^* B_{10}^* \rangle \tag{580}$$

$$= \frac{1}{4} \langle B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{10} - B_{1+}B_{0-}(\tau) B_{10}^* + B_{0+}B_{1-}(\tau) B_{1+}B_{0-} \tag{581}$$

$$+ B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{10} - B_{0+}B_{1-}(\tau) B_{10}^* \rangle \tag{582}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{583}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right) \tag{584}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{585}$$

$$S = \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{586}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = U\exp(-\phi(\tau))\,S \tag{587}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = U^{*}\exp(-\phi(\tau))S \tag{588}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B = \exp(\phi(\tau))\,S \tag{589}$$

$$\left\langle \widetilde{B_{0+}B_{1-}}(\tau)\widetilde{B_{1+}B_{0-}}(0)\right\rangle_B = \left\langle \widetilde{B_{1+}B_{0-}}(\tau)\,\widetilde{B_{0+}B_{1-}}(0)\right\rangle_B \tag{590}$$

$$\left\langle \widetilde{B_{1+}B_{0-}}(\tau)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{591}$$

$$= U^{*1/2}S^{1/2} \tag{592}$$

$$\left\langle \widetilde{B_x}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{10}^{*}+B_{0+}B_{1-}(\tau)B_{1+}B_{0-} \tag{593}$$

$$+B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)\,B_{10}-B_{0+}B_{1-}(\tau)\,B_{10}^{*}\rangle \tag{594}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{1}{4}\langle B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{10}^{*}+B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{0+}B_{1-} \tag{595}$$

$$-B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*}\rangle \tag{596}$$

$$= \frac{1}{4}\left(U^{*}\exp(-\phi(\tau))\,S+\exp(\phi(\tau))\,S-B_{10}^{2}-|B_{10}|^{2}+\exp(\phi(\tau))\,S+U\exp(-\phi(\tau))\,S-B_{10}^{*2}-|B_{10}|^{2}\right) \tag{597}$$

$$= \frac{1}{4}\left(2\Re(U)\exp(-\phi(\tau))\,S+2\exp(\phi(\tau))\,S-2\Re\left(B_{10}^{2}\right)-2\,|B_{10}|^{2}\right) \tag{598}$$

$$= \frac{1}{4}\left(2\Re(U)\exp(-\phi(\tau))\,S+2\exp(\phi(\tau))\,S-2\Re(U^{*})\,S-2S\right) \tag{599}$$

$$= \frac{S}{2}\left(\Re(U)\exp(-\phi(\tau))+\exp(\phi(\tau))-\Re(U^{*})-1\right) \tag{600}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{10}^{*}}{2\mathrm{i}}\,\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}}{2\mathrm{i}}\right\rangle_B \tag{601}$$

$$= -\frac{1}{4}\left\langle\left(B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{10}^{*}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}\right)\right\rangle_B \tag{602}$$

$$= -\frac{1}{4}\langle B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*}-B_{1+}B_{0-}(\tau)B_{0+}B_{1-}+B_{1+}B_{0-}(\tau)B_{1+}B_{0-} \tag{603}$$

$$-B_{1+}B_{0-}(\tau)B_{10}+B_{1+}B_{0-}(\tau)B_{10}^{*}+B_{10}B_{0+}B_{1-}-B_{10}B_{1+}B_{0-}+B_{10}B_{10}-B_{10}B_{10}^{*}-B_{10}^{*}B_{0+}B_{1-}+B_{10}^{*}B_{1+}B_{0-}-B_{10}^{*}B_{10}+B_{10}^{*}B_{10}^{*}\rangle \tag{604}$$

$$= -\frac{1}{4}\langle B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*} \tag{605}$$

$$-B_{1+}B_{0-}(\tau)B_{0+}B_{1-}+B_{1+}B_{0-}(\tau)B_{1+}B_{0-}-B_{1+}B_{0-}(\tau)B_{10}+B_{1+}B_{0-}(\tau)B_{10}^{*}\rangle \tag{606}$$

$$= -\frac{1}{4}\langle B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-}+B_{10}^{*}B_{10}-B_{10}^{*}B_{10}^{*}-B_{1+}B_{0-}(\tau)B_{0+}B_{1-}+B_{1+}B_{0-}(\tau)B_{1+}B_{0-}-B_{10}B_{10}+B_{10}B_{10}^{*}\rangle \tag{607}$$

$$= -\frac{1}{4}\left(U\exp(-\phi(\tau))\,S-\exp(\phi(\tau))\,S-\exp(\phi(\tau))\,S+U^{*}\exp(-\phi(\tau))\,S+2S-2\Re(U^{*})\,S\right) \tag{608}$$

$$= -\frac{S}{4}\left(2\Re(U)\exp(-\phi(\tau))-2\exp(\phi(\tau))+2-2\Re(U)\right) \tag{609}$$

$$= \frac{S}{2}\left(\exp(\phi(\tau))-\Re(U)\exp(-\phi(\tau))-1+\Re(U)\right) \tag{610}$$

$$\left\langle \widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_{1+}B_{0-}(\tau)+B_{0+}B_{1-}(\tau)-B_{10}-B_{10}^{*}}{2}\,\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}}{2\mathrm{i}}\right\rangle_B \tag{611}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_{1+}B_{0-}(\tau)+B_{0+}B_{1-}(\tau)-B_{10}-B_{10}^{*}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}\right)\right\rangle_B \tag{612}$$

$$= \frac{1}{4\mathrm{i}}\langle B_{1+}B_{0-}(\tau)B_{0+}B_{1-}-B_{1+}B_{0-}(\tau)B_{1+}B_{0-}+B_{1+}B_{0-}(\tau)B_{10}-B_{1+}B_{0-}(\tau)B_{10}^{*}+B_{0+}B_{1-}(\tau)B_{0+}B_{1-}-B_{0+}B_{1-}(\tau)B_{1+}B_{0-} \tag{613}$$

$$+B_{0+}B_{1-}(\tau)B_{10}-B_{0+}B_{1-}(\tau)B_{10}^{*}-B_{10}B_{0+}B_{1-}+B_{10}B_{1+}B_{0-}-B_{10}B_{10}+B_{10}B_{10}^{*}-B_{10}^{*}B_{0+}B_{1-}+B_{10}^{*}B_{1+}B_{0-}-B_{10}^{*}B_{10}+B_{10}^{*}B_{10}^{*}\rangle \tag{614}$$

$$= \frac{1}{4\mathrm{i}}\langle B_{1+}B_{0-}(\tau)B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} + B_{1+}B_{0-}(\tau)B_{10} - B_{1+}B_{0-}(\tau)B_{10}^{*} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{10} - B_{0+}B_{1-}(\tau)B_{10}^{*}\rangle \tag{615}$$

$$= \frac{1}{4\mathrm{i}}\langle B_{1+}B_{0-}(\tau)B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} + B_{10}B_{10} - B_{10}B_{10}^{*} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{10}^{*}B_{10} - B_{10}^{*}B_{10}^{*}\rangle \tag{616}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} + B_{10}B_{10} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} - B_{10}^{*}B_{10}^{*}\right\rangle \tag{617}$$

$$= \frac{1}{4\mathrm{i}}\left(\exp(\phi(\tau))\,S - U^{*}\exp(-\phi(\tau))\,S + U\exp(-\phi(\tau))\,S - \exp(\phi(\tau))\,S + U^{*}S - US\right) \tag{618}$$

$$= \frac{1}{4\mathrm{i}}\left(-U^{*}\exp(-\phi(\tau))\,S + U\exp(-\phi(\tau))\,S + U^{*}S - US\right) \tag{619}$$

$$= \frac{S}{4\mathrm{i}}\left(-U^{*}\exp(-\phi(\tau)) + U\exp(-\phi(\tau)) + U^{*} - U\right) \tag{620}$$

$$= \frac{S\,(U - U^{*})}{4\mathrm{i}}\left(\exp(-\phi(\tau)) - 1\right) \tag{621}$$

$$= \frac{2\mathrm{i}\Im(U)\,S}{4\mathrm{i}}\left(\exp(-\phi(\tau)) - 1\right) \tag{622}$$

$$= \frac{\Im(U)\,S}{2}\left(\exp(-\phi(\tau)) - 1\right) \tag{623}$$

$$\left\langle \widetilde{By}(\tau)\,\widetilde{Bx}(0)\right\rangle_{B} = \left\langle \frac{B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^{*}}{2\mathrm{i}}\,\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^{*}}{2}\right\rangle_{B} \tag{624}$$

$$= \frac{1}{4\mathrm{i}}\left\langle \left(B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10} - B_{10}^{*}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^{*}\right)\right\rangle_{B} \tag{625}$$

$$= \frac{1}{4\mathrm{i}}\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)\,B_{10} - B_{0+}B_{1-}(\tau)\,B_{10}^{*} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} \tag{626}$$

$$+ B_{1+}B_{0-}(\tau)\,B_{10} + B_{1+}B_{0-}(\tau)\,B_{10}^{*} + B_{10}B_{1+}B_{0-} + B_{10}B_{0+}B_{1-} - B_{10}B_{10} - B_{10}B_{10}^{*} - B_{10}^{*}B_{1+}B_{0-} - B_{10}^{*}B_{0+}B_{1-} + B_{10}^{*}B_{10} + B_{10}^{*}B_{10}^{*}\rangle \tag{627}$$

$$= \frac{1}{4\mathrm{i}}\langle B_{0+}B_{1-}(\tau)B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)B_{0+}B_{1-} - B_{0+}B_{1-}(\tau)B_{10} - B_{0+}B_{1-}(\tau)B_{10}^{*} - B_{1+}B_{0-}(\tau)B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)B_{0+}B_{1-} + B_{1+}B_{0-}(\tau)B_{10} + B_{1+}B_{0-}(\tau)B_{10}^{*}\rangle \tag{628}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{10}^{*}B_{10} - B_{10}^{*}B_{10}^{*} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{10}B_{10} + B_{10}B_{10}^{*}\right\rangle \tag{629}$$

$$= \frac{1}{4\mathrm{i}}\left\langle B_{0+}B_{1-}(\tau)\,B_{1+}B_{0-} + B_{0+}B_{1-}(\tau)\,B_{0+}B_{1-} - B_{10}^{*}B_{10}^{*} - B_{1+}B_{0-}(\tau)\,B_{1+}B_{0-} - B_{1+}B_{0-}(\tau)\,B_{0+}B_{1-} + B_{10}B_{10}\right\rangle \tag{630}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp(-\phi(\tau))\,S - U^{*}\exp(-\phi(\tau))\,S + B_{10}^{2} - B_{10}^{*2}\right) \tag{631}$$

$$= \frac{1}{4\mathrm{i}}\left(U\exp(-\phi(\tau))\,S - U^{*}\exp(-\phi(\tau))\,S + U^{*}S - US\right) \tag{632}$$

$$= \frac{S\,(U - U^{*})}{4\mathrm{i}}\left(\exp(-\phi(\tau)) - 1\right) \tag{633}$$

$$= \frac{2\mathrm{i}\Im(U)\,S}{4\mathrm{i}}\left(\exp(-\phi(\tau)) - 1\right) \tag{634}$$

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$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} \tag{653}$$

$$\langle B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle \tag{654}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} \tag{655}$$

$$\left\langle B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger\right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}^* \tag{656}$$

$$\langle B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'}\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}^* \tag{657}$$

$$\left\langle \widetilde{B_x}(\tau)\,\widetilde{B_{iz}}(0)\right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}^*\right. \tag{658}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}^*\right) \tag{659}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}^*\right. \tag{660}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) N_{\mathbf{k}'} B_{10}^*\right) \tag{661}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'} + 1)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10}^*\right) + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e\right.\right.\right. \tag{662}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'} + 1)\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* B_{10}^*\right) + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}}\right.\right.\right. \tag{663}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'} + 1)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^* (B_{10} - B_{10}^*) + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)(B_{10} - B_{10}^*)\right) \tag{664}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} 2\mathrm{i}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{665}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} \Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*\right) \tag{666}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} \Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'}\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'} + 1\right)\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) \tag{667}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B = \left\langle \sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2}\right)\right\rangle_B \tag{668}$$

$$= \sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*}{2}\right)\right\rangle_B \tag{669}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-} - B_{10} - B_{10}^*\right)\right\rangle_B \tag{670}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1+}B_{0-} + B_{0+}B_{1-}\right)\right\rangle_B \tag{671}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+}B_{1-} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'} e^{-\mathrm{i}}\right. \tag{672}$$

$$\left\langle (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-}\right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{1+}B_{0-}\right\rangle_B \tag{673}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\right\rangle_B \tag{674}$$

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$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^{*}v_{0\mathbf{k}'}-v_{1\mathbf{k}'}v_{0\mathbf{k}'}^{*}}{\omega_{\mathbf{k}'}^{2}}\right)\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{679}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{680}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{681}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{682}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}N_{\mathbf{k}'}\right)e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{683}$$

$$=-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{684}$$

$$=-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{685}$$

$$\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}\right\rangle_{B}=-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,B_{10}^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{686}$$

$$\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}\right\rangle_{B}=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_{1+}B_{0-}\right\rangle_{B} \tag{687}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{688}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle b_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{689}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{690}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{691}$$

$$=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\,B_{10} \tag{692}$$

$$\left\langle(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}\right\rangle_{B}=(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\,B_{10}^{*} \tag{693}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\,\widetilde{B_{x}}(0)\right\rangle_{B}=\frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,B_{10}^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right. \tag{694}$$

$$\left.+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\,B_{10}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}(N_{\mathbf{k}'}+1)\,B_{10}^{*}\right) \tag{695}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(B_{10}^{*}-B_{10})+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)(B_{10}-B_{10}^{*})\right) \tag{696}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}(B_{10}^{*}-B_{10})-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)(B_{10}^{*}-B_{10})\right) \tag{697}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Im(B_{10})\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)\right) \tag{698}$$

$$\left\langle\widetilde{B_{y}}(\tau)\widetilde{B_{iz}}(0)\right\rangle_{B}=\left\langle\left(\frac{B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{10}^{*}}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\,b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{699}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)+B_{10}-B_{10}^{*}\right)\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\,b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{700}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_{0+}B_{1-}(\tau)-B_{1+}B_{0-}(\tau)\right)\left((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}\,b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{701}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\,b_{\mathbf{k}'}^{\dagger}-B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}+B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^{*}b_{\mathbf{k}'}-B_{1+}B_{0-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'} \tag{702}$$

$$\left\langle B_{0+}B_{1-}(\tau)(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}=-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}(N_{\mathbf{k}'}+1)\,B_{10}^{*} \tag{703}$$

$$\tag{704}$$

$$\tag{705}$$

$$\left\langle B_{0+}B_{1-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right\rangle_B=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^{*} \tag{706}$$

$$\left\langle B_{1+}B_{0-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_B=-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{707}$$

$$\left\langle B_{1+}B_{0-}(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right\rangle_B=\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10} \tag{708}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}^{*}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\right. \tag{709}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^{*}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{710}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}^{*}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\right. \tag{711}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}^{*}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{712}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}+B_{10}^{*}\right)+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(-B_{10}-B_{10}^{*}\right)\right) \tag{713}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}+B_{10}^{*}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(B_{10}+B_{10}^{*}\right)\right) \tag{714}$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\Re\left(B_{10}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\right) \tag{715}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Re\left(B_{10}\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(N_{\mathbf{k}'}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\right) \tag{716}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B=\left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}}{2\mathrm{i}}\right)\right\rangle_B \tag{717}$$

$$=\frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}+B_{10}-B_{10}^{*}\right)\right\rangle_B \tag{718}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{0+}B_{1-}-B_{1+}B_{0-}\right)\right\rangle_B \tag{719}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1-}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1+}B_{0-}\right\rangle \tag{720}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_{0+}B_{1-}\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_{1+}B_{0-}\right\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_{1+}B_{0-}\right\rangle\right) \tag{721}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_{0+}B_{1-}\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_{1+}B_{0-}\right\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\langle b_{\mathbf{k}'}B_{1+}B_{0} \tag{722}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_{0+}B_{1-}\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}B_{1+}B_{0-}\right\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\langle b_{\mathbf{k}'}B_{1+}B_{0} \tag{723}$$

$$\left\langle b_{\mathbf{k}'}^{\dagger}B_{1+}B_{0-}\right\rangle_B=-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}N_{\mathbf{k}'} \tag{724}$$

$$\left\langle b_{\mathbf{k}'}^{\dagger}B_{0+}B_{1-}\right\rangle_B=-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{*}N_{\mathbf{k}'} \tag{725}$$

$$\left\langle b_{\mathbf{k}'}B_{1+}B_{0-}\right\rangle_B=\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10} \tag{726}$$

$$\left\langle b_{\mathbf{k}'}B_{0+}B_{1-}\right\rangle_B=\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}^{*} \tag{727}$$

(728)
(729)
(730)
(731)
(732)
(733)
(734)

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \tfrac{1}{2\mathrm{i}} \textstyle\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(-\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*B_{10}^*N_{\mathbf{k}'}\Big)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(-\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*B_{10}N_{\mathbf{k}'}\Big)+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{10}^*\Big)-e^{-\mathrm{i}\omega} \tag{735}$$

$$= \tfrac{1}{2\mathrm{i}} \textstyle\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big(-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*B_{10}^*N_{\mathbf{k}'}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*B_{10}N_{\mathbf{k}'}\Big)+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{10}^*\Big)\Big)-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v}{} \tag{736}$$

$$= \tfrac{1}{2\mathrm{i}} \textstyle\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big((-g_{i\mathbf{k}'}+v_{i\mathbf{k}'})\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*B_{10}^*N_{\mathbf{k}'}+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})\Big(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)^*B_{10}N_{\mathbf{k}'}\Big)+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big((g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\Big)(N_{\mathbf{k}'}+1)B_{10}^*\Big)\Big)-(g_{i\mathbf{k}'}-v_{i\mathbf{k}'})^*\Big(\Big(\tag{737}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\left(B_{10}+B_{10}^*\right)N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(B_{10}+B_{10}^*)(N_{\mathbf{k}'}+1)\right) \tag{738}$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re\left(B_{10}\right)N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)(N_{\mathbf{k}'}+1)\right) \tag{739}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re\left(B_{10}\right)N_{\mathbf{k}'}\right) \tag{740}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\Re\left(B_{10}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*\Re\left(B_{10}\right)N_{\mathbf{k}'}\right) \tag{741}$$

$$=\mathrm{i}\Re\left(B_{10}\right)\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*N_{\mathbf{k}'}\right) \tag{742}$$

$$ \tag{743}$$

$$ \tag{744}$$

$$ \tag{745}$$

$$ \tag{746}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right\rangle_B=\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{747}$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{748}$$

$$\phi\left(\tau\right)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)+\cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{749}$$

$$\left\langle \widetilde{B_x}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B=\frac{\left|B_{10}\right|^2}{2}\left(\Re\left(U\right)\exp\left(-\phi\left(\tau\right)\right)+\exp\left(\phi\left(\tau\right)\right)-\Re\left(U\right)-1\right) \tag{750}$$

$$\left\langle \widetilde{B_y}\left(\tau\right)\widetilde{B_y}\left(0\right)\right\rangle_B=\frac{\left|B_{10}\right|^2}{2}\left(\exp\left(\phi\left(\tau\right)\right)-\Re\left(U\right)\exp\left(-\phi\left(\tau\right)\right)-1+\Re\left(U\right)\right) \tag{751}$$

$$\left\langle \widetilde{B_x}\left(\tau\right)\widetilde{B_y}\left(0\right)\right\rangle_B=\frac{\Im\left(U\right)\left|B_{10}\right|^2}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right) \tag{752}$$

$$\left\langle \widetilde{B}_y\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B=\frac{\Im\left(U\right)\left|B_{10}\right|^2}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right) \tag{753}$$

$$\left\langle \widetilde{B}_{iz}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B=\mathrm{i}\sum_{\mathbf{k}}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{754}$$

$$\left\langle \widetilde{B}_x\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_B=\mathrm{i}\sum_{\mathbf{k}}\Im\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{755}$$

$$\left\langle \widetilde{B}_{iz}\left(\tau\right)\widetilde{B_y}\left(0\right)\right\rangle_B=\mathrm{i}\Re\left(B_{10}\right)\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{756}$$

$$\left\langle \widetilde{B}_y\left(\tau\right)\widetilde{B_{iz}}\left(0\right)\right\rangle_B=\mathrm{i}\sum_{\mathbf{k}}\Re\left(B_{10}\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(N_{\mathbf{k}}+1\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{757}$$

With the phonon propagator given by:

$$\phi\left(\tau\right)=\int_0^{\infty}\mathrm{d}\omega\frac{J\left(\omega\right)}{\omega^2}F\left(\omega\right)^2G_+\left(\tau\right) \tag{758}$$

defined in terms of $G_{\pm}\left(\tau\right)=\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\pm n\left(\omega\right)e^{-\mathrm{i}\tau\omega}$ with $n\left(\omega\right)=\left(\mathrm{e}^{\beta\omega}-1\right)^{-1}$ the occupation number.

The eigenvalues of the Hamiltonian $\overline{H_S}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \mathrm{Tr}\left(\overline{H_S}\right)\lambda + \mathrm{Det}\left(\overline{H_S}\right) = 0 \tag{759}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_\pm = \frac{\xi\pm\eta}{2}$ and they satisfy $H_S\left|\pm\right\rangle = \lambda_\pm\left|\pm\right\rangle$. Using this notation is possible to write $H_S = \lambda_+\left|+\right\rangle\left\langle+\right| + \lambda_-\left|-\right\rangle\left\langle-\right|$.
The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition:

$$\widetilde{A_i}\left(\tau\right) = e^{\mathrm{i}\overline{H_S}\tau}A_i e^{-\mathrm{i}\overline{H_S}\tau} \tag{760}$$

$$= \sum_w e^{-\mathrm{i}w\tau}A_i\left(w\right) \tag{761}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.
In order to use the equation (761) to descompose the equation (353) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}\left(t'\right)\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}\left(t'\right)\right) \tag{762}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty}\left(-\mathrm{i}\right)^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 ... \int_0^{t_{n-1}}\mathrm{d}t_n H\left(t_1\right)H\left(t_2\right)...H\left(t_n\right) \tag{763}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E\left(t\right)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}\left(t'\right)\right) \equiv \exp\left(-\mathrm{i}tH_E\left(t\right)\right)$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E\left(t\right) = H_E^{(0)}\left(t\right) + H_E^{(1)}\left(t\right) + H_E^{(2)}\left(t\right) + ...$ so we can write:

$$U\left(t\right) = \exp\left(-\mathrm{i}t\left(H_E^{(0)}\left(t\right) + H_E^{(1)}\left(t\right) + H_E^{(2)}\left(t\right) + ...\right)\right) \tag{764}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}\left(t'\right)\right)$ and $U\left(t\right)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}\left(t\right) = \frac{1}{t}\int_0^t \overline{H_S}\left(t'\right)\mathrm{d}t' \tag{765}$$

$$H_E^{(1)}\left(t\right) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\left[\overline{H_S}\left(t'\right), \overline{H_S}\left(t''\right)\right] \tag{766}$$

$$H_E^{(2)}\left(t\right) = \frac{1}{6t}\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\int_0^{t''}\mathrm{d}t'''\left(\left[\left[\overline{H_S}\left(t'\right), \overline{H_S}\left(t''\right)\right], \overline{H_S}\left(t'''\right)\right] + \left[\left[\overline{H_S}\left(t'''\right), \overline{H_S}\left(t''\right)\right], \overline{H_S}\left(t'\right)\right]\right) \tag{767}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A_i}\left(t\right) = e^{\mathrm{i}H_E(t)t}A_i(t)e^{-\mathrm{i}H_E(t)t} \tag{768}$$

$$= \sum_{w(t)} e^{-\mathrm{i}w(t)t}A_i\left(w\left(t\right)\right) \tag{769}$$

$w\left(t\right)$ belongs to the set of differences of eigenvalues that depends of the time. As we can see the eigenvectors are time dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau,t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau,t) = U(t-\tau)U^{\dagger}(t)A_j(t)U(t)U^{\dagger}(t-\tau) \tag{770}$$
$$= e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}e^{\mathrm{i}H_E(t)t}A_j(t)e^{-\mathrm{i}H_E(t)t}e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{771}$$
$$= e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}\sum_{w(t)} e^{-\mathrm{i}w(t)t}A_j(w(t))\,e^{\mathrm{i}(t-\tau)H_E(t-\tau)} \tag{772}$$
$$= \sum_{w(t),w'(t-\tau)} e^{-\mathrm{i}w(t)t}e^{\mathrm{i}w'(t-\tau)}A'_j(w(t),w'(t-\tau)) \tag{773}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $H_S(t-\tau)$ and $H_S(t)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (761) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V} = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta\in\mathrm{V}} \langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta| \tag{774}$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{\mathrm{i}\overline{H_S}\tau} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{775}$$
$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} \tag{776}$$
$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{777}$$
$$= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-| \tag{778}$$

Calculating the transformation (761) directly using the previous relationship we find that:

$$\widetilde{A_i}(\tau) = \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in\mathrm{V}}\langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{779}$$
$$= \langle+|A_i|+\rangle|+\rangle\langle+| + e^{\mathrm{i}\eta\tau}\langle+|A_i|-\rangle|+\rangle\langle-| + e^{-\mathrm{i}\eta\tau}\langle-|A_i|+\rangle|-\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-| \tag{780}$$

Here $\eta = \lambda_+ - \lambda-$. Comparing the RHS of the equations (761) and the explicit expression for $\widetilde{A_i}(\tau)$ and we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$A_i(0) = \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-| \tag{781}$$
$$A_i(w) = \langle+|A_i|-\rangle|+\rangle\langle-| \tag{782}$$
$$A_i(-w) = \langle-|A_i|+\rangle|-\rangle\langle+| \tag{783}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau) = \widetilde{A_i}^{\dagger}(\tau)$ and $\widetilde{B_i}(\tau) = \widetilde{B_i}^{\dagger}(\tau)$ we can use the equation (761) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho}_S}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\overline{S}}(t), \overline{\rho}_S(t)\right] - \frac{1}{2}\sum_{ij}\sum_{w,w'}\gamma_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho}_S(t) - \overline{\rho}_S(t)A_j^{\dagger}(w,w')\right] - \mathrm{i}\sum_{ij}\sum_{w}S_{ij}(w,w',t)\left[A_i, A_j(w,w')\overline{\rho}_S(t) + \overline{\rho}_S(t)A_j^{\dagger}(w,w')\right] \tag{784}$$

where $A_j^{\dagger}(w) = A(-w)$ as expected from the equations (782) and (783). As we can see the equation shown contains the rates and energy shifts $\gamma_{ij}(w,w',t) = 2\Re(K_{ij}(w,w',t))$ and $S_{ij}(w,w',t) = \Im(K_{ij}(w,w',t))$, respectively, defined in terms of the response functions

$$K_{ij}\left(w,w',t\right)=\int_0^t C_i\left(t\right)C_j\left(t-\tau\right)\Lambda_{ij}\left(\tau\right)e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{785}$$
$$=K_{ijww'}\left(t\right) \tag{786}$$

If we extend the upper limit of integration to ∞ in the equation (785) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{\mathrm{d}\overline{\rho}_{\overline{S}}}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H}_{\overline{S}}\left(t\right),\overline{\rho}_{\overline{S}}\left(t\right)\right]-\sum_{ijww'}K_{ijww'}\left(t\right)\left[A_i,A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right)-\overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}\right] \tag{787}$$

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_{\overline{S}}}{\mathrm{d}t}e^{V}=\frac{\mathrm{d}\left(e^{-V}\overline{\rho}_{\overline{S}}e^{V}\right)}{\mathrm{d}t} \tag{788}$$
$$=\frac{\mathrm{d}\rho_{\overline{S}}}{\mathrm{d}t} \tag{789}$$
$$=-\mathrm{i}e^{-V}\left[\overline{H}_{\overline{S}}\left(t\right),\overline{\rho}_{\overline{S}}\left(t\right)\right]e^{V}-\sum_{ijww'}K_{ijww'}\left(t\right)e^{-V}\left[A_i,A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right)-\overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}\right]e^{V} \tag{790}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V}=e^{-V}\overline{A\mathbb{I}B}e^{V} \tag{791}$$
$$=e^{-V}\overline{A}e^{V}e^{-V}\overline{B}e^{V} \tag{792}$$
$$=\left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\overline{B}e^{V}\right) \tag{793}$$
$$=AB \tag{794}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V}=e^{-V}\overline{(AB-BA)}e^{V} \tag{795}$$
$$=e^{-V}\overline{AB}e^{V}-e^{-V}\overline{BA}e^{V} \tag{796}$$
$$=AB-BA \tag{797}$$
$$=[A,B] \tag{798}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_{\overline{S}}}{\mathrm{d}t}=-\mathrm{i}e^{-V}\left[\overline{H}_{\overline{S}}\left(t\right),\overline{\rho}_{\overline{S}}\left(t\right)\right]e^{V}-\sum_{ijww'}K_{ijww'}\left(t\right)e^{-V}\left[A_i,A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right)-\overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}\right]e^{V} \tag{799}$$
$$=-\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right]-\sum_{ijww'}K_{ijww'}\left(t\right)\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}\overline{\rho}_{\overline{S}}\left(t\right)e^{V}-e^{-V}\overline{\rho}_{\overline{S}}\left(t\right)A_{jww'}^{\dagger}e^{V}\right] \tag{800}$$
$$=-\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right]-\sum_{ijww'}K_{ijww'}\left(t\right)\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}e^{V}e^{-V}\overline{\rho}_{\overline{S}}\left(t\right)e^{V}-e^{-V}\overline{\rho}_{\overline{S}}\left(t\right)e^{V}e^{-V}A_{jww'}^{\dagger}e^{V}\right] \tag{801}$$
$$=-\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right]-\sum_{ijww'}K_{ijww'}\left(t\right)\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}e^{V}\rho_{\overline{S}}\left(t\right)-\rho_{\overline{S}}\left(t\right)e^{-V}A_{jww'}^{\dagger}e^{V}\right] \tag{802}$$
$$=-\mathrm{i}\left[H_{\overline{S}}\left(t\right),\rho_{\overline{S}}\left(t\right)\right]-\left(\sum_{ijww'}K_{ijww'}\left(t\right)\left(\left[e^{-V}A_ie^{V},e^{-V}A_{jww'}e^{V}\rho_{\overline{S}}\left(t\right)\right]-\left[e^{-V}A_ie^{V},\rho_{\overline{S}}\left(t\right)e^{-V}A_{jww'}^{\dagger}e^{V}\right]\right)\right) \tag{803}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (784) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (784) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1, 2, 3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{j} g_k \left(b_k^\dagger + b_k\right)\right) |1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k \tag{804}$$

After performing the transformation (24) on the Hamiltonian (804) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)\,|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x \tag{805}$$

$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{806}$$

$$H_B = \sum_k \omega_k b_k^\dagger b_k \tag{807}$$

The Hamiltonian (805) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{808}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{809}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z \tag{810}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2}$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_S}$.

$$\lambda_+ = \frac{\epsilon + \eta}{2} \tag{811}$$

$$\lambda_- = \frac{\epsilon - \eta}{2} \tag{812}$$

$$|+\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}\begin{pmatrix}\epsilon + \eta \\ \Omega_r\end{pmatrix} \tag{813}$$

$$|-\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}\begin{pmatrix}-\Omega_r \\ \epsilon + \eta\end{pmatrix} \tag{814}$$

Using this basis we can find the decomposition matrices using the equations (782)-(783) and the fact that $|+\rangle = \cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle$ and $|-\rangle = -\sin(\theta)\,|1\rangle + \cos(\theta)\,|0\rangle$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\begin{aligned}
\langle +|\,\sigma_x\,|+\rangle &= \begin{pmatrix}\cos(\theta) & \sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} && (815)\\
&= 2\sin(\theta)\cos(\theta) && (816)\\
&= \sin(2\theta) && (817)\\
\langle -|\,\sigma_x\,|-\rangle &= \begin{pmatrix}-\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (818)\\
&= -2\sin(\theta)\cos(\theta) && (819)\\
&= -\sin(2\theta) && (820)\\
\langle -|\,\sigma_x\,|+\rangle &= \begin{pmatrix}-\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} && (821)\\
&= \cos^2(\theta) - \sin^2(\theta) && (822)\\
&= \cos(2\theta) && (823)\\
\langle +|\,\sigma_y\,|+\rangle &= \begin{pmatrix}\cos(\theta) & \sin(\theta)\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} && (824)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (825)\\
&= 0 && (826)\\
\langle -|\,\sigma_y\,|-\rangle &= \begin{pmatrix}-\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (827)\\
&= -\mathrm{i}\sin(\theta)\cos(\theta) + \mathrm{i}\sin(\theta)\cos(\theta) && (828)\\
&= 0 && (829)\\
\langle -|\,\sigma_y\,|+\rangle &= \begin{pmatrix}-\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} && (830)\\
&= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) && (831)\\
&= \mathrm{i} && (832)\\
\langle +|\,\frac{1+\sigma_z}{2}\,|+\rangle &= \begin{pmatrix}\cos(\theta) & \sin(\theta)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} && (833)\\
&= \cos(\theta)\cos(\theta) && (834)\\
&= \cos^2(\theta) && (835)\\
\langle -|\,\frac{1+\sigma_z}{2}\,|-\rangle &= \begin{pmatrix}-\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (836)\\
&= \sin(\theta)\sin(\theta) && (837)\\
&= \sin^2(\theta) && (838)\\
\langle -|\,\frac{1+\sigma_z}{2}\,|+\rangle &= \begin{pmatrix}-\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} && (839)\\
&= -\sin(\theta)\cos(\theta) && (840)\\
&= -\sin(\theta)\cos(\theta) && (841)
\end{aligned}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$\begin{aligned}
A_1(0) &= \sin(2\theta)\left(|+\rangle\langle +| - |-\rangle\langle -|\right) && (842)\\
A_1(\eta) &= \cos(2\theta)\,|-\rangle\langle +| && (843)\\
A_2(0) &= 0 && (844)\\
A_2(\eta) &= \mathrm{i}\,|-\rangle\langle +| && (845)\\
A_3(0) &= \cos^2(\theta)\,|+\rangle\langle +| + \sin^2(\theta)\,|-\rangle\langle -| && (846)\\
A_3(\eta) &= -\sin(\theta)\cos(\theta)\,|-\rangle\langle +| && (847)
\end{aligned}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (419) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\, C_j(t-\tau)\, \Lambda_{ij}(\tau) \tag{848}$$

Using the notation of the master equation (784), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(**??**)-(**??**) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (419) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\,\widetilde{B_1}(0)\,\rho_B\right) \tag{849}$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{850}$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\,\widetilde{B_2}(0)\,\rho_B\right) \tag{851}$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) \tag{852}$$

$$\Lambda'_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)\,(1 - F(\omega))^2\, G_+(\tau) \tag{853}$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)\,(1 - F(\omega))\, \mathrm{i}G_-(\tau) \tag{854}$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau) \tag{855}$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 \tag{856}$$

Finally taking the Hamiltonian (804) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (334) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{857}$$

$$= \frac{g_k\left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)} \tag{858}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (417) under a time-independent approach providing similar results.
Given that the Hamiltonian of this system is time-independent, then $U(t)\,U^\dagger(t-\tau) = U(\tau)$. From the equation (784) and using the fact that

$$\widetilde{A_j}(t-\tau, t) = U(\tau)\, A_j U(-\tau) \tag{859}$$

$$= \sum_w e^{iw\tau} A_i(-w) \tag{860}$$

$$= \sum_w e^{-iw\tau} A_i(w) \tag{861}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_{w}\gamma_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)A_j^{\dagger}(w)\right] \tag{862}$$

$$-\sum_{ij}\sum_{w}S_{ij}(w,t)\left[A_i, A_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)A_j^{\dagger}(w)\right] \tag{863}$$

where $A_j^{\dagger}(w) = A(-w)$, as we can see the equation (863) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2\Re(K_{ij}(w,t))$ and $S_{ij}(w,t) = \Im(K_{ij}(w,t))$, respectively, defined in terms of the response functions

$$K_{ij}(w,t) = \int_0^t \Lambda'_{ij}(\tau)e^{\mathrm{i}w\tau}\mathrm{d}\tau \tag{864}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (109) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_{k}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{865}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (865) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (223) and (**??**) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t) \tag{866}$$

$$\overline{H_{\mathrm{I}}} = \frac{\Omega(t)}{2}\left(B_x\sigma_x + B_y\sigma_y\right) \tag{867}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (**??**)-(**??**) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^{\infty}\frac{J(\omega)}{\omega^2}G_+(\tau)\,\mathrm{d}\omega \tag{868}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (868) can be written as:

$$\phi(\tau) = \int_0^{\infty}\frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega \tag{869}$$

Writing the interaction Hamiltonian (867) in the similar way to the equation (**??**) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (223) with $\delta'|1\rangle\langle 1|=\frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t) \tag{870}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{871}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{872}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{873}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right) \tag{874}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(417) is:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau \left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{875}$$

$$\left.+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right) \tag{876}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (871) and (874) on the equation (876) we obtain that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{877}$$

$$\left.+\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right) \tag{878}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (419) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right) \tag{879}$$

Using the equation (784), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)\right), \rho_S(t)\right] - \frac{1}{2}\sum_w \gamma_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) - \rho_S(t)A_3^\dagger(w)\right] \tag{880}$$

$$\left.-\sum_w S_{33}(w,t)\left[A_3, A_3(w)\rho_S(t) + \rho_S(t)A_3^\dagger(w)\right]\right) \tag{881}$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(\tau) \tag{882}$$

$$\Lambda_{33}(-\tau) = \int_0^\infty \mathrm{d}\omega J(\omega) G_+(-\tau) \tag{883}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (881) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \rho_S(t)\right)] - \sum_w \left(K_{33}(w,t)\left[A_3, A_3(w)\rho_S(t)\right] + K_{33}^*(w,t)\left[\rho_S(t) A_3(w), A_3\right]\right) \tag{884}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$, this Hamiltonian have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2} \tag{885}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right) \tag{886}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2} \tag{887}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(-|1\rangle + |0\rangle\right) \tag{888}$$

The elements of the decomposition matrices are:

$$\langle +| \frac{1+\sigma_z}{2} |+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{889}$$

$$= \frac{1}{2} \tag{890}$$

$$\langle -| \frac{1+\sigma_z}{2} |-\rangle = \frac{1}{2}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} \tag{891}$$

$$= \frac{1}{2} \tag{892}$$

$$\langle -| \frac{1+\sigma_z}{2} |+\rangle = \frac{1}{2}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{893}$$

$$= -\frac{1}{2} \tag{894}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \tag{895}$$

$$= \frac{\mathbb{I}}{2} \tag{896}$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle +| \tag{897}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right) \tag{898}$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle -| \tag{899}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right) \tag{900}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{901}$$

$$- K_{33}^{*}\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^{*}\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] \tag{902}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)G_{+}\left(\tau\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{903}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{904}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)e^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{905}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)\left(n\left(\omega\right)+1\right)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{906}$$

$$= \int_0^{\infty} J\left(\omega\right)\left(n\left(\omega\right)+1\right)\pi\delta\left(\widetilde{\Omega}-\omega\right)\mathrm{d}\omega \tag{907}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right) \tag{908}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)G_{+}\left(\tau\right)e^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{909}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)e^{-\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{910}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)e^{\text{-}\mathrm{i}\widetilde{\Omega}\tau}n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{911}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right)n\left(\omega\right)\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau+\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{912}$$

$$= \int_0^{\infty} J\left(\omega\right)n\left(\omega\right)\pi\delta\left(-\widetilde{\Omega}+\omega\right)\mathrm{d}\omega \tag{913}$$

$$= \pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right) \tag{914}$$

Here we have used $\int_0^{\infty}\mathrm{d}s\, e^{\pm i\varepsilon s} = \pi\delta\left(\varepsilon\right) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3\left(0\right)$ because the spectral density $J\left(\omega\right)$ is equal to zero when $\omega = 0$. Replacing in the equation (901) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z, \left(\sigma_z+\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z-\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{915}$$

$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z+\mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S\left(t\right)\left(\sigma_z-\mathrm{i}\sigma_y\right), \sigma_z\right]\right) \tag{916}$$

This is the same result than the equation ($S17$), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega\left(\mathrm{t}\right)}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\sigma_z, \left(\sigma_z+\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] + n\left(\Omega\left(t\right)\right)\left[\sigma_z, \left(\sigma_z-\mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right]\right) \tag{917}$$

$$- \frac{\pi}{8}J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right)+1\right)\left[\rho_S\left(t\right)\left(\sigma_z+\mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z-\mathrm{i}\sigma_y\right), \sigma_z\right]\right) \tag{918}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{919}$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t)\, |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\, |n\rangle\langle m|, \tag{920}$$

$$H_I = \left(\sum_{n=0} \mu_n(t)\, |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{921}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{922}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t)\, |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{923}$$

At first let's obtain e^V under the transformation (923), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$\begin{aligned}
e^V &= e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} && (924)\\
&= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + \ldots && (925)\\
&= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + \ldots && (926)\\
&= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n|\left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \ldots\right) && (927)\\
&= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} && (928)\\
&= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ && (929)
\end{aligned}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$\begin{aligned}
e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} &= \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} && (930)\\
&= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (931)\\
&= B_{\pm} && (932)
\end{aligned}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{933}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (919):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \tag{934}$$

$$= |0\rangle\langle 0|, \tag{935}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \tag{936}$$

$$= |m\rangle\langle m| B_+ |m\rangle\langle n| n\rangle\langle n| B_-, \tag{937}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{938}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \tag{939}$$

$$= |0\rangle\langle m| B_-\ m \neq 0, \tag{940}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{941}$$

$$= |0\rangle\langle m| B_+\ m \neq 0, \tag{942}$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \tag{943}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B_+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_- \tag{944}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B_+ b_{\mathbf{k}}^\dagger B_- \right) \left(B_+ b_{\mathbf{k}} B_- \right) \tag{945}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{946}$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \tag{947}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \right) \tag{948}$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \right) \tag{949}$$

The transformed Hamiltonians of the equations (920) to (922) written in terms of (934) to (949) are:

$$\overline{H_{\overline{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{950}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{951}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \overline{\sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n|} \tag{952}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\overline{|0\rangle\langle n|} + V_{n0}(t)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0} V_{mn}(t)\overline{|m\rangle\langle n|} \tag{953}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)B_-|0\rangle\langle n| + V_{n0}(t)B_+|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{954}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{955}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_+\right)\left(\left(\sum_{n=0}\mu_n(t)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{956}$$

$$= \left(\mu_0(t)|0\rangle\langle 0| + \sum_{n=1}\mu_n(t)|n\rangle\langle n|B_+\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B_-\right) \tag{957}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}B_+\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)B_- \tag{958}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{959}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{960}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{961}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{962}$$

$$+\sum_{n=0}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} 2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{963}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{964}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{965}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{966}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{967}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\right) \tag{968}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{969}$$

Using the previous functions we have that (966) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}(V_{0n}(t)|0\rangle\langle n|B_- + V_{n0}(t)|n\rangle\langle 0|B_+) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{970}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}R_n|n\rangle\langle n| + \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{971}$$

Now in order to separate the elements of the hamiltonian (971) let's follow the references of the equations (**??**) and (223) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0}\varepsilon_n(t)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| + \sum_{n=1}R_n|n\rangle\langle n| \tag{972}$$

$$\overline{H_I} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|(B_- - B) + V_{n0}(t)|n\rangle\langle 0|(B_+ - B)\right), \tag{973}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{974}$$

Here B is given by (**??**) The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{975}$$

$$\sigma_{nm,y} = \mathrm{i}(|n\rangle\langle m| - |m\rangle\langle n|) \tag{976}$$

$$B_x = \frac{B_+ + B_- - 2B}{2} \tag{977}$$

$$B_y = \frac{B_- - B_+}{2\mathrm{i}} \tag{978}$$

Using this set of hermitian operators to write the Hamiltonians (920)-(922)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0}V_{mn}(t)|m\rangle\langle n| \quad (979)$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \quad (980)$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \quad (981)$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \quad (982)$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \quad (983)$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \quad (984)$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \quad (985)$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \quad (986)$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \quad (987)$$

$$\overline{H_I(t)} = \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(V_{0n}(t)|0\rangle\langle n|(B_- - B) + V_{n0}(t)|n\rangle\langle 0|(B_+ - B)) \quad (988)$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))(B_- - B)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))(B_+ - B)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \quad (989)$$

$$+ \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \quad (990)$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}((B_- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + (B_+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))))\right. \quad (991)$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}((B_+ - B)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - (B_- - B)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))))\right) \quad (992)$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \quad (993)$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}(B_+ + B_- - 2B)\Re(V_{0n}(t)) + \mathrm{i}(B_- - B - B_+ + B)\Im(V_{0n}(t))\right. \quad (994)$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}((B_+ - B - B_- + B)\Re(V_{0n}(t)) + \mathrm{i}(B - B_- + B - B_+)\Im(V_{0n}(t)))\right) + \sum_{n=1}B_{z,n}|n\rangle\langle n| \quad (995)$$

$$= \sum_{n=1}B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \quad (996)$$

$$+\sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \quad (997)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right)+\left\langle\overline{H_I}\right\rangle_{\overline{H_S+H_B}}+O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{998}$$

Taking the equations (242)-(250) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)=C\left(R_1,R_2,...,R_{d-1},B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (250) and using the chain rule we obtain that:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}}+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, && (999)\\ &=0 && (1000)\end{aligned}$$

Let's recall the equations (967) and (969), we can write them in terms of the variational parameters

$$\begin{aligned}B &= \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) && (1001)\\ R_n &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) && (1002)\end{aligned}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\begin{aligned}\frac{\partial B}{\partial v_{\mathbf{k}}} &= -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B && (1003)\\ \frac{\partial R_n}{\partial v_{\mathbf{k}}} &= \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) && (1004)\end{aligned}$$

Introducing this derivates in the equation (999) give us:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right)+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) && (1005)\\ &= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right)-\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) && (1006)\end{aligned}$$

We can obtain the variational parameters:

$$\begin{aligned}v_{\mathbf{k}} &= \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} && (1007)\\ &= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} && (1008)\end{aligned}$$

Now taking $v_{\mathbf{k}}=g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1009}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_{B}$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho\left(0\right)e^{-V}=\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B_{+}\right)\left(|0\rangle\langle 0|\otimes\rho_{B}\right)\left(|0\rangle\langle 0|+\sum_{n=1}|n\rangle\langle n|B_{-}\right) \tag{1010}$$

$$0=|0\rangle\langle 0|\otimes\rho_{B} \tag{1011}$$

$$0=\rho\left(0\right) \tag{1012}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O}\equiv U^{\dagger}(t)OU(t) \tag{1013}$$

$$U(t)\equiv\mathcal{T}\mathrm{exp}\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_{S}}\left(t'\right)\right). \tag{1014}$$

Therefore:

$$\widetilde{\overline{\rho_{S}}}(t)=U^{\dagger}(t)\overline{\rho_{S}}(t)U(t),\text{ where} \tag{1015}$$

$$\overline{\rho_{S}}(t)=\mathrm{Tr}_{B}\left(\bar{\rho}(t)\right) \tag{1016}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_{I}\left(t\right)}=B_{z,0}|0\rangle\langle 0|+\sum_{n=1}\left(\Re\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,x}+\Re\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,y}+B_{z,n}|n\rangle\langle n|\right. \tag{1017}$$

$$\left.+\Im\left(V_{0n}\left(t\right)\right)B_{x}\sigma_{0n,y}-\Im\left(V_{0n}\left(t\right)\right)B_{y}\sigma_{0n,x}\right) \tag{1018}$$

where

$$B_{z,0}=\sum_{\mathbf{k}}g_{\mathbf{k}}\mu_{0}\left(t\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) \tag{1019}$$

$$B_{z,n}=\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_{n}\left(t\right)-\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\text{ if}n\neq 0 \tag{1020}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1021}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1022}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1023}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1024}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1025}$$
$$B_{1n}(t) = B_x \tag{1026}$$
$$B_{2n}(t) = B_y \tag{1027}$$
$$B_{3n}(t) = B_{z,n} \tag{1028}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1029}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1030}$$
$$C_{10}(t) = 0 \tag{1031}$$
$$C_{20}(t) = 0 \tag{1032}$$
$$C_{40}(t) = 0 \tag{1033}$$
$$C_{50}(t) = 0 \tag{1034}$$
$$C_{30}(t) = 1 \tag{1035}$$
$$C_{1n}(t) = \Re(V_{0n}(t)) \tag{1036}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1037}$$
$$C_{3n}(t) = 1 \tag{1038}$$
$$C_{4n}(t) = \Im(V_{0n}(t)) \tag{1039}$$
$$C_{5n}(t) = -\Im(V_{0n}(t)) \tag{1040}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn} \otimes B_{jn}(t)\right) \tag{1041}$$

Here $J = \{1,2,3,4,5\}$.

We write the interaction Hamiltonian transformed under (1013) as:

$$\widetilde{H_I}(t) = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1042}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\, A_i U(t) \tag{1043}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{1044}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\rho_S}(t)\rho_B\right]\right] \mathrm{d}s \tag{1045}$$

Replacing the equation (1042)in (1045)we can obtain:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right), \left[\widetilde{H_I}\left(s\right), \widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1046}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}\left(t\right)\left(\widetilde{A_{jn}}\left(t\right)\otimes\widetilde{B_{jn}}\left(t\right)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}\left(s\right)\left(\widetilde{A_{j'n'}}\left(s\right)\otimes\widetilde{B_{j'n'}}\left(s\right)\right), \widetilde{\rho_S}(t)\rho_B\right]\right]\mathrm{d}s \tag{1047}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}\left(t\right)\left(\widetilde{A_{jn}}\left(t\right)\otimes\widetilde{B_{jn}}\left(t\right)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}\left(s\right)\left(\widetilde{A_{j'n'}}\left(s\right)\otimes\widetilde{B_{j'n'}}\left(s\right)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1048}$$

$$\left. -\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}\left(s\right)\left(\widetilde{A_{j'n'}}\left(s\right)\otimes\widetilde{B_{j'n'}}\left(s\right)\right)\right]\mathrm{d}s \tag{1049}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}\left(t\right)\left(\widetilde{A_{jn}}\left(t\right)\otimes\widetilde{B_{jn}}\left(t\right)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}\left(s\right)\left(\widetilde{A_{j'n'}}\left(s\right)\otimes\widetilde{B_{j'n'}}\left(s\right)\right)\widetilde{\rho_S}(t)\rho_B\right. \tag{1050}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}\left(t\right)\left(\widetilde{A_{jn}}\left(t\right)\otimes\widetilde{B_{jn}}\left(t\right)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}\left(s\right)\left(\widetilde{A_{j'n'}}\left(s\right)\otimes\widetilde{B_{j'n'}}\left(s\right)\right) \tag{1051}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}\left(s\right)\left(\widetilde{A_{j'n'}}\left(s\right)\otimes\widetilde{B_{j'n'}}\left(s\right)\right)\widetilde{\rho_S}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}\left(t\right)\left(\widetilde{A_{jn}}\left(t\right)\otimes\widetilde{B_{jn}}\left(t\right)\right) \tag{1052}$$

$$\left. +\widetilde{\rho_S}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}\left(s\right)\left(\widetilde{A_{j'n'}}\left(s\right)\otimes\widetilde{B_{j'n'}}\left(s\right)\right)\sum_{j\in J}\sum_{n=1} C_{jn}\left(t\right)\left(\widetilde{A_{jn}}\left(t\right)\otimes\widetilde{B_{jn}}\left(t\right)\right)\right)\mathrm{d}s \tag{1053}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}\left(\tau\right) = \left\langle\widetilde{B_{jn}}\left(t\right)\left(t\right)\widetilde{B_{j'n'}}\left(t\right)\left(s\right)\right\rangle_B \tag{1054}$$

$$= \left\langle\widetilde{B_{jn}}\left(\tau\right)\widetilde{B_{j'n'}}\left(0\right)\right\rangle_B \tag{1055}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}\left(t\right)$ and elements related to the system given by $\widetilde{A_{jn}}\left(t\right)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(t\right)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}\left(t\right)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}\left(t\right)\right)$. The correlation functions relevant of the master equation (1053) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\right\rangle_B \tag{1056}$$

$$= \left\langle\widetilde{B_{jn}}\left(0\right)\widetilde{B_{j'n'}}\left(0\right)\right\rangle_B \tag{1057}$$

$$= \Lambda_{jnj'n'}\left(\tau\right) \tag{1058}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}\left(t\right)\rho_B\widetilde{B_{j'n'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\rho_B\right) \tag{1059}$$

$$= \left\langle\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\right\rangle_B \tag{1060}$$

$$= \left\langle\widetilde{B_{j'n'}}\left(-\tau\right)\widetilde{B_{jn}}\left(0\right)\right\rangle_B \tag{1061}$$

$$= \Lambda_{j'n'jn}\left(-\tau\right) \tag{1062}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}\left(s\right)\rho_B\widetilde{B_{jn}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\rho_B\right) \tag{1063}$$

$$= \left\langle\widetilde{B_{jn}}\left(t\right)\widetilde{B_{j'n'}}\left(s\right)\right\rangle_B \tag{1064}$$

$$= \left\langle\widetilde{B_{jn}}\left(\tau\right)\widetilde{B_{j'n'}}\left(0\right)\right\rangle_B \tag{1065}$$

$$= \Lambda_{jnj'n'}\left(\tau\right) \tag{1066}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\rho_B\right) \tag{1067}$$

$$= \left\langle\widetilde{B_{j'n'}}\left(s\right)\widetilde{B_{jn}}\left(t\right)\right\rangle_B \tag{1068}$$

$$= \left\langle\widetilde{B_{j'n'}}\left(-\tau\right)\widetilde{B_{jn}}\left(0\right)\right\rangle_B \tag{1069}$$

$$= \Lambda_{j'n'jn}\left(-\tau\right) \tag{1070}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1046)and (1053) and using the equations (1056)-(1070) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}\left(t\right)C_{j'n'}\left(s\right)\left(\Lambda_{jnj'n'}\left(\tau\right)\widetilde{A_{jn}}\left(t\right)\widetilde{A_{j'n'}}\left(s\right)\widetilde{\overline{\rho_S}}(t) - \Lambda_{j'n'jn}\left(-\tau\right)\widetilde{A_{jn}}\left(t\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'n'}}\left(s\right)\right)\right. \tag{1071}$$

$$\left.+C_{jn}\left(t\right)C_{j'n'}\left(s\right)\left(\Lambda_{j'n'jn}\left(-\tau\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'n'}}\left(s\right)\widetilde{A_{jn}}\left(t\right) - \Lambda_{jnj'n'}\left(\tau\right)\widetilde{A_{j'n'}}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{jn}}\left(t\right)\right)\right)\mathrm{d}s \tag{1072}$$

$$= -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}\left(t\right)C_{j'n'}\left(s\right)\left(\Lambda_{jnj'n'}\left(\tau\right)\left[\widetilde{A_{jn}}\left(t\right),\widetilde{A_{j'n'}}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right] + \Lambda_{j'n'jn}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'n'}}\left(s\right),\widetilde{A_{jn}}\left(t\right)\right]\right)\right) \tag{1073}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'}\left(C_{jn}\left(t\right)C_{j'n'}\left(t-\tau\right)\left(\Lambda_{jnj'n'}\left(\tau\right)\left[A_{jn}\left(t\right),A_{j'n'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}\left(-\tau\right)\left[\overline{\rho_S}(t)A_{j'n'}\left(t-\tau,t\right),A_{jn}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}(t)\right] \tag{1074}$$

For this case we used that $A_{jn}\left(t-\tau,t\right) = U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jn}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0\left(t\right)=0$, $\mu_1\left(t\right)=1$ then we can reproduce a similar expression to (417) as expected.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1075}$$
$$H_S(t) = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{1076}$$
$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right), \tag{1077}$$
$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}}. \tag{1078}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right) \tag{1079}$$

At first let's obtain $e^{\pm V}$ under the transformation (1079), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)$, so the equation (1079) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n|\hat{\varphi}_n} \tag{1080}$$
$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!} + \ldots \tag{1081}$$
$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + \ldots \tag{1082}$$
$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + \ldots \tag{1083}$$
$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \ldots\right) \tag{1084}$$
$$= \sum_n |n\rangle\langle n| e^{\pm\hat{\varphi}_n} \tag{1085}$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*} b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right) = e^{\pm\left(\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)} \tag{1086}$$
$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)}\right) \tag{1087}$$
$$= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) \tag{1088}$$
$$= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1089}$$
$$= \prod_u B_{nu\pm} \tag{1090}$$
$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1091}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1092}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1075):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1093}$$

$$= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, \tag{1094}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \tag{1095}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \tag{1096}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \tag{1097}$$

$$= |0\rangle\langle 0|. \tag{1098}$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1099}$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, \tag{1100}$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \tag{1101}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, \tag{1102}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1103}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), \tag{1104}$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1105}$$

$$\prod_u \left(B_{mu+} B_{nu-}\right) = \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). \tag{1106}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), \tag{1107}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + ...\right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right) \left(|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + ...\right), \tag{1108}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + ..., \tag{1109}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + ...\right) \prod_u B_{0u-} \tag{1110}$$

$$+ |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + ...\right) \prod_u B_{1u-} + ... \tag{1111}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + ...\right) \tag{1112}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + ...\right) + ... \tag{1113}$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + ...\right) \tag{1114}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + ...\right) + ... \tag{1115}$$

The transformed Hamiltonians of the equations (1076) to (1078) written in terms of (1093) to (1118) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1121}$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1122}$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) \tag{1123}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1124}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + g_{0u\mathbf{k}}^{*} b_{u\mathbf{k}}\right) + \ldots\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1125}$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + g_{0u\mathbf{k}}^{*} b_{u\mathbf{k}}\right) \prod_u B_{0u-} \tag{1126}$$

$$+ \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + g_{1u\mathbf{k}}^{*} b_{u\mathbf{k}}\right) \prod_u B_{1u-} + \ldots \tag{1127}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \prod_u B_{0u+} b_{u\mathbf{k}}^{\dagger} \prod_u B_{0u-} + g_{0u\mathbf{k}}^{*} \prod_u B_{0u+} b_{u\mathbf{k}} \prod_u B_{0u-}\right) \tag{1128}$$

$$+ \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \prod_u B_{1u+} b_{u\mathbf{k}}^{\dagger} \prod_u B_{1u-} + g_{1u\mathbf{k}}^{*} \prod_u B_{1u+} b_{u\mathbf{k}} \prod_u B_{1u-}\right) + \ldots \tag{1129}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b_{u\mathbf{k}}^{\dagger} - \frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right) + g_{0u\mathbf{k}}^{*} \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1130}$$

$$+ \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b_{u\mathbf{k}}^{\dagger} - \frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right) + g_{1u\mathbf{k}}^{*} \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots \tag{1131}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} \left(b_{u\mathbf{k}}^{\dagger} - \frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right) + g_{nu\mathbf{k}}^{*} \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1132}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1133}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + v_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)\right) \tag{1134}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}\right. \tag{1135}$$

$$\left. - \left(v_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + v_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)\right) + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*} b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1136}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^{*} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1137}$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}}) b_{u\mathbf{k}}^{\dagger} + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^{*} b_{u\mathbf{k}}\right) \tag{1138}$$

Using the previous functions we have that (1135) can be re-written in the following way:

$$\overline{H} = \sum_{n} \varepsilon_n (t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm} (t) |n\rangle\langle m| \prod_{u} (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}} \tag{1139}$$

$$+ \sum_{n} R_n (t) |n\rangle\langle n| + \sum_{n} B_{z,n} (t) |n\rangle\langle n| \tag{1140}$$

Now in order to separate the elements of the hamiltonian (1140) let's follow the references of the equations (223) and (**??**) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle \prod_{u} (B_{mu+} B_{nu-}) \right\rangle_{\overline{H_0}} = \left\langle \prod_{u\mathbf{k}} \left(D (\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp \left(\frac{1}{2} (-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}}) \right) \right) \right\rangle_{\overline{H_0}} \tag{1141}$$

$$= \left(\prod_{u\mathbf{k}} \exp \left(\frac{1}{2} (-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}}) \right) \right) \left\langle \prod_{u\mathbf{k}} D (\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \right\rangle_{\overline{H_0}} \tag{1142}$$

$$= \left(\prod_{u\mathbf{k}} \exp \left(\frac{(v_{mu\mathbf{k}}^* v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v_{nu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_{u} \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1143}$$

$$\equiv B_{nm} \tag{1144}$$

$$\left\langle \prod_{u} (B_{nu+} B_{mu-}) \right\rangle_{\overline{H_0}} = \left(\prod_{u\mathbf{k}} \exp \left(\frac{(v_{nu\mathbf{k}}^* v_{mu\mathbf{k}} - v_{nu\mathbf{k}} v_{mu\mathbf{k}}^*)}{2\omega_{u\mathbf{k}}^2} \right) \right) \prod_{u} \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1145}$$

$$= B_{nm}^* \tag{1146}$$

Following the reference [4] we define:

$$J_{nm} = \prod_{u} (B_{mu+} B_{nu-}) - B_{nm} \tag{1147}$$

As we can see:

$$J_{nm}^{\dagger} = \left(\prod_{u} (B_{mu+} B_{nu-}) - B_{nm} \right)^{\dagger} \tag{1148}$$

$$= \prod_{u} (B_{nu+} B_{mu-}) - B_{nm}^* \tag{1149}$$

$$= \prod_{u} (B_{nu+} B_{mu-}) - B_{mn} \tag{1150}$$

$$= J_{mn} \tag{1151}$$

We can separate the Hamiltonian (1140) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\overline{S}}(t)} = \sum_{n} (\varepsilon_n (t) + R_n) |n\rangle\langle n| + \sum_{n\neq m} V_{nm} (t) |n\rangle\langle m| B_{nm} \tag{1152}$$

$$\overline{H_{\overline{I}}} = \sum_{n\neq m} V_{nm} (t) |n\rangle\langle m| J_{nm} + \sum_{n} B_{z,n} (t) |n\rangle\langle n|, \tag{1153}$$

$$\overline{H_{\overline{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^{\dagger} b_{u\mathbf{k}} \tag{1154}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\overline{S}}(t)+H_B}\right)}\right)\right)+\left\langle\overline{H_{\overline{I}}}\right\rangle_{\overline{H_{\overline{S}}(t)+H_B}}+O\left(\left\langle\overline{H_{\overline{I}}^2}\right\rangle_{\overline{H_{\overline{S}}(t)+H_B}}\right). \tag{1155}$$

Taking the equations (242)-(250) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\overline{S}}(t)}}\right)=C\left(R_0,R_1,R_2,...,R_{d-1},B_{01},B_{02},...,B_{0(d-1)},...,B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}}=\Re\left(v_{nu\mathbf{k}}\right)+\mathrm{i}\Im\left(v_{nu\mathbf{k}}\right)$. So our approach will be based on the derivation respect to $\Re\left(v_{nu\mathbf{k}}\right)$ and $\Im\left(v_{nu\mathbf{k}}\right)$. The Hamiltonian $\overline{H_{\overline{S}}(t)}$ can be written like:

$$\overline{H_{\overline{S}}(t)} = \sum_{n}\left(\varepsilon_{n}(t)+\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1156}$$

$$+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right) \tag{1157}$$

$$\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1158}$$

$$=\sum_{n}\left(\varepsilon_{n}(t)+\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}}-\frac{g_{nu\mathbf{k}}v_{nu\mathbf{k}}^{*}+g_{nu\mathbf{k}}^{*}v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1159}$$

$$+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right) \tag{1160}$$

$$\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1161}$$

$$=\sum_{n}\left(\varepsilon_{n}(t)+\sum_{u\mathbf{k}}\left(\frac{(\Re(v_{nu\mathbf{k}}))^{2}+(\Im(v_{nu\mathbf{k}}))^{2}}{\omega_{u\mathbf{k}}}-\frac{(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^{*})\Re(v_{nu\mathbf{k}})+\mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^{*}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)| \tag{1162}$$

$$+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right) \tag{1163}$$

$$\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1164}$$

$$v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*}=(\Re(v_{mu\mathbf{k}})-\mathrm{i}\Im(v_{mu\mathbf{k}}))(\Re(v_{nu\mathbf{k}})+\mathrm{i}\Im(v_{nu\mathbf{k}}))-(\Re(v_{mu\mathbf{k}})+\mathrm{i}\Im(v_{mu\mathbf{k}}))(\Re(v_{nu\mathbf{k}})-\mathrm{i}\Im(v_{nu\mathbf{k}})) \tag{1165}$$

$$=(\Re(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}})+\mathrm{i}\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}})-\mathrm{i}\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\Im(v_{nu\mathbf{k}})) \tag{1166}$$

$$-(\Re(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}})-\mathrm{i}\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}})+\mathrm{i}\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}})+\Im(v_{mu\mathbf{k}})\Im(v_{nu\mathbf{k}})) \tag{1167}$$

$$=2\mathrm{i}(\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}})-\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}})) \tag{1168}$$

$$\overline{H_{\overline{S}}(t)}=\sum_{n}\left(\varepsilon_{n}(t)+\sum_{u\mathbf{k}}\left(\frac{(\Re(v_{nu\mathbf{k}}))^{2}+(\Im(v_{nu\mathbf{k}}))^{2}}{\omega_{u\mathbf{k}}}-\frac{(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^{*})\Re(v_{nu\mathbf{k}})+\mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^{*}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)| \tag{1169}$$

$$+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}(\Im(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}})-\Im(v_{mu\mathbf{k}})\Re(v_{nu\mathbf{k}}))}{\omega_{u\mathbf{k}}^{2}}\right)\right) \tag{1170}$$

$$\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1171}$$

$$|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}=(v_{mu\mathbf{k}}-v_{nu\mathbf{k}})(v_{mu\mathbf{k}}-v_{nu\mathbf{k}})^{*} \tag{1172}$$

$$=|v_{mu\mathbf{k}}|^{2}+|v_{nu\mathbf{k}}|^{2}-(v_{nu\mathbf{k}}v_{mu\mathbf{k}}^{*}+v_{nu\mathbf{k}}^{*}v_{mu\mathbf{k}}) \tag{1173}$$

$$=(\Re(v_{mu\mathbf{k}}))^{2}+(\Im(v_{mu\mathbf{k}}))^{2}+(\Re(v_{nu\mathbf{k}}))^{2}+(\Im(v_{nu\mathbf{k}}))^{2} \tag{1174}$$

$$-((\Re(v_{nu\mathbf{k}})+\mathrm{i}\Im(v_{nu\mathbf{k}}))(\Re(v_{mu\mathbf{k}})-\mathrm{i}\Im(v_{mu\mathbf{k}}))+(\Re(v_{nu\mathbf{k}})-\mathrm{i}\Im(v_{nu\mathbf{k}}))(\Re(v_{mu\mathbf{k}})+\mathrm{i}\Im(v_{mu\mathbf{k}}))) \tag{1175}$$

$$=(\Re(v_{mu\mathbf{k}}))^{2}+(\Im(v_{mu\mathbf{k}}))^{2}+(\Re(v_{nu\mathbf{k}}))^{2}+(\Im(v_{nu\mathbf{k}}))^{2}$$

$$-2(\Re(v_{nu\mathbf{k}})\Re(v_{mu\mathbf{k}})+\Im(v_{nu\mathbf{k}})\Im(v_{mu\mathbf{k}})) \tag{1176}$$

$$=(\Re(v_{mu\mathbf{k}})-\Re(v_{nu\mathbf{k}}))^{2}+(\Im(v_{mu\mathbf{k}})-\Im(v_{nu\mathbf{k}}))^{2} \tag{1177}$$

$$R_{n}(t)=\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1178}$$

$$=\sum_{u\mathbf{k}}\left(\frac{(\Re(v_{nu\mathbf{k}}))^{2}+(\Im(v_{nu\mathbf{k}}))^{2}-(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^{*})\Re(v_{nu\mathbf{k}})-\mathrm{i}\Im(v_{nu\mathbf{k}})(g_{nu\mathbf{k}}^{*}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right) \tag{1179}$$

Then we can obtain using the chain rule that:

$$
\begin{aligned}
\frac{\partial R_{n'}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} &= \frac{\partial}{\partial \Re\left(v_{nu\mathbf{k}}\right)} \sum_{u\mathbf{k}} \left(\frac{\left(\Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{nu\mathbf{k}}\right)\right)^2 - 2\Re\left(g_{nu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} \right) && (1184)\\
&= \frac{2\Re\left(v_{nu\mathbf{k}}\right) - 2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} && (1185)\\
&= 2\frac{\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} && (1186)\\
\frac{\partial R_{n'}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} &= \frac{\partial}{\partial \Im\left(v_{nu\mathbf{k}}\right)} \sum_{u\mathbf{k}} \left(\frac{\left(\Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{nu\mathbf{k}}\right)\right)^2 - 2\Re\left(g_{nu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}} \right) && (1187)\\
&= \frac{2\Im\left(v_{nu\mathbf{k}}\right) - 2\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} && (1188)\\
&= 2\frac{\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\delta_{nn'} && (1189)
\end{aligned}
$$

Given that:

$$
\begin{aligned}
\ln B_{mn} &= \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2}\right)\right)\right. && (1190)\\
&\left.\prod_{u} \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right) && (1191)\\
&= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2}\right) && (1192)\\
&+ \sum_{u} \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) && (1193)\\
&= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(\Im\left(v_{nu\mathbf{k}}\right)\Re\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\Re\left(v_{nu\mathbf{k}}\right)\right)}{\omega_{u\mathbf{k}}^2}\right) && (1194)\\
&+ \sum_{u\mathbf{k}} \left(-\frac{1}{2}\frac{\left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)^2 + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) && (1195)\\
\frac{\partial \ln B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} &= \frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) - \left(\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} && (1196)\\
\frac{\partial \ln B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} &= \frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) - \left(\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} && (1197)\\
\frac{\partial \ln B_{mn}}{\partial a} &= \frac{1}{B_{mn}}\frac{\partial B_{mn}}{\partial a} && (1198)\\
\frac{\partial B_{mn}}{\partial a} &= B_{mn}\frac{\partial \ln B_{mn}}{\partial a} && (1199)\\
\frac{\partial B_{mn}}{\partial a} &= \frac{\partial\left(B_{nm}\right)^\dagger}{\partial a} && (1200)
\end{aligned}
$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = B_{mn}\frac{\partial \ln B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} \tag{1201}$$

$$= B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) - \left(\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1202}$$

$$= B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1203}$$

$$\frac{\partial B_{nm}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \left(\frac{\partial B_{mn}}{\partial \Re\left(v_{nu\mathbf{k}}\right)}\right)^\dagger \tag{1204}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)^\dagger \tag{1205}$$

$$= B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1206}$$

$$\frac{\partial B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = B_{mn}\frac{\partial \ln B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} \tag{1207}$$

$$= B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) - \left(\Im\left(v_{nu\mathbf{k}}\right) - \Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1208}$$

$$= B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1209}$$

$$\frac{\partial B_{nm}}{\partial \Im\left(v_{nu\mathbf{k}}\right)} = \left(\frac{\partial B_{mn}}{\partial \Im\left(v_{nu\mathbf{k}}\right)}\right)^\dagger \tag{1210}$$

$$= \left(B_{mn}\right)^\dagger \tag{1211}$$

$$= B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right) + \left(\Im\left(v_{mu\mathbf{k}}\right) - \Im\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1212}$$

Introducing this derivates in the equation (1184) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial \Re\left(v_{nu\mathbf{k}}\right)} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{\Re\left(v_{nu\mathbf{k}}\right) - \Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1213}$$

$$+ \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1214}$$

$$\left.+ \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right) + \left(\Re\left(v_{mu\mathbf{k}}\right) - \Re\left(v_{nu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1215}$$

$$= 0 \tag{1216}$$

We can obtain the variational parameters:

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Re\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\Re\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\Re\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1217}$$

$$=-\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1218}$$

$$\Re\left(v_{nu\mathbf{k}}\right)=\frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2\Re\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1219}$$

$$=\frac{2\Re\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1220}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial\Im\left(v_{nu\mathbf{k}}\right)}=\frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right) \tag{1221}$$

$$+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)-\left(\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right. \tag{1222}$$

$$\left.+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)-\left(\Im\left(v_{nu\mathbf{k}}\right)-\Im\left(v_{mu\mathbf{k}}\right)\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1223}$$

$$=0 \tag{1224}$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Im\left(v_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\Im\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\Im\left(v_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1225}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1226}$$

$$\Im\left(v_{nu\mathbf{k}}\right)=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{\Im\left(g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1227}$$

$$=\frac{2\Im\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1228}$$

$$v_{nu\mathbf{k}}=\Re\left(v_{nu\mathbf{k}}\right)+\mathrm{i}\Im\left(v_{nu\mathbf{k}}\right) \tag{1229}$$

$$=\frac{2\Re\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1230}$$

$$\mathrm{i}\frac{2\Im\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1231}$$

$$=\frac{2\Re\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}\Im\left(g_{nu\mathbf{k}}\right)\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1232}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}\Im\left(v_{mu\mathbf{k}}\right)+\Re\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1233}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}\Re\left(v_{mu\mathbf{k}}\right)+\Im\left(v_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1234}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1235}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1236}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1237}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^V\rho\left(0\right)e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1238}$$

$$0=\left(B_{0+}|0\rangle\langle 0|B_{0-}\right)\otimes\rho_B \tag{1239}$$

$$0=\rho\left(0\right) \tag{1240}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)OU(t) \tag{1241}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}\left(t'\right)\right). \tag{1242}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\overline{\rho_S}(t)U(t), \text{ where} \tag{1243}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1244}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1245}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1246}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1247}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1248}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_mB_n\right)^{\dagger} \tag{1249}$$
$$= B_{n-}^{\dagger}B_{m+}^{\dagger} - B_nB_m \tag{1250}$$
$$= B_{n+}B_{m-} - B_nB_m \tag{1251}$$
$$= B_{nm} \tag{1252}$$

So we can say that the set of matrices (1245) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1245) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} + \sum_n B_{z,n}\left(t\right)|n\rangle\langle n|, \tag{1253}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}\left(t\right)|n\rangle\langle m|B_{nm} + V_{mn}\left(t\right)|m\rangle\langle n|B_{mn}\right) \tag{1254}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}\left(t\right)\right)B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}\Im\left(V_{nm}\left(t\right)\right)B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1255}$$
$$\left. + \Re\left(V_{nm}\left(t\right)\right)B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}\Im\left(V_{nm}\left(t\right)\right)B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1256}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1257}$$
$$\left. + \mathrm{i}\Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1258}$$
$$= \sum_n B_{z,n}\left(t\right)|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1259}$$
$$\left. + \Im\left(V_{nm}\left(t\right)\right)\sigma_{nm,y}B_{nm,x}\right) \tag{1260}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n,m \leq d-1 \wedge \left(n = m \vee n < m\right)\right\} \tag{1261}$$

Now consider the following set of operators,

$$A_{1,nm}\left(t\right)=\sigma_{nm,x}\left(1-\delta_{mn}\right) \tag{1262}$$
$$A_{2,nm}\left(t\right)=\sigma_{nm,y}\left(1-\delta_{mn}\right) \tag{1263}$$
$$A_{3,nm}\left(t\right)=\delta_{mn}|n\rangle\langle m| \tag{1264}$$
$$A_{4,nm}\left(t\right)=A_{2,mn}\left(t\right) \tag{1265}$$
$$A_{5,nm}\left(t\right)=A_{1,nm}\left(t\right) \tag{1266}$$
$$B_{1,nm}\left(t\right)=B_{nm,x} \tag{1267}$$
$$B_{2,nm}\left(t\right)=B_{nm,y} \tag{1268}$$
$$B_{3,nm}\left(t\right)=B_{z,n}\left(t\right) \tag{1269}$$
$$B_{4,nm}\left(t\right)=B_{1,nm}\left(t\right) \tag{1270}$$
$$B_{5,nm}\left(t\right)=B_{2,nm}\left(t\right) \tag{1271}$$
$$C_{1,nm}\left(t\right)=\Re\left(V_{nm}\left(t\right)\right) \tag{1272}$$
$$C_{2,nm}\left(t\right)=C_{1,nm}\left(t\right) \tag{1273}$$
$$C_{3,nm}\left(t\right)=1 \tag{1274}$$
$$C_{4,nm}\left(t\right)=\Im\left(V_{nm}\left(t\right)\right) \tag{1275}$$
$$C_{5,nm}\left(t\right)=-\Im\left(V_{nm}\left(t\right)\right) \tag{1276}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}\left(t\right)$ as:

$$\overline{H_I}=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(A_{jp}\otimes B_{jp}\left(t\right)\right) \tag{1277}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1261).

We write the interaction Hamiltonian transformed under (1241) as:

$$\widetilde{H_I}\left(t\right)=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1278}$$
$$\widetilde{A_{jp}}(t)=U^{\dagger}\left(t\right)A_{jp}U\left(t\right) \tag{1279}$$
$$\widetilde{B_{jp}}\left(t\right)=\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\left(t\right)\left(\mathrm{t}\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{1280}$$

Taking as reference state ρ_B and truncating at second order in $H_I\left(t\right)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t}=-\int_0^t\mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right),\left[\widetilde{H_I}\left(s\right),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1281}$$

Replacing the equation (1278) in (1281) we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right), \left[\widetilde{H_I}\left(s\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1282}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J, p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right), \left[\sum_{j'\in J, p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1283}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J, p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right), \sum_{j'\in J, p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1284}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\right]\mathrm{d}s \tag{1285}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J, p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\sum_{j'\in J, p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1286}$$

$$-\sum_{j\in J, p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right) \tag{1287}$$

$$-\sum_{j'\in J, p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B \sum_{j\in J, p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{1288}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B \sum_{j'\in J, p'\in P} C_{j'p'}\left(s\right)\left(\widetilde{A_{j'p'}}\left(s\right)\otimes\widetilde{B_{j'p'}}\left(s\right)\right)\sum_{j\in J, p\in P} C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1289}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}\left(\tau\right) = \left\langle \widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1290}$$

$$= \left\langle \widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1291}$$

Here $s \to t - \tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right) = \left\langle \widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}\left(t\right)$ and elements related to the system given by $\widetilde{A_{jp}}\left(t\right)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(t\right)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}\left(t\right)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}\left(t\right)\right)$. The correlation functions relevant of the master equation (1289) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1292}$$

$$= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1293}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1294}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1295}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1296}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1297}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1298}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) \tag{1299}$$

$$= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1300}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1301}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1302}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1303}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1304}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1305}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1306}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1282)and (1289) and using the equations (1292)-(1306) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}(t) - \Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'p'}}\left(s\right)\right)\right. \tag{1307}$$

$$\left. + C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right) - \Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}(t)\widetilde{A_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1308}$$

$$= -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[\widetilde{A_{jp}}\left(t\right), \widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}(t)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}(t)\widetilde{A_{j'p'}}\left(s\right), \widetilde{A_{jp}}\left(t\right)\right]\right)\right) \tag{1309}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right), A_{j'p'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}(t)A_{j'p'}\left(t-\tau,t\right), A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right), \overline{\rho_S}(t)\right] \tag{1310}$$

For this case we used that $A_{jp}\left(t-\tau,t\right) = U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation.

VIII. BIBLIOGRAPHY

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