# A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right) = H_S\left(t\right) + H_I + H_B, \tag{1}$$

$$H_S\left(t\right) = \varepsilon_0\left(t\right)|0\rangle\langle 0| + \varepsilon_1\left(t\right)|1\rangle\langle 1| + V_{10}\left(t\right)|1\rangle\langle 0| + V_{01}\left(t\right)|0\rangle\langle 1|, \tag{2}$$

$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

## I. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H\left(t\right)$, the unitary transformation defined by $e^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V\left(t\right) \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}\left(t\right)$ defined as:

$$v_{i\mathbf{k}}\left(t\right) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}\left(t\right). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i\left(t\right) \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V\left(t\right) = |0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V(t)}$ using the notation (6) will give us the following result:

$$
\begin{aligned}
e^{\pm V(t)} &= e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t))} && (10)\\
&= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)))^2}{2!} + ... && (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2(t)}{2!} + ... && (12)\\
&= |0\rangle\langle 0|\left(\mathbb{I} \pm \hat{\varphi}_0(t) + \frac{\hat{\varphi}_0^2(t)}{2!} \pm ...\right) + |1\rangle\langle 1|\left(\mathbb{I} \pm \hat{\varphi}_1(t) + \frac{\hat{\varphi}_1^2(t)}{2!} \pm ...\right) && (13)\\
&= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1(t)} && (14)\\
&= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger} - \alpha_{0\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger} - \alpha_{1\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_0^{\pm}(t) + |1\rangle\langle 1|B_1^{\pm}(t), && (16)\\
B_i^{\pm}(t) &\equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
e^{r(X+Y)} = e^{rX}\, e^{rY}\, e^{-\frac{r^2}{2}[X,Y]}\, e^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \cdots. \tag{18}
$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}'$, $\mathbf{k}$ and $i, j$ we can show making $r = 1$ in (18) the following result:

$$
\begin{aligned}
e^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]} \cdots && (19)\\
&= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{-\frac{1}{2}0} \cdots && (20)\\
&= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. && (21)
\end{aligned}
$$

By induction of this result we can write an expresion of $B_i^{\pm}(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$
\begin{aligned}
D(\pm v_{i\mathbf{k}}(t)) &\equiv e^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, && (22)\\
B_i^{\pm}(t) &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). && (23)
\end{aligned}
$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$
\overline{O(t)} \equiv e^{V(t)} O(t)\, e^{-V(t)}. \tag{24}
$$

We will use the following identities:

$$
\begin{aligned}
\overline{|0\rangle\langle 0|\,(t)} &= e^{V(t)}|0\rangle\langle 0|e^{-V(t)} && (25)\\
&= \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (26)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (27)\\
&= |0\rangle\langle 0|B_0^+(t)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (28)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,B_0^-(t)+|0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,B_1^-(t) && (29)\\
&= |0\rangle\langle 0|, && (30)\\
\overline{|1\rangle\langle 1|\,(t)} &= \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (31)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)+|1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (32)\\
&= |1\rangle\langle 1|B_1^+(t)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (33)\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,B_0^-(t)+B_1^+(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) && (34)\\
&= B_1^+(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) && (35)\\
&= |1\rangle\langle 1|, && (36)\\
\overline{|0\rangle\langle 1|\,(t)} &= e^{V(t)}|0\rangle\langle 1|e^{-V(t)} && (37)\\
&= \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (38)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\,|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (39)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t)+|1\rangle\langle 1|0\rangle\langle 1|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (40)\\
&= |0\rangle\langle 1|B_0^+(t)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (41)\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_0^+(t)\,B_0^-(t)+|0\rangle\langle 1|1\rangle\langle 1|B_0^+(t)\,B_1^-(t) && (42)\\
&= |0\rangle\langle 1|B_0^+(t)\,B_1^-(t)\,, && (43)\\
\overline{|1\rangle\langle 0|\,(t)} &= e^{V(t)}|1\rangle\langle 0|e^{-V(t)} && (44)\\
&= \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (45)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\,|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (46)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|1\rangle\langle 0|B_1^+(t)\right)\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (47)\\
&= |1\rangle\langle 0|0\rangle\langle 0|B_1^+(t)\,B_0^-(t)+|1\rangle\langle 0|1\rangle\langle 1|B_1^+(t)\,B_1^-(t) && (48)\\
&= |1\rangle\langle 0|B_1^+(t)\,B_0^-(t)\,, && (49)\\
\overline{b_{\mathbf{k}}(t)} &= e^{V(t)}b_{\mathbf{k}}e^{-V(t)} && (50)\\
&= \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (51)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0|+|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t)+|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| && (52)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}B_0^-(t)+|0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,b_{\mathbf{k}}B_1^-(t)+|1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,b_{\mathbf{k}}B_0^-(t)+|1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}B_1^-(t) && (53)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (54)\\
&= \left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)b_{\mathbf{k}}-|1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-|0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} && (55)\\
&= b_{\mathbf{k}}-|1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-|0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, && (56)\\
\overline{b_{\mathbf{k}}(t)}^{\dagger} &= e^{V(t)}b_{\mathbf{k}}^{\dagger}e^{-V(t)} && (57)\\
&= \left(|0\rangle\langle 0|B_0^+(t)+|1\rangle\langle 1|B_1^+(t)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^-(t)\right) && (58)\\
&= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle 0|+|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^-(t)+|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^-(t)+|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle 1| && (59)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)+|0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)+|1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)+|1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) && (60)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) && (61)\\
&= b_{\mathbf{k}}^{\dagger}-|1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}-|0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. && (62)
\end{aligned}
$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+}(t)\, b_{\mathbf{k}} B_i^{-}(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \tag{63}$$

$$B_i^{+}(t)\, b_{\mathbf{k}}^{\dagger} B_i^{-}(t) = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}. \tag{64}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|\,(t)} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{65}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|\,(t)} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{66}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|\,(t)} = V_{10}(t)\,|1\rangle\langle 0| B_1^{+}(t)\, B_0^{-}(t), \tag{67}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|\,(t)} = V_{01}(t)\,|0\rangle\langle 1| B_0^{+}(t)\, B_1^{-}(t), \tag{68}$$

$$\overline{\left(g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}}\right)(t)} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\right) + g_{i\mathbf{k}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{69}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{70}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) + g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{72}$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*} b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{73}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{74}$$

$$= |0\rangle\langle 0| B_0^{+}(t)\,|0\rangle\langle 0|\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right)|0\rangle\langle 0| B_0^{-}(t) \tag{75}$$

$$= |0\rangle\langle 0| B_0^{+}(t)\left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_0^{-}(t) \tag{76}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{77}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{78}$$

$$= |1\rangle\langle 1| B_1^{+}(t)\,|1\rangle\langle 1|\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1k}^{*} b_{\mathbf{k}}\right)|1\rangle\langle 1| B_1^{-}(t) \tag{79}$$

$$= |1\rangle\langle 1| B_1^{+}(t)\left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}}\right) B_1^{-}(t) \tag{80}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{81}$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}(t)} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{+}(t) + |1\rangle\langle 1| B_1^{+}(t)\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{-}(t) + |1\rangle\langle 1| B_1^{-}(t)\right) \tag{82}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\left(|0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| B_0^{+}(t)\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_0^{-}(t) + |1\rangle\langle 1| B_1^{+}(t)\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_1^{-}(t)\right) \tag{84}$$

$$= \omega_{\mathbf{k}} \sum_j |j\rangle\langle j| D\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}} \left(D\left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{90}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \omega_{\mathbf{k}}\left(|1\rangle\langle 1|\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right)\right) \tag{91}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} - v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} - v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right) \tag{92}$$

$$= \omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right). \tag{93}$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} \tag{94}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0|B_1^{+}(t)\,B_0^{-}(t) + V_{01}(t)\,|0\rangle\langle 1|B_0^{+}(t)\,B_1^{-}(t), \tag{95}$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} \tag{96}$$

$$= \overline{\sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} + \overline{\sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)} \tag{97}$$

$$= \sum_{\mathbf{k}}|0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) + \sum_{\mathbf{k}}|1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right) + g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{98}$$

$$= \sum_{\mathbf{k},i}|i\rangle\langle i|\left(g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{99}$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} \tag{100}$$

$$= \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + |1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right)\right) \tag{101}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{\mathbf{k}}\left(|1\rangle\langle 1|\left(\frac{|v_{1\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{1\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right) + |0\rangle\langle 0|\left(\frac{|v_{0\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*}(t)\,b_{\mathbf{k}} + v_{0\mathbf{k}}(t)\,b_{\mathbf{k}}^{\dagger}\right)\right)\right). \tag{102}$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_{j}\varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^{+}(t)\,B_{j'}^{-}(t) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}} + \frac{\left|v_{j\mathbf{k}}(t)\right|^{2}}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^{*}\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right). \tag{103}$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}. \tag{104}$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}}\left(\frac{|v_{i\mathbf{k}}(t)|^{2}}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{105}$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right), \tag{106}$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)\,v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right). \tag{107}$$

$\chi_{ij}(t)$ is an imaginary number so $e^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right) & e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \\ \frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2} & e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}e^{\chi_{ij}(t)} \end{pmatrix}, \tag{108}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{109}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{110}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)}. \tag{111}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{112}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{113}$$

So the product of the matrix representation of $b^{\dagger}$ and $b$ with $-\beta$ is:

$$-\beta\omega b^{\dagger}b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{114}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right|, \tag{115}$$

So the density matrix $\rho_B$ written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all $i, j$.

$$e^{-\beta\omega_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}} = \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, , \tag{116}$$

$$e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}} = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \tag{117}$$

The value of $\mathrm{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}}\right)$ is:

$$\begin{aligned}
\mathrm{Tr}\left(e^{-\beta\omega_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (118)\\
&= \sum_{j_{\mathbf{k}}} \left(e^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} && (119)\\
&= \frac{1}{1-e^{-\beta\omega_{\mathbf{k}}}} \quad \text{(by geometric series)} && (120)\\
&\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), && (121)\\
\mathrm{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (122)\\
&= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) && (123)\\
&= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). && (124)
\end{aligned}$$

So the density matrix of the bath is:

$$\begin{aligned}
\rho_B &= \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} && (125)\\
&= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} && (126)\\
&= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. && (127)
\end{aligned}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{128}$$

$$b^\dagger_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. \tag{129}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (128)-(129):

$$\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Tr}\left(B_{iz}\left(t\right)\rho_B\right) \tag{130}$$

$$= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{131}$$

$$= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\rho_B\right) \tag{132}$$

$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right) + \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) \tag{133}$$

$$= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) \tag{134}$$

$$= \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) + \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}} - \mathrm{v}_{i\mathbf{k}}(\mathrm{t})\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), \tag{135}$$

$$\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}\text{)} \tag{136}$$

$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{137}$$

$$= 0, \tag{138}$$

$$\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} \tag{139}$$

$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) \tag{140}$$

$$= 0. \tag{141}$$

we therefore find that:

$$\langle B_{iz}\left(t\right)\rangle_{\overline{H_{\bar{B}}}} = 0. \tag{142}$$

Another important expected value is $B\left(t\right) = \langle B^{\pm}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm}\left(t\right) = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\left\langle B^{\pm}\left(t\right)\right\rangle_{H_B} = \mathrm{Tr}\left(\rho_B B^{\pm}\left(t\right)\right) = \mathrm{Tr}\left(B^{\pm}\left(t\right)\rho_B\right) \tag{143}$$

$$= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) \tag{144}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\rho_B\right) \tag{145}$$

$$= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\left(t\right)\right)\right\rangle. \tag{146}$$

Given that we can write a density operator as:

$$\rho = \int P\left(\alpha\right)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \tag{147}$$

where $P\left(\alpha\right)$ satisfies $\int P\left(\alpha\right)\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P\left(\alpha\right)$ is given by:

$$\langle A\rangle = \mathrm{Tr}\left(A\rho\right) \tag{148}$$

$$= \int P\left(\alpha\right)\langle\alpha\left|A\right|\alpha\rangle\mathrm{d}^2\alpha. \tag{149}$$

We are typically interested in thermal state density operators, for which it can be shown that $P\left(\alpha\right)=\frac{1}{\pi N}e^{-\frac{|\alpha|^2}{N}}$ where $N=\left(e^{\beta\omega}-1\right)^{-1}$ is the average number of excitations in an oscillator of frequency $\omega$ at inverse temperature $\beta=\frac{1}{k_{\mathrm{B}}T}$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D\left(h\right)$ with $h\in\mathbb{C}$ is equal to:

$$\langle D\left(h\right)\rangle=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle\alpha|D\left(h\right)|\alpha\rangle\mathrm{d}^2\alpha \tag{150}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)|0\rangle\mathrm{d}^2\alpha, \tag{151}$$

$$D\left(h\right)D\left(\alpha\right)=D\left(h+\alpha\right)e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)}, \tag{152}$$

$$D\left(-\alpha\right)\left(D\left(h\right)D\left(\alpha\right)\right)=D\left(-\alpha\right)D\left(h+\alpha\right)e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)} \tag{153}$$

$$=D\left(h\right)e^{\frac{1}{2}\left(-\alpha\left(h+\alpha\right)^*+\alpha^*\left(h+\alpha\right)\right)}e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)} \tag{154}$$

$$=D\left(\alpha\right)e^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^*h+|\alpha|^2\right)}e^{\frac{1}{2}\left(h\alpha^*-h^*\alpha\right)} \tag{155}$$

$$=D\left(\alpha\right)e^{\left(h\alpha^*-h^*\alpha\right)}, \tag{156}$$

$$\langle D\left(h\right)\rangle=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|D\left(h\right)e^{\left(h\alpha^*-h^*\alpha\right)}|0\rangle\mathrm{d}^2\alpha \tag{157}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{\left(h\alpha^*-h^*\alpha\right)}\langle 0|D\left(h\right)|0\rangle\mathrm{d}^2\alpha \tag{158}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{\left(h\alpha^*-h^*\alpha\right)}\langle 0|h\rangle\mathrm{d}^2\alpha, \tag{159}$$

$$|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{160}$$

$$\langle D\left(h\right)\rangle=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{\left(h\alpha^*-h^*\alpha\right)}\langle 0|e^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle\mathrm{d}^2\alpha \tag{161}$$

$$=\frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{\left(h\alpha^*-h^*\alpha\right)}e^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{162}$$

$$=\frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int e^{-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha}\mathrm{d}^2\alpha, \tag{163}$$

$$\alpha=x+\mathrm{i}y, \tag{164}$$

$$\langle D\left(h\right)\rangle=\frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{x^2+y^2}{N}+h\left(x-\mathrm{i}y\right)-h^*\left(x+\mathrm{i}y\right)}\mathrm{d}x\mathrm{d}y \tag{165}$$

$$=\frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}e^{-\frac{x^2}{N}+hx-h^*x}\mathrm{d}x\int_{-\infty}^{\infty}e^{-\frac{x^2}{N}+hx-h^*x}\mathrm{d}y, \tag{166}$$

$$-\frac{x^2}{N}+hx-h^*x=-\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) \tag{167}$$

$$=-\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2+\frac{N\left(h^*-h\right)^2}{4}, \tag{168}$$

$$-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y=-\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) \tag{169}$$

$$=-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)-\frac{N\left(h+h^*\right)^2}{4}, \tag{170}$$

$$\sqrt{2\pi}\sigma=\int_{-\infty}^{\infty}e^{-\frac{\left(x-\mu\right)^2}{2\sigma^2}}\mathrm{d}x, \tag{171}$$

$$\langle D(h)\rangle = \frac{e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)}\mathrm{d}x\mathrm{d}y, \tag{172}$$

$$\begin{aligned}
\langle D\left(h\right)\rangle &= \frac{e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}}}{\pi N}\int_{-\infty}^{\infty} e^{-\frac{\left(x+\frac{(Nh^*-Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}x\int_{-\infty}^{\infty} e^{-\frac{\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}y && (173)\\
&= \frac{e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}}}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 && (174)\\
&= e^{-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}} && (175)\\
&= e^{-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}} && (176)\\
&= e^{-|h|^2\left(N+\frac{1}{2}\right)} && (177)\\
&= e^{-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)} && (178)\\
&= e^{-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)} && (179)\\
&= e^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}. && (180)
\end{aligned}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}=\coth\left(\frac{\beta\omega}{2}\right)$. So the value of (145) using (**??**) is given by:

$$B = e^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{181}$$

We will now force $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$. We will also introduce the bath renormalizing driving in $\overline{H_S}\left(t\right)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+\left(t\right)\sigma^+$ and $B_i^-\left(t\right)\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}\left(t\right)$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$.

A final form of the terms of the Hamiltonian $\overline{H}\left(t\right)$ is:

$$\overline{H(t)} = \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+\sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^*b_{\mathbf{k}}+\frac{\left|v_{j\mathbf{k}}(t)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}+g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{182}$$

$$\begin{aligned}
&= \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_{jj'}(t)+\sum_j|j\rangle\langle j|B_{jz}(t)+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+\left(t\right)B_{j'}^-(t)-B_{jj'}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (183)\\
&\equiv \overline{H_{\bar{S}}\left(t\right)}+\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{B}}}. && (184)
\end{aligned}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\begin{aligned}
\langle B_i^+\left(t\right)B_j^-\left(t\right)\rangle &= B_{ij}\left(t\right) && (185)\\
&= \left\langle\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right\rangle && (186)\\
&= \left\langle\prod_{\mathbf{k}}\left(D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle && (187)
\end{aligned}$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \right\rangle \tag{188}$$

$$= \prod_{\mathbf{k}} \left\langle D\left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \tag{189}$$

$$= \prod_{\mathbf{k}} e^{-\frac{1}{2}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} e^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \tag{190}$$

$$= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left( \frac{v_{i\mathbf{k}}^{*}(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)}. \tag{191}$$

From the definition $B_{01}(t) = \langle B_0^{+}(t) B_1^{-}(t) \rangle$ using the displacement operator we have:

$$\begin{aligned}
\langle B_0^{+}(t) B_1^{-}(t) \rangle &= B_{01}(t) && (192)\\
&= \left\langle \prod_{\mathbf{k}} D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D\left( -\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle && (193)\\
&= \left\langle \prod_{\mathbf{k}} \left( D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D\left( -\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle && (194)\\
&= \left\langle \prod_{\mathbf{k}} \left( D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) \right\rangle && (195)\\
&= \prod_{\mathbf{k}} \left( \left\langle D\left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) && (196)\\
&= \prod_{\mathbf{k}} \left( e^{-\frac{1}{2}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} e^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} \right) && (197)\\
&= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)}. && (198)
\end{aligned}$$

We can check:

$$\begin{aligned}
\langle B_0^{+}(t) B_1^{-}(t) \rangle &= B_{01}(t) && (199)\\
&= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left( \frac{v_{0\mathbf{k}}^{*}(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)} && (200)\\
&= e^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \prod_{\mathbf{k}} e^{\frac{1}{2}\left( \frac{v_{1\mathbf{k}}^{*}(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}} \right)^{*}} && (201)\\
&= \langle B_1^{+}(t) B_0^{-}(t) \rangle^{*} && (202)\\
&= B_{10}^{*}(t). && (203)
\end{aligned}$$

The parts of the splitted Hamiltonian are:

$$\begin{aligned}
\overline{H_{\bar{S}}(t)} &\equiv (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + V_{10}(t) B_{10}\sigma^{+} + V_{01}(t) B_{01}\sigma^{-}, && (204)\\
\overline{H_{\bar{I}}} &\equiv V_{10}(t) \left( B_1^{+}(t) B_0^{-}(t) - B_{10}(t) \right) \sigma^{+} + V_{01}(t) \left( B_0^{+}(t) B_1^{-}(t) - B_{01}(t) \right) \sigma^{-} + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t), && (205)\\
\overline{H_{\bar{B}}} &\equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} && (206)\\
&= H_B. && (207)
\end{aligned}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, $H_B$, before transforming to the variational frame.
For the Hamiltonian (205) we can verify the condition $\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$
\begin{aligned}
\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} =& \left\langle \sum_{n\mathbf{k}} \left( \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}\left(t\right) |j\rangle\langle j'| \left( B_{j}^{+}\left(t\right) B_{j'}^{-}\left(t\right) - B_{jj'}\left(t\right) \right) \right\rangle_{\overline{H_{\bar{B}}}} && (208)\\
=& \left\langle \sum_{n\mathbf{k}} \left( \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}\left(t\right) |j\rangle\langle j'| \left( B_{j}^{+}\left(t\right) B_{j'}^{-}\left(t\right) - B_{jj'}\left(t\right) \right) \right\rangle_{\overline{H_{\bar{B}}}} && (209)\\
=& \sum_{n\mathbf{k}} \left( \left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| \left( \left\langle V_{jj'}(t) B_{j}^{+}(t) B_{j'}^{-}(t) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle V_{jj'}(t) B_{jj'}(t) \right\rangle_{\overline{H_{\bar{B}}}} \right) && (210)\\
=& \sum_{n\mathbf{k}} \left( \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}\left(t\right) \left( \left\langle B_{j}^{+}\left(t\right) B_{j'}^{-}\left(t\right) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_{jj'}\left(t\right) \right\rangle_{\overline{H_{\bar{B}}}} \right) && (211)\\
=& \sum_{n\mathbf{k}} \left( \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\left(t\right)\right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}\left(t\right) \left( B_{jj'}\left(t\right) - B_{jj'}\left(t\right) \right) && (212)\\
=& 0. && (213)
\end{aligned}
$$

We used (142) and (**??**) to evaluate the expression. Let's consider the following Hermitian combinations:

$$
\begin{aligned}
B_{x}\left(t\right) &= B_{x}^{\dagger}\left(t\right) && (214)\\
&= \frac{B_{1}^{+}\left(t\right) B_{0}^{-}\left(t\right) + B_{0}^{+}\left(t\right) B_{1}^{-}\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}, && (215)\\
B_{y}\left(t\right) &= B_{y}^{\dagger}\left(t\right) && (216)\\
&= \frac{B_{0}^{+}\left(t\right) B_{1}^{-}\left(t\right) - B_{1}^{+}\left(t\right) B_{0}^{-}\left(t\right) + B_{10}\left(t\right) - B_{01}\left(t\right)}{2\mathrm{i}}, && (217)\\
B_{iz}\left(t\right) &= B_{iz}^{\dagger}\left(t\right) && (218)\\
&= \sum_{\mathbf{k}} \left( \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}} \right). && (219)
\end{aligned}
$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$
\begin{aligned}
\overline{H_{\bar{S}}}\left(t\right) =& \sum_{j\in\{0,1\}} \left(\varepsilon_{j}\left(t\right) + R_{j}\left(t\right)\right) |j\rangle\langle j| + V_{10}\left(t\right) B_{10}\left(t\right) \sigma^{+} + V_{01}\left(t\right) B_{01}\left(t\right) \sigma^{-} && (220)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_{j}\left(t\right) + R_{j}\left(t\right)\right) |j\rangle\langle j| + V_{10}\left(t\right) B_{10}\left(t\right) \frac{\sigma_{x} + \mathrm{i}\sigma_{y}}{2} + V_{01}\left(t\right) B_{01}\left(t\right) \frac{\sigma_{x} - \mathrm{i}\sigma_{y}}{2} && (221)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_{j}\left(t\right) + R_{j}\left(t\right)\right) |j\rangle\langle j| + V_{10}\left(t\right) \left( B_{10}^{\Re}\left(t\right) + \mathrm{i}B_{10}^{\Im}\left(t\right) \right) \frac{\sigma_{x} + \mathrm{i}\sigma_{y}}{2} + V_{01}\left(t\right) \left( B_{10}^{\Re}\left(t\right) - \mathrm{i}B_{10}^{\Im}\left(t\right) \right) \frac{\sigma_{x} - \mathrm{i}\sigma_{y}}{2} && (222)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_{j}(t) + R_{j}(t)\right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left( V_{10}(t) \frac{\sigma_{x} + \mathrm{i}\sigma_{y}}{2} + V_{01}(t) \frac{\sigma_{x} - \mathrm{i}\sigma_{y}}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left( V_{10}(t) \frac{\sigma_{x} + \mathrm{i}\sigma_{y}}{2} - V_{01}(t) \frac{\sigma_{x} - \mathrm{i}\sigma_{y}}{2} \right) && (223)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_{j}\left(t\right) + R_{j}\left(t\right)\right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left( \sigma_{x} \frac{V_{10}(t) + V_{01}(t)}{2} + \mathrm{i}\sigma_{y} \frac{V_{10}(t) - V_{01}(t)}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left( \sigma_{x} \frac{V_{10}(t) - V_{01}(t)}{2} + \mathrm{i}\sigma_{y} \frac{V_{10}(t) + V_{01}(t)}{2} \right) && (224)\\
=& \sum_{j\in\{0,1\}} \left(\varepsilon_{j}\left(t\right) + R_{j}\left(t\right)\right) |j\rangle\langle j| + B_{10}^{\Re}\left(t\right) \left( \sigma_{x} V_{10}^{\Re}\left(t\right) - \sigma_{y} V_{10}^{\Im}\left(t\right) \right) + \mathrm{i}B_{10}^{\Im}\left(t\right) \left( \mathrm{i}\sigma_{x} V_{10}^{\Im}\left(t\right) + \mathrm{i}\sigma_{y} V_{10}^{\Re}\left(t\right) \right) && (225)\\
=& \left(\varepsilon_{0}\left(t\right) + R_{0}\left(t\right)\right) |0\rangle\langle 0| + \left(\varepsilon_{1}\left(t\right) + R_{1}\left(t\right)\right) |1\rangle\langle 1| + B_{10}^{\Re}\left(t\right) \left( \sigma_{x} V_{10}^{\Re}\left(t\right) - \sigma_{y} V_{10}^{\Im}\left(t\right) \right) + \mathrm{i}B_{10}^{\Im}\left(t\right) \left( \mathrm{i}\sigma_{x} V_{10}^{\Im}\left(t\right) + \mathrm{i}\sigma_{y} V_{10}^{\Re}\left(t\right) \right) && (226)\\
=& \left(\varepsilon_{0}(t) + R_{0}(t)\right) |0\rangle\langle 0| + \left(\varepsilon_{1}(t) + R_{1}(t)\right) |1\rangle\langle 1| + \left( \sigma_{x} B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_{y} B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) - \left( \sigma_{x} B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_{y} B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) && (227)\\
=& \left(\varepsilon_{0}(t) + R_{0}(t)\right) |0\rangle\langle 0| + \left(\varepsilon_{1}(t) + R_{1}(t)\right) |1\rangle\langle 1| + \sigma_{x} \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_{y} \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) && (228)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\varepsilon_0(t) + R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), && (229)\\
\overline{H_{\bar{I}}} &= V_{10}(t)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t)\right) + V_{01}(t)\left(\sigma^- B_0^+(t)\,B_1^-(t) - \sigma^- B_{01}(t)\right) + |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) && (230)\\
&= |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t) + \left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t)\right) + \left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\left(\sigma^- B_0^+(t)\,B_1^-(t) - \sigma^- B_{01}(t)\right) && (231)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma^+ B_1^+(t)\,B_0^-(t) - \sigma^+ B_{10}(t) + \sigma^- B_0^+(t)B_1^-(t) - \sigma^- B_{01}(t)\right) + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma^+ B_1^+(t)B_0^-(t) - \sigma^+ B_{10}(t) - \sigma^- B_0^+(t)B_1^-(t) + \sigma^- B_{01}(t)\right) && (232)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+(t)\,B_1^-(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) && (233)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+(t)\,B_1^-(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) && (234)\\
&\quad + \mathrm{i}V_{10}^{\Im}(t)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+(t)\,B_0^-(t) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}(t) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+(t)\,B_1^-(t) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}(t)\right) && (235)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2}\right) && (236)\\
&\quad + \mathrm{i}V_{10}^{\Im}(t)\left(\sigma_x\frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} + \mathrm{i}\sigma_y\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) && (237)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x\frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} - \sigma_y\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) && (238)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+(t)\,B_0^-(t) - B_0^+(t)\,B_1^-(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) && (239)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+ B_0^-(t) - B_0^+ B_1^-(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) && (240)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\left(-B_y(t)\right) - \sigma_y B_x(t)\right) && (241)\\
&= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right). && (242)
\end{aligned}
$$

## II. FREE-ENERGY MINIMIZATION

The true free energy $E_{\mathrm{Free}}(t)$ is bounded by the Bogoliubov inequality:

$$
E_{\mathrm{Free}}(t) \leq E_{\mathrm{Free,B}}(t) \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right)\right) + \left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right). \tag{243}
$$

We will optimize the set of variational parameters $\{v_{\mathbf{k}}(t)\}$ in order to minimize $E_{\mathrm{Free,B}}(t)$ (i.e. to make it as close to the true free energy $E_{\mathrm{Free}}(t)$ as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{\mathbf{k}}(t)\}$:

$$
\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{244}
$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}(t), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$
e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{245}
$$

Then using the fact that $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$
\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{246}
$$

So Eq. (244) becomes:

$$\frac{\partial E_{\text{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{247}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\text{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{248}$$

$$= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\text{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{249}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{250}$$

$$= 0 \quad \text{(by Eq. (244))}. \tag{251}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of $H_B$ that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial \ln\left(\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} \tag{252}$$

$$= 0. \tag{253}$$

This means we need to impose:

$$\frac{\partial \text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{254}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left(\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1| + V_{10}(t)B_{10}(t)\sigma^+ + V_{01}(t)B_{01}(t)\sigma^-\right). \tag{255}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \text{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{256}$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{257}$$

$$\eta \equiv \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{258}$$

The solutions of the equation (256) are:

$$\lambda = \beta\frac{-\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{259}$$

$$= \beta\frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{260}$$

$$= -\beta\frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{261}$$

The value of $\text{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a $2\times 2$ matrix):

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)=\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{\frac{\eta(t)\beta}{2}}+\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{-\frac{\eta(t)\beta}{2}} \tag{262}$$

$$=2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{263}$$

Given that $v_{i\mathbf{k}}\left(t\right)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)=A\left(\varepsilon\left(t\right),\eta\left(t\right)\right)$ to calculate $\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}=\frac{\partial\left(2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} \tag{264}$$

$$=2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\,\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\sinh\left(\frac{\eta(t)\,\beta}{2}\right) \tag{265}$$

$$=-\beta\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{266}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)=0. \tag{267}$$

The derivates included in the expression given are related to:

$$\left\langle B_0^+\left(t\right)B_1^-\left(t\right)\right\rangle=\left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{268}$$

$$=\left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^*e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{269}$$

$$=\left\langle B_1^+\left(t\right)B_0^-\left(t\right)\right\rangle^*, \tag{270}$$

$$R_i\left(t\right)=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right) \tag{271}$$

$$=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}\left(t\right)\right|^2}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*\left(t\right)}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right), \tag{272}$$

$$\left\langle B_0^+\left(t\right)B_1^-\left(t\right)\right\rangle=\left(\prod_{\mathbf{k}}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{273}$$

$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)e^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{274}$$

$$v_{0\mathbf{k}}^*\left(t\right)v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)v_{1\mathbf{k}}^*\left(t\right)=\left(v_{0\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{1\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{1\mathbf{k}}^{\Im}\left(t\right)\right)-\left(v_{0\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{1\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{1\mathbf{k}}^{\Im}\left(t\right)\right) \tag{275}$$

$$=\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)\right) \tag{276}$$

$$-\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)+\mathrm{i}v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)+v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)\right) \tag{277}$$

$$=2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}\left(t\right)v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)v_{1\mathbf{k}}^{\Re}\left(t\right)\right), \tag{278}$$

$$|v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)|^2=(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))^* \tag{279}$$
$$=|v_{1\mathbf{k}}(t)|^2+|v_{0\mathbf{k}}(t)|^2-(v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)+v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)) \tag{280}$$
$$=\left(v_{1\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{0\mathbf{k}}^{\Im}(t)\right)^2-2\left(v_{1\mathbf{k}}^{\Re}(t)v_{0\mathbf{k}}^{\Re}(t)+v_{1\mathbf{k}}^{\Im}(t)v_{0\mathbf{k}}^{\Im}(t)\right) \tag{281}$$
$$=\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2. \tag{282}$$

Rewriting in terms of real and imaginary parts.

$$R_i(t)=\sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{283}$$
$$=\sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \tag{284}$$
$$\langle B_0^+(t)B_1^-(t)\rangle=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{285}$$
$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{2\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{286}$$
$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{287}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}=\frac{\partial\left(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{288}$$
$$=\frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\frac{g_{i\mathbf{k}}^*-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{289}$$
$$=\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \tag{290}$$
$$\frac{\partial|B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)}=\frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{291}$$
$$=-\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^2+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{292}$$
$$=-\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^2}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}|B_{10}(t)|^2, \tag{293}$$
$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}=\frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{294}$$
$$=\frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{295}$$
$$=\frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t))-|V_{10}(t)|^2|B_{10}(t)|^2\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \tag{296}$$

$$= \frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon\,(t) - \varepsilon_i(t) - R_i(t))\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\,|B_{10}(t)\,V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\,(t)} \tag{297}$$

$$= \frac{\varepsilon\,(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon\,(t) - \varepsilon_i\,(t) - R_i\,(t))\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Re}_{i\mathbf{k}}(t)-v^{\Re}_{i'\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\,|B_{10}\,(t)|^2\,|V_{10}\,(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\,(t)} \tag{298}$$

$$= \frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon\,(t) - 4\,(\varepsilon\,(t) - \varepsilon_i(t) - R_i(t)) - \frac{4}{\omega_{\mathbf{k}}}\,|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\,(t)}\right) + \frac{1}{\eta\,(t)}\left(-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon\,(t) + 2\,(\varepsilon\,(t) - \varepsilon_i(t) - R_i(t))\,\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{299}$$

$$\left. +4\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}\,|B_{10}\,(t)|^2\,|V_{10}\,(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{300}$$

From the equation (267) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\,(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}}{\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}} \tag{301}$$

$$= \frac{\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{2g^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}}{\frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(2\,\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right) + 2\,\frac{(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\,\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+2\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}\,|B_{10}(t)|^2\,|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\,\varepsilon(t)}{\eta(t)}}. \tag{302}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-g_{i\mathbf{k}}-g^{*}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}\right)(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t)))+4\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\,|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{303}$$

$$= \frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t)))+4\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\,|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{304}$$

$$= \frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{4|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+4\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\,|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$= \frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))+2\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\,|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{306}$$

Separating (305) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write:

$$\frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2\,(\varepsilon_i(t)+R_i(t))-\varepsilon(t))+2\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\,|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{307}$$

$$v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}} = v^{\Re}_{i\mathbf{k}}(t)\,\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\,(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|V_{10}(t)|^2\,|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\,g^{\Re}_{i\mathbf{k}}\,(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)) \tag{308}$$

$$+2\,\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta\,(t)}\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\,|B_{10}\,(t)|^2\,|V_{10}\,(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{309}$$

$$v^{\Re}_{i\mathbf{k}}(t) = \frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\,(2\varepsilon_i\,(t)+2R_i\,(t)-\varepsilon\,(t))+2\,\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}\,|B_{10}\,(t)|^2\,|V_{10}\,(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\,(t)-2\,(\varepsilon\,(t)-\varepsilon_i\,(t)-R_i\,(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{310}$$

$$v^{\Re}_{i\mathbf{k}}(t) = \frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\,(2\varepsilon_i\,(t)+2R_i\,(t)-\varepsilon\,(t))+2\,\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\,\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}\,|B_{10}\,(t)|^2\,|V_{10}\,(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\,(t)-2\,(\varepsilon\,(t)-\varepsilon_i\,(t)-R_i\,(t))-\frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{311}$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon\left(t\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} = \frac{\partial\left(\varepsilon_{1}\left(t\right)+R_{1}\left(t\right)+\varepsilon_{0}\left(t\right)+R_{0}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{312}$$

$$= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}\left(t\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}\left(t\right)\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{313}$$

$$= 2\frac{v_{i\mathbf{k}}^{\Im}\left(t\right)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{314}$$

$$\frac{\partial\left|B_{10}(t)\right|^{2}}{\partial v_{i\mathbf{k}}^{\Im}(t)} = \frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(\mathrm{v}_{1\mathbf{k}}^{\Re}(t)-\mathrm{v}_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(\mathrm{v}_{1\mathbf{k}}^{\Im}(t)-\mathrm{v}_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)} \tag{315}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{316}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}\left(t\right)-v_{0\mathbf{k}}^{\Im}\left(t\right)\right)}{\partial v_{i\mathbf{k}}^{\Im}\left(t\right)}\left|B_{10}\left(t\right)\right|^{2}, \tag{317}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}\left(t\right)} \tag{318}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} \tag{319}$$

$$= \frac{\varepsilon\left(t\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}(t)\right)\left(\varepsilon_{0}(t)+R_{0}(t)\right)-\left|B_{10}(t)V_{10}(t)\right|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\eta\left(t\right)} \tag{320}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{321}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}(t)-v_{i'\mathbf{k}}^{\Im}(t)\right)\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^{2}}\right)}{\eta\left(t\right)} \tag{322}$$

$$= \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\frac{4\left(\varepsilon_{i}(t)+R_{i}(t)\right)-2\varepsilon(t)-\frac{4\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}}{\eta(t)}+\frac{1}{\eta(t)}\left(2\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Im}(t)\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}^{2}}\right). \tag{323}$$

From the equation (267) and replacing the derivatives obtained we have:

$$\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}} = \tanh\left(\frac{\beta\eta(t)}{2}\right) \tag{324}$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{v_{i\mathbf{k}}^{\Im}(t)\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\right)+\frac{2}{\eta(t)}\left(\frac{g_{i\mathbf{k}}^{*\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}^{*\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. \tag{325}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v^{\Im}_{i\mathbf{k}}(t) - \mathrm{i}\left(g^{*}_{i\mathbf{k}} - g_{i\mathbf{k}}\right)\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \mathrm{i}\left(g^{*}_{i\mathbf{k}} - g_{i\mathbf{k}}\right)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right)\right) + 4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{326}$$

$$= \frac{2\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g^{\Im}_{i\mathbf{k}}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right)\right) + 4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$= \frac{2\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t) - 4\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - 2g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

$$= \frac{\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{329}$$

Separating (329) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Im}_{i\mathbf{k}}(t) - g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) + 2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{330}$$

$$v^{\Im}_{i\mathbf{k}} - g^{\Im}_{i\mathbf{k}} = v^{\Im}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right) \tag{331}$$

$$+ 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{332}$$

$$v^{\Im}_{i\mathbf{k}}(t) = \frac{g^{\Im}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{333}$$

$$v^{\Im}_{i\mathbf{k}}(t) = \frac{g^{\Im}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{334}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t) = v^{\Re}_{i\mathbf{k}}(t) + \mathrm{i}v^{\Im}_{i\mathbf{k}}(t) \tag{335}$$

$$= \frac{g^{\Re}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{336}$$

$$+ \mathrm{i}\frac{g^{\Im}_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{337}$$

$$= \frac{g_{i\mathbf{k}}\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2|V_{10}(t)|^2|B_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{338}$$

Let's obtain the explicit form of $v_{0\mathbf{k}}(\omega_{\mathbf{k}}, t)$ and $v_{1\mathbf{k}}(\omega_{\mathbf{k}}, t)$, at first we have:

$$a_i\left(\omega_{\mathbf{k}},t\right)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{339}$$

$$b_i\left(\omega_{\mathbf{k}},t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}.\tag{340}$$

So the equation (335) written in explicit form is:

$$v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right),\tag{341}$$

$$v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)=g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right).\tag{342}$$

This system of equations has the following solutions:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{343}$$

$$=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+\left(g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{344}$$

$$=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)+v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{345}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)(1-b_1(\omega_{\mathbf{k}},t)b_0(\omega_{\mathbf{k}},t))=g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)\tag{346}$$

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=\frac{g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{347}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+\frac{g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)+g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)b_0\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}b_1\left(\omega_{\mathbf{k}},t\right)\tag{348}$$

$$=\frac{g_1\left(\omega_{\mathbf{k}}\right)a_1\left(\omega_{\mathbf{k}},t\right)+g_0\left(\omega_{\mathbf{k}}\right)a_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{349}$$

For a shorter representation let's define:

$$r_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_i\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{350}$$

$$s_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_{(i+1)\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)b_{i\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{351}$$

So the variational parameters are given by:

$$\begin{pmatrix}v_{0\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\\ v_{1\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\end{pmatrix}\equiv\begin{pmatrix}r_0\left(\omega_{\mathbf{k}},t\right) & s_0\left(\omega_{\mathbf{k}},t\right)\\ r_1\left(\omega_{\mathbf{k}},t\right) & s_1\left(\omega_{\mathbf{k}},t\right)\end{pmatrix}\begin{pmatrix}g_0\left(\omega_{\mathbf{k}}\right)\\ g_1\left(\omega_{\mathbf{k}}\right)\end{pmatrix}.\tag{352}$$

Given that $v_{i\mathbf{k}}\left(\omega_{\mathbf{k}},t\right)\equiv g_i\left(\omega_{\mathbf{k}}\right)F_i\left(\omega_{\mathbf{k}},t\right)$ then we can write:

$$F_0\left(\omega_{\mathbf{k}},t\right)=r_0\left(\omega_{\mathbf{k}},t\right)+\frac{g_1\left(\omega_{\mathbf{k}}\right)}{g_0\left(\omega_{\mathbf{k}}\right)}s_0\left(\omega_{\mathbf{k}},t\right)\tag{353}$$

$$F_1\left(\omega_{\mathbf{k}},t\right)=\frac{g_0\left(\omega_{\mathbf{k}}\right)}{g_1\left(\omega_{\mathbf{k}}\right)}r_1\left(\omega_{\mathbf{k}},t\right)+s_1\left(\omega_{\mathbf{k}},t\right)\tag{354}$$

## III. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}$, where $\rho_B^{\mathrm{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$
\begin{aligned}
\overline{\rho_T(0)} &\equiv e^{V(0)}\rho_T(0)\,e^{-V(0)} && (355)\\
&= \left(|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)\right)(\rho_S(0)\otimes\rho_B)\left(|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)\right), && (356)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|: \quad & |0\rangle\langle 0|0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (357)\\
&= |0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (358)\\
&= |0\rangle\langle 0| \otimes B_0^+(0)\,\rho_B B_0^-(0)\,, && (359)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|: \quad & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (360)\\
&= |1\rangle\langle 1|B_1^+(0)\,\rho_B B_1^-(0) && (361)\\
&= |1\rangle\langle 1| \otimes B_1^+(0)\,\rho_B B_1^-(0)\,, && (362)\\
\text{for } \rho_S(0) = |0\rangle\langle 1|: \quad & |0\rangle\langle 0|B_0^+(0)\,|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (363)\\
&= |0\rangle\langle 1|B_0^+(0)\,\rho_B|1\rangle\langle 1|B_1^-(0) && (364)\\
&= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0)\,\rho_B B_1^-(0) && (365)\\
&= |0\rangle\langle 1| \otimes B_0^+(0)\,\rho_B B_1^-(0)\,, && (366)\\
\text{for } \rho_S(0) = |1\rangle\langle 0|: \quad & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (367)\\
&= |1\rangle\langle 0| \otimes B_1^+(0)\,\rho_B B_0^-(0)\,. && (368)
\end{aligned}
$$

We transform any operator $O$ into the interaction picture in the following way:

$$
\begin{aligned}
\widetilde{O}(t) &\equiv U^\dagger(t)\,O(t)\,U(t)\,, && (369)\\
U(t) &\equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\overline{H_{\bar{S}}}(t')\right). && (370)
\end{aligned}
$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$
\begin{aligned}
\widetilde{\overline{\rho_S}}(t) &= U^\dagger(t)\,\overline{\rho_S}(t)\,U(t)\,, \text{ where} && (371)\\
\overline{\rho_S}(t) &= \mathrm{Tr}_B\left(\bar{\rho}_T(t)\right). && (372)
\end{aligned}
$$

In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$
\begin{pmatrix} A(t)\\ B(t)\\ C(t)\end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2}\\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t)\\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \qquad (373)
$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (**??**):

$$
\begin{aligned}
\overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) && (374)\\
&= B_{0z}(t)\,|0\rangle\langle 0| + B_{1z}(t)\,|1\rangle\langle 1| + V_{10}^{\Re}(t)\,\sigma_x B_x(t) + V_{10}^{\Re}(t)\,\sigma_y B_y(t) + V_{10}^{\Im}(t)\,\sigma_x B_y(t) - V_{10}^{\Im}(t)\,\sigma_y B_x(t) && (375)\\
&= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right). && (376)
\end{aligned}
$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{377}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0)\,\text{-i}\int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]\mathrm{d}t'. \tag{378}$$

Replacing the equation (378) in the equation (377) gives us:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]]\mathrm{d}t'. \tag{379}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 ...\int_0^{t_{n-1}}\mathrm{d}t_n[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdot\cdot]. \tag{380}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 ...\int_0^{t_{n-1}}\mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]] \cdot\cdot]. \tag{381}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\overline{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\,\overline{\rho_S}(0) \tag{382}$$
$$= W(t)\,\overline{\rho_S}(0). \tag{383}$$

in this case

$$W_n(t) = (-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 ...\int_0^{t_{n-1}}\mathrm{d}t_n \mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]] \cdot\cdot]. \tag{384}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{385}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}W(t)\,\overline{\rho_S}(0) \tag{386}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}\,\widetilde{\overline{\rho_S}}(t). \tag{387}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (385) becomes:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\,\widetilde{\overline{\rho_S}}(t) \tag{388}$$
$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]. \tag{389}$$

Replacing $t_1 \to t-\tau$

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right]-\int_0^t \mathrm{d}\tau \mathrm{Tr}_B\left[\overline{H_{\bar{I}}}\left(t\right),\left[\widetilde{\overline{H_{\bar{I}}}}\left(-\tau\right),\overline{\rho_S}\left(t\right)\rho_B\right]\right]. \tag{390}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}\left(t\right)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}\left(t\right)=U^{\dagger}\left(t\right)e^{\mathrm{i}H_Bt}\overline{H_{\bar{I}}}\left(t\right)e^{-\mathrm{i}H_Bt}U\left(t\right). \tag{391}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}\left(t\right)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\begin{aligned}
\widetilde{\overline{H_{\bar{I}}}}\left(t\right)&=\sum_i C_i\left(t\right)\left(\widetilde{A_i}\left(t\right)\otimes\widetilde{B_i}\left(t\right)\right), &(392)\\
\widetilde{A_i}\left(t\right)&=U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}A_i\mathrm{e}^{-\mathrm{i}H_Bt}U\left(t\right) &(393)\\
&=U^{\dagger}\left(t\right)A_iU\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}\mathrm{e}^{-\mathrm{i}H_Bt} &(394)\\
&=U^{\dagger}\left(t\right)A_iU\left(t\right)\mathbb{I} &(395)\\
&=U^{\dagger}\left(t\right)A_iU\left(t\right), &(396)\\
\widetilde{B_i}\left(t\right)&=U^{\dagger}\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt}U\left(t\right) &(397)\\
&=U^{\dagger}\left(t\right)U\left(t\right)\mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt} &(398)\\
&=\mathbb{I}\mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt} &(399)\\
&=\mathrm{e}^{\mathrm{i}H_Bt}B_i\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt}. &(400)
\end{aligned}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}\left(t\right),H_B\right]=0$ because these operators belong to different Hilbert spaces, so $\left[U\left(t\right),\mathrm{e}^{\mathrm{i}H_Bt}\right]=0$.
Using the expression (392) to replace it in the equation (389)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t}&=-\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}\left(t\right),\left[\widetilde{\overline{H_{\bar{I}}}}\left(t'\right),\widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' &(401)\\
&=-\int_0^t \mathrm{Tr}_B\left[\sum_j C_j\left(t\right)\left(\widetilde{A_j}\left(t\right)\otimes\widetilde{B_j}\left(t\right)\right),\left[\sum_i C_i\left(t'\right)\left(\widetilde{A_i}\left(t'\right)\otimes\widetilde{B_i}\left(t'\right)\right),\widetilde{\overline{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' &(402)\\
&=-\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right),\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_S}}(t)\rho_B-\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\right]\mathrm{d}t' &(403)\\
&=-\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_S}}(t)\rho_B-\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right) &(404)\\
&\quad-\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)+\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\Big)\mathrm{d}t'. &(405)
\end{aligned}$$

In order to calculate the correlation functions we define:

$$\mathcal{B}_{ij}\left(t,t'\right)=\mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(t'\right)\rho_B\right). \tag{406}$$

An useful property is

$$\begin{aligned}
\mathcal{B}^*_{ji}\left(t,t'\right)&=\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right)^{\dagger} &(407)\\
&=\mathrm{Tr}_B\left(\rho_B^{\dagger}\widetilde{B_i}^{\dagger}\left(t'\right)\widetilde{B_j}^{\dagger}\left(t\right)\right) &(408)\\
&=\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) &(409)\\
&=\mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) &(410)\\
&=\mathcal{B}_{ij}\left(t',t\right). &(411)
\end{aligned}$$

The correlation functions relevant that appear in the equation (405) are:

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) = \left\langle \widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\right\rangle_B \tag{412}$$

$$= \mathcal{B}_{ji}\left(t,t'\right) \tag{413}$$

$$= \mathcal{B}^*_{ij}\left(t',t\right) \tag{414}$$

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B\widetilde{B_i}\left(t'\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{415}$$

$$= \mathcal{B}_{ij}\left(t',t\right) \tag{416}$$

$$\mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\rho_B\widetilde{B_j}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) \tag{417}$$

$$= \mathcal{B}^*_{ij}\left(t',t\right) \tag{418}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{419}$$

$$= \mathcal{B}_{ij}\left(t',t\right) \tag{420}$$

The cyclic property of the trace was use widely in the development of equations (412) and (420). Replacing in (405)

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big) \tag{421}$$

$$-\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}t'. \tag{422}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\Big) \tag{423}$$

$$+\sum_{ij} C_i(t')C_j(t)\Big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' \tag{424}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\Big) \tag{425}$$

$$+\sum_{ij} C_i(t')C_j(t)\Big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' \tag{426}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t')\Big) \text{ (by permuting i and j because i,j}\in\text{J)} \tag{427}$$

$$+\sum_{ij} C_i(t')C_j(t)\Big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' \tag{428}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}(t')\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{B_j}(t)\rho_B\widetilde{B_i}(t') \tag{429}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\rho_B\widetilde{B_i}(t')\widetilde{B_j}(t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}(t')\rho_B\widetilde{B_j}(t)\Big)\Big)\mathrm{d}t' \tag{430}$$

$$= -\int_0^t \Big(\sum_{ij} C_j(t)C_i(t')\Big(\widetilde{A_j}(t)\widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\mathcal{B}_{ji}(t,t') - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\mathcal{B}_{ij}(t',t) \tag{431}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}(t')\widetilde{A_j}(t)\mathcal{B}_{ij}(t',t) - \widetilde{A_i}(t')\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\mathcal{B}_{ji}(t,t')\Big)\Big)\mathrm{d}t' \tag{432}$$

$$= -\int_0^t \left(\sum_{ij} C_j\left(t\right)C_i\left(t'\right)\left(\mathcal{B}_{ji}\left(t,t'\right)\left[\widetilde{A_j}\left(t\right),\widetilde{A_i}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \mathcal{B}_{ij}\left(t',t\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t'\right),\widetilde{A_j}\left(t\right)\right]\right)\right)\mathrm{d}t' \tag{433}$$

$$= -\int_0^t \Big(\sum_{ij} C_i(t)C_j(t')\Big(\mathcal{B}_{ij}(t,t')\big[\widetilde{A_i}(t),\widetilde{A_j}(t')\widetilde{\widetilde{\rho_S}}(t)\big] + \mathcal{B}_{ji}(t',t)\big[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t'),\widetilde{A_i}(t)\big]\Big)\Big)\mathrm{d}t' \text{ (exchanging i and j)} \tag{434}$$

$$= -\int_0^t \left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}t' \tag{435}$$

$$= -\int_0^t \left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] - \mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right)\right]\right)\right)\mathrm{d}t' \tag{436}$$

We could identify the following commutators in the equation deduced:

$$\mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right) - \mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right) = \mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho}}_s\left(t\right)\right], \tag{437}$$

$$\mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right) - \mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right) = \mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]. \tag{438}$$

Returning to the Schroedinger picture we have:

$$U(t)\,\widetilde{A}_i(t)\,\widetilde{A}_j(t')\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) = U(t)\,\widetilde{A}_i(t)\,U^\dagger(t)\,U(t)\,\widetilde{A}_j(t')\,U^\dagger(t)\,U(t)\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t), \tag{439}$$

$$= \left(U(t)\,\widetilde{A}_i(t)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{A}_j(t')\,U^\dagger(t)\right)\left(U(t)\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t)\right), \tag{440}$$

$$= A_i(t)\,\widetilde{A}_j(t',t)\,\overline{\rho_S}(t)\,. \tag{441}$$

This procedure applying to the relevant commutators give us:

$$U(t)\left[\widetilde{A}_i(t),\widetilde{A}_j(t')\,\widetilde{\overline{\rho_S}}(t)\right]U^\dagger(t) = \left(U(t)\,\widetilde{A}_i(t)\,\widetilde{A}_j(t')\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) - U(t)\,\widetilde{A}_j(t')\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A}_i(t)\,U^\dagger(t)\right) \tag{442}$$

$$= A_i(t)\,\widetilde{A}_j(t',t)\,\overline{\rho_S}(t) - \widetilde{A}_j(t',t)\,\overline{\rho_S}(t)\,A_i \tag{443}$$

$$= \left[A_i(t),\widetilde{A}_j(t',t)\,\overline{\rho_S}(t)\right]. \tag{444}$$

Introducing this transformed commutators in the equation (436) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathcal{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}s C_i(t)\,C_j(t')\left(\mathcal{B}_{ij}(t,t')\left[A_i(t),\widetilde{A}_j(t',t)\,\overline{\rho_S}(t)\right] + \mathcal{B}^*_{ij}(t,t')\left[\overline{\rho_S}(t)\widetilde{A}_j(t',t),A_i\right]\right), \tag{445}$$

$$t' = t-\tau \text{ (Change of variables in the integration process)}, \tag{446}$$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)\,C_j(t')\left(\mathcal{B}_{ij}(t,t')\left[A_i(t),\widetilde{A}_j(t',t)\overline{\rho_S}(t)\right] + \mathcal{B}^*_{ij}(t,t')\left[\overline{\rho_S}(t)\widetilde{A}_j(t',t),A_i(t)\right]\right). \tag{447}$$

where $i,j \in \{1,2,3,4,5.6\}$ and $t' = t-\tau$.
Here $\widetilde{A}_j(t-\tau,t) = U(t)\,U^\dagger(t-\tau)\,A_j(t)\,U(t-\tau)\,U^\dagger(t)$ where $U(t)$ is given by (370). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. In order to write in a simplified way we define the following notation:

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B\left(\widetilde{B}_i(t)\,\widetilde{B}_j(t')\,\rho_B\right) \tag{448}$$

$$= \mathrm{Tr}_B\left(e^{iH_Bt}B_i(t)\,e^{-iH_Bt}e^{iH_Bt'}B_j(t')\,e^{-iH_Bt'}\rho_B\right) \tag{449}$$

$$e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!} \tag{450}$$

$$e^{-iH_Bt'}e^{-\beta H_B} = \sum_{m=0}^{\infty}\frac{(-iH_Bt')^m}{m!}\sum_{n=0}^{\infty}\frac{(-\beta H_B)^n}{n!} \tag{451}$$

$$= \sum_{m,n}\frac{(-iH_Bt')^m}{m!}\frac{(-\beta H_B)^n}{n!} \tag{452}$$

$$= \sum_{m,n}\frac{(-it')^m}{m!}\frac{(-\beta)^n}{n!}H_B^mH_B^n \tag{453}$$

$$= \sum_{m,n}\frac{(-it')^m}{m!}\frac{(-\beta)^n}{n!}H_B^nH_B^m \text{ (because the powers of a matrix commute)} \tag{454}$$

$$= \sum_{m,n}\frac{(-\beta)^n}{n!}H_B^n\frac{(-it')^m}{m!}H_B^m \tag{455}$$

$$= \sum_{m,n}\frac{(-\beta H_B)^n}{n!}\frac{(-it'H_B)^m}{m!} \tag{456}$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_B t')^m}{m!} \tag{457}$$

$$= e^{-\beta H_B} e^{-iH_B t'} \tag{458}$$

$$0 = e^{-iH_B t'} e^{-\beta H_B} - e^{-\beta H_B} e^{-iH_B t'} \text{ (then } e^{-iH_B t'} \text{ and } \rho_B \text{ commute)} \tag{459}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left( e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \rho_B e^{-iH_B t'} \right) \text{ (by permuting } e^{-iH_B t'} \text{ and } \rho_B\text{)} \tag{460}$$

$$= \mathrm{Tr}_B \left( \left( e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \right) \rho_B e^{-iH_B t'} \right) \text{ (by associative property)} \tag{461}$$

$$= \mathrm{Tr}_B \left( e^{-iH_B t'} \left( e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B t'} B_j(t') \right) \rho_B \right) \text{ (by cyclic property of the trace)} \tag{462}$$

$$= \mathrm{Tr}_B \left( \left( e^{-iH_B t'} e^{iH_B t} \right) B_i(t) \left( e^{-iH_B t} e^{iH_B t'} \right) B_j(t') \rho_B \right) \text{ (by associative property)} \tag{463}$$

$$[iH_B t, -iH_B t'] = iH_B t(-iH_B t') - (-iH_B t') iH_B t \tag{464}$$

$$= tt' H_B^2 - tt' H_B^2 \tag{465}$$

$$= 0 \text{ (so } iH_B t \text{ and } -iH_B t' \text{ commute)} \tag{466}$$

$$e^{-iH_B t'} e^{iH_B t} = e^{iH_B t - iH_B t'} \text{ (by the Zassenhaus formula because } iH_B t \text{ and } -iH_B t' \text{ commute)} \tag{467}$$

$$= e^{iH_B (t-t')} \tag{468}$$

$$= e^{iH_B \tau} \tag{469}$$

$$e^{iH_B t'} e^{-iH_B t} = e^{-iH_B t + iH_B t'} \text{ (by the Zassenhaus formula because } -iH_B t \text{ and } iH_B t' \text{ commute)} \tag{470}$$

$$= e^{iH_B (-t+t')} \tag{471}$$

$$= e^{-iH_B \tau} \tag{472}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left( e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(t') \rho_B \right) \tag{473}$$

$$B_i(t,\tau) \equiv e^{iH_B \tau} B_i(t) e^{-iH_B \tau} \tag{474}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left( e^{iH_B (t-t')} B_i(t) e^{-iH_B (t-t')} B_j(t') \rho_B \right) \tag{475}$$

$$t' = t - \tau \tag{476}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left( e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(t') \rho_B \right) \tag{477}$$

$$= \mathrm{Tr}_B \left( B_i(t,\tau) B_j(t',0) \rho_B \right) \tag{478}$$

For the following results $i, j \in \{3, 6\}$, calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \mathrm{Tr}_B \left( B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \rho_B \right) \tag{479}$$

$$= \int \mathrm{d}^2\alpha P(\alpha) \langle \alpha | B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) | \alpha \rangle \tag{480}$$

$$= \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \langle \alpha | B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) | \alpha \rangle \, \mathrm{d}^2\alpha, \tag{481}$$

$$q_{j\mathbf{k}}(t) = g_{j\mathrm{mod}2\mathbf{k}} - v_{j\mathrm{mod}2\mathbf{k}}(t) \tag{482}$$

$$B_{j\mathrm{mod}2z}(t,\tau) = \sum_{\mathbf{k}} \left( q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), \tag{483}$$

$$B_{j\mathrm{mod}2z}(t',0) = \sum_{\mathbf{k}'} \left( q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right), \tag{484}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \mathrm{Tr}_B \left( B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \rho_B \right) \tag{485}$$

$$= \mathrm{Tr}_B \left( \sum_{\mathbf{k}} \left( q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left( q_{j\mathbf{k}'}(t') b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(t') b_{\mathbf{k}'} \right) \rho_B \right) \tag{486}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\,(t)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\,(t)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j\mathbf{k}'}\,(t')\,b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\,(t')b_{\mathbf{k}'}\right)\rho_B\right) \tag{487}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\,(t)\,b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\,(t)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\,(t')\,b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\,(t')b_{\mathbf{k}'}\right)\rho_B\right) \tag{488}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\,(t)\,b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\,(t)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\,(t')b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\,(t')b_{\mathbf{k}}\right)\rho_B\right), \tag{489}$$

$$\left\langle\widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\,(t)\,b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\,(t)\,b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\,(t')\,b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\,(t')\,b_{\mathbf{k}'}\right)\rho_B\right) \tag{490}$$

$$+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\,(t)\,b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\,(t)\,b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\,(t')\,b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\,(t')\,b_{\mathbf{k}}\right)\rho_B\right) \tag{491}$$

$$0=\mathrm{Tr}_B\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\,(t)\,b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\,(t)\,b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\,(t')\,b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\,(t')\,b_{\mathbf{k}'}\right)\rho_B\right) \tag{492}$$

$$\left\langle\widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_B=0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\,(t)\,b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\,(t)\,b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\,(t')\,b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\,(t')\,b_{\mathbf{k}}\right)\rho_B\right) \tag{493}$$

$$=\mathrm{Tr}_B\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\left(b_{\mathbf{k}}^{\dagger}\right)^2\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{2}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B \tag{494}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\,b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{495}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right)+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right) \tag{496}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle+q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{497}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{498}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{499}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{500}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{501}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{502}$$

$$+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{503}$$

$$=\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right. \tag{504}$$

$$\left.+q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{505}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right. \tag{506}$$

$$\left.+q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right|0\right\rangle+q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{507}$$

$$=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\sum_{\mathbf{k}}\left(\left\langle 0\left|q_{j\mathbf{k}}(t)\,q_{j\mathbf{k}}^{*}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|\alpha_{\mathbf{k}}|^2\right)+q_{j\mathbf{k}}^{*}(t)\,q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\right)\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{508}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}\,(t)\,q_{j\mathbf{k}}^{*}\,(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle+\left\langle 0\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{509}$$

$$+\sum_{\mathbf{k}}\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}q_{j\mathbf{k}}^{*}\,(t)\,q_{j\mathbf{k}}\,(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\left\langle 0\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|0\right\rangle+\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{510}$$

$$1=\frac{1}{\pi N}\int\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{511}$$

$$b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left|0\right\rangle=0, \tag{512}$$

$$b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\left|0\right\rangle=\left|0\right\rangle, \tag{513}$$

$$\left\langle \widetilde{B_{j\text{mod}2z}}(t)\widetilde{B_{j\text{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}} \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left(\left(q_{j\mathbf{k}}(t)\, q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\rangle \tag{514}$$

$$+q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|0\right\rangle\right)\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{515}$$

$$= \frac{1}{\pi N}\int\sum_{\mathbf{k}}\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(\left(q_{j\mathbf{k}}(t)q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)|\alpha_{\mathbf{k}}|^2 + q^*_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{516}$$

$$\int_0^{2\pi}\int_0^{+\infty} r^2\mathrm{e}^{-\frac{r^2}{N}}r\mathrm{d}r\mathrm{d}\theta = \int |\alpha_{\mathbf{k}}|^2\mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\mathrm{d}^2\alpha_{\mathbf{k}} \tag{517}$$

$$= \pi N^2 \tag{518}$$

$$\left\langle \widetilde{B_{j\text{mod}2z}}(t)\widetilde{B_{j\text{mod}2z}}(t')\right\rangle_B = \sum_{\mathbf{k}}\left(\left(q_{j\mathbf{k}}(t)\, q^*_{j\mathbf{k}}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)N + q^*_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{519}$$

$$\left\langle \widetilde{B_{j\text{mod}2z}}(t)\widetilde{B_{j'\text{mod}2z}}(t')\right\rangle_B = \mathrm{Tr}_B\left(B_{j\text{mod}2z}(t,\tau)\, B_{j'\text{mod}2z}(t',0)\,\rho_B\right) \tag{520}$$

$$= \int \mathrm{d}^2\alpha P(\alpha)\left\langle\alpha\left|B_{j\text{mod}2z}(t,\tau)\, B_{j'\text{mod}2z}(t',0)\right|\alpha\right\rangle \tag{521}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\left|B_{j\text{mod}2z}(t,\tau)\, B_{j'\text{mod}2z}(t',0)\right|\alpha\right\rangle\mathrm{d}^2\alpha \tag{522}$$

$$= \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}'} + q^*_{j'\mathbf{k}'}(t')\, b_{\mathbf{k}'}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{523}$$

$$= \langle\alpha|\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}'} + q^*_{j'\mathbf{k}'}(t')b_{\mathbf{k}'}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{524}$$

$$+ \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}'}(t')\, b^\dagger_{\mathbf{k}} + q^*_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{525}$$

$$= \langle\alpha|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left(q_{j\mathbf{k}}(t)\, b^\dagger_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q^*_{j\mathbf{k}}(t)\, b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j'\mathbf{k}}(t')\, b^\dagger_{\mathbf{k}} + q^*_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}\right)|\alpha\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{526}$$

$$= \left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\, b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{527}$$

$$+ \left\langle\alpha\left|\sum_{\mathbf{k}}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right|\alpha\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{528}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{529}$$

$$+ \sum_{\mathbf{k}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle\alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{530}$$

$$\left\langle b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right\rangle_B = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\, D(-\alpha_{\mathbf{k}})\, b^\dagger_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{531}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{532}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + \alpha_{\mathbf{k}}b^\dagger_{\mathbf{k}} + b_{\mathbf{k}}\alpha^*_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{533}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{534}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\rangle\,\mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|b_{\mathbf{k}}b^\dagger_{\mathbf{k}}\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{535}$$

$$= N+1, \tag{536}$$

$$\left\langle b^\dagger_{\mathbf{k}}b_{\mathbf{k}}\right\rangle_B = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|\left(b^\dagger_{\mathbf{k}}+\alpha^*_{\mathbf{k}}\right)(b_{\mathbf{k}}+\alpha_{\mathbf{k}})\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{537}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}|\alpha_{\mathbf{k}}|^2\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{538}$$

$$= N, \tag{539}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j'\mathrm{mod}2z}}\left(t'\right)\right\rangle_B = \sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)q^*_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N + q^*_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N+1\right)\right) \tag{540}$$

$$= \sum_{\mathbf{k}} 2N\left(q_{j\mathbf{k}}\left(t\right)q^*_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Re} + \sum_{\mathbf{k}} q^*_{j\mathbf{k}}\left(t\right)q_{j'\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{541}$$

$$D\left(h'\right)D\left(h\right) = \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}D\left(h'+h\right), \tag{542}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \mathrm{Tr}_B\left(\mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}D\left(h'+h\right)\rho_B\right) \tag{543}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\mathrm{Tr}_B\left(D\left(h'+h\right)\rho_B\right) \tag{544}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\frac{1}{\pi N}\int \mathrm{d}^2\alpha P\left(\alpha\right)\left\langle\alpha\left|D\left(h'+h\right)\right|\alpha\right\rangle \tag{545}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\mathrm{e}^{-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{546}$$

$$h' = h\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{547}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_B = \mathrm{e}^{\frac{1}{2}\left(hh^*\mathrm{e}^{\mathrm{i}\omega\tau}-h^*h\mathrm{e}^{-\mathrm{i}\omega\tau}\right)}\mathrm{e}^{-\frac{|h+h\mathrm{e}^{\mathrm{i}\omega\tau}|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{548}$$

$$\frac{1}{2}\left|h\right|^2\left(\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\right) = \frac{1}{2}\left(hh^*\mathrm{e}^{\mathrm{i}\omega\tau}-h^*h\mathrm{e}^{-\mathrm{i}\omega\tau}\right) \tag{549}$$

$$= \frac{1}{2}\left|h\right|^2\left(\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)-\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{550}$$

$$= \frac{1}{2}\left|h\right|^2\left(2\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{551}$$

$$= \mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right), \tag{552}$$

$$-\frac{|h+h\mathrm{e}^{\mathrm{i}\omega\tau}|^2}{2} = -\left|h\right|^2\frac{|1+\mathrm{e}^{\mathrm{i}\omega\tau}|^2}{2} \tag{553}$$

$$= -\left|h\right|^2\frac{\left(1+2\cos\left(\omega\tau\right)+\cos^2\left(\omega\tau\right)\right)+\sin^2\left(\omega\tau\right)}{2} \tag{554}$$

$$= -\left|h\right|^2\frac{2+2\cos\left(\omega\tau\right)}{2} \tag{555}$$

$$= -\left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right), \tag{556}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(h\right)\right\rangle_B = \mathrm{e}^{\mathrm{i}|h|^2\sin(\omega\tau)}\mathrm{e}^{-|h|^2(1+\cos(\omega\tau))\coth\left(\frac{\beta\omega}{2}\right)} \tag{557}$$

$$= \mathrm{e}^{\mathrm{i}|\mathrm{h}|^2\sin(\omega\tau)-|h|^2(1+\cos(\omega\tau))\coth\left(\frac{\beta\omega}{2}\right)} \tag{558}$$

$$= \mathrm{e}^{-|h|^2\left(-\mathrm{i}\sin(\omega\tau)+\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)}\mathrm{e}^{-|h|^2\coth\left(\frac{\beta\omega}{2}\right)} \tag{559}$$

$$= \left\langle D\left(h\right)\right\rangle_B\mathrm{e}^{-\phi(\tau)}, \tag{560}$$

$$\mathrm{e}^{-\phi(\tau)} = \mathrm{e}^{-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}, \tag{561}$$

$$\phi\left(\tau\right) = \left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right), \tag{562}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\mathrm{e}^{-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{563}$$

$$h' = v\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{564}$$

$$m_{\mathbf{k}}\left(t\right) = \frac{v_{1\mathbf{k}}\left(t\right)-v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}, \tag{565}$$

$$\Gamma_{\mathbf{k}}\left(t\right) = \frac{v^*_{1\mathbf{k}}\left(t\right)v_{0\mathbf{k}}\left(t\right)-v_{1\mathbf{k}}\left(t\right)v^*_{0\mathbf{k}}\left(t\right)}{\omega^2_{\mathbf{k}}} \tag{566}$$

$$\left\langle \widetilde{B_1^+B_0^-}\left(t\right)\widetilde{B_1^+B_0^-}\left(t'\right)\right\rangle_B = \left\langle B_1^+B_0^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)\right\rangle_B \tag{567}$$

$$= \left\langle B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\right\rangle_B \tag{568}$$

$$= \mathrm{Tr}_B\left(B_{10}\left(t,\tau\right)B_{10}\left(t',0\right)\rho_B\right) \tag{569}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t')\right)\right)\rho_B\right) \tag{570}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(m_{\mathbf{k}}(t')\right)\right)\rho_B\right) \tag{571}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')-\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}\right)^*m_{\mathbf{k}}(t')\right)-\frac{|m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}+m_{\mathbf{k}}(t')|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{572}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')\right)^{\Im}-\frac{|\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\mathrm{e}^{\mathrm{i}\omega\tau}+\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{573}$$

$$=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')\right)^{\Im}-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{574}$$

$$\left\langle\widetilde{B_0^+B_1^-}(t)\widetilde{B_0^+B_1^-}(t')\right\rangle_B=\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\left(\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m^*_{\mathbf{k}}(t')\right)^{\Im}-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{575}$$

$$\langle D(h)\,b\rangle_B=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle\alpha\,|D(h)\,b|\,\alpha\rangle \tag{576}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle\alpha\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,\alpha\rangle \tag{577}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle 0\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,0\rangle \tag{578}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha)|\,0\rangle \tag{579}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,(b+\alpha)|\,0\rangle \tag{580}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,(b+\alpha)|\,0\rangle \tag{581}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,b|\,0\rangle+\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,\alpha|\,0\rangle \tag{582}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,\alpha|\,0\rangle \tag{583}$$

$$=\frac{1}{\pi N}\int\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{584}$$

$$=hN\,\langle D(h)\rangle_B\,, \tag{585}$$

$$\left\langle D(h)\,b^\dagger\right\rangle_B=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle\alpha\,\middle|D(h)\,b^\dagger\middle|\,\alpha\right\rangle \tag{586}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\,\middle|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \tag{587}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\,\middle|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \tag{588}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\,\middle|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \tag{589}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\,\middle|D(-\alpha)\,D(h)\,D(\alpha)\left(b^\dagger+\alpha^*\right)\middle|\,0\right\rangle \tag{590}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\,\middle|D(h)\left(b^\dagger+\alpha^*\right)\middle|\,0\right\rangle \tag{591}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\,\middle|D(h)\,b^\dagger\middle|\,0\right\rangle+\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,\alpha^*|\,0\rangle \tag{592}$$

$$=\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)|\,1\rangle+\frac{1}{\pi N}\int\mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\langle 0\,|D(h)|\,0\rangle \tag{593}$$

$$=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h|1\right\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0\left|D\left(h\right)\right|0\right\rangle, \tag{594}$$

$$\left\langle -h\right| =\mathrm{e}^{-\frac{|-h^*|^2}{2}}\sum_n\frac{\left(-h^*\right)^n}{\sqrt{n!}}\left\langle n\right| \tag{595}$$

$$\left\langle -h|1\right\rangle =\mathrm{e}^{-\frac{|-h^*|^2}{2}}\sum_n\frac{\left(-h^*\right)^n}{\sqrt{n!}}\left\langle n|1\right\rangle \tag{596}$$

$$\left\langle -h|1\right\rangle =\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right), \tag{597}$$

$$\left\langle D\left(h\right)b^\dagger\right\rangle_B=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right)+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|-h^*|^2}{2}} \tag{598}$$

$$\left\langle D\left(h\right)b^\dagger\right\rangle_B=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right)+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|-h^*|^2}{2}} \tag{599}$$

$$=-h^*\left\langle D\left(h\right)\right\rangle_B\left(N+1\right), \tag{600}$$

$$\left\langle bD\left(h\right)\right\rangle_B=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle \alpha\left|bD\left(h\right)\right|\alpha\right\rangle \tag{601}$$

$$=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}} \tag{602}$$

$$=h\left\langle D\left(h\right)\right\rangle_B\left(N+1\right), \tag{603}$$

$$\left\langle b^\dagger D\left(h\right)\right\rangle_B=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle \alpha\left|b^\dagger D\left(h\right)\right|\alpha\right\rangle \tag{604}$$

$$=\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h+\frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}} \tag{605}$$

$$=-h^*\left\langle D\left(h\right)\right\rangle_B N. \tag{606}$$

The correlation functions can be found readily as:

$$B_1^+B_0^-\left(t,\tau\right)=\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), \tag{607}$$

$$B_0^+B_1^-\left(t,\tau\right)=\prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), \tag{608}$$

$$B_{10}\left(t\right)=\mathrm{e}^{\chi_{10}(t)}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), \tag{609}$$

$$B_x\left(t,\tau\right)=\frac{B_1^+B_0^-\left(t,\tau\right)+B_0^+B_1^-\left(t,\tau\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}, \tag{610}$$

$$B_y\left(t,\tau\right)=\frac{B_0^+B_1^-\left(t,\tau\right)-B_1^+B_0^-\left(t,\tau\right)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}, \tag{611}$$

$$B_{i\mathrm{mod}2z}\left(t,\tau\right)=\sum_{\mathbf{k}}\left(q_{i\mathbf{k}}\left(t\right)b_{\mathbf{k}}^\dagger\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{i\mathbf{k}}^*\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{612}$$

$$\left\langle \widetilde{B_{i\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j\mathrm{mod}2z}}\left(t'\right)\right\rangle_B=\left\langle B_{i\mathrm{mod}2z}\left(t,\tau\right)B_{j\mathrm{mod}2z}\left(t',0\right)\right\rangle_B \tag{613}$$

$$=\left\langle\sum_{\mathbf{k}}\left(q_{i\mathbf{k}}\left(t\right)b_{\mathbf{k}}^\dagger\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{i\mathbf{k}}^*\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^\dagger+q_{j\mathbf{k}}^*\left(t'\right)b_{\mathbf{k}}\right)\right\rangle_B \tag{614}$$

$$=\sum_{\mathbf{k}}q_{i\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^*\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}q_{i\mathbf{k}}^*\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{615}$$

$$\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_x}\left(t'\right)\right\rangle_B=\left\langle B_x\left(t,\tau\right)B_x\left(t',0\right)\right\rangle_B \tag{616}$$

$$= \left\langle \left( \frac{B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left( \frac{B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t',0) - B_{10}(t') - B_{01}(t')}{2} \right) \right\rangle_B \tag{617}$$

$$= \frac{1}{4} \left\langle \left( B_1^+ B_0^- (t,\tau) + B_0^+ B_1^- (t,\tau) - B_{10}(t) - B_{01}(t) \right) \left( B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t',0) - B_{10}(t') - B_{01}(t') \right) \right\rangle_B \tag{618}$$

$$= \frac{1}{4} \Big\langle B_1^+ B_0^- (t,\tau)\, B_1^+ B_0^- (t',0) + B_1^+ B_0^- (t,\tau)\, B_0^+ B_1^- (t',0) - B_1^+ B_0^- (t,\tau)\, B_{10}(t') - B_1^+ B_0^- (\tau)\, B_{01}(t') \tag{619}$$

$$+ B_0^+ B_1^- (t,\tau)\, B_1^+ B_0^- (t',0) + B_0^+ B_1^- (t,\tau)\, B_0^+ B_1^- (t',0) - B_0^+ B_1^- (t,\tau)\, B_{10}(t') - B_0^+ B_1^- (t,\tau)\, B_{01}(t') \tag{620}$$

$$- B_{10}(t) B_1^+ B_0^- (t',0) - B_{10}(t) B_0^+ B_1^- (t',0) + B_{10}(t)\, B_{10}(t') + B_{10}(t)\, B_{01}(t') - B_{01}(t)\, B_1^+ B_0^- (t',0) \tag{621}$$

$$- B_{01}(t)\, B_0^+ B_1^- (t',0) + B_{01}(t)\, B_{10}(t') + B_{01}(t)\, B_{01}(t') \Big\rangle \tag{622}$$

$$= \frac{1}{4} \Big\langle B_1^+ B_0^- (t,\tau)\, B_1^+ B_0^- (t',0) + B_1^+ B_0^- (t,\tau)\, B_0^+ B_1^- (t',0) + B_0^+ B_1^- (t,\tau)\, B_1^+ B_0^- (t',0) \tag{623}$$

$$+ B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) \Big\rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(t') + B_{10}(t'))}{4}, \tag{624}$$

$$U_{10}(t,t') = \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t) m^*_{\mathbf{k}}(t') \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Im}}, \tag{625}$$

$$\left\langle B_1^+ B_0^- (t,\tau) B_0^+ B_1^- (t',0) \right\rangle_B = \left\langle \Pi_{\mathbf{k}} \left( D(m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}) \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \Pi_{\mathbf{k}} \left( D(-m_{\mathbf{k}}(t')) \mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \tag{626}$$

$$= \mathrm{e}^{\chi_{10}(t) + \chi_{01}(t')} \left\langle \prod_{\mathbf{k}} \left( D\left( m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right) \prod_{\mathbf{k}} \left( D\left( -m_{\mathbf{k}}(t') \right) \right) \right\rangle_B \tag{627}$$

$$= \mathrm{e}^{\chi_{10}(t) + \chi_{01}(t')} \prod_{\mathbf{k}} \left\langle \left( D\left( m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left( -m_{\mathbf{k}}(t') \right) \right) \right\rangle_B \tag{628}$$

$$= \mathrm{e}^{\chi_{10}(t) + \chi_{01}(t')} U^*_{10}(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{629}$$

$$\left\langle B_0^+ B_1^- (t,\tau) B_1^+ B_0^- (t',0) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left( D\left( -m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \prod_{\mathbf{k}} \left( D(m_{\mathbf{k}}(t')) \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \tag{630}$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}} \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle D\left( -m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left( m_{\mathbf{k}}(t') \right) \right\rangle_B \tag{631}$$

$$= \mathrm{e}^{\chi_{01}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} \left\langle D\left( -m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left( m_{\mathbf{k}}(t') \right) \right\rangle_B \tag{632}$$

$$= \mathrm{e}^{\chi_{01}(t) + \chi_{10}(t')} U^*_{10}(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left| (v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{633}$$

$$\left\langle B_1^+ B_0^- (t,\tau) B_1^+ B_0^- (t',0) \right\rangle_B = \mathrm{e}^{\chi_{10}(t) + \chi_{10}(t')} U_{10}(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{634}$$

$$\left\langle B_0^+ B_1^- (t,\tau) B_0^+ B_1^- (t',0) \right\rangle_B = \mathrm{e}^{\chi_{01}(t) + \chi_{01}(t')} U_{10}(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{635}$$

$$\left\langle \widetilde{B_x}(t) \widetilde{B_x}(t') \right\rangle_B = \frac{1}{4} \Big\langle B_1^+ B_0^- (t,\tau)\, B_1^+ B_0^- (t',0) + B_1^+ B_0^- (t,\tau)\, B_0^+ B_1^- (t',0) + B_0^+ B_1^- (t,\tau)\, B_1^+ B_0^- (t',0) \tag{636}$$

$$+ B_0^+ B_1^- (t,\tau)\, B_0^+ B_1^- (t',0) \Big\rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(t') + B_{10}(t'))}{4}, \tag{637}$$

$$= \frac{1}{4} \Bigg( 2U_{10}(t,t') \left( \mathrm{e}^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{638}$$

$$+ 2U^*_{10}(t,t') \left( \mathrm{e}^{\chi_{10}(t) + \chi_{01}(t')} \right)^{\Re} \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \Bigg) \tag{639}$$

$$- \left( \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} |m_{\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \left( \mathrm{e}^{\chi_{01}(t')} \mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} \left( |m_{\mathbf{k}}(t')|^2 \right) \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^{\Re} \tag{640}$$

$$= \frac{1}{2} \Bigg( \left( \mathrm{e}^{\chi_{10}(t) + \chi_{10}(t')} \right)^{\Re} U_{10}(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{641}$$

$$+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{642}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^{2}\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{643}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B=\left\langle B_y\left(t,\tau\right)B_y\left(t',0\right)\right\rangle_B \tag{644}$$

$$=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{645}$$

$$\left.+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{646}$$

$$=\left\langle\left(\frac{B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\left(\frac{B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_B \tag{647}$$

$$=-\frac{1}{4}\left\langle\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)\right)\left(B_0^+B_1^-\left(t',0\right)-B_1^+B_0^-\left(t',0\right)+B_{10}\left(t'\right)-B_{01}\left(t'\right)\right)\right\rangle_B \tag{648}$$

$$=-\frac{1}{4}\Big\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)-B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_{10}(t')-B_0^+B_1^-(\tau)B_{01}(t')-B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0) \tag{649}$$

$$+B_1^+B_0^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)-B_1^+B_0^-\left(t,\tau\right)B_{10}\left(t'\right)+B_1^+B_0^-\left(t,\tau\right)B_{01}\left(t'\right)+B_{10}(t)B_0^+B_1^-\left(t',0\right)-B_{10}(t)B_1^+B_0^-\left(t',0\right) \tag{650}$$

$$+B_{10}(t)B_{10}\left(t'\right)-B_{10}\left(t\right)B_{01}\left(t'\right)-B_{01}\left(t\right)B_0^+B_1^-\left(t',0\right)+B_{01}\left(t\right)B_1^+B_0^-\left(t',0\right)-B_{01}\left(t\right)B_{10}\left(t'\right)+B_{01}\left(t\right)B_{01}\left(t'\right)\Big\rangle \tag{651}$$

$$=-\frac{1}{4}\left\langle B_0^+B_1^-\left(t,\tau\right)B_0^+B_1^-\left(t',0\right)-B_0^+B_1^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)-B_1^+B_0^-\left(t,\tau\right)B_0^+B_1^-\left(t',0\right)+B_1^+B_0^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)\right\rangle \tag{652}$$

$$+\left(B_{01}\left(t\right)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Im} \tag{653}$$

$$=-\frac{1}{4}\left(2\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{654}$$

$$\left.-2\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{655}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{656}$$

$$=-\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{657}$$

$$\left.-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{658}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{659}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B=\left\langle B_x(t,\tau)B_y\left(t',0\right)\right\rangle_B \tag{660}$$

$$=\left\langle\left(\frac{B_1^+B_0^-\left(t,\tau\right)+B_0^+B_1^-\left(t,\tau\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\right)\left(\frac{B_0^+B_1^-\left(t',0\right)-B_1^+B_0^-\left(t',0\right)+B_{10}\left(t'\right)-B_{01}\left(t'\right)}{2\mathrm{i}}\right)\right\rangle_B \tag{661}$$

$$=\frac{1}{4i}\Big\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)B_{10}(t')-B_1^+B_0^-(t,\tau)B_{01}(t')+B_{01}(t)B_1^+B_0^-(t',0) \tag{662}$$

$$+B_0^+B_1^-\left(t,\tau\right)B_0^+B_1^-\left(t',0\right)-B_0^+B_1^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)+B_0^+B_1^-\left(t,\tau\right)B_{10}\left(t'\right)-B_0^+B_1^-\left(t,\tau\right)B_{01}\left(t'\right)+B_{01}(t)B_{01}\left(t'\right) \tag{663}$$

$$-B_{10}(t)B_0^+B_1^-\left(t',0\right)+B_{10}(t)B_1^+B_0^-\left(t',0\right)-B_{10}(t)B_{10}\left(t'\right)+B_{10}(t)B_{01}\left(t'\right)-B_{01}(t)B_0^+B_1^-\left(t',0\right)-B_{01}(t)B_{10}\left(t'\right)\Big\rangle_B \tag{664}$$

$$=\frac{1}{4i}\Big\langle B_1^+B_0^-\left(t,\tau\right)B_0^+B_1^-\left(t',0\right)-B_1^+B_0^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)+B_1^+B_0^-\left(t,\tau\right)B_{10}\left(t'\right)-B_1^+B_0^-\left(t,\tau\right)B_{01}\left(t'\right) \tag{665}$$

$$+B_0^+B_1^-\left(t,\tau\right)B_0^+B_1^-\left(t',0\right)-B_0^+B_1^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right)+B_0^+B_1^-\left(t,\tau\right)B_{10}\left(t'\right)-B_0^+B_1^-\left(t,\tau\right)B_{01}\left(t'\right) \tag{666}$$

$$-B_{10}\left(t\right)B_0^+B_1^-\left(t',0\right)+B_{10}\left(t\right)B_1^+B_0^-\left(t',0\right)-B_{10}\left(t\right)B_{10}\left(t'\right)+B_{10}\left(t\right)B_{01}\left(t'\right)-B_{01}\left(t\right)B_0^+B_1^-\left(t',0\right) \tag{667}$$

$$+B_{01}\left(t\right)B_1^+B_0^-\left(t',0\right)-B_{01}\left(t\right)B_{10}\left(t'\right)+B_{01}\left(t\right)B_{01}\left(t'\right)\Big\rangle_B \tag{668}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\right\rangle \tag{669}$$

$$+ \frac{1}{4i}\left(B_{10}(t) + B_{01}(t)\right)\left(B_{10}(t') - B_{01}(t')\right) \tag{670}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right. \tag{671}$$

$$\left. -B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right\rangle + \frac{1}{4i}\left(B_{10}(t) + B_{01}(t)\right)\left(B_{10}(t') - B_{01}(t')\right) \tag{672}$$

$$= \frac{1}{4i}\left\langle B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)\,B_0^+B_1^-(t',0)\right. \tag{673}$$

$$\left. -B_0^+B_1^-(t,\tau)\,B_1^+B_0^-(t',0)\right\rangle + \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{674}$$

$$= \frac{1}{4i}\left(\left(e^{\chi_{10}(t)+\chi_{01}(t')} - e^{\chi_{01}(t)+\chi_{10}(t')}\right)U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{675}$$

$$\left. + \left(e^{\chi_{01}(t)+\chi_{01}(t')} - e^{\chi_{10}(t)+\chi_{10}(t')}\right)U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{676}$$

$$+ \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{677}$$

$$= \frac{1}{2}\left(\left(e^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{678}$$

$$\left. + \left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{679}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B = \left\langle\left(\frac{B_0^+B_1^-(t,\tau) - B_1^+B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2i}\right)\left(\frac{B_1^+B_0^-(t',0) + B_0^+B_1^-(t',0) - B_{10}(t') - B_{01}(t')}{2}\right)\right\rangle_B \tag{680}$$

$$= \frac{1}{4i}\left\langle\left(B_0^+B_1^-(t,\tau) - B_1^+B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)\right)\left(B_1^+B_0^-(t',0) + B_0^+B_1^-(t',0) - B_{10}(t') - B_{01}(t')\right)\right\rangle_B \tag{681}$$

$$= \frac{1}{4i}\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_0^+B_1^-(t,\tau)B_{10}(t') - B_0^+B_1^-(t,\tau)B_{01}(t') + B_{01}(t)B_{01}(t')\right. \tag{682}$$

$$- B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0) - B_1^+B_0^-(t,\tau)\,B_0^+B_1^-(t',0) + B_1^+B_0^-(t,\tau)\,B_{10}(t') + B_1^+B_0^-(t,\tau)\,B_{01}(t') \tag{683}$$

$$+ B_{10}(t)\,B_0^+B_1^-(t',0) - B_{10}(t)\,B_{10}(t') - B_{10}(t)\,B_{01}(t') - B_{01}(t)\,B_1^+B_0^-(t',0) - B_{01}(t)\,B_0^+B_1^-(t',0) \tag{684}$$

$$\left. + B_{01}(t)\,B_{10}(t') + B_{10}(t)\,B_1^+B_0^-(t',0)\right\rangle \tag{685}$$

$$= \frac{1}{4i}\left\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0) + B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) - B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0) - B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)\right\rangle \tag{686}$$

$$+ \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{687}$$

$$= \frac{1}{4i}\left(e^{\chi_{01}(t)+\chi_{10}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{688}$$

$$+ e^{\chi_{01}(t)+\chi_{01}(t')}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{689}$$

$$- e^{\chi_{10}(t)+\chi_{10}(t')}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{690}$$

$$\left. - e^{\chi_{10}(t)+\chi_{01}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{691}$$

$$= \frac{1}{4i}\left(2i\left(e^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^*(t,t')\prod_{\mathbf{k}} e^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} - \left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{692}$$

$$\left. + 2i\left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}} e^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))e^{i\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{693}$$

$$= \left(B_{10}(t)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re} + \frac{1}{2}\Bigg(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}\left(t'\right)}\right)^{\Im} U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{694}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}\left(t'\right)}\right)^{\Im} U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}\left(t'\right)-v_{0\mathbf{k}}\left(t'\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \tag{695}$$

$$\left\langle b^{\dagger}D\left(h\right)\right\rangle_{B} = -h^{*}\left\langle D\left(h\right)\right\rangle_{B} N \tag{696}$$

$$\left\langle bD\left(h\right)\right\rangle_{B} = h\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{697}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{B} = -h^{*}\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{698}$$

$$\left\langle D\left(h\right)b\right\rangle_{B} = h\left\langle D\left(h\right)\right\rangle_{B} N \tag{699}$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t\right) = q_{j\mathbf{k}}\left(t\right) \tag{700}$$

$$\left\langle B_{1}^{+}B_{0}^{-}\left(t,\tau\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} = \prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} \tag{701}$$

$$= \mathrm{e}^{\chi_{10}(t)}\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{702}$$

$$= \mathrm{e}^{\chi_{10}(t)}q_{i\mathbf{k}'}\left(t'\right)\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{703}$$

$$= \mathrm{e}^{\chi_{10}(t)}q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{704}$$

$$= q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{705}$$

$$= -q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right) \tag{706}$$

$$\left\langle B_{1}^{+}B_{0}^{-}\left(t,\tau\right)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right\rangle_{B} = q_{i\mathbf{k}'}^{*}\left(t'\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^{2}}\right)}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}\left\langle\prod_{\mathbf{k}}D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle \tag{707}$$

$$= q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right), \tag{708}$$

$$\left\langle B_{0}^{+}B_{1}^{-}(t,\tau)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} = -q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right), \tag{709}$$

$$\left\langle B_{0}^{+}B_{1}^{-}(t,\tau)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right\rangle_{B} = q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right), \tag{710}$$

$$q_{i\mathbf{k}'}\left(0\right) = g_{i\mathbf{k}'} - v_{i\mathbf{k}'} \tag{711}$$

$$\left\langle B_{x}\left(t,\tau\right)B_{i\mathrm{mod}2z}\left(t',0\right)\right\rangle_{B} = \left\langle\left(\frac{B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}}{2}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{712}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}\right)\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{713}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)\right)\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{714}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big\langle B_{1+}B_{0-}\left(\tau\right)q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+B_{0+}B_{1-}\left(\tau\right)q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+B_{1+}B_{0-}\left(\tau\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'} \tag{715}$$

$$+B_{0+}B_{1-}\left(\tau\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\Big\rangle_{B} \tag{716}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right) \tag{717}$$

$$+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right)\Big) \tag{718}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right) \tag{719}$$

$$+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right)\Big) \tag{720}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}\left(t\right)+\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{01}\left(t\right)\Big) \tag{721}$$

$$+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{01}\left(t\right)\right)\Big) \tag{722}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}\left(t\right)-\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{01}\left(t\right)\Big) \tag{723}$$

$$+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{01}\left(t\right)\right)\Big) \tag{724}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(B_{10}\left(t\right)-B_{01}\left(t\right)\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(B_{10}\left(t\right) \tag{725}$$

$$-B_{01}(t))) \tag{726}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'} 2\mathrm{i}B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{727}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{728}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)(m_{\mathbf{k}'}(t))^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right), \tag{729}$$

$$= \mathrm{i}\sum_{\mathbf{k}'} B_{10}^{\Im}(t)\left(q_{i\mathbf{k}'}^{*}(t')N_{\mathbf{k}'}m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)(m_{\mathbf{k}'}(t))^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) \tag{730}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_x(t',0)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_B \tag{731}$$

$$= \sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_B \tag{732}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')\right)\right\rangle_B \tag{733}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)\right)\right\rangle_B \tag{734}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)+q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0) \tag{735}$$

$$+q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)\Big\rangle, \tag{736}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\right\rangle_B = q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\right\rangle_B \tag{737}$$

$$= q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B \tag{738}$$

$$= q_{i\mathbf{k}'}(t)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D(m_{\mathbf{k}'}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}'}(t')}{2}}\right)\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B \tag{739}$$

$$= q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D(m_{\mathbf{k}'}(t'))\right\rangle_B \tag{740}$$

$$= q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D(m_{\mathbf{k}'}(t'))\right\rangle_B \tag{741}$$

$$= q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left\langle b_{\mathbf{k}'}^{\dagger}D(m_{\mathbf{k}'}(t'))\right\rangle_B\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{742}$$

$$= q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D(m_{\mathbf{k}}(t'))\right\rangle_B\left(-m_{\mathbf{k}'}^{*}(t')\left\langle D(m_{\mathbf{k}'}(t'))\right\rangle_B N_{\mathbf{k}'}\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{743}$$

$$= -m_{\mathbf{k}'}^{*}(t')q_{i\mathbf{k}'}(t)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle\prod_{\mathbf{k}}D(m_{\mathbf{k}}(t'))\right\rangle_B N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{744}$$

$$= -m_{\mathbf{k}'}^{*}(t')q_{i\mathbf{k}'}(t)B_{10}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{745}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)\right\rangle_B = m_{\mathbf{k}'}^{*}(t')q_{i\mathbf{k}'}(t)B_{01}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{746}$$

$$\left\langle q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\right\rangle_B = q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_1^+B_0^-(t',0)\right\rangle_B \tag{747}$$

$$= q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D(m_{\mathbf{k}}(t'))\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\right)\right\rangle_B \tag{748}$$

$$= q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}\left\langle b_{\mathbf{k}'}D(m_{\mathbf{k}'}(t'))\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}(D(m_{\mathbf{k}}(t')))\right\rangle_B \tag{749}$$

$$= q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left\langle D(m_{\mathbf{k}'}(t'))\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}(D(m_{\mathbf{k}}(t')))\right\rangle_B \tag{750}$$

$$= q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}}m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left\langle D(m_{\mathbf{k}'}(t'))\right\rangle_B\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}(D(m_{\mathbf{k}}(t')))\right\rangle_B \tag{751}$$

$$= q^*_{i\mathbf{k}'}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)B_{10}(t'), \tag{752}$$

$$\left\langle (q_{i\mathbf{k}'}(t))^* b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} B_{0+}B_1^-(t',0)\right\rangle_B = q^*_{i\mathbf{k}'}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(-m_{\mathbf{k}'}(t'))(N_{\mathbf{k}'}+1)B_{01}(t'), \tag{753}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_x(t',0)\right\rangle_B = \frac{1}{2}\sum_{\mathbf{k}'}\Big(-m^*_{\mathbf{k}'}(t')q_{i\mathbf{k}'}(t)B_{10}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - (-m^*_{\mathbf{k}'}(t'))q_{i\mathbf{k}'}(t)B_{01}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{754}$$

$$+q^*_{i\mathbf{k}'}(t)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)B_{10}(t') + q^*_{i\mathbf{k}'}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(-m_{\mathbf{k}'}(t'))(N_{\mathbf{k}'}+1)B_{01}(t')\Big) \tag{755}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t')\big(B_{01}(t')-B_{10}(t')\big) + q^*_{i\mathbf{k}'}(t)m_{\mathbf{k}'}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'} \tag{756}$$

$$+1)\left(B_{10}(t')-B_{01}(t')\right)\Big) \tag{757}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t')\big(B_{01}(t')-B_{10}(t')\big) - q^*_{i\mathbf{k}'}(t)m_{\mathbf{k}'}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'} \tag{758}$$

$$+1)\left(B_{01}(t')-B_{10}(t')\right)\Big) \tag{759}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}B_{01}^{\Im}(t')\Big(q_{i\mathbf{k}'}(t)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t') - q^*_{i\mathbf{k}'}(t)m_{\mathbf{k}'}(t')\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}(N_{\mathbf{k}'}+1)\Big), \tag{760}$$

$$\left\langle B_y(t,\tau)B_{i\mathrm{mod}2z}(t',0)\right\rangle_B = \left\langle\left(\frac{B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}+q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right)\right\rangle_B \tag{761}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)\right)\left(q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}+q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right)\right\rangle_B \tag{762}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)\right)\left(q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}+q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right)\right\rangle_B \tag{763}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle B_0^+B_1^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'} - B_1^+B_0^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'} + B_0^+B_1^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'} \tag{764}$$

$$-B_1^+B_0^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\Big\rangle, \tag{765}$$

$$\left\langle B_0^+B_1^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}\right\rangle_B = -q_{i\mathbf{k}'}(t')\left(-m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}(t), \tag{766}$$

$$\left\langle B_0^+B_1^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right\rangle_B = q^*_{i\mathbf{k}'}(t')\left(-m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t), \tag{767}$$

$$\left\langle B_1^+B_0^-(t,\tau)q_{i\mathbf{k}'}(t')b^\dagger_{\mathbf{k}'}\right\rangle_B = -q_{i\mathbf{k}'}(t')\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}(t), \tag{768}$$

$$\left\langle B_1^+B_0^-(t,\tau)q^*_{i\mathbf{k}'}(t')b_{\mathbf{k}'}\right\rangle_B = q^*_{i\mathbf{k}'}(t')m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}N_{\mathbf{k}'}B_{10}(t), \tag{769}$$

$$\left\langle B_y(t,\tau)B_{i\mathrm{mod}2z}(t',0)\right\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}(B_{01}(t)+B_{10}(t))\Big(q_{i\mathbf{k}'}(t')\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1) - q^*_{i\mathbf{k}'}(t')m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}N_{\mathbf{k}'}\Big) \tag{770}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(q^*_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t) - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)\Big)(B_{10}(t)+B_{01}(t)), \tag{771}$$

$$= \frac{2}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t)(B_{10}(t))^{\Re} - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)(B_{10}(t))^{\Re}\Big), \tag{772}$$

$$= \frac{(B_{10}(t))^{\Re}}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t) - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)\Big), \tag{773}$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_y(t',0)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_B \tag{774}$$

$$= \frac{1}{2\mathrm{i}}\Big\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\big(B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t') \tag{775}$$

$$-B_{01}(t')\big)\Big\rangle_B \tag{776}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)\right)\right\rangle_B \tag{777}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0) - q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0) \tag{778}$$

$$+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)-q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\Big\rangle \quad (779)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}\left(t',0\right)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}\left(t',0\right)\Big\rangle \quad (780)$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}\left(t',0\right)\right\rangle-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle\Big\rangle \quad (781)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}\left(t',0\right)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}\left(t',0\right)\Big\rangle+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}(t)\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}\left(t',0\right)\rangle \quad (782)$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle \quad (783)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}\left(t',0\right)\Big\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}\left(t',0\right)\Big\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}(t)\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}\left(t',0\right)\rangle \quad (784)$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle\Big) \quad (785)$$

$$\Big\langle b_{\mathbf{k}'}^{\dagger}B_{1}^{+}B_{0}^{-}\left(t',0\right)\Big\rangle_{B}=-m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}, \quad (786)$$

$$\Big\langle b_{\mathbf{k}'}^{\dagger}B_{0}^{+}B_{1}^{-}\left(t',0\right)\Big\rangle_{B}=m_{\mathbf{k}'}^{*}\left(t'\right)B_{01}\left(t'\right)N_{\mathbf{k}'}, \quad (787)$$

$$\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}=m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right), \quad (788)$$

$$\left\langle b_{\mathbf{k}'}B_{0}^{+}B_{1}^{-}\left(t',0\right)\right\rangle_{B}=-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right), \quad (789)$$

$$\left\langle B_{i\mathrm{mod}2z}(t,\tau)B_{y}\left(t',0\right)\right\rangle_{B}=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-\left(-m_{\mathbf{k}'}^{*}\left(t'\right)\right)B_{01}\left(t'\right)N_{\mathbf{k}'}\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \quad (790)$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)\right)-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\Big) \quad (791)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-q_{i\mathbf{k}'}\left(t\right)\left(-m_{\mathbf{k}'}^{*}\left(t'\right)\right)B_{01}\left(t'\right)N_{\mathbf{k}'}+q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\right) \quad (792)$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(q_{i\mathbf{k}'}^{*}\left(t\right)\left(-m_{\mathbf{k}'}\left(t'\right)\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)-q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)\right)\Big) \quad (793)$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(B_{10}\left(t'\right)+B_{01}\left(t'\right)\right)\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big) \quad (794)$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}^{\Re}\left(t'\right)N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^{\Re}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\Big) \quad (795)$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^{\Re}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}^{\Re}\left(t'\right)N_{\mathbf{k}'}\Big) \quad (796)$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)B_{10}^{\Re}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)B_{10}^{\Re}\left(t'\right)N_{\mathbf{k}'}\Big) \quad (797)$$

$$=\mathrm{i}B_{10}^{\Re}\left(t'\right)\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}\left(t\right)m_{\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\Big). \quad (798)$$

The correlation functions are equal to:

$$\Big\langle\widetilde{B_{i\mathrm{mod}2z}}\left(t\right)\widetilde{B_{j\mathrm{mod}2z}}\left(t'\right)\Big\rangle_{B}=\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)^{*}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \quad (799)$$

$$\Big\langle\widetilde{B_{x}}\left(t\right)\widetilde{B_{x}}\left(t'\right)\Big\rangle_{B}=\frac{1}{2}\Bigg(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (800)$$

$$+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^{2}}{2\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg) \quad (801)$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^{2}\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \quad (802)$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B=-\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(\mathrm{v}_{1\mathbf{k}}(\mathrm{t})-\mathrm{v}_{0\mathbf{k}}(\mathrm{t}))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')-\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{803}$$

$$\left.-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{804}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}-\frac{\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im} \tag{805}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{806}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{807}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^{*}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{808}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{809}$$

$$\left\langle\widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_x}(t')\right\rangle_B=\mathrm{i}\sum_{\mathbf{k}}B_{01}^{\Im}(t')\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^{*}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{810}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_{i\mathrm{mod}2z}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^{*}N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{811}$$

$$\left\langle\widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_y}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t')\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{812}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_{i\mathrm{mod}2z}}(t')\right\rangle_B=\mathrm{i}B_{10}^{\Re}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^{*}N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))(N_{\mathbf{k}}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\right). \tag{813}$$

Let's consider the following expression related to the sum of coupling constants for a bath over all the frequences:

$$L_i(\omega)\equiv\sum_{\mathbf{k}}g_{i\mathbf{k}}\sqrt{\delta(\omega-\omega_{\mathbf{k}})}. \tag{814}$$

Under this definition we have the following expression for a function $f(\omega)\in L^2$:

$$\int_0^{\infty}f(\omega)L_i(\omega)L_j^{*}(\omega)\mathrm{d}\omega\approx\int_0^{\infty}f(\omega)\sum_{\mathbf{k}}g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sum_{\mathbf{k}'}g_j(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega \tag{815}$$

$$=\int_0^{\infty}f(\omega)\sum_{\mathbf{k},\mathbf{k}'}g_i(\omega_{\mathbf{k}})g_j(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega, \tag{816}$$

$$\int_0^{\infty}f(\omega)\sum_{\mathbf{k}}g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\mathrm{d}\omega=\sum_{\mathbf{k}}\int_0^{\infty}f(\omega)g_i(\omega_{\mathbf{k}})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\mathrm{d}\omega. \tag{817}$$

Now we will approach to the function $\sqrt{\delta(\omega-\omega_{\mathbf{k}})}$ using the normal distribution, so:

$$\delta\left(\omega-\omega_{\mathbf{k}}\right)=\lim_{\sigma\to0^{+}}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\sigma^{2}}} \tag{818}$$

$$\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}=\lim_{\sigma\to0^{+}}\sqrt{\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\sigma^{2}}}} \tag{819}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{4\sigma^{2}}} \tag{820}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\left(\sqrt{2}\sigma\right)^{2}}} \tag{821}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right). \tag{822}$$

So we can obtain that:

$$\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega=\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega \tag{823}$$

$$=\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{824}$$

$$=\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{825}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega \tag{826}$$

$$=\sum_{\mathbf{k}}\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)\left(\lim_{\sigma\to0^{+}}\int_{0}^{\infty}f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega\right) \tag{827}$$

$$=\sum_{\mathbf{k}}\left(\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\text{ (with }f\left(\omega\right)\text{,}g_{i}\left(\omega\right)\in L^{2}\text{)} \tag{828}$$

$$=\lim_{\sigma\to0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\text{ (with }f\left(\omega\right)\text{,}g_{i}\left(\omega\right)\in L^{2}\text{)} \tag{829}$$

$$=0\,\text{(because the sum }\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\text{ is finite).} \tag{830}$$

Then we can proof the following:

$$\int_{0}^{\infty}f(\omega)L_{i}(\omega)L_{j}^{*}\left(\omega\right)\mathrm{d}\omega\approx\int_{0}^{\infty}f\left(\omega\right)\sum_{\mathbf{k}}g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sum_{\mathbf{k}'}g_{j}^{*}\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{831}$$

$$=\int_{0}^{\infty}f\left(\omega\right)\sum_{\mathbf{k},\mathbf{k}'}g_{i}\left(\omega_{\mathbf{k}}\right)g_{j}^{*}\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{832}$$

$$=\sum_{\mathbf{k}\neq\mathbf{k}'}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega+\sum_{\mathbf{k}}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{833}$$

$$=0+\sum_{\mathbf{k}}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{834}$$

$$=\sum_{\mathbf{k}}\int_{0}^{\infty}f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{835}$$

$$=\sum_{\mathbf{k}}f(\omega_{\mathbf{k}})g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}}) \tag{836}$$

if $i=j$ we recover the spectral density defined in the usual way when we integrate for a function $f\left(\omega\right)$ that belongs to the set $L^{2}$:

$$\sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) = \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) \delta(\omega - \omega_{\mathbf{k}}) \mathrm{d}\omega \tag{837}$$

$$= \int_0^\infty f(\omega) J_{ii}(\omega) \mathrm{d}\omega \tag{838}$$

$$= \int_0^\infty f(\omega) |L_i(\omega)|^2 \mathrm{d}\omega. \tag{839}$$

where

$$J_{ii}(\omega) = \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \tag{840}$$

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t). \tag{841}$$

In this case $g_i(\omega)$ and $v_i(\omega, t)$ are the continuous version of $g_i(\omega_{\mathbf{k}})$ and $v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t)$ respectively.
The integral version of the correlation functions can be obtained as follows:

$$\left\langle \widetilde{B_{iz}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t')) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \tag{842}$$

$$= \sum_{\mathbf{k}} \left( g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}}, t)) g_{j\mathbf{k}}^* (1 - F_j(\omega_{\mathbf{k}}, t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}, t))^* g_{j\mathbf{k}} (1 - F_j(\omega_{\mathbf{k}}, t')) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \tag{843}$$

$$\approx \int_0^\infty \left( L_i(\omega) L_j^*(\omega) (1 - F_i(\omega, t)) \left(1 - F_j^*(\omega, t')\right) \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) + L_i^*(\omega) L_j(\omega) \left(1 - F_i^*(\omega, t)\right) \left(1 - F_j(\omega, t')\right) \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega) + 1) \right) \mathrm{d}\omega, \tag{844}$$

$$\chi_{10}(t) = \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \tag{845}$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \tag{846}$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{g_{1\mathbf{k}}^* g_{0\mathbf{k}} F_1^*(\omega_{\mathbf{k}}, t) F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} g_{0\mathbf{k}}^* F_1(\omega_{\mathbf{k}}, t) F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \tag{847}$$

$$\approx \int_0^\infty \frac{L_0(\omega) L_1^*(\omega) F_1^*(\omega, t) F_0(\omega, t) - L_1(\omega) L_0^*(\omega) F_1(\omega, t) F_0^*(\omega, t)}{2\omega^2} \mathrm{d}\omega, \tag{848}$$

$$U_{10}(t, t') = \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i} \left( \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{849}$$

$$= \mathrm{e}^{\mathrm{i} \sum_{\mathbf{k}} \left( \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{850}$$

$$= \mathrm{e}^{\mathrm{i} \left( \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{851}$$

$$= \mathrm{e}^{\mathrm{i} \left( \sum_{\mathbf{k}} \frac{(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t))(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{852}$$

$$\approx \mathrm{e}^{\mathrm{i} \left( \int_0^\infty \frac{(L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t))(L_1(\omega) F_1(\omega, t') - L_0(\omega) F_0(\omega, t'))^* \mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2} \mathrm{d}\omega \right)^{\Im}}, \tag{853}$$

$$B_{10}(t) = \left( \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \left( \mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right), \tag{854}$$

$$= \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{855}$$

$$\approx \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\frac{1}{2} \int_0^\infty \left| \frac{L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t)}{\omega} \right|^2 \coth\left( \frac{\beta\omega}{2} \right) \mathrm{d}\omega} \tag{856}$$

$$\xi^+(t, t') = \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left( \frac{\beta\omega_{\mathbf{k}}}{2} \right)} \tag{857}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{858}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{859}$$

$$\approx \mathrm{e}^{-\int_0^\infty \frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}+L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t')|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{860}$$

$$\xi^-(t,t') = \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{861}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{862}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t'))|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{863}$$

$$\approx \mathrm{e}^{-\int_0^\infty \frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}-(L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t'))|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{864}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}(t,t')\xi^+(t,t') + \left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*(t,t')\xi^-(t,t')\right) - (B_{10}(t))^{\Re}(B_{01}(t'))^{\Re} \tag{865}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B = -\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}(t,t')\xi^+(t,t') - \left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*(t,t')\xi^-(t,t')\right) + (B_{01}(t))^{\Im}(B_{10}(t'))^{\Im} \tag{866}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*(t,t')\xi^-(t,t') + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\xi^+(t,t')\right) + (B_{10}(t))^{\Re}(B_{10}(t'))^{\Im} \tag{867}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^*(t,t')\xi^-(t,t') + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\xi^+(t,t')\right) + (B_{10}(t))^{\Im}(B_{10}(t'))^{\Re} \tag{868}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B = \mathrm{i}B_{01}^{\Im}(t')\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^* N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} - (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{869}$$

$$= \mathrm{i}B_{01}^{\Im}(t')\sum_{\mathbf{k}}\left(-g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}},t))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right. \tag{870}$$

$$\left.+g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}},t))\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')}{\omega_{\mathbf{k}}}\right)^* N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{871}$$

$$Q(\omega,t) = \frac{L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t)}{\omega} \tag{872}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B \approx \mathrm{i}B_{01}^{\Im}(t')\int_0^\infty\left(L_i(\omega)(1-F_i(\omega,t))Q^*(\omega,t')N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - L_i^*(\omega)(1-F_i^*(\omega,t))Q(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega, \tag{873}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_{iz}}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^* N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) - (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{874}$$

$$\approx \mathrm{i}B_{01}^{\Im}(t)\int_0^\infty\left(L_i^*(\omega)(1-F_i^*(\omega,t'))Q(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - L_i(\omega)(1-F_i(\omega,t'))Q^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{875}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_y}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Re}(t')\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1) - \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^* N_{\mathbf{k}}\right), \tag{876}$$

$$\approx \mathrm{i}B_{10}^{\Re}(t')\int_0^\infty\left(L_i^*(\omega)(1-F_i^*(\omega,t'))Q(\omega,t')(N(\omega)+1)\mathrm{e}^{-\mathrm{i}\omega\tau} - L_i(\omega)(1-F_i(\omega,t'))Q^*(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{877}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_{iz}}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Re}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^* N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) - (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))(N_{\mathbf{k}}+1)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\right) \tag{878}$$

$$\approx i B_{10}^{\Re}(t)\int_0^\infty\left(L_i^*(\omega)(1-F_i^*(\omega,t'))Q(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - L_i(\omega)(1-F_i(\omega,t'))Q^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega. \tag{879}$$

The integral version of $F_0(\omega,t)$ and $F_1(\omega,t)$ are:

$$a_i(\omega_{\mathbf{k}},t) = \frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t))\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{880}$$

$$b_i(\omega_{\mathbf{k}},t)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{881}$$

$$r_i(\omega_{\mathbf{k}},t)=\frac{a_i(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)},\tag{882}$$

$$s_i(\omega_{\mathbf{k}},t)=\frac{a_{(i+1)\mathrm{mod}2}(\omega_{\mathbf{k}},t)\,b_{i\mathrm{mod}2}(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}.\tag{883}$$

$$F_0(\omega,t)=r_0(\omega_{\mathbf{k}},t)+\frac{g_1(\omega_{\mathbf{k}})}{g_0(\omega_{\mathbf{k}})}s_0(\omega_{\mathbf{k}},t)\tag{884}$$

$$\approx r_0(\omega,t)+\frac{g_1(\omega)}{g_0(\omega)}s_0(\omega,t)\tag{885}$$

$$=r_0(\omega,t)+\frac{L_1(\omega)}{L_0(\omega)}s_0(\omega,t),\tag{886}$$

$$F_1(\omega,t)=\frac{g_0(\omega_{\mathbf{k}})}{g_1(\omega_{\mathbf{k}})}r_1(\omega_{\mathbf{k}},t)+s_1(\omega_{\mathbf{k}},t)\tag{887}$$

$$\approx\frac{g_0(\omega)}{g_1(\omega)}r_1(\omega,t)+s_1(\omega,t)\tag{888}$$

$$=\frac{L_0(\omega)}{L_1(\omega)}r_1(\omega,t)+s_1(\omega,t).\tag{889}$$

The expression showed are well defined because the relevant products present in the correlations functions are of the form:

$$\int_0^\infty f(\omega)L_j(\omega)F_j(\omega,t)L_i^*(\omega)F_i^*(\omega,t)\,\mathrm{d}\omega=\int_0^\infty f(\omega)L_j(\omega)\left(r_j(\omega,t)+\frac{L_i(\omega)}{L_j(\omega)}s_j(\omega,t)\right)L_i^*(\omega)\left(r_i^*(\omega,t)+\frac{L_j^*(\omega)}{L_i^*(\omega)}s_i^*(\omega,t)\right)\mathrm{d}\omega\tag{890}$$

$$=\int_0^\infty f(\omega)\left(L_j(\omega)\,r_j(\omega,t)+L_i(\omega)\,s_j(\omega,t)\right)\left(L_i^*(\omega)\,r_i^*(\omega,t)+L_j^*(\omega)\,s_i^*(\omega,t)\right)\mathrm{d}\omega\tag{891}$$

$$=\int_0^\infty f(\omega)\left(L_j(\omega)\,L_i^*(\omega)\,r_j(\omega,t)\,r_i^*(\omega,t)+\left|L_j(\omega)\right|^2 r_j(\omega,t)\,s_i^*(\omega,t)\right.\tag{892}$$

$$\left.+\left|L_i(\omega)\right|^2 s_j(\omega,t)\,r_i^*(\omega,t)+L_i(\omega)\,L_j^*(\omega)\,s_j(\omega,t)\,s_i^*(\omega,t)\right)\mathrm{d}\omega\tag{893}$$

here $f(\omega)\in L^2$. As we could proof these integral are convergent.
So the integral version of the correlation functions $\mathcal{B}_{ij}(t,t')$ is can be written in a neater form as:

$$\mathcal{B}(t,t')\equiv\begin{pmatrix}\mathcal{B}_{11}(t,t') & \mathcal{B}_{12}(t,t') & \mathcal{B}_{13}(t,t') & \mathcal{B}_{12}(t,t') & \mathcal{B}_{11}(t,t') & \mathcal{B}_{16}(t,t')\\ \mathcal{B}_{21}(t,t') & \mathcal{B}_{22}(t,t') & \mathcal{B}_{23}(t,t') & \mathcal{B}_{22}(t,t') & \mathcal{B}_{21}(t,t') & \mathcal{B}_{26}(t,t')\\ \mathcal{B}_{31}(t,t') & \mathcal{B}_{32}(t,t') & \mathcal{B}_{33}(t,t') & \mathcal{B}_{32}(t,t') & \mathcal{B}_{31}(t,t') & \mathcal{B}_{36}(t,t')\\ \mathcal{B}_{21}(t,t') & \mathcal{B}_{22}(t,t') & \mathcal{B}_{23}(t,t') & \mathcal{B}_{22}(t,t') & \mathcal{B}_{21}(t,t') & \mathcal{B}_{26}(t,t')\\ \mathcal{B}_{11}(t,t') & \mathcal{B}_{12}(t,t') & \mathcal{B}_{13}(t,t') & \mathcal{B}_{12}(t,t') & \mathcal{B}_{11}(t,t') & \mathcal{B}_{16}(t,t')\\ \mathcal{B}_{61}(t,t') & \mathcal{B}_{62}(t,t') & \mathcal{B}_{63}(t,t') & \mathcal{B}_{62}(t,t') & \mathcal{B}_{61}(t,t') & \mathcal{B}_{66}(t,t')\end{pmatrix},\tag{894}$$

$$\mathcal{B}_{11}(t,t')=\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^{+}(t,t')+\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}(t,t')\,\xi_{10}^{-}(t,t')\right)-B_{10}^{\Re}(t)\,B_{01}^{\Re}(t'),\tag{895}$$

$$\mathcal{B}_{22}(t,t')=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^{+}(t,t')-\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}(t,t')\,\xi_{10}^{-}(t,t')\right)+B_{01}^{\Im}(t)B_{10}^{\Im}(t'),\tag{896}$$

$$\mathcal{B}_{12}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}(t,t')\,\xi_{10}^{-}(t,t')+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^{+}(t,t')\right)+B_{10}^{\Re}(t)\,B_{10}^{\Im}(t'),\tag{897}$$

$$\mathcal{B}_{21}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^{*}(t,t')\,\xi_{10}^{-}(t,t')+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^{+}(t,t')\right)+B_{10}^{\Im}(t)\,B_{10}^{\Re}(t'),\tag{898}$$

$$
\begin{aligned}
\mathcal{B}_{ij}\left(t,t'\right) &= \int_0^\infty \left(P_i\left(\omega,t\right)P_j^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_i^*\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\{3,6\},\\
\mathcal{B}_{i1}\left(t,t'\right) &= \mathrm{i}B_{01}^{\Im}\left(t'\right)\int_0^\infty \left(P_i\left(\omega,t\right)Q_{10}^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i^*\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\},\\
\mathcal{B}_{1i}\left(t,t'\right) &= \mathrm{i}B_{01}^{\Im}\left(t\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\},\\
\mathcal{B}_{i2}\left(t,t'\right) &= \mathrm{i}B_{10}^{\Re}\left(t'\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\},\\
\mathcal{B}_{2i}\left(t,t'\right) &= \mathrm{i}B_{10}^{\Re}\left(t\right)\int_0^\infty \left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\},\\
\zeta_{ij}\left(t,t'\right) &= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)},\\
\xi_{ij}^{\pm}\left(t,t'\right) &= \mathrm{e}^{-\int_0^\infty \frac{\left|\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega},\\
P_i\left(\omega,t\right) &= L_{i\mathrm{mod}2}\left(\omega\right)\left(1-F_{i\mathrm{mod}2}\left(\omega,t\right)\right),\\
Q_{ij}(\omega,t) &= \frac{L_i\left(\omega\right)F_1\left(\omega,t\right)-L_j\left(\omega\right)F_j\left(\omega,t\right)}{\omega},\\
a_i\left(\omega,t\right) &= \frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)},\\
b_i\left(\omega,t\right) &= \frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)},\\
r_i\left(\omega,t\right) &= \frac{a_i\left(\omega,t\right)}{1-b_0\left(\omega,t\right)b_1\left(\omega,t\right)},\\
s_i\left(\omega,t\right) &= \frac{a_{(i+1)\mathrm{mod}2}\left(\omega,t\right)b_{i\mathrm{mod}2}\left(\omega,t\right)}{1-b_0\left(\omega,t\right)b_1\left(\omega,t\right)},\\
F_0\left(\omega,t\right) &= r_0\left(\omega,t\right)+\frac{L_1\left(\omega\right)}{L_0\left(\omega\right)}s_0\left(\omega,t\right),\\
F_1\left(\omega,t\right) &= \frac{L_0\left(\omega\right)}{L_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right).
\end{aligned}
$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda+\mathrm{Det}\left(\overline{H_{\bar{S}}}\right)=0. \tag{899}$$

The solutions of this equation written in terms of $\eta$ and $\xi$ as defined in the previous section are given by $\lambda_\pm=\frac{\xi\pm\eta}{2}$ and they satisfy $H_S\left|\pm\right\rangle=\lambda_\pm\left|\pm\right\rangle$. Using this notation is possible to write $H_{\bar{S}}=\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$
\begin{aligned}
\widetilde{A_i}\left(\tau\right) &= e^{\mathrm{i}\overline{H_{\bar{S}}}\tau}A_ie^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} &(900)\\
&= \sum_w e^{-\mathrm{i}w\tau}\mathcal{A}_i\left(w\right). &(901)
\end{aligned}
$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w\in\{0,\pm\eta\}$.

In order to use the equation (901) to descompose the equation (370) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right) \tag{902}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty}(-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1) H(t_2) ... H(t_n). \tag{903}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right) \equiv \exp(-\mathrm{i}tH_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...\right)\right). \tag{904}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \tag{905}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{906}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{907}$$

In this case the Fourier decomposition using the expansion of $H_E(t)$ is:

$$\widetilde{A_i}(t) = U^\dagger(t) A_i(t) U(t) \tag{908}$$

$$\widetilde{A_i}(t) = e^{\mathrm{i}H_E(t)t} A_i(t) e^{-\mathrm{i}H_E(t)t} \tag{909}$$

$$= \sum_{w(t)} e^{-\mathrm{i}w(t)t}\mathcal{A}_i(t, w(t)). \tag{910}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau, t)$ using the Magnus expansion generates:

$$\widetilde{A_j}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_j(t) U(t-\tau) U^\dagger(t) \tag{911}$$

$$= e^{-\mathrm{i}tH_E(t)} e^{\mathrm{i}(t-\tau)H_E(t-\tau)} A_j(t) e^{-\mathrm{i}(t-\tau)H_E(t-\tau)} e^{\mathrm{i}tH_E(t)} \tag{912}$$

$$= e^{-\mathrm{i}tH_E(t)} \left(\sum_{w'(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathcal{A}_j(t, w(t-\tau))\right) e^{\mathrm{i}tH_E(t)} \tag{913}$$

$$= \sum_{w(t), w'(t-\tau)} e^{\mathrm{i}w'(t)t} e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathcal{A}_j(t, w(t-\tau), w'(t)) \tag{914}$$

$$= \sum_{w(t), w'(t-\tau)} e^{\mathrm{i}w'(t)t} e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathcal{A}_j(t, w(t-\tau), w'(t)) \tag{915}$$

$$= \sum_{w(t), w'(t-\tau)} e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_j(t, w(t-\tau), w'(t)) \tag{916}$$

where $w'(t-\tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}(t-\tau)$ and $\overline{H_E}(t)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (901) for a general $2 \times 2$ matrix in a given time let's write the matrix $A_i$ in the base $\mathrm{V} = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta \in \mathrm{V}} \langle\alpha|A_i|\beta\rangle|\alpha\rangle\langle\beta|. \tag{917}$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{\mathrm{i}\overline{H_E}\tau} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} \tag{918}$$
$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} \tag{919}$$
$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{920}$$
$$= e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|. \tag{921}$$

Calculating the transformation (901) directly using the previous relationship we find that:

$$U^\dagger(\tau)\,A_i(\tau)U(\tau) = \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right)\left(\sum_{\alpha,\beta\in\mathrm{V}}\langle\alpha|A_i(\tau)|\beta\rangle|\alpha\rangle\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-\tau}|-\rangle\langle-|\right) \tag{922}$$
$$= \langle+|A_i(\tau)|+\rangle|+\rangle\langle+| + e^{\mathrm{i}\eta\tau}\langle+|A_i(\tau)|-\rangle|+\rangle\langle-| + e^{-\mathrm{i}\eta\tau}\langle-|A_i(\tau)|+\rangle|-\rangle\langle+| + \langle-|A_i(\tau)|-\rangle|-\rangle\langle-|. \tag{923}$$
$$= \mathcal{A}_i(0) + \mathcal{A}_i(-w)\,e^{\mathrm{i}w\tau} + \mathcal{A}_i(w)\,e^{-\mathrm{i}w\tau} \tag{924}$$

Here $w = \lambda_+ - \lambda-$. Comparing the RHS of the equations (901) and the explicit expression for $\widetilde{A_i}(\tau)$ in (909), we obtain the form of the expansion matrices of the Fourier decomposition for a general $2\times2$ matrix:

$$\mathcal{A}_i(0) = \langle+|A_i(\tau)|+\rangle|+\rangle\langle+| + \langle-|A_i(\tau)|-\rangle|-\rangle\langle-|, \tag{925}$$
$$\mathcal{A}_i(-w) = \langle+|A_i(\tau)|-\rangle|+\rangle\langle-|, \tag{926}$$
$$\mathcal{A}_i(w) = \langle-|A_i(\tau)|+\rangle|-\rangle\langle+|. \tag{927}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}(\tau) = \widetilde{A_i}^\dagger(\tau)$ and $\widetilde{B_i}(\tau) = \widetilde{B_i}^\dagger(\tau)$ we can use the equation (901) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big] + \mathcal{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \tag{928}$$
$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{929}$$
$$-\mathcal{B}_{ji}(-\tau)\left[A_i, \overline{\rho_S}(t)\,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_j(w(t-\tau),w'(t))\right]\Big) \tag{930}$$

Given that $\mathcal{A}_j(w(t-\tau), w'(t)) = \mathcal{A}_j^\dagger(-w(t-\tau), -w'(t))$ from the Fourier decomposition (901) then we can rearrange the precedent sum in the following way with the trace respect to the bath:

$$\mathcal{B}_{ij}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(s)\,\rho_B\right) \tag{931}$$
$$= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)\,\widetilde{B_j}(0)\,\rho_B\right). \tag{932}$$

Let's define:

$$\mathcal{A}_j \left(w\left(t-\tau\right), w'\left(t\right)\right) = \mathcal{A}_{jww'}\left(t-\tau, t\right) \tag{933}$$

The master equation can be re-written in the following form:

$$\begin{aligned}
\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} =& -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] && (934)\\
&+ \sum_{ijww'}\mathcal{B}_{ji}\left(-\tau\right)\left[A_i, \overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\right] && (935)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] && (936)\\
&+ \sum_{ijww'}\mathcal{B}_{ji}\left(-\tau\right)\left[A_i, \overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\right] && (937)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] && (938)\\
&+ \sum_{ijww'}\mathcal{B}_{ji}\left(-\tau\right)\left[A_i, \overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\right] && (939)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathcal{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] && (940)\\
&- \mathcal{B}_{ji}\left(-\tau\right)\left[A_i, \overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\right]\Big) && (941)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \textstyle\sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathrm{Tr}_B\Big(\left[A_i,\widetilde{B_i}(\tau)\widetilde{B_j}(0)\rho_B e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right] && (942)\\
&- \left[A_i,\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}(t-\tau,t)\right]\Big) && (943)
\end{aligned}$$

Given that if we define:

$$D_{ijww'}\left(t-\tau,t\right) = C_i\left(t\right)C_j\left(t-\tau\right)\mathcal{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right) \tag{944}$$

then

$$\begin{aligned}
D^{\dagger}_{ijww'}\left(t-\tau,t\right) &= \left(C_i\left(t\right)C_j\left(t-\tau\right)\mathcal{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_{jww'}\left(t-\tau,t\right)\right)^{\dagger} && (945)\\
&= \mathcal{B}^{*}_{ij}\left(\tau\right)C_i\left(t\right)C_j\left(t-\tau\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}^{\dagger}_{jww'}\left(t-\tau,t\right) && (946)
\end{aligned}$$

We used the fact that $C_i\left(t\right), C_j\left(t-\tau\right)$ are real. Now let's consider the following trace recalling that $\mathrm{Tr}\left(A\right)^{*} = \mathrm{Tr}\left(A^{\dagger}\right)$ so:

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(-\tau\right)\widetilde{B_i}\left(0\right)\rho_B\right) = \mathrm{Tr}_B\left(e^{-\mathrm{i}\tau H_B(\tau)}B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\right) \tag{947}$$

$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B e^{-\mathrm{i}\tau H_B(\tau)}\right) \text{ (by cyclic permutivity of trace)} \tag{948}$$

$$= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i e^{-\mathrm{i}\tau H_B(\tau)}\rho_B\right) \text{ (by commutativity of } e^{-\mathrm{i}\tau H_B(\tau)} \text{ and } \rho_B) \tag{949}$$

$$= \mathrm{Tr}_B\left(B_j\widetilde{B_i}\left(\tau\right)\rho_B\right) \text{ (by definition of time evolution)} \tag{950}$$

$$= \mathrm{Tr}_B\left(B_j\widetilde{B_i}\left(\tau\right)\rho_B\right) \tag{951}$$

$$= \mathrm{Tr}_B\left(\rho_B B_j\widetilde{B_i}\left(\tau\right)\right) \tag{952}$$

$$= \mathrm{Tr}_B\left(\left(\widetilde{B_i}\left(\tau\right)B_j\rho_B\right)^\dagger\right) \text{ (by definition of adjoint)} \tag{953}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)B_j\rho_B\right)^* \tag{954}$$

$$= \mathcal{B}^*_{ij}\left(\tau\right) \tag{955}$$

So we can write the master equation like:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \tag{956}$$

$$-\mathcal{B}^*_{ij}\left(\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathcal{A}^\dagger_j\left(w\left(t-\tau\right),w'\left(t\right)\right)\right]\Big) \tag{957}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\left(\left[A_i, D_{ijww'}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)D^\dagger_{ijww'}\left(t-\tau,t\right)\right]\right) \tag{958}$$

Let's define the response matrix in the following way.

$$\mathcal{D}_{ijww'}\left(t\right) = \int_0^t \mathrm{d}\tau D_{ijww'}\left(t-\tau,t\right) \tag{959}$$

Then the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i,\mathcal{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathcal{D}^\dagger_{ijww'}\left(t\right)\right]\right) \tag{960}$$

If we extend the upper limit of integration to $\infty$ in the equation (959) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V}\frac{\mathrm{d}\overline{\rho}_S(t)}{\mathrm{d}t}e^V = \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^V\right)}{\mathrm{d}t} \tag{961}$$

$$= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} \tag{962}$$

$$= -\mathrm{i}e^{-\mathrm{V}}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]e^V - \sum_{ijww'}\int_0^t \mathrm{d}\tau\Big(e^{-V}\left[A_i, D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right]e^V - e^{-V}\Big[A_i,\overline{\rho_S}(t)D^\dagger_{ijww'}(t-\tau,t)\Big]e^V\Big). \tag{963}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V} = e^{-V}\overline{A\mathbb{I}B}e^{V} \quad (964)$$
$$= e^{-V}\overline{A}e^{V}e^{-V}\bar{B}e^{V} \quad (965)$$
$$= \left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\bar{B}e^{V}\right) \quad (966)$$
$$= AB. \quad (967)$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V} = e^{-V}\overline{(AB-BA)}e^{V} \quad (968)$$
$$= e^{-V}\overline{AB}e^{V} - e^{-V}\overline{BA}e^{V} \quad (969)$$
$$= AB - BA \quad (970)$$
$$= [A,B]. \quad (971)$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right]e^{V} - e^{-V}\sum_{ijww'}\left(\left[A_i, \mathcal{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\mathcal{D}^{\dagger}_{ijww'}(t)\right]\right)e^{V} \quad (972)$$
$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right]e^{V} - \sum_{ijww'}\left(e^{-V}\left[A_i, \mathcal{D}_{ijww'}(t)\overline{\rho_S}(t)\right]e^{V} - e^{-V}\left[A_i, \overline{\rho_S}(t)\mathcal{D}^{\dagger}_{ijww'}(t)\right]e^{V}\right) \quad (973)$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t), \rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V}, e^{-V}\mathcal{D}_{ijww'}(t)\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V}, e^{-V}\overline{\rho_S}(t)\mathcal{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right) \quad (974)$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t), \rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V}, e^{-V}\mathcal{D}_{ijww'}(t)e^{V}e^{-V}\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V}, e^{-V}\overline{\rho_S}(t)e^{V}e^{-V}\mathcal{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right) \quad (975)$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t), \rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V}, e^{-V}\mathcal{D}_{ijww'}(t)e^{V}\rho_S(t)\right] - \left[e^{-V}A_ie^{V}, \rho_S(t)e^{-V}\mathcal{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right). \quad (976)$$

## IV. LIMIT CASES

In order to show the plausibility of the master equation (960) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

### A. Time-independent variational quantum master equation

At first let's show that the master equation (960) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1,2,3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \quad (977)$$

After performing the transformation (24) on the Hamiltonian (977) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, \quad (978)$$
$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), \quad (979)$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \quad (980)$$

The Hamiltonian (978) differs from the transformed Hamiltonian $H_S$ of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{981}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{982}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \tag{983}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \tag{984}$$

$$\lambda_\pm = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} \tag{985}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{986}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{987}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \tag{988}$$

$$\lambda_\pm = \frac{R \pm \eta}{2}. \tag{989}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{990}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{991}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle$. Summarizing these results we can write:

$$\lambda_{+} = \frac{\epsilon + \eta}{2}, \tag{992}$$
$$\lambda_{-} = \frac{\epsilon - \eta}{2}, \tag{993}$$
$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \tag{994}$$
$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, \tag{995}$$
$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}, \tag{996}$$
$$\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}. \tag{997}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{998}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x) + 1}. \tag{999}$$

So the value of $\tan(\theta)$ using (999) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}} + 1} \tag{1000}$$
$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{1001}$$
$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{1002}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (926)-(927) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\begin{aligned}
\langle +|\sigma_x|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1003)\\
&= 2\sin(\theta)\cos(\theta) && (1004)\\
&= \sin(2\theta)\,, && (1005)\\
\langle -|\sigma_x|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1006)\\
&= -2\sin(\theta)\cos(\theta) && (1007)\\
&= -\sin(2\theta)\,, && (1008)\\
\langle -|\sigma_x|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1009)\\
&= \cos^2(\theta) - \sin^2(\theta) && (1010)\\
&= \cos(2\theta)\,, && (1011)\\
\langle +|\sigma_y|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1012)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (1013)\\
&= 0, && (1014)\\
\langle -|\sigma_y|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1015)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (1016)\\
&= 0, && (1017)\\
\langle -|\sigma_y|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1018)\\
&= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) && (1019)\\
&= \mathrm{i}. && (1020)
\end{aligned}$$

$$\begin{aligned}
\langle +|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1021)\\
&= \cos(\theta)\cos(\theta) && (1022)\\
&= \cos^2(\theta)\,, && (1023)\\
\langle -|\frac{1+\sigma_z}{2}|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (1024)\\
&= \sin(\theta)\sin(\theta) && (1025)\\
&= \sin^2(\theta)\,, && (1026)\\
\langle -|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (1027)\\
&= -\sin(\theta)\cos(\theta) && (1028)\\
&= -\sin(\theta)\cos(\theta)\,. && (1029)
\end{aligned}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)(|+\rangle\langle+| - |-\rangle\langle-|), \tag{1030}$$
$$A_1(\eta) = \cos(2\theta)|-\rangle\langle+|, \tag{1031}$$
$$A_2(0) = 0, \tag{1032}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{1033}$$
$$A_3(0) = \cos^2(\theta)|+\rangle\langle+| + \sin^2(\theta)|-\rangle\langle-|, \tag{1034}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)|-\rangle\langle+|. \tag{1035}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (963) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (944) is equivalent to:

$$\mathcal{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau, t) \tag{1036}$$
$$= \int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{\mathrm{i}\tau w(t-\tau)} e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} \mathcal{A}_j(w(t-\tau), w'(t)) \tag{1037}$$
$$= \int_0^t \mathrm{d}\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)} \mathcal{A}_j(w, w'). \tag{1038}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (411) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau). \tag{1039}$$

So the response matrix can be rewritten as:

$$\mathcal{D}_{ijww'}(t) = \left(\int_0^t \mathrm{d}\tau \Lambda'_{ij}(\tau) e^{\mathrm{i}\tau w} e^{-\mathrm{i}t\left(w-w'\right)}\right) \mathcal{A}_j(w, w') \tag{1040}$$

Let's define the response function like:

$$K_{ij}(w, w', t) = \int_0^t C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1041}$$
$$= \int_0^t \Lambda'_{ij}(\tau) e^{\mathrm{i}w\tau} e^{-\mathrm{i}t\left(w-w'\right)} \mathrm{d}\tau \tag{1042}$$
$$= K_{ijww'}(t). \tag{1043}$$

Then we have the following equivalence:

$$\mathcal{D}_{ijww'}(t) = K_{ijww'}(t) \mathcal{A}_j(w, w') \tag{1044}$$
$$= K_{ijww'}(t) \mathcal{A}_{jww'} \tag{1045}$$

We can proof that

$$
\begin{aligned}
\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} &= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i,\mathcal{D}_{ijww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)\mathcal{D}^{\dagger}_{ijww'}\left(t\right)\right]\right) && (1046)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(\left[A_i,K_{ijww'}\left(t\right)\mathcal{A}_{jww'}\overline{\rho_S}\left(t\right)\right] - \left[A_i,\overline{\rho_S}\left(t\right)K^{*}_{ijww'}\left(t\right)\mathcal{A}^{\dagger}_{jww'}\right]\right) && (1047)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_S}\left(t\right)\right] - \sum_{ijww'}\left(K_{ijww'}\left(t\right)\left[A_i,\mathcal{A}_{jww'}\overline{\rho_S}\left(t\right)\right] - K^{*}_{ijww'}\left(t\right)\left[A_i,\overline{\rho_S}\left(t\right)\mathcal{A}^{\dagger}_{jww'}\right]\right) && (1048)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t)+\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\mathcal{A}_{jww'}\overline{\rho_S}(t)\right]-\left(K^{\Re}_{ijww'}(t)-\mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i,\overline{\rho_S}(t)\mathcal{A}^{\dagger}_{jww'}\right]\right) && (1049)\\
&= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijww'}K^{\Re}_{ijww'}(t)\left[A_i,\mathcal{A}_{jww'}\overline{\rho_S}(t)-\overline{\rho_S}(t)\mathcal{A}^{\dagger}_{jww'}\right]-\mathrm{i}\sum_{ijww'}K^{\Im}_{ijww'}(t)\left[A_i,\mathcal{A}_{jww'}\overline{\rho_S}(t)+\overline{\rho_S}(t)\mathcal{A}^{\dagger}_{jww'}\right] && (1050)
\end{aligned}
$$

Using the notation of the master equation (960), we can say that $C_1\left(t\right) = \frac{\Omega}{2} = C_2\left(t\right)$ and $C_3\left(t\right) = 1$, being $\Omega$ a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B\left(t\right) = B$. Taking the equations(**??**)-(**??**) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (411) are equal to:

$$\left\langle\widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B = \sum_{\mathbf{k}} \left( \left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right)\left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right)^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right)^* \left(g_{1\mathbf{k}} - v_{1\mathbf{k}}\right) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{1051}$$

$$= \sum_{\mathbf{k}} \left|g_{1\mathbf{k}} - v_{1\mathbf{k}}\right|^2 \left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{1052}$$

$$\approx \int_0^\infty J_1(\omega)\left(1-F_1(\omega)\right)^2 \left(\mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) + \mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right) \mathrm{d}\omega \tag{1053}$$

$$G_{\pm}(\omega,\tau) = \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) + \mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right) \tag{1054}$$

$$\left\langle\widetilde{B_{1z}}(t)\widetilde{B_{1z}}(s)\right\rangle_B \approx \int_0^\infty J_1(\omega)\left(1-F_1(\omega)\right)^2 G_+(\omega,t)\,\mathrm{d}\omega \tag{1055}$$

$$\chi_{10}(t) = 0\,(\text{because } v_{0\mathbf{k}}(t) = 0 \text{ for all } \mathbf{k}) \tag{1056}$$

$$U_{10}(t,s) = \prod_{\mathbf{k}} \exp\left(i\left(\frac{\left(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)\right)\left(v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)\right)^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{1057}$$

$$= \prod_{\mathbf{k}} \exp\left(i\left(\frac{v_{1\mathbf{k}}^2(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2}\right)^{\Im}\right) \tag{1058}$$

$$= \prod_{\mathbf{k}} \exp\left(i\frac{v_{1\mathbf{k}}^2 \sin\left(\omega_{\mathbf{k}}\tau\right)}{\omega_{\mathbf{k}}^2}\right) \tag{1059}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\left(\prod_{\mathbf{k}} \exp\left(i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\right)\prod_{\mathbf{k}} \exp\left(-\tfrac{|v_{1\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \prod_{\mathbf{k}} \exp\left(-i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\right)\prod_{\mathbf{k}} \exp\left(-\tfrac{\left|v_{1\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-v_{1\mathbf{k}}\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right.\right.\right. \tag{1060}$$

$$-\left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left(\left|\tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\left(\exp\left(-\tfrac{1}{2}\sum_{\mathbf{k}}\left(\left|\tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1061}$$

$$= \tfrac{1}{2}\left(\prod_{\mathbf{k}} \exp\left(i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{|v_{1\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \prod_{\mathbf{k}} \exp\left(-i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{\left|v_{1\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-v_{1\mathbf{k}}\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1062}$$

$$-\left(\exp\left(-\sum_{\mathbf{k}}\tfrac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1063}$$

$$|v_{1\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \pm v_{1\mathbf{k}}|^2 = v_{1\mathbf{k}}^2|\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \pm 1|^2 \tag{1064}$$

$$= v_{1\mathbf{k}}^2|\cos(\omega_{\mathbf{k}}\tau) + i\sin(\omega_{\mathbf{k}}\tau) \pm 1|^2 \tag{1065}$$

$$= v_{1\mathbf{k}}^2\left(\left(1 \pm \cos\left(\omega_{\mathbf{k}}\tau\right)\right)^2 + \sin^2\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{1066}$$

$$= 2v_{1\mathbf{k}}^2\left(1 \pm \cos\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{1067}$$

$$B \equiv \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{1068}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\left(\exp\left(\sum_{\mathbf{k}} i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{|v_{1\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(\sum_{\mathbf{k}} -i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{\left|v_{1\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-v_{1\mathbf{k}}\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1069}$$

$$-\left(\exp\left(-\sum_{\mathbf{k}}\tfrac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{1070}$$

$$\phi(\tau) = \sum_{\mathbf{k}}\frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right) + \cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{1071}$$

$$\approx \int_0^\infty \frac{J_1(\omega)F_1^2(\omega)}{\omega^2}\left(-\mathrm{i}\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega \tag{1072}$$

$$= \int_0^\infty \frac{J_1(\omega)F_1^2(\omega)}{\omega^2}G_+(\omega,\tau)\,\mathrm{d}\omega \tag{1073}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(s)\right\rangle_B = \tfrac{1}{2}\left(\exp\left(\sum_{\mathbf{k}} i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{2v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(\sum_{\mathbf{k}} -i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{2v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) - B^2 \tag{1074}$$

$$= \tfrac{1}{2}\left(\exp\left(\sum_{\mathbf{k}} i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{v_{1\mathbf{k}}^2(1+\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(\sum_{\mathbf{k}} -i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{v_{1\mathbf{k}}^2(1-\cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) - B^2 \tag{1075}$$

$$= \tfrac{1}{2}\left(\exp\left(-\sum_{\mathbf{k}}\tfrac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}} i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \tfrac{v_{1\mathbf{k}}^2 \cos(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right) + \exp\left(-\sum_{\mathbf{k}}\tfrac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2}\coth\left(\tfrac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}} -i\tfrac{v_{1\mathbf{k}}^2 \sin(\omega\right.\right.\right. \tag{1076}$$

$$B^2$$

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{1118}$$
$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2\right) \tag{1119}$$
$$= \frac{\Omega_r^2}{4}\left(\cosh(\phi(\tau)) - 1\right) \tag{1120}$$
$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{1121}$$
$$= \frac{\Omega_r^2}{8}\left(e^{\phi(\tau)} - e^{-\phi(\tau)}\right), \tag{1122}$$
$$\Lambda'_{33}(\tau) = \int_0^\infty \mathrm{d}\omega J(\omega)(1-F(\omega))^2 G_+(\tau), \tag{1123}$$
$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)(1-F(\omega))\mathrm{i}G_-(\tau), \tag{1124}$$
$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \tag{1125}$$
$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \tag{1126}$$
$$= \Lambda'_{13}(\tau) \tag{1127}$$
$$= \Lambda'_{31}(\tau) \tag{1128}$$
$$= 0. \tag{1129}$$

Finally taking the Hamiltonian (977) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (335) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{1130}$$
$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{1131}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (447) under a time-independent approach providing similar results.
Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (960) and using the fact that

$$\widetilde{A_j}(t-\tau, t) = U(\tau)A_jU(-\tau) \tag{1132}$$
$$= \sum_w e^{\mathrm{i}w\tau}\mathcal{A}_j(-w) \tag{1133}$$
$$= \sum_w e^{-\mathrm{i}w\tau}\mathcal{A}_j(w). \tag{1134}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \overline{\rho_S}\left(t\right)\right] - \frac{1}{2}\sum_{ij}\sum_{w}\gamma_{ij}\left(w,t\right)\left[A_i, \mathcal{A}_j\left(w\right)\overline{\rho}_S\left(t\right) - \overline{\rho}_S\left(t\right)\mathcal{A}_j^{\dagger}\left(w\right)\right] \tag{1135}$$

$$-\sum_{ij}\sum_{w}S_{ij}\left(w,t\right)\left[A_i, \mathcal{A}_j\left(w\right)\overline{\rho}_S\left(t\right) + \overline{\rho}_S\left(t\right)\mathcal{A}_j^{\dagger}\left(w\right)\right]. \tag{1136}$$

where $\mathcal{A}_j^{\dagger}\left(w\right) = \mathcal{A}_j\left(-w\right)$, as we can see the equation (1136) contains the rates and energy shifts $\gamma_{ij}\left(w,t\right) = 2K_{ij}^{\Re}\left(w,t\right)$ and $S_{ij}\left(w,t\right) = K_{ij}^{\Im}\left(w,t\right)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}\left(w,t\right) = \int_0^t \Lambda'_{ij}\left(\tau\right)e^{\mathrm{i}w\tau}\mathrm{d}\tau.$$

The fact $\mathcal{A}_j^{\dagger}\left(w\right) = \mathcal{A}_j\left(-w\right)$ can be verified directly for a $2\times 2$ matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{\mathrm{i}\overline{H_S}(t-\tau)} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)(t-\tau)} \tag{1137}$$

$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|(t-\tau)}e^{\mathrm{i}\lambda_-|-\rangle\langle-|(t-\tau)} \tag{1138}$$

$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1139}$$

$$= e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|. \tag{1140}$$

Where $\lambda_+, \lambda_-$are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$ . Calculating the transformation (901) of (925)-(927) directly using the previous relationship we find that:

$$\widetilde{A_i(0)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1141}$$

$$= \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|, \tag{1142}$$

$$\widetilde{A_i\left(w\right)}\left(t-\tau\right) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|-\rangle|+\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1143}$$

$$= \langle+|A_i|-\rangle|+\rangle\langle-|e^{\mathrm{i}w(t-\tau)}, \tag{1144}$$

$$\widetilde{A_i\left(-w\right)}\left(t-\tau\right) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle-|A_i|+\rangle|-\rangle\langle+|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{1145}$$

$$= \langle-|A_i|+\rangle|-\rangle\langle+|e^{-\mathrm{i}w(t-\tau)}. \tag{1146}$$

Here $w = \lambda_+ - \lambda -$. So we can see that for the equation (911) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w \neq w'$ then $A'_j\left(w,w'\right) = 0$ so:

$$\widetilde{A_j}\left(t-\tau,t\right)=U\left(t\right)U^{\dagger}\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right) \tag{1147}$$

$$=U\left(t\right)\left(\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}A_j\left(w(t-\tau)\right)\right)U^{\dagger}\left(t\right) \tag{1148}$$

$$=\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}U\left(t\right)A_j\left(w(t-\tau)\right)U^{\dagger}\left(t\right) \tag{1149}$$

$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w(t-\tau),w'\left(t\right)\right) \tag{1150}$$

$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_{jww'} \tag{1151}$$

$$==\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w\right)\delta_{ww'} \tag{1152}$$

$$=\sum_{w}e^{-\mathrm{i}(t-\tau)w}e^{\mathrm{i}tw}A_j\left(w\right) \tag{1153}$$

$$=\sum_{w}e^{\mathrm{i}\tau w}A_j\left(w\right) \tag{1154}$$

$$=U^{\dagger}\left(-\tau\right)A_jU\left(-\tau\right) \tag{1155}$$

So using now as reference the equation (1050) and $A_j'\left(w,w'\right)=0$ we can deduce that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijw}K_{ij}^{\Re}(w,t)\Big[A_i,A_j(w)\overline{\rho_S}(t)-\overline{\rho_S}(t)A_j^{\dagger}(w)\Big]-\mathrm{i}\sum_{ijw}K_{ij}^{\Im}(w,t)\Big[A_i,A_j(w)\overline{\rho_S}(t)+\overline{\rho_S}(t)A_j^{\dagger}(w)\Big] \tag{1156}$$

### B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (106) that $B_z=0$. The Hamiltonian studied is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega\left(t\right)}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{1157}$$

If $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then $B\left(\tau\right)=B$, so $B$ is independent of the time. In order to reproduce the Hamiltonian of the equation (1157) using the Hamiltonian of the equation (1) we can say that $\delta=\varepsilon_1\left(t\right)$, $\varepsilon_0\left(t\right)=0$, $V_{10}\left(t\right)=\frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then, in this case and using the equation (**??**) and (**??**) we obtain the following transformed Hamiltonians:

$$\overline{H_S}=\left(\delta+R_1\right)|1\rangle\langle 1|+\frac{B\sigma_x}{2}\Omega\left(t\right), \tag{1158}$$

$$\overline{H_{\mathrm{I}}}=\frac{\Omega\left(t\right)}{2}\left(B_x\sigma_x+B_y\sigma_y\right). \tag{1159}$$

In this case $R_1=\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}=g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right)=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta+R_1=\delta'$. If $F\left(\omega_{\mathbf{k}}\right)=1$ and using the equations (1118)-(1126) we can deduce that the only terms that survive are $\Lambda_{11}\left(\tau\right)$ and $\Lambda_{22}\left(\tau\right)$. The phonon propagator for this case is:

$$\phi\left(\tau\right)=\int_0^{\infty}\frac{J\left(\omega\right)}{\omega^2}G_{+}\left(\tau\right)\mathrm{d}\omega. \tag{1160}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (1160) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega. \tag{1161}$$

Writing the interaction Hamiltonian (1159) in the similar way to the equation (**??**) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (**??**) with $\delta'|1\rangle\langle 1|=\frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t). \tag{1162}$$

As we can see the function $B$ is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{1163}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1164}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{1165}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right). \tag{1166}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(447) is:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t)\right] - \sum_{i=1}^{2}\int_0^t \mathrm{d}\tau\left(C_i(t)C_i(t-\tau)\Lambda_{ii}(\tau)\left[A_i, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]\right. \tag{1167}$$

$$\left.+C_i(t)C_i(t-\tau)\Lambda_{ii}(-\tau)\left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_i\right]\right). \tag{1168}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau,t) = \widetilde{\sigma_i}(t-\tau,t)$, also using the equations (1163) and (1166) on the equation (1168) we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{1169}$$

$$\left.+\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau,t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau,t), \sigma_y\right]\Lambda_y(\tau)\right). \tag{1170}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau,t)\rho_S(t)\right]^\dagger = \left[\rho_S(t)\widetilde{A_i}(t-\tau,t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation $(21)$ of the reference [2] extended in the hermitian conjugate.

### C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference $[3]$ we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (411) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}}\right). \tag{1171}$$

Using the equation (960), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)\right), \rho_S\left(t\right)\right] - \frac{1}{2}\sum_w \gamma_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) - \rho_S\left(t\right)A_3^\dagger\left(w\right)\right] \tag{1172}$$

$$-\sum_w S_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) + \rho_S\left(t\right)A_3^\dagger\left(w\right)\right]\Bigg). \tag{1173}$$

The correlation functions are relevant if $F\left(\omega\right) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}\left(\tau\right) = \int_0^\infty \mathrm{d}\omega J\left(\omega\right) G_+\left(\tau\right), \tag{1174}$$

$$\Lambda_{33}\left(-\tau\right) = \int_0^\infty \mathrm{d}\omega J\left(\omega\right) G_+\left(-\tau\right). \tag{1175}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (1173) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)\right)\right] - \sum_w \left(K_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right)\right] + K_{33}^*\left(w,t\right)\left[\rho_S\left(t\right)A_3\left(w\right), A_3\right]\right). \tag{1176}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3\left(w\right)$, we have to remember that $H_S = \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{1177}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right), \tag{1178}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{1179}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{1180}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1181}$$

$$= \frac{1}{2}, \tag{1182}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{1183}$$

$$= \frac{1}{2}, \tag{1184}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1185}$$

$$= -\frac{1}{2}. \tag{1186}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{1187}$$

$$= \frac{\mathbb{I}}{2}, \tag{1188}$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle+| \tag{1189}$$

$$= \frac{1}{4}(\sigma_\mathrm{z} + \mathrm{i}\sigma_\mathrm{y}), \tag{1190}$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle-| \tag{1191}$$

$$= \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y). \tag{1192}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}[\sigma_x, \rho_S(t))] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)\right] \tag{1193}$$

$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right]. \tag{1194}$$

Calculating the response functions extending the upper limit of $\tau$ to $\infty$, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{1195}$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega) + 1)\mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega) e^{\mathrm{i}\tau\omega}\right) \mathrm{d}\tau \mathrm{d}\omega \tag{1196}$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{\mathrm{i}\widetilde{\Omega}\tau}(n(\omega) + 1)\mathrm{e}^{-\mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1197}$$

$$= \int_0^\infty \int_0^\infty J(\omega)(n(\omega) + 1)\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1198}$$

$$= \int_0^\infty J(\omega)(n(\omega) + 1)\pi\delta\left(\widetilde{\Omega} - \omega\right) \mathrm{d}\omega \tag{1199}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right) + 1\right), \tag{1200}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{-\mathrm{i}\widetilde{\Omega}\tau} \mathrm{d}\tau \mathrm{d}\omega \tag{1201}$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega) + 1)\mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega) e^{\mathrm{i}\tau\omega}\right) \mathrm{d}\tau \mathrm{d}\omega \tag{1202}$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-\mathrm{i}\widetilde{\Omega}\tau} n(\omega) \mathrm{e}^{\mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1203}$$

$$= \int_0^\infty \int_0^\infty J(\omega) n(\omega) \mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau + \mathrm{i}\tau\omega} \mathrm{d}\tau \mathrm{d}\omega \tag{1204}$$

$$= \int_0^\infty J(\omega) n(\omega) \pi\delta\left(-\widetilde{\Omega} + \omega\right) \mathrm{d}\omega \tag{1205}$$

$$= \pi J\left(\widetilde{\Omega}\right) n\left(\widetilde{\Omega}\right). \tag{1206}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s} = \pi\delta(\varepsilon) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1193) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S(t))\right] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S(t)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)\right]\right) \tag{1207}$$

$$-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S(t)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S(t)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right). \tag{1208}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x, \rho_S(t))\right] - \frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right)+1\right)[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)] + n\left(\Omega(t)\right)[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)]\right) \tag{1209}$$

$$-\frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right)+1\right)\left[\rho_S(t)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\Omega(t)\right)\left[\rho_S(t)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right). \tag{1210}$$

## V. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1211}$$

$$H_S(t) = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{1212}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b^{\dagger}_{u\mathbf{k}} + g^{*}_{nu\mathbf{k}}b_{u\mathbf{k}}\right), \tag{1213}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b^{\dagger}_{u\mathbf{k}}b_{u\mathbf{k}}. \tag{1214}$$

### A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}}b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}}b_{u\mathbf{k}}\right) \tag{1215}$$

At first let's obtain $e^{\pm V}$ under the transformation (1215), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}}\omega^{-1}_{u\mathbf{k}}\left(f_{nu\mathbf{k}}b^{\dagger}_{u\mathbf{k}} - f^{*}_{nu\mathbf{k}}b_{u\mathbf{k}}\right)$, so the equation (1215) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm\sum_n |n\rangle\langle n|\hat{\varphi}_n} \tag{1216}$$
$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n\right)^2}{2!} + ... \tag{1217}$$
$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + ... \tag{1218}$$
$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2}{2!} + ... \tag{1219}$$
$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + ...\right) \tag{1220}$$
$$= \sum_n |n\rangle\langle n| e^{\pm\hat{\varphi}_n} \tag{1221}$$

Given that $\left[f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}, f_{nu'\mathbf{k}'}b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*}b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}^{'}, \mathbf{k}$ and $u, u'$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right) = e^{\pm\left(\alpha_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} = \prod_u e^{\pm\sum_{\mathbf{k}}\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)} \tag{1222}$$
$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)}\right) \tag{1223}$$
$$= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) \tag{1224}$$
$$= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1225}$$
$$= \prod_u B_{nu\pm} \tag{1226}$$
$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) \tag{1227}$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V}e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1228}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1211):

$$
\begin{aligned}
\overline{|0\rangle\langle 0|} &= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), &(1229)\\
&= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, &(1230)\\
&= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, &(1231)\\
&= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} &(1232)\\
&= |0\rangle\langle 0| \prod_u \mathbb{I} &(1233)\\
&= |0\rangle\langle 0|. &(1234)\\
\overline{|m\rangle\langle n|} &= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), &(1235)\\
&= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, &(1236)\\
&= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, &(1237)\\
&= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right), \ m \neq n, &(1238)\\
&= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), &(1239)\\
&= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), &(1240)\\
&= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). &(1241)\\
\Pi_u (B_{mu+} B_{nu-}) &= \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). &(1242)\\
\overline{\textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} &= \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), &(1243)\\
&= \left(|0\rangle\langle 0| \textstyle\prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right) \left(\textstyle\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right) \left(|0\rangle\langle 0| \textstyle\prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), &(1244)\\
&= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, &(1245)\\
&= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots &(1246)\\
&= |0\rangle\langle 0| \left(\textstyle\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) &(1247)\\
&+ |1\rangle\langle 1| \left(\textstyle\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots &(1248)\\
&= |0\rangle\langle 0| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) &(1249)\\
&+ |1\rangle\langle 1| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots &(1250)\\
&= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots &(1251)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) &(1252)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) &(1253)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) &(1254)\\
&= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) &(1255)
\end{aligned}
$$

The transformed Hamiltonians of the equations (1212) to (1214) written in terms of (1229) to (1253) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1256}$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1257}$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) \tag{1258}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1259}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b^\dagger_{u\mathbf{k}} + g^*_{0u\mathbf{k}} b_{u\mathbf{k}}\right) + ...\right)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \tag{1260}$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} b^\dagger_{u\mathbf{k}} + g^*_{0u\mathbf{k}} b_{u\mathbf{k}}\right) \prod_u B_{0u-} + \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} b^\dagger_{u\mathbf{k}} + g^*_{1u\mathbf{k}} b_{u\mathbf{k}}\right) \prod_u B_{1u-} + ... \tag{1261}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \prod_u B_{0u+} b^\dagger_{u\mathbf{k}} \prod_u B_{0u-} + g^*_{0u\mathbf{k}} \prod_u B_{0u+} b_{u\mathbf{k}} \prod_u B_{0u-}\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \prod_u B_{1u+} b^\dagger_{u\mathbf{k}} \prod_u B_{1u-} + g^*_{1u\mathbf{k}} \prod_u B_{1u+} b_{u\mathbf{k}} \prod_u B_{1u-}\right) + ... \tag{1262}$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{0u\mathbf{k}} \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{1u\mathbf{k}} \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + ... \tag{1263}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + g^*_{nu\mathbf{k}} \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1264}$$

$$= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}} b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1265}$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{1266}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) \tag{1267}$$

$$+ \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + g^*_{nu\mathbf{k}} b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1268}$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1269}$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left(\left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right) b^\dagger_{u\mathbf{k}} + \left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}\right)^* b_{u\mathbf{k}}\right) \tag{1270}$$

Using the previous functions we have that (1267) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+}B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \tag{1271}$$

$$\tag{1272}$$

Now in order to separate the elements of the hamiltonian (1272) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\left\langle\prod_u(B_{mu+}B_{nu-})\right\rangle_{\overline{H_0}}=\left\langle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\right\rangle_{\overline{H_0}} \quad (1273)$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}}+\alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}})\right)\right)\left\langle\prod_{u\mathbf{k}}D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\right\rangle_{\overline{H_0}} \quad (1274)$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \quad (1275)$$

$$\equiv B_{nm} \quad (1276)$$

$$\left\langle\prod_u(B_{nu+}B_{mu-})\right\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v^*_{mu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \quad (1277)$$

$$=B^*_{nm} \quad (1278)$$

Following the reference [4] we define:

$$J_{nm}=\prod_u(B_{mu+}B_{nu-})-B_{nm} \quad (1279)$$

As we can see:

$$J^\dagger_{nm}=\left(\prod_u(B_{mu+}B_{nu-})-B_{nm}\right)^\dagger \quad (1280)$$

$$=\prod_u(B_{nu+}B_{mu-})-B^*_{nm} \quad (1281)$$

$$=\prod_u(B_{nu+}B_{mu-})-B_{mn} \quad (1282)$$

$$=J_{mn} \quad (1283)$$

We can separate the Hamiltonian (1272) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)}=\sum_n(\varepsilon_n(t)+R_n)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|B_{nm} \quad (1284)$$

$$\overline{H_{\bar{I}}}=\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|J_{nm}+\sum_n B_{z,n}(t)|n\rangle\langle n|, \quad (1285)$$

$$\overline{H_{\bar{B}}}=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b^\dagger_{u\mathbf{k}}b_{u\mathbf{k}} \quad (1286)$$

### B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A\leq A_{\mathrm{B}}\equiv-\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right)+\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}+O\left(\left\langle\overline{H^2_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_B}}\right). \quad (1287)$$

Taking the equations (244)-(252) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right)=C\left(R_0,R_1,...,R_{d-1},B_{01},...,B_{0(d-1)},...,B_{(d-2)(d-1)}\right)$, where each $R_i$ and $B_{kj}$ depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}}=v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}$. So our approach will be based on the derivation respect to $v^{\Re}_{nu\mathbf{k}}$ and $v^{\Im}_{nu\mathbf{k}}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)} = \sum_n \left( \varepsilon_n(t) + \sum_{u\mathbf{k}} \left( \frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left( g_{nu\mathbf{k}} \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \right) |n\rangle\langle n| \tag{1288}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \prod_{u\mathbf{k}} \exp\left( \frac{(v^*_{mu\mathbf{k}} v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}} \coth\left( \frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1289}$$

$$= \sum_n \left( \varepsilon_n(t) + \sum_{u\mathbf{k}} \left( \frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \frac{g_{nu\mathbf{k}} v^*_{nu\mathbf{k}} + g^*_{nu\mathbf{k}} v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1290}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \prod_{u\mathbf{k}} \exp\left( \frac{(v^*_{mu\mathbf{k}} v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}} \coth\left( \frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1291}$$

$$= \sum_n \left( \varepsilon_n(t) + \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g^*_{nu\mathbf{k}}) v^{\Re}_{nu\mathbf{k}} + \mathrm{i} v^{\Im}_{nu\mathbf{k}} (g^*_{nu\mathbf{k}} - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1292}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \prod_{u\mathbf{k}} \exp\left( \frac{(v^*_{mu\mathbf{k}} v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}} \coth\left( \frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1293}$$

$$v^*_{mu\mathbf{k}} v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v^*_{nu\mathbf{k}} = \left( v^{\Re}_{mu\mathbf{k}} - \mathrm{i} v^{\Im}_{mu\mathbf{k}} \right) \left( v^{\Re}_{nu\mathbf{k}} + \mathrm{i} v^{\Im}_{nu\mathbf{k}} \right) - \left( v^{\Re}_{mu\mathbf{k}} + \mathrm{i} v^{\Im}_{mu\mathbf{k}} \right) \left( v^{\Re}_{nu\mathbf{k}} - \mathrm{i} v^{\Im}_{nu\mathbf{k}} \right) \tag{1294}$$

$$= \left( v^{\Re}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} + \mathrm{i} v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - \mathrm{i} v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} + v^{\Im}_{mu\mathbf{k}} v^{\Im}_{nu\mathbf{k}} \right) \tag{1295}$$

$$- \left( v^{\Re}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - \mathrm{i} v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} + \mathrm{i} v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} + v^{\Im}_{mu\mathbf{k}} v^{\Im}_{nu\mathbf{k}} \right) \tag{1296}$$

$$= 2\mathrm{i} \left( v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} \right) \tag{1297}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left( \varepsilon_n(t) + \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}} - \frac{(g_{nu\mathbf{k}} + g^*_{nu\mathbf{k}}) v^{\Re}_{nu\mathbf{k}} + \mathrm{i} v^{\Im}_{nu\mathbf{k}} (g^*_{nu\mathbf{k}} - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \right) |n\rangle\langle n| \tag{1298}$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \prod_{u\mathbf{k}} \exp\left( \frac{\mathrm{i}\left( v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} \right)}{\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}} \coth\left( \frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1299}$$

$$|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2 = (v_{mu\mathbf{k}} - v_{nu\mathbf{k}})(v_{mu\mathbf{k}} - v_{nu\mathbf{k}})^* \tag{1300}$$

$$= |v_{mu\mathbf{k}}|^2 + |v_{nu\mathbf{k}}|^2 - (v_{nu\mathbf{k}} v^*_{mu\mathbf{k}} + v^*_{nu\mathbf{k}} v_{mu\mathbf{k}}) \tag{1301}$$

$$= \left(v^{\Re}_{mu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}}\right)^2 + \left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - \left( v^{\Re}_{nu\mathbf{k}} + \mathrm{i} v^{\Im}_{nu\mathbf{k}} \right) \left( v^{\Re}_{mu\mathbf{k}} - \mathrm{i} v^{\Im}_{mu\mathbf{k}} \right) \tag{1302}$$

$$- \left( v^{\Re}_{nu\mathbf{k}} - \mathrm{i} v^{\Im}_{nu\mathbf{k}} \right) \left( v^{\Re}_{mu\mathbf{k}} + \mathrm{i} v^{\Im}_{mu\mathbf{k}} \right) \tag{1303}$$

$$= \left(v^{\Re}_{mu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}}\right)^2 + \left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2\left( v^{\Re}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} + v^{\Im}_{nu\mathbf{k}} v^{\Im}_{mu\mathbf{k}} \right) \tag{1304}$$

$$= \left( v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}} \right)^2 + \left( v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}} \right)^2 \tag{1305}$$

$$R_n(t) = \sum_{u\mathbf{k}} \left( \frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left( g_{nu\mathbf{k}} \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + g^*_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \tag{1306}$$

$$= \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - (g_{nu\mathbf{k}} + g^*_{nu\mathbf{k}}) v^{\Re}_{nu\mathbf{k}} - \mathrm{i} v^{\Im}_{nu\mathbf{k}} (g^*_{nu\mathbf{k}} - g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}} \right) \tag{1307}$$

$$= \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2 g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2 g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \tag{1308}$$

$$B_{mn} = \left( \prod_{u\mathbf{k}} \exp\left( \frac{(v^*_{mu\mathbf{k}} v_{nu\mathbf{k}} - v_{mu\mathbf{k}} v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}} \coth\left( \frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1309}$$

$$= \left( \prod_{u\mathbf{k}} \exp\left( \frac{\mathrm{i}\left( v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} \right)}{\omega^2_{u\mathbf{k}}} \right) \right) \prod_u \exp\left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{\left( v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}} \right)^2 + \left( v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}} \right)^2}{\omega^2_{u\mathbf{k}}} \coth\left( \frac{\beta_u \omega_{u\mathbf{k}}}{2} \right) \right) \tag{1310}$$

Then we can obtain using the chain rule that:

$$
\begin{aligned}
\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1311)\\
&= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1312)\\
&= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1313)\\
\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left( \frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1314)\\
&= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1315)\\
&= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1316)
\end{aligned}
$$

Given that:

$$
\begin{aligned}
\ln B_{mn} &= \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) && (1317)\\
&= \sum_{u\mathbf{k}} \ln\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln\exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1318)\\
&= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1319)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1320)\\
\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1321)\\
\frac{\partial \ln B_{mn}}{\partial a} &= \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} && (1322)\\
\frac{\partial B_{mn}}{\partial a} &= B_{mn} \frac{\partial \ln B_{mn}}{\partial a} && (1323)\\
\frac{\partial B_{mn}}{\partial a} &= \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} && (1324)
\end{aligned}
$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1325}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1326}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1327}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1328}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1329}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1330}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1331}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1332}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1333}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1334}$$

$$= (B_{mn})^{\dagger} \tag{1335}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1336}$$

Introducing this derivates in the equation (1311) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1337}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1338}$$

$$= 0 \tag{1339}$$

We can obtain the variational parameters:

$$- 2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1340}$$

$$= -\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1341}$$

$$v^{\Re}_{nu\mathbf{k}} = \frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}}+v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1342}$$

$$= \frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1343}$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1344}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1345}$$

$$= 0 \tag{1346}$$

Rearranging we obtain

$$-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1347}$$

$$=-2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1348}$$

$$v_{nu\mathbf{k}}^{\Im}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1349}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1350}$$

$$v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im} \tag{1351}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1352}$$

$$\mathrm{i}\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1353}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1354}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1355}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1356}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1357}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1358}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1359}$$

## C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^{V}\rho(0)\,e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)(|0\rangle\langle 0|\otimes\rho_B)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1360}$$

$$0=\left(B_0^{+}|0\rangle\langle 0|B_0^{-}\right)\otimes\rho_B \tag{1361}$$

$$0=\rho(0) \tag{1362}$$

We transform any operator $O$ into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1363}$$
$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1364}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\, \overline{\rho_S}(t)\, U(t)\,, \text{ where} \tag{1365}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1366}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\!\langle m| + |m\rangle\!\langle n| \tag{1367}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\!\langle m| - |m\rangle\!\langle n|\right) \tag{1368}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1369}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1370}$$

We can proof that $B_{nm} = B_{mn}^{\dagger}$

$$B_{mn}^{\dagger} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1371}$$
$$= B_{n-}^{\dagger}B_{m+}^{\dagger} - B_n B_m \tag{1372}$$
$$= B_{n+}B_{m-} - B_n B_m \tag{1373}$$
$$= B_{nm} \tag{1374}$$

So we can say that the set of matrices (1367) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1367) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)\, |n\rangle\!\langle m| B_{nm} + \sum_n B_{z,n}(t)\, |n\rangle\!\langle n|, \tag{1375}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\!\langle n| + \sum_{n<m}\left(V_{nm}(t)\, |n\rangle\!\langle m| B_{nm} + V_{mn}(t)\, |m\rangle\!\langle n| B_{mn}\right) \tag{1376}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\!\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V_{nm}^{\Im}(t)\, B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1377}$$
$$\left. + \Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V_{nm}^{\Im}(t)\, B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1378}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\!\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right) \sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1379}$$
$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right) \sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right) \sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1380}$$
$$= \sum_n B_{z,n}(t)\, |n\rangle\!\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) \sigma_{nm,x} B_{nm,x} - \Im\left(V_{nm}(t)\right) \sigma_{nm,x} B_{nm,y} + \Re\left(V_{nm}(t)\right) \sigma_{nm,y} B_{nm,y}\right. \tag{1381}$$
$$\left. + \Im\left(V_{nm}(t)\right) \sigma_{nm,y} B_{nm,x}\right) \tag{1382}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1383}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1-\delta_{mn}) \tag{1384}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1-\delta_{mn}) \tag{1385}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1386}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1387}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1388}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1389}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1390}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1391}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1392}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1393}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1394}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1395}$$
$$C_{3,nm}(t) = 1 \tag{1396}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1397}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1398}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J, p\in P} C_{jp}(t)\left(A_{jp}\otimes B_{jp}(t)\right) \tag{1399}$$

Here $J = \{1,2,3,4,5\}$and $P$ the set defined in (1383).

We write the interaction Hamiltonian transformed under (1363) as:

$$\widetilde{H_I}(t) = \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes \widetilde{B_{jp}}(t)\right) \tag{1400}$$
$$\widetilde{A_{jp}}(t) = U^{\dagger}(t)\,A_{jp}U(t) \tag{1401}$$
$$\widetilde{B_{jp}}(t) = \mathrm{e}^{\mathrm{i}H_B t}B_{jp}(t)\,(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{1402}$$

Taking as reference state $\rho_B$ and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1403}$$

Replacing the equation (1400) in (1403) we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1404}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right),\widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1405}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right),\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1406}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1407}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1408}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1409}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1410}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1411}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1412}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1413}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1411) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1414}$$
$$= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1415}$$
$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1416}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1417}$$
$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1418}$$
$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1419}$$
$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1420}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) \tag{1421}$$
$$= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1422}$$
$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1423}$$
$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1424}$$
$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1425}$$
$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1426}$$
$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1427}$$
$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1428}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1404)and (1411) and using the equations (1414)-(1428) we can re-write:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_S}}\left(t\right)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',p,p'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right) - \Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\right)\right. \tag{1429}$$

$$\left. + C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right) - \Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{jp}}\left(t\right)\right)\right)\mathrm{d}s \tag{1430}$$

$$= -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[\widetilde{A_{jp}}\left(t\right), \widetilde{A_{j'p'}}\left(s\right)\widetilde{\overline{\rho_S}}\left(t\right)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right), \widetilde{A_{jp}}\left(t\right)\right]\right)\right) \tag{1431}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\int_0^t \sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right), A_{j'p'}\left(t-\tau, t\right)\overline{\rho_S}\left(t\right)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}\left(t\right)A_{j'p'}\left(t-\tau, t\right), A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right), \overline{\rho_S}\left(t\right)\right] \tag{1432}$$

For this case we used that $A_{jp}\left(t-\tau, t\right) = U\left(t\right)U^\dagger\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^\dagger\left(t\right)$. This is a non-Markovian equation.

## VI. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1433}$$
$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1434}$$
$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1435}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1436}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

### A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1437}$$

At first let's obtain $e^V$ under the transformation (1437), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{1438}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + ... \tag{1439}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + ... \tag{1440}$$
$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + ...\right) \tag{1441}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \tag{1442}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+} \tag{1443}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1444}$$
$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1445}$$
$$= B_{\pm} \tag{1446}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_{-}$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{+}\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^{-}\right) \tag{1447}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1433):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1448}$$

$$= |0\rangle\langle 0|, \tag{1449}$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle n|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1450}$$

$$= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, \tag{1451}$$

$$= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, \tag{1452}$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|0\rangle\langle m|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right), \tag{1453}$$

$$= |0\rangle\langle m|B^{-}\ m \neq 0, \tag{1454}$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)|m\rangle\langle 0|\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1455}$$

$$= |0\rangle\langle m|B^{+}\ m \neq 0, \tag{1456}$$

$$\overline{\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1457}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}B^{+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B^{-} \tag{1458}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B^{+}b_{\mathbf{k}}^{\dagger}B^{-}\right)\left(B^{+}b_{\mathbf{k}}B^{-}\right) \tag{1459}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1460}$$

$$= |0\rangle\langle 0|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1461}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right) \tag{1462}$$

$$= \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1463}$$

The transformed Hamiltonians of the equations (1434) to (1436) written in terms of (1448) to (1463) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1464}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|} \tag{1465}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \overline{\sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n|} \tag{1466}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)\overline{|0\rangle\langle n|} + V_{n0}(t)\overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0} V_{mn}(t)\overline{|m\rangle\langle n|} \tag{1467}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)B^{-}|0\rangle\langle n| + V_{n0}(t)B^{+}|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1468}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1469}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(\left(\sum_{n=0}\mu_n(t)|n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1470}$$

$$= \left(\mu_0(t)|0\rangle\langle 0| + \sum_{n=1}\mu_n(t)|n\rangle\langle n|B^{+}\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1471}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}B^{+}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)B^{-} \tag{1472}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1473}$$

$$\overline{H_B} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1474}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1475}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1}|n\rangle\langle n|\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1476}$$

$$+\sum_{n=0}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) - \sum_{n=1}\mu_n(t)|n\rangle\langle n|\sum_{\mathbf{k}} 2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1477}$$

$$= \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t)|0\rangle\langle n|B^{-} + V_{n0}(t)|n\rangle\langle 0|B^{+}\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1478}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + \sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1479}$$

$$+\sum_{n=1}|n\rangle\langle n|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1480}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1481}$$

$$= \sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t)g_{\mathbf{k}}\right) \tag{1482}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1483}$$

Using the previous functions we have that (1480) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n (t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n} (t) |0\rangle\langle n| B^- + V_{n0} (t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn} (t) |m\rangle\langle n| \tag{1484}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0 (t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1485}$$

Now in order to separate the elements of the hamiltonian (1485) let's follow the references of the equations (**??**) and (**??**) to separate the hamiltonian like:

$$\overline{H_S (t)} = \sum_{n=0} \varepsilon_n (t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n} (t) |0\rangle\langle n| + V_{n0} (t) |n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn} (t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \tag{1486}$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0 (t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \left(V_{0n} (t) |0\rangle\langle n| \left(B^- - B\right) + V_{n0} (t) |n\rangle\langle 0| \left(B^+ - B\right)\right), \tag{1487}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{1488}$$

Here B is given by:

$$\begin{aligned} B &= \left\langle B^+ \right\rangle \\ &= \left\langle B^- \right\rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1489}$$

$$\sigma_{nm,y} = \mathrm{i} \left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1490}$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \tag{1491}$$

$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \tag{1492}$$

Using this set of hermitian operators to write the Hamiltonians (1434)-(1436)

$$\overline{H_S(t)} = \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \tag{1493}$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)|m\rangle\langle n| + V_{nm}(t)|n\rangle\langle m|) \tag{1494}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1495}$$

$$= \sum_{0<m<n}((\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t)))|m\rangle\langle n| + (\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t)))|n\rangle\langle m|) + \varepsilon_0(t)|0\rangle\langle 0| \tag{1496}$$

$$+ B\sum_{n=1}(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1497}$$

$$= \sum_{0<m<n}\left((\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t)))\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1498}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1499}$$

$$= \sum_{0<m<n}(\Re(V_{nm}(t))\sigma_{nm,x} + \Im(V_{nm}(t))\sigma_{nm,y}) + B\sum_{n=1}(\Re(V_{0n}(t))\sigma_{0n,x} + \Im(V_{mn}(t))\sigma_{0n,y}) \tag{1500}$$

$$+ \varepsilon_0(t)|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)|n\rangle\langle n| \tag{1501}$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^+ - B\right)\right) \tag{1502}$$

$$= \sum_{n=1}\left((\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\left(B^- - B\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\left(B^+ - B\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1503}$$

$$+ \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1504}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))) + \left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t)))\right)\right. \tag{1505}$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B\right)(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))) - \left(B^- - B\right)(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t)))\right)\right) \tag{1506}$$

$$+ \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1507}$$

$$= \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^+ + B^- - 2B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B^- - B - B^+ + B\right)\Im(V_{0n}(t))\right. \tag{1508}$$

$$\left.+\frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B - B^- + B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B - B^- + B - B^+\right)\Im(V_{0n}(t))\right)\right) + \sum_{n=1} B_{z,n}|n\rangle\langle n| \tag{1509}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}(\sigma_{0n,x}(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))) \tag{1510}$$

$$+\sigma_{0n,y}(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t)))) \tag{1511}$$

### B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta(\overline{H_S+H_B})}\right)\right) + \langle \overline{H_I}\rangle_{\overline{H_S+H_B}} + O\left(\left\langle \overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1512}$$

Taking the equations (244)-(252) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right) = C\left(R_1, R_2, ..., R_{d-1}, B\right)$, where each $R_i$ and $B$ depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (252) and using the chain rule we obtain that:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, \tag{1513}$$

$$= 0 \tag{1514}$$

Let's recall the equations (1481) and (1483), we can write them in terms of the variational parameters

$$B = \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) \tag{1515}$$

$$R_n = \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) \tag{1516}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B \tag{1517}$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1518}$$

Introducing this derivates in the equation (1513) give us:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} = \frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right) + \sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}} - 2\mu_n\left(t\right)g_{\mathbf{k}}\right) \tag{1519}$$

$$= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) \tag{1520}$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - \frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1521}$$

$$= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} \tag{1522}$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n} - B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1523}$$

### C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho(0)\,e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(|0\rangle\langle 0| \otimes \rho_B\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1524}$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \tag{1525}$$

$$0 = \rho(0) \tag{1526}$$

We transform any operator $O$ into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\,OU(t) \tag{1527}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_{0}^{t}\mathrm{d}t'\overline{H_S}(t')\right). \tag{1528}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\,,\ \text{where} \tag{1529}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1530}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle 0| + \sum_{n=1}\left(\Re\left(V_{0n}(t)\right)B_x\sigma_{0n,x} + \Re\left(V_{0n}(t)\right)B_y\sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right. \tag{1531}$$

$$\left.+\Im\left(V_{0n}(t)\right)B_x\sigma_{0n,y} - \Im\left(V_{0n}(t)\right)B_y\sigma_{0n,x}\right) \tag{1532}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}}\mu_0(t)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1533}$$

$$B_{z,n} = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\ \text{if} n \neq 0 \tag{1534}$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \tag{1535}$$
$$A_{2n}(t) = \sigma_{0n,y} \tag{1536}$$
$$A_{3n}(t) = |n\rangle\langle n| \tag{1537}$$
$$A_{4n}(t) = A_{2n}(t) \tag{1538}$$
$$A_{5n}(t) = A_{1n}(t) \tag{1539}$$
$$B_{1n}(t) = B_x \tag{1540}$$
$$B_{2n}(t) = B_y \tag{1541}$$
$$B_{3n}(t) = B_{z,n} \tag{1542}$$
$$B_{4n}(t) = B_{1n}(t) \tag{1543}$$
$$B_{5n}(t) = B_{2n}(t) \tag{1544}$$
$$C_{10}(t) = 0 \tag{1545}$$
$$C_{20}(t) = 0 \tag{1546}$$
$$C_{40}(t) = 0 \tag{1547}$$
$$C_{50}(t) = 0 \tag{1548}$$
$$C_{30}(t) = 1 \tag{1549}$$
$$C_{1n}(t) = \Re\left(V_{0n}(t)\right) \tag{1550}$$
$$C_{2n}(t) = C_{1n}(t) \tag{1551}$$
$$C_{3n}(t) = 1 \tag{1552}$$
$$C_{4n}(t) = \Im\left(V_{0n}(t)\right) \tag{1553}$$
$$C_{5n}(t) = -\Im\left(V_{0n}(t)\right) \tag{1554}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn} \otimes B_{jn}(t)\right) \tag{1555}$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1527) as:

$$\widetilde{H_I}(t) = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t) \otimes \widetilde{B_{jn}}(t)\right) \tag{1556}$$
$$\widetilde{A_i}(t) = U^{\dagger}(t)\, A_i U(t) \tag{1557}$$
$$\widetilde{B_i}(t) = \mathrm{e}^{\mathrm{i}H_B t} B_i(\mathrm{t})\, \mathrm{e}^{-\mathrm{i}H_B t} \tag{1558}$$

Taking as reference state $\rho_B$ and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right] \mathrm{d}s \tag{1559}$$

Replacing the equation (1556)in (1559)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1560}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1561}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1562}$$

$$\left. -\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1563}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1564}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1565}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1566}$$

$$\left. +\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1567}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1568}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1569}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1567) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1570}$$

$$= \left\langle \widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1571}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1572}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1573}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1574}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1575}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1576}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1577}$$

$$= \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1578}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1579}$$

$$= \Lambda_{jnj'n'}(\tau) \tag{1580}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1581}$$

$$= \left\langle \widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1582}$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1583}$$

$$= \Lambda_{j'n'jn}(-\tau) \tag{1584}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1560)and (1567) and using the equations (1570)-(1584) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t) - \Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1585}$$

$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t) - \Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1586}$$

$$= -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t)\,,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,,\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1587}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t)\,,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right] + \Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t)\,,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1588}$$

For this case we used that $A_{jn}(t-\tau,t) = U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n = 2$ (two sites), $\mu_0(t) = 0$, $\mu_1(t) = 1$ then we can reproduce a similar expression to (447) as expected.

## VII. BIBLIOGRAPHY

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