A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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We start with a time-dependent Hamiltonian of the form:

$$H\left(t\right)=H_S\left(t\right)+H_I+H_B, \tag{1}$$
$$H_S\left(t\right)=\varepsilon_0\left(t\right)|0\rangle\langle 0|+\varepsilon_1\left(t\right)|1\rangle\langle 1|+V_{10}\left(t\right)|1\rangle\langle 0|+V_{01}\left(t\right)|0\rangle\langle 1|, \tag{2}$$
$$H_I=|0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)+|1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$
$$H_B=\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle=\delta_{ij}. \tag{5}$$

I. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H\left(t\right)$, the unitary transformation defined by $\mathrm{e}^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V\left(t\right)\equiv|0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+|1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}\left(t\right)$ defined as:

$$v_{i\mathbf{k}}\left(t\right)=\omega_{\mathbf{k}}\alpha_{i\mathbf{k}}\left(t\right). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i\left(t\right)\equiv\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{i\mathbf{k}}^{*}\left(t\right)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$
$$V\left(t\right)=|0\rangle\langle 0|\hat{\varphi}_0\left(t\right)+|1\rangle\langle 1|\hat{\varphi}_1\left(t\right). \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $\mathrm{e}^{\pm V(t)}$ using the notation (6) will give us the following result:

$$
\begin{aligned}
\mathrm{e}^{\pm V(t)} &= \mathrm{e}^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t)+|1\rangle\langle 1|\hat{\varphi}_1(t))} && (10)\\
&= \mathbb{I} \pm \left(|0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right)\right) + \frac{\left(\pm\left(|0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right)\right)\right)^2}{2!} + \ldots && (11)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| \pm \left(|0\rangle\langle 0|\hat{\varphi}_0\left(t\right) + |1\rangle\langle 1|\hat{\varphi}_1\left(t\right)\right) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2\left(t\right)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2\left(t\right)}{2!} + \ldots && (12)\\
&= |0\rangle\langle 0|\left(\mathbb{I} \pm \hat{\varphi}_0\left(t\right) + \frac{\hat{\varphi}_0^2\left(t\right)}{2!} \pm \ldots\right) + |1\rangle\langle 1|\left(\mathbb{I} \pm \hat{\varphi}_1\left(t\right) + \frac{\hat{\varphi}_1^2\left(t\right)}{2!} \pm \ldots\right) && (13)\\
&= |0\rangle\langle 0|\mathrm{e}^{\pm\hat{\varphi}_0(t)} + |1\rangle\langle 1|\mathrm{e}^{\pm\hat{\varphi}_1(t)} && (14)\\
&= |0\rangle\langle 0|\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger} - \alpha_{0\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} + |1\rangle\langle 1|\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}(t)b_{\mathbf{k}}^{\dagger} - \alpha_{1\mathbf{k}}^{*}(t)b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_0^{\pm}\left(t\right) + |1\rangle\langle 1|B_1^{\pm}\left(t\right), && (16)\\
B_i^{\pm}\left(t\right) &\equiv \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\ \mathrm{e}^{rY}\ \mathrm{e}^{-\frac{r^2}{2}[X,Y]}\ \mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\ \mathrm{e}^{\frac{-r^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\cdots. \tag{18}
$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^{\dagger} - \frac{v_{j\mathbf{k}'}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$ for all $\mathbf{k}', \mathbf{k}, i, j$ we can show plugging $r = 1$ in (18) the following result:

$$
\begin{aligned}
\mathrm{e}^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\cdots && (19)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{-\frac{1}{2}0}\cdots && (20)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathbb{I} && (21)\\
&= \mathrm{e}^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}\mathrm{e}^{\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. && (22)
\end{aligned}
$$

By induction of this result we can write an expresion of $B_i^{\pm}\left(t\right)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D\left(\pm v_{i\mathbf{k}}\left(t\right)\right)$:

$$
\begin{aligned}
D\left(\pm v_{i\mathbf{k}}\left(t\right)\right) &\equiv \mathrm{e}^{\pm\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, && (23)\\
B_i^{\pm}\left(t\right) &= \prod_{\mathbf{k}} D\left(\pm\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right). && (24)
\end{aligned}
$$

this will help us to write operators $O\left(t\right)$ transformed in the variational frame as:

$$
\overline{O\left(t\right)} \equiv \mathrm{e}^{V(t)}O\left(t\right)\mathrm{e}^{-V(t)}. \tag{25}
$$

We will use the following identities:

$$\overline{|0\rangle\langle 0|\,(t)} = \mathrm{e}^{V(t)}|0\rangle\langle 0|\mathrm{e}^{-V(t)} \tag{26}$$

$$= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{27}$$

$$= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{28}$$

$$= |0\rangle\langle 0|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{29}$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) \tag{30}$$

$$= |0\rangle\langle 0|, \tag{31}$$

$$\overline{|1\rangle\langle 1|\,(t)} = \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{32}$$

$$= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{33}$$

$$= |1\rangle\langle 1|B_1^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{34}$$

$$= |1\rangle\langle 1|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) \tag{35}$$

$$= B_1^+\,(t)\,|1\rangle\langle 1|1\rangle\langle 1|B_1^-\,(t) \tag{36}$$

$$= |1\rangle\langle 1|, \tag{37}$$

$$\overline{|0\rangle\langle 1|\,(t)} = \mathrm{e}^{V(t)}|0\rangle\langle 1|\mathrm{e}^{-V(t)} \tag{38}$$

$$= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{39}$$

$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{40}$$

$$= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+\,(t) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{41}$$

$$= |0\rangle\langle 1|B_0^+\,(t)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{42}$$

$$= |0\rangle\langle 1|0\rangle\langle 0|B_0^+\,(t)\,B_0^-\,(t) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t) \tag{43}$$

$$= |0\rangle\langle 1|B_0^+\,(t)\,B_1^-\,(t)\,, \tag{44}$$

$$\overline{|1\rangle\langle 0|\,(t)} = \mathrm{e}^{V(t)}|1\rangle\langle 0|\mathrm{e}^{-V(t)} \tag{45}$$

$$= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{46}$$

$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\,|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{47}$$

$$= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\,(t)\right)\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{48}$$

$$= |1\rangle\langle 0|0\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+\,(t)\,B_1^-\,(t) \tag{49}$$

$$= |1\rangle\langle 0|B_1^+\,(t)\,B_0^-\,(t)\,, \tag{50}$$

$$\overline{b_{\mathbf{k}}\,(t)} = \mathrm{e}^{V(t)}b_{\mathbf{k}}\mathrm{e}^{-V(t)} \tag{51}$$

$$= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{52}$$

$$= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| \tag{53}$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)\,b_{\mathbf{k}}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)\,b_{\mathbf{k}}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)\,b_{\mathbf{k}}B_0^-(t) + |1\rangle\langle 1|B_1^+(t)\,b_{\mathbf{k}}B_1^-(t) \tag{54}$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{55}$$

$$= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{56}$$

$$= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{57}$$

$$\overline{b_{\mathbf{k}}\,(t)}^{\dagger} = \mathrm{e}^{V(t)}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V(t)} \tag{58}$$

$$= \left(|0\rangle\langle 0|B_0^+\,(t) + |1\rangle\langle 1|B_1^+\,(t)\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^-\,(t) + |1\rangle\langle 1|B_1^-\,(t)\right) \tag{59}$$

$$= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t)|1\rangle\langle 1| \tag{60}$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_0^-(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^{\dagger}B_1^-(t) \tag{61}$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}\,(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}\,(t)}{\omega_{\mathbf{k}}}\right) \tag{62}$$

$$= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^{*}\,(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^{*}\,(t)}{\omega_{\mathbf{k}}}. \tag{63}$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+}(t)\, b_{\mathbf{k}} B_i^{-}(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \tag{64}$$

$$B_i^{+}(t)\, b_{\mathbf{k}}^{\dagger} B_i^{-}(t) = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}. \tag{65}$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t)\,|0\rangle\langle 0|\,(t)} = \varepsilon_0(t)\,|0\rangle\langle 0|, \tag{66}$$

$$\overline{\varepsilon_1(t)\,|1\rangle\langle 1|\,(t)} = \varepsilon_1(t)\,|1\rangle\langle 1|, \tag{67}$$

$$\overline{V_{10}(t)\,|1\rangle\langle 0|\,(t)} = V_{10}(t)\,|1\rangle\langle 0|B_1^{+}(t)\,B_0^{-}(t), \tag{68}$$

$$\overline{V_{01}(t)\,|0\rangle\langle 1|\,(t)} = V_{01}(t)\,|0\rangle\langle 1|B_0^{+}(t)\,B_1^{-}(t), \tag{69}$$

$$\overline{\left(g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}\right)(t)} = g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\right)+g_{\mathbf{ik}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{70}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{71}$$

$$= g_{i\mathbf{k}}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left((|0\rangle\langle 0|+|1\rangle\langle 1|)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) \tag{72}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|0\rangle\langle 0| - g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| - g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|1\rangle\langle 1| \tag{73}$$

$$= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^{*}b_{\mathbf{k}} - \left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0| - \left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, \tag{74}$$

$$\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0|B_0^{+}(t)+|1\rangle\langle 1|B_1^{+}(t)\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}(t)+|1\rangle\langle 1|B_1^{-}(t)\right) \tag{75}$$

$$= |0\rangle\langle 0|B_0^{+}(t)\,|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^{-}(t) \tag{76}$$

$$= |0\rangle\langle 0|B_0^{+}(t)\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_0^{-}(t) \tag{77}$$

$$= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)+g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{78}$$

$$\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)(t)} = \left(|0\rangle\langle 0|B_0^{+}(t)+|1\rangle\langle 1|B_1^{+}(t)\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}(t)+|1\rangle\langle 1|B_1^{-}(t)\right) \tag{79}$$

$$= |1\rangle\langle 1|B_1^{+}(t)\,|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^{-}(t) \tag{80}$$

$$= |1\rangle\langle 1|B_1^{+}(t)\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_1^{-}(t) \tag{81}$$

$$= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)+g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right), \tag{82}$$

$$\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}(t)} = \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}(t)+|1\rangle\langle 1|B_1^{+}(t)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{-}(t)+|1\rangle\langle 1|B_1^{-}(t)\right) \tag{83}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{84}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_0^{-}(t)+|1\rangle\langle 1|B_1^{+}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_1^{-}(t)\right) \tag{85}$$

$$= \omega_{\mathbf{k}}\sum_j |j\rangle\langle j|D\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right) \tag{86}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I}+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) \tag{87}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{88}$$

$$= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\right) \tag{89}$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (92)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \quad (93)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right). \quad (94)$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)|0\rangle\langle 0|} + \overline{\varepsilon_1(t)|1\rangle\langle 1|} + \overline{V_{10}(t)|1\rangle\langle 0|} + \overline{V_{01}(t)|0\rangle\langle 1|} \quad (95)$$

$$= \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| B_1^+(t) B_0^-(t) + V_{01}(t)|0\rangle\langle 1| B_0^+(t) B_1^-(t), \quad (96)$$

$$\overline{H_I} = \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \quad (97)$$

$$= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} \quad (98)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (99)$$

$$= \sum_{\mathbf{k},i} |i\rangle\langle i| \left(g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (100)$$

$$\overline{H_B} = \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \quad (101)$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) \right) \quad (102)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^{\dagger} \right) \right) \right). \quad (103)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t)|j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t)|j\rangle\langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} + \frac{\left| v_{j\mathbf{k}}(t) \right|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right). \quad (104)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}. \quad (105)$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (106)$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^* b_{\mathbf{k}} \right), \quad (107)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right). \quad (108)$$

$\chi_{ij}(t)$ is an imaginary number so $\mathrm{e}^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) & \mathrm{e}^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \tag{109}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{110}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{111}$$

We reduced the length of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{\mathrm{e}^{-\beta H_B}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_B}\right)}. \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{113}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{114}$$

So the product of the matrix representation of $b^{\dagger}$ and b with $-\beta$ is:

$$-\beta\omega b^{\dagger}b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right|, \tag{116}$$

The density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j.

$$
\begin{aligned}
\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} &= \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, , &(117)\\
\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} &= \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. &(118)
\end{aligned}
$$

The value of $\mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)$ is:

$$
\begin{aligned}
\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &(119)\\
&= \sum_{j_{\mathbf{k}}} \left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} &(120)\\
&= \frac{1}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \quad \text{(by geometric series)} &(121)\\
&\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), &(122)\\
\mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) &= \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &(123)\\
&= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &(124)\\
&= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). &(125)
\end{aligned}
$$

So the density matrix of the bath is:

$$
\begin{aligned}
\rho_B &= \frac{\mathrm{e}^{-\beta H_B}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H_B}\right)} &(126)\\
&= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} &(127)\\
&= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. &(128)
\end{aligned}
$$

Now, given that creation and annihilation satisfy:

$$
\begin{aligned}
b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle &= \sqrt{j_{\mathbf{k}}}\, |j_{\mathbf{k}} - 1\rangle, &(129)\\
b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle &= \sqrt{j_{\mathbf{k}} + 1} \mid j_{\mathbf{k}} + 1\rangle. &(130)
\end{aligned}
$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (129)-(130):

$$\begin{aligned}
\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Tr}\left(B_{iz}(t)\,\rho_B\right) && (131)\\
&= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\,b_{\mathbf{k}}\right)\right)\rho_B\right) && (132)\\
&= \sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\,b_{\mathbf{k}}\rho_B\right) && (133)\\
&= \sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\,\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) && (134)\\
&= \sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\,b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right) && (135)\\
&= \sum_{\mathbf{k}}(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t}))\,\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right)+\sum_{\mathbf{k}}(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}(\mathrm{t}))^{*}\,\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}\right), && (136)\\
\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\quad\text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}) && (137)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) && (138)\\
&= 0, && (139)\\
\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) &= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\quad\text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}) && (140)\\
&= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) && (141)\\
&= 0. && (142)
\end{aligned}$$

we therefore find that:

$$\langle B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = 0. \tag{143}$$

Another important expected value is $B(t) = \langle B^{\pm}(t)\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm}(t) = {}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\begin{aligned}
\left\langle B^{\pm}(t)\right\rangle_{H_B} &= \mathrm{Tr}\left(\rho_B B^{\pm}(t)\right) = \mathrm{Tr}\left(B^{\pm}(t)\,\rho_B\right) && (144)\\
&= \mathrm{Tr}\left(\mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) && (145)\\
&= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}(t)\right)\rho_B\right) && (146)\\
&= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}(t)\right)\right\rangle. && (147)
\end{aligned}$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha. \tag{148}$$

where $P(\alpha)$ satisfies $\int P(\alpha)\,\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P(\alpha)$ is given by:

$$\begin{aligned}
\langle A\rangle &= \mathrm{Tr}\left(A\rho\right) && (149)\\
&= \int P(\alpha)\,\langle\alpha|A|\alpha\rangle\mathrm{d}^2\alpha. && (150)
\end{aligned}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N}\mathrm{e}^{-\frac{|\alpha|^2}{N}}$ where $N = \left(\mathrm{e}^{\beta\omega} - 1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = \frac{1}{k_{\mathrm{B}}T}$.

Using the integral representation (150) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle\alpha|D(h)|\alpha\rangle \mathrm{d}^2\alpha \tag{151}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(-\alpha)D(h)D(\alpha)|0\rangle \mathrm{d}^2\alpha, \tag{152}$$

$$D(h)D(\alpha) = D(h+\alpha)\,\mathrm{e}^{\frac{1}{2}(h\alpha^* - h^*\alpha)}, \tag{153}$$

$$D(-\alpha)(D(h)D(\alpha)) = D(-\alpha)D(h+\alpha)\,\mathrm{e}^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{154}$$

$$= D(h)\,\mathrm{e}^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))}\mathrm{e}^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{155}$$

$$= D(h)\,\mathrm{e}^{\frac{1}{2}\left(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2\right)}\mathrm{e}^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \tag{156}$$

$$= D(h)\,\mathrm{e}^{(h\alpha^* - h^*\alpha)}, \tag{157}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle 0|D(h)\,\mathrm{e}^{(h\alpha^* - h^*\alpha)}|0\rangle \mathrm{d}^2\alpha \tag{158}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^* - h^*\alpha)}\langle 0|D(h)|0\rangle \mathrm{d}^2\alpha \tag{159}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^* - h^*\alpha)}\langle 0|h\rangle \mathrm{d}^2\alpha, \tag{160}$$

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{161}$$

$$\langle D(h)\rangle = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^* - h^*\alpha)}\langle 0|\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha \tag{162}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{(h\alpha^* - h^*\alpha)}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{163}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha}\mathrm{d}^2\alpha, \tag{164}$$

$$\alpha = x + \mathrm{i}y, \tag{165}$$

$$\langle D(h)\rangle = \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2+y^2}{N} + h(x-\mathrm{i}y) - h^*(x+\mathrm{i}y)}\mathrm{d}x\mathrm{d}y \tag{166}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{N} + hx - h^*x}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{N} + hx - h^*x}\mathrm{d}y, \tag{167}$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N}\left(x^2 - Nhx + Nh^*x\right) \tag{168}$$

$$= -\frac{1}{N}\left(x + \frac{(Nh^* - Nh)}{2}\right)^2 + \frac{N(h^* - h)^2}{4}, \tag{169}$$

$$-\frac{y^2}{N} - \mathrm{i}hy - \mathrm{i}h^*y = -\frac{1}{N}\left(y^2 + \mathrm{i}Nhy + \mathrm{i}Nh^*y\right) \tag{170}$$

$$= -\frac{1}{N}\left(y^2 + \frac{\mathrm{i}N(h + h^*)}{2}\right) - \frac{N(h + h^*)^2}{4}, \tag{171}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x, \tag{172}$$

$$\langle D(h)\rangle = \frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}\mathrm{d}x\mathrm{d}y, \tag{173}$$

$$\begin{aligned}\langle D\left(h\right)\rangle &= \frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}}\mathrm{d}y &(174)\\ &= \frac{\mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}}}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 &(175)\\ &= \mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}} &(176)\\ &= \mathrm{e}^{-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}} &(177)\\ &= \mathrm{e}^{-|h|^2\left(N+\frac{1}{2}\right)} &(178)\\ &= \mathrm{e}^{-|h|^2\left(\frac{1}{\mathrm{e}^{\beta\omega}-1}+\frac{1}{2}\right)} &(179)\\ &= \mathrm{e}^{-\frac{|h|^2}{2}\left(\frac{\mathrm{e}^{\beta\omega}+1}{\mathrm{e}^{\beta\omega}-1}\right)} &(180)\\ &= \mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}. &(181)\end{aligned}$$

In the last line we used $\frac{\mathrm{e}^{\beta\omega}+1}{\mathrm{e}^{\beta\omega}-1}=\coth\left(\frac{\beta\omega}{2}\right)$. So the value of (146) using (181) is given by:

$$B = \mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{182}$$

We will now force $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$. We will also introduce the bath renormalizing driving in $\overline{H_S}\left(t\right)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+\left(t\right)\sigma^+$ and $B_i^-\left(t\right)\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}\left(t\right)$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\left(t\right)\rangle_{\overline{H_{\bar{B}}}}=0$.

A final form of the terms of the Hamiltonian $\overline{H}\left(t\right)$ is:

$$\begin{aligned}\overline{H(t)} &= \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_j^+(t)B_{j'}^-(t)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}+\sum_{j\mathbf{k}}|j\rangle\langle j|\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}(t)\right)^*b_{\mathbf{k}}+\frac{\left|v_{j\mathbf{k}}(t)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{j\mathbf{k}}\frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}+g_{j\mathbf{k}}^*\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) &(183)\\ &= \sum_j\varepsilon_j(t)|j\rangle\langle j|+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|B_{jj'}(t)+\sum_j|j\rangle\langle j|B_{jz}(t)+\sum_{j\neq j'}V_{jj'}(t)|j\rangle\langle j'|\left(B_j^+\left(t\right)B_{j'}^-(t)-B_{jj'}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}} &(184)\\ &\equiv \overline{H_{\bar{S}}\left(t\right)}+\overline{H_{\bar{I}}}\left(t\right)+\overline{H_{\bar{B}}}. &(185)\end{aligned}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\begin{aligned}\langle B_i^+\left(t\right)B_j^-\left(t\right)\rangle &= B_{ij}\left(t\right) &(186)\\ &= \left\langle\prod_{\mathbf{k}}D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}}D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right\rangle &(187)\\ &= \left\langle\prod_{\mathbf{k}}\left(D\left(\frac{v_{i\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)D\left(-\frac{v_{j\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right)\right\rangle &(188)\end{aligned}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \tag{189}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{190}$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{-\frac{1}{2}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \tag{191}$$

$$= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \tag{192}$$

From the definition $B_{01}(t) = \langle B_0^+(t)\, B_1^-(t) \rangle$ using the displacement operator we have:

$$\begin{aligned}
\langle B_0^+(t)\, B_1^-(t)\rangle &= B_{01}(t) & (193)\\
&= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle & (194)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle & (195)\\
&= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle & (196)\\
&= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) & (197)\\
&= \prod_{\mathbf{k}} \left(\mathrm{e}^{-\frac{1}{2}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) & (198)\\
&= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. & (199)
\end{aligned}$$

We can check:

$$\begin{aligned}
\langle B_0^+(t)\, B_1^-(t)\rangle &= B_{01}(t) & (200)\\
&= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} & (201)\\
&= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)^*} & (202)\\
&= \langle B_1^+(t)\, B_0^-(t)\rangle^* & (203)\\
&= B_{10}^*(t)\,. & (204)
\end{aligned}$$

The parts of the splitted Hamiltonian with $\sigma^+ \equiv |1\rangle\langle 0|$ and $\sigma^- \equiv |0\rangle\langle 1|$ are:

$$\begin{aligned}
\overline{H_{\bar{S}}(t)} &\equiv (\varepsilon_0(t) + R_0(t))\,|0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t))\,|1\rangle\langle 1| + V_{10}(t)\,B_{10}\sigma^+ + V_{01}(t)\,B_{01}\sigma^-, & (205)\\
\overline{H_{\bar{I}}} &\equiv V_{10}(t)\left(B_1^+(t)\,B_0^-(t) - B_{10}(t)\right)\sigma^+ + V_{01}(t)\left(B_0^+(t)\,B_1^-(t) - B_{01}(t)\right)\sigma^- + |0\rangle\langle 0|B_{0z}(t) + |1\rangle\langle 1|B_{1z}(t)\,, & (206)\\
\overline{H_{\bar{B}}} &\equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} & (207)\\
&= H_B. & (208)
\end{aligned}$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.
For the Hamiltonian (206) we can verify the condition $\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$\begin{aligned}
\left\langle \overline{H_{\bar{I}}} \right\rangle_{\overline{H_{\bar{B}}}} &= \left\langle \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^{+}(t) B_{j'}^{-}(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} && (209)\\
&= \left\langle \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^{+}(t) B_{j'}^{-}(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} && (210)\\
&= \sum_{n\mathbf{k}} \left(\left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| \left(\left\langle V_{jj'}(t) B_j^{+}(t) B_{j'}^{-}(t) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle V_{jj'}(t) B_{jj'}(t) \right\rangle_{\overline{H_{\bar{B}}}} \right) && (211)\\
&= \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(\left\langle B_j^{+}(t) B_{j'}^{-}(t) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_{jj'}(t) \right\rangle_{\overline{H_{\bar{B}}}} \right) && (212)\\
&= \sum_{n\mathbf{k}} \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)\right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j\neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(B_{jj'}(t) - B_{jj'}(t) \right) && (213)\\
&= 0. && (214)
\end{aligned}$$

We used (143) and (199) to evaluate the expression. Let's consider the following Hermitian combinations:

$$\begin{aligned}
B_x(t) &= B_x^{\dagger}(t) && (215)\\
&= \frac{B_1^{+}(t) B_0^{-}(t) + B_0^{+}(t) B_1^{-}(t) - B_{10}(t) - B_{01}(t)}{2}, && (216)\\
B_y(t) &= B_y^{\dagger}(t) && (217)\\
&= \frac{B_0^{+}(t) B_1^{-}(t) - B_1^{+}(t) B_0^{-}(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, && (218)\\
B_{iz}(t) &= B_{iz}^{\dagger}(t) && (219)\\
&= \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right). && (220)
\end{aligned}$$

Writing the equations (205) and (206) using the previous combinations we obtain that:

$$\begin{aligned}
\overline{H_{\bar{S}}}(t) &= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \sigma^{+} + V_{01}(t) B_{01}(t) \sigma^{-} && (221)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) B_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} && (222)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right) |j\rangle\langle j| + V_{10}(t) \left(B_{10}^{\Re}(t) + \mathrm{i}B_{10}^{\Im}(t) \right) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) \left(B_{10}^{\Re}(t) - \mathrm{i}B_{10}^{\Im}(t) \right) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} && (223)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left(V_{10}(t) \frac{\sigma_x + \mathrm{i}\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - \mathrm{i}\sigma_y}{2} \right) && (224)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + \mathrm{i}\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) && (225)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t) + R_j(t)\right) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\mathrm{i}\sigma_x V_{10}^{\Im}(t) + \mathrm{i}\sigma_y V_{10}^{\Re}(t) \right) && (226)\\
&= \left(\varepsilon_0(t) + R_0(t)\right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right) |1\rangle\langle 1| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + \mathrm{i}B_{10}^{\Im}(t) \left(\mathrm{i}\sigma_x V_{10}^{\Im}(t) + \mathrm{i}\sigma_y V_{10}^{\Re}(t) \right) && (227)\\
&= \left(\varepsilon_0(t) + R_0(t)\right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right) |1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) - \left(\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) && (228)\\
&= \left(\varepsilon_0(t) + R_0(t)\right) |0\rangle\langle 0| + \left(\varepsilon_1(t) + R_1(t)\right) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) && (229)
\end{aligned}$$

$$= \left(\varepsilon_0\left(t\right) + R_0\left(t\right)\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right) + R_1\left(t\right)\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right) - B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right) - \sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right) + B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right), \tag{230}$$

$$\overline{H_{\bar{I}}}\left(t\right) = V_{10}\left(t\right)\left(\sigma^+B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+B_{10}\left(t\right)\right) + V_{01}\left(t\right)\left(\sigma^-B_0^+\left(t\right)B_1^-\left(t\right) - \sigma^-B_{01}\left(t\right)\right) + |0\rangle\langle 0|B_{0z}\left(t\right) + |1\rangle\langle 1|B_{1z}\left(t\right) \tag{231}$$

$$= |0\rangle\langle 0|B_{0z}\left(t\right) + |1\rangle\langle 1|B_{1z}\left(t\right) + \left(V_{10}^{\Re}\left(t\right) + \mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^+B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+B_{10}\left(t\right)\right) + \left(V_{10}^{\Re}\left(t\right) - \mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^-B_0^+\left(t\right)B_1^-\left(t\right) - \sigma^-B_{01}\left(t\right)\right) \tag{232}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma^+B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+B_{10}\left(t\right) + \sigma^-B_0^+\left(t\right)B_1^-\left(t\right) - \sigma^-B_{01}\left(t\right)\right) + \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma^+B_1^+\left(t\right)B_0^-\left(t\right) - \sigma^+B_{10}\left(t\right) - \sigma^-B_0^+\left(t\right)B_1^-\left(t\right) + \sigma^-B_{01}\left(t\right)\right) \tag{233}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+\left(t\right)B_0^-\left(t\right) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+\left(t\right)B_1^-\left(t\right) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{234}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+\left(t\right)B_0^-\left(t\right) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}\left(t\right) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+\left(t\right)B_1^-\left(t\right) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{235}$$

$$+ \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_1^+\left(t\right)B_0^-\left(t\right) - \frac{\sigma_x + \mathrm{i}\sigma_y}{2}B_{10}\left(t\right) - \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_0^+\left(t\right)B_1^-\left(t\right) + \frac{\sigma_x - \mathrm{i}\sigma_y}{2}B_{01}\left(t\right)\right) \tag{236}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2} + \mathrm{i}\sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) - B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2}\right) \tag{237}$$

$$+ \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma_x\frac{B_1^+\left(t\right)B_0^-\left(t\right) - B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2} + \mathrm{i}\sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{238}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_xB_x(t) + \sigma_yB_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x\frac{B_1^+(t)B_0^-(t) - B_0^+(t)B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} - \sigma_y\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{239}$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_xB_x(t) + \sigma_yB_y(t)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+(t)B_0^-(t) - B_0^+(t)B_1^-(t) - B_{10}(t) + B_{01}(t)}{2\mathrm{i}} - \sigma_y\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right) \tag{240}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_xB_x\left(t\right) + \sigma_yB_y\left(t\right)\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^+B_0^-\left(t\right) - B_0^+B_1^-\left(t\right) - B_{10}\left(t\right) + B_{01}\left(t\right)}{2\mathrm{i}} - \sigma_y\frac{B_1^+\left(t\right)B_0^-\left(t\right) + B_0^+\left(t\right)B_1^-\left(t\right) - B_{10}\left(t\right) - B_{01}\left(t\right)}{2}\right) \tag{241}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right) + \sigma_yB_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^2\sigma_x\left(-B_y\left(t\right)\right) - \sigma_yB_x\left(t\right)\right) \tag{242}$$

$$= \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_xB_x\left(t\right) + \sigma_yB_y\left(t\right)\right) + V_{10}^{\Im}\left(t\right)\left(\sigma_xB_y\left(t\right) - \sigma_yB_x\left(t\right)\right). \tag{243}$$

II. FREE-ENERGY MINIMIZATION

The true free energy $E_{\text{Free}}\left(t\right)$ is bounded by the Bogoliubov inequality:

$$E_{\text{Free}}\left(t\right) \leq E_{\text{Free,B}}\left(t\right) \equiv -\frac{1}{\beta}\ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right)\right) + \left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right). \tag{244}$$

We will optimize the set of variational parameters $\{v_{\mathbf{k}}\left(t\right)\}$ in order to minimize $E_{\text{Free,B}}\left(t\right)$ (i.e. to make it as close to the true free energy $E_{\text{Free}}\left(t\right)$ as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{\mathbf{k}}\left(t\right)\}$:

$$\frac{\partial E_{\text{Free,B}}\left(t\right)}{\partial v_{i\mathbf{k}}\left(t\right)} = 0. \tag{245}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\left(t\right), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = \mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}. \tag{246}$$

Then using the fact that $\overline{H_{\bar{S}}}\left(t\right)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) = \text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right). \tag{247}$$

So Eq. (245) becomes:

$$\frac{\partial E_{\text{Free,B}}(t)}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{248}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{249}$$

$$= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)+\ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{250}$$

$$= -\frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta}\frac{\partial \ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}(t)} \tag{251}$$

$$= 0 \quad \text{(by Eq. (245))}. \tag{252}$$

But since $\overline{H_{\bar{B}}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial \ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} \tag{253}$$

$$= 0. \tag{254}$$

This means we need to impose:

$$\frac{\partial \text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{255}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}(t) = -\beta\left(\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0| + \left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1| + V_{10}(t)B_{10}(t)\sigma^+ + V_{01}(t)B_{01}(t)\sigma^-\right). \tag{256}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}\left(-\beta\overline{H_{\bar{S}}}(t)\right) + \text{Det}\left(-\beta\overline{H_{\bar{S}}}(t)\right) = 0. \tag{257}$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{258}$$

$$\eta \equiv \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}. \tag{259}$$

The solutions of the equation (257) are:

$$\lambda = \beta\frac{-\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right) \pm \sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}}(t)\right)}}{2} \tag{260}$$

$$= \beta\frac{-\varepsilon(t) \pm \eta(t)}{2} \tag{261}$$

$$= -\beta\frac{\varepsilon(t) \mp \eta(t)}{2}. \tag{262}$$

The value of Tr $\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)=\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{\frac{\eta(t)\beta}{2}}+\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\mathrm{e}^{-\frac{\eta(t)\beta}{2}} \tag{263}$$

$$=2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right). \tag{264}$$

Given that $v_{i\mathbf{k}}(t)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.
Using the chain rule on the function Tr $\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)=A(\varepsilon(t),\eta(t))$ to calculate $\frac{\partial\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}=\frac{\partial\left(2\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \tag{265}$$

$$=2\left(-\frac{\beta}{2}\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\cosh\left(\frac{\eta(t)\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\right)\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\sinh\left(\frac{\eta(t)\beta}{2}\right) \tag{266}$$

$$=-\beta\mathrm{e}^{-\frac{\varepsilon(t)\beta}{2}}\left(\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right)-\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right)\right). \tag{267}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\cosh\left(\frac{\eta(t)\beta}{2}\right)-\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\sinh\left(\frac{\eta(t)\beta}{2}\right)=0. \tag{268}$$

The derivates included in the expression given are related to:

$$\left\langle B_0^+(t)B_1^-(t)\right\rangle=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{269}$$

$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)^*\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{270}$$

$$=\left\langle B_1^+(t)B_0^-(t)\right\rangle^*, \tag{271}$$

$$R_i(t)=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}(t)\right|^2}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right) \tag{272}$$

$$=\sum_{\mathbf{k}}\left(\frac{\left|v_{i\mathbf{k}}(t)\right|^2}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}-g_{i\mathbf{k}}^*\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \tag{273}$$

$$\left\langle B_0^+(t)B_1^-(t)\right\rangle=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}\right)}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{274}$$

$$=\left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}}\right)\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{275}$$

$$v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)=\left(v_{0\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right)-\left(v_{0\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{0\mathbf{k}}^{\Im}(t)\right)\left(v_{1\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{1\mathbf{k}}^{\Im}(t)\right) \tag{276}$$

$$=\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-\mathrm{i}v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)+v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Im}(t)\right) \tag{277}$$

$$-\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)+\mathrm{i}v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)+v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Im}(t)\right) \tag{278}$$

$$=2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right), \tag{279}$$

$$
\begin{aligned}
\left|v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right|^{2} &= \left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)^{*} && (280)\\
&= \left|v_{1\mathbf{k}}(t)\right|^{2}+\left|v_{0\mathbf{k}}(t)\right|^{2}-\left(v_{1\mathbf{k}}(t)\,v_{0\mathbf{k}}^{*}(t)+v_{1\mathbf{k}}^{*}(t)\,v_{0\mathbf{k}}(t)\right) && (281)\\
&= \left(v_{1\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)\right)^{2}+\left(v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{0\mathbf{k}}^{\Im}(t)\right)^{2}-2\left(v_{1\mathbf{k}}^{\Re}(t)\,v_{0\mathbf{k}}^{\Re}(t)+v_{1\mathbf{k}}^{\Im}(t)\,v_{0\mathbf{k}}^{\Im}(t)\right) && (282)\\
&= \left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}. && (283)
\end{aligned}
$$

Rewriting in terms of real and imaginary parts.

$$
\begin{aligned}
R_i(t) &= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}}-\left(g_{i\mathbf{k}}\frac{v_{i\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{i\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}}\right)\right) && (284)\\
&= \sum_{\mathbf{k}}\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\,\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\,\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), && (285)\\
\left\langle B_0^{+}(t)\,B_1^{-}(t)\right\rangle &= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{v_{0\mathbf{k}}^{*}(t)v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^{*}(t)}{2\omega_{\mathbf{k}}^{2}}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) && (286)\\
&= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{2\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{2\omega_{\mathbf{k}}^{2}}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) && (287)\\
&= \left(\prod_{\mathbf{k}}\mathrm{e}^{\frac{\mathrm{i}\left(v_{0\mathbf{k}}^{\Re}(t)v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)v_{1\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}}}\right)\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right), && (288)
\end{aligned}
$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$
\begin{aligned}
\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\left(\varepsilon_1(t)+R_1+\varepsilon_0(t)+R_0(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (289)\\
&= \frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}(t)\,\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}(t)\,\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (290)\\
&= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}, && (291)\\
\frac{\partial\left|B_{10}(t)\right|^{2}}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (292)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}(t)-v_{0\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} && (293)\\
&= -\frac{2\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}(t)-v_{0\mathbf{k}}^{\Re}(t)\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}\left|B_{10}(t)\right|^{2}, && (294)\\
\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} &= \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}(t)} && (295)\\
&= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} && (296)\\
&= \frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_1(t)+R_1(t)\right)\left(\varepsilon_0(t)+R_0(t)\right)-\left|V_{10}(t)\right|^{2}\left|B_{10}(t)\right|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} && (297)
\end{aligned}
$$

$$=\frac{\varepsilon(t)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}(t)-R_{i}(t)\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\omega^{2}_{\mathbf{k}}}\frac{\partial\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)}{\partial v^{\Re}_{i\mathbf{k}}(t)}\left|B_{10}(t)\,V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{298}$$

$$=\frac{\varepsilon\left(t\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)\left(\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v^{\Re}_{i\mathbf{k}}(t)-v^{\Re}_{i'\mathbf{k}}(t)\right)}{\omega^{2}_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{299}$$

$$=\frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon\left(t\right)-4\left(\varepsilon\left(t\right)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}(t)V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\left(t\right)}\right)+\frac{1}{\eta\left(t\right)}\left(-\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon\left(t\right)+2\left(\varepsilon\left(t\right)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right. \tag{300}$$

$$\left.+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^{2}_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{301}$$

From the equation (268) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}}{\frac{\partial\eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)}} \tag{302}$$

$$=\frac{\frac{2v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{2g^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}}{\frac{v^{\Re}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega^{2}_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g^{\Re}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{303}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-g_{i\mathbf{k}}-g^{*}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{304}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}(t)-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$=\frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{307}$$

Separating (306) such that the terms with $v_{i\mathbf{k}}\left(t\right)$ are located at one side of the equation permit us to write:

$$\frac{\left(v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Re}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t))-\frac{2|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}\left(2\left(\varepsilon_{i}(t)+R_{i}(t)\right)-\varepsilon(t)\right)+2\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{308}$$

$$v^{\Re}_{i\mathbf{k}}(t)-g^{\Re}_{i\mathbf{k}}=v^{\Re}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right) \tag{309}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta\left(t\right)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{310}$$

$$v^{\Re}_{i\mathbf{k}}(t)=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{311}$$

$$v^{\Re}_{i\mathbf{k}}(t)=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}\left(t\right)-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{g^{\Re}_{i\mathbf{k}}}\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\left(t\right)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{312}$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon(t)}{\partial v^{\Im}_{i\mathbf{k}}(t)} = \frac{\partial\left(\varepsilon_1(t) + R_1(t) + \varepsilon_0(t) + R_0(t)\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)} \tag{313}$$

$$= \frac{\partial\left(\left(\frac{\left(v^{\Re}_{i\mathbf{k}}(t)\right)^2 + \left(v^{\Im}_{i\mathbf{k}}(t)\right)^2}{\omega_{\mathbf{k}}} - v^{\Re}_{i\mathbf{k}}(t)\,\frac{g_{i\mathbf{k}}+g^*_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \mathrm{i}v^{\Im}_{i\mathbf{k}}(t)\,\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)} \tag{314}$$

$$= 2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}} - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{315}$$

$$\frac{\partial\left|B_{10}(t)\right|^2}{\partial v^{\Im}_{i\mathbf{k}}(t)} = \frac{\partial\left(\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(\mathrm{v}^{\Re}_{1\mathbf{k}}(t)-\mathrm{v}^{\Re}_{0\mathbf{k}}(t)\right)^2+\left(\mathrm{v}^{\Im}_{1\mathbf{k}}(t)-\mathrm{v}^{\Im}_{0\mathbf{k}}(t)\right)^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)} \tag{316}$$

$$= -\frac{2\left(v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\,\frac{\partial\left(v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{1\mathbf{k}}(t)-v^{\Re}_{0\mathbf{k}}(t)\right)^2+\left(v^{\Im}_{1\mathbf{k}}(t)-v^{\Im}_{0\mathbf{k}}(t)\right)^2}{\omega^2_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{317}$$

$$= -\frac{2\left(v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\,\frac{\partial\left(v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}\left|B_{10}(t)\right|^2, \tag{318}$$

$$\frac{\partial \eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v^{\Re}_{i\mathbf{k}}(t)} \tag{319}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial \mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)} - 4\frac{\partial \mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{320}$$

$$= \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t)) - |B_{10}(t)|^2|V_{10}(t)|^2\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{\eta(t)} \tag{321}$$

$$= \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial\left((\varepsilon_1(t)+R_1(t))(\varepsilon_0(t)+R_0(t)) - |B_{10}(t)V_{10}(t)|^2\right)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{\eta(t)} \tag{322}$$

$$= \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Im}_{1\mathbf{k}}(t)-v^{\Im}_{0\mathbf{k}}(t)\right)}{\omega^2_{\mathbf{k}}}\,\frac{\partial\left(v^{\Im}_{1\mathbf{k}}(t)-v^{\Im}_{0\mathbf{k}}(t)\right)}{\partial v^{\Im}_{i\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{323}$$

$$\delta_{1i} - \delta_{0i} = \frac{\partial\left(v^{\Im}_{1\mathbf{k}}(t) - v^{\Im}_{0\mathbf{k}}(t)\right)}{\partial v^{\Im}_{i\mathbf{k}}} \tag{324}$$

$$\frac{\partial \eta(t)}{\partial v^{\Re}_{i\mathbf{k}}(t)} = \frac{\varepsilon(t)\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left((\varepsilon(t)-\varepsilon_i(t)-R_i(t))\left(2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-v^{\Im}_{i'\mathbf{k}}(t)\right)|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega^2_{\mathbf{k}}}\right)}{\eta(t)} \tag{325}$$

$$= \frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\frac{4(\varepsilon_i(t)+R_i(t)) - 2\varepsilon(t) - \frac{4|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}}{\eta(t)} + \frac{1}{\eta(t)}\left(2\frac{g^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\varepsilon(t) - 4(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} + 4\frac{v^{\Im}_{i'\mathbf{k}}(t)|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega^2_{\mathbf{k}}}\right). \tag{326}$$

From the equation (268) and replacing the derivatives obtained we have:

$$\frac{\frac{\partial\varepsilon(t)}{\partial v^{\Im}_{i\mathbf{k}}(t)}}{\frac{\partial\eta(t)}{\partial v^{\Im}_{i\mathbf{k}}(t)}} = \tanh\left(\frac{\beta\eta(t)}{2}\right) \tag{327}$$

$$= \frac{2\frac{v^{\Im}_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \mathrm{i}\frac{g^*_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{v^{\Im}_{i\mathbf{k}}(t)\left(\frac{2\varepsilon(t) - 4(\varepsilon(t)-\varepsilon_i(t)-R_i(t)) - \frac{4}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\right) + \frac{2}{\eta(t)}\left(\frac{g^{*\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t) - 2(\varepsilon(t)-\varepsilon_i(t)-R_i(t))\frac{g^{*\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}} + 2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega^2_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. \tag{328}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v^{\Im}_{i\mathbf{k}}(t)-\mathrm{i}\left(g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$=\frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{330}$$

$$=\frac{2\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{4|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+4\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{331}$$

$$=\frac{\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{332}$$

Separating (332) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Im}_{i\mathbf{k}}(t)-g^{\Im}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Im}_{i\mathbf{k}}(t)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)+2\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{333}$$

$$v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}=v^{\Im}_{i\mathbf{k}}(t)\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right) \tag{334}$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{335}$$

$$v^{\Im}_{i\mathbf{k}}(t)=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{336}$$

$$v^{\Im}_{i\mathbf{k}}(t)=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{337}$$

The variational parameters are:

$$v_{i\mathbf{k}}(t)=v^{\Re}_{i\mathbf{k}}(t)+\mathrm{i}v^{\Im}_{i\mathbf{k}}(t) \tag{338}$$

$$=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{339}$$

$$+\mathrm{i}\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \tag{340}$$

$$=\frac{g_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}(t)-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}(t)\right)-\frac{2|V_{10}(t)|^{2}|B_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{341}$$

Let's obtain the explicit form of $v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)$ and $v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)$, at first we have:

$$a_i(\omega_{\mathbf{k}},t)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t)+2R_i(t)-\varepsilon(t)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{342}$$

$$b_i(\omega_{\mathbf{k}},t)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_i(t)-R_i(t)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}.\tag{343}$$

So the equation (338) written in explicit form is:

$$v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)\,,\tag{344}$$

$$v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)=g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)\,.\tag{345}$$

This system of equations has the following solutions:

$$\begin{aligned}
v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)&=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t) &(346)\\
&=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+\left(g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)\right)b_0(\omega_{\mathbf{k}},t) &(347)\\
&=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)+v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t) &(348)\\
v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)(1-b_1(\omega_{\mathbf{k}},t)b_0(\omega_{\mathbf{k}},t))&=g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t) &(349)\\
v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)&=\frac{g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}, &(350)\\
v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)&=g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+\frac{g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)+g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)\,b_0(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}b_1(\omega_{\mathbf{k}},t) &(351)\\
&=\frac{g_1(\omega_{\mathbf{k}})\,a_1(\omega_{\mathbf{k}},t)+g_0(\omega_{\mathbf{k}})\,a_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}. &(352)
\end{aligned}$$

For a shorter representation let's define:

$$r_i(\omega_{\mathbf{k}},t)=\frac{a_i(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)},\tag{353}$$

$$s_i(\omega_{\mathbf{k}},t)=\frac{a_{(i+1)\mathrm{mod}2}(\omega_{\mathbf{k}},t)\,b_{i\mathrm{mod}2}(\omega_{\mathbf{k}},t)}{1-b_0(\omega_{\mathbf{k}},t)\,b_1(\omega_{\mathbf{k}},t)}.\tag{354}$$

So the variational parameters are given by:

$$\begin{pmatrix}v_{0\mathbf{k}}(\omega_{\mathbf{k}},t)\\ v_{1\mathbf{k}}(\omega_{\mathbf{k}},t)\end{pmatrix}\equiv\begin{pmatrix}r_0(\omega_{\mathbf{k}},t) & s_0(\omega_{\mathbf{k}},t)\\ r_1(\omega_{\mathbf{k}},t) & s_1(\omega_{\mathbf{k}},t)\end{pmatrix}\begin{pmatrix}g_0(\omega_{\mathbf{k}})\\ g_1(\omega_{\mathbf{k}})\end{pmatrix}.\tag{355}$$

Given that $v_{i\mathbf{k}}(\omega_{\mathbf{k}},t)\equiv g_i(\omega_{\mathbf{k}})\,F_i(\omega_{\mathbf{k}},t)$ then we can write:

$$F_0(\omega_{\mathbf{k}},t)=r_0(\omega_{\mathbf{k}},t)+\frac{g_1(\omega_{\mathbf{k}})}{g_0(\omega_{\mathbf{k}})}s_0(\omega_{\mathbf{k}},t)\tag{356}$$

$$F_1(\omega_{\mathbf{k}},t)=\frac{g_0(\omega_{\mathbf{k}})}{g_1(\omega_{\mathbf{k}})}r_1(\omega_{\mathbf{k}},t)+s_1(\omega_{\mathbf{k}},t)\tag{357}$$

III. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\mathrm{Thermal}}$, where $\rho_B^{\mathrm{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$
\begin{aligned}
\overline{\rho_T(0)} &\equiv e^{V(0)}\rho_T(0)\,e^{-V(0)} && (358)\\
&= \left(|0\rangle\langle 0|B_0^+(0) + |1\rangle\langle 1|B_1^+(0)\right)(\rho_S(0)\otimes\rho_B)\left(|0\rangle\langle 0|B_0^-(0) + |1\rangle\langle 1|B_1^-(0)\right), && (359)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|: \quad & |0\rangle\langle 0|0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (360)\\
&= |0\rangle B_0^+(0)\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (361)\\
&= |0\rangle\langle 0| \otimes B_0^+(0)\,\rho_B B_0^-(0), && (362)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|: \quad & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (363)\\
&= |1\rangle\langle 1|B_1^+(0)\,\rho_B B_1^-(0) && (364)\\
&= |1\rangle\langle 1| \otimes B_1^+(0)\,\rho_B B_1^-(0), && (365)\\
\text{for } \rho_S(0) = |0\rangle\langle 1|: \quad & |0\rangle\langle 0|B_0^+(0)\,|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^-(0) && (366)\\
&= |0\rangle\langle 1|B_0^+(0)\,\rho_B|1\rangle\langle 1|B_1^-(0) && (367)\\
&= |0\rangle\langle 1|1\rangle\langle 1|B_0^+(0)\,\rho_B B_1^-(0) && (368)\\
&= |0\rangle\langle 1| \otimes B_0^+(0)\,\rho_B B_1^-(0), && (369)\\
\text{for } \rho_S(0) = |1\rangle\langle 0|: \quad & |1\rangle\langle 1|B_1^+(0)\,|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^-(0) && (370)\\
&= |1\rangle\langle 0| \otimes B_1^+(0)\,\rho_B B_0^-(0). && (371)
\end{aligned}
$$

We transform any operator O into the interaction picture in the following way:

$$
\begin{aligned}
\widetilde{O}(t) &\equiv U^\dagger(t)\,O(t)\,U(t), && (372)\\
U(t) &\equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\overline{H_{\bar{S}}}(t')\right). && (373)
\end{aligned}
$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$
\begin{aligned}
\widetilde{\overline{\rho_S}}(t) &= U^\dagger(t)\,\overline{\rho_S}(t)\,U(t), \text{ where} && (374)\\
\overline{\rho_S}(t) &= \mathrm{Tr}_B\left(\bar{\rho}_T(t)\right). && (375)
\end{aligned}
$$

In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i(t)$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$
\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \quad (376)
$$

In this case $|1\rangle\langle 1| = \frac{I+\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I-\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |1\rangle\langle 1| - |0\rangle\langle 0|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}(t)$ as pointed in the equation (243):

$$
\begin{aligned}
\overline{H_{\bar{I}}}(t) &= \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x(t) + \sigma_y B_y(t)\right) + V_{10}^{\Im}(t)\left(\sigma_x B_y(t) - \sigma_y B_x(t)\right) && (377)\\
&= B_{0z}(t)\,|0\rangle\langle 0| + B_{1z}(t)\,|1\rangle\langle 1| + V_{10}^{\Re}(t)\,\sigma_x B_x(t) + V_{10}^{\Re}(t)\,\sigma_y B_y(t) + V_{10}^{\Im}(t)\,\sigma_x B_y(t) - V_{10}^{\Im}(t)\,\sigma_y B_x(t) && (378)\\
&= \sum_i C_i(t)\left(A_i \otimes B_i(t)\right). && (379)
\end{aligned}
$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\partial\widetilde{\overline{\rho_T}}(t)}{\partial t} = \text{-i}[\widetilde{\overline{H_{\bar{I}}}}(t), \widetilde{\overline{\rho_T}}(t)]. \tag{380}$$

This equation has the formal solution

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0)\text{ -i}\int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]\mathrm{d}t'. \tag{381}$$

Replacing the equation (381) in the equation (380) gives us:

$$\frac{\partial\widetilde{\overline{\rho_T}}(t)}{\partial t} = -\mathrm{i}[\widetilde{\overline{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\overline{H_{\bar{I}}}}(t), [\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_T}}(t')]]\mathrm{d}t'. \tag{382}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\overline{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_n[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots[\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]]\cdots]. \tag{383}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\overline{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty}(-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_n\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots[\widetilde{\overline{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]]\cdots]. \tag{384}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0)\otimes\rho_B$. Consider the following notation:

$$\widetilde{\overline{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\overline{\rho_S}(0) \tag{385}$$
$$= W(t)\overline{\rho_S}(0). \tag{386}$$

in this case

$$W_n(t) = (-\mathrm{i})^n\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2...\int_0^{t_{n-1}}\mathrm{d}t_n\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t_1), [\widetilde{\overline{H_{\bar{I}}}}(t_2), \cdots[\widetilde{\overline{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]]\cdots]. \tag{387}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)\overline{\rho_S}(0) \tag{388}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}W(t)\overline{\rho_S}(0) \tag{389}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right)W(t)^{-1}\widetilde{\overline{\rho_S}}(t). \tag{390}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\overline{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t)\approx\mathbb{I}$ then the equation (388) becomes:

$$\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t} = \dot{W}_2(t)\widetilde{\overline{\rho_S}}(t) \tag{391}$$
$$= -\int_0^t \mathrm{d}t_1\mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]. \tag{392}$$

We had impose the Markovian condition on the (392) using the difference between the environment memory and the timescale of significant evolution of the system S. If this is not the case then the master equation would have the form:

$$\frac{\partial \widetilde{\overline{\rho_S}}(t)}{\partial t} = -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B \left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t_1), \widetilde{\overline{\rho_S}}(t_1)\rho_B\right]\right]. \tag{393}$$

We can use the Markovian approximation to justify the approximation $\overline{\rho_S}(s) \to \overline{\rho_S}(t)$. Replacing $t_1 \to t-\tau$ in (392):

$$\frac{\partial \overline{\rho_S}(t)}{\partial t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B \left[\overline{H_{\bar{I}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t-\tau), \overline{\rho_S}(t)\rho_B\right]\right]. \tag{394}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}(t)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}\,\overline{H_{\bar{I}}}(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t). \tag{395}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\begin{aligned}
\widetilde{\overline{H_{\bar{I}}}}(t) &= \sum_i C_i(t)\left(\widetilde{A_i}(t)\otimes\widetilde{B_i}(t)\right), && (396)\\
\widetilde{A_i}(t) &= U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U(t) && (397)\\
&= U^\dagger(t)\,A_i U(t)\,\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} && (398)\\
&= U^\dagger(t)\,A_i U(t)\,\mathbb{I} && (399)\\
&= U^\dagger(t)\,A_i U(t), && (400)\\
\widetilde{B_i}(t) &= U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) && (401)\\
&= U^\dagger(t)\,U(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} && (402)\\
&= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} && (403)\\
&= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}. && (404)
\end{aligned}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (396) to replace it in the equation (392)

$$\begin{aligned}
\frac{\partial \widetilde{\overline{\rho_S}}(t)}{\partial t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}t' && (405)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \left[\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}t' && (406)\\
&= -\int_0^t \mathrm{Tr}_B\Big[\textstyle\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big), \sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\overline{\rho_S}}(t)\rho_B - \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\Big]\mathrm{d}t' && (407)\\
&= -\int_0^t \mathrm{Tr}_B\Big(\textstyle\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\overline{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big) && (408)\\
&\quad - \textstyle\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(t')\big(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}t'. && (409)
\end{aligned}$$

In order to calculate the correlation functions we define:

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(t')\,\rho_B\right). \tag{410}$$

An useful property is

$$\mathcal{B}_{ji}^{*}\left(t,t'\right) = \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right)^{\dagger} \tag{411}$$

$$= \mathrm{Tr}_B\left(\rho_B^{\dagger}\widetilde{B_i}^{\dagger}\left(t'\right)\widetilde{B_j}^{\dagger}\left(t\right)\right) \tag{412}$$

$$= \mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) \tag{413}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{414}$$

$$= \mathcal{B}_{ij}\left(t',t\right). \tag{415}$$

The correlation functions relevant that appear in the equation (409) are:

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) = \left\langle\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\right\rangle_B \tag{416}$$

$$= \mathcal{B}_{ji}\left(t,t'\right) \tag{417}$$

$$= \mathcal{B}_{ij}^{*}\left(t',t\right) \tag{418}$$

$$\mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\rho_B\widetilde{B_i}\left(t'\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{419}$$

$$= \mathcal{B}_{ij}\left(t',t\right) \tag{420}$$

$$\mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\rho_B\widetilde{B_j}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_j}\left(t\right)\widetilde{B_i}\left(t'\right)\rho_B\right) \tag{421}$$

$$= \mathcal{B}_{ij}^{*}\left(t',t\right) \tag{422}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t'\right)\widetilde{B_j}\left(t\right)\rho_B\right) \tag{423}$$

$$= \mathcal{B}_{ij}\left(t',t\right) \tag{424}$$

The cyclic property of the trace was use widely in the development of equations (416) and (424). Replacing in (409)

$$\frac{\partial\widetilde{\widetilde{\rho_S}}(t)}{\partial t} = -\int_0^t \mathrm{Tr}_B\Big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i\big(t'\big)\big(\widetilde{A_i}\big(t'\big)\otimes\widetilde{B_i}\big(t'\big)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i\big(t'\big)\big(\widetilde{A_i}\big(t'\big)\otimes\widetilde{B_i}\big(t'\big)\big) \tag{425}$$

$$-\sum_i C_i\big(t'\big)\big(\widetilde{A_i}\big(t'\big)\otimes\widetilde{B_i}\big(t'\big)\big)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_i C_i\big(t'\big)\big(\widetilde{A_i}\big(t'\big)\otimes\widetilde{B_i}\big(t'\big)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\Big)\mathrm{d}t'. \tag{426}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i\big(t'\big)\big(\widetilde{A_j}(t)\widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}\big(t'\big)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{B_j}(t)\rho_B\widetilde{B_i}\big(t'\big)\big) \tag{427}$$

$$+\sum_{ij} C_i\big(t'\big)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{A_j}(t)\rho_B\widetilde{B_i}\big(t'\big)\widetilde{B_j}(t) - \widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}\big(t'\big)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{428}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ji} C_j(t)C_i\big(t'\big)\big(\widetilde{A_j}(t)\widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}\big(t'\big)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{B_j}(t)\rho_B\widetilde{B_i}\big(t'\big)\big) \tag{429}$$

$$+\sum_{ij} C_i\big(t'\big)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{A_j}(t)\rho_B\widetilde{B_i}\big(t'\big)\widetilde{B_j}(t) - \widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}\big(t'\big)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{430}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i\big(t'\big)\big(\widetilde{A_j}(t)\widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}\big(t'\big)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{B_j}(t)\rho_B\widetilde{B_i}\big(t'\big)\big)\ \text{(by permuting i and j because i,j}\in\text{J)} \tag{431}$$

$$+\sum_{ij} C_i\big(t'\big)C_j(t)\big(\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{A_j}(t)\rho_B\widetilde{B_i}\big(t'\big)\widetilde{B_j}(t) - \widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}\big(t'\big)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{432}$$

$$= -\int_0^t \mathrm{Tr}_B\Big(\sum_{ij} C_j(t)C_i\big(t'\big)\big(\widetilde{A_j}(t)\widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{B_j}(t)\widetilde{B_i}\big(t'\big)\rho_B - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{B_j}(t)\rho_B\widetilde{B_i}\big(t'\big) \tag{433}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{A_j}(t)\rho_B\widetilde{B_i}\big(t'\big)\widetilde{B_j}(t) - \widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\widetilde{B_i}\big(t'\big)\rho_B\widetilde{B_j}(t)\big)\Big)\mathrm{d}t' \tag{434}$$

$$= -\int_0^t\Big(\sum_{ij} C_j(t)C_i\big(t'\big)\big(\widetilde{A_j}(t)\widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\mathcal{B}_{ji}\big(t,t'\big) - \widetilde{A_j}(t)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\mathcal{B}_{ij}\big(t',t\big) \tag{435}$$

$$+\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_i}\big(t'\big)\widetilde{A_j}(t)\mathcal{B}_{ij}\big(t',t\big) - \widetilde{A_i}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}(t)\mathcal{B}_{ji}\big(t,t'\big)\big)\Big)\mathrm{d}t' \tag{436}$$

$$= -\int_0^t\left(\sum_{ij} C_j\left(t\right)C_i\left(t'\right)\left(\mathcal{B}_{ji}\left(t,t'\right)\left[\widetilde{A_j}\left(t\right),\widetilde{A_i}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \mathcal{B}_{ij}\left(t',t\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_i}\left(t'\right),\widetilde{A_j}\left(t\right)\right]\right)\right)\mathrm{d}t' \tag{437}$$

$$= -\int_0^t\Big(\sum_{ij} C_i(t)C_j\big(t'\big)\big(\mathcal{B}_{ij}\big(t,t'\big)\big[\widetilde{A_i}(t),\widetilde{A_j}\big(t'\big)\widetilde{\widetilde{\rho_S}}(t)\big] + \mathcal{B}_{ji}\big(t',t\big)\big[\widetilde{\widetilde{\rho_S}}(t)\widetilde{A_j}\big(t'\big),\widetilde{A_i}(t)\big]\big)\Big)\mathrm{d}t'\ \text{(exchanging i and j)} \tag{438}$$

$$= -\int_0^t\left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] + \mathcal{B}_{ij}^{*}\left(t,t'\right)\left[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]\right)\right)\mathrm{d}t' \tag{439}$$

$$= -\int_0^t\left(\sum_{ij} C_i\left(t\right)C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\right] - \mathcal{B}_{ij}^{*}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right)\right]\right)\right)\mathrm{d}t'. \tag{440}$$

We could identify the following commutators in the equation deduced:

$$\mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)-\mathcal{B}_{ij}\left(t,t'\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)=\mathcal{B}_{ij}\left(t,t'\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_s}}\left(t\right)\right], \tag{441}$$

$$\mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)\widetilde{A_i}\left(t\right)-\mathcal{B}^*_{ij}\left(t,t'\right)\widetilde{A_i}\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(s\right)=\mathcal{B}^*_{ij}\left(t,t'\right)\left[\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_j}\left(t'\right),\widetilde{A_i}\left(t\right)\right]. \tag{442}$$

Returning to the Schrödinger picture we have:

$$\begin{aligned} U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right) &= U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{A_j}\left(t'\right)U^\dagger\left(t\right)U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right), &(443)\\ &= \left(U\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{A_j}\left(t'\right)U^\dagger\left(t\right)\right)\left(U\left(t\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right)\right), &(444)\\ &= A_i\left(t\right)\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right). &(445)\end{aligned}$$

This procedure applying to the relevant commutators give us:

$$\begin{aligned} U\left(t\right)\left[\widetilde{A_i}\left(t\right),\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)\right]U^\dagger\left(t\right) &= \left(U\left(t\right)\widetilde{A_i}\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)U^\dagger\left(t\right)-U\left(t\right)\widetilde{A_j}\left(t'\right)\widetilde{\overline{\rho_S}}\left(t\right)\widetilde{A_i}\left(t\right)U^\dagger\left(t\right)\right) &(446)\\ &= A_i\left(t\right)\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)-\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)A_i &(447)\\ &= \left[A_i\left(t\right),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]. &(448)\end{aligned}$$

Introducing this transformed commutators in the equation (440) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathcal{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\begin{aligned} U\left(t\right)\frac{\partial\widetilde{\overline{\rho_S}}\left(t\right)}{\partial t}U^\dagger\left(t\right) &= -\sum_{ij}\int_0^t \mathrm{d}s C_i(t)\,C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[A_i\left(t\right),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]+\mathcal{B}^*_{ij}\left(t,t'\right)\left[\overline{\rho_S}\left(t\right)\widetilde{A_j}\left(t',t\right),A_i\right]\right), &(449)\\ t' &= t-\tau \text{ (Change of variables in the integration process)}, &(450)\\ U\left(t\right)\frac{\partial\widetilde{\overline{\rho_S}}\left(t\right)}{\partial t}U^\dagger\left(t\right) &= -\sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)\,C_j\left(t'\right)\left(\mathcal{B}_{ij}\left(t,t'\right)\left[A_i(t),\widetilde{A_j}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]+\mathcal{B}^*_{ij}\left(t,t'\right)\left[\overline{\rho_S}(t)\widetilde{A_j}\left(t',t\right),A_i\left(t\right)\right]\right). &(451)\end{aligned}$$

where $i,j\in\{1,2,3,4,5.6\}$ and $t'=t-\tau$.

Here $\widetilde{A_j}\left(t-\tau,t\right)=U\left(t\right)U^\dagger\left(t-\tau\right)A_jU\left(t-\tau\right)U^\dagger\left(t\right)$ where $U\left(t\right)$ is given by (373). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I\left(t\right)$. In order to write in a simplified way we define the following notation:

$$\begin{aligned} \mathcal{B}_{ij}\left(t,t'\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(t'\right)\rho_B\right) &(452)\\ &= \mathrm{Tr}_B\left(\mathrm{e}^{iH_Bt}B_i\left(t\right)\mathrm{e}^{-iH_Bt}\mathrm{e}^{iH_Bt'}B_j\left(t'\right)\mathrm{e}^{-iH_Bt'}\rho_B\right) &(453)\\ \mathrm{e}^A &= \sum_{k=0}^{\infty}\frac{A^k}{k!} &(454)\\ \mathrm{e}^{-iH_Bt'}\mathrm{e}^{-\beta H_B} &= \sum_{m=0}^{\infty}\frac{\left(-iH_Bt'\right)^m}{m!}\sum_{n=0}^{\infty}\frac{\left(-\beta H_B\right)^n}{n!} &(455)\end{aligned}$$

$$= \sum_{m,n} \frac{(-iH_B t')^m}{m!} \frac{(-\beta H_B)^n}{n!} \tag{456}$$

$$= \sum_{m,n} \frac{(-it')^m}{m!} \frac{(-\beta)^n}{n!} H_B^m H_B^n \tag{457}$$

$$= \sum_{m,n} \frac{(-it')^m}{m!} \frac{(-\beta)^n}{n!} H_B^n H_B^m \text{ (because the powers of a matrix commute)} \tag{458}$$

$$= \sum_{m,n} \frac{(-\beta)^n}{n!} H_B^n \frac{(-it')^m}{m!} H_B^m \tag{459}$$

$$= \sum_{m,n} \frac{(-\beta H_B)^n}{n!} \frac{(-it' H_B)^m}{m!} \tag{460}$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_B t')^m}{m!} \tag{461}$$

$$= \mathrm{e}^{-\beta H_B} \mathrm{e}^{-iH_B t'} \tag{462}$$

$$0 = \mathrm{e}^{-iH_B t'} \mathrm{e}^{-\beta H_B} - \mathrm{e}^{-\beta H_B} \mathrm{e}^{-iH_B t'} \text{ (then } \mathrm{e}^{-iH_B t'} \text{ and } \rho_B \text{ commute)} \tag{463}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left(\mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \rho_B \mathrm{e}^{-iH_B t'} \right) \text{ (by permuting } \mathrm{e}^{-iH_B t'} \text{ and } \rho_B) \tag{464}$$

$$= \mathrm{Tr}_B \left(\left(\mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \right) \rho_B \mathrm{e}^{-iH_B t'} \right) \text{ (by associative property)} \tag{465}$$

$$= \mathrm{Tr}_B \left(\mathrm{e}^{-iH_B t'} \left(\mathrm{e}^{iH_B t} B_i(t) \mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} B_j(t') \right) \rho_B \right) \text{ (by cyclic property of the trace)} \tag{466}$$

$$= \mathrm{Tr}_B \left(\left(\mathrm{e}^{-iH_B t'} \mathrm{e}^{iH_B t} \right) B_i(t) \left(\mathrm{e}^{-iH_B t} \mathrm{e}^{iH_B t'} \right) B_j(t') \rho_B \right) \text{ (by associative property)} \tag{467}$$

$$[iH_B t, -iH_B t'] = iH_B t(-iH_B t') - (-iH_B t') iH_B t \tag{468}$$

$$= tt' H_B^2 - tt' H_B^2 \tag{469}$$

$$= 0 \text{ (so } iH_B t \text{ and } -iH_B t' \text{ commute)} \tag{470}$$

$$\mathrm{e}^{-iH_B t'} \mathrm{e}^{iH_B t} = \mathrm{e}^{iH_B t - iH_B t'} \text{ (by the Zassenhaus formula because } iH_B t \text{ and } -iH_B t' \text{ commute)} \tag{471}$$

$$= \mathrm{e}^{iH_B (t-t')} \tag{472}$$

$$= \mathrm{e}^{iH_B \tau} \tag{473}$$

$$\mathrm{e}^{iH_B t'} \mathrm{e}^{-iH_B t} = \mathrm{e}^{-iH_B t + iH_B t'} \text{ (by the Zassenhaus formula because } -iH_B t \text{ and } iH_B t' \text{ commute)} \tag{474}$$

$$= \mathrm{e}^{iH_B (-t+t')} \tag{475}$$

$$= \mathrm{e}^{-iH_B \tau} \tag{476}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left(\mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} B_j(t') \rho_B \right) \tag{477}$$

$$B_i(t,\tau) \equiv \mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} \tag{478}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left(\mathrm{e}^{iH_B (t-t')} B_i(t) \mathrm{e}^{-iH_B (t-t')} B_j(t') \rho_B \right) \tag{479}$$

$$t' = t - \tau \tag{480}$$

$$\mathcal{B}_{ij}(t,t') = \mathrm{Tr}_B \left(\mathrm{e}^{iH_B \tau} B_i(t) \mathrm{e}^{-iH_B \tau} B_j(t') \rho_B \right) \tag{481}$$

$$= \mathrm{Tr}_B \left(B_i(t,\tau) B_j(t',0) \rho_B \right). \tag{482}$$

For the following results $i, j \in \{3, 6\}$, calculating the correlation functions allow us to obtain:

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \mathrm{Tr}_B \left(B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \rho_B \right) \tag{483}$$

$$= \int \mathrm{d}^2\alpha P(\alpha) \left\langle \alpha \left| B_{j\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \right| \alpha \right\rangle \tag{484}$$

$$=\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|B_{j\mathrm{mod}2z}\left(t,\tau\right)B_{j\mathrm{mod}2z}\left(t',0\right)\right|\alpha\right\rangle \mathrm{d}^2\alpha, \tag{485}$$

$$q_{j\mathbf{k}}\left(t\right)=g_{j\mathrm{mod}2\mathbf{k}}-v_{j\mathrm{mod}2\mathbf{k}}\left(t\right) \tag{486}$$

$$B_{j\mathrm{mod}2z}\left(t,\tau\right)=\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{487}$$

$$B_{j\mathrm{mod}2z}\left(t',0\right)=\sum_{\mathbf{k}'}\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right), \tag{488}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_{B}=\mathrm{Tr}_{B}\left(B_{j\mathrm{mod}2z}\left(t,\tau\right)B_{j\mathrm{mod}2z}\left(t',0\right)\rho_{B}\right) \tag{489}$$

$$=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_{B}\right) \tag{490}$$

$$=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_{B}\right) \tag{491}$$

$$=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_{B}\right) \tag{492}$$

$$+\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_{B}\right), \tag{493}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_{B}=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_{B}\right) \tag{494}$$

$$+\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_{B}\right) \tag{495}$$

$$0=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{j\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\rho_{B}\right) \tag{496}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}(t')\right\rangle_{B}=0+\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}\left(t\right)b_{\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}\left(t\right)b_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}^{\dagger}+q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}\right)\rho_{B}\right) \tag{497}$$

$$=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}(t')\left(b_{\mathbf{k}}^{\dagger}\right)^{2}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right.\right. \tag{498}$$

$$\left.\left.+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}^{*}(t')b_{\mathbf{k}}^{2}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_{B}\right) \tag{499}$$

$$=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_{B}\right)+\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_{B}\right) \tag{500}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}\left(t\right)q_{j\mathbf{k}}^{*}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_{B}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_{B}\right)+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}\left(t\right)q_{j\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_{B}\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_{B}\right) \tag{501}$$

$$=\sum_{\mathbf{k}}\frac{1}{\pi N_{\mathbf{k}}}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}}\left(q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')\left\langle \alpha_{\mathbf{k}}\left|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right|\alpha_{\mathbf{k}}\right\rangle \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+q_{j\mathbf{k}}^{*}(t)q_{j\mathbf{k}}(t')\left\langle \alpha_{\mathbf{k}}\left|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right|\alpha_{\mathbf{k}}\right\rangle\right. \tag{502}$$

$$\left.\times\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{d}^2\alpha_{\mathbf{k}} \tag{503}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')\left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D(-\alpha_{\mathbf{k}})\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}q_{j\mathbf{k}}^{*}(t) \tag{504}$$

$$\times\, q_{j\mathbf{k}}\left(t'\right)\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}}\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{505}$$

$$=\sum_{\mathbf{k}}q_{j\mathbf{k}}(t)q_{j\mathbf{k}}^{*}(t')\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N_{\mathbf{k}}}\int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}}\left\langle 0\left|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\right|0\right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{506}$$

$$+\sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \left| D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}}^{\dagger} D(\alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{507}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}^{*}(t') \left(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \left| \left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right)(b_{\mathbf{k}} + \alpha_{\mathbf{k}}) \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{508}$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t') \left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \left| (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) \left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right), \tag{509}$$

$$= \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}^{*}(t')\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \left| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right. \tag{510}$$

$$\left. + q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \right) \tag{511}$$

$$= \sum_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left(q_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}^{*}(t')\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \left\langle 0 \left| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle + q_{j\mathbf{k}}(t) \left\langle 0 \left| b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} \right| 0 \right\rangle \right. \tag{512}$$

$$\times\, q_{j\mathbf{k}}^{*}(t')\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(t') \left\langle 0 \left| b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*} \right| 0 \right\rangle \tag{513}$$

$$\left. \times \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \mathrm{d}^2\alpha_{\mathbf{k}} \tag{514}$$

$$= \sum_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} q_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}^{*}(t')\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\left\langle 0 \left| |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle + \left\langle 0 \left| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \right| 0 \right\rangle \right) \mathrm{d}^2\alpha_{\mathbf{k}} + \sum_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \tag{515}$$

$$\times\, q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left(\left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right| 0 \right\rangle + \left\langle 0 \left| |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \right) \mathrm{d}^2\alpha_{\mathbf{k}}, \tag{516}$$

$$1 = \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \mathrm{d}^2\alpha_{\mathbf{k}}, \tag{517}$$

$$b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} |0\rangle = 0, \tag{518}$$

$$b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} |0\rangle = |0\rangle, \tag{519}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \sum_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left(\left(q_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}^{*}(t') \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left\langle 0 \left| |\alpha_{\mathbf{k}}|^2 \right| 0 \right\rangle \right. \tag{520}$$

$$\left. + q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t') \left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \left\langle 0 \left| b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right| 0 \right\rangle \right) \right) \mathrm{d}^2\alpha_{\mathbf{k}} \tag{521}$$

$$= \sum_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left(\left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^{*}(t') \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(t') \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) |\alpha_{\mathbf{k}}|^2 + q_{j\mathbf{k}}^{*}(t) q_{j\mathbf{k}}(t') \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \mathrm{d}^2\alpha_{\mathbf{k}}, \tag{522}$$

$$\int_0^{2\pi} \int_0^{+\infty} r^2 \mathrm{e}^{-\frac{r^2}{N}} r \mathrm{d}r \mathrm{d}\theta = \int |\alpha_{\mathbf{k}}|^2 \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \mathrm{d}^2\alpha_{\mathbf{k}} \tag{523}$$

$$= \pi N_{\mathbf{k}}^2 \tag{524}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \sum_{\mathbf{k}} \left(\left(q_{j\mathbf{k}}(t)\, q_{j\mathbf{k}}^{*}(t') \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) N_{\mathbf{k}} + q_{j\mathbf{k}}^{*}(t)\, q_{j\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \tag{525}$$

$$\left\langle \widetilde{B_{j\mathrm{mod}2z}}(t) \widetilde{B_{j'\mathrm{mod}2z}}(t') \right\rangle_B = \mathrm{Tr}_B \left(B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0)\, \rho_B \right) \tag{526}$$

$$= \int \mathrm{d}^2\alpha P(\alpha) \left\langle \alpha \left| B_{j\mathrm{mod}2z}(t,\tau)\, B_{j'\mathrm{mod}2z}(t',0) \right| \alpha \right\rangle \tag{527}$$

$$= \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left(q_{j'\mathbf{k}'}(t')\, b_{\mathbf{k}'}^{\dagger} + q_{j'\mathbf{k}'}^{*}(t')\, b_{\mathbf{k}'} \right) \rho_B \right) \tag{528}$$

$$= \mathrm{Tr}_B \left(\sum_{\mathbf{k}\neq\mathbf{k}'} \left(q_{j\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(q_{j'\mathbf{k}'}(t')\, b_{\mathbf{k}'}^{\dagger} + q_{j'\mathbf{k}'}^{*}(t') b_{\mathbf{k}'} \right) \rho_B \right) \tag{529}$$

$$+ \mathrm{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(q_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}^{\dagger} + q_{j'\mathbf{k}}^{*}(t') b_{\mathbf{k}} \right) \rho_B \right) \tag{530}$$

$$= \sum_{\mathbf{k}} \mathrm{Tr}_B \left(\left(q_{j\mathbf{k}}(t)\, b_{\mathbf{k}}^{\dagger} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^{*}(t) b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \left(q_{j'\mathbf{k}}(t')\, b_{\mathbf{k}}^{\dagger} + q_{j'\mathbf{k}}^{*}(t') b_{\mathbf{k}} \right) \rho_B \right) \tag{531}$$

$$= \sum_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} q_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}^{*}(t')\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \mathrm{d}^2\alpha_{\mathbf{k}} + \sum_{\mathbf{k}} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} q_{j\mathbf{k}}^{*}(t)\, q_{j'\mathbf{k}}(t')\, b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \tag{532}$$

$$\times \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{d}^2\alpha_{\mathbf{k}} \tag{533}$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle \alpha_{\mathbf{k}} \middle| b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \sum_{\mathbf{k}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{534}$$

$$\times \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle \alpha_{\mathbf{k}} \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} \middle| \alpha_{\mathbf{k}} \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}}, \tag{535}$$

$$\left\langle b_{\mathbf{k}} b^\dagger_{\mathbf{k}} \right\rangle_B = \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \middle| D(-\alpha_{\mathbf{k}})\, b_{\mathbf{k}} D(\alpha_{\mathbf{k}})\, D(-\alpha_{\mathbf{k}})\, b^\dagger_{\mathbf{k}} D(\alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{536}$$

$$= \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \middle| (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) \left(b^\dagger_{\mathbf{k}} + \alpha^*_{\mathbf{k}}\right) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{537}$$

$$= \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \alpha_{\mathbf{k}} b^\dagger_{\mathbf{k}} + b_{\mathbf{k}} \alpha^*_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{538}$$

$$= \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} + |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{539}$$

$$= \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \middle| |\alpha_{\mathbf{k}}|^2 \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N}} \left\langle 0 \middle| b_{\mathbf{k}} b^\dagger_{\mathbf{k}} \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{540}$$

$$= N_{\mathbf{k}} + 1, \tag{541}$$

$$\left\langle b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \right\rangle_B = \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} \left\langle 0 \middle| \left(b^\dagger_{\mathbf{k}} + \alpha^*_{\mathbf{k}}\right) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) \middle| 0 \right\rangle \mathrm{d}^2\alpha_{\mathbf{k}} \tag{542}$$

$$= \frac{1}{\pi N_{\mathbf{k}}} \int \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{N_{\mathbf{k}}}} |\alpha_{\mathbf{k}}|^2 \mathrm{d}^2\alpha_{\mathbf{k}} \tag{543}$$

$$= N_{\mathbf{k}}, \tag{544}$$

$$\left\langle \widetilde{B_{j \bmod 2z}}(t)\, \widetilde{B_{j' \bmod 2z}}(t') \right\rangle_B = \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \tag{545}$$

$$= \sum_{\mathbf{k}} 2N_{\mathbf{k}} \left(q_{j\mathbf{k}}(t)\, q^*_{j'\mathbf{k}}(t')\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right)^{\Re} + \sum_{\mathbf{k}} q^*_{j\mathbf{k}}(t)\, q_{j'\mathbf{k}}(t')\, \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{546}$$

$$D(h')\, D(h) = \mathrm{e}^{\frac{1}{2}\left(h'h^* - h'^*h\right)} D(h' + h), \tag{547}$$

$$\left\langle D(h')\, D(h) \right\rangle_B = \mathrm{Tr}_B \left(\mathrm{e}^{\frac{1}{2}\left(h'h^* - h'^*h\right)} D(h' + h)\, \rho_B \right) \tag{548}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^* - h'^*h\right)} \mathrm{Tr}_B \left(D(h' + h)\, \rho_B \right) \tag{549}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^* - h'^*h\right)} \frac{1}{\pi N} \int \mathrm{d}^2\alpha P(\alpha) \left\langle \alpha \middle| D(h' + h) \middle| \alpha \right\rangle \tag{550}$$

$$= \mathrm{e}^{\frac{1}{2}\left(h'h^* - h'^*h\right)} \mathrm{e}^{-\frac{|h+h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}, \tag{551}$$

$$h' = h\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{552}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right) D(h) \right\rangle_B = \mathrm{e}^{\frac{1}{2}\left(hh^*\mathrm{e}^{\mathrm{i}\omega\tau} - h^*h\mathrm{e}^{-\mathrm{i}\omega\tau}\right)} \mathrm{e}^{-\frac{|h+h\mathrm{e}^{\mathrm{i}\omega\tau}|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}, \tag{553}$$

$$\frac{1}{2}|h|^2 \left(\mathrm{e}^{\mathrm{i}\omega\tau} - \mathrm{e}^{-\mathrm{i}\omega\tau}\right) = \frac{1}{2}\left(hh^*\mathrm{e}^{\mathrm{i}\omega\tau} - h^*h\mathrm{e}^{-\mathrm{i}\omega\tau}\right) \tag{554}$$

$$= \frac{1}{2}|h|^2 \left(\cos(\omega\tau) + \mathrm{i}\sin(\omega\tau) - \cos(\omega\tau) + \mathrm{i}\sin(\omega\tau)\right) \tag{555}$$

$$= \frac{1}{2}|h|^2 \left(2\mathrm{i}\sin(\omega\tau)\right) \tag{556}$$

$$= \mathrm{i}\,|h|^2 \sin(\omega\tau), \tag{557}$$

$$-\frac{|h + h\mathrm{e}^{\mathrm{i}\omega\tau}|^2}{2} = -|h|^2 \frac{|1 + \mathrm{e}^{\mathrm{i}\omega\tau}|^2}{2} \tag{558}$$

$$= -|h|^2 \frac{\left(1 + 2\cos(\omega\tau) + \cos^2(\omega\tau)\right) + \sin^2(\omega\tau)}{2} \tag{559}$$

$$= -|h|^2 \frac{2 + 2\cos(\omega\tau)}{2} \tag{560}$$

$$= -|h|^2 \left(1 + \cos(\omega\tau)\right), \tag{561}$$

$$\left\langle D\left(h\mathrm{e}^{\mathrm{i}\omega\tau}\right) D(h) \right\rangle_B = \mathrm{e}^{\mathrm{i}|h|^2 \sin(\omega\tau)} \mathrm{e}^{-|h|^2 (1+\cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)} \tag{562}$$

$$= \mathrm{e}^{\mathrm{i}|\mathrm{h}|^2\sin(\omega\tau)-|h|^2(1+\cos(\omega\tau))\coth\left(\frac{\beta\omega}{2}\right)} \tag{563}$$

$$= \mathrm{e}^{-|h|^2\left(-\mathrm{i}\sin(\omega\tau)+\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)}\mathrm{e}^{-|h|^2\coth\left(\frac{\beta\omega}{2}\right)} \tag{564}$$

$$= \langle D(h)\rangle_B\,\mathrm{e}^{-\phi(\tau)}, \tag{565}$$

$$\mathrm{e}^{-\phi(\tau)} = \mathrm{e}^{-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)}, \tag{566}$$

$$\phi(\tau) = |h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right), \tag{567}$$

$$\langle D(h')\,D(h)\rangle_B = \mathrm{e}^{\frac{1}{2}\left(h'h^*-h'^*h\right)}\mathrm{e}^{-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{568}$$

$$h' = v\mathrm{e}^{\mathrm{i}\omega\tau}, \tag{569}$$

$$m_{\mathbf{k}}(t) = \frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \tag{570}$$

$$\Gamma_{\mathbf{k}}(t) = \frac{v_{1\mathbf{k}}^*(t)\,v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\,v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \tag{571}$$

$$\left\langle \widetilde{B_1^+B_0^-}(t)\,\widetilde{B_1^+B_0^-}(t')\right\rangle_B = \left\langle B_1^+B_0^-(t,\tau)\,B_1^+B_0^-(t',0)\right\rangle_B \tag{572}$$

$$= \langle B_{10}(t,\tau)\,B_{10}(t',0)\rangle_B \tag{573}$$

$$= \mathrm{Tr}_B\left(B_{10}(t,\tau)\,B_{10}(t',0)\,\rho_B\right) \tag{574}$$

$$= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t')\right)\right)\rho_B\right) \tag{575}$$

$$= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega\tau}\right)D\left(m_{\mathbf{k}}(t')\right)\right)\rho_B\right) \tag{576}$$

$$= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m_{\mathbf{k}}^*(t')-\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}\right)^*m_{\mathbf{k}}(t')\right)-\frac{|m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}+m_{\mathbf{k}}(t')|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{577}$$

$$= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m_{\mathbf{k}}^*(t')\right)^{\Im}-\frac{\left|\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\mathrm{e}^{\mathrm{i}\omega\tau}+\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{578}$$

$$= \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m_{\mathbf{k}}^*(t')\right)^{\Im}-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{579}$$

$$\left\langle \widetilde{B_0^+B_1^-}(t)\widetilde{B_0^+B_1^-}(t')\right\rangle_B = \mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\prod_{\mathbf{k}}\left(\mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega\tau}m_{\mathbf{k}}^*(t')\right)^{\Im}-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{580}$$

$$\langle D(h)\,b\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle\alpha\,|D(h)\,b|\,\alpha\rangle \tag{581}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle\alpha\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,\alpha\rangle \tag{582}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle 0\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,0\rangle \tag{583}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha)|\,0\rangle \tag{584}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,(b+\alpha)|\,0\rangle \tag{585}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,(b+\alpha)|\,0\rangle \tag{586}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,b|\,0\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,\alpha|\,0\rangle \tag{587}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\langle 0\,|D(h)\,\alpha|\,0\rangle \tag{588}$$

$$
\begin{aligned}
&= \frac{1}{\pi N}\int \alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha && (589)\\
&= hN\left\langle D\left(h\right)\right\rangle_B, && (590)\\
\left\langle D\left(h\right)b^\dagger\right\rangle_B &= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle \alpha\left|D\left(h\right)b^\dagger\right|\alpha\right\rangle && (591)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)b^\dagger D\left(\alpha\right)\right|0\right\rangle && (592)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)b^\dagger D\left(\alpha\right)\right|0\right\rangle && (593)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)D\left(-\alpha\right)b^\dagger D\left(\alpha\right)\right|0\right\rangle && (594)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle 0\left|D\left(-\alpha\right)D\left(h\right)D\left(\alpha\right)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle && (595)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D\left(h\right)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle && (596)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D\left(h\right)b^\dagger\right|0\right\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D\left(h\right)\alpha^*\right|0\right\rangle && (597)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D\left(h\right)\right|1\right\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0\left|D\left(h\right)\right|0\right\rangle && (598)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h|1\right\rangle + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0\left|D\left(h\right)\right|0\right\rangle, && (599)\\
\left\langle -h\right| &= \mathrm{e}^{-\frac{|-h^*|^2}{2}}\sum_n \frac{\left(-h^*\right)^n}{\sqrt{n!}}\left\langle n\right| && (600)\\
\left\langle -h|1\right\rangle &= \mathrm{e}^{-\frac{|-h^*|^2}{2}}\sum_n \frac{\left(-h^*\right)^n}{\sqrt{n!}}\left\langle n|1\right\rangle && (601)\\
\left\langle -h|1\right\rangle &= \mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right), && (602)\\
\left\langle D\left(h\right)b^\dagger\right\rangle_B &= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right) + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|-h^*|^2}{2}} && (603)\\
\left\langle D\left(h\right)b^\dagger\right\rangle_B &= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|-h^*|^2}{2}}\left(-h^*\right) + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|-h^*|^2}{2}} && (604)\\
&= -h^*\left\langle D\left(h\right)\right\rangle_B\left(N+1\right), && (605)\\
\left\langle bD\left(h\right)\right\rangle_B &= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle \alpha\left|bD\left(h\right)\right|\alpha\right\rangle && (606)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}} && (607)\\
&= h\left\langle D\left(h\right)\right\rangle_B\left(N+1\right), && (608)\\
\left\langle b^\dagger D\left(h\right)\right\rangle_B &= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\left\langle \alpha\left|b^\dagger D\left(h\right)\right|\alpha\right\rangle && (609)\\
&= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h + \frac{1}{\pi N}\int \mathrm{d}^2\alpha \mathrm{e}^{-\frac{|\alpha|^2}{2}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}} && (610)\\
&= -h^*\left\langle D\left(h\right)\right\rangle_B N. && (611)
\end{aligned}
$$

The correlation functions can be found readily as:

$$B_1^+B_0^-\left(t,\tau\right) = \prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), \tag{612}$$

$$B_0^+B_1^-\left(t,\tau\right) = \prod_{\mathbf{k}}\left(D\left(-m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}}\right), \tag{613}$$

$$B_{10}(t) = \mathrm{e}^{\chi_{10}(t)} \left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} |m_{\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right), \tag{614}$$

$$B_x(t,\tau) = \frac{B_1^+ B_0^-(t,\tau) + B_0^+ B_1^-(t,\tau) - B_{10}(t) - B_{01}(t)}{2}, \tag{615}$$

$$B_y(t,\tau) = \frac{B_0^+ B_1^-(t,\tau) - B_1^+ B_0^-(t,\tau) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}, \tag{616}$$

$$B_{i\mathrm{mod}2z}(t,\tau) = \sum_{\mathbf{k}} \left(q_{i\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{i\mathbf{k}}^*(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right), \tag{617}$$

$$\left\langle \widetilde{B_{i\mathrm{mod}2z}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \left\langle B_{i\mathrm{mod}2z}(t,\tau) B_{j\mathrm{mod}2z}(t',0) \right\rangle_B \tag{618}$$

$$= \left\langle \sum_{\mathbf{k}} \left(q_{i\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + q_{i\mathbf{k}}^*(t) b_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t') b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*(t') b_{\mathbf{k}} \right) \right\rangle_B \tag{619}$$

$$= \sum_{\mathbf{k}} q_{i\mathbf{k}}(t) q_{j\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} q_{i\mathbf{k}}^*(t) q_{j\mathbf{k}}(t') \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \tag{620}$$

$$\left\langle \widetilde{B_x}(t) \widetilde{B_x}(t') \right\rangle_B = \left\langle B_x(t,\tau) B_x(t',0) \right\rangle_B \tag{621}$$

$$= \left\langle \left(\frac{B_1^+ B_0^-(t,\tau) + B_0^+ B_1^-(t,\tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left(\frac{B_1^+ B_0^-(t',0) + B_0^+ B_1^-(t',0) - B_{10}(t') - B_{01}(t')}{2} \right) \right\rangle_B \tag{622}$$

$$= \frac{1}{4} \left\langle \left(B_1^+ B_0^-(t,\tau) + B_0^+ B_1^-(t,\tau) - B_{10}(t) - B_{01}(t) \right) \left(B_1^+ B_0^-(t',0) + B_0^+ B_1^-(t',0) - B_{10}(t') - B_{01}(t') \right) \right\rangle_B \tag{623}$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^-(t,\tau) B_1^+ B_0^-(t',0) + B_1^+ B_0^-(t,\tau) B_0^+ B_1^-(t',0) - B_1^+ B_0^-(t,\tau) B_{10}(t') - B_1^+ B_0^-(\tau) B_{01}(t') \right. \tag{624}$$

$$+ B_0^+ B_1^-(t,\tau) B_1^+ B_0^-(t',0) + B_0^+ B_1^-(t,\tau) B_0^+ B_1^-(t',0) - B_0^+ B_1^-(t,\tau) B_{10}(t') - B_0^+ B_1^-(t,\tau) B_{01}(t') \tag{625}$$

$$- B_{10}(t) B_1^+ B_0^-(t',0) - B_{10}(t) B_0^+ B_1^-(t',0) + B_{10}(t) B_{10}(t') + B_{10}(t) B_{01}(t') - B_{01}(t) B_1^+ B_0^-(t',0) \tag{626}$$

$$\left. - B_{01}(t) B_0^+ B_1^-(t',0) + B_{01}(t) B_{10}(t') + B_{01}(t) B_{01}(t') \right\rangle \tag{627}$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^-(t,\tau) B_1^+ B_0^-(t',0) + B_1^+ B_0^-(t,\tau) B_0^+ B_1^-(t',0) + B_0^+ B_1^-(t,\tau) B_1^+ B_0^-(t',0) \right. \tag{628}$$

$$\left. + B_0^+ B_1^-(t,\tau) B_0^+ B_1^-(t',0) \right\rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(t') + B_{10}(t'))}{4}, \tag{629}$$

$$U_{10}(t,t') = \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\left(m_{\mathbf{k}}(t) m_{\mathbf{k}}^*(t') \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)^{\Im}}, \tag{630}$$

$$\left\langle B_1^+ B_0^-(t,\tau) B_0^+ B_1^-(t',0) \right\rangle_B = \left\langle \prod\nolimits_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \prod\nolimits_{\mathbf{k}} \left(D(-m_{\mathbf{k}}(t')) \mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \tag{631}$$

$$= \mathrm{e}^{\chi_{10}(t) + \chi_{01}(t')} \left\langle \prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right) \prod_{\mathbf{k}} \left(D\left(-m_{\mathbf{k}}(t') \right) \right) \right\rangle_B \tag{632}$$

$$= \mathrm{e}^{\chi_{10}(t) + \chi_{01}(t')} \prod_{\mathbf{k}} \left\langle \left(D\left(m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left(-m_{\mathbf{k}}(t') \right) \right) \right\rangle_B \tag{633}$$

$$= \mathrm{e}^{\chi_{10}(t) + \chi_{01}(t')} U_{10}^*(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left| (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} - (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{634}$$

$$\left\langle B_0^+ B_1^-(t,\tau) B_1^+ B_0^-(t',0) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left(D\left(-m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t)}{2}} \right) \prod_{\mathbf{k}} \left(D\left(m_{\mathbf{k}}(t') \right) \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \right) \right\rangle_B \tag{635}$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\Gamma_{\mathbf{k}}(t')}{2}} \mathrm{e}^{\frac{\Gamma_{\mathbf{k}}(t')}{2}} \left\langle D\left(-m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left(m_{\mathbf{k}}(t') \right) \right\rangle_B \tag{636}$$

$$= \mathrm{e}^{\chi_{01}(t) + \chi_{10}(t')} \prod_{\mathbf{k}} \left\langle D\left(-m_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) D\left(m_{\mathbf{k}}(t') \right) \right\rangle_B \tag{637}$$

$$= \mathrm{e}^{\chi_{01}(t) + \chi_{10}(t')} U_{10}^*(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left| (v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + (v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')) \right|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{638}$$

$$\left\langle B_1^+ B_0^-(t,\tau) B_1^+ B_0^-(t',0) \right\rangle_B = \mathrm{e}^{\chi_{10}(t) + \chi_{10}(t')} U_{10}(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{639}$$

$$\left\langle B_0^+ B_1^-(t,\tau) B_0^+ B_1^-(t',0) \right\rangle_B = \mathrm{e}^{\chi_{01}(t) + \chi_{01}(t')} U_{10}(t,t') \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{640}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B = \frac{1}{4}\Big\langle B_1^+B_0^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right) + B_1^+B_0^-\left(t,\tau\right)B_0^+B_1^-\left(t',0\right) + B_0^+B_1^-\left(t,\tau\right)B_1^+B_0^-\left(t',0\right) \tag{641}$$

$$+B_0^+B_1^-\left(t,\tau\right)B_0^+B_1^-\left(t',0\right)\Big\rangle - \frac{\left(B_{01}\left(t\right)+B_{10}\left(t\right)\right)\left(B_{01}\left(t'\right)+B_{10}\left(t'\right)\right)}{4}, \tag{642}$$

$$= \frac{1}{4}\left(2U_{10}\left(t,t'\right)\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{643}$$

$$\left.+2U_{10}^*\left(t,t'\right)\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{644}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{645}$$

$$= \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{646}$$

$$\left.+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{647}$$

$$-\left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}|m_{\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(|m_{\mathbf{k}}(t')|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{648}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B = \left\langle B_y\left(t,\tau\right)B_y\left(t',0\right)\right\rangle_B \tag{649}$$

$$= \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{650}$$

$$\left.+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{651}$$

$$= \left\langle\left(\frac{B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\left(\frac{B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_B \tag{652}$$

$$= -\frac{1}{4}\left\langle\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)\right)\left(B_0^+B_1^-\left(t',0\right)-B_1^+B_0^-\left(t',0\right)+B_{10}(t')-B_{01}(t')\right)\right\rangle_B \tag{653}$$

$$= -\frac{1}{4}\Big\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-\left(t',0\right) - B_0^+B_1^-(t,\tau)B_1^+B_0^-\left(t',0\right) + B_0^+B_1^-(t,\tau)B_{10}(t') - B_0^+B_1^-(\tau)B_{01}(t') - B_1^+B_0^-(t,\tau) \tag{654}$$

$$\times B_0^+B_1^-\left(t',0\right) + B_1^+B_0^-(t,\tau)B_1^+B_0^-\left(t',0\right) - B_1^+B_0^-(t,\tau)B_{10}(t') + B_1^+B_0^-(t,\tau)B_{01}(t') + B_{10}(t)B_0^+B_1^-\left(t',0\right) - B_{10}(t) \tag{655}$$

$$\times B_1^+B_0^-\left(t',0\right) + B_{10}(t)B_{10}(t') - B_{10}\left(t\right)B_{01}\left(t'\right) - B_{01}\left(t\right)B_0^+B_1^-\left(t',0\right) + B_{01}\left(t\right)B_1^+B_0^-\left(t',0\right) - B_{01}\left(t\right)B_{10}\left(t'\right) \tag{656}$$

$$+B_{01}\left(t\right)B_{01}\left(t'\right)\Big\rangle \tag{657}$$

$$= -\frac{1}{4}\Big\langle B_0^+B_1^-(t,\tau)B_0^+B_1^-\left(t',0\right) - B_0^+B_1^-(t,\tau)B_1^+B_0^-\left(t',0\right) - B_1^+B_0^-(t,\tau)B_0^+B_1^-\left(t',0\right) + B_1^+B_0^-(t,\tau)B_1^+B_0^-\left(t',0\right)\Big\rangle \tag{658}$$

$$+\left(B_{01}\left(t\right)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Im} \tag{659}$$

$$= -\frac{1}{4}\left(2\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{660}$$

$$\left.-2\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{661}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{\mathrm{v}_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{\mathrm{v}_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{662}$$

$$= -\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{663}$$

$$\left.-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{664}$$

$$+\left(\mathrm{e}^{\chi_{01}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Im} \tag{665}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B = \left\langle B_x(t,\tau)B_y(t',0)\right\rangle_B \tag{666}$$

$$=\left\langle\left(\frac{B_1^+B_0^-(t,\tau)+B_0^+B_1^-(t,\tau)-B_{10}(t)-B_{01}(t)}{2}\right)\left(\frac{B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\right)\right\rangle_B \tag{667}$$

$$=\frac{1}{4i}\Big\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)B_{10}(t')-B_1^+B_0^-(t,\tau)B_{01}(t')+B_{01}(t)B_1^+B_0^-(t',0) \tag{668}$$

$$+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)-B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_{10}(t')-B_0^+B_1^-(t,\tau)B_{01}(t')+B_{01}(t)B_{01}(t') \tag{669}$$

$$-B_{10}(t)B_0^+B_1^-(t',0)+B_{10}(t)B_1^+B_0^-(t',0)-B_{10}(t)B_{10}(t')+B_{10}(t)B_{01}(t')-B_{01}(t)B_0^+B_1^-(t',0)-B_{01}(t)B_{10}(t')\Big\rangle_B \tag{670}$$

$$=\frac{1}{4i}\Big\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)+B_1^+B_0^-(t,\tau)B_{10}(t')-B_1^+B_0^-(t,\tau)B_{01}(t') \tag{671}$$

$$+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)-B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_{10}(t')-B_0^+B_1^-(t,\tau)B_{01}(t') \tag{672}$$

$$-B_{10}(t)B_0^+B_1^-(t',0)+B_{10}(t)B_1^+B_0^-(t',0)-B_{10}(t)B_{10}(t')+B_{10}(t)B_{01}(t')-B_{01}(t)B_0^+B_1^-(t',0) \tag{673}$$

$$+B_{01}(t)B_1^+B_0^-(t',0)-B_{01}(t)B_{10}(t')+B_{01}(t)B_{01}(t')\Big\rangle_B \tag{674}$$

$$=\frac{1}{4i}\Big\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)-B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\Big\rangle \tag{675}$$

$$+\frac{1}{4i}\left(B_{10}(t)+B_{01}(t)\right)\left(B_{10}(t')-B_{01}(t')\right) \tag{676}$$

$$=\frac{1}{4i}\Big\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) \tag{677}$$

$$-B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\Big\rangle+\frac{1}{4i}\left(B_{10}(t)+B_{01}(t)\right)\left(B_{10}(t')-B_{01}(t')\right) \tag{678}$$

$$=\frac{1}{4i}\Big\langle B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0) \tag{679}$$

$$-B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)\Big\rangle+\left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{680}$$

$$=\frac{1}{4i}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}-\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)U_{10}^*(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{681}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}-\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{682}$$

$$+\left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{683}$$

$$=\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{684}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\left(B_{10}(t)\right)^{\Re}\left(B_{10}(t')\right)^{\Im} \tag{685}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B=\left\langle\left(\frac{B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\left(\frac{B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_B \tag{686}$$

$$=\frac{1}{4\mathrm{i}}\left\langle\left(B_0^+B_1^-(t,\tau)-B_1^+B_0^-(t,\tau)+B_{10}(t)-B_{01}(t)\right)\left(B_1^+B_0^-(t',0)+B_0^+B_1^-(t',0)-B_{10}(t')-B_{01}(t')\right)\right\rangle_B \tag{687}$$

$$=\frac{1}{4\mathrm{i}}\Big\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)-B_0^+B_1^-(t,\tau)B_{10}(t')-B_0^+B_1^-(t,\tau)B_{01}(t')+B_{01}(t)B_{01}(t') \tag{688}$$

$$-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)-B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)+B_1^+B_0^-(t,\tau)B_{10}(t')+B_1^+B_0^-(t,\tau)B_{01}(t') \tag{689}$$

$$+B_{10}(t)B_0^+B_1^-(t',0)-B_{10}(t)B_{10}(t')-B_{10}(t)B_{01}(t')-B_{01}(t)B_1^+B_0^-(t',0)-B_{01}(t)B_0^+B_1^-(t',0) \tag{690}$$

$$+B_{01}(t)B_{10}(t')+B_{10}(t)B_1^+B_0^-(t',0)\Big\rangle \tag{691}$$

$$=\frac{1}{4i}\Big\langle B_0^+B_1^-(t,\tau)B_1^+B_0^-(t',0)+B_0^+B_1^-(t,\tau)B_0^+B_1^-(t',0)-B_1^+B_0^-(t,\tau)B_1^+B_0^-(t',0)-B_1^+B_0^-(t,\tau)B_0^+B_1^-(t',0)\Big\rangle \tag{692}$$

$$+\left(B_{10}(t)\right)^{\Im}\left(B_{10}(t')\right)^{\Re} \tag{693}$$

$$=\frac{1}{4i}\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}U_{10}^*(t,t')\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{694}$$

$$+\,\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{695}$$

$$-\,\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{696}$$

$$\left.-\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\left(B_{10}(t)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re} \tag{697}$$

$$=\frac{1}{4i}\left(2i\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{698}$$

$$\left.+2i\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\left(B_{10}(t)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re} \tag{699}$$

$$=\left(B_{10}(t)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re}+\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^{*}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t'))\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{700}$$

$$\left.+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}\left(t,t'\right)\prod_{\mathbf{k}}\mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{701}$$

$$\left\langle b^{\dagger}D\left(h\right)\right\rangle_{B}=-h^{*}\left\langle D\left(h\right)\right\rangle_{B}N \tag{702}$$

$$\left\langle bD\left(h\right)\right\rangle_{B}=h\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{703}$$

$$\left\langle D\left(h\right)b^{\dagger}\right\rangle_{B}=-h^{*}\left\langle D\left(h\right)\right\rangle_{B}\left(N+1\right) \tag{704}$$

$$\left\langle D\left(h\right)b\right\rangle_{B}=h\left\langle D\left(h\right)\right\rangle_{B}N \tag{705}$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}}\left(t\right)=q_{j\mathbf{k}}\left(t\right) \tag{706}$$

$$\left\langle B_1^{+}B_0^{-}\left(t,\tau\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}=\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^2}\right)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B} \tag{707}$$

$$=\mathrm{e}^{\chi_{10}(t)}\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{708}$$

$$=\mathrm{e}^{\chi_{10}(t)}q_{i\mathbf{k}'}\left(t'\right)\left\langle D\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{709}$$

$$=\mathrm{e}^{\chi_{10}(t)}q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{710}$$

$$=q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right)\right\rangle_{B} \tag{711}$$

$$=-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right) \tag{712}$$

$$\left\langle B_1^{+}B_0^{-}\left(t,\tau\right)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t'\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}(t)v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^2}\right)}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}\left\langle\prod_{\mathbf{k}}D\left(m_{\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle \tag{713}$$

$$=q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right), \tag{714}$$

$$\left\langle B_0^{+}B_1^{-}(t,\tau)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}=-q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right), \tag{715}$$

$$\left\langle B_0^{+}B_1^{-}(t,\tau)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right), \tag{716}$$

$$q_{i\mathbf{k}'}\left(0\right)=g_{i\mathbf{k}'}-v_{i\mathbf{k}'} \tag{717}$$

$$\left\langle B_x\left(t,\tau\right)B_{i\bmod 2z}\left(t',0\right)\right\rangle_{B}=\left\langle\left(\frac{B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}}{2}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{718}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)-B_{10}-B_{10}^{*}\right)\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{719}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(B_{1+}B_{0-}\left(\tau\right)+B_{0+}B_{1-}\left(\tau\right)\right)\left(q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(0\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{720}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle B_{1+}B_{0-}\left(\tau\right)q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+B_{0+}B_{1-}\left(\tau\right)q_{i\mathbf{k}'}\left(0\right)b_{\mathbf{k}'}^{\dagger}+B_{1+}B_{0-}\left(\tau\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right. \tag{721}$$

$$\left.+B_{0+}B_{1-}\left(\tau\right)q_{i\mathbf{k}'}^{*}\left(0\right)b_{\mathbf{k}'}\right\rangle_{B} \tag{722}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}(t)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}(t)\right. \tag{723}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right)\right) \tag{724}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right)+q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right)\right. \tag{725}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\left(t\right)+q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right)\right) \tag{726}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}\left(t\right)+\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{01}\left(t\right)\right)\right. \tag{727}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{01}\left(t\right)\right)\right) \tag{728}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{10}\left(t\right)-\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}B_{01}\left(t\right)\right)\right. \tag{729}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{10}\left(t\right)-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{01}\left(t\right)\right)\right) \tag{730}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left(-q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\left(m_{\mathbf{k}'}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}(B_{10}(t)-B_{01}(t))+q_{i\mathbf{k}'}^{*}(t')\,N_{\mathbf{k}'}m_{\mathbf{k}'}(t)\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}(B_{10}(t)\right. \tag{731}$$

$$\left.-B_{01}\left(t\right)\right)) \tag{732}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}2\mathrm{i}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{733}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\right) \tag{734}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right), \tag{735}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(t\right)\left(q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-q_{i\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left(m_{\mathbf{k}'}\left(t\right)\right)^{*}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right) \tag{736}$$

$$\left\langle B_{i\mathrm{mod}2z}\left(t,\tau\right)B_{x}\left(t',0\right)\right\rangle_{B}=\left\langle\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_{B} \tag{737}$$

$$=\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)\,b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}(t',0)+B_{0}^{+}B_{1}^{-}(t',0)-B_{10}(t')-B_{01}(t')}{2}\right)\right\rangle_{B} \tag{738}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}(t)\,b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)-B_{10}\left(t'\right)-B_{01}\left(t'\right)\right)\right\rangle_{B} \tag{739}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(q_{i\mathbf{k}'}\left(t\right)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}\left(t',0\right)+B_{0}^{+}B_{1}^{-}\left(t',0\right)\right)\right\rangle_{B} \tag{740}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)+q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right. \tag{741}$$

$$\left.+q_{i\mathbf{k}'}^{*}\left(t\right)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\left(t',0\right)\right\rangle, \tag{742}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B}=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\left(t',0\right)\right\rangle_{B} \tag{743}$$

$$=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B} \tag{744}$$

$$=q_{i\mathbf{k}'}\left(t\right)\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(m_{\mathbf{k}'}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}'}\left(t'\right)}{2}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B} \tag{745}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{746}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{747}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{748}$$

$$=q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\left(-m_{\mathbf{k}'}^{*}\left(t'\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}N_{\mathbf{k}'}\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{749}$$

$$=-m_{\mathbf{k}'}^{*}\left(t'\right)q_{i\mathbf{k}'}\left(t\right)\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle\prod_{\mathbf{k}}D\left(m_{\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{750}$$

$$=-m_{\mathbf{k}'}^{*}\left(t'\right)q_{i\mathbf{k}'}\left(t\right)B_{10}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{751}$$

$$\left\langle q_{i\mathbf{k}'}(t)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^{+}B_1^{-}\left(t',0\right)\right\rangle_{B}=m_{\mathbf{k}'}^{*}\left(t'\right)q_{i\mathbf{k}'}\left(t\right)B_{01}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{752}$$

$$\left\langle q_{i\mathbf{k}'}^{*}(t)b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^{+}B_0^{-}\left(t',0\right)\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_1^{+}B_0^{-}\left(t',0\right)\right\rangle_{B} \tag{753}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\right)\right\rangle_{B} \tag{754}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}\left\langle b_{\mathbf{k}'}D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B} \tag{755}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B} \tag{756}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\mathrm{e}^{\frac{\Gamma_{\mathbf{k}}\left(t'\right)}{2}}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(m_{\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(m_{\mathbf{k}}\left(t'\right)\right)\right)\right\rangle_{B} \tag{757}$$

$$=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right), \tag{758}$$

$$\left\langle\left(q_{i\mathbf{k}'}(t)\right)^{*}b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_1^{-}\left(t',0\right)\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-m_{\mathbf{k}'}\left(t'\right)\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right), \tag{759}$$

$$\left\langle B_{i\mathrm{mod}2z}\left(t,\tau\right)B_x\left(t',0\right)\right\rangle_{B}=\frac{1}{2}\sum_{\mathbf{k}'}\Big(-m_{\mathbf{k}'}^{*}\left(t'\right)q_{i\mathbf{k}'}(t)B_{10}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-\left(-m_{\mathbf{k}'}^{*}\left(t'\right)\right)q_{i\mathbf{k}'}(t)B_{01}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{760}$$

$$+q_{i\mathbf{k}'}^{*}(t)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t'\right)+q_{i\mathbf{k}'}^{*}(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-m_{\mathbf{k}'}\left(t'\right)\right)\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t'\right)\Big) \tag{761}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}\left(t\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}^{*}\left(t'\right)\left(B_{01}\left(t'\right)-B_{10}\left(t'\right)\right)+q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{762}$$

$$\times\left(N_{\mathbf{k}'}+1\right)\left(B_{10}\left(t'\right)-B_{01}\left(t'\right)\right)\Big) \tag{763}$$

$$=\frac{1}{2}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}\left(t\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}^{*}\left(t'\right)\left(B_{01}\left(t'\right)-B_{10}\left(t'\right)\right)-q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{764}$$

$$\times\left(N_{\mathbf{k}'}+1\right)\left(B_{01}\left(t'\right)-B_{10}\left(t'\right)\right)\Big) \tag{765}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}B_{01}^{\Im}\left(t'\right)\left(q_{i\mathbf{k}'}\left(t\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}^{*}\left(t'\right)-q_{i\mathbf{k}'}^{*}\left(t\right)m_{\mathbf{k}'}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\right), \tag{766}$$

$$\left\langle B_y\left(t,\tau\right)B_{i\mathrm{mod}2z}\left(t',0\right)\right\rangle_{B}=\left\langle\left(\frac{B_0^{+}B_1^{-}(t,\tau)-B_1^{+}B_0^{-}(t,\tau)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{767}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^{+}B_1^{-}(t,\tau)-B_1^{+}B_0^{-}(t,\tau)+B_{10}\left(t\right)-B_{01}\left(t\right)\right)\left(q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{768}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^{+}B_1^{-}\left(t,\tau\right)-B_1^{+}B_0^{-}\left(t,\tau\right)\right)\left(q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right)\right\rangle_{B} \tag{769}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle B_0^{+}B_1^{-}\left(t,\tau\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}-B_1^{+}B_0^{-}\left(t,\tau\right)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}+B_0^{+}B_1^{-}\left(t,\tau\right)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'} \tag{770}$$

$$-B_1^{+}B_0^{-}\left(t,\tau\right)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\Big\rangle, \tag{771}$$

$$\left\langle B_0^{+}B_1^{-}(t,\tau)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}=-q_{i\mathbf{k}'}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{01}\left(t\right), \tag{772}$$

$$\left\langle B_0^{+}B_1^{-}(t,\tau)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t'\right)\left(-m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}\left(t\right), \tag{773}$$

$$\left\langle B_1^{+}B_0^{-}(t,\tau)q_{i\mathbf{k}'}\left(t'\right)b_{\mathbf{k}'}^{\dagger}\right\rangle_{B}=-q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)B_{10}\left(t\right), \tag{774}$$

$$\left\langle B_1^{+}B_0^{-}(t,\tau)q_{i\mathbf{k}'}^{*}\left(t'\right)b_{\mathbf{k}'}\right\rangle_{B}=q_{i\mathbf{k}'}^{*}\left(t'\right)m_{\mathbf{k}'}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}N_{\mathbf{k}'}B_{10}\left(t\right), \tag{775}$$

$$\left\langle B_y\left(t,\tau\right)B_{i\mathrm{mod}2z}\left(t',0\right)\right\rangle_{B}=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(B_{01}(t)+B_{10}(t)\right)\left(q_{i\mathbf{k}'}\left(t'\right)\left(m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^{*}\left(N_{\mathbf{k}'}+1\right)-q_{i\mathbf{k}'}^{*}\left(t'\right)m_{\mathbf{k}'}(t)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}N_{\mathbf{k}'}\right) \tag{776}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(q_{i\mathbf{k}'}^{*}\left(t'\right)\left(N_{\mathbf{k}'}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}^{*}(t)-q_{i\mathbf{k}'}^{*}\left(t'\right)N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)\right)\left(B_{10}\left(t\right)+B_{01}\left(t\right)\right), \tag{777}$$

$$= \frac{2}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t)(B_{10}(t))^{\Re} - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)(B_{10}(t))^{\Re}\Big), \tag{778}$$

$$= \frac{(B_{10}(t))^{\Re}}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t')(N_{\mathbf{k}'}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}m^*_{\mathbf{k}'}(t) - q^*_{i\mathbf{k}'}(t')N_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}m_{\mathbf{k}'}(t)\Big), \tag{779}$$

$$\langle B_{i\mathrm{mod}2z}(t,\tau)B_y(t',0)\rangle_B = \Big\langle \textstyle\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t)b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(\frac{B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t')-B_{01}(t')}{2\mathrm{i}}\Big)\Big\rangle_B \tag{780}$$

$$= \frac{1}{2\mathrm{i}}\Big\langle\sum_{\mathbf{k}'}\Big(q_{i\mathbf{k}'}(t)\,b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)+B_{10}(t') \tag{781}$$

$$-B_{01}(t')\Big)\Big\rangle_B \tag{782}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle\Big(q_{i\mathbf{k}'}(t)\,b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+q^*_{i\mathbf{k}'}(t)\,b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\Big)\Big(B_0^+B_1^-(t',0)-B_1^+B_0^-(t',0)\Big)\Big\rangle_B \tag{783}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle q_{i\mathbf{k}'}(t)\,b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)-q_{i\mathbf{k}'}(t)\,b^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0) \tag{784}$$

$$+q^*_{i\mathbf{k}'}(t)\,b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-(t',0)-q^*_{i\mathbf{k}'}(t)\,b_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-(t',0)\Big\rangle \tag{785}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big\langle\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-(t',0)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-(t',0)\Big\rangle \tag{786}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\langle b_{\mathbf{k}'}B_0^+B_1^-(t',0)\rangle-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\langle b_{\mathbf{k}'}B_1^+B_0^-(t',0)\rangle\Big\rangle \tag{787}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-(t',0)\Big\rangle-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-(t',0)\Big\rangle+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\langle b_{\mathbf{k}'}B_0^+B_1^-(t',0)\rangle \tag{788}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\langle b_{\mathbf{k}'}B_1^+B_0^-(t',0)\rangle \tag{789}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-(t',0)\Big\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-(t',0)\Big\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\langle b_{\mathbf{k}'}B_0^+B_1^-(t',0)\rangle \tag{790}$$

$$-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\langle b_{\mathbf{k}'}B_1^+B_0^-(t',0)\rangle\Big) \tag{791}$$

$$\Big\langle b^\dagger_{\mathbf{k}'}B_1^+B_0^-(t',0)\Big\rangle_B = -m^*_{\mathbf{k}'}(t')\,B_{10}(t')\,N_{\mathbf{k}'}, \tag{792}$$

$$\Big\langle b^\dagger_{\mathbf{k}'}B_0^+B_1^-(t',0)\Big\rangle_B = m^*_{\mathbf{k}'}(t')\,B_{01}(t')\,N_{\mathbf{k}'}, \tag{793}$$

$$\langle b_{\mathbf{k}'}B_1^+B_0^-(t',0)\rangle_B = m_{\mathbf{k}'}(t')\,(N_{\mathbf{k}'}+1)\,B_{10}(t'), \tag{794}$$

$$\langle b_{\mathbf{k}'}B_0^+B_1^-(t',0)\rangle_B = -m_{\mathbf{k}'}(t')\,(N_{\mathbf{k}'}+1)\,B_{01}(t'), \tag{795}$$

$$\langle B_{i\mathrm{mod}2z}(t,\tau)B_y(t',0)\rangle_B = \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\left(-\left(-m^*_{\mathbf{k}'}(t')\right)B_{01}(t')\,N_{\mathbf{k}'}\right)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\left(-m^*_{\mathbf{k}'}(t')\,B_{10}(t')\,N_{\mathbf{k}'}\right) \tag{796}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\left(-m_{\mathbf{k}'}(t')\,(N_{\mathbf{k}'}+1)\,B_{01}(t')\right)-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\,m_{\mathbf{k}'}(t')\,(N_{\mathbf{k}'}+1)\,B_{10}(t')\Big) \tag{797}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-q_{i\mathbf{k}'}(t)\left(-m^*_{\mathbf{k}'}(t')\right)B_{01}(t')\,N_{\mathbf{k}'}+q_{i\mathbf{k}'}(t)\,m^*_{\mathbf{k}'}(t')\,B_{10}(t')\,N_{\mathbf{k}'}\right) \tag{798}$$

$$+\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(q^*_{i\mathbf{k}'}(t)\left(-m_{\mathbf{k}'}(t')\right)(N_{\mathbf{k}'}+1)\,B_{01}(t')-q^*_{i\mathbf{k}'}(t)\,m_{\mathbf{k}'}(t')\,(N_{\mathbf{k}'}+1)\,B_{10}(t')\right)\Big) \tag{799}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(B_{10}(t')+B_{01}(t')\right)\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\,m^*_{\mathbf{k}'}(t')\,N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\,m_{\mathbf{k}'}(t')\,(N_{\mathbf{k}'}+1)\Big) \tag{800}$$

$$= \frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\,m^*_{\mathbf{k}'}(t')\,B^{\Re}_{10}(t')\,N_{\mathbf{k}'}-\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\,m_{\mathbf{k}'}(t')\,B^{\Re}_{10}(t')\,(N_{\mathbf{k}'}+1)\Big) \tag{801}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\,m_{\mathbf{k}'}(t')\,B^{\Re}_{10}(t')\,(N_{\mathbf{k}'}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\,m^*_{\mathbf{k}'}(t')\,B^{\Re}_{10}(t')\,N_{\mathbf{k}'}\Big) \tag{802}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\,m_{\mathbf{k}'}(t')\,B^{\Re}_{10}(t')\,(N_{\mathbf{k}'}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\,m^*_{\mathbf{k}'}(t')\,B^{\Re}_{10}(t')\,N_{\mathbf{k}'}\Big) \tag{803}$$

$$= \mathrm{i}B^{\Re}_{10}(t')\sum_{\mathbf{k}'}\Big(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}q^*_{i\mathbf{k}'}(t)\,m_{\mathbf{k}'}(t')\,(N_{\mathbf{k}'}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}'}\tau}q_{i\mathbf{k}'}(t)\,m^*_{\mathbf{k}'}(t')\,N_{\mathbf{k}'}\Big). \tag{804}$$

The correlation functions are equal to:

$$\left\langle \widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_{j\mathrm{mod}2z}}\left(t'\right)\right\rangle_B = \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t'\right)\right)^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^* \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\left(t'\right)\right) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right), \tag{805}$$

$$\left\langle \widetilde{B_x}\left(t\right)\widetilde{B_x}\left(t'\right)\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re} U_{10}\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{806}$$

$$\left. + \left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{807}$$

$$- \left(\mathrm{e}^{\chi_{10}(t)}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re}\left(\mathrm{e}^{\chi_{01}(t')}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left(\left|\frac{v_{1\mathbf{k}}(t')}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right|^2\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)^{\Re} \tag{808}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_y}\left(t'\right)\right\rangle_B = -\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(\mathrm{v}_{1\mathbf{k}}(\mathrm{t})-\mathrm{v}_{0\mathbf{k}}(\mathrm{t})\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')-\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right. \tag{809}$$

$$\left. - \left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re} U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(v_{0\mathbf{k}}(t)-v_{1\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{810}$$

$$+ \left(\mathrm{e}^{\chi_{01}(t)}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im}\left(\mathrm{e}^{\chi_{10}(t')}\left(\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{\mathrm{v}_{1\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}-\frac{\mathrm{v}_{0\mathbf{k}}(\mathrm{t}')}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right)^{\Im} \tag{811}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_y}\left(t'\right)\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im} U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}\right. \tag{812}$$

$$\left. \times U_{10}\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Re}\left(B_{10}\left(t'\right)\right)^{\Im} \tag{813}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_x}\left(t'\right)\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im} U_{10}^*\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}\right. \tag{814}$$

$$\left. \times U_{10}\left(t,t'\right)\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|\left(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)\right)\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) + \left(B_{10}(t)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re} \tag{815}$$

$$\left\langle \widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_x}\left(t'\right)\right\rangle_B = \mathrm{i}\sum_{\mathbf{k}} B_{01}^{\Im}\left(t'\right)\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}\left(t'\right) - v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\right)^* - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^* \frac{v_{1\mathbf{k}}\left(t'\right) - v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{816}$$

$$\left\langle \widetilde{B_x}(t)\widetilde{B_{i\mathrm{mod}2z}}\left(t'\right)\right\rangle_B = \mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t'\right)\right)^* N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t'\right)\right)\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{817}$$

$$\left\langle \widetilde{B_{i\mathrm{mod}2z}}(t)\widetilde{B_y}\left(t'\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(t'\right)\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^*\left(\frac{v_{1\mathbf{k}}\left(t'\right) - v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right) - \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}\left(t'\right) - v_{0\mathbf{k}}\left(t'\right)}{\omega_{\mathbf{k}}}\right)^* N_{\mathbf{k}}\right), \tag{818}$$

$$\left\langle \widetilde{B_y}(t)\widetilde{B_{i\mathrm{mod}2z}}\left(t'\right)\right\rangle_B = \mathrm{i}B_{10}^{\Re}(t)\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t'\right)\right)^* N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t'\right)\right)\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\right). \tag{819}$$

Let’s consider the following expression related to the sum of coupling constants for a bath over all the frequences:

$$L_i\left(\omega\right) \equiv \sum_{\mathbf{k}} g_{i\mathbf{k}}\sqrt{\delta\left(\omega - \omega_{\mathbf{k}}\right)}. \tag{820}$$

Under this definition we have the following expression for a function $f\left(\omega\right) \in L^2$:

$$\int_0^\infty f\left(\omega\right) L_i\left(\omega\right) L_j^*\left(\omega\right)\mathrm{d}\omega \approx \int_0^\infty f\left(\omega\right)\sum_{\mathbf{k}} g_i\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega - \omega_{\mathbf{k}}\right)}\sum_{\mathbf{k}'} g_j\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega - \omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{821}$$

$$= \int_0^\infty f\left(\omega\right)\sum_{\mathbf{k},\mathbf{k}'} g_i\left(\omega_{\mathbf{k}}\right) g_j\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega - \omega_{\mathbf{k}}\right)}\sqrt{\delta\left(\omega - \omega_{\mathbf{k}'}\right)}\mathrm{d}\omega, \tag{822}$$

$$\int_0^\infty f\left(\omega\right)\sum_{\mathbf{k}} g_i\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega - \omega_{\mathbf{k}}\right)}\mathrm{d}\omega = \sum_{\mathbf{k}}\int_0^\infty f\left(\omega\right) g_i\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega - \omega_{\mathbf{k}}\right)}\mathrm{d}\omega. \tag{823}$$

Now we will approach to the function $\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}$ using the normal distribution, so:

$$\delta\left(\omega-\omega_{\mathbf{k}}\right) = \lim_{\sigma\to 0^{+}} \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\sigma^{2}}} \tag{824}$$

$$\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)} = \lim_{\sigma\to 0^{+}} \sqrt{\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\sigma^{2}}}} \tag{825}$$

$$= \lim_{\sigma\to 0^{+}} \sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{4\sigma^{2}}} \tag{826}$$

$$= \lim_{\sigma\to 0^{+}} \sqrt{\sqrt{2\pi}\sigma}\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{2}\sigma}\mathrm{e}^{-\frac{\left(\omega-\omega_{\mathbf{k}}\right)^{2}}{2\left(\sqrt{2}\sigma\right)^{2}}} \tag{827}$$

$$= \lim_{\sigma\to 0^{+}} \sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right). \tag{828}$$

So we can obtain that:

$$\sum_{\mathbf{k}}\int_{0}^{\infty} f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega = \sum_{\mathbf{k}}\int_{0}^{\infty} f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\mathrm{d}\omega \tag{829}$$

$$= \sum_{\mathbf{k}}\int_{0}^{\infty} f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to 0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{830}$$

$$= \sum_{\mathbf{k}}\int_{0}^{\infty} f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)\left(\lim_{\sigma\to 0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\right)\mathrm{d}\omega \tag{831}$$

$$= \lim_{\sigma\to 0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}\int_{0}^{\infty} f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega \tag{832}$$

$$= \sum_{\mathbf{k}}\left(\lim_{\sigma\to 0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)\left(\lim_{\sigma\to 0^{+}}\int_{0}^{\infty} f\left(\omega\right)g_{i}\left(\omega_{\mathbf{k}}\right)N\left(x;\omega_{\mathbf{k}},\sqrt{2}\sigma\right)\mathrm{d}\omega\right) \tag{833}$$

$$= \sum_{\mathbf{k}}\left(\lim_{\sigma\to 0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\right)f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\ \text{(with } f\left(\omega\right)\text{,}g_{i}\left(\omega\right)\in L^{2}\text{)} \tag{834}$$

$$= \lim_{\sigma\to 0^{+}}\sqrt{2}\sqrt{\sqrt{2\pi}\sigma}\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\ \text{(with } f\left(\omega\right)\text{,}g_{i}\left(\omega\right)\in L^{2}\text{)} \tag{835}$$

$$= 0\,\text{(because the sum }\sum_{\mathbf{k}}f\left(\omega_{\mathbf{k}}\right)g_{i}\left(\omega_{\mathbf{k}}\right)\text{ is finite).} \tag{836}$$

Then we can proof the following:

$$\int_{0}^{\infty}f(\omega)L_{i}(\omega)L_{j}^{*}\left(\omega\right)\mathrm{d}\omega \approx \int_{0}^{\infty} f\left(\omega\right)\sum_{\mathbf{k}}g_{i}\left(\omega_{\mathbf{k}}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sum_{\mathbf{k}'}g_{j}^{*}\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{837}$$

$$= \int_{0}^{\infty} f\left(\omega\right)\sum_{\mathbf{k},\mathbf{k}'}g_{i}\left(\omega_{\mathbf{k}}\right)g_{j}^{*}\left(\omega_{\mathbf{k}'}\right)\sqrt{\delta\left(\omega-\omega_{\mathbf{k}}\right)}\sqrt{\delta\left(\omega-\omega_{\mathbf{k}'}\right)}\mathrm{d}\omega \tag{838}$$

$$= \sum_{\mathbf{k}\neq\mathbf{k}'}\int_{0}^{\infty} f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}'})\sqrt{\delta(\omega-\omega_{\mathbf{k}})}\sqrt{\delta(\omega-\omega_{\mathbf{k}'})}\mathrm{d}\omega+\sum_{\mathbf{k}}\int_{0}^{\infty} f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{839}$$

$$= 0+\sum_{\mathbf{k}}\int_{0}^{\infty} f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{840}$$

$$= \sum_{\mathbf{k}}\int_{0}^{\infty} f(\omega)g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})\mathrm{d}\omega \tag{841}$$

$$= \sum_{\mathbf{k}}f(\omega_{\mathbf{k}})g_{i}(\omega_{\mathbf{k}})g_{j}^{*}(\omega_{\mathbf{k}}) \tag{842}$$

if $i = j$ we recover the spectral density defined in the usual way when we integrate for a function $f(\omega)$ that belongs to the set L^2:

$$\sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) = \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_i(\omega_{\mathbf{k}}) g_i^*(\omega_{\mathbf{k}}) \delta(\omega - \omega_{\mathbf{k}}) \mathrm{d}\omega \tag{843}$$

$$= \int_0^\infty f(\omega) J_{ii}(\omega) \mathrm{d}\omega \tag{844}$$

$$= \int_0^\infty f(\omega) |L_i(\omega)|^2 \, \mathrm{d}\omega. \tag{845}$$

where

$$J_{ii}(\omega) = \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \, \delta(\omega - \omega_{\mathbf{k}}), \tag{846}$$

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t) = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t). \tag{847}$$

In this case $g_i(\omega)$ and $v_i(\omega, t)$ are the continuous version of $g_i(\omega_{\mathbf{k}})$ and $v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t)$ respectively.
The integral version of the correlation functions can be obtained as follows:

$$\left\langle \widetilde{B_{iz}}(t) \widetilde{B_{j\mathrm{mod}2z}}(t') \right\rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t')) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \tag{848}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}}, t)) g_{j\mathbf{k}}^* (1 - F_j(\omega_{\mathbf{k}}, t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}, t))^* g_{j\mathbf{k}} (1 - F_j(\omega_{\mathbf{k}}, t')) \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \tag{849}$$

$$\approx \int_0^\infty \Big(L_i(\omega) L_j^*(\omega) (1 - F_i(\omega, t)) \left(1 - F_j^*(\omega, t')\right) \mathrm{e}^{\mathrm{i}\omega\tau} N(\omega) + L_i^*(\omega) L_j(\omega) \left(1 - F_i^*(\omega, t)\right) \left(1 - F_j(\omega, t')\right) \mathrm{e}^{-\mathrm{i}\omega\tau} (N(\omega) + 1) \Big) \mathrm{d}\omega, \tag{850}$$

$$\chi_{10}(t) = \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \tag{851}$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \tag{852}$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g_{1\mathbf{k}}^* g_{0\mathbf{k}} F_1^*(\omega_{\mathbf{k}}, t) F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} g_{0\mathbf{k}}^* F_1(\omega_{\mathbf{k}}, t) F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \tag{853}$$

$$\approx \int_0^\infty \frac{L_0(\omega) L_1^*(\omega) F_1^*(\omega, t) F_0(\omega, t) - L_1(\omega) L_0^*(\omega) F_1(\omega, t) F_0^*(\omega, t)}{2\omega^2} \mathrm{d}\omega, \tag{854}$$

$$U_{10}(t, t') = \prod_{\mathbf{k}} \mathrm{e}^{\mathrm{i} \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{855}$$

$$= \mathrm{e}^{\mathrm{i} \sum_{\mathbf{k}} \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{856}$$

$$= \mathrm{e}^{\mathrm{i} \left(\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{857}$$

$$= \mathrm{e}^{\mathrm{i} \left(\sum_{\mathbf{k}} \frac{(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t))(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t') - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t'))^* \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}}{\omega_{\mathbf{k}}^2} \right)^{\Im}} \tag{858}$$

$$\approx \mathrm{e}^{\mathrm{i} \left(\int_0^\infty \frac{(L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t))(L_1(\omega) F_1(\omega, t') - L_0(\omega) F_0(\omega, t'))^* \mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2} \mathrm{d}\omega \right)^{\Im}}, \tag{859}$$

$$B_{10}(t) = \left(\prod_{\mathbf{k}} \mathrm{e}^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \left(\mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right), \tag{860}$$

$$= \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{861}$$

$$\approx \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\frac{1}{2} \int_0^\infty \left| \frac{L_1(\omega) F_1(\omega, t) - L_0(\omega) F_0(\omega, t)}{\omega} \right|^2 \coth\left(\frac{\beta\omega}{2}\right) \mathrm{d}\omega} \tag{862}$$

$$\xi^+(t, t') = \prod_{\mathbf{k}} \mathrm{e}^{-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau} + v_{1\mathbf{k}}(t') - v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{863}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{864}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}+g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{865}$$

$$\approx \mathrm{e}^{-\int_0^\infty \frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}+L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t')|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{866}$$

$$\xi^-\left(t,t'\right) = \prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')\right)\right|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{867}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(s)\right)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{868}$$

$$= \mathrm{e}^{-\sum_{\mathbf{k}} \frac{|(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t)-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t))\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-\left(g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}},t')-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}},t')\right)|^2}{2\omega_{\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{869}$$

$$\approx \mathrm{e}^{-\int_0^\infty \frac{|(L_1(\omega)F_1(\omega,t)-L_0(\omega)F_0(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}-\left(L_1(\omega)F_1(\omega,t')-L_0(\omega)F_0(\omega,t')\right)|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{870}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_x}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)^{\Re}U_{10}(t,t')\xi^+\left(t,t'\right)+\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)\right)-\left(B_{10}\left(t\right)\right)^{\Re}\left(B_{01}\left(t'\right)\right)^{\Re} \tag{871}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_y}(t')\right\rangle_B = -\frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}\left(t,t'\right)\xi^+\left(t,t'\right)-\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Re}U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)\right)+\left(B_{01}\left(t\right)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Im} \tag{872}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_y}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}\left(t,t'\right)\xi^+\left(t,t'\right)\right)+\left(B_{10}\left(t\right)\right)^{\Re}\left(B_{10}\left(t'\right)\right)^{\Im} \tag{873}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_x}(t')\right\rangle_B = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}U_{10}^*\left(t,t'\right)\xi^-\left(t,t'\right)+\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}U_{10}\left(t,t'\right)\xi^+\left(t,t'\right)\right)+\left(B_{10}\left(t\right)\right)^{\Im}\left(B_{10}\left(t'\right)\right)^{\Re} \tag{874}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B = \mathrm{i}B_{01}^{\Im}(t')\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right), \tag{875}$$

$$= \mathrm{i}B_{01}^{\Im}\left(t'\right)\sum_{\mathbf{k}}\left(-g_{i\mathbf{k}}^*\left(1-F_i\left(\omega_{\mathbf{k}},t\right)\right)^*\frac{g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}},t'\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}},t'\right)}{\omega_{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right. \tag{876}$$

$$\left.+g_{i\mathbf{k}}\left(1-F_i\left(\omega_{\mathbf{k}},t\right)\right)\left(\frac{g_{1\mathbf{k}}F_1\left(\omega_{\mathbf{k}},t'\right)-g_{0\mathbf{k}}F_0\left(\omega_{\mathbf{k}},t'\right)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{877}$$

$$Q\left(\omega,t\right) = \frac{L_1\left(\omega\right)F_1\left(\omega,t\right)-L_0\left(\omega\right)F_0\left(\omega,t\right)}{\omega} \tag{878}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_x}(t')\right\rangle_B \approx \mathrm{i}B_{01}^{\Im}\left(t'\right)\int_0^\infty\left(L_i\left(\omega\right)\left(1-F_i\left(\omega,t\right)\right)Q^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-L_i^*\left(\omega\right)\left(1-F_i^*\left(\omega,t\right)\right)Q\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, \tag{879}$$

$$\left\langle\widetilde{B_x}(t)\widetilde{B_{iz}}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Im}(t)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t')\right)\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{880}$$

$$\approx \mathrm{i}B_{01}^{\Im}\left(t\right)\int_0^\infty\left(L_i^*\left(\omega\right)\left(1-F_i^*\left(\omega,t'\right)\right)Q\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-L_i\left(\omega\right)\left(1-F_i\left(\omega,t'\right)\right)Q^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{881}$$

$$\left\langle\widetilde{B_{iz}}(t)\widetilde{B_y}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Re}(t')\sum_{\mathbf{k}}\left(\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t')-v_{0\mathbf{k}}(t')}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right), \tag{882}$$

$$\approx \mathrm{i}B_{10}^{\Re}\left(t'\right)\int_0^\infty\left(L_i^*\left(\omega\right)\left(1-F_i^*(\omega,t')\right)Q(\omega,t')\left(N(\omega)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-L_i(\omega)\left(1-F_i(\omega,t')\right)Q^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{883}$$

$$\left\langle\widetilde{B_y}(t)\widetilde{B_{iz}}(t')\right\rangle_B = \mathrm{i}B_{10}^{\Re}\left(t\right)\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t'))^*N_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t')\right)(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}(t)-v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\right) \tag{884}$$

$$\approx i B_{10}^{\Re}\left(t\right)\int_0^\infty\left(L_i^*\left(\omega\right)\left(1-F_i^*\left(\omega,t'\right)\right)Q\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-L_i\left(\omega\right)\left(1-F_i\left(\omega,t'\right)\right)Q^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega. \tag{885}$$

The integral version of $F_0\left(\omega,t\right)$ and $F_1\left(\omega,t\right)$ are:

$$a_i\left(\omega_{\mathbf{k}},t\right) = \frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{886}$$

$$b_i\left(\omega_{\mathbf{k}},t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega_{\mathbf{k}}}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2\left|B_{10}(t)\right|^2\left|V_{10}(t)\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)},\tag{887}$$

$$r_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_i\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)},\tag{888}$$

$$s_i\left(\omega_{\mathbf{k}},t\right)=\frac{a_{(i+1)\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)b_{i\mathrm{mod}2}\left(\omega_{\mathbf{k}},t\right)}{1-b_0\left(\omega_{\mathbf{k}},t\right)b_1\left(\omega_{\mathbf{k}},t\right)}.\tag{889}$$

$$F_0\left(\omega,t\right)=r_0\left(\omega_{\mathbf{k}},t\right)+\frac{g_1\left(\omega_{\mathbf{k}}\right)}{g_0\left(\omega_{\mathbf{k}}\right)}s_0\left(\omega_{\mathbf{k}},t\right)\tag{890}$$

$$\approx r_0\left(\omega,t\right)+\frac{g_1\left(\omega\right)}{g_0\left(\omega\right)}s_0\left(\omega,t\right)\tag{891}$$

$$=r_0\left(\omega,t\right)+\frac{L_1\left(\omega\right)}{L_0\left(\omega\right)}s_0\left(\omega,t\right),\tag{892}$$

$$F_1\left(\omega,t\right)=\frac{g_0\left(\omega_{\mathbf{k}}\right)}{g_1\left(\omega_{\mathbf{k}}\right)}r_1\left(\omega_{\mathbf{k}},t\right)+s_1\left(\omega_{\mathbf{k}},t\right)\tag{893}$$

$$\approx\frac{g_0\left(\omega\right)}{g_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right)\tag{894}$$

$$=\frac{L_0\left(\omega\right)}{L_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right).\tag{895}$$

The expressions showed are well defined because the relevant products present in the correlations functions are of the form:

$$\int_0^\infty f(\omega)L_j(\omega)F_j(\omega,t)L_i^*(\omega)F_i^*(\omega,t)\,\mathrm{d}\omega=\int_0^\infty f(\omega)L_j(\omega)\left(r_j(\omega,t)+\frac{L_i(\omega)}{L_j(\omega)}s_j(\omega,t)\right)L_i^*(\omega)\left(r_i^*(\omega,t)+\frac{L_j^*(\omega)}{L_i^*(\omega)}s_i^*(\omega,t)\right)\mathrm{d}\omega\tag{896}$$

$$=\int_0^\infty f\left(\omega\right)\left(L_j\left(\omega\right)r_j\left(\omega,t\right)+L_i\left(\omega\right)s_j\left(\omega,t\right)\right)\left(L_i^*\left(\omega\right)r_i^*\left(\omega,t\right)+L_j^*\left(\omega\right)s_i^*\left(\omega,t\right)\right)\mathrm{d}\omega\tag{897}$$

$$=\int_0^\infty f\left(\omega\right)\Big(L_j\left(\omega\right)L_i^*\left(\omega\right)r_j\left(\omega,t\right)r_i^*\left(\omega,t\right)+\left|L_j\left(\omega\right)\right|^2r_j\left(\omega,t\right)s_i^*\left(\omega,t\right)\tag{898}$$

$$+\left|L_i\left(\omega\right)\right|^2s_j\left(\omega,t\right)r_i^*\left(\omega,t\right)+L_i\left(\omega\right)L_j^*\left(\omega\right)s_j\left(\omega,t\right)s_i^*\left(\omega,t\right)\Big)\,\mathrm{d}\omega.\tag{899}$$

here $f\left(\omega\right)\in L^2$. As we could proof these integral are convergent.
So the integral version of the correlation functions $\mathcal{B}_{ij}\left(t,t'\right)$ is can be written in a neater form as:

$$\mathcal{B}\left(t,t'\right)\equiv\begin{pmatrix}\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{13}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{16}\left(t,t'\right)\\\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{23}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{26}\left(t,t'\right)\\\mathcal{B}_{31}\left(t,t'\right)&\mathcal{B}_{32}\left(t,t'\right)&\mathcal{B}_{33}\left(t,t'\right)&\mathcal{B}_{32}\left(t,t'\right)&\mathcal{B}_{31}\left(t,t'\right)&\mathcal{B}_{36}\left(t,t'\right)\\\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{23}\left(t,t'\right)&\mathcal{B}_{22}\left(t,t'\right)&\mathcal{B}_{21}\left(t,t'\right)&\mathcal{B}_{26}\left(t,t'\right)\\\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{13}\left(t,t'\right)&\mathcal{B}_{12}\left(t,t'\right)&\mathcal{B}_{11}\left(t,t'\right)&\mathcal{B}_{16}\left(t,t'\right)\\\mathcal{B}_{61}\left(t,t'\right)&\mathcal{B}_{62}\left(t,t'\right)&\mathcal{B}_{63}\left(t,t'\right)&\mathcal{B}_{62}\left(t,t'\right)&\mathcal{B}_{61}\left(t,t'\right)&\mathcal{B}_{66}\left(t,t'\right)\end{pmatrix},\tag{900}$$

$$\mathcal{B}_{11}(t,t')=\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}^{\Re}\left(t\right)B_{01}^{\Re}\left(t'\right),\tag{901}$$

$$\mathcal{B}_{22}(t,t')=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)+B_{01}^{\Im}\left(t\right)B_{10}^{\Im}\left(t'\right),\tag{902}$$

$$\mathcal{B}_{12}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right),\tag{903}$$

$$\mathcal{B}_{21}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Im}\left(t\right)B_{10}^{\Re}\left(t'\right),\tag{904}$$

$$\mathcal{B}_{ij}\left(t,t'\right)=\int_0^{\infty}\left(P_i\left(\omega,t\right)P_j^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_i^*\left(\omega,t\right)P_j\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\{3,6\}, \tag{905}$$

$$\mathcal{B}_{i1}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t'\right)\int_0^{\infty}\left(P_i\left(\omega,t\right)Q_{10}^*\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i^*\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{906}$$

$$\mathcal{B}_{1i}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t\right)\int_0^{\infty}\left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{907}$$

$$\mathcal{B}_{i2}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t'\right)\int_0^{\infty}\left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{908}$$

$$\mathcal{B}_{2i}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t\right)\int_0^{\infty}\left(P_i^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_i\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{909}$$

$$\zeta_{ij}\left(t,t'\right)=\mathrm{e}^{\mathrm{i}\Im\left(\int_0^{\infty}\frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)}, \tag{910}$$

$$\xi_{ij}^{\pm}\left(t,t'\right)=\mathrm{e}^{-\int_0^{\infty}\frac{\left|\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{911}$$

$$P_i\left(\omega,t\right)=L_{i\mathrm{mod}2}\left(\omega\right)\left(1-F_{i\mathrm{mod}2}\left(\omega,t\right)\right), \tag{912}$$

$$Q_{ij}(\omega,t)=\frac{L_i\left(\omega\right)F_1\left(\omega,t\right)-L_j\left(\omega\right)F_j\left(\omega,t\right)}{\omega}, \tag{913}$$

$$a_i\left(\omega,t\right)=\frac{\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i\left(t\right)+2R_i\left(t\right)-\varepsilon\left(t\right)\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)}, \tag{914}$$

$$b_i\left(\omega,t\right)=\frac{2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{1}{\omega}\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2\coth\left(\frac{\beta\omega}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_i\left(t\right)-R_i\left(t\right)\right)-\frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega}{2}\right)}{\omega}\right)}, \tag{915}$$

$$r_i\left(\omega,t\right)=\frac{a_i\left(\omega,t\right)}{1-b_0\left(\omega,t\right)b_1\left(\omega,t\right)}, \tag{916}$$

$$s_i\left(\omega,t\right)=\frac{a_{(i+1)\mathrm{mod}2}\left(\omega,t\right)b_{i\mathrm{mod}2}\left(\omega,t\right)}{1-b_0\left(\omega,t\right)b_1\left(\omega,t\right)}, \tag{917}$$

$$F_0\left(\omega,t\right)=r_0\left(\omega,t\right)+\frac{L_1\left(\omega\right)}{L_0\left(\omega\right)}s_0\left(\omega,t\right), \tag{918}$$

$$F_1\left(\omega,t\right)=\frac{L_0\left(\omega\right)}{L_1\left(\omega\right)}r_1\left(\omega,t\right)+s_1\left(\omega,t\right). \tag{919}$$

The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\widetilde{A_i}\left(\tau\right)=e^{\mathrm{i}\overline{H_{\bar{S}}}\tau}A_i\mathrm{e}^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} \tag{920}$$

$$=\sum_w\mathrm{e}^{-\mathrm{i}w\tau}A_i\left(w\right). \tag{921}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w\in\{0,\pm\eta\}$.

In order to use the equation (921) to descompose the equation (373) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U\left(t\right)\equiv\mathcal{T}\exp\left(-\mathrm{i}\int_0^t\mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right)$ like:

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_{\bar{S}}}(t')\right) \tag{922}$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1) H(t_2) ... H(t_n). \tag{923}$$

Here $0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_{\text{eff}}(t)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right) \equiv \exp(-\mathrm{i}tH_{\text{eff}}(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_{\text{eff}}(t) = H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_{\text{eff}}^{(0)}(t) + H_{\text{eff}}^{(1)}(t) + H_{\text{eff}}^{(2)}(t) + ...\right)\right). \tag{924}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \tag{925}$$

$$H_E^{(1)}(t) = -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \tag{926}$$

$$H_E^{(2)}(t) = \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). \tag{927}$$

We can summarize that:

$$\widetilde{O}(t) \equiv U^\dagger(t)\, O(t)\, U(t), \tag{928}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_T}(t')\right) \tag{929}$$

$$= \exp(-\mathrm{i}tH_{T,\text{eff}}(t)), \text{ where} \tag{930}$$

$$H_{X,\text{eff}}(t) \equiv \frac{1}{t}\int_0^t \overline{H_X}(t')\mathrm{d}t' - \frac{\mathrm{i}}{2t}\int_0^t \int_0^{t'} \left[\overline{H_X}(t'), \overline{H_X}(t'')\right]\mathrm{d}t'\mathrm{d}t'' + \frac{1}{6t}\int_0^t \int_0^{t'} \int_0^{t''} \left(\left[\left[\overline{H_X}(t'), \overline{H_X}(t'')\right], \overline{H_X}(t''')\right]\right. \tag{931}$$

$$\left. + \left[\left[\overline{H_X}(t'''), \overline{H_X}(t'')\right], \overline{H_X}(t')\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t''' + \cdots \tag{932}$$

In order to show the explicit form of the matrices present in the RHS of the equation (921) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $W(t) = \left\{\left|H_{\bar{S},\text{eff},1}(t)\right\rangle, \left|H_{\bar{S},\text{eff},0}(t)\right\rangle\right\}$, formed by the time-dependent eigenvectors of $H_{\bar{S},\text{eff}}(t)$ in the following way:

$$A_i = \sum_{j,j'} \left\langle H_{\bar{S},\text{eff},j}(t-\tau) \left| A_i(t) \right| H_{\bar{S},\text{eff},j'}(t-\tau)\right\rangle \left|H_{\bar{S},\text{eff},j}(t-\tau)\right\rangle\!\left\langle H_{\bar{S},\text{eff},j'}(t-\tau)\right|. \tag{933}$$

Let's obtain $U^\dagger(t')\, A_i U(t')$ in explicit form:

$$U^\dagger(t')A_iU(t') = U^\dagger(t')\left(\sum_{j,j'}\left\langle H_{\bar{S},\text{eff},j}(t-\tau)\left|A_i\right| H_{\bar{S},\text{eff},j'}(t-\tau)\right\rangle \left|H_{\bar{S},\text{eff},j}(t-\tau)\right\rangle\!\left\langle H_{\bar{S},\text{eff},j'}(t-\tau)\right|\right) U(t') \tag{934}$$

$$= \sum_{j,j'} \left\langle H_{\bar{S},\text{eff},j}(t-\tau)\left|A_i\right| H_{\bar{S},\text{eff},j'}(t-\tau)\right\rangle U^\dagger(t') \left|H_{\bar{S},\text{eff},j}(t-\tau)\right\rangle\!\left\langle H_{\bar{S},\text{eff},j'}(t-\tau)\right| U(t') \tag{935}$$

$$
\begin{aligned}
&= \sum_{j,j'} \left\langle H_{\bar{S},\mathrm{eff},j}(t-\tau) \left| A_i \right| H_{\bar{S},\mathrm{eff},j'}(t-\tau) \right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\lambda_j(t-\tau)} \left| H_{\bar{S},\mathrm{eff},j}(t-\tau) \middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau) \right| \mathrm{e}^{-\mathrm{i}(t-\tau)\lambda_{j'}(t-\tau)} && (936)\\
&= \sum_{j,j'} \left\langle H_{\bar{S},\mathrm{eff},j}(t-\tau) \left| A_i \right| H_{\bar{S},\mathrm{eff},j'}(t-\tau) \right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\left(\lambda_j(t-\tau)-\lambda_{j'}(t-\tau)\right)} \left| H_{\bar{S},\mathrm{eff},j}(t-\tau) \middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau) \right|, && (937)
\end{aligned}
$$

$$
M_{jj'}(t-\tau) = \left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right) \left| A_i \right| H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right) \right\rangle \left| H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right) \middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right) \right|, \quad (938)
$$

$$
U^{\dagger}(t')A_iU(t') = M_{00}\left(t-\tau\right) + M_{01}\left(t-\tau\right)\mathrm{e}^{\mathrm{i}(t-\tau)(\lambda_0(t-\tau)-\lambda_1(t-\tau))} + M_{10}\left(t-\tau\right)\mathrm{e}^{\mathrm{i}(t-\tau)(\lambda_1(t-\tau)-\lambda_0(t-\tau))} + M_{11}\left(t-\tau\right), \quad (939)
$$

$$
w(t-\tau) = \lambda_1\left(t-\tau\right) - \lambda_0\left(t-\tau\right), \quad (940)
$$

$$
\begin{aligned}
U^{\dagger}(t')A_iU(t') &= M_{00}\left(t-\tau\right) + M_{01}\left(t-\tau\right)\mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} + M_{10}\mathrm{e}^{\mathrm{i}(t-\tau)w(t-\tau)} + M_{11} && (941)\\
&= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i\left(w\left(t-\tau\right)\right) && (942)\\
&= A_i\left(0\right) + A_i\left(w\left(t-\tau\right)\right)\mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} + A_i\left(-w\left(t-\tau\right)\right)\mathrm{e}^{\mathrm{i}(t-\tau)w(t-\tau)}. && (943)
\end{aligned}
$$

By direct comparison we obtain that:

$$
\begin{aligned}
A_i\left(w\left(t-\tau\right)\right) &= \left\langle H_{\bar{S},\mathrm{eff},0}\left(t-\tau\right) \left| A_i \right| H_{\bar{S},\mathrm{eff},1}\left(t-\tau\right) \right\rangle \left| H_{\bar{S},\mathrm{eff},0}\left(t-\tau\right) \middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},1}\left(t-\tau\right) \right|, && (944)\\
A_i\left(-w\left(t-\tau\right)\right) &= \left\langle H_{\bar{S},\mathrm{eff},1}\left(t-\tau\right) \left| A_i \right| H_{\bar{S},\mathrm{eff},0}\left(t-\tau\right) \right\rangle \left| H_{\bar{S},\mathrm{eff},1}\left(t-\tau\right) \middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},0}\left(t-\tau\right) \right|, && (945)\\
A_i\left(0\right) &= \sum_{j} \left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right) \left| A_i \right| H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right) \right\rangle \left| H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right) \middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right) \right|. && (946)
\end{aligned}
$$

These matrix have the following property $A_i\left(w\left(t-\tau\right)\right) = A_i^{\dagger}\left(-w\left(t-\tau\right)\right)$. Now in order to perform the double Fourier decomposition let's recall:

$$
\widetilde{A_i}\left(t,t'\right) \equiv U\left(t\right)U^{\dagger}\left(t'\right)A_iU\left(t'\right)U^{\dagger}\left(t\right). \quad (947)
$$

In this case the decomposition can be written as:

$$
\begin{aligned}
\widetilde{A_i}\left(t,t-\tau\right) &\equiv U\left(t\right)U^{\dagger}\left(t-\tau\right)A_iU\left(t-\tau\right)U^{\dagger}\left(t\right) && (948)\\
&= U\left(t\right)\left(\sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i\left(w\left(t-\tau\right)\right)\right)U^{\dagger}\left(t\right). && (949)
\end{aligned}
$$

Now writting $A_i\left(w\left(t-\tau\right)\right)$ in terms of the eigenstates of $H_{\bar{S},\mathrm{eff}}\left(t\right)$ we find:

$$
A_i\left(w\left(t-\tau\right)\right) = \sum_{j,j'} \left\langle H_{\bar{S},\mathrm{eff},j}\left(t\right) \left| A_i\left(w\left(t-\tau\right)\right) \right| H_{\bar{S},\mathrm{eff},j'}\left(t\right) \right\rangle \left| H_{\bar{S},\mathrm{eff},j}\left(t\right) \middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},j'}\left(t\right) \right|. \quad (950)
$$

Then the time evolution is given by:

$$\widetilde{A_i}(t,t-\tau) = U(t)\left(\sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} A_i(w(t-\tau))\right) U^\dagger(t) \tag{951}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} U(t) A_i(w(t-\tau)) U^\dagger(t) \tag{952}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} U(t)\left(\sum_{j,j'} \left\langle H_{\bar{S},\mathrm{eff},j}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},j'}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},j}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j'}(t)\right|\right) U^\dagger(t) \tag{953}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{j,j'} \left\langle H_{\bar{S},\mathrm{eff},j}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},j'}(t)\right\rangle U(t) \left|H_{\bar{S},\mathrm{eff},j}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j'}(t)\right| U^\dagger(t) \tag{954}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{j,j'} \left\langle H_{\bar{S},\mathrm{eff},j}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},j'}(t)\right\rangle \mathrm{e}^{-\mathrm{i}t\lambda_j(t)} \left|H_{\bar{S},\mathrm{eff},j}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j'}(t)\right| \mathrm{e}^{\mathrm{i}t\lambda_{j'}(t)}, \tag{955}$$

$$w'(t) = \lambda_1(t) - \lambda_0(t), \tag{956}$$

$$\widetilde{A_i}(t,t-\tau) = \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{957}$$

$$+ \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right| \tag{958}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{959}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|\Big) \tag{960}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \sum_{w'(t)} \mathrm{e}^{\mathrm{i}tw'(t)} A_{iww'}(t-\tau,t) \tag{961}$$

$$= \sum_{w(t-\tau),w'(t)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \mathrm{e}^{\mathrm{i}tw'(t)} A_{iww'}(t-\tau,t) \tag{962}$$

$$= \sum_{w(t-\tau),w'(t)} \mathrm{e}^{\mathrm{i}\tau w(t-\tau)} \mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)} A_{iww'}(t-\tau,t). \tag{963}$$

$$= \sum_{w(t-\tau)} \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{964}$$

$$+ \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right| \tag{965}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{966}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|\Big) \tag{967}$$

$$= \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(0) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| + \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(0) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right| \tag{968}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{969}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right| \tag{970}$$

$$+ \mathrm{e}^{-\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{971}$$

$$+ \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right| \tag{972}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{973}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|\Big) \tag{974}$$

$$+ e^{\mathrm{i}(t-\tau)w(t-\tau)} \Big(\left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(-w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{975}$$

$$+ \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(-w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right| \tag{976}$$

$$+ \mathrm{e}^{-\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},1}(t)\right| A_i(-w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},0}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right| \tag{977}$$

$$+ \mathrm{e}^{\mathrm{i}tw'(t)} \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(-w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|\Big). \tag{978}$$

Directly we can find that the decomposition matrices are:

$$A_{i0w'}(t-\tau,t) = \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(0) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|, \tag{979}$$

$$A_{iww'}(t-\tau,t) = \left\langle H_{\bar{S},\mathrm{eff},0}(t)\right| A_i(w(t-\tau)) \left| H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|, \tag{980}$$

$$A_{iw(-w')}\left(t-\tau,t\right)=\left\langle H_{\bar{S},\mathrm{eff},1}\left(t\right)\left|A_{i}\left(w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},0}\left(t-\tau\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},1}\left(t-\tau\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}\left(t-\tau\right)\right|,\tag{981}$$

$$A_{iw0}\left(t-\tau,t\right)=\sum_{j}\left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\left|A_{i}\left(w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right|,\tag{982}$$

$$A_{i00}\left(t-\tau,t\right)=\sum_{j}\left\langle H_{\bar{S},\mathrm{eff},j}\left(t\right)\left|A_{i}\left(0\right)\right|H_{\bar{S},\mathrm{eff},j}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t\right)\right|,\tag{983}$$

$$A_{i0(-w')}\left(t-\tau,t\right)=\left\langle H_{\bar{S},\mathrm{eff},1}\left(t\right)\left|A_{i}\left(0\right)\right|H_{\bar{S},\mathrm{eff},0}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},1}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}\left(t\right)\right|,\tag{984}$$

$$A_{i(-w)0}\left(t-\tau,t\right)=\sum_{j}\left\langle H_{\bar{S},\mathrm{eff},j}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},j}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t\right)\right|,\tag{985}$$

$$A_{i(-w)w'}\left(t-\tau,t\right)=\left\langle H_{\bar{S},\mathrm{eff},0}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},1}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},0}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}\left(t\right)\right|,\tag{986}$$

$$A_{i(-w)(-w')}\left(t-\tau,t\right)=\left\langle H_{\bar{S},\mathrm{eff},1}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},0}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},1}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}\left(t-\tau\right)\right|.\tag{987}$$

Let's prove that $A_{jww'}\left(t-\tau,t\right)=A^{\dagger}_{j(-w)(-w')}\left(t-\tau,t\right)$:

$$\left(\left\langle H_{\bar{S},\mathrm{eff},j}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},j'}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\right|\right)^{\dagger}\tag{988}$$

$$=\left\langle H_{\bar{S},\mathrm{eff},j}\left(t\right)\left|A_{i}\left(-w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},j'}\left(t\right)\right\rangle^{*}\left|H_{\bar{S},\mathrm{eff},j'}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right|\tag{989}$$

$$=\left\langle H_{\bar{S},\mathrm{eff},j'}\left(t\right)\left|A^{\dagger}_{i}\left(-w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},j}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j'}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right|\tag{990}$$

$$=\left\langle H_{\bar{S},\mathrm{eff},j'}\left(t\right)\left|A_{i}\left(w\left(t-\tau\right)\right)\right|H_{\bar{S},\mathrm{eff},j}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j'}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right|.\tag{991}$$

It can be seen that the index $-w$ and $-w'$ change to the functions w and w'.

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}\left(\tau\right)=\widetilde{A_i}^{\dagger}\left(\tau\right)$ and $\widetilde{B_i}\left(\tau\right)=\widetilde{B_i}^{\dagger}\left(\tau\right)$ we can use the equation (921) to write the master equation in the following neater form:

$$U\left(t\right)\frac{\partial\widetilde{\rho_S}\left(t\right)}{\partial t}U^{\dagger}\left(t\right)=-\sum_{ij}\int_{0}^{t}\mathrm{d}\tau C_i(t)\,C_j(t-\tau)\left(\mathcal{B}_{ij}(\tau)\left[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\right]+\mathcal{B}_{ji}(-\tau)\left[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t)\,,A_i\right]\right)\tag{992}$$

$$=-\sum_{ijww'}\int_{0}^{t}\mathrm{d}\tau C_i(t)\,C_j(t-\tau)\Big(\mathcal{B}_{ij}(t,t')\left[A_i,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right]\tag{993}$$

$$-\mathcal{B}^{*}_{ij}(t,t')\left[A_i,\overline{\rho_S}\left(t\right)\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)\right]\Big).\tag{994}$$

Given that $A_{jww'}\left(t-\tau,t\right)=A^{\dagger}_{j(-w)(-w')}\left(t-\tau,t\right)$ and $w\left(t-\tau\right),w'\left(t\right)$ belong to the set of differences of eigenvalues of $H_{\bar{S},\mathrm{eff}}\left(t-\tau\right)$ and $H_{\bar{S},\mathrm{eff}}\left(t\right)$ denoted by J_t and $J_{t-\tau}$ respectively that depends of the time we can take an application where $w\left(t-\tau\right)\to-w\left(t-\tau\right)$ and $w'\left(t\right)\to-w'\left(t\right)$ such that the sum:

$$\sum_{ww'}\int_{0}^{t}\mathrm{d}\tau\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right)=\sum_{ww'}\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j(-w)(-w')}\left(t-\tau,t\right)\tag{995}$$

$$=\sum_{ww'}\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right).\tag{996}$$

is invariant because if $\left(w\left(t-\tau\right),w'\left(t\right)\right)\in J_{t-\tau}\times J_t$ then $\left(-w\left(t-\tau\right),-w'\left(t\right)\right)\in J_{t-\tau}\times J_t$ where J_t denotes the set of differences of eigenvalues at time t. So the master equation can be written as:

$$U\left(t\right)\frac{\partial\widetilde{\rho_S}\left(t\right)}{\partial t}U^{\dagger}\left(t\right)=-\sum_{ijww'}\int_{0}^{t}\mathrm{d}\tau C_i(t)\,C_j(t-\tau)\Big(\mathcal{B}_{ij}\left(t,t'\right)\left[A_i,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\,\overline{\rho_S}(t)\right]\tag{997}$$

$$+\mathcal{B}^{*}_{ij}\left(t,t'\right)\left[\overline{\rho_S}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}\left(t-\tau,t\right),A_i\right]\Big)\tag{998}$$

With the definition:

$$L_{ijww'}(t,t') \equiv \int_0^t C_i(t)\, C_j(t')\, \mathcal{B}_{ij}(t,t')\, \mathrm{e}^{\mathrm{i}\tau w(t')} \mathrm{e}^{-\mathrm{i}t\left(w(t')-w'(t)\right)} A_{jww'}(t-\tau,t)\, \mathrm{d}\tau. \tag{999}$$

We can show that:

$$\begin{aligned}
L^{\dagger}_{ijww'}(t,t') &= \int_0^t \left(C_i(t)\, C_j(t')\, \mathcal{B}_{ij}(t,t')\, \mathrm{e}^{\mathrm{i}\tau w(t')} \mathrm{e}^{-\mathrm{i}t\left(w(t')-w'(t)\right)} A_{jww'}(t',t)\, \mathrm{d}\tau\right)^{\dagger} && (1000)\\
&= \int_0^t C_i^*(t)\, C_j^*(t')\, \mathcal{B}_{ij}^*(t,t')\, \mathrm{e}^{-\mathrm{i}\tau w^*(t')} \mathrm{e}^{\mathrm{i}t\left(w(t')-w'(t)\right)^*} A^{\dagger}_{jww'}(t',t)\, \mathrm{d}\tau && (1001)\\
&= \int_0^t C_i(t)\, C_j(t')\, \mathcal{B}_{ij}^*(t,t')\, \mathrm{e}^{-\mathrm{i}\tau w(t')} \mathrm{e}^{\mathrm{i}t\left(w(t')-w'(t)\right)} A^{\dagger}_{jww'}(t',t)\, \mathrm{d}\tau \;(t'=t-\tau). && (1002)
\end{aligned}$$

So we can write the master equation as:

$$\begin{aligned}
U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}(t) =& -\sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathcal{B}_{ij}(t,t-\tau)\Big[A_i, \mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\overline{\rho_S}(t)\Big] && (1003)\\
&-\mathcal{B}_{ij}^*(t,t-\tau)\Big[A_i, \overline{\rho_S}(t)\, \mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}(t-\tau,t)\Big]\Big) && (1004)\\
=& -\sum_{ijww'}\Big(\left[A_i, L_{ijww'}(t)\,\overline{\rho_{\bar{S}}}(t)\right] + \Big[\overline{\rho_{\bar{S}}}(t)\, L^{\dagger}_{ijww'}(t), A_i\Big]\Big). && (1005)
\end{aligned}$$

If we extend the upper limit of integration to ∞ in the equation (1002) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We require to get a general form of the term $U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}(t)$ present in the LHS of (1005) so performing the algebra we will obtain:

$$\begin{aligned}
U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}(t) =& U(t)\left(\frac{\partial\left(U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\right)}{\partial t}\right)U^{\dagger}(t) && (1006)\\
=& U(t)\left(\frac{\partial U^{\dagger}(t)}{\partial t}\overline{\rho_S}(t)U(t)\right)U^{\dagger}(t) + U(t)\left(U^{\dagger}(t)\,\frac{\partial\overline{\rho_S}(t)}{\partial t}U(t)\right)U^{\dagger}(t) + U(t)\left(U^{\dagger}(t)\overline{\rho_S}(t)\,\frac{\partial U(t)}{\partial t}\right)U^{\dagger}(t) && (1007)\\
=& U(t)\,\frac{\partial U^{\dagger}(t)}{\partial t}\overline{\rho_S}(t)\left(U(t)\,U^{\dagger}(t)\right) + \left(U(t)\,U^{\dagger}(t)\right)\frac{\partial\overline{\rho_S}(t)}{\partial t}\left(U(t)\,U^{\dagger}(t)\right) + \left(U(t)U^{\dagger}(t)\right)\overline{\rho_S}(t)\,\frac{\partial U(t)}{\partial t}U^{\dagger}(t) && (1008)\\
=& U(t)\,\frac{\partial U^{\dagger}(t)}{\partial t}\overline{\rho_S}(t)\,\mathbb{I} + \mathbb{I}\frac{\partial\overline{\rho_S}(t)}{\partial t}\mathbb{I} + \mathbb{I}\overline{\rho_S}(t)\,\frac{\partial U(t)}{\partial t}U^{\dagger}(t) && (1009)\\
=& \frac{\partial\overline{\rho_S}(t)}{\partial t} + U(t)\,\frac{\partial U^{\dagger}(t)}{\partial t}\overline{\rho_S}(t) + \overline{\rho_S}(t)\,\frac{\partial U(t)}{\partial t}U^{\dagger}(t) && (1010)\\
=& \frac{\partial\overline{\rho_S}(t)}{\partial t} + U(t)\,\frac{\partial U^{\dagger}(t)}{\partial t}\overline{\rho_S}(t) + \overline{\rho_S}(t)\left(U(t)\,\frac{\partial U^{\dagger}(t)}{\partial t}\right)^{\dagger}. && (1011)
\end{aligned}$$

In order to continue the reduction of (1010) we introduce the derivative of the exponential map as:

$$\mathrm{ad}_X(Y) \equiv [X,Y], \tag{1012}$$
$$(\mathrm{ad}_X)^0(Y) \equiv Y, \tag{1013}$$
$$(\mathrm{ad}_X)^{k+1}(Y) \equiv \mathrm{ad}_X\left((\mathrm{ad}_X)^k(Y)\right), \tag{1014}$$
$$\frac{1-\mathrm{e}^{-\mathrm{ad}_X}}{\mathrm{ad}_X}(Y) \equiv Y - \frac{[X,Y]}{2!} + \frac{[X,[X,Y]]}{3!} - \frac{[X,[X,[X,Y]]]}{4!} + \cdots \tag{1015}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{ad}_X)^k(Y), \tag{1016}$$
$$\frac{\partial \mathrm{e}^{X(t)}}{\partial t} = \mathrm{e}^{X(t)}\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{X(t)}}}{\mathrm{ad}_{X(t)}}\left(\frac{\partial X(t)}{\partial t}\right)\right). \tag{1017}$$

Using the expansion $U(t) = \mathrm{e}^{-\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}$ and $U^\dagger(t) = \mathrm{e}^{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}$ then we can obtain:

$$U(t)\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^\dagger(t) = \frac{\partial\overline{\rho_S}(t)}{\partial t} + \left(U(t)\frac{\partial U^\dagger(t)}{\partial t}\right)\overline{\rho_S}(t) + \overline{\rho_S}(t)\left(U(t)\frac{\partial U^\dagger(t)}{\partial t}\right)^\dagger \tag{1018}$$
$$= \frac{\partial\overline{\rho_S}(t)}{\partial t} + \left(U(t)U^\dagger(t)\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right)\right)\right)\overline{\rho_S}(t) + \overline{\rho_S}(t)\left(U(t)U^\dagger(t)\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\right.\right. \tag{1019}$$
$$\left.\left.\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right)\right)\right)^\dagger \tag{1020}$$
$$= \frac{\partial\overline{\rho_S}(t)}{\partial t} + \mathbb{I}\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right)\right)\overline{\rho_S}(t) + \overline{\rho_S}(t)\left(\mathbb{I}\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right)\right)\right)^\dagger \tag{1021}$$
$$= \frac{\partial\overline{\rho_S}(t)}{\partial t} + \left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right)\right)\overline{\rho_S}(t) + \overline{\rho_S}(t)\left(\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right)\right)\right)^\dagger. \tag{1022}$$

The form $\mathrm{ad}_X Y = [X,Y]$ is a bilinear form that satisfies:

$$\mathrm{ad}_{aX} bY = [aX, bY] \tag{1023}$$
$$= aXbY - bYaX \tag{1024}$$
$$= ab[X,Y]. \tag{1025}$$

Let's prove by induction the following property of ad. Let be X, Y a pair of operators and a, b scalars numbers then:

$$(\mathrm{ad}_{aX})^k bY = a^k b(\mathrm{ad}_X)^k Y \text{ (for } k \in \mathbb{N}). \tag{1026}$$

Proof: for $k = 0$ we obtain using (1013) that $(\mathrm{ad}_{aX})^0 bY = bY$ and this is the same result obtained for (1026) taking $(\mathrm{ad}_X)^0 \equiv \mathbb{I}$. This is our case base. By induction hyphothesis consider that the proposition is true for $k \in \mathbb{N}$. The induction step follows from:

$$(\mathrm{ad}_{aX})^{k+1} bY = (\mathrm{ad}_{aX})\left((\mathrm{ad}_{aX})^k bY\right) \text{ (by (1014))} \tag{1027}$$
$$= (\mathrm{ad}_{aX})\left(a^k b(\mathrm{ad}_X)^k Y\right) \text{ (by induction hypothesis)} \tag{1028}$$
$$= \left[aX, a^k b(\mathrm{ad}_X)^k Y\right] \text{ (by definition of ad)} \tag{1029}$$
$$= a^{k+1}b\left[X, (\mathrm{ad}_X)^k Y\right] \text{ (by commutator properties)} \tag{1030}$$
$$= a^{k+1}b\,\mathrm{ad}_X\left((\mathrm{ad}_X)^k Y\right) \text{ (by ad operator properties)} \tag{1031}$$
$$= a^{k+1}b(\mathrm{ad}_X)^{k+1} Y \text{ (by definition of power of ad).} \tag{1032}$$

By the principle of mathematical induction we conclude that the proposition given is true for all $k \in \mathbb{N}$.

We can reduce the term $\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right)$ using the previous proposition.

$$
\begin{aligned}
\frac{1-\mathrm{e}^{-\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}}{\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}}\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{ad}_{\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)}\right)^k\left(\frac{\partial\left(\mathrm{i}tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right) \text{ (by (1015))} && (1033)\\
&= \mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\left(\frac{\partial\left(tH_{\overline{S},\mathrm{eff}}(t)\right)}{\partial t}\right) \text{ (by (1026))} && (1034)\\
&= \mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\left(H_{\overline{S},\mathrm{eff}}(t)+t\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right) \text{ (by derivative properties)} && (1035)\\
&= \mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k H_{\overline{S},\mathrm{eff}}(t)+\mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k t\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t} && (1036)\\
&\text{(by bilinear properties of ad)} && (1037)\\
&= \mathrm{i}H_{\overline{S},\mathrm{eff}}(t)+\mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k t\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t} \text{(by } (\mathrm{ad}_X)^0 X = X \text{ and for } k\in\mathbb{N}^* && (1038)\\
&(\mathrm{ad}_X)^k X = 0). && (1039)
\end{aligned}
$$

Then we will obtain:

$$
\begin{aligned}
U(t)\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^\dagger(t) &= \frac{\partial\overline{\rho_S}(t)}{\partial t}+\mathrm{i}\left(H_{\overline{S},\mathrm{eff}}(t)+t\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)\overline{\rho_S}(t)-\mathrm{i}\overline{\rho_S}(t)\left(H_{\overline{S},\mathrm{eff}}(t)+t\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\right. && (1040)\\
&\left.\times(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)^\dagger && (1041)\\
&= \frac{\partial\overline{\rho_S}(t)}{\partial t}+\mathrm{i}H_{\overline{S},\mathrm{eff}}(t)\,\overline{\rho_S}(t)+\mathrm{i}t\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)\overline{\rho_S}(t)-\mathrm{i}\overline{\rho_S}(t)\,H^\dagger_{\overline{S},\mathrm{eff}}(t) && (1042)\\
&-\mathrm{i}t\overline{\rho_S}(t)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)^\dagger && (1043)\\
&= \frac{\partial\overline{\rho_S}(t)}{\partial t}+\mathrm{i}H_{\overline{S},\mathrm{eff}}(t)\,\overline{\rho_S}(t)-\mathrm{i}\overline{\rho_S}(t)\,H_{\overline{S},\mathrm{eff}}(t)+\mathrm{i}t\left(\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)\overline{\rho_S}(t)-\overline{\rho_S}(t)\right. && (1044)\\
&\left.\times\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)^\dagger\right) && (1045)\\
&= \frac{\partial\overline{\rho_S}(t)}{\partial t}+\mathrm{i}\left[H_{\overline{S},\mathrm{eff}}(t),\overline{\rho_S}(t)\right]+\mathrm{i}t\left(\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)\overline{\rho_S}(t)-\overline{\rho_S}(t)\right. && (1046)\\
&\left.\times\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)^\dagger\right). && (1047)
\end{aligned}
$$

We will proof another property useful related to ad. Let X, Y two hermitic operators and $k \in \mathbb{N}$ then:

$$\left((\mathrm{ad}_X)^k Y\right)^\dagger = (-1)^k\left((\mathrm{ad}_X)^k Y\right). \tag{1048}$$

Proof: given that X, Y are hermitic then $X = X^\dagger$ and $Y = Y^\dagger$, for $k = 1$:

$$
\begin{aligned}
\left(\mathrm{ad}_X Y\right)^{\dagger} &= \left([X,Y]\right)^{\dagger} && (1049)\\
&= \left(XY - YX\right)^{\dagger} && (1050)\\
&= Y^{\dagger}X^{\dagger} - X^{\dagger}Y^{\dagger} && (1051)\\
&= \left[Y^{\dagger}, X^{\dagger}\right] && (1052)\\
&= -\left[X^{\dagger}, Y^{\dagger}\right] && (1053)\\
&= -\left[X, Y\right] \text{ (by hermiticity of } X, Y) && (1054)\\
&= -\left(\mathrm{ad}_X Y\right) && (1055)\\
&= \left(-1\right)^{1}\left(\mathrm{ad}_X Y\right). && (1056)
\end{aligned}
$$

This is our case base. Suppouse that the proposition is true for k then the induction step for $k+1$ is:

$$
\begin{aligned}
\left(\left(\mathrm{ad}_X\right)^{k+1} Y\right)^{\dagger} &= \left(\mathrm{ad}_X\left(\left(\mathrm{ad}_X\right)^{k} Y\right)\right)^{\dagger} && (1057)\\
&= \left(\left[X, \left(\mathrm{ad}_X\right)^{k} Y\right]\right)^{\dagger} && (1058)\\
&= \left[\left(\left(\mathrm{ad}_X\right)^{k} Y\right)^{\dagger}, X^{\dagger}\right] && (1059)\\
&= \left[\left(-1\right)^{k}\left(\left(\mathrm{ad}_X\right)^{k} Y\right), X\right] \text{ (by inductive step)} && (1060)\\
&= -\left(-1\right)^{k}\left[X, \left(\mathrm{ad}_X\right)^{k} Y\right] \text{ (rearranging the commutator)} && (1061)\\
&= \left(-1\right)^{k+1}\left(\mathrm{ad}_X\right)^{k+1} Y \text{ (using definition of ad to a power).} && (1062)
\end{aligned}
$$

By the principle of mathematical induction we can deduce that the proposition is true for all $k \in \mathbb{N}$.
Recalling that $H_{\overline{S},\mathrm{eff}}\left(t\right) = H^{\dagger}_{\overline{S},\mathrm{eff}}\left(t\right)$ and $\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t} = \left(\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)^{\dagger}$ then we can rewrite further:

$$
\begin{aligned}
\left(\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(\mathrm{i}t\right)^{k}\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}\frac{\partial H_{\overline{S},\mathrm{eff}}\left(t\right)}{\partial t}\right)^{\dagger} &= \sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(-\mathrm{i}t\right)^{k}\left(\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}\frac{\partial H_{\overline{S},\mathrm{eff}}\left(t\right)}{\partial t}\right)^{\dagger} && (1063)\\
&= \sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(-\mathrm{i}t\right)^{k}\left(-1\right)^{k}\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}\frac{\partial H_{\overline{S},\mathrm{eff}}\left(t\right)}{\partial t} && (1064)\\
&= \sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(\mathrm{i}t\right)^{k}\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}\frac{\partial H_{\overline{S},\mathrm{eff}}\left(t\right)}{\partial t}. && (1065)
\end{aligned}
$$

Introducing in $U\left(t\right)\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}\left(t\right)$ we get:

$$
\begin{aligned}
U\left(t\right)\frac{\partial\widetilde{\overline{\rho_S}}\left(t\right)}{\partial t}U^{\dagger}\left(t\right) = \frac{\partial\overline{\rho_S}\left(t\right)}{\partial t} + \mathrm{i}\left[H_{\overline{S},\mathrm{eff}}\left(t\right), \overline{\rho_S}\left(t\right)\right] + \mathrm{i}t\left(\left(\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(\mathrm{i}t\right)^{k}\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}\frac{\partial H_{\overline{S},\mathrm{eff}}\left(t\right)}{\partial t}\right)\overline{\rho_S}\left(t\right) - \overline{\rho_S}\left(t\right)\right. && (1066)\\
\left.\times\left(\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(\mathrm{i}t\right)^{k}\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right)\right) && (1067)\\
= \frac{\partial\overline{\rho_S}(t)}{\partial t} + \mathrm{i}\left[H_{\overline{S},\mathrm{eff}}(t), \overline{\rho_S}(t)\right] + \mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(\mathrm{i}t\right)^{k}\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right), \overline{\rho_S}(t)\right]. && (1068)
\end{aligned}
$$

Applying the inverse transformation we will obtain that:

$$\begin{aligned}
\mathrm{e}^{-V}\overline{AB}\mathrm{e}^{V} &= \mathrm{e}^{-V}\overline{A\mathbb{I}B}\mathrm{e}^{V} && (1069)\\
&= \mathrm{e}^{-V}\overline{A}\mathrm{e}^{V}\mathrm{e}^{-V}\bar{B}\mathrm{e}^{V} && (1070)\\
&= \left(\mathrm{e}^{-V}\overline{A}\mathrm{e}^{V}\right)\left(\mathrm{e}^{-V}\bar{B}\mathrm{e}^{V}\right) && (1071)\\
&= AB. && (1072)
\end{aligned}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$\begin{aligned}
\mathrm{e}^{-V}\overline{[A,B]}\mathrm{e}^{V} &= \mathrm{e}^{-V}\overline{(AB-BA)}\mathrm{e}^{V} && (1073)\\
&= \mathrm{e}^{-V}\overline{AB}\mathrm{e}^{V} - \mathrm{e}^{-V}\overline{BA}\mathrm{e}^{V} && (1074)\\
&= AB - BA && (1075)\\
&= [A,B]. && (1076)
\end{aligned}$$

From the notation we can deduce that $\overline{\underline{A\,(t)}} = \mathrm{e}^{-V(t)}\left(\mathrm{e}^{V(t)}A\,(t)\,\mathrm{e}^{-V(t)}\right)$ is:

$$\begin{aligned}
\overline{\underline{A\,(t)}} &= \mathrm{e}^{-V(t)}\left(\mathrm{e}^{V(t)}A\,(t)\,\mathrm{e}^{-V(t)}\right)\mathrm{e}^{V(t)} && (1077)\\
&= \left(\mathrm{e}^{-V(t)}\mathrm{e}^{V(t)}\right)A\,(t)\left(\mathrm{e}^{-V(t)}\mathrm{e}^{V(t)}\right) && (1078)\\
&= A\,(t)\,. && (1079)
\end{aligned}$$

Let $\underline{A\,(t)} \equiv \mathrm{e}^{-V(t)}A\,(t)\,\mathrm{e}^{V(t)}$ then we will obtain that:

$$\begin{aligned}
\mathrm{e}^{-V(t)}\left(U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}(t)\right)\mathrm{e}^{V(t)} &= -\mathrm{e}^{-V(t)}\sum_{ijww'}\left(\left[A_i, L_{ijww'}\,(t)\,\overline{\rho_S}\,(t)\right] - \left[A_i, \overline{\rho_S}\,(t)\,L^{\dagger}_{ijww'}\,(t)\right]\right)\mathrm{e}^{V(t)} && (1080)\\
&= -\sum_{ijww'}\left(\mathrm{e}^{-V(t)}\left[A_i, L_{ijww'}\,(t)\,\overline{\rho_S}\,(t)\right]\mathrm{e}^{V(t)} - \mathrm{e}^{-V(t)}\left[A_i, \overline{\rho_S}\,(t)\,L^{\dagger}_{ijww'}\,(t)\right]\mathrm{e}^{V(t)}\right) && (1081)\\
&= \sum_{ijww'}\left(\left[\mathrm{e}^{-V(t)}A_i\mathrm{e}^{V(t)}, \mathrm{e}^{-V(t)}\overline{\rho_S}(t)\,L^{\dagger}_{ijww'}(t)\,\mathrm{e}^{V(t)}\right] - \left[\mathrm{e}^{-V(t)}A_i\mathrm{e}^{V(t)}, \mathrm{e}^{-V(t)}L_{ijww'}(t)\,\overline{\rho_S}(t)\,\mathrm{e}^{V(t)}\right]\right) && (1082)\\
&= -\sum_{ijww'}\left(\left[\underline{A_i\,(t)}, \underline{L_{ijww'}\,(t)}\rho_S\,(t)\right] - \left[\underline{A_i\,(t)}, \rho_S\,(t)\,\underline{L^{\dagger}_{ijww'}\,(t)}\right]\right). && (1083)
\end{aligned}$$

We will obtain a reduced form of the term $\mathrm{e}^{-V(t)}\left(U\,(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}\,(t)\right)\mathrm{e}^{V(t)}$, as we can see there is a time dependence related to $\overline{\rho_S}\,(t)$ and $V\,(t)$ that requires to be written in shorter terms, in our case we have:

$$\begin{aligned}
\mathrm{e}^{-V(t)}\left(U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}(t)\right)\mathrm{e}^{V(t)} &= \mathrm{e}^{-V(t)}\left(\frac{\partial\overline{\rho_S}(t)}{\partial t} + \mathrm{i}\left[H_{\overline{S},\mathrm{eff}}(t)\,,\overline{\rho_S}(t)\right] + \mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\,(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right), \overline{\rho_S}(t)\right]\right)\mathrm{e}^{V(t)} && (1084)\\
&= \mathrm{e}^{-V(t)}\,\frac{\partial\overline{\rho_S}(t)}{\partial t}\mathrm{e}^{V(t)} + \mathrm{i}\underline{\left[H_{\overline{S},\mathrm{eff}}(t)\,,\overline{\rho_S}(t)\right]} + \mathrm{i}t\underline{\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\,(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right), \overline{\rho_S}(t)\right]} && (1085)\\
&= \mathrm{e}^{-V(t)}\,\frac{\partial\overline{\rho_S}(t)}{\partial t}\mathrm{e}^{V(t)} + \mathrm{i}\left[\underline{H_{\overline{S},\mathrm{eff}}(t)}, \underline{\overline{\rho_S}(t)}\right] + \mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\,(\mathrm{i}t)^k\,\underline{\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}}\right), \underline{\overline{\rho_S}(t)}\right] && (1086)\\
&= \mathrm{e}^{-V(t)}\,\frac{\partial\overline{\rho_S}(t)}{\partial t}\mathrm{e}^{V(t)} + \mathrm{i}\left[\underline{H_{\overline{S},\mathrm{eff}}(t)}, \rho_S(t)\right] + \mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\,(\mathrm{i}t)^k\,\underline{\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}}\right), \rho_S(t)\right]. && (1087)
\end{aligned}$$

We will deal with the algebra of each term separately:

$$
\begin{aligned}
\mathrm{e}^{-V(t)}\frac{\partial\overline{\rho_S}(t)}{\partial t}\mathrm{e}^{V(t)} &= \mathrm{e}^{-V(t)}\left(\frac{\partial}{\partial t}\left(\mathrm{e}^{V(t)}\rho_S(t)\,\mathrm{e}^{-V(t)}\right)\right)\mathrm{e}^{V(t)} && (1088)\\
&= \mathrm{e}^{-V(t)}\left(\frac{\partial \mathrm{e}^{V(t)}}{\partial t}\rho_S(t)\,\mathrm{e}^{-V(t)} + \mathrm{e}^{V(t)}\frac{\partial\rho_S(t)}{\partial t}\mathrm{e}^{-V(t)} + \mathrm{e}^{V(t)}\rho_S(t)\,\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\right)\mathrm{e}^{V(t)} && (1089)\\
&= \mathrm{e}^{-V(t)}\frac{\partial \mathrm{e}^{V(t)}}{\partial t}\rho_S(t)\,\mathrm{e}^{-V(t)}\mathrm{e}^{V(t)} + \mathrm{e}^{-V(t)}\mathrm{e}^{V(t)}\frac{\partial\rho_S(t)}{\partial t}\mathrm{e}^{-V(t)}\mathrm{e}^{V(t)} + \mathrm{e}^{-V(t)}\mathrm{e}^{V(t)}\rho_S(t)\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\mathrm{e}^{V(t)} && (1090)\\
&= \mathrm{e}^{-V(t)}\frac{\partial \mathrm{e}^{V(t)}}{\partial t}\rho_S(t) + \frac{\partial\rho_S(t)}{\partial t} + \rho_S(t)\,\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\mathrm{e}^{V(t)}. && (1091)
\end{aligned}
$$

For the equation (1091) we need to recall that $V(t)$ is a hermitic operator so we can write the term $\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\mathrm{e}^{V(t)}$ as:

$$
\begin{aligned}
\left(\mathrm{e}^{V(t)}\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\right)^\dagger &= \left(\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\right)^\dagger\left(\mathrm{e}^{V(t)}\right)^\dagger && (1092)\\
&= \frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\mathrm{e}^{V(t)}. && (1093)
\end{aligned}
$$

We can see that we only require to deal with $\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\mathrm{e}^{V(t)}$, using the formula of the derivative of an exponential map (1017) we can obtain:

$$
\begin{aligned}
\left(\mathrm{e}^{V(t)}\frac{\partial \mathrm{e}^{-V(t)}}{\partial t}\right)^\dagger &= \left(\mathrm{e}^{V(t)}\mathrm{e}^{-V(t)}\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{-V(t)}}}{\mathrm{ad}_{-V(t)}}\left(-\frac{\partial V(t)}{\partial t}\right)\right)\right)^\dagger && (1094)\\
&= \left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{-V(t)}}}{\mathrm{ad}_{-V(t)}}\left(-\frac{\partial V(t)}{\partial t}\right)\right)^\dagger && (1095)\\
&= \left(-\frac{\partial V(t)}{\partial t} - \frac{\left[-V(t),-\frac{\partial V(t)}{\partial t}\right]}{2!} + \frac{\left[-V(t),\left[-V(t),-\frac{\partial V(t)}{\partial t}\right]\right]}{3!} - \cdots\right)^\dagger && (1096)\\
&= \left(-\frac{\partial V(t)}{\partial t} - \frac{\left[V(t),\frac{\partial V(t)}{\partial t}\right]}{2!} - \frac{\left[V(t),\left[V(t),\frac{\partial V(t)}{\partial t}\right]\right]}{3!} - \frac{\left[V(t),\left[V(t),\left[V(t),\frac{\partial V(t)}{\partial t}\right]\right]\right]}{4!} - \cdots\right)^\dagger && (1097)\\
&= -\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\left(\mathrm{ad}_{V(t)}\right)^k\left(\frac{\partial V(t)}{\partial t}\right)\right)^\dagger && (1098)\\
&= -\sum_{k=0}^{\infty}\frac{1}{(k+1)!}(-1)^k\left(\left(\mathrm{ad}_{V(t)}\right)^k\left(\frac{\partial V(t)}{\partial t}\right)\right) && (1099)\\
&= -\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right)\right), && (1100)\\
\mathrm{e}^{-V(t)}\frac{\partial \mathrm{e}^{V(t)}}{\partial t} &= \mathrm{e}^{-V(t)}\mathrm{e}^{V(t)}\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right)\right) && (1101)\\
&= \frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right), && (1102)\\
\mathrm{e}^{-V(t)}\frac{\partial\overline{\rho_S}(t)}{\partial t}\mathrm{e}^{V(t)} &= \left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right)\right)\rho_S(t) - \rho_S(t)\left(\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right)\right) + \frac{\partial\rho_S(t)}{\partial t} && (1103)\\
&= \left[\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right),\rho_S(t)\right] + \frac{\partial\rho_S(t)}{\partial t}. && (1104)
\end{aligned}
$$

The term $\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\underline{\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}}\right),\rho_S(t)\right]$ can be reduced as well using the following property $\underline{\left(\mathrm{ad}_X\right)^k Y} = \left(\mathrm{ad}_{\underline{X}}\right)^k\underline{Y}$. For $k=0$

$$\underline{(\mathrm{ad}_X)^0 Y} = \mathbb{I}\underline{Y} \tag{1105}$$
$$= \underline{Y}. \tag{1106}$$

For $k = 1$ we have:

$$\underline{(\mathrm{ad}_X)^1 Y} = \underline{\mathrm{ad}_X Y} \tag{1107}$$
$$= \underline{[X, Y]} \tag{1108}$$
$$= [\underline{X}, \underline{Y}] \tag{1109}$$
$$= \mathrm{ad}_{\underline{X}}\underline{Y}. \tag{1110}$$

The inductive step is given for k as $\underline{(\mathrm{ad}_X)^k Y} = \left(\mathrm{ad}_{\underline{X}}\right)^k \underline{Y}$, in the case $k+1$ we have:

$$\underline{(\mathrm{ad}_X)^{k+1} Y} = \underline{\mathrm{ad}_X \left((\mathrm{ad}_X)^k Y\right)} \tag{1111}$$
$$= \underline{\left[X, (\mathrm{ad}_X)^k Y\right]} \tag{1112}$$
$$= \left[\underline{X}, \underline{(\mathrm{ad}_X)^k Y}\right] \tag{1113}$$
$$= \left[\underline{X}, \left(\mathrm{ad}_{\underline{X}}\right)^k \underline{Y}\right] \tag{1114}$$
$$= \mathrm{ad}_{\underline{X}} \left(\left(\mathrm{ad}_{\underline{X}}\right)^k \underline{Y}\right) \tag{1115}$$
$$= \left(\mathrm{ad}_{\underline{X}}\right)^{k+1} \underline{Y}. \tag{1116}$$

So we can continue rewrite:

$$\underline{\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k \frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}} = \left(\mathrm{ad}_{\underline{H_{\overline{S},\mathrm{eff}}(t)}}\right)^k \underline{\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}}. \tag{1117}$$

The final transformation is:

$$\mathrm{e}^{-V(t)}\left(U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^\dagger(t)\right)\mathrm{e}^{V(t)} = \frac{\partial\rho_S(t)}{\partial t} + \left[\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right), \rho_S(t)\right] + \mathrm{i}t\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{\underline{H_{\overline{S},\mathrm{eff}}(t)}}\right)^k \underline{\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}}, \rho_S(t)\right] \tag{1118}$$
$$+\,\mathrm{i}\left[\underline{H_{\overline{S},\mathrm{eff}}(t)}, \rho_S(t)\right] \tag{1119}$$
$$= -\sum_{ijww'}\left(\left[\underline{A_i(t)}, \underline{L_{ijww'}(t)}\rho_S(t)\right] - \left[\underline{A_i(t)}, \rho_S(t)\,\underline{L^\dagger_{ijww'}(t)}\right]\right). \tag{1120}$$

Our master equation in the variationally optimized frame and the lab frame are respectively:

$$\frac{\partial\overline{\rho_S}(t)}{\partial t} = -\mathrm{i}\left[H_{\overline{S},\mathrm{eff}}(t), \overline{\rho_S}(t)\right] - \mathrm{i}t\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}, \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, L_{ijww'}(t)\,\overline{\rho_S}(t)\right]\right. \tag{1121}$$
$$\left.+\left[\overline{\rho_S}(t)\,L^\dagger_{ijww'}(t), A_i\right]\right), \tag{1122}$$
$$\frac{\partial\rho_S(t)}{\partial t} = -\mathrm{i}\left[\underline{H_{\overline{S},\mathrm{eff}}(t)}, \rho_S(t)\right] - \sum_{ijww'}\left(\left[\underline{A_i(t)}, \underline{L_{ijww'}(t)}\rho_S(t)\right] - \left[\underline{A_i(t)}, \rho_S(t)\,\underline{L^\dagger_{ijww'}(t)}\right]\right) - \left[\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right), \rho_S(t)\right] \tag{1123}$$
$$-\,\mathrm{i}t\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{\underline{H_{\overline{S},\mathrm{eff}}(t)}}\right)^k\underline{\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}}, \rho_S(t)\right]. \tag{1124}$$

IV. LIMIT CASES

In order to show the plausibility of the master equation (1121) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-dependent VPQME for 2LS with real-valued system Hamiltonian and real-valued uniform coupling

This hamiltonian has as particular feature that the coupling constants are real, so we know that $g_{\mathbf{k}} = g_{\mathbf{k}}^*$ then:

$$H_T(t) = H_S(t) + H_I + H_B, \tag{1125}$$
$$H_S(t) = \sum_i \varepsilon_i(t)|i\rangle\langle i| + \sum_{i\neq j} V_{ij}(t)|i\rangle\langle j|, \tag{1126}$$
$$H_I = \sum_i |i\rangle\langle i| \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right), \tag{1127}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{1128}$$

The transformed hamiltonian is:

$$\overline{H_{\bar{S}}}(t) \equiv \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right), \tag{1129}$$

We can summarize the principal results of the elements of the variational parameters and the transformed hamiltonians as:

$$\overline{H_{\bar{S}}}(t) \equiv \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| + \sigma_x B_{10}(t)V_{10}(t) - \sigma_y B_{10}(t)V_{10}(t), \tag{1130}$$
$$R_i(t) = \int_0^\infty \frac{J(\omega)}{\omega}\left(F_i^2(\omega,t) - 2F_i(\omega,t)\right)\mathrm{d}\omega, \tag{1131}$$
$$\chi_{ij}(t) = 0, \tag{1132}$$
$$B_{ij}(t) = \mathrm{e}^{-\frac{1}{2}\int_0^\infty \frac{J(\omega)\left(F_i(\omega,t)-F_j(\omega,t)\right)^2}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{1133}$$
$$F_i(\omega,t) = \frac{\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)\right)\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{F_{i'}(\omega,t)g(\omega)}{\omega}B_{10}^2(t)V_{10}^2(t)\coth(\beta\omega/2)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i(t)\right) - \frac{2B_{10}^2(t)V_{10}^2(t)\coth(\beta\omega/2)}{\omega}\right)}, \tag{1134}$$
$$\eta(t) \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}, \tag{1135}$$
$$\varepsilon(t) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right), \tag{1136}$$
$$J(\omega) \equiv \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2\delta(\omega - \omega_{\mathbf{k}}). \tag{1137}$$

The Fourier decomposition remains without change:

$$L_{ijww'}(t,t') \equiv \int_0^t C_i(t)C_j(t')\mathcal{B}_{ij}(t,t')\mathrm{e}^{\mathrm{i}\tau w(t')}\mathrm{e}^{-\mathrm{i}t\left(w(t')-w'(t)\right)}A_{jww'}(t,t')\mathrm{d}\tau, \tag{1138}$$
$$t' = t - \tau, \tag{1139}$$

$$A \equiv \begin{pmatrix}\sigma_x & \sigma_y & \frac{I+\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I-\sigma_z}{2}\end{pmatrix}, \tag{1140}$$

$$C(t) \equiv \begin{pmatrix}V_{10}(t) & V_{10}(t) & 1 & 0 & 0 & 1\end{pmatrix}, \tag{1141}$$

$$A_{j00}(t,t') = \sum_i \left\langle H_{\bar{S},\mathrm{eff},i}(t)\left|A_{j0}(t')\right|H_{\bar{S},\mathrm{eff},i}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},i}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},i}(t)\right|, \tag{1142}$$

$$A_{j0w'}(t,t') = \left\langle H_{\bar{S},\mathrm{eff},0}(t)\left|A_{j0}(t')\right|H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|, \tag{1143}$$

$$A_{jw0}(t,t') = \sum_i \left\langle H_{\bar{S},\mathrm{eff},i}(t)\left|A_{jw}(t')\right|H_{\bar{S},\mathrm{eff},i}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},i}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},i}(t)\right|, \tag{1144}$$

$$A_{jww'}(t,t') = \left\langle H_{\bar{S},\mathrm{eff},0}(t)\left|A_{jw}(t')\right|H_{\bar{S},\mathrm{eff},1}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t)\right|, \tag{1145}$$

$$A_{jw(-w')}(t,t') = \left\langle H_{\bar{S},\mathrm{eff},1}(t)\left|A_{jw}(t')\right|H_{\bar{S},\mathrm{eff},0}(t-\tau)\right\rangle \left|H_{\bar{S},\mathrm{eff},1}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},0}(t)\right|, \tag{1146}$$

$$A_{j(-w)(-w')}(t,t') = A^{\dagger}_{jww'}(t,t') \tag{1147}$$

$$A_{j0}(t') = \sum_i \left\langle H_{\bar{S},\mathrm{eff},i}(t')\left|A_{j}(t)\right|H_{\bar{S},\mathrm{eff},i}(t')\right\rangle \left|H_{\bar{S},\mathrm{eff},i}(t')\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},i}(t')\right|, \tag{1148}$$

$$A_{jw}(t') = \left\langle H_{\bar{S},\mathrm{eff},0}(t')\left|A_{j}(t)\right|H_{\bar{S},\mathrm{eff},1}(t')\right\rangle \left|H_{\bar{S},\mathrm{eff},0}(t')\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},1}(t')\right|, \tag{1149}$$

$$A_{j(-w)}(t') = A^{\dagger}_{jw}(t'). \tag{1150}$$

The effective hamiltonian is:

$$H_{\bar{S},\mathrm{eff}}(t) \equiv \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t' - \frac{\mathrm{i}}{2t}\int_0^t\int_0^{t'}\left[\overline{H_{\bar{S}}}(t'),\overline{H_{\bar{S}}}(t'')\right]\mathrm{d}t'\mathrm{d}t'' + \frac{1}{6t}\int_0^t\int_0^{t'}\int_0^{t''}\left(\left[\left[\overline{H_{\bar{S}}}(t'),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}(t''')\right]\right. \tag{1151}$$

$$\left. + \left[\left[\overline{H_{\bar{S}}}(t'''),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}(t')\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t''' + \cdots \tag{1152}$$

The correlation functions are:

$$\mathcal{B}(t,t') \equiv \begin{pmatrix}
\mathcal{B}_{11}(t,t') & 0 & 0 & 0 & \mathcal{B}_{11}(t,t') & 0\\
0 & \mathcal{B}_{22}(t,t') & \mathcal{B}_{23}(t,t') & \mathcal{B}_{22}(t,t') & 0 & \mathcal{B}_{26}(t,t')\\
0 & \mathcal{B}_{32}(t,t') & \mathcal{B}_{33}(t,t') & \mathcal{B}_{32}(t,t') & 0 & \mathcal{B}_{36}(t,t')\\
0 & \mathcal{B}_{22}(t,t') & \mathcal{B}_{23}(t,t') & \mathcal{B}_{22}(t,t') & 0 & \mathcal{B}_{26}(t,t')\\
\mathcal{B}_{11}(t,t') & 0 & 0 & 0 & \mathcal{B}_{11}(t,t') & 0\\
0 & \mathcal{B}_{62}(t,t') & \mathcal{B}_{63}(t,t') & \mathcal{B}_{62}(t,t') & 0 & \mathcal{B}_{66}(t,t')
\end{pmatrix}, \tag{1153}$$

$$v^*_{i\mathbf{k}}(t) = v_{i\mathbf{k}}(t), \tag{1154}$$

$$\chi_{10}(t) = \sum_{\mathbf{k}}\left(\frac{v^*_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)\,v^*_{j\mathbf{k}}(t)}{2\omega^2_{\mathbf{k}}}\right) \tag{1155}$$

$$= \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}(t)}{2\omega^2_{\mathbf{k}}}\right) \tag{1156}$$

$$= 0, \tag{1157}$$

$$B_{10}(t) = B^*_{10}(t), \tag{1158}$$

$$Q_{ij}(\omega,t) = Q^*_{ij}(\omega,t), \tag{1159}$$

$$\zeta_{ij}(t,t') = \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1160}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1161}$$

$$= \zeta_{ji}(t,t'), \tag{1162}$$

$$\mathcal{B}_{11}(t,t') = \frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}(t,t')\,\xi^+_{10}(t,t') + \Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta^*_{10}(t,t')\,\xi^-_{10}(t,t')\right) - B^{\Re}_{10}(t)\,B^{\Re}_{01}(t') \tag{1163}$$

$$=\frac{1}{2}\left(\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}\left(t\right)B_{01}\left(t'\right) \tag{1164}$$

$$=\frac{1}{2}\left(\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)+\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right)-B_{10}\left(t\right)B_{01}\left(t'\right), \tag{1165}$$

$$\mathcal{B}_{22}(t,t')=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\,\xi_{10}^{+}(t,t')-\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}(t,t')\,\xi_{10}^{-}(t,t')\right)+B_{01}^{\Im}(t)\,B_{10}^{\Im}(t') \tag{1166}$$

$$=-\frac{1}{2}\left(\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\Re\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right) \tag{1167}$$

$$=-\frac{1}{2}\left(\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)-\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)\right), \tag{1168}$$

$$\mathcal{B}_{12}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) \tag{1169}$$

$$=\frac{1}{2}\left(\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)B_{10}^{\Im}\left(t'\right) \tag{1170}$$

$$=\frac{1}{2}\left(0\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+0\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Re}\left(t\right)0 \tag{1171}$$

$$=0, \tag{1172}$$

$$\mathcal{B}_{21}(t,t')=\frac{1}{2}\left(\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+B_{10}^{\Im}\left(t\right)B_{10}^{\Re}\left(t'\right) \tag{1173}$$

$$=\frac{1}{2}\left(\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}^{*}\left(t,t'\right)\xi_{10}^{-}\left(t,t'\right)+\Im\left(\mathrm{e}^{0+0}\right)\zeta_{10}\left(t,t'\right)\xi_{10}^{+}\left(t,t'\right)\right)+0B_{10}^{\Re}\left(t'\right) \tag{1174}$$

$$=0, \tag{1175}$$

$$\mathcal{B}_{i2}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t'\right)\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\} \tag{1176}$$

$$=\mathrm{i}B_{10}\left(t'\right)\int_{0}^{\infty}\left(P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\} \tag{1177}$$

$$=\mathrm{i}B_{10}\left(t'\right)\int_{0}^{\infty}P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{1178}$$

$$\mathcal{B}_{2i}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t\right)\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{1179}$$

$$=\mathrm{i}B_{10}\left(t\right)\int_{0}^{\infty}\left(P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{1180}$$

$$=\mathrm{i}B_{10}\left(t\right)\int_{0}^{\infty}P_{i}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)\left(N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i\in\{3,6\}, \tag{1181}$$

$$P_{i}\left(\omega,t\right)=P_{i}^{*}\left(\omega,t\right), \tag{1182}$$

$$\mathcal{B}_{ij}\left(t,t'\right)=\int_{0}^{\infty}\left(P_{i}\left(\omega,t\right)P_{j}^{*}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_{i}^{*}\left(\omega,t\right)P_{j}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\{3,6\}, \tag{1183}$$

$$=\int_{0}^{\infty}\left(P_{i}\left(\omega,t\right)P_{j}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)+P_{i}\left(\omega,t\right)P_{j}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\{3,6\}, \tag{1184}$$

$$=\int_{0}^{\infty}P_{i}\left(\omega,t\right)P_{j}\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}\left(N\left(\omega\right)+\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega, i,j\in\{3,6\}, \tag{1185}$$

$$\mathcal{B}_{i1}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t'\right)\int_{0}^{\infty}\left(P_{i}\left(\omega,t\right)Q_{10}^{*}\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}^{*}\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1186}$$

$$=\mathrm{i}0\int_{0}^{\infty}\left(P_{i}\left(\omega,t\right)Q_{10}^{*}\left(\omega,t'\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}^{*}\left(\omega,t\right)Q_{10}\left(\omega,t'\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1187}$$

$$=0, i\in\{3,6\}, \tag{1188}$$

$$\mathcal{B}_{1i}\left(t,t'\right)=\mathrm{i}B_{01}^{\Im}\left(t\right)\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1189}$$

$$=\mathrm{i}0\int_{0}^{\infty}\left(P_{i}^{*}\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_{i}\left(\omega,t'\right)Q_{10}^{*}\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1190}$$

$$
\begin{aligned}
&= 0, i \in \{3,6\}, && (1191)\\
\zeta_{ij}(t,t') &= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty \frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)}, && (1192)\\
\xi_{ij}^{\pm}(t,t') &= \mathrm{e}^{-\int_0^\infty \frac{\left|\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm\left(L_i(\omega)F_i(\omega,t')-L_j(\omega)F_j(\omega,t')\right)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, && (1193)\\
P_i(\omega,t) &= L_{i\mathrm{mod}2}(\omega)\left(1-F_{i\mathrm{mod}2}(\omega,t)\right), && (1194)\\
Q_{ij}(\omega,t) &= \frac{L_i(\omega)F_1(\omega,t)-L_j(\omega)F_j(\omega,t)}{\omega}. && (1195)
\end{aligned}
$$

B. Time-independent variational quantum master equation

At first let's show that the master equation (1121) reproduces the results of the reference [1], for the latter case we have that $i,j \in \{1,2,3\}$ and $\omega \in (0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \quad (1196)$$

After performing the transformation (25) on the Hamiltonian (1196) it's possible to split that result in the following set of Hamiltonians:

$$
\begin{aligned}
\overline{H_S} &= (\delta + R)|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, && (1197)\\
\overline{H_I} &= B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), && (1198)\\
H_B &= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. && (1199)
\end{aligned}
$$

The Hamiltonian (1197) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$, where $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$ (this base for the Pauli matrix is different from the base assumed in [1] which is $\sigma_z' = |X\rangle\langle X| - |0\rangle\langle 0|$) by a term proportional to the identity given by $-\frac{\delta}{2}\mathbb{I}$ which is independent of the variational parameters, this can be seen in the following way, with $\epsilon = \delta + R$:

$$
\begin{aligned}
(\delta + R)|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} &= (\delta + R)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| - \frac{\delta}{2}|1\rangle\langle 1| && (1200)\\
&= \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1201)\\
&= \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1202)\\
&= \frac{R}{2}|1\rangle\langle 1| + \left(\frac{\delta}{2} + \frac{R}{2}\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| && (1203)\\
&= \frac{R}{2}|1\rangle\langle 1| + \frac{R}{2}|0\rangle\langle 0| + \left(\frac{\delta}{2} + \frac{R}{2}\right)|1\rangle\langle 1| - \frac{R}{2}|0\rangle\langle 0| - \frac{\delta}{2}|0\rangle\langle 0| && (1204)\\
&= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}|1\rangle\langle 1| - \frac{\delta + R}{2}|0\rangle\langle 0| && (1205)\\
&= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) && (1206)\\
&= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) && (1207)\\
&= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. && (1208)
\end{aligned}
$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then Tr $\left(\overline{H_{\bar{S}}}\right) = R$ and Det $\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by:

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \tag{1209}$$

$$\lambda_\pm = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} \tag{1210}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{1211}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{1212}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \tag{1213}$$

$$\lambda_\pm = \frac{R \pm \eta}{2}. \tag{1214}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} \frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & -\frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{1215}$$

so the eigenvector $|+\rangle = a|1\rangle + b|0\rangle$ satisfies $\frac{\Omega_r}{2}a - \frac{\epsilon+\eta}{2}b = 0$, so $a = \frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} \frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & -\frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{1216}$$

so the eigenvector $|-\rangle = a|1\rangle + b|0\rangle$ satisfies $\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = -\frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$.
Summarizing these results we can write:

$$\lambda_+ = \frac{R + \eta}{2}, \tag{1217}$$

$$\lambda_- = \frac{R - \eta}{2}, \tag{1218}$$

$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \tag{1219}$$

$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, \tag{1220}$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}, \tag{1221}$$

$$\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}. \tag{1222}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{1223}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{1224}$$

So the value of $\tan(\theta)$ using (1224) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{\epsilon^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{\epsilon^2+\Omega_r^2}}+1} \tag{1225}$$

$$= \frac{\frac{\Omega_r}{\sqrt{\epsilon^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{\epsilon^2+\Omega_r^2}}{\sqrt{\epsilon^2+\Omega_r^2}}} \tag{1226}$$

$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{1227}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (944)-(946) and the fact that $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1228}$$

$$= 2\sin(\theta)\cos(\theta) \tag{1229}$$

$$= \sin(2\theta), \tag{1230}$$

$$\langle -|\sigma_x|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{1231}$$

$$= -2\sin(\theta)\cos(\theta) \tag{1232}$$

$$= -\sin(2\theta), \tag{1233}$$

$$\langle -|\sigma_x|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1234}$$

$$= \cos^2(\theta) - \sin^2(\theta) \tag{1235}$$

$$= \cos(2\theta), \tag{1236}$$

$$\langle +|\sigma_y|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1237}$$

$$= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) \tag{1238}$$

$$= 0, \tag{1239}$$

$$\langle -|\sigma_y|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{1240}$$

$$= -\mathrm{i}\sin(\theta)\cos(\theta) + \mathrm{i}\sin(\theta)\cos(\theta) \tag{1241}$$

$$= 0, \tag{1242}$$

$$\langle -|\sigma_y|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1243}$$
$$= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) \tag{1244}$$
$$= \mathrm{i}, \tag{1245}$$
$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1246}$$
$$= \cos(\theta)\cos(\theta) \tag{1247}$$
$$= \cos^2(\theta), \tag{1248}$$
$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} \tag{1249}$$
$$= \sin(\theta)\sin(\theta) \tag{1250}$$
$$= \sin^2(\theta), \tag{1251}$$
$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{1252}$$
$$= -\sin(\theta)\cos(\theta) \tag{1253}$$
$$= -\sin(\theta)\cos(\theta). \tag{1254}$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), \tag{1255}$$
$$A_1(\eta) = \cos(2\theta)\,|-\rangle\langle+|, \tag{1256}$$
$$A_2(0) = 0, \tag{1257}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{1258}$$
$$A_3(0) = \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-|, \tag{1259}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)\,|-\rangle\langle+|. \tag{1260}$$

Now to prove the fact that the model of the “Time-independent variational quantum master equation” is a special case the master equation (1121) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (999) is equivalent to:

$$L_{ijww'}(t,t-\tau) \equiv \int_0^t C_i(t)\,C_j(t-\tau)\,\mathcal{B}_{ij}(t,t-\tau)\,\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t,t-\tau)\,\mathrm{d}\tau, \tag{1261}$$
$$= \int_0^t C_i(t)\,C_j(t-\tau)\,\mathcal{B}_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}\tau w}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}A_j(w,w')\,\mathrm{d}\tau. \tag{1262}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda_{ij}(\tau)$ relate with the correlation functions defined in the equation (415) in the following way:

$$\Lambda_{ij}(\tau) = C_i(t)\,C_j(t-\tau)\,\mathcal{B}_{ij}(\tau). \tag{1263}$$

So the response matrix can be rewritten as:

$$L_{ijww'}(t,t-\tau) = \left(\int_0^t \mathrm{d}\tau\Lambda_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}\tau w}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}\right)A_j(w,w'). \tag{1264}$$

Let’s define the response function like:

$$K_{ij}\left(w,w',t\right)=\int_{0}^{t}C_{i}\left(t\right)C_{j}\left(t-\tau\right)\mathcal{B}_{ij}\left(\tau\right)\mathrm{e}^{\mathrm{i}w\tau}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{1265}$$
$$=\int_{0}^{t}\Lambda_{ij}\left(\tau\right)\mathrm{e}^{\mathrm{i}w\tau}\mathrm{e}^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau \tag{1266}$$
$$=K_{ijww'}\left(t\right). \tag{1267}$$

Then we have the following equivalence:

$$L_{ijww'}\left(t\right)=K_{ijww'}\left(t\right)A_{j}\left(w,w'\right) \tag{1268}$$
$$=K_{ijww'}\left(t\right)A_{jww'}. \tag{1269}$$

Recalling the time-independent nature of the hamiltonian and the variational transformation then we can conclude that (1152) has the following form:

$$\left[\overline{H_{\bar{S}}}(t_1),\overline{H_{\bar{S}}}(t_2)\right]=0 \text{ (because } \overline{H_{\bar{S}}} \text{ is time independent)}, \tag{1270}$$
$$H_{\bar{S},\mathrm{eff}}(t)=\frac{1}{t}\int_{0}^{t}\overline{H_{\bar{S}}}\left(t'\right)\mathrm{d}t'-\frac{\mathrm{i}}{2t}\int_{0}^{t}\int_{0}^{t'}\left[\overline{H_{\bar{S}}}(t'),\overline{H_{\bar{S}}}(t'')\right]\mathrm{d}t'\mathrm{d}t''+\frac{1}{6t}\int_{0}^{t}\int_{0}^{t'}\int_{0}^{t''}\left(\left[\left[\overline{H_{\bar{S}}}(t'),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}(t''')\right]\right. \tag{1271}$$
$$\left.+\left[\left[\overline{H_{\bar{S}}}(t'''),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}\left(t'\right)\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t'''+\cdots \tag{1272}$$
$$=\frac{1}{t}\int_{0}^{t}\overline{H_{\bar{S}}}(t')\,\mathrm{d}t'-\frac{\mathrm{i}}{2t}\int_{0}^{t}\int_{0}^{t'}0\mathrm{d}t'\mathrm{d}t''+\frac{1}{6t}\int_{0}^{t}\int_{0}^{t'}\int_{0}^{t''}\left(\left[0,\overline{H_{\bar{S}}}(t''')\right]+\left[0,\overline{H_{\bar{S}}}(t')\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t'''+\cdots \tag{1273}$$
$$=\frac{1}{t}\int_{0}^{t}\overline{H_{\bar{S}}}\mathrm{d}t' \tag{1274}$$
$$=\frac{1}{t}\overline{H_{\bar{S}}}t'\mid_{0}^{t} \tag{1275}$$
$$=\frac{(t-0)}{t}\overline{H_{\bar{S}}} \tag{1276}$$
$$=\overline{H_{\bar{S}}}. \tag{1277}$$

Now we have that $\frac{\partial V(t)}{\partial t}=0$ and $\underline{H_{\overline{S},\mathrm{eff}}\left(t\right)}=\underline{\overline{H_{\bar{S}}}}=H_{\bar{S}}$ and $\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}=\frac{\partial\overline{H_{\bar{S}}}}{\partial t}=0$ so the equations (1121) and (1123) for the time independent case are:

$$\frac{\partial\overline{\rho_{S}}\left(t\right)}{\partial t}=-\mathrm{i}\left[\overline{H_{\bar{S}}},\overline{\rho_{S}}\left(t\right)\right]-\mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(k+1\right)!}\left(\mathrm{i}t\right)^{k}\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^{k}0\right),\overline{\rho_{S}}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_{i},L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]\right. \tag{1278}$$
$$\left.+\left[\overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right),A_{i}\right]\right) \tag{1279}$$
$$=-\mathrm{i}\left[\overline{H_{\bar{S}}},\overline{\rho_{S}}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_{i},L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left[\overline{\rho_{\bar{S}}}\left(t\right)L_{ijww'}^{\dagger}\left(t\right),A_{i}\right]\right), \tag{1280}$$
$$\frac{\partial\rho_{S}\left(t\right)}{\partial t}=-\mathrm{i}\left[H_{\bar{S}},\rho_{S}\left(t\right)\right]-\sum_{ijww'}\left(\left[\underline{A_{i}\left(t\right)},\underline{L_{ijww'}\left(t\right)}\rho_{S}\left(t\right)\right]-\left[\underline{A_{i}\left(t\right)},\rho_{S}\left(t\right)\underline{L_{ijww'}\left(t\right)}\right]\right). \tag{1281}$$

We can proof that:

$$\frac{\partial \overline{\rho_S}(t)}{\partial t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, L_{ijww'}(t)\,\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\,L^{\dagger}_{ijww'}(t)\right]\right) \tag{1282}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, K_{ijww'}(t)\,A_{jww'}\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\,K^{*}_{ijww'}(t)\,A^{\dagger}_{jww'}\right]\right) \tag{1283}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(K_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho_S}(t)\right] - K^{*}_{ijww'}(t)\left[A_i, \overline{\rho_S}(t)\,A^{\dagger}_{jww'}\right]\right) \tag{1284}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t) + \mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i, A_{jww'}\overline{\rho_S}(t)\right] - \left(K^{\Re}_{ijww'}(t) - \mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i, \overline{\rho_S}(t)A^{\dagger}_{jww'}\right]\right) \tag{1285}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'} K^{\Re}_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho_S}(t) - \overline{\rho_S}(t)A^{\dagger}_{jww'}\right] - \mathrm{i}\sum_{ijww'} K^{\Im}_{ijww'}(t)\left[A_i, A_{jww'}\overline{\rho_S}(t) + \overline{\rho_S}(t)A^{\dagger}_{jww'}\right] \tag{1286}$$

For the time-independent [1] we have the following correlations obtained from the general model, we take account from the fact that $L_0(\omega) = 0$, $\Im(L_1(\omega)) = 0$, $\Im(F_1(\omega)) = 0$, $F_0(\omega) = 0$ and $\int_0^{\infty} |L_1(\omega)|^2 f(\omega)\,\mathrm{d}\omega = \int_0^{\infty} J(\omega) f(\omega)\,\mathrm{d}\omega$ for $f(\omega) \in L^2$. We can drop the time vector (t, t') and instead write the correlation functions as function of τ, we will drop t and t' from the expressions that contain them given the time independence of the hamiltonian:

$$\chi_{ij}(t) = \int_0^{\infty} \frac{L_1^*(\omega)\,L_0(\omega)\,F_1^*(\omega,t)\,F_0(\omega,t) - L_1(\omega)\,L_0^*(\omega)\,F_1(\omega,t)\,F_0^*(\omega,t)}{2\omega^2}\mathrm{d}\omega, \tag{1287}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{10}(t')}\right)\zeta_{10}(t,t')\,\xi^{+}_{10}(t,t') + \Re\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta^{*}_{10}(t,t')\,\xi^{-}_{10}(t,t')\right) - B^{\Re}_{10}(t)\,B^{\Re}_{01}(t'), \tag{1288}$$

$$= \frac{1}{2}\left(\mathrm{e}^{\chi_{10}+\chi_{10}}\zeta_{10}\xi^{+}_{10} + \mathrm{e}^{\chi_{10}+\chi_{01}}\zeta^{*}_{10}\xi^{-}_{10}\right) - B^2, \tag{1289}$$

$$\chi_{ij} = \int_0^{\infty} \frac{L_1(\omega)\,L_0(\omega)\,F_1(\omega)\,F_0(\omega) - L_1(\omega)\,L_0(\omega)\,F_1(\omega)\,F_0(\omega)}{2\omega^2}\mathrm{d}\omega, \tag{1290}$$

$$= 0 \tag{1291}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\mathrm{e}^{0}\zeta_{10}\xi^{+}_{10} + \mathrm{e}^{0}\zeta^{*}_{10}\xi^{-}_{10}\right) - B^2, \tag{1292}$$

$$\zeta_{10} = \mathrm{e}^{\mathrm{i}\Im\left(\int_0^{\infty}\frac{(L_1(\omega)F_1(\omega) - L_0(\omega)F_0(\omega))(L_1(\omega)F_1(\omega) - L_0(\omega)F_0(\omega))^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1293}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^{\infty}\frac{(L_1(\omega)F_1(\omega))(L_1(\omega)F_1(\omega))^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1294}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^{\infty}\frac{J(\omega)F^2(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)} \tag{1295}$$

$$= \mathrm{e}^{\mathrm{i}\Im\left(\int_0^{\infty}\frac{J(\omega)F^2(\omega)}{\omega^2}(\cos(\omega\tau) + \mathrm{i}\sin(\omega\tau))\mathrm{d}\omega\right)} \tag{1296}$$

$$= \mathrm{e}^{\mathrm{i}\int_0^{\infty}\frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega} \tag{1297}$$

$$\mathcal{B}_{11}(\tau) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\int_0^{\infty}\frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\xi^{+}_{10} + \mathrm{e}^{-\mathrm{i}\int_0^{\infty}\frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\xi^{-}_{10}\right) - B^2, \tag{1298}$$

$$\xi^{\pm}_{10} = \mathrm{e}^{-\int_0^{\infty}\frac{\left|(L_1(\omega)F_1(\omega) - L_0(\omega)F_0(\omega))\mathrm{e}^{\mathrm{i}\omega\tau} \pm L_1(\omega)F_1(\omega) \mp L_0(\omega)F_0(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1299}$$

$$= \mathrm{e}^{-\int_0^{\infty}\frac{\left|L_1(\omega)F_1(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} \pm L_1(\omega)F_1(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1300}$$

$$= \mathrm{e}^{-\int_0^{\infty}\frac{J(\omega)F^2(\omega)\left|\mathrm{e}^{\mathrm{i}\omega\tau} \pm 1\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1301}$$

$$\left|\mathrm{e}^{\mathrm{i}\omega\tau} \pm 1\right|^2 = 2\left(1 \pm \cos(\omega\tau)\right) \tag{1302}$$

$$\xi^{\pm}_{10} = \mathrm{e}^{-\int_0^{\infty}\frac{J(\omega)F^2(\omega)(1 \pm \cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{1303}$$

$$\mathcal{B}_{11}(\tau) = -B^2 + \frac{1}{2}\left(e^{i\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)d\omega} e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1+\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}\right. \tag{1304}$$

$$\left. + e^{-i\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)d\omega} e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1-\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}\right) \tag{1305}$$

$$= -B^2 + \frac{e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}}{2}\left(e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-i\sin(\omega\tau)\right)}{\omega^2}d\omega}\right. \tag{1306}$$

$$\left. + e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(-\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)+i\sin(\omega\tau)\right)}{\omega^2}d\omega}\right) \tag{1307}$$

$$B = e^{-\frac{1}{2}\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}, \tag{1308}$$

$$G_+(\omega) = e^{i\omega\tau}N(\omega) + e^{-i\omega\tau}(N(\omega)+1) \tag{1309}$$

$$= (\cos(\omega\tau) + i\sin(\omega\tau))N(\omega) + (\cos(\omega\tau) - i\sin(\omega\tau))(N(\omega)+1) \tag{1310}$$

$$= \cos(\omega\tau)(2N(\omega)+1) - i\sin(\omega\tau) \tag{1311}$$

$$= \cos(\omega\tau)\left(\frac{2}{e^{\beta\omega}-1}+1\right) - i\sin(\omega\tau) \tag{1312}$$

$$= \cos(\omega\tau)\left(\frac{1+e^{\beta\omega}}{e^{\beta\omega}-1}\right) - i\sin(\omega\tau) \tag{1313}$$

$$= \cos(\omega\tau)\left(\frac{e^{-\beta\omega/2}+e^{\beta\omega/2}}{-e^{-\beta\omega/2}+e^{\beta\omega/2}}\right) - i\sin(\omega\tau) \tag{1314}$$

$$= \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - i\sin(\omega\tau), \tag{1315}$$

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}G_+(\omega,\tau)\,d\omega \tag{1316}$$

$$= \int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - i\sin(\omega\tau)\right)d\omega, \tag{1317}$$

$$\mathcal{B}_{11}(\tau) = -B^2 + \frac{e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}}{2}\left(e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-i\sin(\omega\tau)\right)}{\omega^2}d\omega}\right. \tag{1318}$$

$$\left. + e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(-\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)+i\sin(\omega\tau)\right)}{\omega^2}d\omega}\right) \tag{1319}$$

$$= \frac{B^2}{2}\left(e^{-\phi(\tau)} + e^{\phi(\tau)} - 2\right) \tag{1320}$$

$$\mathcal{B}_{22}(\tau) = -\frac{1}{2}\left(\Re\left(e^{\chi_{01}(t)+\chi_{01}(t')}\right)\zeta_{10}(t,t')\xi_{10}^+(t,t') - \Re\left(e^{\chi_{10}(t)+\chi_{01}(t')}\right)\zeta_{10}^*(t,t')\xi_{10}^-(t,t')\right) + B_{01}^{\Im}(t)B_{10}^{\Im}(t'), \tag{1321}$$

$$= -\frac{1}{2}\left(\left(e^{\chi_{01}+\chi_{01}}\right)\zeta_{10}\xi_{10}^+ - \left(e^{\chi_{10}+\chi_{01}}\right)\zeta_{10}^*\xi_{10}^-\right), \tag{1322}$$

$$= \frac{1}{2}\left(\zeta_{10}^*\xi_{10}^- - \zeta_{10}\xi_{10}^+\right), \tag{1323}$$

$$= \frac{1}{2}\left(e^{-i\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)d\omega} e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1-\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}\right. \tag{1324}$$

$$\left. - e^{i\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\sin(\omega\tau)d\omega} e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)(1+\cos(\omega\tau))}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}\right) \tag{1325}$$

$$= \frac{1}{2}e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)d\omega}\left(e^{\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-i\sin(\omega\tau)\right)}{\omega^2}d\omega}\right. \tag{1326}$$

$$\left. - e^{-\int_0^\infty \frac{J(\omega)F^2(\omega)\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-i\sin(\omega\tau)\right)}{\omega^2}d\omega}\right) \tag{1327}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} - \mathrm{e}^{-\phi(\tau)}\right), \tag{1328}$$

$$\mathcal{B}_{12}(\tau) = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{10}(t)+\chi_{01}(t')}\right)^{\Im}\zeta_{10}^{*}(t,t')\xi_{10}^{-}(t,t') + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}\zeta_{10}(t,t')\xi_{10}^{+}(t,t')\right) + B_{10}^{\Re}(t)B_{10}^{\Im}(t') \tag{1329}$$

$$= 0, \tag{1330}$$

$$\mathcal{B}_{21}(\tau) = \frac{1}{2}\left(\left(\mathrm{e}^{\chi_{01}(t)+\chi_{10}(t')}\right)^{\Im}\zeta_{10}^{*}(t,t')\xi_{10}^{-}(t,t') + \left(\mathrm{e}^{\chi_{01}(t)+\chi_{01}(t')}\right)^{\Im}\zeta_{10}(t,t')\xi_{10}^{+}(t,t')\right) + B_{10}^{\Im}(t)B_{10}^{\Re}(t'), \tag{1331}$$

$$= 0, \tag{1332}$$

$$\mathcal{B}_{ij}(t,t') = \int_0^\infty \left(P_i(\omega,t)\,P_j^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_i^*(\omega,t)\,P_j(\omega,t')\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega, i,j \in \{3,6\}, \tag{1333}$$

$$\mathcal{B}_{66}(t,t') = \int_0^\infty \left(P_6(\omega,t)\,P_6^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_6^*(\omega,t)\,P_6(\omega,t')\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{1334}$$

$$P_6(\omega,t) = L_{6\mathrm{mod}2}(\omega)\left(1 - F_{6\mathrm{mod}2}(\omega,t)\right), \tag{1335}$$

$$= L_0(\omega)\left(1 - F_0(\omega,t)\right), \tag{1336}$$

$$= 0, \tag{1337}$$

$$\mathcal{B}_{66}(\tau) = 0, \tag{1338}$$

$$\mathcal{B}_{36}(t,t') = \int_0^\infty \left(P_3(\omega,t)\,P_6^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)\,P_6(\omega,t')\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{1339}$$

$$= 0, \tag{1340}$$

$$\mathcal{B}_{63}(t,t') = \int_0^\infty \left(P_6(\omega,t)\,P_3^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_6^*(\omega,t)\,P_3(\omega,t')\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{1341}$$

$$= 0, \tag{1342}$$

$$\mathcal{B}_{33}(t,t') = \int_0^\infty \left(P_3(\omega,t)\,P_3^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)\,P_3(\omega,t')\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{1343}$$

$$= \int_0^\infty \left(P_3(\omega,t)\,P_3^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + P_3^*(\omega,t)\,P_3(\omega,t')\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{1344}$$

$$P_3(\omega,t) = L_{3\mathrm{mod}2}(\omega)\left(1 - F_{3\mathrm{mod}2}(\omega,t)\right), \tag{1345}$$

$$= L_1(\omega)\left(1 - F_1(\omega,t)\right), \tag{1346}$$

$$P_3(\omega,t)\,P_3^*(\omega,t') = L_1(\omega)\left(1 - F_1(\omega)\right)L_1^*(\omega)\left(1 - F_1(\omega)\right), \tag{1347}$$

$$= \left|L_1(\omega)\right|^2\left(1 - F_1(\omega)\right)^2 \tag{1348}$$

$$\mathcal{B}_{33}(t,t') = \int_0^\infty \left|L_1(\omega)\right|^2\left(1 - F_1(\omega)\right)^2\left(\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega) + \mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{1349}$$

$$= \int_0^\infty J(\omega)\left(1 - F(\omega)\right)^2 G_+(\omega)\,\mathrm{d}\omega \tag{1350}$$

$$\mathcal{B}_{i1}(t,t') = \mathrm{i}B_{01}^{\Im}(t')\int_0^\infty \left(P_i(\omega,t)Q_{10}^*(\omega,t')N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_i^*(\omega,t)Q_{10}(\omega,t')\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega, i \in \{3,6\} \tag{1351}$$

$$= 0, \tag{1352}$$

$$\mathcal{B}_{1i}(t,t') = \mathrm{i}B_{01}^{\Im}(t)\int_0^\infty \left(P_i^*(\omega,t')Q_{10}(\omega,t)N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau} - P_i(\omega,t')Q_{10}^*(\omega,t)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega, i \in \{3,6\}, \tag{1353}$$

$$= 0, \tag{1354}$$

$$\mathcal{B}_{62}(t,t') = \mathrm{i}B_{10}^{\Re}(t')\int_0^\infty \left(P_6^*(\omega,t')\,Q_{10}(\omega,t')\left(N(\omega)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau} - P_6(\omega,t')\,Q_{10}^*(\omega,t')\,\mathrm{e}^{\mathrm{i}\omega\tau}N(\omega)\right)\mathrm{d}\omega \tag{1355}$$

$$= 0, \tag{1356}$$

$$\mathcal{B}_{26}(t,t') = \mathrm{i}B_{10}^{\Re}(t)\int_0^\infty \left(P_6^*(\omega,t')\,Q_{10}(\omega,t)\,N(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau} - P_6(\omega,t')\,Q_{10}^*(\omega,t)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N(\omega)+1\right)\right)\mathrm{d}\omega \tag{1357}$$

$$= 0, \tag{1358}$$

$$\mathcal{B}_{32}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t'\right)\int_0^\infty\left(P_3^*\left(\omega,t'\right)Q_{10}\left(\omega,t'\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_3\left(\omega,t'\right)Q_{10}^*\left(\omega,t'\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega \tag{1359}$$

$$=\mathrm{i}B\int_0^\infty\left(P_3^*\left(\omega\right)Q_{10}\left(\omega\right)\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P_3\left(\omega\right)Q_{10}^*\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega, \tag{1360}$$

$$Q_{ij}(\omega,t)=\frac{L_i\left(\omega\right)F_j\left(\omega,t\right)-L_i\left(\omega\right)F_j\left(\omega,t\right)}{\omega}, \tag{1361}$$

$$Q_{10}(\omega,t)=\frac{L_1\left(\omega\right)F_1\left(\omega,t\right)}{\omega}, \tag{1362}$$

$$\mathcal{B}_{32}\left(t,t'\right)=\mathrm{i}B\int_0^\infty\Bigg(L_1^*\left(\omega\right)\left(1-F_1\left(\omega,t\right)\right)\frac{L_1\left(\omega\right)F_1\left(\omega,t\right)}{\omega}\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau} \tag{1363}$$

$$-L_1\left(\omega\right)\left(1-F_1\left(\omega,t\right)\right)\frac{L_1^*\left(\omega\right)F_1\left(\omega,t\right)}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\Bigg)\mathrm{d}\omega \tag{1364}$$

$$=\mathrm{i}B\int_0^\infty\left|L_1\left(\omega\right)\right|^2\left(\left(1-F_1\left(\omega\right)\right)\frac{F_1\left(\omega\right)}{\omega}\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-\left(1-F_1\left(\omega\right)\right)\frac{F_1\left(\omega\right)}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega \tag{1365}$$

$$=\mathrm{i}B\int_0^\infty J\left(\omega\right)\left(1-F\left(\omega,t\right)\right)\frac{F\left(\omega\right)}{\omega}\left(\left(N\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}N\left(\omega\right)\right)\mathrm{d}\omega \tag{1366}$$

$$=\mathrm{i}B\int_0^\infty J\left(\omega\right)\left(1-F\left(\omega,t\right)\right)\frac{F\left(\omega\right)}{\omega}G_-\left(\omega\right)\mathrm{d}\omega \tag{1367}$$

$$\mathcal{B}_{23}\left(t,t'\right)=\mathrm{i}B_{10}^{\Re}\left(t\right)\int_0^\infty\left(P_3^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_3\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1368}$$

$$=\mathrm{i}B\int_0^\infty\left(P_3^*\left(\omega,t'\right)Q_{10}\left(\omega,t\right)N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-P_3\left(\omega,t'\right)Q_{10}^*\left(\omega,t\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1369}$$

$$=\mathrm{i}B\int_0^\infty\left(L_1^*(\omega)(1-F_1(\omega,t))\frac{L_1(\omega)F_1(\omega,t)}{\omega}N(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}-L_1(\omega)(1-F_1(\omega,t))\frac{L_1^*(\omega)F_1(\omega,t)}{\omega}\mathrm{e}^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega \tag{1370}$$

$$=\mathrm{i}B\int_0^\infty J\left(\omega\right)\left(1-F_1\left(\omega,t\right)\right)\frac{F_1\left(\omega,t\right)}{\omega}\left(N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}-\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1371}$$

$$=-\mathrm{i}B\int_0^\infty J\left(\omega\right)\left(1-F_1\left(\omega,t\right)\right)\frac{F_1\left(\omega,t\right)}{\omega}\left(-N\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}+\mathrm{e}^{-\mathrm{i}\omega\tau}\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{1372}$$

$$=-\mathcal{B}_{32}\left(t,t'\right) \tag{1373}$$

$$\zeta_{ij}\left(t,t'\right)=\mathrm{e}^{\mathrm{i}\Im\left(\int_0^\infty\frac{\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\left(L_i(\omega)F_i\left(\omega,t'\right)-L_j(\omega)F_j\left(\omega,t'\right)\right)^*\mathrm{e}^{\mathrm{i}\omega\tau}}{\omega^2}\mathrm{d}\omega\right)}, \tag{1374}$$

$$\xi_{ij}^{\pm}\left(t,t'\right)=\mathrm{e}^{-\int_0^\infty\frac{\left|\left(L_i(\omega)F_i(\omega,t)-L_j(\omega)F_j(\omega,t)\right)\mathrm{e}^{\mathrm{i}\omega\tau}\pm L_i(\omega)F_i\left(\omega,t'\right)\mp L_j(\omega)F_j\left(\omega,t'\right)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{1375}$$

$$P_i\left(\omega,t\right)=L_{i\mathrm{mod}2}\left(\omega\right)\left(1-F_{i\mathrm{mod}2}\left(\omega,t\right)\right), \tag{1376}$$

$$Q_{ij}(\omega,t)=\frac{L_i\left(\omega\right)F_j\left(\omega,t\right)-L_i\left(\omega\right)F_j\left(\omega,t\right)}{\omega}, \tag{1377}$$

$$\mathcal{B}\left(\tau\right)\equiv\begin{pmatrix}\mathcal{B}_{11}\left(\tau\right) & 0 & 0 & 0 & \mathcal{B}_{11}\left(\tau\right) & 0\\ 0 & \mathcal{B}_{22}\left(\tau\right) & \mathcal{B}_{23}\left(\tau\right) & \mathcal{B}_{22}\left(\tau\right) & 0 & 0\\ 0 & -\mathcal{B}_{23}\left(\tau\right) & \mathcal{B}_{33}\left(\tau\right) & -\mathcal{B}_{23}\left(\tau\right) & 0 & 0\\ 0 & \mathcal{B}_{22}\left(\tau\right) & \mathcal{B}_{23}\left(\tau\right) & \mathcal{B}_{22}\left(\tau\right) & 0 & 0\\ \mathcal{B}_{11}\left(\tau\right) & 0 & 0 & 0 & \mathcal{B}_{11}\left(\tau\right) & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}. \tag{1378}$$

The correlation functions as we can see in [1] can be obtained using the following definition:

$$\Lambda_{ij}\left(\tau\right)\equiv C_iC_j\mathcal{B}_{ij}\left(\tau\right) \tag{1379}$$

Also the matrix C(t) can be decomposed as:

$$C_1 = C_2 = \frac{\Omega}{2}, \tag{1380}$$
$$C_3 = C_6 = 1, \tag{1381}$$
$$C_4 = C_4 = 0 \tag{1382}$$

Let's recall that $\Omega_r = B\Omega$. So the correlation functions Λ are:

$$\Lambda_{11}(\tau) = C_1 C_1 \mathcal{B}_{11}(\tau) \tag{1383}$$
$$= \left(\frac{\Omega}{2}\right)^2 \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{1384}$$

$$= \frac{(\Omega B)^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right) \tag{1385}$$
$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1386}$$
$$\Lambda_{22}(\tau) = C_2 C_2 \mathcal{B}_{22}(\tau) \tag{1387}$$
$$= \left(\frac{\Omega}{2}\right)^2 \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} - \mathrm{e}^{-\phi(\tau)}\right) \tag{1388}$$
$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} - \mathrm{e}^{-\phi(\tau)}\right), \tag{1389}$$
$$\Lambda_{33}(\tau) = C_3 C_3 \mathcal{B}_{33}(\tau) \tag{1390}$$
$$= (1)^2 \int_0^\infty J(\omega)\left(1 - F(\omega)\right)^2 G_+(\omega)\,\mathrm{d}\omega \tag{1391}$$
$$= \int_0^\infty J(\omega)\left(1 - F(\omega)\right)^2 G_+(\omega)\,\mathrm{d}\omega, \tag{1392}$$
$$\Lambda_{23}(\tau) = C_2 C_3 \mathcal{B}_{23}(\tau) \tag{1393}$$
$$= \frac{\Omega}{2} 1\mathrm{i}B \int_0^\infty J(\omega)\left(1 - F(\omega, t)\right) \frac{F(\omega)}{\omega} G_-(\omega)\,\mathrm{d}\omega \tag{1394}$$
$$= \mathrm{i}\frac{\Omega_r}{2} \int_0^\infty J(\omega)\left(1 - F(\omega, t)\right) \frac{F(\omega)}{\omega} G_-(\omega)\,\mathrm{d}\omega, \tag{1395}$$
$$\Lambda_{12}(\tau) = \Lambda_{13}(\tau) = \Lambda_{16}(\tau) \tag{1396}$$
$$= \Lambda_{21}(\tau) = \Lambda_{26}(\tau) \tag{1397}$$
$$= \Lambda_{31}(\tau) = \Lambda_{36}(\tau) = 0. \tag{1398}$$

Now let's define:

$$K_{ijw}(t) = \int_0^t C_i C_j \mathcal{B}_{ij}(\tau)\,\mathrm{e}^{\mathrm{i}w\tau}\mathrm{d}\tau. \tag{1399}$$

So

$$L_{ijw}(t) = A_{jw} K_{ijw}(t). \tag{1400}$$

Now for a time-independent hamiltonian is possible to show that for the decomposition matrix $A_j(w(t)) = A_j(w)$:

$$U^\dagger(t)\, A_j(t)\, U(t) = \sum_w \mathrm{e}^{-\mathrm{i}wt} A_j(w). \tag{1401}$$

It means that a decomposition matrix of $\widetilde{A_j}\left(t\right)$ associated to the eigenvector under evolution for the same time-independent hamiltonian $U\left(t\right)A_j\left(w\right)U^{\dagger}\left(t\right)$ generates the same decomposition matrix multiplied by a phase $\mathrm{e}^{\mathrm{i}wt}$. It means that the decomposition matrix $A_{jww'}$ for a time-independent hamiltonian fulfill $A_{jww'}=A_j\left(w\right)\delta_{ww'}$ so only if $w=w'$ then the response function is relevant for taking account and it's equal to:

$$K_{ijww}\left(t\right)=\int_0^t C_iC_j\mathcal{B}_{ij}\left(\tau\right)\mathrm{e}^{\mathrm{i}w\tau}\mathrm{d}\tau \tag{1402}$$

$$\equiv K_{ijw}\left(t\right). \tag{1403}$$

Finally taking the Hamiltonian (1196) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}\left(t\right)=\frac{\Omega}{2}$, $\varepsilon_0\left(t\right)=0$ and $\varepsilon_1\left(t\right)=\delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right)=-\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right)=\epsilon$. Now $\eta=\sqrt{\epsilon^2+\Omega_r^2}$ and using the equation (338) we have that:

$$f_k=\frac{g_k\left(1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1-\frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon-\frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{1404}$$

$$=\frac{g_k\left(1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1-\frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1-\frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{1405}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (451) under a time-independent approach providing similar results.

The master equation for this special case is:

$$L_{ijww'}(t)=\delta_{ww'}A_{jw}K_{ijw}\left(t\right), \tag{1406}$$

$$\frac{\partial\overline{\rho_{\bar{S}}}(t)}{\partial t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_i,L_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left[\overline{\rho_{\bar{S}}}\left(t\right)L^{\dagger}_{ijww'}\left(t\right),A_i\right]\right) \tag{1407}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijw}\left(\left[A_i,L_{ijw}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left[\overline{\rho_{\bar{S}}}\left(t\right)L^{\dagger}_{ijw}\left(t\right),A_i\right]\right) \tag{1408}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijw}\left(\left[A_i,A_{jw}K_{ijw}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left[\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw}K^{*}_{ijw}\left(t\right),A_i\right]\right) \tag{1409}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijw}\left(\left(K^{\Re}_{ijw}\left(t\right)+\mathrm{i}K^{\Im}_{ijw}\left(t\right)\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}\left(t\right)\right]+\left(K^{\Re}_{ijw}\left(t\right)-\mathrm{i}K^{\Im}_{ijw}\left(t\right)\right)\left[\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw},A_i\right]\right) \tag{1410}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(K^{\Re}_{ijw}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]+\left[\overline{\rho_{\bar{S}}}(t)A^{\dagger}_{jw},A_i\right]\right)+\mathrm{i}K^{\Im}_{ijw}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]-\left[\overline{\rho_{\bar{S}}}(t)A^{\dagger}_{jw},A_i\right]\right)\right) \tag{1411}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijw}\left(K^{\Re}_{ijw}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]-\left[A_i,\overline{\rho_{\bar{S}}}(t)A^{\dagger}_{jw}\right]\right)+\mathrm{i}K^{\Im}_{ijw}(t)\left(\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}(t)\right]+\left[A_i,\overline{\rho_{\bar{S}}}(t)A^{\dagger}_{jw}\right]\right)\right) \tag{1412}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijw}\left(K^{\Re}_{ijw}\left(t\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}\left(t\right)-\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw}\right]+\mathrm{i}K^{\Im}_{ijw}\left(t\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}\left(t\right)+\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw}\right]\right) \tag{1413}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijw}K^{\Re}_{ijw}\left(t\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}\left(t\right)-\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw}\right]-\mathrm{i}\sum_{ijw}K^{\Im}_{ijw}\left(t\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}\left(t\right)+\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw}\right], \tag{1414}$$

$$\gamma_{ij}\left(w,t\right)\equiv 2K^{\Re}_{ijw}\left(t\right) \tag{1415}$$

$$S_{ij}\left(w,t\right)\equiv K^{\Im}_{ijw}\left(t\right) \tag{1416}$$

$$A_j\left(\omega\right)\equiv A_{jw} \tag{1417}$$

$$\frac{\partial\overline{\rho_{\bar{S}}}(t)}{\partial t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\frac{1}{2}\sum_{ijw}\gamma_{ij}\left(w,t\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}\left(t\right)-\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw}\right]-\mathrm{i}\sum_{ijw}S_{ij}\left(w,t\right)\left[A_i,A_{jw}\overline{\rho_{\bar{S}}}\left(t\right)+\overline{\rho_{\bar{S}}}\left(t\right)A^{\dagger}_{jw}\right] \tag{1418}$$

$$=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_{\bar{S}}}(t)\right]-\frac{1}{2}\sum_{ijw}\gamma_{ij}(w,t)\left[A_i,A_j(w)\,\overline{\rho_{\bar{S}}}(t)-\overline{\rho_{\bar{S}}}(t)\,A^{\dagger}_j(w)\right]-\mathrm{i}\sum_{ijw}S_{ij}(w,t)\left[A_i,A_j(w)\,\overline{\rho_{\bar{S}}}(t)+\overline{\rho_{\bar{S}}}(t)\,A^{\dagger}_j(w)\right]. \tag{1419}$$

C. Time-dependent polaron quantum master equation

Following the reference [1] , when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*} b_{\mathbf{k}}\right)\right) |1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1420}$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (1420) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (1197) and (1198) we obtain ther following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1)|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t), \tag{1421}$$

$$\overline{H_{\mathrm{I}}} = \frac{\Omega(t)}{2}\left(B_x\sigma_x + B_y\sigma_y\right). \tag{1422}$$

In this case $R_1 = \sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right) = \sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (1383)-(1398) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^{\infty}\frac{J(\omega)}{\omega^2}G_+(\tau)\,\mathrm{d}\omega. \tag{1423}$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)$ so (1423) can be written as:

$$\phi(\tau) = \int_0^{\infty}\frac{J(\omega)}{\omega^2}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right)\mathrm{d}\omega. \tag{1424}$$

Writing the interaction Hamiltonian (1422) in the similar way to the equation (1198) allow us to to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (1197) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [4]. Then $\overline{H_S}$is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t). \tag{1425}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{1426}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{1427}$$

$$\Lambda_{22}(\tau) = \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{1428}$$

$$= \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right). \tag{1429}$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively.
The effective hamiltonian is given by:

$$H_{\bar{S},\mathrm{eff}}\left(t\right)\equiv\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}(t')\mathrm{d}t'-\frac{\mathrm{i}}{2t}\int_0^t\int_0^{t'}\left[\overline{H_{\bar{S}}}(t')\,,\overline{H_{\bar{S}}}(t'')\right]\mathrm{d}t'\mathrm{d}t''+\frac{1}{6t}\int_0^t\int_0^{t'}\int_0^{t''}\left(\left[\left[\overline{H_{\bar{S}}}(t')\,,\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}(t''')\right]\right. \tag{1430}$$

$$\left.+\left[\left[\overline{H_{\bar{S}}}(t'''),\overline{H_{\bar{S}}}(t'')\right],\overline{H_{\bar{S}}}\left(t'\right)\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t'''+\cdots, \tag{1431}$$

$$[\overline{H_{\bar{S}}}(t),\overline{H_{\bar{S}}}(t')]=\left[\frac{\delta'}{2}\sigma_z+\frac{\Omega_r(t)\,\sigma_x}{2},\frac{\delta'}{2}\sigma_z+\frac{\Omega_r(t')\,\sigma_x}{2}\right] \tag{1432}$$

$$=\left(\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t\right)\sigma_x}{2}\right)\left(\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t'\right)\sigma_x}{2}\right)-\left(\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t'\right)\sigma_x}{2}\right)\left(\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t\right)\sigma_x}{2}\right) \tag{1433}$$

$$=\left(\left(\frac{\delta'}{2}\sigma_z\right)^2+\frac{\delta'}{2}\sigma_z\frac{\Omega_r\left(t'\right)\sigma_x}{2}+\frac{\Omega_r\left(t\right)\sigma_x}{2}\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t\right)\sigma_x}{2}\frac{\Omega_r\left(t'\right)\sigma_x}{2}\right)-\left(\left(\frac{\delta'}{2}\sigma_z\right)^2+\frac{\delta'}{2}\sigma_z\frac{\Omega_r\left(t\right)\sigma_x}{2}\right. \tag{1434}$$

$$\left.+\frac{\Omega_r\left(t'\right)\sigma_x}{2}\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t'\right)\sigma_x}{2}\frac{\Omega_r\left(t\right)\sigma_x}{2}\right) \tag{1435}$$

$$=\left(\frac{\delta'}{2}\sigma_z\frac{\Omega_r\left(t'\right)\sigma_x}{2}+\frac{\Omega_r\left(t\right)\sigma_x}{2}\frac{\delta'}{2}\sigma_z\right)-\left(\frac{\delta'}{2}\sigma_z\frac{\Omega_r\left(t\right)\sigma_x}{2}+\frac{\Omega_r\left(t'\right)\sigma_x}{2}\frac{\delta'}{2}\sigma_z\right) \tag{1436}$$

$$=\frac{\Omega_r\left(t'\right)\delta'}{4}\mathrm{i}\sigma_y-\frac{\Omega_r\left(t\right)\delta'}{4}\mathrm{i}\sigma_y+\frac{\Omega_r\left(t\right)\delta'}{4}\mathrm{i}\sigma_y-\frac{\Omega_r\left(t'\right)\delta'}{4}\mathrm{i}\sigma_y \tag{1437}$$

$$=0, \tag{1438}$$

$$H_{\bar{S},\mathrm{eff}}(t)=\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}\left(t'\right)\mathrm{d}t'-\frac{\mathrm{i}}{2t}\int_0^t\int_0^{t'}0\mathrm{d}t'\mathrm{d}t''+\frac{1}{6t}\int_0^t\int_0^{t'}\int_0^{t''}\left(\left[0,\overline{H_{\bar{S}}}\left(t'''\right)\right]+\left[0,\overline{H_{\bar{S}}}\left(t'\right)\right]\right)\mathrm{d}t'\mathrm{d}t''\mathrm{d}t'''+\cdots \tag{1439}$$

$$=\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}\left(t'\right)\mathrm{d}t' \tag{1440}$$

$$=\frac{1}{t}\int_0^t\left(\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t'\right)\sigma_x}{2}\right)\mathrm{d}t', \tag{1441}$$

$$U\left(t\right)=\mathrm{e}^{-\mathrm{i}t\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}(t')\mathrm{d}t'} \tag{1442}$$

$$=\mathrm{e}^{-\mathrm{i}\int_0^t\overline{H_{\bar{S}}}(t')\mathrm{d}t'}. \tag{1443}$$

In general we can deduce that $\left[\frac{\delta'}{2}\sigma_z+\frac{g(t)\sigma_x}{2},\frac{\delta'}{2}\sigma_z+\frac{g(t')\sigma_x}{2}\right]=0$.

The master equation for this section based on the equation (451) is:

$$U\left(t\right)\frac{\partial\widetilde{\rho_S}\left(t\right)}{\partial t}U^\dagger\left(t\right)=-\sum_{i=1}^{2}\int_0^t\mathrm{d}\tau\left(C_i\left(t\right)C_i\left(t-\tau\right)\Lambda_{ii}\left(\tau\right)\left[A_i,\widetilde{A_i}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\right. \tag{1444}$$

$$\left.+C_i\left(t\right)C_i\left(t-\tau\right)\Lambda_{ii}\left(-\tau\right)\left[\rho_S\left(t\right)\widetilde{A_i}\left(t-\tau,t\right),A_i\right]\right). \tag{1445}$$

Replacing $C_i\left(t\right)=\frac{\Omega(t)}{2}$ and $\widetilde{A_i}\left(t-\tau,t\right)=\widetilde{\sigma_i}\left(t-\tau,t\right)$, also using the equations (1426) and (1429) on the equation (1445) we obtain that:

$$U\left(t\right)\frac{\partial\widetilde{\rho_S}\left(t\right)}{\partial t}U^\dagger\left(t\right)=-\frac{\Omega\left(t\right)}{4}\int_0^t\mathrm{d}\tau\Omega\left(t-\tau\right)\left(\left[\sigma_x,\widetilde{\sigma_x}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\Lambda_x\left(\tau\right)+\left[\sigma_y,\widetilde{\sigma_y}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\Lambda_y\left(\tau\right)\right. \tag{1446}$$

$$\left.+\left[\rho_S\left(t\right)\widetilde{\sigma_x}\left(t-\tau,t\right),\sigma_x\right]\Lambda_x\left(\tau\right)+\left[\rho_S\left(t\right)\widetilde{\sigma_y}\left(t-\tau,t\right),\sigma_y\right]\Lambda_y\left(\tau\right)\right). \tag{1447}$$

Now let's focus on the LHS, as shown in (1121):

$$U(t)\frac{\partial\widetilde{\rho_S}(t)}{\partial t}U^\dagger(t)=\frac{\partial\overline{\rho_S}(t)}{\partial t}+\mathrm{i}\left[\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}(t')\,\mathrm{d}t',\overline{\rho_S}(t)\right]+\mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\frac{\partial\left(\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}(t')\,\mathrm{d}t'\right)}{\partial t}\right),\overline{\rho_S}(t)\right], \tag{1448}$$

$$=\frac{\partial\overline{\rho_S}(t)}{\partial t}+\frac{\mathrm{i}}{t}\left[\int_0^t\overline{H_{\bar{S}}}(t')\,\mathrm{d}t',\overline{\rho_S}(t)\right]+\mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\frac{\partial\left(\frac{1}{t}\int_0^t\overline{H_{\bar{S}}}(t')\,\mathrm{d}t'\right)}{\partial t}\right),\overline{\rho_S}(t)\right]. \tag{1449}$$

The term that can be reduced is:

$$\left(\mathrm{ad}_{\frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \frac{\partial\left(\frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'\right)}{\partial t} = \left(\mathrm{ad}_{\frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \left(-\frac{\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t'}{t^2} + \frac{\overline{H_{\bar{S}}}(t)}{t}\right) \tag{1450}$$

$$= \left(\mathrm{ad}_{\frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \left(-\frac{\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t'}{t^2}\right) + \left(\mathrm{ad}_{\frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \left(\frac{\overline{H_{\bar{S}}}(t)}{t}\right) \tag{1451}$$

$$= -\left(\frac{1}{t}\right)^k \frac{1}{t^2}\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \left(\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t'\right) + \left(\frac{1}{t}\right)^k \frac{1}{t}\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \left(\overline{H_{\bar{S}}}(t)\right) \tag{1452}$$

$$= -\frac{1}{t^2}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t'\delta_{0k} + \left(\frac{1}{t}\right)^k \frac{1}{t}\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \left(\overline{H_{\bar{S}}}(t)\right), \tag{1453}$$

$$\mathrm{i}t\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{\frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k \frac{\partial\left(\frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t'\right)}{\partial t}\right) \tag{1454}$$

$$= \mathrm{i}t\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(-\frac{1}{t^2}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t'\delta_{0k} + \left(\frac{1}{t}\right)^k\frac{1}{t}\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right)\right)\right) \tag{1455}$$

$$= -\frac{\mathrm{i}}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t' + \mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\frac{1}{t}\right)^k\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right) \tag{1456}$$

$$= -\frac{\mathrm{i}}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t' + \mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i})^k\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right), \tag{1457}$$

$$U(t)\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^\dagger(t) = \frac{\partial\overline{\rho_S}(t)}{\partial t} + \frac{\mathrm{i}}{t}\left[\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \overline{\rho_S}(t)\right] + \left[\mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i})^k\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right) - \frac{\mathrm{i}}{t}\right. \tag{1458}$$

$$\left.\times\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \overline{\rho_S}(t)\right] \tag{1459}$$

$$= \frac{\partial\overline{\rho_S}(t)}{\partial t} + \mathrm{i}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i})^k\left(\mathrm{ad}_{\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right), \overline{\rho_S}(t)\right] \tag{1460}$$

$$= \frac{\partial\overline{\rho_S}(t)}{\partial t} + \mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] + \mathrm{i}\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{ad}_{\mathrm{i}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right), \overline{\rho_S}(t)\right], \tag{1461}$$

$$\overline{H_{\bar{S}}}(t) = \frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)}{2}\sigma_x, \tag{1462}$$

$$\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t' = \int_0^t\left(\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)}{2}\sigma_x\right)\mathrm{d}t' \tag{1463}$$

$$= \frac{\delta'}{2}\sigma_z t + \sigma_x\int_0^t \frac{\Omega_r(t')}{2}\mathrm{d}t', \tag{1464}$$

$$\left[\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', \overline{H_{\bar{S}}}(t)\right] = \left[\frac{\delta'}{2}\sigma_z t + \sigma_x\int_0^t\frac{\Omega_r(t')}{2}\mathrm{d}t', \frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)}{2}\sigma_x\right] \tag{1465}$$

$$= \left[\frac{\delta'}{2}\sigma_z t, \frac{\delta'}{2}\sigma_z\right] + \left[\frac{\delta'}{2}\sigma_z t, \frac{\Omega_r(t)}{2}\sigma_x\right] + \left[\sigma_x\int_0^t\frac{\Omega_r(t')}{2}\mathrm{d}t', \frac{\delta'}{2}\sigma_z\right] + \left[\sigma_x\int_0^t\frac{\Omega_r(t')}{2}\mathrm{d}t', \frac{\Omega_r(t)}{2}\sigma_x\right] \tag{1466}$$

$$= \left(\frac{\delta'}{2}\right)^2 t\left[\sigma_z, \sigma_z\right] + \frac{\delta'}{2}t\frac{\Omega_r(t)}{2}\left[\sigma_z, \sigma_x\right] + \frac{\delta'}{2}\int_0^t\frac{\Omega_r(t')}{2}\mathrm{d}t'\left[\sigma_x, \sigma_z\right] + \frac{\Omega_r(t)}{2}\int_0^t\frac{\Omega_r(t')}{2}\mathrm{d}t'\left[\sigma_x, \sigma_x\right] \tag{1467}$$

$$= \frac{\delta'}{2}t\frac{\Omega_r(t)}{2}\left[\sigma_z, \sigma_x\right] + \frac{\delta'}{2}\int_0^t\frac{\Omega_r(t')}{2}\mathrm{d}t'\left[\sigma_x, \sigma_z\right] \tag{1468}$$

$$= \frac{\delta'}{2}t\frac{\Omega_r(t)}{2}2\mathrm{i}\sigma_y - \frac{\delta'}{2}\int_0^t\frac{\Omega_r(t')}{2}\mathrm{d}t'2\mathrm{i}\sigma_y \tag{1469}$$

$$= \mathrm{i}\frac{\delta'}{2}\left(t\Omega_r(t) - \int_0^t\Omega_r(t')\,\mathrm{d}t'\right)\sigma_y \tag{1470}$$

$$= \mathrm{i}\frac{\delta'}{2}\left(\int_0^t t'\frac{\mathrm{d}\Omega_r(t')}{\mathrm{d}t'}\mathrm{d}t'\right)\sigma_y. \tag{1471}$$

Neglecting the term $\int_0^t t'\frac{\mathrm{d}\Omega_r(t')}{\mathrm{d}t'}\mathrm{d}t'$ we can conclude that $\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{ad}_{\mathrm{i}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right) = 0$ so we infer the following equality $\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{ad}_{\mathrm{i}\int_0^t \overline{H_{\bar{S}}}(t')\mathrm{d}t'}\right)^k\left(\overline{H_{\bar{S}}}(t)\right), \overline{\rho_S}(t)\right] = 0$ then:

$$U(t)\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^\dagger(t) \approx \frac{\partial\overline{\rho_S}(t)}{\partial t} + \mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right]. \tag{1472}$$

So we can conclude that:

$$\frac{\partial \overline{\rho_S}(t)}{\partial t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t)\,,\overline{\rho_S}(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau \Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\,\rho_S(t)\right]\Lambda_x(\tau) + \left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\,\rho_S(t)\right]\Lambda_y(\tau)\right. \tag{1473}$$

$$\left. + \left[\rho_S\left(t\right)\widetilde{\sigma_x}\left(t-\tau,t\right),\sigma_x\right]\Lambda_x\left(\tau\right) + \left[\rho_S\left(t\right)\widetilde{\sigma_y}\left(t-\tau,t\right),\sigma_y\right]\Lambda_y\left(\tau\right)\right). \tag{1474}$$

As we can see:

$$\left[A_j, \widetilde{A_i}\left(t-\tau,t\right)\rho_S\left(t\right)\right]^{\dagger} = \left[\left(\widetilde{A_i}\left(t-\tau,t\right)\rho_S\left(t\right)\right)^{\dagger}, A_j^{\dagger}\right] \tag{1475}$$

$$= \left[\rho_S^{\dagger}\left(t\right)\widetilde{A_i}^{\dagger}\left(t-\tau,t\right), A_j^{\dagger}\right] \tag{1476}$$

$$= \left[\rho_S\left(t\right)\left(U\left(t\right)U^{\dagger}\left(t-\tau\right)A_i U\left(t-\tau\right)U^{\dagger}\left(t\right)\right)^{\dagger}, A_j\right] \tag{1477}$$

$$= \left[\rho_S\left(t\right)\left(\left(U^{\dagger}\left(t\right)\right)^{\dagger}\left(U\left(t-\tau\right)\right)^{\dagger}A_i^{\dagger}\left(U^{\dagger}\left(t-\tau\right)\right)^{\dagger}\left(U\left(t\right)\right)^{\dagger}\right), A_j\right] \tag{1478}$$

$$= \left[\rho_S\left(t\right)\left(U\left(t\right)U^{\dagger}\left(t-\tau\right)A_i U\left(t-\tau\right)U^{\dagger}\left(t\right)\right), A_j\right] \tag{1479}$$

$$= \left[\rho_S\left(t\right)\widetilde{A_i}\left(t-\tau,t\right)A_j\right]. \tag{1480}$$

So the result obtained is the same master equation (21) of the reference [4] extended in the hermitian conjugate of $\left[\sigma_x, \widetilde{\sigma_x}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\Lambda_x\left(\tau\right) + \left[\sigma_y, \widetilde{\sigma_y}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\Lambda_y\left(\tau\right)$:

$$\Lambda_i^{*}\left(\tau\right) = \Lambda_i\left(\tau\right)\;\; i\in\{x,y\}, \tag{1481}$$

$$\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\,\rho_S(t)\right]\Lambda_x(\tau)\right)^{\dagger} = \left[\rho_S\left(t\right)\widetilde{\sigma_x}\left(t-\tau,t\right),\sigma_x\right]\Lambda_x\left(\tau\right), \tag{1482}$$

$$\left(\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\,\rho_S(t)\right]\Lambda_y(\tau)\right)^{\dagger} = \left[\rho_S\left(t\right)\widetilde{\sigma_y}\left(t-\tau,t\right),\sigma_y\right]\Lambda_y\left(\tau\right), \tag{1483}$$

$$\frac{\partial \overline{\rho_S}\left(t\right)}{\partial t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t)\,,\overline{\rho_S}(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau \Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\,\rho_S(t)\right]\Lambda_x(\tau) + \left[\sigma_y,\right.\right. \tag{1484}$$

$$\left.\widetilde{\sigma_y}(t-\tau,t)\,\rho_S(t)\right]\Lambda_y(\tau) + \left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau,t)\,\rho_S(t)\right]\Lambda_x(\tau)\right)^{\dagger} + \left(\left[\sigma_y, \widetilde{\sigma_y}(t-\tau,t)\right.\right. \tag{1485}$$

$$\left.\left.\times\rho_S\left(t\right)\right]\Lambda_y\left(\tau\right)\right)^{\dagger}\Big). \tag{1486}$$

D. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F\left(\omega\right)=0$, so there is no transformation in this case. As we can see from the definition (415) the only term that survives is $\Lambda_{33}\left(\tau\right)$. Taking $\bar{h}=1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega\left(t\right)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*} b_{\mathbf{k}}\right). \tag{1487}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta=0$. Then the hamiltonian in interest is:

$$H = \frac{\Omega\left(t\right)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*} b_{\mathbf{k}}\right). \tag{1488}$$

We have no transformation so $\overline{\rho_S}\left(t\right) = \rho_S\left(t\right)$ and we can verify:

$$U\left(t\right)\frac{\partial\widetilde{\rho_S}\left(t\right)}{\partial t}U^{\dagger}\left(t\right)=\frac{\partial\rho_S\left(t\right)}{\partial t}+\mathrm{i}\left[H_S\left(t\right),\rho_S\left(t\right)\right]+\mathrm{i}\left[\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{\left(k+1\right)!}\left(\mathrm{ad}_{\mathrm{i}\int_0^t H_S\left(t'\right)\mathrm{d}t'}\right)^k\left(H_S\left(t\right)\right),\rho_S\left(t\right)\right],\tag{1489}$$

$$H_S\left(t\right)=\frac{\Omega\left(t\right)}{2}\left(|1\rangle\langle 0|+|0\rangle\langle 1|\right)\tag{1490}$$

$$=\sigma_x\frac{\Omega\left(t\right)}{2},\tag{1491}$$

$$\int_0^t H_S\left(t'\right)\mathrm{d}t'=\left(|1\rangle\langle 0|+|0\rangle\langle 1|\right)\int_0^t\frac{\Omega\left(t'\right)}{2}\mathrm{d}t'\tag{1492}$$

$$=\sigma_x\int_0^t\frac{\Omega\left(t'\right)}{2}\mathrm{d}t',\tag{1493}$$

$$\left[\int_0^t H_S(t')\,\mathrm{d}t',H_S(t)\right]=\left[\sigma_x\int_0^t\frac{\Omega\left(t'\right)}{2}\mathrm{d}t',\sigma_x\frac{\Omega\left(t\right)}{2}\right]\tag{1494}$$

$$=\frac{\Omega\left(t\right)}{2}\int_0^t\frac{\Omega\left(t'\right)}{2}\mathrm{d}t'\left[\sigma_x,\sigma_x\right]\tag{1495}$$

$$=0,\tag{1496}$$

$$U\left(t\right)\frac{\partial\widetilde{\rho_S}\left(t\right)}{\partial t}U^{\dagger}\left(t\right)=\mathrm{i}\left[H_S(t),\rho_S(t)\right]+\mathrm{i}\left[\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{\left(k+1\right)!}\left(\mathrm{ad}_{\mathrm{i}\int_0^t H_S(t')\mathrm{d}t'}\right)^{k-1}\left(\mathrm{ad}_{\mathrm{i}\int_0^t H_S(t')\mathrm{d}t'}\right)(H_S(t)),\rho_S(t)\right]\tag{1497}$$

$$+\frac{\partial\rho_S\left(t\right)}{\partial t}\tag{1498}$$

$$=\frac{\partial\rho_S\left(t\right)}{\partial t}+\mathrm{i}\left[H_S\left(t\right),\rho_S\left(t\right)\right]+\mathrm{i}\left[\sum_{k=1}^{\infty}\frac{\left(-1\right)^k}{\left(k+1\right)!}\left(\mathrm{ad}_{\mathrm{i}\int_0^t H_S(t')\mathrm{d}t'}\right)^{k-1}0,\rho_S\left(t\right)\right]\tag{1499}$$

$$=\frac{\partial\rho_S\left(t\right)}{\partial t}+\mathrm{i}\left[H_S\left(t\right),\rho_S\left(t\right)\right]+\mathrm{i}\left[0,\rho_S\left(t\right)\right]\tag{1500}$$

$$=\frac{\partial\rho_S\left(t\right)}{\partial t}+\mathrm{i}\left[H_S\left(t\right),\rho_S\left(t\right)\right].\tag{1501}$$

Using the equation (1121) and the precedent equations allow us to write:

$$\frac{\partial\rho_S\left(t\right)}{\partial t}=-\mathrm{i}\left[H_S\left(t\right),\rho_S\left(t\right)\right]-\frac{1}{2}\sum_w\gamma_{33}\left(w,t\right)\left[A_3,A_3\left(w\right)\rho_S\left(t\right)-\rho_S\left(t\right)A_3^{\dagger}\left(w\right)\right]\tag{1502}$$

$$\left.-\sum_w S_{33}\left(w,t\right)\left[A_3,A_3\left(w\right)\rho_S\left(t\right)+\rho_S\left(t\right)A_3^{\dagger}\left(w\right)\right]\right).\tag{1503}$$

The correlation functions are relevant if $F\left(\omega\right)=0$ for the weak-coupling approximation are:

$$\Lambda_{33}\left(\tau\right)=\int_0^{\infty}\mathrm{d}\omega J\left(\omega\right)G_+\left(\tau\right),\tag{1504}$$

$$\Lambda_{33}\left(-\tau\right)=\int_0^{\infty}\mathrm{d}\omega J\left(\omega\right)G_+\left(-\tau\right).\tag{1505}$$

In our case $A_3=\frac{\mathbb{I}+\sigma_z}{2}$, the equation (1503) can be transformed in

$$\frac{\partial\rho_S\left(t\right)}{\partial t}=-\mathrm{i}\left[H_S\left(t\right),\rho_S\left(t\right)\right)]-\sum_w\left(K_{33}\left(w,t\right)\left[A_3,A_3\left(w\right)\rho_S\left(t\right)\right]+K_{33}^{*}\left(w,t\right)\left[\rho_S\left(t\right)A_3\left(w\right),A_3\right]\right).\tag{1506}$$

The relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$. To find the matrices $A_3\left(w\right)$, we have to remember that $H_S=\frac{\Omega(t)}{2}\left(|1\rangle\langle 0|+|0\rangle\langle 1|\right)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_{+} = \frac{\widetilde{\Omega}}{2}, \tag{1507}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right), \tag{1508}$$

$$\lambda_{-} = -\frac{\widetilde{\Omega}}{2}, \tag{1509}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{1510}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1511}$$

$$= \frac{1}{2}, \tag{1512}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} \tag{1513}$$

$$= \frac{1}{2}, \tag{1514}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1515}$$

$$= -\frac{1}{2}. \tag{1516}$$

The decomposition matrices are:

$$A_3\left(0\right) = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \tag{1517}$$

$$= \frac{\mathbb{I}}{2}, \tag{1518}$$

$$A_3\left(\eta\right) = -\frac{1}{2}|-\rangle\langle +| \tag{1519}$$

$$= \frac{1}{4}\left(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}\right), \tag{1520}$$

$$A_3\left(-\eta\right) = -\frac{1}{2}|+\rangle\langle -| \tag{1521}$$

$$= \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right). \tag{1522}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\partial \rho_S\left(t\right)}{\partial t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S\left(t\right)\right)] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S\left(t\right)\right] \tag{1523}$$

$$- K_{33}^{*}\left(\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z + \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right] - K_{33}^{*}\left(-\widetilde{\Omega}, t\right)\left[\rho_S\left(t\right)\frac{1}{4}\left(\sigma_z - \mathrm{i}\sigma_y\right), \frac{\sigma_z}{2}\right]. \tag{1524}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^{\infty}\int_0^{\infty} J\left(\omega\right) G_{+}\left(\tau\right) \mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1525}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right) \mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\left(\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega} + n\left(\omega\right)\mathrm{e}^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1526}$$

$$= \int_0^{\infty}\int_0^{\infty} J\left(\omega\right) \mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau}\left(n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1527}$$

$$= \int_0^\infty \int_0^\infty J(\omega)(n(\omega)+1)\,\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1528}$$

$$= \int_0^\infty J(\omega)(n(\omega)+1)\,\pi\delta\left(\widetilde{\Omega}-\omega\right)\mathrm{d}\omega \tag{1529}$$

$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right)+1\right), \tag{1530}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty \int_0^\infty J(\omega)\,G_+(\tau)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1531}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega}+n(\omega)\,\mathrm{e}^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1532}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau}n(\omega)\,\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1533}$$

$$= \int_0^\infty \int_0^\infty J(\omega)\,n(\omega)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau+\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1534}$$

$$= \int_0^\infty J(\omega)\,n(\omega)\,\pi\delta\left(-\widetilde{\Omega}+\omega\right)\mathrm{d}\omega \tag{1535}$$

$$= \pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right). \tag{1536}$$

Here we have used $\int_0^\infty \mathrm{d}s\,\mathrm{e}^{\pm i\varepsilon s} = \pi\delta(\varepsilon) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1523) lead us to obtain:

$$\frac{\partial\rho_S(t)}{\partial t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x,\rho_S(t)\right)] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\,\rho_S(t)\right]+n\left(\widetilde{\Omega}\right)\left[\sigma_z,(\sigma_z-\mathrm{i}\sigma_y)\,\rho_S(t)\right]\right) \tag{1537}$$

$$-\frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S(t)\,(\sigma_z+\mathrm{i}\sigma_y)\,,\sigma_z\right]+n\left(\widetilde{\Omega}\right)\left[\rho_S(t)\,(\sigma_z-\mathrm{i}\sigma_y)\,,\sigma_z\right]\right). \tag{1538}$$

This is the same result than the equation ($S17$), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\partial\rho_S(t)}{\partial t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x,\rho_S(t)\right)] - \frac{\pi}{8}J(\Omega(t))\left((n(\Omega(t))+1)[\sigma_z,(\sigma_z+\mathrm{i}\sigma_y)\rho_S(t)]+n(\Omega(t))[\sigma_z,(\sigma_z-\mathrm{i}\sigma_y)\,\rho_S(t)]\right) \tag{1539}$$

$$-\frac{\pi}{8}J(\Omega(t))\left((n(\Omega(t))+1)\left[\rho_S(t)\,(\sigma_z+\mathrm{i}\sigma_y)\,,\sigma_z\right]+n(\Omega(t))\left[\rho_S(t)\,(\sigma_z-\mathrm{i}\sigma_y)\,,\sigma_z\right]\right). \tag{1540}$$

V. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality on $\overline{H}(t)$ and $\overline{H_0}(t)$ and its partition function given by $Z(t)$ and $Z_0(t)$ respectively as:

$$Z(t) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}(t)}\right), \tag{1541}$$

$$Z_0(t) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}_0(t)}\right). \tag{1542}$$

where the transformed hamiltonians $\overline{H}(t)$ and $\overline{H_0}(t)$ are defined as:

$$\overline{H}(t) \equiv \overline{H_{\bar{I}}}(t) + \overline{H_0}(t), \tag{1543}$$

$$\overline{H_0}(t) \equiv \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}. \tag{1544}$$

For any operator $A(t)$ we define the expected value respect to $\overline{H_0}(t)$ as:

$$\langle A(t)\rangle_{\overline{H_0}(t)} \equiv \frac{\operatorname{Tr}\left(A(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}. \tag{1545}$$

The terms $\overline{H_{\bar{S}}}(t)$, $\overline{H_{\bar{B}}}$ and $\overline{H_{\bar{I}}}(t)$ are related to the variational transformation performed in [1, 2], this transformation allowed us to construct $\overline{H_{\bar{I}}}(t)$ such that $\langle\overline{H_{\bar{I}}}(t)\rangle_{\overline{H_0}(t)} = 0$. The diagonalization of $\overline{H_0}(t)$ in terms of it's eigenstates and eigenvalues, such that $\overline{H_0}(t)|n\rangle = E_{0,n}(t)|n\rangle$ being $|n\rangle$ an eigenstate of $\overline{H_0}(t)$ with eigenvalue $E_{0,n}(t)$ is $\overline{H_0}(t) = \sum_n E_{0,n}(t)|n\rangle\langle n|$, with $\langle n|n'\rangle = \delta_{nn'}$. A simple form of $\mathrm{e}^{-\beta\overline{H_0}(t)}$ can be found as follows:

$$\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\mathrm{e}^{rY}\mathrm{e}^{-\frac{r^2}{2}[X,Y]}\mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\cdots \text{ (Zassenhaus formula)}, \tag{1546}$$

$$\mathrm{e}^{X+Y} = \mathrm{e}^{X}\mathrm{e}^{Y}\mathrm{e}^{-\frac{1^2}{2}0}\mathrm{e}^{\frac{1^3}{6}(2[Y,0]+[X,0])}\cdots \text{ (setting } r=1 \text{ and } [X,Y]=0 \text{ in (1546))} \tag{1547}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}\mathbb{I} \tag{1548}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}, \tag{1549}$$

$$\mathrm{e}^{-\beta\overline{H_0}(t)} = \mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by the diagonalization of } \overline{H_0}(t)) \tag{1550}$$

$$= \prod_n \mathrm{e}^{-\beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by (1549) and } [|n\rangle\langle n|, |n'\rangle\langle n'|] = 0) \tag{1551}$$

$$= \prod_n \sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}(t)|n\rangle\langle n|\right)^j}{j!} \text{ (by the exponential formula)} \tag{1552}$$

$$= \prod_n \left(\mathbb{I} + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j |n\rangle\langle n|}{j!}\right) \text{ (using } (aA)^j = a^j A^j \text{ and } (|n\rangle\langle n|)^2 = |n\rangle\langle n|) \tag{1553}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j |n\rangle\langle n|}{j!}\right) \tag{1554}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left(\sum_{j=0}^{\infty} \frac{\left(-\beta E_{0,n}(t)\right)^j}{j!}\right)\right) \text{ (introducing } |n\rangle\langle n| \text{ inside the sum)} \tag{1555}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \text{ (by the exponential formula)} \tag{1556}$$

$$= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right). \tag{1557}$$

We will prove by induction a neat form for (1557), we will show that:

$$\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \mathbb{I} + \sum_{j=1}^{n} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \tag{1558}$$

For $n=1$ the formula is trivial, in the case $n=2$ we obtain that:

$$\prod_{j=1}^{2} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2|\right) \tag{1559}$$

$$= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1| + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2| \text{ (by } \langle i|j\rangle = \delta_{ij}) \tag{1560}$$

$$= \mathbb{I} + \sum_{j=1}^{2} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \tag{1561}$$

It was our case base, our induction step is (1558). In the case $n+1$ we will have:

$$\prod_{j=1}^{n+1}\left(\mathbb{I}+\left(\mathrm{e}^{-\beta E_{0,j}(t)}-1\right)|j\rangle\langle j|\right)=\left(\prod_{j=1}^{n}\left(\mathbb{I}+\left(\mathrm{e}^{-\beta E_{0,j}(t)}-1\right)|j\rangle\langle j|\right)\right)\left(\mathbb{I}+\left(\mathrm{e}^{-\beta E_{0,n+1}(t)}-1\right)|n+1\rangle\langle n+1|\right) \tag{1562}$$

$$=\left(\mathbb{I}+\sum_{n}\left(\mathrm{e}^{-\beta E_{0,n}(t)}-1\right)|n\rangle\langle n|\right)\left(\mathbb{I}+\left(\mathrm{e}^{-\beta E_{0,n+1}(t)}-1\right)|n+1\rangle\langle n+1|\right) \text{ (by induction step)} \tag{1563}$$

$$=\mathbb{I}+\left(\mathrm{e}^{-\beta E_{0,n+1}(t)}-1\right)|n+1\rangle\langle n+1|+\sum_{n}\left(\mathrm{e}^{-\beta E_{0,n}(t)}-1\right)|n\rangle\langle n| \text{ (by } \langle i|j\rangle=\delta_{ij}) \tag{1564}$$

$$=\mathbb{I}+\sum_{j=1}^{n+1}\left(\mathrm{e}^{-\beta E_{0,j}(t)}-1\right)|j\rangle\langle j|. \tag{1565}$$

By mathematical induction we proved that (1558) is true for all $n\in\mathbb{N}$. Given that the resolution of the identity is $\mathbb{I}=\sum_n |n\rangle\langle n|$ then we find that:

$$\mathrm{e}^{-\beta\overline{H_0}(t)}=\prod_{n}\left(\mathbb{I}+\left(\mathrm{e}^{-\beta E_{0,n}(t)}-1\right)|n\rangle\langle n|\right) \tag{1566}$$

$$=\mathbb{I}+\sum_{n}\left(\mathrm{e}^{-\beta E_{0,n}(t)}-1\right)|n\rangle\langle n| \text{ (by (1558))} \tag{1567}$$

$$=\mathbb{I}+\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|-\sum_{n}|n\rangle\langle n| \text{ (separating the terms of the sum)} \tag{1568}$$

$$=\cancel{\mathbb{I}}+\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|-\cancel{\mathbb{I}} \text{ (by the resolution of the identity } \mathbb{I}=\sum_{n}|n\rangle\langle n|) \tag{1569}$$

$$=\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|. \tag{1570}$$

The partition function $Z_0(t)$ is equal to:

$$Z_0(t)=\mathrm{Tr}\left(\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \text{ (by (1570))} \tag{1571}$$

$$=\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(|n\rangle\langle n|\right) \text{ (by trace linearity)} \tag{1572}$$

$$=\sum_{n}\mathrm{e}^{-\beta E_{0,n}(t)} \text{ (because } \mathrm{Tr}\left(|n\rangle\langle n|\right)=1). \tag{1573}$$

The explicit form of the average value $\langle A(t)\rangle_{\overline{H_0}(t)}$ can be found from the partition function $Z_0(t)$:

$$\langle A(t)\rangle_{\overline{H_0}(t)}=\frac{\mathrm{Tr}\left(A(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{Z_0(t)} \text{ (by (1545))} \tag{1574}$$

$$=\frac{\mathrm{Tr}\left(\sum_n A(t)\,\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \text{ (by (1570))} \tag{1575}$$

$$=\frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A(t)\,|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)} \text{ (by commutativity in scalar product)} \tag{1576}$$

$$=\frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A(t)\,|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} \text{ (by (1573))} \tag{1577}$$

$$=\frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(A(t)\,|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} \text{ (by trace linearity).} \tag{1578}$$

At first we show a double sequence of inequalities of order M,N which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$Z\left(t\right) \geq Z_0\left(t\right) \mathrm{e}^{-\left\langle \overline{H_{\overline{I}}}\left(t\right)\right\rangle_{\overline{H_0\left(t\right)}}} \left(1 + F_M\left(\overrightarrow{u}\left(t\right);\alpha\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right);\alpha\right)\right). \tag{1579}$$

where the function $F_N\left(\overrightarrow{u}\left(t\right);\alpha\right)$ is defined as:

$$F_N\left(\overrightarrow{w}\left(t\right);\alpha\right) \equiv \mathrm{e}^{-\alpha} \sum_{k=2}^{2N-1} \left(-\beta\right)^k \frac{w_k\left(t\right)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!}. \tag{1580}$$

In this case α is a parameter that can be optimized, $\beta \equiv \frac{1}{\mathrm{k_B}T}$, $\overrightarrow{w}\left(t\right)$ is a vector such that $\overrightarrow{w}\left(t\right) = \left(w_1, w_2, ...\right)$ and $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian $\overline{H_{\overline{I}D}}\left(t\right)$ respect to the basis of eigenstates of $\overline{H_0}\left(t\right)$ as:

$$\overline{H_{\overline{I}D}}\left(t\right) \equiv \sum_n \left\langle n \left| \overline{H_{\overline{I}}}\left(t\right) \right| n \right\rangle |n\rangle\langle n|. \tag{1581}$$

We will prove an important property related to $\overline{H_{\overline{I}D}}\left(t\right)$ which is a Hamiltonian written as a linear combination of a set of orthonormal operators. Let's consider a ring R with two operations $+$ and $\cdot$, if there exist $a, b \in R$ such that $a \cdot b = 0$ and $b \cdot a = 0$ then for any $k \in \mathbb{N}$ we have $\left(a+b\right)^k = a^k + b^k$ where $a^k = a^{k-1} \cdot a$ is a recursive definition of the power of an element written in terms of $\cdot$. At first we prove that this result yields for any $k \in \mathbb{N}$ by induction, the case $k = 1$ is trivial so we will focus on the case $k = 2$, we have that:

$$\begin{aligned}
\left(a+b\right)^2 &= \left(a+b\right) \cdot \left(a+b\right) \text{ (by definition of the power of an element)} && (1582)\\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b \text{ (by distributive multiplication respect addition)} && (1583)\\
&= a^2 + a \cdot b + b \cdot a + b^2 \text{ (by definition of the power of an element)} && (1584)\\
&= a^2 + 0 + 0 + b^2 \text{ (because } a \cdot b = b \cdot a = 0\text{)} && (1585)\\
&= a^2 + b^2. && (1586)
\end{aligned}$$

It was the base case. By induction step we will consider that $\left(a+b\right)^k = a^k + b^k$ with $k \geq 2$, now for $k+1$ we will have that:

$$\begin{aligned}
\left(a+b\right)^{k+1} &= \left(a+b\right)^k \cdot \left(a+b\right) \text{ (by definition of the power of an element)} && (1587)\\
&= \left(a^k + b^k\right) \cdot \left(a+b\right) \text{ (by induction step)} && (1588)\\
&= a^k \cdot a + a^k \cdot b + b^k \cdot a + b^k \cdot b \text{ (by distributive multiplication respect addition)} && (1589)\\
&= a^{k+1} + a^{k-1} \cdot a \cdot b + b^{k-1} \cdot b \cdot a + b^{k+1} \text{ (by recursive definition of } a^k\text{)} && (1590)\\
&= a^{k+1} + a^{k-1} \cdot \left(a \cdot b\right) + b^{k-1} \cdot \left(b \cdot a\right) + b^{k+1} \text{ (by associativity on } R \text{ respect } \cdot\text{)} && (1591)\\
&= a^{k+1} + a^{k-1} \cdot \left(0\right) + b^{k-1} \cdot \left(0\right) + b^{k+1} \text{ (because } a \cdot b = b \cdot a = 0\text{)} && (1592)\\
&= a^{k+1} + b^{k+1}. && (1593)
\end{aligned}$$

By the principle of mathematical induction we can conclude that the proposition is true for all $k \in \mathbb{N}$. Now we will extend the result, let $a_1, ..., a_n \in R$ such that $a_i \cdot a_j = 0$ for all $i \neq j$ then $\left(a_1 + ... + a_n\right)^k = a_1^k + ... + a_n^k$. The case $n = 1$ is trivial as well so we will focus on $n = 2$, this case was proved in the precedent lines so it will be our base case. By induction step we will consider that $\left(a_1 + ... + a_n\right)^k = a_1^k + ... + a_n^k$ with $n \geq 2$, now for $n+1$ we will have that:

$$\begin{aligned}
a_{n+1} \cdot \left(a_1 + ... + a_n\right) &= a_{n+1} \cdot a_1 + ... + a_{n+1} \cdot a_n \text{ (by distributive multiplication respect addition)} && (1594)\\
&= 0 \text{ (because } a_i \cdot a_j = 0 \text{ for all } i \neq j \text{),} && (1595)\\
\left(a_1 + ... + a_n + a_{n+1}\right)^k &= \left(\left(a_1 + ... + a_n\right) + a_{n+1}\right)^k \text{ (by associative property of +)} && (1596)\\
&= \left(a_1 + ... + a_n\right)^k + a_{n+1}^k \text{ (by (1593) and (1595))} && (1597)\\
&= a_1^k + ... + a_n^k + a_{n+1}^k \text{ (by inductive step).} && (1598)
\end{aligned}$$

So we can conclude by mathematical induction that the proposition is true for all $n \in \mathbb{N}$. We can prove the following property for $\left(\overline{H_{\overline{I}}}_D(t)\right)^k$:

$$\left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle |n\rangle\langle n| \left\langle n'\left|\overline{H_{\overline{I}}}(t)\right|n'\right\rangle |n'\rangle\langle n'| = \left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle \left\langle n'\left|\overline{H_{\overline{I}}}(t)\right|n'\right\rangle |n\rangle\langle n|n'\rangle\langle n'| \tag{1599}$$

$$= \left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle \left\langle n'\left|\overline{H_{\overline{I}}}(t)\right|n'\right\rangle |n\rangle\langle n'|\delta_{nn'} \text{ (by } \delta \text{ properties)}, \tag{1600}$$

$$\left(\overline{H_{\overline{I}}}_D(t)\right)^k = \left(\sum_n \left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle |n\rangle\langle n|\right)^k \text{ (by (1581))} \tag{1601}$$

$$= \sum_n \left(\left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle |n\rangle\langle n|\right)^k \text{ (by (1598) and (1600))}, \tag{1602}$$

$$(aA)^k = a^k A^k \text{ (by the property of the power of a matrix)}, \tag{1603}$$

$$(|n\rangle\langle n|)^k = |n\rangle\langle n| \text{ (because } |n\rangle\langle n| \text{ is a projector and } k \in \mathbb{N}^*\text{)}, \tag{1604}$$

$$\left(\overline{H_{\overline{I}}}_D(t)\right)^k = \sum_n \left(\left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle\right)^k |n\rangle\langle n| \text{ (by (1603) and (1604))}. \tag{1605}$$

The vectors $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ are defined as $\overrightarrow{u}(t) \equiv (u_1(t), u_2(t), ...)$ and $\overrightarrow{v}(t) \equiv (v_1(t), v_2(t), ...)$. We can define the elements of $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ in terms of the matrix $\overline{H_{\overline{I}}}_D(t)$:

$$u_k(t) \equiv \left\langle \left(\overline{H_{\overline{I}}}_D(t) - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k \right\rangle_{\overline{H_0(t)}} \tag{1606}$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \mathrm{Tr}\left(\left(\sum_n \left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle |n\rangle\langle n| - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k |n\rangle\langle n|\right)}{Z_0(t)} \text{(by (1578))}, \tag{1607}$$

$$\left(\sum_n \langle n|\overline{H_{\overline{I}}}(t)|n\rangle |n\rangle\langle n| - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\sum_n \langle n|\overline{H_{\overline{I}}}(t)|n\rangle |n\rangle\langle n|\right)^j \left(\left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k-j} \text{(by binomial theorem)} \tag{1608}$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\sum_n \left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle^j |n\rangle\langle n|\right) \left(\left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k-j} \text{ (by (1605))} \tag{1609}$$

$$= \sum_n \left(\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle^j \left(\left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^{k-j}\right) |n\rangle\langle n| \text{ (exchange of } \textstyle\sum) \tag{1610}$$

$$= \sum_n \left(\left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k |n\rangle\langle n| \text{ (by binomial theorem)}, \tag{1611}$$

$$u_k(t) = \frac{\sum_n e^{-\beta E_{0,n}(t)} \mathrm{Tr}\left(\sum_{n'} \left(\left\langle n'\left|\overline{H_{\overline{I}}}(t)\right|n'\right\rangle - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k |n'\rangle\langle n'|n\rangle\langle n|\right)}{Z_0(t)} \tag{1612}$$

$$= \frac{\sum_{nn'} e^{-\beta E_{0,n}(t)} \mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{\overline{I}}}(t)\right|n'\right\rangle - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k |n'\rangle\langle n|\langle n'|n\rangle\right)}{Z_0(t)} \tag{1613}$$

$$= \frac{\sum_{nn'} e^{-\beta E_{0,n}(t)} \mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{\overline{I}}}(t)\right|n'\right\rangle - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k |n'\rangle\langle n|\delta_{nn'}\right)}{Z_0(t)} \tag{1614}$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left(\left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k \mathrm{Tr}(|n\rangle\langle n|)}{Z_0(t)} \text{ (by } \delta \text{ properties)} \tag{1615}$$

$$= \frac{\sum_n e^{-\beta E_{0,n}(t)} \left(\left\langle n\left|\overline{H_{\overline{I}}}(t)\right|n\right\rangle - \left\langle \overline{H_{\overline{I}}}(t)\right\rangle_{\overline{H_0(t)}}\right)^k 1}{Z_0(t)} \text{ (by } \mathrm{Tr}(|n\rangle\langle n|) = 1) \tag{1616}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left(\langle n \left|\overline{H_I}(t)\right| n\rangle - \langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)}\right)^k}{Z_0(t)}, \tag{1617}$$

$$v_k(t) \equiv \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t) - \langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)}\right)^k \right| n \right\rangle}{Z_0(t)}. \tag{1618}$$

By construction $\langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)} = 0$, so we summarize the double inequality that generalizes the Bogoliubov inequality and it's coefficients as:

$$Z(t) \geq Z_0(t)\left(1 + F_M(\overrightarrow{u}(t)) + F_N(\overrightarrow{v}(t) - \overrightarrow{u}(t))\right), \tag{1619}$$

$$Z(t) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}(t)}\right), \tag{1620}$$

$$Z_0(t) = \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}, \tag{1621}$$

$$F_N(\overrightarrow{u}(t); \alpha) = \mathrm{e}^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!}, \tag{1622}$$

$$u_k(t) = \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \langle n \left|\overline{H_I}(t)\right| n\rangle^k}{Z_0(t)}, \tag{1623}$$

$$v_k(t) = \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)\right)^k \right| n \right\rangle}{Z_0(t)}. \tag{1624}$$

As we can see the expression (1623) was written in shorter terms, we want to do the same for (1624) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for $v_k(t)$:

$$\left(\overline{H_0}(t) - E_{0,n}(t)\right)|n\rangle = \overline{H_0}(t)|n\rangle - E_{0,n}(t)|n\rangle \text{ (by distributive property)} \tag{1625}$$

$$= E_{0,n}(t)|n\rangle - E_{0,n}(t)|n\rangle \text{ (by } \overline{H_0}(t)|n\rangle = E_{0,n}(t)|n\rangle) \tag{1626}$$

$$= 0, \tag{1627}$$

$$\langle n|\left(\overline{H_0}(t) - E_{0,n}\right) = \langle n|\overline{H_0}(t) - \langle n|E_{0,n}(t) \text{ (by distributive property)} \tag{1628}$$

$$= \langle n|E_{0,n}(t) - \langle n|E_{0,n}(t) \text{ (by } \langle n|\overline{H_0}(t) = \langle n|E_{0,n}(t)) \tag{1629}$$

$$= 0. \tag{1630}$$

At first we calculated $v_1(t)$ using the definition (1624) :

$$v_1(t) = \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \langle n \left|\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)\right| n\rangle \text{ (by (1624))} \tag{1631}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \langle n \left|\overline{H_0}(t) - E_{0,n}(t)\right| n\rangle + \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \langle n \left|\overline{H_I}(t)\right| n\rangle \text{ (Distributive law)} \tag{1632}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left(\langle n \left|\overline{H_0}(t)\right| n\rangle - \langle n \left|E_{0,n}(t)\right| n\rangle\right) + \langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)} \text{ (by (1545))} \tag{1633}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left(\langle n \left|E_{0,n}(t)\right| n\rangle - \langle n \left|E_{0,n}(t)\right| n\rangle\right) + \langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)} \text{ (by } \overline{H_0}(t)|n\rangle = E_{0,n}(t)|n\rangle) \tag{1634}$$

$$= 0 + \langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)} \tag{1635}$$

$$= 0 \text{ (by construction } \langle\overline{H_I}(t)\rangle_{\overline{H_0}(t)} = 0). \tag{1636}$$

For $k \geq 2$ and $k \in \mathbb{N}$ we calculated:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^k \right| n \right\rangle \tag{1637}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{1638}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{1639}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{1640}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle . \tag{1641}$$

In general we can write a formula for $v_k(t)$ that implies an expected value of a dependent expression of $\overline{H_{\overline{I}}}(t)$ and $\overline{H_0}(t)$:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{1642}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) + \overline{H_{\overline{I}}}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{1643}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \text{ (by (1543))} \tag{1644}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j}(t)\, E_{0,n}^j(t)\right) \overline{H_{\overline{I}}}(t) \right| n \right\rangle \text{ (by binomial theorem)} \tag{1645}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t) \overline{H_{\overline{I}}}(t) E_{0,n}^j(t) \right| n \right\rangle \text{ (exchange } \Sigma \text{ and } \langle n | \cdots | n \rangle) \tag{1646}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \text{ (by } E_{0,n}(t)\,|n\rangle = \overline{H_0}(t)\,|n\rangle) \tag{1647}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{1648}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \text{ (by (1545))} \tag{1649}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \text{ (rewriting using (1543)).} \tag{1650}$$

The formula (1650) is well defined taking as example $k = 2, 3$.

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{2-2-j} \overline{H_{\overline{I}}}(t)\; \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \tag{1651}$$

$$= (-1)^0 \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\; \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \tag{1652}$$

$$= \left\langle \overline{H_{\overline{I}}}^2(t) \right\rangle_{\overline{H_0}(t)} . \tag{1653}$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{3-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{1654}$$

$$= \left\langle \sum_{j=0}^{1} (-1)^j \binom{1}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{1-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{1655}$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}(t)+\overline{H_0}(t)\right)^1 \overline{H_{\overline{I}}}(t)\,\overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}(t)+\overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\,\overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \tag{1656}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right) \overline{H_{\overline{I}}}(t)\,\mathbb{I} - \overline{H_{\overline{I}}}(t)\,\mathbb{I}\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{1657}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right) \overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{1658}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{1659}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t)\left(\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\right) \right\rangle_{\overline{H_0}(t)} \tag{1660}$$

$$= \left\langle \overline{H_{\overline{I}}}(t)^3 + \overline{H_{\overline{I}}}(t)\left[\overline{H_0}(t), \overline{H_{\overline{I}}}(t)\right] \right\rangle_{\overline{H_0}(t)} \ \text{(because } \left[\overline{H_0}(t), \overline{H_{\overline{I}}}(t)\right] = \overline{H_0}(t)\,\overline{H_{\overline{I}}}(t) - \overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\text{).} \tag{1661}$$

So we summarize:

$$\overline{H_{\overline{I}D}}(t) = \sum_n \left\langle n \left| \overline{H_{\overline{I}}}(t) \right| n \right\rangle |n\rangle\langle n|, \tag{1662}$$

$$u_k(t) = \left\langle \left(\overline{H_{\overline{I}D}}(t)\right)^k \right\rangle_{\overline{H_0}(t)}, \tag{1663}$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{1664}$$

The free energy $E_{\text{free}}(t)$ and free energy $E_{\text{free},1}(t)$ at first order are respectively:

$$E_{\text{free}}(t) \equiv -\frac{1}{\beta}\ln(Z(t)), \tag{1665}$$

$$E_{\text{free},1}(t) \equiv -\frac{1}{\beta}\ln(Z_0(t)). \tag{1666}$$

It is well-known that the function $f(x) = -\ln(x)$ is a decreasing function so we can transform (1579):

$$E_{\text{free}}(t) = -\frac{1}{\beta}\ln(Z(t)) \ \text{(by (1665))} \tag{1667}$$

$$\leq -\frac{1}{\beta}\ln\left(Z_0(t)\left(1 + F_M\left(\overrightarrow{u}(t);\alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t);\alpha\right)\right)\right) \tag{1668}$$

$$= -\frac{1}{\beta}\ln(Z_0(t)) - \frac{1}{\beta}\ln\left(1 + F_M\left(\overrightarrow{u}(t);\alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t);\alpha\right)\right) \tag{1669}$$

$$= E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_M\left(\overrightarrow{u}(t);\alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t);\alpha\right)\right) \ \text{(by (1666))} \tag{1670}$$

$$\equiv E_{\text{free,MN}}(t). \tag{1671}$$

here $E_{\text{free,MN}}(t)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of M, N order. In our approach we will set $N = M$, so the inequality (1671) of N, N order is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{u}(t);\alpha\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t);\alpha\right)\right) \tag{1672}$$

$$= E_{\text{free,NN}}(t). \tag{1673}$$

A weaker form of the inequality (1673) is obtained making $\overrightarrow{u}(t) = 0$ as suggest [3]:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{v}(t);\alpha\right)\right) \tag{1674}$$

$$\equiv E_{\text{free,N}}(t). \tag{1675}$$

The algebraic equation associated with $\alpha_{\text{opt}}(t)$ such that $E_{\text{free,N}}(t)$ is closer to $E_{\text{free}}(t)$ follows from the fact that in the optimal parameter $\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha}|_{\alpha=\alpha_{\text{opt}(t)}} = 0$, calculating this derivative we have:

$$\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1 + F_N\left(\overrightarrow{v}(t);\alpha\right)\right)\right) \tag{1676}$$

$$= -\frac{1}{\beta}\frac{\frac{\partial}{\partial\alpha}\left(F_N\left(\overrightarrow{v}(t);\alpha\right)\right)}{1 + F_N\left(\overrightarrow{v}(t);\alpha\right)} \tag{1677}$$

$$= 0. \tag{1678}$$

The precedent equation is equivalent to make the numerator equal to 0:

$$\frac{\partial F_N(\overrightarrow{v}(t);\alpha)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{1679}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\partial}{\partial\alpha}\frac{\alpha^i}{i!}\text{(by product rule)} \tag{1680}$$

$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} \tag{1681}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{1682}$$

$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{j=0}^{2N-2-k}\frac{\alpha^j}{j!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right)\text{ (setting } j = i-1) \tag{1683}$$

$$= \mathrm{e}^{-\alpha}\left(-\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha^{2N-1-k}}{(2N-1-k)!}\right)\text{(do the substraction leaving } i = 2N-1-k) \tag{1684}$$

$$= 0. \tag{1685}$$

Then the optimal value $\alpha_{\text{opt}}(t)$ will satisfy the following equation:

$$G\left(\alpha_{\text{opt}}(t)\right) \equiv \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \tag{1686}$$

$$= 0. \tag{1687}$$

The elements presented are the required to find variational parameters of the system using the inequality (1675) and the self consistent equation (SCE) (1686) to a particular order required.

VI. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2(t), v_3(t), v_4(t), v_5(t)$ using the (1664) because the order $N = 3$ requires to obtain the elements $v_k(t)$ until $k = 2N-1 = 5$. We already have $v_2(t), v_3(t)$, so we will find $v_4(t)$ and $v_5(t)$:

$$v_4(t)=\sum_{j=0}^{4-2}(-1)^j\binom{4-2}{j}\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^{4-2-j}\overline{H_I}(t)\,\overline{H_0}^{\,j}(t)\right\rangle_{\overline{H_0}(t)}\tag{1688}$$

$$=\sum_{j=0}^{2}(-1)^j\binom{2}{j}\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^{2-j}\overline{H_I}(t)\,\overline{H_0}^{\,j}(t)\right\rangle_{\overline{H_0}(t)}\tag{1689}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\,\overline{H_0}^{\,0}(t)\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^1\overline{H_I}(t)\,\overline{H_0}^{\,1}(t)\right\rangle_{\overline{H_0}(t)}+\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)\right.\right.\tag{1690}$$

$$\left.\left.+\overline{H_0}(t)\right)^0\overline{H_I}(t)\,\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1691}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\,\mathbb{I}\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^1\overline{H_I}(t)\,\overline{H_0}^{\,1}(t)\right\rangle_{\overline{H_0}(t)}+\left\langle\overline{H_I}^{\,2}(t)\,\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1692}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\right\rangle_{\overline{H_0}(t)}-2\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t)\right\rangle_{\overline{H_0}(t)}+\left\langle\overline{H_I}^{\,2}(t)\,\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1693}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)-2\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}^{\,2}(t)\,\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1694}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}^{\,2}(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_0}^{\,2}(t)\right)\overline{H_I}(t)-2\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}^{\,2}(t)\right.\tag{1695}$$

$$\left.\times\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1696}$$

$$=\left\langle\overline{H_I}^{\,4}(t)+\overline{H_I}^{\,2}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}^{\,2}(t)+\overline{H_I}(t)\,\overline{H_0}^{\,2}(t)\,\overline{H_I}(t)-2\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t)\right.\tag{1697}$$

$$\left.+\overline{H_I}^{\,2}(t)\,\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1698}$$

$$=\left\langle\overline{H_I}^{\,4}(t)+\overline{H_I}^{\,2}(t)\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^{\,2}(t)+\overline{H_I}(t)\overline{H_0}^{\,2}(t)\,\overline{H_I}(t)-2\overline{H_I}^{\,3}(t)\overline{H_0}(t)-2\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\right.\tag{1699}$$

$$\left.+\overline{H_I}^{\,2}(t)\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1700}$$

$$=\left\langle\overline{H_I}^{\,4}(t)+\overline{H_I}^{\,2}(t)\overline{H_0}(t)\,\overline{H_I}(t)+\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^{\,2}(t)+\overline{H_I}(t)\overline{H_0}^{\,2}(t)\,\overline{H_I}(t)-\overline{H_I}^{\,3}(t)\overline{H_0}(t)-\overline{H_I}^{\,3}(t)\overline{H_0}(t)+\overline{H_I}^{\,2}(t)\right.\tag{1701}$$

$$\left.\times\overline{H_0}^{\,2}(t)-\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\right\rangle_{\overline{H_0}(t)}\tag{1702}$$

$$=\left\langle\overline{H_I}^{\,4}(t)+\overline{H_I}^{\,2}(t)\overline{H_0}(t)\overline{H_I}(t)-\overline{H_I}^{\,3}(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}(t)\,\overline{H_I}^{\,2}(t)-\overline{H_I}^{\,3}(t)\overline{H_0}(t)+\overline{H_I}(t)\overline{H_0}^{\,2}(t)\,\overline{H_I}(t)-\overline{H_I}(t)\right.\tag{1703}$$

$$\left.\times\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_I}^{\,2}(t)\,\overline{H_0}^{\,2}(t)-\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}(t)\right\rangle_{\overline{H_0}(t)}\ \text{(rewriting (1702))}\tag{1704}$$

$$=\left\langle\overline{H_I}^{\,4}(t)+\overline{H_I}(t)\Big(\Big(\left(\overline{H_I}(t)\overline{H_0}(t)\right)\overline{H_I}(t)-\overline{H_I}(t)\left(\overline{H_I}(t)\overline{H_0}(t)\right)\Big)+\left(\overline{H_0}(t)\overline{H_I}^{\,2}(t)-\overline{H_I}^{\,2}(t)\overline{H_0}(t)\right)+\Big(\overline{H_0}(t)\Big(\overline{H_0}(t)\right.\tag{1705}$$

$$\left.\times\overline{H_I}(t)\Big)-\left(\overline{H_0}(t)\,\overline{H_I}(t)\right)\overline{H_0}(t)\Big)+\Big(\left(\overline{H_I}(t)\,\overline{H_0}(t)\right)\overline{H_0}(t)-\overline{H_0}(t)\left(\overline{H_I}(t)\,\overline{H_0}(t)\right)\Big)\Big)\right\rangle_{\overline{H_0}(t)}\tag{1706}$$

$$=\left\langle\overline{H_I}^{\,4}(t)+\overline{H_I}(t)\Big(\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_I}(t)\right]+\left[\overline{H_0}(t),\overline{H_I}^{\,2}(t)\right]+\left[\overline{H_0}(t),\overline{H_0}(t)\overline{H_I}(t)\right]+\left[\overline{H_I}(t)\overline{H_0}(t),\overline{H_0}(t)\right]\Big)\right\rangle_{\overline{H_0}(t)},\tag{1707}$$

$$v_5(t)=\sum_{j=0}^{5-2}(-1)^j\binom{5-2}{j}\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^{5-2-j}\overline{H_I}(t)\,\overline{H_0}^{\,j}(t)\right\rangle_{\overline{H_0}(t)}\tag{1708}$$

$$=\sum_{j=0}^{3}(-1)^j\binom{3}{j}\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^{3-j}\overline{H_I}(t)\,\overline{H_0}^{\,j}(t)\right\rangle_{\overline{H_0}(t)}\tag{1709}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)\overline{H_0}^{\,0}(t)-3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\overline{H_0}(t)+3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^{\,2}(t)\right.\tag{1710}$$

$$\left.-\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^0\overline{H_I}(t)\,\overline{H_0}^{\,3}(t)\right\rangle_{\overline{H_0}(t)}\tag{1711}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)-3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^2\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^{\,2}(t)\right.\tag{1712}$$

$$\left.-\overline{H_I}^{\,2}(t)\,\overline{H_0}^{\,3}(t)\right\rangle_{\overline{H_0}(t)}\tag{1713}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)-3\overline{H_I}(t)\left(\overline{H_I}^{\,2}(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)+\overline{H_0}^{\,2}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t)+3\overline{H_I}(t)\right.\tag{1714}$$

$$\left.\times\left(\overline{H_I}(t)+\overline{H_0}(t)\right)\overline{H_I}(t)\,\overline{H_0}^{\,2}(t)-\overline{H_I}^{\,2}(t)\,\overline{H_0}^{\,3}(t)\right\rangle_{\overline{H_0}(t)}\tag{1715}$$

$$=\left\langle\overline{H_I}(t)\left(\overline{H_I}(t)+\overline{H_0}(t)\right)^3\overline{H_I}(t)-3\overline{H_I}(t)\left(\overline{H_I}^{\,2}(t)+\overline{H_I}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)+\overline{H_0}^{\,2}(t)\right)\overline{H_I}(t)\,\overline{H_0}(t)-\overline{H_I}^{\,2}(t)\right.\tag{1716}$$

$$\left.\times\overline{H_0}^{\,3}(t)+3\overline{H_I}^{\,3}(t)\,\overline{H_0}^{\,2}(t)+3\overline{H_I}(t)\,\overline{H_0}(t)\,\overline{H_I}(t)\,\overline{H_0}^{\,2}(t)\right\rangle_{\overline{H_0}(t)}\tag{1717}$$

$$
\begin{aligned}
&=\Big\langle\overline{H_{\bar{I}}}(t)\Big(\overline{H_{\bar{I}}}^{3}(t)+\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)+\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)+\overline{H_{\bar{I}}}(t) \qquad (1718)\\
&\quad\times\overline{H_0}^{2}(t)+\overline{H_0}^{3}(t)\Big)\,\overline{H_{\bar{I}}}(t)-3\overline{H_{\bar{I}}}(t)\Big(\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_0}^{2}(t)\Big)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)+3\overline{H_{\bar{I}}}(t)\big(\overline{H_{\bar{I}}}(t) \qquad (1719)\\
&\quad+\overline{H_0}(t)\big)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}^{2}(t)-\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}^{3}(t)\Big\rangle_{\overline{H_0}(t)} \qquad (1720)\\
&=\Big\langle\overline{H_{\bar{I}}}^{5}(t)+\overline{H_{\bar{I}}}^{3}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{3}(t)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}^{2}(t)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t) \qquad (1721)\\
&\quad\times\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}^{3}(t)\,\overline{H_{\bar{I}}}(t)-3\overline{H_{\bar{I}}}^{4}(t)\,\overline{H_0}(t)-3\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)-3\overline{H_{\bar{I}}}(t) \qquad (1722)\\
&\quad\times\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)-3\overline{H_{\bar{I}}}(t)\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)\overline{H_0}(t)+3\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}^{2}(t)+3\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\overline{H_0}^{2}(t)-\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}^{3}(t)\Big\rangle_{\overline{H_0}(t)} \qquad (1723)\\
&=\Big\langle\overline{H_{\bar{I}}}^{5}(t)+\overline{H_{\bar{I}}}(t)\Big(\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)+\overline{H_0}(t)\overline{H_{\bar{I}}}^{3}(t)+\overline{H_0}^{2}(t)\overline{H_{\bar{I}}}^{2}(t)+\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t) \qquad (1724)\\
&\quad+\overline{H_{\bar{I}}}(t)\,\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)+\overline{H_0}^{3}(t)\,\overline{H_{\bar{I}}}(t)-3\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)-3\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)-3\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)-3\overline{H_0}^{2}(t) \qquad (1725)\\
&\quad\times\overline{H_{\bar{I}}}(t)\overline{H_0}(t)+3\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}^{2}(t)+3\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}^{2}(t)-\overline{H_{\bar{I}}}(t)\,\overline{H_0}^{3}(t)\Big)\Big\rangle_{\overline{H_0}(t)} \qquad (1726)\\
&=\Big\langle\overline{H_{\bar{I}}}^{5}(t)+\overline{H_{\bar{I}}}(t)\Big(\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)-\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_{\bar{I}}}^{3}(t)-\overline{H_{\bar{I}}}^{3}(t)\overline{H_0}(t) \qquad (1727)\\
&\quad+\overline{H_0}(t)\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}(t)\overline{H_0}(t)+\overline{H_0}^{2}(t)\overline{H_{\bar{I}}}^{2}(t)-\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)+\overline{H_0}^{3}(t)\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}(t)\overline{H_0}^{3}(t) \qquad (1728)\\
&\quad+\overline{H_{\bar{I}}}(t)\overline{H_0}^{2}(t)\overline{H_{\bar{I}}}(t)-\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)+2\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}^{2}(t)-2\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)+3\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}^{2}(t)-3 \qquad (1729)\\
&\quad\times\overline{H_0}^{2}(t)\,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)+\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}^{2}(t)-\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\Big)\Big\rangle_{\overline{H_0}(t)}\ \text{(rewriting (1726))} \qquad (1730)\\
&=\Big\langle\overline{H_{\bar{I}}}^{5}(t)+\overline{H_{\bar{I}}}(t)\Big(\Big(\Big(\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\Big)\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}(t)\Big(\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\Big)\Big)+\Big(\big(\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\big)\overline{H_{\bar{I}}}^{2}(t)-\overline{H_{\bar{I}}}^{2}(t)\,\big(\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\big)\Big) \qquad (1731)\\
&\quad+\Big(\overline{H_0}(t)\overline{H_{\bar{I}}}^{3}(t)-\overline{H_{\bar{I}}}^{3}(t)\,\overline{H_0}(t)\Big)+\big(\overline{H_0}(t)\big(\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}(t)\big)-\big(\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\overline{H_{\bar{I}}}(t)\big)\overline{H_0}(t)\big)+\Big(\overline{H_0}(t)\Big(\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)\Big) \qquad (1732)\\
&\quad-\Big(\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)\Big)\overline{H_0}(t)\Big)+\Big(\overline{H_0}^{3}(t)\overline{H_{\bar{I}}}(t)-\overline{H_{\bar{I}}}(t)\overline{H_0}^{3}(t)\Big)+\big(\big(\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\big)\big(\overline{H_0}(t)\overline{H_{\bar{I}}}(t)\big)-\big(\overline{H_0}(t)\overline{H_{\bar{I}}}(t)\big)\big(\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\big)\big) \qquad (1733)\\
&\quad+2\overline{H_{\bar{I}}}(t)\big(\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\big)\overline{H_0}(t)+3\overline{H_0}(t)\big(\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\big)\overline{H_0}(t)+\Big(\Big(\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)\Big)\big(\overline{H_0}(t)\big) \qquad (1734)\\
&\quad-\big(\overline{H_0}(t)\big)\Big(\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\Big)\Big)\Big)\Big\rangle_{\overline{H_0}(t)}\ \text{(factorizing to introduce commutators)} \qquad (1735)\\
&=\Big\langle\overline{H_{\bar{I}}}^{5}(t)+\overline{H_{\bar{I}}}(t)\Big(\Big[\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\,,\overline{H_{\bar{I}}}(t)\Big]+\Big[\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,,\overline{H_{\bar{I}}}^{2}(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_{\bar{I}}}^{3}(t)\Big]+\big[\overline{H_0}(t)\,,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\big] \qquad (1736)\\
&\quad+\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_{\bar{I}}}^{2}(t)\Big]+\Big[\overline{H_0}^{3}(t)\,,\overline{H_{\bar{I}}}(t)\Big]+\big[\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_{\bar{I}}}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+2\overline{H_{\bar{I}}}(t) \qquad (1737)\\
&\quad\times\big[\overline{H_{\bar{I}}}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}\ \text{(put the terms required in commutators).} \qquad (1738)
\end{aligned}
$$

Summarizing we have that:

$$
\begin{aligned}
v_2(t)&=\Big\langle\overline{H_{\bar{I}}}^{2}(t)\Big\rangle_{\overline{H_0}(t)}, \qquad (1739)\\
v_3(t)&=\Big\langle\overline{H_{\bar{I}}}^{3}(t)+\overline{H_{\bar{I}}}(t)\,\big[\overline{H_0}(t)\,,\overline{H_{\bar{I}}}(t)\big]\Big\rangle_{\overline{H_0}(t)}, \qquad (1740)\\
v_4(t)&=\Big\langle\overline{H_{\bar{I}}}^{4}(t)+\overline{H_{\bar{I}}}(t)\Big(\big[\overline{H_{\bar{I}}}(t)\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\big]+\Big[\overline{H_0}(t),\overline{H_{\bar{I}}}^{2}(t)\Big]+\big[\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\big]+\big[\overline{H_{\bar{I}}}(t)\overline{H_0}(t),\overline{H_0}(t)\big]\Big)\Big\rangle_{\overline{H_0}(t)}, \qquad (1741)\\
v_5(t)&=\Big\langle\overline{H_{\bar{I}}}^{5}(t)+\overline{H_{\bar{I}}}(t)\Big(\Big[\overline{H_{\bar{I}}}^{2}(t)\overline{H_0}(t)\,,\overline{H_{\bar{I}}}(t)\Big]+\Big[\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\,,\overline{H_{\bar{I}}}^{2}(t)\Big]+\Big[\overline{H_0}(t)\,,\overline{H_{\bar{I}}}^{3}(t)\Big]+\big[\overline{H_0}(t)\,,\overline{H_{\bar{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\bar{I}}}(t)\big] \qquad (1742)\\
&\quad+\Big[\overline{H_0}(t)\,,\overline{H_0}(t)\overline{H_{\bar{I}}}^{2}(t)\Big]+\Big[\overline{H_0}^{3}(t)\,,\overline{H_{\bar{I}}}(t)\Big]+\big[\overline{H_{\bar{I}}}(t)\overline{H_0}(t)\,,\overline{H_0}(t)\overline{H_{\bar{I}}}(t)\big]+3\overline{H_0}(t)\,\big[\overline{H_{\bar{I}}}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+2\overline{H_{\bar{I}}}(t) \qquad (1743)\\
&\quad\times\big[\overline{H_{\bar{I}}}(t)\,,\overline{H_0}(t)\big]\,\overline{H_0}(t)+\Big[\overline{H_{\bar{I}}}^{2}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\Big]\Big)\Big\rangle_{\overline{H_0}(t)}. \qquad (1744)
\end{aligned}
$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_{\bar{S}}}(t) \equiv (\varepsilon_0(t)+R_0(t))\,|0\rangle\langle 0| + (\varepsilon_1(t)+R_1(t))\,|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \quad (1745)$$

$$\overline{H_{\bar{I}}}(t) \equiv \sum_i B_{iz}(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t)), \quad (1746)$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (1747)$$

$$= H_B. \quad (1748)$$

In this case $\varepsilon_j(t)$, $R_j(t)$ for $j \in \{0,1\}$, $B_{10}^{\Re}(t)$, $B_{10}^{\Im}(t)$, $V_{10}^{\Re}(t)$ and $V_{10}^{\Im}(t)$ are scalars and the other operators are:

$$\sigma_x \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \quad (1749)$$

$$\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1750)$$

$$\sigma_y \equiv -\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1| \quad (1751)$$

$$\equiv \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad (1752)$$

$$\sigma_z \equiv |1\rangle\langle 1| - |0\rangle\langle 0| \quad (1753)$$

$$\equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (1754)$$

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\,b_{\mathbf{k}}\right) & \mathrm{e}^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} \\ \frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)-B_{10}(t)-B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{i\mathbf{k}}(t)-v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \quad (1755)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2}\left(\frac{v_{i\mathbf{k}}^{*}(t)\,v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)\,v_{j\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}^2}\right), \quad (1756)$$

$$B_i^{+}(t)\,B_j^{-}(t) = \mathrm{e}^{\chi_{ij}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right), \quad (1757)$$

$$D(\pm v_{\mathbf{k}}(t)) \equiv \mathrm{e}^{\pm\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^{*}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (1758)$$

As we can see they verify the relationship $\sigma_x\sigma_y = \mathrm{i}\sigma_z$. The explicit form of $\overline{H_{\bar{I}}}^2(t)$ is:

$$\overline{H_{\bar{I}}}^2(t) = \left(\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\right)\left(\sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + V_{10}^{\Re}(t)(\sigma_x B_x(t)\right. \quad (1759)$$

$$\left. + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\,(\sigma_x B_y(t) - \sigma_y B_x(t))\right) \quad (1760)$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i| \sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + V_{10}^{\Re}(t)\sum_i B_{iz}(t)|i\rangle\langle i|\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\sum_i B_{iz}(t)|i\rangle\langle i|\,(\sigma_x B_y(t) \quad (1761)$$

$$- \sigma_y B_x(t)) + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))\sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + \left(V_{10}^{\Re}(t)\right)^2(\sigma_x B_x(t) + \sigma_y B_y(t))(\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (1762)$$

$$+ V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x B_x(t) + \sigma_y B_y(t))(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_{i'} B_{i'z}(t)|i'\rangle\langle i'| + V_{10}^{\Im}(t) \quad (1763)$$

$$\times V_{10}^{\Re}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\,(\sigma_x B_x(t) + \sigma_y B_y(t)) + \left(V_{10}^{\Im}(t)\right)^2(\sigma_x B_y(t) - \sigma_y B_x(t))\,(\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (1764)$$

$$= \sum_i B_{iz}^2(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i (B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x + B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i (B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x - B_{iz}(t) \tag{1765}$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i (\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2 (\sigma_x^2 B_x^2(t) + \sigma_x\sigma_y B_x(t)B_y(t) + \sigma_y \tag{1766}$$

$$\times \sigma_x B_y(t)B_x(t) + \sigma_y^2 B_y^2(t)) + V_{10}^{\Im}(t)\sum_i (\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (\sigma_x^2 B_y^2(t) + \sigma_y^2 B_x^2(t) \tag{1767}$$

$$-\sigma_x\sigma_y B_y(t)B_x(t) - \sigma_y\sigma_x B_x(t)B_y(t)) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x^2 B_y(t)B_x(t) + \sigma_x\sigma_y B_y^2(t) - \sigma_y\sigma_x B_x^2(t) - \sigma_y^2 B_x(t)B_y(t) \tag{1768}$$

$$+\sigma_x^2 B_x(t)B_y(t) - \sigma_x\sigma_y B_x^2(t) + \sigma_y\sigma_x B_y^2(t) - \sigma_y^2 B_y(t)B_x(t))\,, \tag{1769}$$

$$\sigma_x\sigma_y = \mathrm{i}\sigma_z \text{ (by Pauli matrices properties)}, \tag{1770}$$

$$\sigma_j^2 = \mathbb{I} \text{ (for } j \in \{x, y, x\}), \tag{1771}$$

$$\overline{H_I}^2(t) = \sum_i B_{iz}^2(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i (B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x + B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i (B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x - B_{iz}(t) \tag{1772}$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i (\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2 (B_x^2(t) + \mathrm{i}\sigma_z B_x(t)B_y(t) - \mathrm{i}\sigma_z \tag{1773}$$

$$\times B_y(t)B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t)\sum_i (\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_y^2(t) + B_x^2(t) - \mathrm{i}\sigma_z \tag{1774}$$

$$\times B_y(t)\,B_x(t) + \mathrm{i}\sigma_z B_x(t)\,B_y(t))\,. \tag{1775}$$

To introduce the direct calculation of the expected values recall that the hamiltonian $\overline{H_0}\,(t)$ is a direct sum of the hamiltonians of two Hilbert spaces given by $\overline{H_{\bar{S}}}\,(t)$ and $\overline{H_{\bar{B}}}$, so we can write the hamiltonian $\overline{H_0}\,(t)$ as:

$$\overline{H_0}\,(t) = \overline{H_{\bar{S}}}\,(t) \otimes \mathbb{I}_{\bar{B}} + \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}. \tag{1776}$$

where $\mathbb{I}_{\bar{B}}$ and $\mathbb{I}_{\bar{S}}$ are the identity of the systems $\bar{B}$ and $\bar{S}$ respectively.
We can show that:

$$\begin{aligned}
\left[\overline{H_{\bar{S}}}\,(t) \otimes \mathbb{I}_{\bar{B}}, \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}\right] &= \overline{H_{\bar{S}}}\,(t) \otimes \mathbb{I}_{\bar{B}} \cdot \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} \cdot \overline{H_{\bar{S}}}\,(t) \otimes \mathbb{I}_{\bar{B}} && (1777)\\
&= \overline{H_{\bar{S}}}\,(t)\,\mathbb{I}_{\bar{S}} \otimes \mathbb{I}_{\bar{B}}\overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}}\overline{H_{\bar{S}}}\,(t) \otimes \overline{H_{\bar{B}}}\mathbb{I}_{\bar{B}} && (1778)\\
&= \overline{H_{\bar{S}}}\,(t) \otimes \overline{H_{\bar{B}}} - \overline{H_{\bar{S}}}\,(t) \otimes \overline{H_{\bar{B}}} \text{ (by definition of identity operator)} && (1779)\\
&= 0. && (1780)
\end{aligned}$$

Let's introduce the following partition functions $Z_{\bar{S}}\,(t)$ and $Z_{\bar{B}}$ related to the systems $\bar{S}$ and $\bar{B}$ respectively.:

$$Z_{\bar{S}}\,(t) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right), \tag{1781}$$

$$Z_{\bar{B}} \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \tag{1782}$$

Using (1549), (1777) and $\mathrm{Tr}\,(\mathrm{A} \otimes \mathrm{B}) = \mathrm{Tr}\,(\mathrm{A})\,\mathrm{Tr}\,(\mathrm{B})$ we can infer that the partition function $Z_0\,(t)$ can be factorized as:

$$\begin{aligned}
Z_0\,(t) &= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0(t)}}\right). && (1783)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}\right)}\right) \text{ (by (1544))}, && (1784)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by (1549))} && (1785)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)} \otimes \mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (because } \bar{S} \text{ and } \bar{B} \text{ are disjoint Hilbert spaces)} && (1786)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by Tr}(A \otimes B)\text{=Tr}(A)\text{Tr}(B)), && (1787)\\
&= Z_{\bar{S}}\,(t)\,Z_{\bar{B}} \text{ (by (1781) and (1782)))}. && (1788)
\end{aligned}$$

For an operator $J(t)$ that can be factorized as $J(t) = S(t) \otimes B(t)$ with $S(t) \in \operatorname{gen}\left(\overline{H_{\bar{S}}}(t)\right)$ and $B(t) \in \operatorname{gen}\left(\overline{H_{\bar{B}}}\right)$, being $\operatorname{gen}(\mathrm{A})$ the vectorial space generated by the eigenvectors of the operator A, we calculate it's expected value respect to $\overline{H_0}(t)$ using a simple way as follows:

$$\langle J(t)\rangle_{\overline{H_0(t)}} = \frac{\operatorname{Tr}\left(J(t)\,\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \text{ (by (1545))} \tag{1789}$$

$$= \frac{\operatorname{Tr}\left((S(t)\otimes B(t))\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\otimes\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \text{ (by } J(t) = S(t)\otimes B(t) \text{ and } \mathrm{e}^{-\beta\overline{H_0}(t)} = \mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\otimes\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\text{)} \tag{1790}$$

$$= \frac{\operatorname{Tr}\left(\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\otimes\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \text{ (rearranging and factorizing)} \tag{1791}$$

$$= \frac{\operatorname{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\operatorname{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)}{\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \text{ (by Tr}(A\otimes B)\text{=Tr}(A)\text{Tr}(B)\text{)} \tag{1792}$$

$$= \frac{\operatorname{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)}\frac{\operatorname{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)}{\operatorname{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \tag{1793}$$

$$= \langle S(t)\rangle_{\overline{H_{\bar{S}}(t)}}\,\langle B(t)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1545)).} \tag{1794}$$

The factorization of $\left\langle \overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_0(t)}}$ in terms of expected values of elements from $\operatorname{gen}\left(\overline{H_{\bar{S}}}(t)\right)$ and $\operatorname{gen}\left(\overline{H_{\bar{B}}}\right)$ is:

$$\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_0(t)}} = \sum_i \langle |i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar{B}}}} + V_{10}^{\Re}(t)\sum_i\left(\langle |i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} + \langle |i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}}\right) \tag{1795}$$

$$+V_{10}^{\Im}(t)\sum_i\left(\langle |i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}} - \langle |i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}}\right) + V_{10}^{\Re}(t)\sum_i\left(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}\right. \tag{1796}$$

$$\times\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} + \left.\langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}\right) + \left(V_{10}^{\Re}(t)\right)^2\left(\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}}\right. \tag{1797}$$

$$-\mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} + \left.\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar{B}}}}\right) + V_{10}^{\Im}(t)\sum_i\left(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} - \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}\right. \tag{1798}$$

$$\times\left.\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}\right) + \left(V_{10}^{\Im}(t)\right)^2\left(\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}} - \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}\right. \tag{1799}$$

$$\times\left.\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar{B}}}}\right). \tag{1800}$$

In order to obtain the expected values of $\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_0(t)}}$ respect to the part related to the bath we need to calculate the following expected values that appear in the equation (1795) and can be obtained using the bath and system terms. The expected values relevant for calculations are $\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_{iz}(t)\,B_x(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_{iz}(t)\,B_y(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_y(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}$, $\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar{B}}}}$, $\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar{B}}}}$, $\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar{B}}}}$ and $\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar{B}}}}$. Recalling the form of the hamiltonian $\overline{H_{\bar{B}}} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$ we can extend the result (1788), introducing the notation:

$$A_1\otimes\cdots\otimes A_n \equiv \bigotimes_k A_k, \tag{1801}$$

$$Z_{\mathbf{k}} \equiv \operatorname{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) \tag{1802}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{-1} \tag{1803}$$

$$= f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{1804}$$

with the creation $b_{\mathbf{k}}$ and annihilation $b_{\mathbf{k}}^\dagger$ operators defined in terms of their actions as:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{1805}$$

$$b_{\mathbf{k}}^\dagger \mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}+1}\mid j_{\mathbf{k}}+1\rangle. \tag{1806}$$

being $|j_{\mathbf{k}}\rangle$ an eigenstate of $H_{\mathbf{k}} \equiv \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$. With this notation we can write the partition function as:

$$Z_{\bar{B}} = \mathrm{Tr}\left(\mathrm{e}^{-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right), \tag{1807}$$

$$\mathrm{e}^{-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}, \tag{1808}$$

$$Z_{\bar{B}} = \mathrm{Tr}\left(\bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) \text{ (by (1808))} \tag{1809}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right) \text{ (by } \mathrm{Tr}\left(A \otimes B\right) = \mathrm{Tr}\left(A\right) \mathrm{Tr}\left(B\right)) \tag{1810}$$

$$= \prod_{\mathbf{k}} Z_{\mathbf{k}} \text{ (by (1808)).} \tag{1811}$$

For a function $f\left(t\right)$ which can be factorized as:

$$f\left(t\right) \equiv \prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right). \tag{1812}$$

with $f_{\mathbf{k}}\left(t\right) \in \mathrm{gen}\left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right)$, it's expected value is given by:

$$\left\langle f\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(f\left(t\right) \mathrm{e}^{-\beta \overline{H_{\bar{B}}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \overline{H_{\bar{B}}}}\right)} \tag{1813}$$

$$= \frac{\mathrm{Tr}\left(\prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right) \bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \text{ (by (1808) and (1812))} \tag{1814}$$

$$= \frac{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} f_{\mathbf{k}}\left(t\right) \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \tag{1815}$$

$$= \frac{\prod_{\mathbf{k}} \mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right) \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\prod_{\mathbf{k}} \mathrm{Tr}\left(\mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \tag{1816}$$

$$= \prod_{\mathbf{k}} \frac{\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right) \mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}\right)} \tag{1817}$$

$$= \prod_{\mathbf{k}} \left\langle f_{\mathbf{k}}\left(t\right)\right\rangle_{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}. \tag{1818}$$

It means that for an operator that can be factorized in terms of functions generated by $\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}$ for each $\mathbf{k}$ we only require to calculate the expected value respect to the Hilbert space where the operator belongs. This process lead us to the following explicit forms of the expected values relevant for our calculations:

$$\left\langle B_{iz}^{2}(t)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \left(\sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right)\right)^{*} b_{\mathbf{k}}\right)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1755))}, \tag{1819}$$

$$= \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right)^{2} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}}\right)\left(\left(g_{i\mathbf{k}'}\right.\right.\right. \tag{1820}$$

$$\left.\left.\left. - v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\left(t\right)\right)^{*} b_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by square expansion properties)}, \tag{1821}$$

$$= \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{1822}$$

$$\times \left\langle \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^{*} b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1818))}, \tag{1823}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1824}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1825}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (1806))}, \tag{1826}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \mathrm{Tr}\left(|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1827}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by trace properties)}, \tag{1828}$$

$$= 0, \tag{1829}$$

$$\left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1830}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1831}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (1805))}, \tag{1832}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \mathrm{Tr}\left(|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1833}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by trace properties)}, \tag{1834}$$

$$= 0, \tag{1835}$$

$$\left\langle B_{iz}^{2}(t) \right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{1836}$$

$$\times \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) \left\langle b_{\mathbf{k}'}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^{*} \left\langle b_{\mathbf{k}'} \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{1837}$$

$$= \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k} \neq \mathbf{k}'} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \cdot 0 + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \cdot 0 \right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) \cdot 0 \right. \tag{1838}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^{*} \cdot 0 \right) \text{ (by (1824) and (1830))} \tag{1839}$$

$$= \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{1840}$$

$$= \sum_{\mathbf{k}} \left\langle \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{2} \left(b_{\mathbf{k}}^{\dagger}\right)^{2} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(\left(g_{i\mathbf{k}} \right.\right.\right. \tag{1841}$$

$$\left.\left.\left. - v_{i\mathbf{k}}(t)\right)^{*}\right)^{2} b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{1842}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{2} \left\langle \left(b_{\mathbf{k}}^{\dagger}\right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left|g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right|^{2} \left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right)^{2} \left\langle b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}}, \tag{1843}$$

$$\left\langle \left(b_{\mathbf{k}}^{\dagger}\right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger}\right)^{2} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1844}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger}\right)^{2} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1845}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)} |j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (1806) applied twice)} \tag{1846}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)} \mathrm{Tr}\left(|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1847}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{1848}$$

$$= 0, \tag{1849}$$

$$\left\langle b_{\mathbf{k}}^{2}\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{2} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1850}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} |j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (1805) applied twice)} \tag{1851}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} \mathrm{Tr}\left(|j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{1852}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{1853}$$

$$= 0, \tag{1854}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{1855}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{1856}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{1857}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}}+1} |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{1858}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} \sqrt{j_{\mathbf{k}}+1} \sqrt{j_{\mathbf{k}}+1} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{1859}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} j_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(j_{\mathbf{k}}+1\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{1860}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{1861}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \mathrm{Tr}\left(|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{1862}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right) \text{ (by properties of trace)} \tag{1863}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \sum_{j_{\mathbf{k}}=0}^{\infty} \left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}} \left(2j_{\mathbf{k}}+1\right), \tag{1864}$$

$$\sum_{j_{\mathbf{k}}=0}^{\infty} x^{j_{\mathbf{k}}}(2j_{\mathbf{k}}+1) = \frac{1+x}{(1-x)^2}, \tag{1865}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\frac{\mathrm{e}^{-\beta\omega_{\mathbf{k}}}+1}{(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}})^2}\ \text{(setting } x=\mathrm{e}^{-\beta\omega_{\mathbf{k}}} \text{ in (1865) and by (1855))}, \tag{1866}$$

$$= \frac{1+\mathrm{e}^{-\beta\omega_{\mathbf{k}}}}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \tag{1867}$$

$$= \frac{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}\frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}+\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}}{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}\frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}-\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}} \tag{1868}$$

$$= \frac{\cosh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\sinh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{1869}$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{1870}$$

$$\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\ \text{(by (1844), (1850) and (1870))}, \tag{1871}$$

$$\left\langle B_{iz}(t)\,B_x(t)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)-B_{10}(t)-B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{1872}$$

$$= \frac{1}{2}\left\langle \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\right.\right. \tag{1873}$$

$$\left.\left.\times\prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2}\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{1874}$$

$$= \frac{1}{2}\left\langle \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\,b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\right.\right. \tag{1875}$$

$$\left.\left.\times\prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\ \text{(by (1824) and (1830))}, \tag{1876}$$

$$\left\langle F(h)\right\rangle_{\overline{H_{\bar{B}}}} \equiv \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\langle\alpha|F(h)|\alpha\rangle \mathrm{d}^2\alpha\ \text{(using the coherent representation with } N=\left(\mathrm{e}^{\beta\omega}-1\right)^{-1}\text{)}, \tag{1877}$$

$$D(\alpha_{\mathbf{k}}) \equiv \mathrm{e}^{\left(\frac{\alpha_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{\alpha_{\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\ \text{(displacement operator definition)}, \tag{1878}$$

$$|\alpha\rangle \equiv D(\alpha)|0\rangle\ \text{(displacement operator properties)}, \tag{1879}$$

$$\langle\alpha| \equiv \langle 0|D(-\alpha), \tag{1880}$$

$$D(-\alpha)D(h)D(\alpha) \equiv D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\ \text{(displacement operator properties)}, \tag{1881}$$

$$D(0) \equiv \mathbb{I}\ \text{(identity written in terms of the displacement operator)}, \tag{1882}$$

$$D(-\alpha)D(0)D(\alpha) = D(0)\,\mathrm{e}^{0\cdot\alpha^*-0^*\cdot\alpha} \tag{1883}$$

$$= D(0) \tag{1884}$$

$$= \mathbb{I}, \tag{1885}$$

$$D(-\alpha)\,b^{\dagger}D(\alpha) = b^{\dagger}+\alpha^*\ \text{(displacement operator properties)}, \tag{1886}$$

$$D(-\alpha)\,bD(\alpha) = b+\alpha\ \text{(displacement operator properties)}, \tag{1887}$$

$$\left\langle D(h)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}\ \text{(expected value displacement operator)}, \tag{1888}$$

$$\left\langle b^{\dagger}D(h)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\left|b^{\dagger}D(h)\right|\alpha\right\rangle \mathrm{d}^2\alpha\ \text{(by (1877))} \tag{1889}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,b^{\dagger}D(h)\,D(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha\ \text{(by (1879) and (1880))} \tag{1890}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,b^{\dagger}\mathbb{I}D(h)\,D(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha\ \text{(inserting identity operator)} \tag{1891}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(D(-\alpha)\,b^{\dagger}D(\alpha)\right)\left(D(-\alpha)\,D(h)\,D(\alpha)\right)\right|0\right\rangle \mathrm{d}^2\alpha\ \text{(by associative property)} \tag{1892}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(b^\dagger+\alpha^*\right)D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (1886) and (1881))} \tag{1893}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|b^\dagger D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha^* D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \tag{1894}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} 0D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\left|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha^* D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \tag{1895}$$

$$= \frac{1}{\pi N}\int 0\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0\left|D(h)\right|0\right\rangle \mathrm{d}^2\alpha \tag{1896}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0\left|D(h)\right|0\right\rangle \mathrm{d}^2\alpha \tag{1897}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \text{ (by (1879))} \tag{1898}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (because } \left\langle 0|h\right\rangle\text{=}\mathrm{e}^{-\frac{|h|^2}{2}}\text{)}, \tag{1899}$$

$$x = \alpha^{\Re}\in\mathbb{R}, \tag{1900}$$

$$y = \alpha^{\Im}\in\mathbb{R}, \tag{1901}$$

$$\alpha = x+\mathrm{i}y, \tag{1902}$$

$$\left\langle b^\dagger D(h)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \tag{1903}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(x-\mathrm{i}y\right)\mathrm{d}x\mathrm{d}y \text{ (by (1900) and (1901))} \tag{1904}$$

$$= -h^*\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}N \tag{1905}$$

$$= -h^*\left\langle D(h)\right\rangle_{\overline{H_{\bar{B}}}}N, \tag{1906}$$

$$|h\rangle = \mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle\,, \tag{1907}$$

$$\left\langle bD(h)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|bD(h)\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (by (1880) and (1877))} \tag{1908}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,b\mathbb{I}D(h)\,D(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (1879) and (1880))} \tag{1909}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(D(-\alpha)\,bD(\alpha)\right)\left(D(-\alpha)\,D(h)\,D(\alpha)\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} \tag{1910}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|(b+\alpha)\,D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (1887) and (1881))} \tag{1911}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|bD(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha \tag{1912}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|b\right|h\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \;\left(D(h)\left|0\right\rangle = |h\rangle\right) \tag{1913}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|b\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle\,\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \text{ (by (1907))} \tag{1914}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=1}^{\infty}\frac{h^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle\,\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha\text{(by (1805))} \tag{1915}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=1}^{\infty}\frac{h^n}{\sqrt{n!}}\sqrt{n}\delta_{0,n-1}\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha\text{(by } \left\langle n|n'\right\rangle = \delta_{nn'}\text{)} \tag{1916}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\frac{h^1}{\sqrt{1!}}\sqrt{1}\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha \tag{1917}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}h\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (because } \left\langle 0|h\right\rangle\text{=}\mathrm{e}^{-\frac{|h|^2}{2}}\text{)} \tag{1918}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\left(\alpha+h\right)\mathrm{d}^2\alpha \tag{1919}$$

$$= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}\left(x+\mathrm{i}y+h\right)\mathrm{d}x\mathrm{d}y \tag{1920}$$

$$= he^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}(N+1) \tag{1921}$$
$$= h\langle D(h)\rangle_{\overline{H_{\tilde{B}}}}(N+1), \tag{1922}$$
$$\langle D(h)b\rangle_{\overline{H_{\tilde{B}}}} = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle\alpha|D(h)b|\alpha\rangle d^2\alpha \text{ (by (1877))} \tag{1923}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|D(-\alpha)D(h)\mathbb{I}bD(\alpha)|0\rangle d^2\alpha \text{ (by (1879) and (1880))} \tag{1924}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\langle 0|(D(-\alpha)D(h)D(\alpha))(D(-\alpha)bD(\alpha))|0\rangle d^2\alpha \text{ (by associative property)} \tag{1925}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|D(h)e^{h\alpha^*-h^*\alpha}(b+\alpha)\middle|0\right\rangle d^2\alpha \text{ (by (1887) and (1881))} \tag{1926}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|D(h)e^{h\alpha^*-h^*\alpha}b\middle|0\right\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|D(h)e^{h\alpha^*-h^*\alpha}\alpha\middle|0\right\rangle d^2\alpha \tag{1927}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\langle 0|D(h)b|0\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha\langle 0|h\rangle d^2\alpha \tag{1928}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\langle 0|D(h)0d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha e^{-\frac{|h|^2}{2}}d^2\alpha \text{ (by (1805))} \tag{1929}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha e^{-\frac{|h|^2}{2}}d^2\alpha \tag{1930}$$
$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\left(x^2+y^2\right)/N}e^{h(x-iy)-h^*(x+iy)}(x+iy)\,dxdy \tag{1931}$$
$$= hNe^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)} \tag{1932}$$
$$= hN\langle D(h)\rangle_B, \tag{1933}$$
$$\left\langle D(h)b^\dagger\right\rangle_{\overline{H_{\tilde{B}}}} = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle\alpha\middle|D(h)b^\dagger\middle|\alpha\right\rangle d^2\alpha \text{ (by (1877))} \tag{1934}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|D(-\alpha)D(h)\mathbb{I}b^\dagger D(\alpha)\middle|0\right\rangle d^2\alpha \text{ (by (1879) and (1880))} \tag{1935}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|(D(-\alpha)D(h)D(\alpha))\left(D(-\alpha)b^\dagger D(\alpha)\right)\middle|0\right\rangle d^2\alpha \text{ (by associative property)} \tag{1936}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|D(h)e^{h\alpha^*-h^*\alpha}\left(b^\dagger+\alpha^*\right)\middle|0\right\rangle d^2\alpha \text{ (by (1887) and (1881))} \tag{1937}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|D(h)e^{h\alpha^*-h^*\alpha}b^\dagger\middle|0\right\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}\left\langle 0\middle|\alpha^*D(h)e^{h\alpha^*-h^*\alpha}\middle|0\right\rangle d^2\alpha \tag{1938}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\left\langle 0\middle|D(h)b^\dagger\middle|0\right\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*\langle 0|h\rangle d^2\alpha \tag{1939}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\left\langle 0\middle|D(h)b^\dagger\middle|0\right\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*\langle 0|h\rangle d^2\alpha \tag{1940}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\langle -h|\sqrt{0+1}|1\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*\langle 0|h\rangle d^2\alpha \tag{1941}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\langle -h|\sqrt{0+1}|1\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*e^{-\frac{|h|^2}{2}}d^2\alpha \text{ (by (1880))}, \tag{1942}$$
$$\langle h| = e^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(h^*)^n}{\sqrt{n!}}\langle n|, \tag{1943}$$
$$\left\langle D(h)b^\dagger\right\rangle_{\overline{H_{\tilde{B}}}} = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}e^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(-h^*)^n}{\sqrt{n!}}\langle n|1\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*e^{-\frac{|h|^2}{2}}d^2\alpha \text{ (by (1943))} \tag{1944}$$
$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}e^{-\frac{|h|^2}{2}}\frac{(-h^*)^1}{\sqrt{1!}}d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}\alpha^*e^{-\frac{|h|^2}{2}}d^2\alpha \text{ (by } \langle n|n'\rangle = \delta_{nn'}) \tag{1945}$$
$$= \frac{1}{\pi N}\int(\alpha^*-h^*)e^{-\frac{|\alpha|^2}{N}}e^{h\alpha^*-h^*\alpha}e^{-\frac{|h|^2}{2}}d^2\alpha \tag{1946}$$
$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\left(x^2+y^2\right)/N}e^{h(x-iy)-h^*(x+iy)}(x-iy-h^*)\,dxdy \tag{1947}$$
$$= -h^*\langle D(h)\rangle_B(N+1), \tag{1948}$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\left\langle \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right.\right. \quad (1949)$$

$$\left.\left.+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{(replacing the definitions in (1755))} \quad (1950)$$

$$= \frac{1}{2}\left\langle \mathrm{e}^{\chi_{10}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)+\mathrm{e}^{\chi_{01}(t)}\right. \quad (1951)$$

$$\left.\times\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad (1952)$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\langle b_{\mathbf{k}}\right. \quad (1953)$$

$$\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\Bigg)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right.\right.\right. \quad (1954)$$

$$\left.\left.\left.-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right), \quad (1955)$$

$$\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (1888))}, \quad (1956)$$

$$N_{\mathbf{k}} = \left(\mathrm{e}^{\beta\omega_{\mathbf{k}}}-1\right)^{-1}, \quad (1957)$$

$$\langle b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} = \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1922))}, \quad (1958)$$

$$\left\langle b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} = -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1906))}, \quad (1959)$$

$$\left\langle\prod_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (1956) and (1818))}, \quad (1960)$$

$$\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1818))} \quad (1961)$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}'}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1818))} \quad (1962)$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\prod_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \quad (1963)$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (1956))}, \quad (1964)$$

$$\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} = \langle b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1818))} \quad (1965)$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}'}\right)\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1958))} \quad (1966)$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\prod_{\mathbf{k}}\langle D\left(\alpha_{\mathbf{k}}\right)\rangle_{\overline{H_{\bar{B}}}} \quad (1967)$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \text{ (by (1956))}, \quad (1968)$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\langle b_{\mathbf{k}}\right. \quad (1969)$$

$$\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\Big\rangle_{\overline{H_{\bar{B}}}}\Bigg)+\frac{e^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\Big\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\Big\rangle_{\overline{H_{\bar{B}}}}\right. \tag{1970}$$

$$\left.\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\Big\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)\Big\rangle_{\overline{H_{\bar{B}}}}\right)\quad\text{(by (1755))} \tag{1971}$$

$$=\frac{e^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{1972}$$

$$\times\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\frac{e^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)(-N_{\mathbf{k}}\right. \tag{1973}$$

$$\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1) \tag{1974}$$

$$\left.\times e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{1975}$$

$$=\frac{1}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}e^{\chi_{10}(t)}e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\right. \tag{1976}$$

$$\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\,e^{\chi_{10}(t)}e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\frac{1}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)(-N_{\mathbf{k}}\right. \tag{1977}$$

$$\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}e^{\chi_{01}(t)}e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\Bigg)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \tag{1978}$$

$$\left.\times(N_{\mathbf{k}}+1)\,e^{\chi_{01}(t)}e^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right) \tag{1979}$$

$$=\frac{1}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(-\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{10}\left(t\right)\right)+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right. \tag{1980}$$

$$\times\left(N_{\mathbf{k}}+1\right)B_{10}(t))+\frac{1}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}B_{01}(t)\right)+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\right. \tag{1981}$$

$$\left.\times\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)B_{01}(t)\right)\quad\text{(by (1755) using the definition } B_{ij}\left(t\right)) \tag{1982}$$

$$=\frac{B_{10}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{1983}$$

$$+\frac{B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{1984}$$

$$=\frac{B_{10}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{1985}$$

$$-\frac{B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right)+\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right) \tag{1986}$$

$$=\frac{B_{10}(t)-B_{01}(t)}{2}\left(\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-\sum_{\mathbf{k}}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{1987}$$

$$\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\,b_{\mathbf{k}}\right)\frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)+B_{10}(t)-B_{01}(t)}{2i}\right\rangle_{\overline{H_{\bar{B}}}} \tag{1988}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t)\, B_{1}^{-}(t) - B_{1}^{+}(t)\, B_{0}^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} + \frac{\left(B_{10}(t) - B_{01}(t) \right)}{2\mathrm{i}} \tag{1989}$$

$$\times \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right) \right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{(by the properties of expected value)} \tag{1990}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} + \frac{\left(B_{10}(t) - B_{01}(t) \right)}{2\mathrm{i}} \cdot 0 \tag{1991}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \text{((1968) and (1964))} \tag{1992}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \right) \left(\mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \tag{1993}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right. \tag{1994}$$

$$\left. + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} b_{\mathbf{k}} \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{1995}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) \mathrm{e}^{\chi_{01}(t)} \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} - \mathrm{e}^{\chi_{10}(t)} \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \right. \tag{1996}$$

$$\times \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \mathrm{e}^{\chi_{01}(t)} \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{1997}$$

$$\left. \times \mathrm{e}^{\chi_{10}(t)} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \right) \tag{1998}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) \mathrm{e}^{\chi_{01}(t)} \left(- \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) - \mathrm{e}^{\chi_{10}(t)} \right. \tag{1999}$$

$$\times \left(- \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right) \right) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right) \right)^{*} \mathrm{e}^{\chi_{01}(t)} \tag{2000}$$

$$\times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \left(N_{\mathbf{k}} + 1 \right) \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \left(N_{\mathbf{k}} + 1 \right) \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \tag{2001}$$

$$\left. \times \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right) \text{(by (1960), (1968) and (1964))} \tag{2002}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right) \right) \left(- \left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{01}\left(t\right) \right) - \left(- \left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{10}\left(t\right) \right) \right. \tag{2003}$$

$$\times \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \left(N_{\mathbf{k}} + 1 \right) B_{01}(t) - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \left(N_{\mathbf{k}} + 1 \right) B_{10}(t) \tag{2004}$$

$$\left. \times \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \right) \text{ (by (1755) using the definition } B_{ij}\left(t\right)\text{)} \tag{2005}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{01}\left(t\right) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{10}\left(t\right) \right. \tag{2006}$$

$$\left. - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \left(N_{\mathbf{k}} + 1 \right) B_{01}(t) - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \left(N_{\mathbf{k}} + 1 \right) B_{10}(t) \right) \tag{2007}$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t) \right)^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \left(N_{\mathbf{k}} + 1 \right) \right), \tag{2008}$$

$$\langle B_{x}(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} = \left\langle \frac{B_{1}^{+}(t)\, B_{0}^{-}(t) + B_{0}^{+}(t)\, B_{1}^{-}(t) - B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right) \right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\left(t\right) \right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{2009}$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2} \quad (2010)$$

$$\times \left\langle \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by expected value properties and (1968))} \quad (2011)$$

$$= \left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2}\cdot 0 \quad (2012)$$

$$= \frac{1}{2}\left\langle \left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad (2013)$$

$$= \frac{1}{2}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^\dagger\right.\right. \quad (2014)$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \quad (2015)$$

$$\langle D(\alpha_{\mathbf{k}})\,b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}} = \alpha_{\mathbf{k}}N_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}, \quad (2016)$$

$$\left\langle D(\alpha_{\mathbf{k}})\,b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = -\alpha_{\mathbf{k}}^*(N_{\mathbf{k}}+1)\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}, \quad (2017)$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle D(\alpha_{\mathbf{k}})\,b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} \quad (2018)$$

$$= -\alpha_{\mathbf{k}}^*(N_{\mathbf{k}}+1)\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D(\alpha_{\mathbf{k}'})\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (2017))} \quad (2019)$$

$$= -\alpha_{\mathbf{k}}^*(N_{\mathbf{k}}+1)\prod_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \quad (2020)$$

$$= -\alpha_{\mathbf{k}}^*(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad \text{(by (1956))}, \quad (2021)$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} = \langle D(\alpha_{\mathbf{k}})\,b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} \quad (2022)$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D(\alpha_{\mathbf{k}'})\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (2016))} \quad (2023)$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\prod_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \quad (2024)$$

$$= \alpha_{\mathbf{k}}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad \text{(by (1956))}, \quad (2025)$$

$$\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\right. \quad (2026)$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} \quad (2027)$$

$$\left.\times\mathrm{e}^{\chi_{10}(t)}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \quad \text{(by (1755))} \quad (2028)$$

$$= \frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right. \quad (2029)$$

$$+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{10}(t)} \quad (2030)$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^*\left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^* \quad (2031)$$

$$\times\left(\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\Bigg)\quad\text{(by (1933) and (1948))}\tag{2032}$$

$$=\frac{1}{2}\sum_{\mathbf{k}}\Bigg(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)B_{10}\left(t\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\tag{2033}$$

$$\times(N_{\mathbf{k}}+1)B_{01}(t)+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{10}(t)+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\tag{2034}$$

$$\times N_{\mathbf{k}}B_{01}\left(t\right)\Bigg)\tag{2035}$$

$$=\frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}(N_{\mathbf{k}}+1)\right)\tag{2036}$$

$$\left\langle B_{y}\left(t\right)B_{iz}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\tag{2037}$$

$$=\left\langle\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\tag{2038}$$

$$\times\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\quad\text{(by (1933) and (1948))}\tag{2039}$$

$$=\left\langle\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\quad\text{(by (1830) and (1824))}\tag{2040}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)-\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\right.\right.\tag{2041}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\quad\text{(by (1755))}\tag{2042}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\mathrm{e}^{\chi_{01}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\right.\tag{2043}$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\tag{2044}$$

$$\left.\times\mathrm{e}^{\chi_{01}(t)}-\mathrm{e}^{\chi_{10}(t)}(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)\quad\text{(by expected value properties)}\tag{2045}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\Bigg(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\mathrm{e}^{\chi_{01}(t)}\left(-\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\tag{2046}$$

$$-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)}\tag{2047}$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\tag{2048}$$

$$\times\left(\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\Bigg)\quad\text{(by (1933) and (1948))}\tag{2049}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\Bigg(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)B_{01}\left(t\right)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\tag{2050}$$

$$\times(N_{\mathbf{k}}+1)B_{10}(t)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{01}(t)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\tag{2051}$$

$$\times N_{\mathbf{k}}B_{10}\left(t\right)\Bigg)\tag{2052}$$

$$= \frac{B_{01}(t)+B_{10}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}}+1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} \right), \quad (2053)$$

$$\mathrm{Var}_{\overline{H_{\bar{B}}}}(A) \equiv \left\langle A^2 \right\rangle_{\overline{H_{\bar{B}}}} - \langle A \rangle^2_{\overline{H_{\bar{B}}}} \text{ (definition of variance)}, \quad (2054)$$

$$\mathrm{Var}(aX+b) = a^2 \mathrm{Var}(X) \text{ (properties of variance)}, \quad (2055)$$

$$\langle B_x(t) \rangle_{\overline{H_{\bar{B}}}} = 0 \text{ (expected value of obtained in [2])}, \quad (2056)$$

$$\langle B_y(t) \rangle_{\overline{H_{\bar{B}}}} = 0 \text{ (expected value of obtained in [2])}, \quad (2057)$$

$$\left\langle B_x^2(t) \right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}(B_x(t)) + \langle B_x(t) \rangle^2_{\overline{H_{\bar{B}}}} \text{ (by (2054))} \quad (2058)$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(\frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \text{ (because } \langle B_x(t) \rangle_{\overline{H_{\bar{B}}}} = 0) \quad (2059)$$

$$= \frac{1}{4} \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t) \right) \quad (2060)$$

$$= \frac{1}{4} \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) \right) \text{ (by (2055))} \quad (2061)$$

$$= \frac{1}{4} \left(\left\langle \left(B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{10}(t) + B_{01}(t) \right)^2 \right) \text{ (by (2054))} \quad (2062)$$

$$= \frac{1}{4} \left(\left\langle \left(B_1^+(t) B_0^-(t) \right)^2 + B_1^+(t) B_0^-(t) B_0^+(t) B_1^-(t) + B_0^+(t) B_1^-(t) B_1^+(t) B_0^-(t) + \left(B_0^+(t) B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} \right. \quad (2063)$$

$$\left. - \left(B_{10}(t) + B_{01}(t) \right)^2 \right) \quad (2064)$$

$$= \frac{1}{4} \left(\left\langle \left(B_1^+(t) B_0^-(t) \right)^2 + 2\mathbb{I} + \left(B_0^+(t) B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{10}(t) + B_{01}(t) \right)^2 \right) \text{ (by } B_j^{\pm}(t) B_j^{\mp}(t) = \mathbb{I}), \quad (2065)$$

$$(D(h))^2 = D(h) D(h) \quad (2066)$$

$$= D(h+h) \, \mathrm{e}^{\frac{1}{2}\left(\frac{h^* h - h h^*}{\omega^2} \right)} \text{ (by displacement operator properties)} \quad (2067)$$

$$= D(2h), \quad (2068)$$

$$\left\langle \left(B_i^+(t) B_j^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \left(\prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} \quad (2069)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(2\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} \right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (2068))} \quad (2070)$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} \mathrm{e}^{-2\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \text{ (by (1956))} \quad (2071)$$

$$= \left(\prod_{\mathbf{k}} \mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}} \mathrm{e}^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right)^2 \left(\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right)^2 \quad (2072)$$

$$= B_{ij}^2(t) \left| B_{ij}(t) \right|^2 \text{ (by (1755))}, \quad (2073)$$

$$\left\langle B_x^2(t) \right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4} \left(\left\langle \left(B_1^+(t) B_0^-(t) \right)^2 + 2\mathbb{I} + \left(B_0^+(t) B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{10}(t) + B_{01}(t) \right)^2 \right) \text{ (by (2054))} \quad (2074)$$

$$= \frac{1}{4} \left(\left\langle \left(B_1^+(t) B_0^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + 2 \langle \mathbb{I} \rangle_{\overline{H_{\bar{B}}}} + \left\langle \left(B_0^+(t) B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{10}(t) + B_{01}(t) \right)^2 \right) \quad (2075)$$

$$= \frac{1}{4} \left(\left\langle \left(B_1^+(t) B_0^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + 2 + \left\langle \left(B_0^+(t) B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{10}(t) + B_{01}(t) \right)^2 \right) \quad (2076)$$

$$= \frac{1}{4} \left(B_{10}^2(t) \left| B_{10}^2(t) \right| + 2 + B_{01}^2(t) \left| B_{01}^2(t) \right| - \left(B_{10}^2(t) + 2 B_{10}(t) B_{01}(t) + B_{01}^2(t) \right) \right) \text{ (by (2073))} \quad (2077)$$

$$= \frac{1}{4} \left(B_{10}^2(t) \left| B_{10}^2(t) \right| + 2 + B_{01}^2(t) \left| B_{10}^2(t) \right| - \left(B_{10}^2(t) + 2 \left| B_{10}^2(t) \right| + B_{01}^2(t) \right) \right) \quad (2078)$$

$$= \frac{1}{4} \left(B_{10}^2(t) + B_{01}^2(t) - 2 \right) \left(\left| B_{10}^2(t) \right| - 1 \right), \quad (2079)$$

$$\left\langle B_y^2(t) \right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Var}_{\overline{H_{\bar{B}}}}(B_y(t)) + \langle B_y(t) \rangle^2_{\overline{H_{\bar{B}}}} \quad (2080)$$

$$= \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(\frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \right) \text{ (by } \langle B_y(t) \rangle_{\overline{H_{\bar{B}}}} = 0 \text{ and (1755))} \quad (2081)$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t) + B_{10}(t) - B_{01}(t)\right) \tag{2082}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right) \tag{2083}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{2084}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - 2\mathbb{I} + \left(B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{2085}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - 2\langle\mathbb{I}\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)^2\right) \tag{2086}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{01}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right) \text{ (by (2073))} \tag{2087}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{01}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + 2B_{01}(t)\,B_{10}(t) - B_{10}^2(t)\right) \tag{2088}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\,|B_{10}(t)|^2 - 2 + B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + 2\,|B_{10}(t)|^2 - B_{10}^2(t)\right) \tag{2089}$$

$$= -\frac{1}{4}\left(B_{10}^2(t) + B_{01}^2(t) + 2\right)\left(|B_{10}(t)|^2 - 1\right), \tag{2090}$$

$$\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_x(t)\,\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{2091}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\,\langle B_x(t)\rangle_{\overline{H_{\bar{B}}}} \tag{2092}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\cdot 0 \text{ (by } \langle B_x(t)\rangle_{\overline{H_{\bar{B}}}} = 0) \tag{2093}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{2094}$$

$$= \left\langle \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1755))} \tag{2095}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle\left(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)\right)\left(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\right)\big\rangle_{\overline{H_{\bar{B}}}} - \big\langle\left(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\right)\big\rangle_{\overline{H_{\bar{B}}}} \tag{2096}$$

$$\times\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{2097}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle B_1^+(t)\,B_0^-(t)\,B_0^+(t)B_1^-(t) - B_1^+(t)\,B_0^-(t)\,B_1^+(t)\,B_0^-(t) + B_0^+(t)B_1^-(t)B_0^+(t)B_1^-(t) - B_0^+(t)B_1^-(t) \tag{2098}$$

$$\times B_1^+(t)B_0^-(t)\big\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{2099}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\mathbb{I} - \left(B_1^+(t)\,B_0^-(t)\right)^2 + \left(B_0^+(t)\,B_1^-(t)\right)^2 - \mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{2100}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{2101}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{01}(t) + B_{10}(t)\right)\right) \tag{2102}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}^2(t) - B_{10}^2(t)\right)\right) \tag{2103}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\,|B_{10}(t)|^2 - B_{10}^2(t)\,|B_{10}(t)|^2 - B_{01}^2(t) + B_{10}^2(t)\right) \text{ (by (2073))} \tag{2104}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t) - B_{10}^2(t)\right)\left(|B_{10}(t)|^2 - 1\right), \tag{2105}$$

$$\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (1755))} \tag{2106}$$

$$= \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_y(t)\,\frac{B_{10}(t) + B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{2107}$$

$$= \frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t) + B_{01}(t)}{2}\,\langle B_y(t)\rangle_{\overline{H_{\bar{B}}}} \tag{2108}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar B}}} - \frac{B_{10}(t)+B_{01}(t)}{2}\cdot 0 \text{ (by } \langle B_y(t)\rangle_{\overline{H_{\bar B}}}=0) && (2109)\\
&= \frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar B}}} && (2110)\\
&= \frac{1}{2}\left\langle \frac{B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar B}}} \text{ (by (1755))} && (2111)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_0^+(t)B_1^-(t)-B_1^+(t)B_0^-(t)\right)\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right)\right\rangle_{\overline{H_{\bar B}}} + \frac{(B_{10}(t)-B_{01}(t))}{4\mathrm{i}}\left\langle\left(B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)\right)\right\rangle_{\overline{H_{\bar B}}} && (2112)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar B}}} + \frac{(B_{10}(t)-B_{01}(t))\,(B_{10}(t)+B_{01}(t))}{4\mathrm{i}} && (2113)\\
&= \frac{1}{4\mathrm{i}}\left\langle\left(B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\right)\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar B}}} + \frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} && (2114)\\
&= \frac{1}{4\mathrm{i}}\left\langle B_0^+(t)\,B_1^-(t)\,B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\,B_0^+(t)\,B_1^-(t)-B_1^+(t)\,B_0^-(t)\,B_1^+(t)\,B_0^-(t)-B_1^+(t)\,B_0^-(t)\,B_0^+(t)\,B_1^-(t)\right\rangle_{\overline{H_{\bar B}}} && (2115)\\
&\quad + \frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} && (2116)\\
&= \frac{1}{4\mathrm{i}}\left\langle \mathbb{I}+\left(B_0^+(t)\,B_1^-(t)\right)^2-\left(B_1^+(t)\,B_0^-(t)\right)^2-\mathbb{I}\right\rangle_{\overline{H_{\bar B}}} + \frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} && (2117)\\
&= \frac{1}{4\mathrm{i}}\left\langle \left(B_0^+(t)\,B_1^-(t)\right)^2-\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar B}}} + \frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} && (2118)\\
&= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\,|B_{10}(t)|^2-B_{10}^2(t)\,|B_{10}(t)|^2\right) + \frac{B_{10}^2(t)-B_{01}^2(t)}{4\mathrm{i}} \text{ (by (2073))} && (2119)\\
&= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\,|B_{10}(t)|^2-B_{10}^2(t)\,|B_{10}(t)|^2+B_{10}^2(t)-B_{01}^2(t)\right) && (2120)\\
&= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)-B_{10}^2(t)\right)\left(|B_{10}(t)|^2-1\right). && (2121)
\end{aligned}
$$

Summarizing the expected values obtained in the precedent lines we have:

$$
\begin{aligned}
\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar B}}} &= \sum_{\mathbf{k}} |g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), && (2122)\\
\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}} &= \frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right), && (2123)\\
\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}} &= \frac{B_{10}(t)+B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)\right), && (2124)\\
\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} &= \frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)\right), && (2125)\\
\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} &= \frac{B_{01}(t)+B_{10}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*(N_{\mathbf{k}}+1)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^*\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\right), && (2126)\\
\langle B_x^2(t)\rangle_{\overline{H_{\bar B}}} &= \frac{1}{4}\left(B_{10}^2(t)+B_{01}^2(t)-2\right)\left(\left|B_{10}^2(t)\right|-1\right), && (2127)\\
\langle B_y^2(t)\rangle_{\overline{H_{\bar B}}} &= -\frac{1}{4}\left(B_{10}^2(t)+B_{01}^2(t)+2\right)\left(|B_{10}(t)|^2-1\right), && (2128)\\
\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar B}}} &= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)-B_{10}^2(t)\right)\left(|B_{10}(t)|^2-1\right), && (2129)\\
\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar B}}} &= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)-B_{10}^2(t)\right)\left(|B_{10}(t)|^2-1\right). && (2130)
\end{aligned}
$$

The density matrix associated to $\rho_{\overline{S}}=\frac{\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}}(t)}\right)}\equiv\sum\rho_{\overline{S},ij}|i\rangle\langle j|$ has the following element

$$\rho_{\overline{S},00}=\frac{1}{2}-\frac{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}}, \tag{2131}$$

$$\rho_{\overline{S},01}=-\frac{B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}}, \tag{2132}$$

$$\rho_{\overline{S},10}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}}, \tag{2133}$$

$$\rho_{\overline{S},11}=\frac{1}{2}+\frac{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)\right|^{2}\left|V_{10}\left(t\right)\right|^{2}}}. \tag{2134}$$

The expected values respect to the system $\overline{S}$ of relevance for calculating $\left\langle \overline{H_{I}}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)}}$ are $\langle |i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle |i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle |i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle \sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle \sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$ and $\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}$, we took account that $\sigma_x\sigma_y = \mathrm{i}\sigma_z$ and $\sigma_y\sigma_x = -\mathrm{i}\sigma_z$. The values needed for our calculation are:

$$\langle|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}-\frac{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{2135}$$

$$\langle|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}+\frac{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{2\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{2136}$$

$$\langle|0\rangle\langle 0|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{2137}$$

$$\langle|1\rangle\langle 1|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}^{*}\left(t\right)V_{10}^{*}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{2138}$$

$$\langle|0\rangle\langle 0|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}\right)}{\sqrt{\left(\sum_{i}\left(-1\right)^{i}\left(\varepsilon_{i}\left(t\right)+R_{i}\left(t\right)\right)\right)^{2}+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^{2}}}, \tag{2139}$$

$$\langle |1\rangle\langle 1|\sigma_y\rangle_{\overline{H_{\bar{S}}}(t)} = \frac{\mathrm{i}B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{2140}$$

$$\langle \sigma_x|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}}(t)} = -\frac{B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{2141}$$

$$\langle \sigma_x|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}}(t)} = -\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{2142}$$

$$\langle \sigma_y|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}}(t)} = \frac{\mathrm{i}B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{2143}$$

$$\langle \sigma_y|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}}(t)} = -\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{2144}$$

$$\langle \sigma_z\rangle_{\overline{H_{\bar{S}}}(t)} = \frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}. \tag{2145}$$

Our next step is to find $v_3\left(t\right)$, the commutator $\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]$ is a central point for our calculations and it is equal to:

$$\left[\overline{H_0}(t),\overline{H_{\bar{I}}}(t)\right]=\Big[\left(\varepsilon_0(t)+R_0(t)\right)|0\rangle\langle 0|+\left(\varepsilon_1(t)+R_1(t)\right)|1\rangle\langle 1|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{2146}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\Big]\ \text{(by (1745) and (1746))} \tag{2147}$$

$$=\Bigg[\sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, \tag{2148}$$

$$\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\Bigg]\ \text{(introduced }\textstyle\sum\text{ to decrease the size)} \tag{2149}$$

$$=\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|\sum_{i'}B_{i'z}(t)|i'\rangle\langle i'|+\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i| \tag{2150}$$

$$\times V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{2151}$$

$$\times V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)\right. \tag{2152}$$

$$\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)\right. \tag{2153}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{2154}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{i'}(\varepsilon_{i'}(t)+R_{i'}(t))\,|i'\rangle\langle i'|-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) \tag{2155}$$

$$+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{2156}$$

$$\times\sigma_x\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{2157}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{2158}$$

$$+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{2159}$$

$$=\sum_{i,i'}(\varepsilon_i(t)+R_i(t))\delta_{ii'}B_{i'z}(t)|i\rangle\langle i'|+\sum_i(\varepsilon_i(t)+R_i(t))V_{10}^{\Re}(t)\,|i\rangle\langle i|\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i(\varepsilon_i(t)+R_i(t))V_{10}^{\Im}(t)|i\rangle\langle i|(\sigma_x B_y(t)-\sigma_y \tag{2160}$$

$$\times B_x(t))+\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i B_{iz}(t)\,\sigma_x|i\rangle\langle i|+V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\big(\sigma_x^2 B_x(t)+\sigma_x\sigma_y B_y(t)\big)+V_{10}^{\Im}(t) \tag{2161}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\big(\sigma_x^2 B_y(t)-\sigma_x\sigma_y B_x(t)\big)-\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i B_{iz}(t)\,\sigma_y|i\rangle\langle i|-V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t) \tag{2162}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\big(\sigma_y\sigma_x B_x(t)+\sigma_y^2 B_y(t)\big)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\big(\sigma_y\sigma_x B_y(t)-\sigma_y^2 B_x(t)\big)+\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\,|i\rangle\langle i| \tag{2163}$$

$$+V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_{i,i'}(\varepsilon_i(t)+R_i(t))\delta_{ii'}B_{i'z}(t)|i\rangle\langle i'|-\Big(B_{10}^{\Re}(t) \tag{2164}$$

$$\times V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x+\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y-V_{10}^{\Re}(t)\sum_i(\varepsilon_i(t)+R_i(t))(\sigma_x B_x(t) \tag{2165}$$

$$+\sigma_y B_y(t))|i\rangle\langle i|-V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\big(\sigma_x^2 B_x(t)+\sigma_y\sigma_x B_y(t)\big)+V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)(\sigma_x\sigma_y B_x(t) \tag{2166}$$

$$+\sigma_y^2 B_y(t))-V_{10}^{\Re}(t)\sum_{\mathbf{k}}(\sigma_x B_x(t)+\sigma_y B_y(t))\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)\sum_i(\varepsilon_i(t)+R_i(t))(\sigma_x B_y(t)-\sigma_y B_x(t))\,|i\rangle\langle i|-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{2167}$$

$$\left.-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\big(\sigma_x^2 B_y(t)-\sigma_y\sigma_x B_x(t)\big)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\big(\sigma_x\sigma_y B_y(t)-\sigma_y^2 B_x(t)\big)-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y \tag{2168}$$

$$\times B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{2169}$$

$$=\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))+\sigma_x\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) \tag{2170}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|+\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\;(B_x(t)+\mathrm{i}\sigma_z B_y(t))+\sigma_x\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t) \tag{2171}$$

$$\times(B_y(t)-\mathrm{i}\sigma_z B_x(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\,(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{2172}$$

$$-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{2173}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{2174}$$

$$\left.+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{2175}$$

$$\times\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)+V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{2176}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-V_{10}^{\Im}(t)\,(B_y(t)+\mathrm{i}\sigma_z B_x(t))\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big) \tag{2177}$$

$$+V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{2178}$$

We will obtain a neat form of $\left[\,\overline{H_0}\,(t)\,,\overline{H_{\bar{I}}}\,(t)\right]$ as we will see:

$$[|0\rangle\langle 0|,\sigma_x]=|0\rangle\langle 0|\,(|0\rangle\langle 1|+|1\rangle\langle 0|)-(|0\rangle\langle 1|+|1\rangle\langle 0|)\,|0\rangle\langle 0| \tag{2179}$$

$$=|0\rangle\langle 1|-|1\rangle\langle 0| \tag{2180}$$

$$=-\mathrm{i}\sigma_y, \tag{2181}$$

$$
\begin{aligned}
[|0\rangle\langle 0|, \sigma_x] &= |0\rangle\langle 0| \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) - \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) |0\rangle\langle 0| && (2182)\\
&= |0\rangle\langle 1| - |1\rangle\langle 0| && (2183)\\
&= -\mathrm{i}\sigma_y, && (2184)\\
[|1\rangle\langle 1|, \sigma_x] &= |1\rangle\langle 1| \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) - \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) |1\rangle\langle 1| && (2185)\\
&= |1\rangle\langle 0| - |0\rangle\langle 1| && (2186)\\
&= \mathrm{i}\sigma_y, && (2187)\\
[|i\rangle\langle i|, \sigma_x] &= (-1)^{i+1}\, \mathrm{i}\sigma_y, && (2188)\\
[|0\rangle\langle 0|, \sigma_y] &= |0\rangle\langle 0| \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) - \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) |0\rangle\langle 0| && (2189)\\
&= \mathrm{i}|0\rangle\langle 1| + \mathrm{i}|1\rangle\langle 0| && (2190)\\
&= \mathrm{i}\sigma_x, && (2191)\\
[|1\rangle\langle 1|, \sigma_y] &= |1\rangle\langle 1| \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) - \left(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|\right) |1\rangle\langle 1| && (2192)\\
&= -\mathrm{i}|1\rangle\langle 0| - \mathrm{i}|0\rangle\langle 1| && (2193)\\
&= -\mathrm{i}\sigma_x, && (2194)\\
[|i\rangle\langle i|, \sigma_y] &= (-1)^{i}\, \mathrm{i}\sigma_x, && (2195)\\
\left[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)\right] &= \sum_{i,i'} (\varepsilon_i(t) + R_i(t))(\delta_{ii'} - \delta_{ii'})\, B_{i'z}(t) |i\rangle\langle i'| + V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t))((|i\rangle\langle i|\sigma_x - \sigma_x|i\rangle\langle i|)\, B_y(t) - (|i\rangle\langle i|\sigma_y && (2196)\\
&\quad -\sigma_y|i\rangle\langle i|)\, B_x(t)) + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t)\, (\sigma_x|i\rangle\langle i| - |i\rangle\langle i|\sigma_x) + V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t)\right. && (2197)\\
&\quad \left.-B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \left(\sigma_x^2 B_x(t) + \sigma_x\sigma_y B_y(t) - \sigma_x^2 B_x(t) - \sigma_y\sigma_x B_y(t)\right) + V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) && (2198)\\
&\quad \times \left(\sigma_x^2 B_y(t) - \sigma_x\sigma_y B_x(t) - \sigma_x^2 B_y(t) + \sigma_y\sigma_x B_x(t)\right) - \left(B_{10}^{\Re}(t)\, V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\, V_{10}^{\Re}(t)\right) \sum_i B_{iz}\,(t)\, (\sigma_y|i\rangle\langle i| && (2199)\\
&\quad -|i\rangle\langle i|\sigma_y) - V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \left(\sigma_y\sigma_x B_x(t) + \sigma_y^2 B_y(t) - \sigma_x\sigma_y B_x(t) - \sigma_y^2 B_y(t)\right) - V_{10}^{\Im}(t) && (2200)\\
&\quad \times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \left(\sigma_y\sigma_x B_y(t) - \sigma_y^2 B_x(t) - \sigma_x\sigma_y B_y(t) + \sigma_y^2 B_x(t)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{iz}(t)\right. && (2201)\\
&\quad \left.-B_{iz}(t)\, b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\, (\sigma_x B_x(t) + \sigma_y B_y(t)) - (\sigma_x B_x(t) + \sigma_y B_y(t))\, \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) + V_{10}^{\Im}(t) && (2202)\\
&\quad \times \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\, (\sigma_x B_y\,(t) - \sigma_y B_x\,(t)) - (\sigma_x B_y(t) - \sigma_y B_x(t))\, \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) && (2203)\\
&= V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t))\, ([|i\rangle\langle i|, \sigma_x]\, B_y(t) - [|i\rangle\langle i|, \sigma_y]\, B_x(t)) + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t) && (2204)\\
&\quad \times [\sigma_x, |i\rangle\langle i|] + V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) [\sigma_x, \sigma_y]\, B_y(t) + V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) && (2205)\\
&\quad \times [\sigma_y, \sigma_x]\, B_x(t) - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) [\sigma_y, |i\rangle\langle i|] - V_{10}^{\Re}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) && (2206)\\
&\quad \times [\sigma_y, \sigma_x]\, B_x(t) - V_{10}^{\Im}(t) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) [\sigma_y, \sigma_x]\, B_y(t) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{iz}(t)\right] |i\rangle\langle i| + V_{10}^{\Re}(t) && (2207)\\
&\quad \times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\,(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\,(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y\,(t)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x\,(t)\right]\right), && (2208)\\
\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{iz}(t)\right] &= \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t))\, b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t))^{*}\, b_{\mathbf{k}'}\right)\right] && (2209)\\
&= \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t))^{*}\, b_{\mathbf{k}}\right) + \sum_{\mathbf{k}' \neq \mathbf{k}} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t))\, b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\,(t))^{*}\, b_{\mathbf{k}'}\right)\right] && (2210)\\
&= \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right] + \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_{\mathbf{k}' \neq \mathbf{k}} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))\, b_{\mathbf{k}'}^{\dagger} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^{*}\, b_{\mathbf{k}'}\right)\right] && (2211)\\
&= (g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t)) \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right] + (g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t))^{*} \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, b_{\mathbf{k}}\right] && (2212)\\
&= (g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t))\, b_{\mathbf{k}}^{\dagger} \left[b_{\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right] + (g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t))^{*} \left[b_{\mathbf{k}}^{\dagger}, b_{\mathbf{k}}\right] b_{\mathbf{k}} && (2213)
\end{aligned}
$$

$$= (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}, \tag{2214}$$

$$[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)] = V_{10}^{\Im}(t) \sum_i (\varepsilon_i(t) + R_i(t))\, (-1)^{i+1}\, \mathrm{i}\, (\sigma_y B_y(t) + \sigma_x B_x(t)) + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t)\, (-1)^i\, \mathrm{i}\sigma_y \tag{2215}$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)\, (-1)^{i+1}\, \mathrm{i}\sigma_x + 2\mathrm{i} B_{10}^{\Re}(t) \left(\left(V_{10}^{\Re}(t)\right)^2 + \left(V_{10}^{\Im}(t)\right)^2\right) \sigma_z B_y(t) + 2\mathrm{i} B_{10}^{\Im}(t) \tag{2216}$$

$$\times \left(\left(V_{10}^{\Re}(t)\right)^2 + \left(V_{10}^{\Im}(t)\right)^2\right) \sigma_z B_x(t) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{iz}(t)\right] |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] \sigma_x + \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right. \tag{2217}$$

$$\left.\times \sigma_y\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right]\right) \tag{2218}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i}\, (\sigma_y B_y(t) + \sigma_x B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))\, (-1)^{i+1} + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t)\, (-1)^i\, \mathrm{i}\sigma_y \tag{2219}$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)\, (-1)^{i+1}\, \mathrm{i}\sigma_x + 2\mathrm{i}\, |V_{10}(t)|^2\, \sigma_z \left(B_{10}^{\Re}(t)\, B_y(t) + B_{10}^{\Im}(t)\, B_x(t)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \tag{2220}$$

$$\times \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \tag{2221}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right]\right) \tag{2222}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i}\, (\sigma_y B_y(t) + \sigma_x B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))\, (-1)^{i+1} + \left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \sum_i B_{iz}(t)\, (-1)^i\, \mathrm{i}\sigma_y \tag{2223}$$

$$- \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t)\, (-1)^{i+1}\, \mathrm{i}\sigma_x + 2\mathrm{i}\, |V_{10}(t)|^2\, \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} |i\rangle\langle i| \tag{2224}$$

$$\times \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right) + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \tag{2225}$$

$$\times \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right]\right) \tag{2226}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i}\, (\sigma_y B_y(t) + \sigma_x B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))(-1)^{i+1} + 2\mathrm{i}\, |V_{10}(t)|^2\, \sigma_z \left(B_{10}^{\Re}(t)\, B_y(t) + B_{10}^{\Im}(t)\, B_x(t)\right) + \sum_i B_{iz}(t) \tag{2227}$$

$$\times (-1)^i \left(\left(B_{10}^{\Re}(t)\, V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\, V_{10}^{\Im}(t)\right) \mathrm{i}\sigma_y + \left(B_{10}^{\Re}(t)\, V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\, V_{10}^{\Re}(t)\right) \mathrm{i}\sigma_x\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger}\right. \tag{2228}$$

$$\left.- (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right. \tag{2229}$$

$$\left.- \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right]\right) \tag{2230}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i}\, (\sigma_y B_y(t) + \sigma_x B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))(-1)^{i+1} + 2\mathrm{i}\, |V_{10}(t)|^2\, \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_i B_{iz}(t) \tag{2231}$$

$$\times (-1)^i \left(\left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) (|1\rangle\langle 0| - |0\rangle\langle 1|) + \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \mathrm{i}\, (|1\rangle\langle 0| + |0\rangle\langle 1|)\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \tag{2232}$$

$$\times \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \tag{2233}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right]\right) \tag{2234}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i}\, (\sigma_y B_y(t) + \sigma_x B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))(-1)^{i+1} + 2\mathrm{i}\, |V_{10}(t)|^2\, \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_i B_{iz}(t) \tag{2235}$$

$$\times (-1)^i \left(\left((B_{10}(t)\, V_{10}(t))^{\Re} + \mathrm{i}\, (B_{10}(t)\, V_{10}(t))^{\Im}\right) |1\rangle\langle 0| + \left((B_{10}(t)\, V_{10}(t))^{\Re} - \mathrm{i}\, (B_{10}(t)\, V_{10}(t))^{\Im}\right) |0\rangle\langle 1|\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \tag{2236}$$

$$\times \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \tag{2237}$$

$$\times \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right]\right) \tag{2238}$$

$$= V_{10}^{\Im}(t)\, \mathrm{i}\, (\sigma_y B_y(t) + \sigma_x B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t))(-1)^{i+1} + 2\mathrm{i}\, |V_{10}(t)|^2\, \sigma_z \left(B_{10}^{\Re}(t) B_y(t) + B_{10}^{\Im}(t) B_x(t)\right) + \sum_i B_{iz}(t) \tag{2239}$$

$$\times (-1)^i \left(B_{10}(t)\, V_{10}(t)\, |1\rangle\langle 0| + B_{10}^{*}(t)\, V_{10}^{*}(t)\, |0\rangle\langle 1|\right) + \sum_{i,\mathbf{k}} \omega_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^{*}\, b_{\mathbf{k}}\right) |i\rangle\langle i| + V_{10}^{\Re}(t) \tag{2240}$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right] + \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t) \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\sigma_x \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_y(t)\right] - \sigma_y \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_x(t)\right]\right), \tag{2241}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{x}(t)\right]=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{10}\left(t\right)-B_{01}\left(t\right)}{2}\right] \tag{2242}$$

$$=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)}{2}\right]+\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},-\frac{B_{10}\left(t\right)+B_{01}\left(t\right)}{2}\right] \tag{2243}$$

$$=\frac{1}{2}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right]+\frac{1}{2}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)\right] \tag{2244}$$

$$=\frac{1}{2}\left(\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right]+\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right]\right) \tag{2245}$$

$$=\frac{1}{2}\left(\mathrm{e}^{\chi_{10}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\right. \tag{2246}$$

$$\left.\times\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right), \tag{2247}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{y}(t)\right]=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)+B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}\right] \tag{2248}$$

$$=\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)-B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)}{2\mathrm{i}}\right]+\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},\frac{B_{10}\left(t\right)-B_{01}\left(t\right)}{2\mathrm{i}}\right] \tag{2249}$$

$$=\frac{1}{2\mathrm{i}}\left(\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{0}^{+}\left(t\right)B_{1}^{-}\left(t\right)\right]-\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},B_{1}^{+}\left(t\right)B_{0}^{-}\left(t\right)\right]\right) \tag{2250}$$

$$=\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\chi_{01}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)-\mathrm{e}^{\chi_{10}(t)}\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\frac{v_{1\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}\left(t\right)}{\omega_{\mathbf{k}}}\right)\right]\right. \tag{2251}$$

$$\left.\times\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\frac{v_{1\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}\left(t\right)}{\omega_{\mathbf{k}'}}\right)\right). \tag{2252}$$

We will focus on the term $\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]$:

$$D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)=b_{\mathbf{k}}+\alpha_{\mathbf{k}}\,\text{(by properties of the displacement operator)}\,, \tag{2253}$$

$$D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)=b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\,\text{(by properties of the displacement operator)}\,, \tag{2254}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]=b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)-D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{2255}$$

$$=b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)-D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(-\alpha_{\mathbf{k}}\right)D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\,\text{(introducing }\mathbb{I}=D\left(-\alpha\right)D\left(\alpha\right)\text{)} \tag{2256}$$

$$=b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\left(D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right)-\left(D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(-\alpha_{\mathbf{k}}\right)\right)D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}} \tag{2257}$$

$$=b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}\right)D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}} \tag{2258}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}\right)\left(D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(-\alpha_{\mathbf{k}}\right)\right)D\left(\alpha_{\mathbf{k}}\right) \tag{2259}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}-\alpha_{\mathbf{k}}\right)D\left(\alpha_{\mathbf{k}}\right) \tag{2260}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^{2}\right)-\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^{2}\right)D\left(\alpha_{\mathbf{k}}\right) \tag{2261}$$

$$=D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)-\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}-\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)D\left(\alpha_{\mathbf{k}}\right) \tag{2262}$$

$$=D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right) \tag{2263}$$

$$=\left[D\left(\alpha_{\mathbf{k}}\right),b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right]+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right) \tag{2264}$$

$$=-\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right), \tag{2265}$$

$$\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}},D\left(\alpha_{\mathbf{k}}\right)\right]=\frac{1}{2}\left(\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right) \tag{2266}$$

$$=\frac{1}{2}\left(\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}+\alpha_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)+\alpha_{\mathbf{k}}^{*}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right) \tag{2267}$$

$$= \frac{1}{2}\left(\alpha_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right) b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} D\left(\alpha_{\mathbf{k}}\right) b_{\mathbf{k}} + \alpha_{\mathbf{k}} D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) + \alpha_{\mathbf{k}}^{*} D\left(\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\right) \tag{2268}$$

$$= \frac{D\left(\alpha_{\mathbf{k}}\right)}{2}\left(\alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \alpha_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*}\right) + \alpha_{\mathbf{k}}^{*}\left(b_{\mathbf{k}} + \alpha_{\mathbf{k}}\right)\right) \tag{2269}$$

$$= \frac{D\left(\alpha_{\mathbf{k}}\right)}{2}\left(\alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \left|\alpha_{\mathbf{k}}\right|^{2} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \left|\alpha_{\mathbf{k}}\right|^{2}\right) \tag{2270}$$

$$= D\left(\alpha_{\mathbf{k}}\right)\left(\alpha_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + \alpha_{\mathbf{k}}^{*} b_{\mathbf{k}} + \left|\alpha_{\mathbf{k}}\right|^{2}\right), \tag{2271}$$

$$\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{i}^{+}(t) B_{j}^{-}(t)\right] = \mathrm{e}^{\chi_{ij}(t)}\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\right] \prod_{\mathbf{k}' \neq \mathbf{k}} D\left(\frac{v_{i\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \tag{2272}$$

$$= \mathrm{e}^{\chi_{ij}(t)} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)\left(\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) \tag{2273}$$

$$\times \prod_{\mathbf{k}' \neq \mathbf{k}} D\left(\frac{v_{i\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \tag{2274}$$

$$= \mathrm{e}^{\chi_{ij}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{i\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\left(\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) \tag{2275}$$

$$= B_{i}^{+}(t) B_{j}^{-}(t)\left(\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right), \tag{2276}$$

$$v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t) \equiv v_{ij\mathbf{k}}, \tag{2277}$$

$$v_{ij\mathbf{k}} = -v_{ji\mathbf{k}}, \tag{2278}$$

$$\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{x}(t)\right] = \frac{1}{2}\left(\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{1}^{+}(t) B_{0}^{-}(t)\right] + \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{0}^{+}(t) B_{1}^{-}(t)\right]\right) \tag{2279}$$

$$= \frac{1}{2}\left(B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) + B_{0}^{+}(t) B_{1}^{-}(t)\left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. \tag{2280}$$

$$\left.\left. + \left|\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{2281}$$

$$= \frac{1}{2}\left(B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) + B_{0}^{+}(t) B_{1}^{-}(t)\left(-\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. \tag{2282}$$

$$\left.\left. + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{2283}$$

$$= \frac{1}{2}\left(\left(B_{1}^{+}(t) B_{0}^{-}(t) - B_{0}^{+}(t) B_{1}^{-}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right) + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{1}^{+}(t) B_{0}^{-}(t) + B_{0}^{+}(t) B_{1}^{-}(t)\right)\right) \tag{2284}$$

$$= \frac{1}{2}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{1}^{+}(t) B_{0}^{-}(t) + B_{0}^{+}(t) B_{1}^{-}(t)\right) - \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right)\right) \tag{2285}$$

$$= \frac{1}{2}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(2 B_{x}(t) + B_{10}(t) + B_{01}(t)\right) - \left(2\mathrm{i} B_{y}(t) - B_{10}(t) + B_{01}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right)\right) \tag{2286}$$

$$= \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{x}(t) + B_{10}^{\Re}\right) - \mathrm{i}\left(B_{y}(t) - B_{10}^{\Im}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right), \tag{2287}$$

$$\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{y}(t)\right] = \frac{1}{2\mathrm{i}}\left(\left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{0}^{+}(t) B_{1}^{-}(t)\right] - \left[b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, B_{1}^{+}(t) B_{0}^{-}(t)\right]\right) \tag{2288}$$

$$= \frac{1}{2\mathrm{i}}\left(B_{0}^{+}(t) B_{1}^{-}(t)\left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) - B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. \tag{2289}$$

$$\left.\left. + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{2290}$$

$$= \frac{1}{2\mathrm{i}}\left(B_{0}^{+}(t) B_{1}^{-}(t)\left(-\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}} + \left|\frac{v_{01\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right) - B_{1}^{+}(t) B_{0}^{-}(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right.\right. \tag{2291}$$

$$\left.\left. + \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right)\right) \tag{2292}$$

$$= \frac{1}{2\mathrm{i}}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t)\right) - \left(B_{0}^{+}(t) B_{1}^{-}(t) + B_{1}^{+}(t) B_{0}^{-}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*} b_{\mathbf{k}}\right)\right) \tag{2293}$$

$$= \frac{1}{2\mathrm{i}}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 (2\mathrm{i}B_y(t) - B_{10}(t) + B_{01}(t)) - (2B_x(t) + B_{10}(t) + B_{01}(t))\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) \quad (2294)$$

$$= \frac{1}{2\mathrm{i}}\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \left(2\mathrm{i}B_y(t) - 2\mathrm{i}B_{10}^{\Im}(t)\right) - \left(2B_x(t) + 2B_{10}^{\Re}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) \quad (2295)$$

$$= \left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2 \left(B_y(t) - B_{10}^{\Im}(t)\right) + \mathrm{i}\left(B_x(t) + B_{10}^{\Re}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right). \quad (2296)$$

The term that we will rewrite is defined as:

$$A_{T\mathbf{k}}(t) \equiv V_{10}^{\Re}(t)\left(\sigma_x\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right] + \sigma_y\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right]\right) + V_{10}^{\Im}(t)\left(\sigma_x\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right] - \sigma_y\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right]\right) \quad (2297)$$

$$= \left(V_{10}^{\Re}(t)\sigma_x - V_{10}^{\Im}(t)\sigma_y\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right] + \left(V_{10}^{\Re}(t)\sigma_y + V_{10}^{\Im}(t)\sigma_x\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right] \quad (2298)$$

$$= \left(V_{10}^{\Re}(t)(|1\rangle\langle 0| + |0\rangle\langle 1|) - V_{10}^{\Im}(t)(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right] + \left(V_{10}^{\Re}(t)(-\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1|) + V_{10}^{\Im}(t)(|1\rangle\langle 0| + |0\rangle\langle 1|)\right) \quad (2299)$$

$$\times\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right] \quad (2300)$$

$$= \left(|1\rangle\langle 0|\left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right) + |0\rangle\langle 1|\left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right] + \left(|1\rangle\langle 0|\left(-\mathrm{i}V_{10}^{\Re}(t) + V_{10}^{\Im}(t)\right) + |0\rangle\langle 1|\left(\mathrm{i}V_{10}^{\Re}(t)\right.\right. \quad (2301)$$

$$\left.\left. + V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right] \quad (2302)$$

$$= \left(|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right] - \mathrm{i}\left(|1\rangle\langle 0|\left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right) + |0\rangle\langle 1|\left(-V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right] \quad (2303)$$

$$= \left(|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right] - \mathrm{i}\left(|1\rangle\langle 0|\left(V_{10}^{\Re}(t) + \mathrm{i}V_{10}^{\Im}(t)\right) - |0\rangle\langle 1|\left(V_{10}^{\Re}(t) - \mathrm{i}V_{10}^{\Im}(t)\right)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right] \quad (2304)$$

$$= \left(|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_x(t)\right] - \mathrm{i}\left(|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t)\right)\left[b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}, B_y(t)\right] \quad (2305)$$

$$= \left(|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t)\right)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(B_x(t) + B_{10}^{\Re}(t)\right) - \mathrm{i}\left(B_y(t) - B_{10}^{\Im}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^*\right)b_{\mathbf{k}}\right) \quad (2306)$$

$$- \mathrm{i}\left(|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t)\right)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(B_y(t) - B_{10}^{\Im}(t)\right) + \mathrm{i}\left(B_x(t) + B_{10}^{\Re}(t)\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right), \quad (2307)$$

$$B_x(t) + B_{10}^{\Re}(t) = \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} + B_{10}^{\Re}(t) \quad (2308)$$

$$= \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - 2B_{10}^{\Re}(t)}{2} + B_{10}^{\Re}(t) \quad (2309)$$

$$= \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}, \quad (2310)$$

$$B_y(t) - B_{10}^{\Im}(t) = \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} - B_{10}^{\Im}(t) \quad (2311)$$

$$= \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + 2\mathrm{i}B_{10}^{\Im}(t)}{2\mathrm{i}} - B_{10}^{\Im}(t) \quad (2312)$$

$$= \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2\mathrm{i}}, \quad (2313)$$

$$A_{T\mathbf{k}}(t) = \left(|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t)\right)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}\right) - \mathrm{i}\left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2\mathrm{i}}\right)\right. \quad (2314)$$

$$\left.\times\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) - \mathrm{i}\left(|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t)\right)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2\mathrm{i}}\right)\right. \quad (2315)$$

$$\left. + \mathrm{i}\left(\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) \quad (2316)$$

$$= \left(|1\rangle\langle 0|V_{10}(t) + |0\rangle\langle 1|V_{10}^*(t)\right)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)}{2}\right) - \left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right.\right. \quad (2317)$$

$$\left.\left. + \left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^* b_{\mathbf{k}}\right)\right) - \left(|1\rangle\langle 0|V_{10}(t) - |0\rangle\langle 1|V_{10}^*(t)\right)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2\left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)}{2}\right) - \left(b_{\mathbf{k}}^{\dagger}\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}} + b_{\mathbf{k}}\right.\right. \quad (2318)$$

$$\times\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\Bigg)\left(\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)}{2}\right)\Bigg) \tag{2319}$$

$$=(|1\rangle\langle 0|V_{10}(t)+|0\rangle\langle 1|V_{10}^{*}(t))\Bigg(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)}{2}\right)-\left(\frac{B_0^{+}(t)B_1^{-}(t)-B_1^{+}(t)B_0^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\Bigg) \tag{2320}$$

$$+(|1\rangle\langle 0|V_{10}(t)-|0\rangle\langle 1|V_{10}^{*}(t))\Bigg(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)B_0^{-}(t)-B_0^{+}(t)B_1^{-}(t)}{2}\right)+\left(\frac{B_1^{+}(t)B_0^{-}(t)+B_0^{+}(t)B_1^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\Bigg) \tag{2321}$$

$$=|1\rangle\langle 0|V_{10}\,(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}\,(t)\,B_0^{-}\,(t)+B_0^{+}\,(t)\,B_1^{-}\,(t)}{2}\right)-\left(\frac{B_0^{+}\,(t)\,B_1^{-}\,(t)-B_1^{+}\,(t)\,B_0^{-}\,(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right. \tag{2322}$$

$$\left.+\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)\,B_0^{-}(t)-B_0^{+}(t)\,B_1^{-}(t)}{2}\right)+\left(\frac{B_1^{+}(t)\,B_0^{-}(t)+B_0^{+}(t)\,B_1^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)+|0\rangle\langle 1|V_{10}^{*}\,(t) \tag{2323}$$

$$\times\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\left(\frac{B_1^{+}(t)\,B_0^{-}(t)+B_0^{+}(t)\,B_1^{-}(t)}{2}\right)-\left(\frac{B_0^{+}(t)\,B_1^{-}(t)-B_1^{+}(t)\,B_0^{-}(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)-\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}\right.\right. \tag{2324}$$

$$\left.\left.\times\left(\frac{B_1^{+}\,(t)\,B_0^{-}\,(t)-B_0^{+}\,(t)\,B_1^{-}\,(t)}{2}\right)+\left(\frac{B_1^{+}\,(t)\,B_0^{-}\,(t)+B_0^{+}\,(t)\,B_1^{-}\,(t)}{2}\right)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)\right) \tag{2325}$$

$$=|1\rangle\langle 0|V_{10}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}B_1^{+}\,(t)B_0^{-}\,(t)+B_1^{+}\,(t)B_0^{-}\,(t)\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)+|0\rangle\langle 1|V_{10}^{*}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}B_0^{+}\,(t)B_1^{-}\,(t)\right. \tag{2326}$$

$$\left.-B_0^{+}\,(t)\,B_1^{-}\,(t)\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right) \tag{2327}$$

$$=|1\rangle\langle 0|V_{10}\,(t)\,B_1^{+}(t)B_0^{-}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)+|0\rangle\langle 1|V_{10}^{*}\,(t)\,B_0^{+}(t)B_1^{-}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}-\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\right.\right. \tag{2328}$$

$$\left.\left.+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right). \tag{2329}$$

Inserting the precedent term in the sum of $\left[\,\overline{H_0}(t),\overline{H_{\overline{I}}}(t)\right]$ help us to obtain:

$$\left[\,\overline{H_0}(t),\overline{H_{\overline{I}}}(t)\right]=V_{10}^{\Im}(t)\,\mathrm{i}\,(\sigma_yB_y(t)+\sigma_xB_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\,(-1)^{i+1}+2\mathrm{i}\,|V_{10}(t)|^{2}\,\sigma_z\left(B_{10}^{\Re}(t)\,B_y(t)+B_{10}^{\Im}(t)\,B_x(t)\right) \tag{2330}$$

$$+\sum_i B_{iz}(t)(-1)^{i}(B_{10}(t)V_{10}(t)\,|1\rangle\langle 0|+B_{10}^{*}(t)V_{10}^{*}(t)\,|0\rangle\langle 1|)+\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}b_{\mathbf{k}}\right) \tag{2331}$$

$$\times|i\rangle\langle i|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(|1\rangle\langle 0|V_{10}(t)\,B_1^{+}(t)B_0^{-}(t)\left(\left|\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)+|0\rangle\langle 1|V_{10}^{*}(t)\right. \tag{2332}$$

$$\left.\times B_0^{+}\,(t)\,B_1^{-}\,(t)\left(\left|\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right|^{2}-\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}+\left(\frac{v_{10\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}}\right)^{*}b_{\mathbf{k}}\right)\right)\right). \tag{2333}$$

The term $\overline{H_{\overline{I}}}\,(t)\left[\,\overline{H_0}\,(t)\,,\overline{H_{\overline{I}}}\,(t)\right]$ is given by:

$$\overline{H_{\overline{I}}}(t)[\,\overline{H_0}(t),\overline{H_{\overline{I}}}(t)]=\left(\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\right)\left(\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\right. \tag{2334}$$

$$\times(\sigma_xB_x(t)+\sigma_yB_y(t))+\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{2335}$$

$$\times\sum_i B_{iz}(t)|i\rangle\langle i|+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_zB_y(t))+\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{2336}$$

$$\times V_{10}^{\Im}(t)(B_y(t)-\mathrm{i}\sigma_zB_x(t))-\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{2337}$$

$$\times V_{10}^{\Re}(t)(-\mathrm{i}\sigma_zB_x(t)+B_y(t))-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)(-\mathrm{i}\sigma_zB_y(t)-B_x(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t) \tag{2338}$$

$$\times|i\rangle\langle i|+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x \tag{2339}$$

$$\times\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y \tag{2340}$$

$$\times B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V^{\Re}_{10}(t)(B_x(t)-\mathrm{i}\sigma_z B_y(t))\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)+V^{\Re}_{10}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\,\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t) \tag{2341}$$

$$+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\Big(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\Big)\sum_i(\varepsilon_i(t)+R_i(t)) \tag{2342}$$

$$\times|i\rangle\langle i|-V^{\Im}_{10}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))+(\mathrm{i}\sigma_z B_y(t)-B_x(t))\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big)V^{\Im}_{10}(t)-V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\Big) \tag{2343}$$

$$=\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{2344}$$

$$+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\left(B^{\Re}_{10}(t)\,V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\,V^{\Im}_{10}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\left(B^{\Re}_{10}(t)\,V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\,V^{\Im}_{10}(t)\right)V^{\Re}_{10}(t)\,(B_x(t) \tag{2345}$$

$$+\mathrm{i}\sigma_z B_y(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B^{\Re}_{10}(t)\,V^{\Re}_{10}(t)-B^{\Im}_{10}(t)\,V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)\right. \tag{2346}$$

$$\left.+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)\,(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i| \tag{2347}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{2348}$$

$$\times V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}\,V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_i B_{iz}(t)\,|i\rangle\langle i|\,\sigma_x \tag{2349}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i| \tag{2350}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Re}_{10}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{2351}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Re}_{10}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)-\sum_i B_{iz}(t)|i\rangle\langle i|V^{\Re}_{10}(t) \tag{2352}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Im}_{10}(t) \tag{2353}$$

$$\times(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Im}_{10}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right) \tag{2354}$$

$$-\sum_i B_{iz}(t)\,|i\rangle\langle i|V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|V^{\Re}_{10}(t)\,(\sigma_x B_x(t) \tag{2355}$$

$$+\sigma_y B_y(t))+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x \tag{2356}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|+V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Re}_{10}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t)) \tag{2357}$$

$$+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_y \tag{2358}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i B_{iz}(t)|i\rangle\langle i|-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{2359}$$

$$-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{2360}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{2361}$$

$$\times V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Re}_{10}(t)\,(\sigma_x B_x(t) \tag{2362}$$

$$+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_y\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)-V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t) \tag{2363}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V^{\Re}_{10}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{2364}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Re}_{10}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B^{\Re}_{10}(t)\,V^{\Im}_{10}(t)+B^{\Im}_{10}(t)\,V^{\Re}_{10}(t)\right)-V^{\Re}_{10}(t) \tag{2365}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t) \tag{2366}$$

$$+R_i(t))|i\rangle\langle i|-V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V^{\Im}_{10}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t)-B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)+V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{2367}$$

$$\times V^{\Im}_{10}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t)+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)-V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))V^{\Im}_{10}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}} \tag{2368}$$

$$+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i| \tag{2369}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{2370}$$

$$-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{2371}$$

$$\times V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i|-V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{2372}$$

$$-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{2373}$$

$$\times V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{2374}$$

$$\times V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{2375}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{2376}$$

$$-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t)) \tag{2377}$$

$$\times|i\rangle\langle i|-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Re}(t) \tag{2378}$$

$$\times(\mathrm{i}\sigma_z B_x(t)+B_y(t))\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{2379}$$

$$-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))\,|i\rangle\langle i|-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Im}(t)\,(B_y(t) \tag{2380}$$

$$+\mathrm{i}\sigma_z B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{2381}$$

$$-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{2382}$$

$$=V_{10}^{\Re}(t)\sum_i(\varepsilon_i(t)+R_i(t))(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t))+V_{10}^{\Im}(t)\sum_i(\varepsilon_i(t)+R_i(t))(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y \tag{2383}$$

$$\times B_{iz}(t)B_x(t))+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_{i\neq i'}B_{iz}(t)\,B_{i'z}(t)\,|i\rangle\langle i'|+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\sum_i(|i\rangle\langle i|B_{iz}(t) \tag{2384}$$

$$\times B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t))+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\sum_i(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_x(t))-\left(B_{10}^{\Re}(t)\right. \tag{2385}$$

$$\left.\times V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i\neq i'}B_{iz}(t)B_{i'z}(t)\,|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\sum_i(-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+B_{iz}(t) \tag{2386}$$

$$\times B_y(t)\,|i\rangle\langle i|)+\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\sum_i(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)+|i\rangle\langle i|B_{iz}(t)\,B_x(t))+\sum_{i,\mathbf{k}}|i\rangle\langle i|B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t) \tag{2387}$$

$$+V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\right. \tag{2388}$$

$$\left.\times\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}^2(t)\,|i\rangle\langle i|\sigma_x+\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}^2(t)\,|i\rangle\langle i|\sigma_y-\sum_{i,\mathbf{k}}|i\rangle\langle i| \tag{2389}$$

$$\times B_{iz}^2(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)\sum_{i,i'}(\varepsilon_{i'}(t)+R_{i'}(t))\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|B_{iz}(t)B_x(t)+|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_y(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\right. \tag{2390}$$

$$\left.\times V_{10}^{\Im}(t)\right)\sum_i(|i\rangle\langle i|B_{iz}(t)\,B_x(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t))+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+|i\rangle\langle i| \tag{2391}$$

$$\times B_{iz}(t)\,B_y(t))-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-V_{10}^{\Im}(t)\sum_{i\neq i'}(\varepsilon_{i'}(t)+R_{i'}(t))\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|\right. \tag{2392}$$

$$\left.\times B_{iz}(t)B_y(t)-|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_x(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i(|i\rangle\langle i|B_{iz}(t)\,B_y(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_x(t))+V_{10}^{\Im}(t) \tag{2393}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)-|i\rangle\langle i|B_{iz}(t)\,B_x(t))-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)B_y(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-|i\rangle\langle i|\sigma_y\right. \tag{2394}$$

$$\left.\times B_{iz}(t)B_x(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i(\varepsilon_i(t)+R_i(t))\left(\sigma_x|i\rangle\langle i|\sigma_x B_x^2(t)+\sigma_x|i\rangle\langle i|\sigma_y B_x(t)B_y(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)B_x(t)+\sigma_y|i\rangle\langle i|\right. \tag{2395}$$

$$\left.\times\sigma_y B_y^2(t)\right)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(\varepsilon_i(t)+R_i(t))\left(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B_x^2(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_y B_y(t)\,B_x(t)\right) \tag{2396}$$

$$+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i(|i\rangle\langle i|B_x(t)\,B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{2397}$$

$$\times\left(\sigma_x B_x^2(t)+\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)-\sigma_x B_y^2(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(B_x(t)\,B_y(t)-\mathrm{i}\sigma_z B_y^2(t)\right. \tag{2398}$$

$$\left.-\mathrm{i}\sigma_z B_x^2(t)-B_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|\,B_x(t)\,B_{iz}(t)+|i\rangle\langle i|\,B_y(t)\,B_{iz}(t)\right)-V_{10}^{\Re}(t) \tag{2399}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(-\sigma_y B_x^2(t)+\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B_y^2(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\right. \tag{2400}$$

$$\left.\times V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\left(-\sigma_y B_x(t)B_y(t)+\sigma_x B_y^2(t)-\sigma_x B_x^2(t)-\sigma_y B_y(t)B_x(t)\right)+V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)+\sigma_y|i\rangle\langle i|B_y(t)\right)\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t) \tag{2401}$$

$$+\left(V_{10}^{\Re}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}} \tag{2402}$$

$$\times\left(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{2403}$$

$$\times\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)\,B_{iz}(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_y B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i|\right. \tag{2404}$$

$$\left.\times\sigma_y B_y(t)B_{iz}(t)\right)-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\sigma_y|i\rangle\langle i|B_y(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_x^2(t)\right. \tag{2405}$$

$$\left.-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)+|i\rangle\langle i|B_y^2(t)\right)-\left(V_{10}^{\Re}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_x B_x^2(t)+\sigma_y B_y(t)\,B_x(t)\right. \tag{2406}$$

$$\left.-\sigma_y B_x(t)\,B_y(t)+\sigma_x B_y^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_y B_x^2(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B_y^2(t)\right) \tag{2407}$$

$$-\left(V_{10}^{\Re}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right) \tag{2408}$$

$$\times\left(|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)-|i\rangle\langle i|B_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\left(\sigma_x B_x(t)\right. \tag{2409}$$

$$\left.\times B_y(t)+\sigma_y B_y^2(t)+\sigma_y B_x^2(t)-\sigma_x B_y(t)B_x(t)\right)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_y B_x(t)B_y(t)-\sigma_x B_y^2(t)-\sigma_x B_x^2(t)\right. \tag{2410}$$

$$\left.-\sigma_y B_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_x^2(t)-B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_i\left(\varepsilon_i(t)\right. \tag{2411}$$

$$\left.+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B_y(t)B_x(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x^2(t)+\sigma_x|i\rangle\langle i|\sigma_y B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_y B_x(t)B_y(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\right. \tag{2412}$$

$$\left.\times\sigma_x B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B_y(t)B_x(t)+\sigma_y|i\rangle\langle i|\sigma_y B_x^2(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i\left(|i\rangle\langle i|B_y(t)\right. \tag{2413}$$

$$\left.\times B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\left(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)+\sigma_y B_y^2(t)+\sigma_x B_x(t)B_y(t)\right) \tag{2414}$$

$$+\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(B_y^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_x^2(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{2415}$$

$$\times\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_{iz}(t)-|i\rangle\langle i|B_x(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(-\sigma_y B_y(t)\,B_x(t)-\sigma_x B_x^2(t)+\sigma_x B_y^2(t)\right. \tag{2416}$$

$$\left.-\sigma_y B_x(t)\,B_y(t)\right)-\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(-\sigma_y B_y^2(t)-\sigma_x B_x(t)\,B_y(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_y B_x^2(t)\right)+V_{10}^{\Im}(t) \tag{2417}$$

$$\times\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\right)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z\right. \tag{2418}$$

$$\left.\times B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+B_x(t)\right. \tag{2419}$$

$$\left.\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_x B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\right. \tag{2420}$$

$$\left.+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i\left(\sigma_x|i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_y B_x(t)B_{iz}(t)\right)-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{2421}$$

$$-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_y(t)B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-|i\rangle\langle i|B_x(t)B_y(t)\right)-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{2422}$$

$$\left.-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)-\sigma_y B_y^2(t)-\sigma_x B_x(t)B_y(t)\right)+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\left(\sigma_y B_y(t)B_x(t)\right. \tag{2423}$$

$$\left.+\sigma_x B_x^2(t)+\sigma_x B_y^2(t)-\sigma_y B_x(t)B_y(t)\right)-\sum_{\mathbf{k}}V_{10}^{\Im}(t)V_{10}^{\Re}(t)\,\omega_{\mathbf{k}}\left(B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) \tag{2424}$$

$$-\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|B_y^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+|i\rangle\langle i|B_x^2(t)\right)-\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)\right. \tag{2425}$$

$$\left.-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\left(\sigma_x B_y^2(t)-\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)+\sigma_x B_x^2(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\left(\sigma_y B_y^2(t)\right. \tag{2426}$$

$$\left.+\sigma_x B_x(t)B_y(t)-\sigma_x B_y(t)B_x(t)+\sigma_y B_x^2(t)\right)-\left(V_{10}^{\Im}(t)\right)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(B_y^2(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)B_y(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)B_x(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_x^2(t)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right). \tag{2427}$$

Now let's obtain the form of $\overline{H_I}^3(t)$:

$$\overline{H_I}^3(t)=\left(\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\right)\left(\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x \tag{2428}$$

$$+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t) \tag{2429}$$

$$\times B_{iz}(t))+\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) \tag{2430}$$

$$+\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_z B_y(t)\,B_x(t)+\mathrm{i}\sigma_z B_x(t)\,B_y(t)\right)\Bigg) \tag{2431}$$

$$=\sum_i B_{iz}(t)\,|i\rangle\langle i|\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t) \tag{2432}$$

$$\times\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\sum_i|i\rangle\langle i| \tag{2433}$$

$$\times B_{iz}(t)\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\right)+\sum_i B_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i| \tag{2434}$$

$$\times B_x(t)B_{iz}(t))+\sum_i B_{iz}(t)\,|i\rangle\langle i|\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)\right)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{2435}$$

$$\times\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y \tag{2436}$$

$$\times B_y(t))\,V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) \tag{2437}$$

$$+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Re}(t) \tag{2438}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)\right. \tag{2439}$$

$$\left.+B_x^2(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Re}(t) \tag{2440}$$

$$\times\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t) \tag{2441}$$

$$|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{2442}$$

$$\times\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y \tag{2443}$$

$$\times|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_z B_y(t)\,B_x(t)+\mathrm{i}\sigma_z B_x(t)\,B_y(t)\right) \tag{2444}$$

$$=\sum_i B_{iz}^3(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i\left(B_{iz}^2(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}^2(t)B_y(t)\,|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}(t)\sum_i\left(B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_y\right) \tag{2445}$$

$$+V_{10}^{\Re}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_x(t)B_{iz}(t)+|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_x^2(t)+\mathrm{i}|i\rangle\langle i|\sigma_z\right. \tag{2446}$$

$$\times B_{iz}(t)B_x(t)B_y(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)B_x(t)+|i\rangle\langle i|B_{iz}(t)B_y^2(t)\Big)+V_{10}^{\Im}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)-|i'\rangle\langle i'|\sigma_y\right. \tag{2447}$$

$$\times|i\rangle\langle i|B_{i'z}(t)\,B_x(t)B_{iz}(t)\Big)+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_y^2(t)+|i\rangle\langle i|B_{iz}(t)B_x^2(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)B_y(t)B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\right. \tag{2448}$$

$$\times B_x(t)B_y(t)\Big)+V_{10}^{\Re}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}^2(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i(B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_x+B_x(t)B_{iz}(t) \tag{2449}$$

$$\times B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_x+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(B_x(t)B_{iz}(t)B_y(t)\sigma_x|i\rangle\langle i|\sigma_x \tag{2450}$$

$$-B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_x-B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_x^2(t)B_{iz}(t)\right. \tag{2451}$$

$$+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y^2(t)B_{iz}(t)\Big)+\left(V_{10}^{\Re}(t)\right)^3\left(\sigma_x B_x^3(t)+\sigma_y B_x^2(t)B_y(t)-\sigma_y B_x(t)\right. \tag{2452}$$

$$\times B_y(t)B_x(t)+\sigma_x B_x(t)B_y^2(t)+\sigma_y B_y(t)B_x^2(t)-\sigma_x B_y(t)B_x(t)B_y(t)+\sigma_x B_y^2(t)B_x(t)+\sigma_y B_y^3(t)\Big)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i\left(|i\rangle\langle i|B_x(t)\right. \tag{2453}$$

$$\times B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_y^2(t)B_{iz}(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_y(t)B_x(t)B_{iz}(t)\Big)+V_{10}^{\Re}(t)\left(V_{10}^{\Im}(t)\right)^2\left(\sigma_x B_x(t)B_y^2(t)+\sigma_x\right. \tag{2454}$$

$$\times B_x^3(t) - \sigma_y B_x(t)B_y(t)B_x(t) + \sigma_y B_x^2(t)B_y(t) + \sigma_y B_y^3(t) + \sigma_y B_y(t)B_x^2(t) + \sigma_x B_y^2(t)B_x(t) - \sigma_x B_y(t)B_x(t)B_y(t)\big) + V_{10}^{\Im}(t)\sum_i (\sigma_x|i\rangle\langle i| \tag{2455}$$

$$\times B_y(t)B_{iz}^2(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}^2(t)) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x|i\rangle\langle i|\sigma_x B_y(t)B_{iz}(t)B_x(t) + \sigma_x|i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)B_y(t) - \sigma_y|i\rangle\langle i|\sigma_x B_y(t)B_{iz}(t) \tag{2456}$$

$$\times B_y(t) - \sigma_x|i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)B_y(t)) + \left(V_{10}^{\Im}(t)\right)^2 (\sigma_x|i\rangle\langle i|\sigma_x B_y(t)B_{iz}(t)B_y(t) - \sigma_x|i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)B_x(t) - \sigma_y|i\rangle\langle i|\sigma_x B_x(t)B_{iz}(t) \tag{2457}$$

$$\times B_y(t) + \sigma_y|i\rangle\langle i|\sigma_y B_x(t)B_{iz}(t)B_x(t)) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i \left(|i\rangle\langle i|B_y(t)B_x(t)B_{iz}(t) + \mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)B_{iz}(t) + \mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t) - |i\rangle\langle i|\right. \tag{2458}$$

$$\times B_x(t)B_y(t)B_{iz}(t)) + V_{10}^{\Im}(t)\left(V_{10}^{\Re}(t)\right)^2 (\sigma_x B_y(t)B_x^2(t) + \sigma_y B_y(t)B_x(t)B_y(t) - \sigma_y B_y^2(t)B_x(t) + \sigma_x B_y^3(t) - \sigma_y B_x^3(t) + \sigma_x B_x^2(t)B_y(t) \tag{2459}$$

$$-\sigma_x B_x(t)B_y(t)B_x(t) - \sigma_y B_x(t)B_y^2(t)) + \left(V_{10}^{\Im}(t)\right)^2 \sum_i \left(|i\rangle\langle i|B_y^2(t)B_{iz}(t) - \mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_x(t)B_{iz}(t) + \mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)\right. \tag{2460}$$

$$+|i\rangle\langle i|B_x^2(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^3 (\sigma_x B_y^3(t) + \sigma_x B_y(t)B_x^2(t) - \sigma_y B_y^2(t)B_x(t) + \sigma_y B_y(t)B_x(t)B_y(t) - \sigma_y B_x(t)B_y^2(t) - \sigma_y B_x^3(t) - \sigma_x \tag{2461}$$

$$\times B_x(t)B_y(t)\, B_x(t) + \sigma_x B_x^2(t)B_y(t)) \,. \tag{2462}$$

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of d-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{2463}$$

$$H_S(t) = \sum_n \varepsilon_n(t)\,|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|, \tag{2464}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}\right), \tag{2465}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \tag{2466}$$

where $n, m \in \{0, ..., d-1\}$ and $u \in \{1, ..., v\}$.

A. Variational Transformation

We consider the following operator:

$$V(t) = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega_{u\mathbf{k}}^{-1} \left(v_{nu\mathbf{k}}(t)\, b_{u\mathbf{k}}^\dagger - v_{nu\mathbf{k}}^*(t)\, b_{u\mathbf{k}}\right). \tag{2467}$$

At first let's obtain $\mathrm{e}^{\pm V}$ under the transformation (2467), consider $\hat{\varphi}_n(t) = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(v_{nu\mathbf{k}}(t)\, b_{u\mathbf{k}}^\dagger - v_{nu\mathbf{k}}^*(t)\, b_{u\mathbf{k}}\right)$, so the equation (2467) can be written as $V(t) = \sum_n |n\rangle\langle n|\hat{\varphi}_n(t)$, then we have:

$$\mathrm{e}^{\pm V(t)} = \mathrm{e}^{\pm \sum_n |n\rangle\langle n|\hat{\varphi}_n(t)} \tag{2468}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n(t) + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n(t)\right)^2}{2!} \pm \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n(t)\right)^3}{3!} + \frac{\left(\sum_n |n\rangle\langle n|\hat{\varphi}_n(t)\right)^4}{4!} + ... \tag{2469}$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n(t) + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2(t)}{2!} \pm \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^3(t)}{3!} + ... \text{ (by } (|n\rangle\langle n|)^k = |n\rangle\langle n| \text{for } k \in \mathbb{N}^*) \tag{2470}$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n|\hat{\varphi}_n(t) + \frac{\sum_n |n\rangle\langle n|\hat{\varphi}_n^2(t)}{2!} + ... \text{ (by resolution of the identity } \sum_n |n\rangle\langle n| = \mathbb{I}) \tag{2471}$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I}_B \pm \hat{\varphi}_n(t) + \frac{\hat{\varphi}_n^2(t)}{2!} + ...\right) \text{ (factorizing the identity } \mathbb{I}_B \text{ of the bath)} \tag{2472}$$

$$= \sum_n |n\rangle\langle n| e^{\pm\hat{\varphi}_n(t)} \text{ (using the definition of exponential of a matrix).} \tag{2473}$$

Given that $\left[v_{nu\mathbf{k}}(t)\, b_{u\mathbf{k}}^{\dagger} - v_{nu\mathbf{k}}^{*}(t)\, b_{u\mathbf{k}}, v_{nu'\mathbf{k}'}(t)\, b_{u'\mathbf{k}'}^{\dagger} - v_{nu'\mathbf{k}'}^{*}(t)\, b_{u'\mathbf{k}'}\right] = 0$ because if $\mathbf{k}' \neq \mathbf{k}$ or $u \neq u'$ then $u\mathbf{k} \neq u'\mathbf{k}'$ so the conmutator is related to terms that belong to different Hilbert spaces so their conmutator is zero. If $\mathbf{k}' = \mathbf{k}$ and $u = u'$ then we have the following conmutator $\left[v_{nu\mathbf{k}}(t)\, b_{u\mathbf{k}}^{\dagger} - v_{nu\mathbf{k}}^{*}(t)\, b_{u\mathbf{k}}, v_{nu\mathbf{k}}(t)\, b_{u\mathbf{k}}^{\dagger} - v_{nu\mathbf{k}}^{*}(t)\, b_{u\mathbf{k}}\right] = 0$. We can proof using the Zassenhaus formula, the precedent result and defining $D(\pm\alpha_{nu\mathbf{k}}(t)) \equiv e^{\pm\left(\alpha_{nu\mathbf{k}}(t)b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*}(t)b_{u\mathbf{k}}\right)}$ in the same way than (24) with $\alpha_{nu\mathbf{k}}(t) = \frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}$ then:

$$e^{\pm\sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(v_{nu\mathbf{k}}(t)b_{u\mathbf{k}}^{\dagger} - v_{nu\mathbf{k}}^{*}(t)b_{u\mathbf{k}}\right)} = \prod_u e^{\pm\sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(v_{nu\mathbf{k}}(t)b_{u\mathbf{k}}^{\dagger} - v_{nu\mathbf{k}}^{*}(t)b_{u\mathbf{k}}\right)} \tag{2474}$$

$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm\omega_{u\mathbf{k}}^{-1}\left(v_{nu\mathbf{k}}(t)b_{u\mathbf{k}}^{\dagger} - v_{nu\mathbf{k}}^{*}(t)b_{u\mathbf{k}}\right)}\right) \text{ (by the Zassenhaus formula)} \tag{2475}$$

$$= \prod_u \left(\prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}(t))\right) \text{ (by the displacement operator)} \tag{2476}$$

$$= \prod_{u\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}(t)), \tag{2477}$$

$$B_{nu\pm}(t) \equiv \prod_{\mathbf{k}} D(\pm\alpha_{nu\mathbf{k}}(t)). \tag{2478}$$

$$e^{\pm\sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(v_{nu\mathbf{k}}(t)b_{u\mathbf{k}}^{\dagger} - v_{nu\mathbf{k}}^{*}(t)b_{u\mathbf{k}}\right)} = \prod_u B_{nu\pm}(t). \tag{2479}$$

As we can see $e^{-V(t)} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}(t)$ and $e^{V(t)} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}(t)$, which implies that $e^{-V(t)}e^{V(t)} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$\overline{A(t)} = e^{V(t)} A(t)\, e^{-V(t)} \tag{2480}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}(t)\right) A(t) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}(t)\right). \tag{2481}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (2463):

$$\overline{|0\rangle\langle 0|}(t) = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}(t)\right) |0\rangle\langle 0| \left(\sum_{n'} |n'\rangle\langle n'| \prod_u B_{n'u-}(t)\right) \tag{2482}$$

$$= \sum_{n,n'} |n\rangle\langle n'| \delta_{n0}\delta_{0n'} \prod_u B_{nu+}(t) \prod_u B_{n'u-}(t) \tag{2483}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+}(t) \prod_u B_{0u-}(t) \tag{2484}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+}(t)\, B_{0u-}(t) \text{ (by the independence of } u \neq u' \text{ and commutativity)} \tag{2485}$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \text{ (because } B_{0u+}(t)\, B_{0u-}(t) = \mathbb{I}_u \equiv \mathbb{I}) \tag{2486}$$

$$= |0\rangle\langle 0|. \tag{2487}$$

$$\overline{|m\rangle\langle n|}\,(t) = \left(\sum_j |j\rangle\langle j| \prod_u B_{ju+}(t)\right) |m\rangle\langle n| \left(\sum_{n'} |n'\rangle\langle n'| \prod_u B_{n'u-}(t)\right) \tag{2488}$$

$$= \sum_{j,n'} |j\rangle\langle n'| \delta_{jm}\delta_{nn'} \prod_u B_{ju+}(t) \prod_u B_{n'u-}(t) \tag{2489}$$

$$= |m\rangle\langle n| \prod_u B_{mu+}(t) \prod_u B_{nu-}(t) \tag{2490}$$

$$= |m\rangle\langle n| \prod_u \left(B_{mu+}(t) B_{nu-}(t)\right) \text{ (by independence of } u \neq u' \text{ and commutativity)}, \tag{2491}$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D(\alpha_{mu\mathbf{k}}(t)) \prod_{\mathbf{k}} D(-\alpha_{nu\mathbf{k}}(t))\right) \text{ (by definition of } B_{nu\pm}(t)) \tag{2492}$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}}(t)) D(-\alpha_{nu\mathbf{k}}(t))\right) \text{ (by independence of } \mathbf{k} \neq \mathbf{k}'), \tag{2493}$$

$$D(\alpha) D(-\beta) = D(\alpha - \beta)\, \mathrm{e}^{\frac{1}{2}(\alpha^*\beta - \alpha\beta^*)} \text{ (by displacement operator properties)}, \tag{2494}$$

$$\overline{|m\rangle\langle n|}\,(t) = |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D(\alpha_{mu\mathbf{k}}(t) - \alpha_{nu\mathbf{k}}(t))\, \mathrm{e}^{\frac{1}{2}(\alpha^*_{mu\mathbf{k}}(t)\alpha_{nu\mathbf{k}}(t) - \alpha_{mu\mathbf{k}}(t)\alpha^*_{nu\mathbf{k}}(t))}\right) \text{ (by (2494))}, \tag{2495}$$

$$\prod_u (B_{mu+}(t) B_{nu-}(t)) = \prod_{u\mathbf{k}} D(\alpha_{mu\mathbf{k}}(t) - \alpha_{nu\mathbf{k}}(t))\, \mathrm{e}^{\frac{1}{2}(\alpha^*_{mu\mathbf{k}}(t)\alpha_{nu\mathbf{k}}(t) - \alpha_{mu\mathbf{k}}(t)\alpha^*_{nu\mathbf{k}}(t))}, \tag{2496}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}}\,(t) = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}(t)\right) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}(t)\right) \tag{2497}$$

$$= \left(|0\rangle\langle 0| \prod_u B_{0u+}(t) + \ldots\right) \left(\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right) \left(|0\rangle\langle 0| \prod_u B_{0u-}(t) + \ldots\right) \tag{2498}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+}(t) \sum_{u'\mathbf{k}} \omega_{u'\mathbf{k}} b^\dagger_{u'\mathbf{k}} b_{u'\mathbf{k}} \prod_u B_{0u-}(t) + |1\rangle\langle 1| \prod_u B_{1u+}(t) \sum_{u'\mathbf{k}} \omega_{u'\mathbf{k}} b^\dagger_{u'\mathbf{k}} b_{u'\mathbf{k}} \prod_u B_{1u-}(t) + \ldots \tag{2499}$$

$$= |0\rangle\langle 0| \prod_u B_{0u+}(t) \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-}(t) + |1\rangle\langle 1| \prod_u B_{1u+}(t) \tag{2500}$$

$$\times \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-}(t) + \ldots \tag{2501}$$

$$= |0\rangle\langle 0| \left(\prod_u B_{0u+}(t) \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-}(t) + \prod_u B_{0u+}(t) \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-}(t) + \ldots\right) \tag{2502}$$

$$+ |1\rangle\langle 1| \left(\prod_u B_{1u+}(t) \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-}(t) + \prod_u B_{1u+}(t) \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-}(t) + \ldots\right) + \ldots, \tag{2503}$$

$$\prod_u B_{ju+}(t) \left(b^\dagger_{u'\mathbf{k}} b_{u'\mathbf{k}}\right) \prod_u B_{ju-}(t) = B_{ju'+}(t)\, b^\dagger_{u'\mathbf{k}} b_{u'\mathbf{k}} B_{ju'-}(t) \prod_{u\neq u'} B_{ju+}(t) \prod_{u\neq u'} B_{ju-}(t) \tag{2504}$$

$$= \left(\prod_{\mathbf{k}'} D(\alpha_{ju\mathbf{k}'}(t))\, b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_{\mathbf{k}'} D(-\alpha_{ju\mathbf{k}'}(t))\right) \prod_{u\neq u'} B_{ju+}(t) B_{ju-}(t) \tag{2505}$$

$$= \left(D(\alpha_{ju\mathbf{k}}(t))\, b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} D(-\alpha_{ju\mathbf{k}}(t)) \left(\prod_{\mathbf{k}'\neq\mathbf{k}} D(\alpha_{ju\mathbf{k}'}(t)) \prod_{\mathbf{k}'\neq\mathbf{k}} D(-\alpha_{ju\mathbf{k}'}(t))\right)\right) \prod_{u\neq u'} \mathbb{I} \tag{2506}$$

$$= \left(D(\alpha_{ju\mathbf{k}}(t))\, b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} D(-\alpha_{ju\mathbf{k}}(t)) \left(\prod_{\mathbf{k}'\neq\mathbf{k}} D(\alpha_{ju\mathbf{k}'}(t)) D(-\alpha_{ju\mathbf{k}'}(t))\right)\right) \tag{2507}$$

$$= D(\alpha_{ju\mathbf{k}}(t))\, b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} D(-\alpha_{ju\mathbf{k}}(t)) \left(\prod_{\mathbf{k}'\neq\mathbf{k}} \mathbb{I}\right) \tag{2508}$$

$$= \left(D(\alpha_{ju\mathbf{k}}(t))\, b^\dagger_{u\mathbf{k}} D(-\alpha_{ju\mathbf{k}}(t))\right) \left(D(\alpha_{ju\mathbf{k}}(t))\, b_{u\mathbf{k}} D(-\alpha_{ju\mathbf{k}}(t))\right) \tag{2509}$$

$$= \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{ju\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right) \left(b_{u\mathbf{k}} - \frac{v_{ju\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right), \tag{2510}$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}}\,(t) = |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}(t)}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}(t)}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}(t)}{\omega_{1\mathbf{k}}}\right)\left(b_{1\mathbf{k}} - \frac{v_{01\mathbf{k}}(t)}{\omega_{1\mathbf{k}}}\right) + \ldots\right) \tag{2511}$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}(t)}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}(t)}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}(t)}{\omega_{1\mathbf{k}}}\right)\left(b_{1\mathbf{k}} - \frac{v_{11\mathbf{k}}(t)}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots \tag{2512}$$

$$= |0\rangle\langle 0|\left(\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right)+|1\rangle\langle 1|\left(\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right)+\ldots \tag{2513}$$

$$= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\omega_{u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right) \tag{2514}$$

$$= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\omega_{u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}b_{u\mathbf{k}}+\left|\frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right|^{2}\right)\right) \tag{2515}$$

$$= \sum_{nu\mathbf{k}}|n\rangle\langle n|\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{|v_{nu\mathbf{k}}(t)|^{2}}{\omega_{u\mathbf{k}}}-v_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}-v_{nu\mathbf{k}}^{*}(t)\,b_{u\mathbf{k}}\right) \tag{2516}$$

$$= \sum_{u\mathbf{k}}|n\rangle\langle n|\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{|v_{nu\mathbf{k}}(t)|^{2}}{\omega_{u\mathbf{k}}}-v_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}-v_{nu\mathbf{k}}^{*}(t)\,b_{u\mathbf{k}}\right) \tag{2517}$$

$$= \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{|v_{nu\mathbf{k}}(t)|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}(t)\,b_{u\mathbf{k}}\right)\right). \tag{2518}$$

The transformed Hamiltonians of the equations (2464) to (2466) written in terms of (2482) to (2515) are:

$$\overline{H_S(t)} = \overline{\sum_{n}\varepsilon_n(t)\,|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|} \tag{2519}$$

$$= \overline{\sum_{n}\varepsilon_n(t)\,|n\rangle\langle n|}+\overline{\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|} \tag{2520}$$

$$= \sum_{n}\varepsilon_n(t)\,|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)\,|n\rangle\langle m|\prod_{u}\left(B_{nu+}(t)\,B_{mu-}(t)\right), \tag{2521}$$

$$\overline{H_I(t)} = \left(\sum_{n}|n\rangle\langle n|\prod_{u}B_{nu+}(t)\right)\left(\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_{n}|n\rangle\langle n|\prod_{u}B_{nu-}(t)\right) \tag{2522}$$

$$= \left(\sum_{n}|n\rangle\langle n|\prod_{u}B_{nu+}(t)\right)\left(\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)+\ldots\right)\left(\sum_{n}|n\rangle\langle n|\prod_{u}B_{nu-}(t)\right) \tag{2523}$$

$$= \prod_{u}B_{0u+}(t)\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_{u}B_{0u-}(t)+\prod_{u}B_{1u+}(t)\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{2524}$$

$$\times\prod_{u}B_{1u-}(t)+\prod_{u}B_{2u+}(t)\sum_{u\mathbf{k}}|2\rangle\langle 2|\left(g_{2u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{2u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_{u}B_{2u-}(t)+\ldots \tag{2525}$$

$$=\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\prod_{u}B_{0u+}(t)\,b_{u\mathbf{k}}^{\dagger}\prod_{u}B_{0u-}(t)+g_{0u\mathbf{k}}^{*}\prod_{u}B_{0u+}(t)\,b_{u\mathbf{k}}\prod_{u}B_{0u-}(t)\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\prod_{u}B_{1u+}(t)\right. \tag{2526}$$

$$\left.b_{u\mathbf{k}}^{\dagger}\prod_{u}B_{1u-}(t)+g_{1u\mathbf{k}}^{*}\prod_{u}B_{1u+}(t)\,b_{u\mathbf{k}}\prod_{u}B_{1u-}(t)\right)+\ldots \tag{2527}$$

$$= \sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}\right)+g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}\right)\right. \tag{2528}$$

$$\left.+g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right)+\ldots \tag{2529}$$

$$= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}\right)+g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right) \tag{2530}$$

$$= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right) \tag{2531}$$

$$\overline{H_B}(t) = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{|v_{nu\mathbf{k}}(t)|^{2}}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}(t)\,b_{u\mathbf{k}}\right)\right). \tag{2532}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H\left(t\right)}=\sum_{n}\varepsilon_{n}\left(t\right)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|\prod_{u}\left(B_{nu+}\left(t\right)B_{mu-}\left(t\right)\right)+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{\left|v_{nu\mathbf{k}}\left(t\right)\right|^{2}}{\omega_{u\mathbf{k}}}\right. \tag{2533}$$

$$\left.-\left(v_{nu\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\right)\right)+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}\left(t\right)}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}\left(t\right)}{\omega_{u\mathbf{k}}}\right)\right). \tag{2534}$$

Let's define the following functions:

$$R_{n}\left(t\right)=\sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\left(t\right)\right|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}\left(t\right)}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}\left(t\right)}{\omega_{u\mathbf{k}}}\right)\right), \tag{2535}$$

$$B_{z,n}\left(t\right)=\sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\left(t\right)\right)b_{u\mathbf{k}}^{\dagger}+\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\left(t\right)\right)^{*}b_{u\mathbf{k}}\right). \tag{2536}$$

Using the previous functions we have that (2533) can be re-written in the following way:

$$\overline{H(t)}=\sum_{n}(\varepsilon_{n}(t)+R_{n}(t))|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_{u}(B_{nu+}(t)B_{mu-}(t))+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{n}B_{z,n}(t)|n\rangle\langle n|. \tag{2537}$$

Now in order to separate the elements of the hamiltonian (2537) we need to consider the expected term of the form:

$$\begin{aligned}
\left\langle \prod_{u}\left(B_{mu+}\left(t\right)B_{nu-}\left(t\right)\right)\right\rangle_{\overline{H_{0}}} &= \left\langle \prod_{u\mathbf{k}}\left(D\left(\alpha_{mu\mathbf{k}}\left(t\right)-\alpha_{nu\mathbf{k}}\left(t\right)\right)\mathrm{e}^{\frac{1}{2}\left(\alpha_{mu\mathbf{k}}^{*}(t)\alpha_{nu\mathbf{k}}(t)-\alpha_{mu\mathbf{k}}(t)\alpha_{nu\mathbf{k}}^{*}(t)\right)}\right)\right\rangle_{\overline{H_{0}(t)}} && (2538)\\
&= \left(\prod_{u\mathbf{k}}\mathrm{e}^{\frac{1}{2}\left(\alpha_{mu\mathbf{k}}^{*}(t)\alpha_{nu\mathbf{k}}(t)-\alpha_{mu\mathbf{k}}(t)\alpha_{nu\mathbf{k}}^{*}(t)\right)}\right)\left\langle \prod_{u\mathbf{k}}D\left(\alpha_{mu\mathbf{k}}\left(t\right)-\alpha_{nu\mathbf{k}}\left(t\right)\right)\right\rangle_{\overline{H_{0}(t)}} && (2539)\\
&= \prod_{u\mathbf{k}}\mathrm{e}^{\frac{v_{mu\mathbf{k}}^{*}(t)v_{nu\mathbf{k}}(t)-v_{mu\mathbf{k}}(t)v_{nu\mathbf{k}}^{*}(t)}{2\omega_{u\mathbf{k}}^{2}}}\prod_{u}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)\right|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)} && (2540)\\
&\equiv B_{mn}\left(t\right), && (2541)\\
\left\langle \prod_{u}\left(B_{nu+}\left(t\right)B_{mu-}\left(t\right)\right)\right\rangle_{\overline{H_{0}}} &= \prod_{u\mathbf{k}}\mathrm{e}^{\frac{v_{nu\mathbf{k}}^{*}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}^{*}(t)}{2\omega_{u\mathbf{k}}^{2}}}\prod_{u}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)\right|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)} && (2542)\\
&= B_{nm}\left(t\right) && (2543)\\
&= B_{mn}^{*}\left(t\right). && (2544)
\end{aligned}$$

Following the reference [6] we define:

$$J_{nm}\left(t\right)=\prod_{u}\left(B_{nu+}\left(t\right)B_{mu-}\left(t\right)\right)-B_{nm}\left(t\right). \tag{2545}$$

As we can see:

$$\begin{aligned}
J_{nm}^{\dagger}\left(t\right) &= \left(\prod_{u}\left(B_{nu+}\left(t\right)B_{mu-}\left(t\right)\right)-B_{nm}\left(t\right)\right)^{\dagger} && (2546)\\
&= \prod_{u}\left(B_{mu+}\left(t\right)B_{nu-}\left(t\right)\right)-B_{nm}^{*}\left(t\right) && (2547)\\
&= \prod_{u}\left(B_{mu+}\left(t\right)B_{nu-}\left(t\right)\right)-B_{mn}\left(t\right) && (2548)\\
&= J_{mn}\left(t\right). && (2549)
\end{aligned}$$

We can separate the Hamiltonian (2537) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H(t)} = \sum_n (\varepsilon_n(t) + R_n(t))|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|\prod_u (B_{nu+}(t)B_{mu-}(t)) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n B_{z,n}(t)|n\rangle\langle n| \tag{2550}$$

$$= \sum_n (\varepsilon_n(t) + R_n(t))|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m| \left(\prod_u (B_{nu+}(t)B_{mu-}(t)) - B_{nm}(t)\right) + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|B_{nm}(t) \tag{2551}$$

$$+ \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n B_{z,n}(t)|n\rangle\langle n| \tag{2552}$$

$$\overline{H_{\bar{S}}(t)} = \sum_n \left(\varepsilon_n(t) + R_n\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|B_{nm}(t), \tag{2553}$$

$$\overline{H_{\bar{I}}(t)} = \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|J_{nm}(t) + \sum_n B_{z,n}(t)\,|n\rangle\langle n|, \tag{2554}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}, \tag{2555}$$

$$\overline{H(t)} = \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}(t)} + \overline{H_{\bar{B}}}. \tag{2556}$$

B. Free-energy minimization

The true free energy $E_{\text{Free}}(t)$ is bounded by the Bogoliubov inequality:

$$E_{\text{Free}}(t) \leq E_{\text{Free,B}}(t) \equiv -\frac{1}{\beta}\ln\left(\text{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right)\right) + \left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}}\right). \tag{2557}$$

We will optimize the set of variational parameters $\{v_{\mathbf{k}}(t)\}$ in order to minimize $E_{\text{Free,B}}(t)$ (i.e. to make it as close to the true free energy $E_{\text{Free}}(t)$ as possible). Neglecting the higher order terms and using the fact that $\left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} = 0$ because:

$$\langle J_{nm}(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle\prod_u \left(B_{nu+}(t)\,B_{mu-}(t)\right) - B_{nm}(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{2558}$$

$$= \left\langle\prod_u \left(B_{nu+}(t)\,B_{mu-}(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \langle B_{nm}(t)\rangle_{\overline{H_{\bar{B}}}} \tag{2559}$$

$$= B_{nm}(t) - B_{nm}(t) \tag{2560}$$

$$= 0, \tag{2561}$$

$$\langle B_{z,n}(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle\sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}(t)\right)b_{u\mathbf{k}}^\dagger + \left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}(t)\right)^* b_{u\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{2562}$$

$$= \sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}(t)\right)\left\langle b_{u\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}(t)\right)^*\langle b_{u\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right) \tag{2563}$$

$$= \sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}(t)\right)\cdot 0 + \left(g_{nu\mathbf{k}} - v_{nu\mathbf{k}}(t)\right)^*\cdot 0\right) \tag{2564}$$

$$= 0, \tag{2565}$$

$$\left\langle\overline{H_{\bar{I}}}(t)\right\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} = \sum_{n\neq m} V_{nm}(t)\,\langle|n\rangle\langle m|\rangle_{\overline{H_{\bar{S}}(t)}}\,\langle J_{nm}(t)\rangle_{\overline{H_{\bar{B}}}} + \sum_n \langle|n\rangle\langle n|\rangle_{\overline{H_{\bar{S}}(t)}}\,\langle B_{z,n}(t)\rangle_{\overline{H_{\bar{B}}}} \tag{2566}$$

$$= \sum_{n\neq m} V_{nm}(t)\,\langle|n\rangle\langle m|\rangle_{\overline{H_{\bar{S}}(t)}}\cdot 0 + \sum_n \langle|n\rangle\langle n|\rangle_{\overline{H_{\bar{S}}(t)}}\cdot 0 \tag{2567}$$

$$= 0. \tag{2568}$$

we can obtain the following condition to obtain the set $\{v_{\mathbf{k}}(t)\}$:

$$\frac{\partial E_{\text{Free,B}}\left(t\right)}{\partial v_{i\mathbf{k}}\left(t\right)}=0. \tag{2569}$$

Given that the numbers $v_{nu\mathbf{k}}\left(t\right)$ are complex then we can separate them as $v_{nu\mathbf{k}}\left(t\right)=v_{nu\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(t\right)$. So our approach will be based on the derivation respect to $v_{nu\mathbf{k}}^{\Re}\left(t\right)$ and $v_{nu\mathbf{k}}^{\Im}\left(t\right)$. The Hamiltonian $\overline{H_{\bar{S}}\left(t\right)}$ can be written like:

$$\begin{aligned}
\overline{H_{\bar{S}}(t)}=&\sum_{n}\left(\varepsilon_{n}(t)+\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}(t)|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}(t)}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_{u\mathbf{k}}\mathrm{e}^{\frac{v_{nu\mathbf{k}}^{*}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}^{*}(t)}{2\omega_{u\mathbf{k}}^{2}}} && (2570)\\
&\times\prod_{u}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{nu\mathbf{k}}(t)-v_{mu\mathbf{k}}(t)|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)} && (2571)\\
=&\sum_{n}\left(\varepsilon_{n}(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{nu\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{u\mathbf{k}}}-\frac{v_{nu\mathbf{k}}^{\Re}(t)\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^{*}\right)+\mathrm{i}v_{nu\mathbf{k}}^{\Im}(t)\left(g_{nu\mathbf{k}}^{*}-g_{nu\mathbf{k}}\right)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m| && (2572)\\
&\times\prod_{u\mathbf{k}}\mathrm{e}^{\frac{v_{nu\mathbf{k}}^{*}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}^{*}(t)}{2\omega_{u\mathbf{k}}^{2}}}\prod_{u}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{nu\mathbf{k}}(t)-v_{mu\mathbf{k}}(t)|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)} && (2573)
\end{aligned}$$

The following expressions appears in the equation (2573) and they are shown in explicit form:

$$\begin{aligned}
v_{nu\mathbf{k}}^{*}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}^{*}(t)=&\left(v_{nu\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{nu\mathbf{k}}^{\Im}(t)\right)\left(v_{mu\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{mu\mathbf{k}}^{\Im}(t)\right)-\left(v_{nu\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{nu\mathbf{k}}^{\Im}(t)\right)\left(v_{mu\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{mu\mathbf{k}}^{\Im}(t)\right) && (2574)\\
=&v_{nu\mathbf{k}}^{\Re}(t)\,v_{mu\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{nu\mathbf{k}}^{\Re}(t)\,v_{mu\mathbf{k}}^{\Im}(t)-\mathrm{i}v_{nu\mathbf{k}}^{\Im}(t)\,v_{mu\mathbf{k}}^{\Re}(t)+v_{nu\mathbf{k}}^{\Im}(t)\,v_{mu\mathbf{k}}^{\Im}(t)-\left(v_{nu\mathbf{k}}^{\Re}(t)\,v_{mu\mathbf{k}}^{\Re}(t)\right. && (2575)\\
&\left.-\mathrm{i}v_{nu\mathbf{k}}^{\Re}(t)\,v_{mu\mathbf{k}}^{\Im}(t)+\mathrm{i}v_{nu\mathbf{k}}^{\Im}(t)\,v_{mu\mathbf{k}}^{\Re}(t)+v_{nu\mathbf{k}}^{\Im}(t)\,v_{mu\mathbf{k}}^{\Im}(t)\right) && (2576)\\
=&2\mathrm{i}\left(v_{nu\mathbf{k}}^{\Re}(t)\,v_{mu\mathbf{k}}^{\Im}(t)-v_{nu\mathbf{k}}^{\Im}(t)\,v_{mu\mathbf{k}}^{\Re}(t)\right), && (2577)\\
|v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)|^{2}=&\left(v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)\right)\left(v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)\right)^{*} && (2578)\\
=&|v_{mu\mathbf{k}}(t)|^{2}+|v_{nu\mathbf{k}}(t)|^{2}-\left(v_{nu\mathbf{k}}(t)\,v_{mu\mathbf{k}}^{*}(t)+v_{nu\mathbf{k}}^{*}(t)\,v_{mu\mathbf{k}}(t)\right) && (2579)\\
=&\left(v_{mu\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{mu\mathbf{k}}^{\Im}(t)\right)^{2}+\left(v_{nu\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{nu\mathbf{k}}^{\Im}(t)\right)^{2}-\left(v_{nu\mathbf{k}}^{\Re}(t)+\mathrm{i}v_{nu\mathbf{k}}^{\Im}(t)\right)\left(v_{mu\mathbf{k}}^{\Re}(t)-\mathrm{i}v_{mu\mathbf{k}}^{\Im}(t)\right) && (2580)\\
&-\left(v_{nu\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}v_{nu\mathbf{k}}^{\Im}\left(t\right)\right)\left(v_{mu\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{mu\mathbf{k}}^{\Im}\left(t\right)\right) && (2581)\\
=&\left(v_{mu\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{mu\mathbf{k}}^{\Im}(t)\right)^{2}+\left(v_{nu\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{nu\mathbf{k}}^{\Im}(t)\right)^{2}-2\left(v_{nu\mathbf{k}}^{\Re}(t)\,v_{mu\mathbf{k}}^{\Re}(t)+v_{nu\mathbf{k}}^{\Im}(t)\,v_{mu\mathbf{k}}^{\Im}(t)\right) && (2582)\\
=&\left(v_{mu\mathbf{k}}^{\Re}\left(t\right)-v_{nu\mathbf{k}}^{\Re}\left(t\right)\right)^{2}+\left(v_{mu\mathbf{k}}^{\Im}\left(t\right)-v_{nu\mathbf{k}}^{\Im}\left(t\right)\right)^{2}. && (2583)
\end{aligned}$$

So we can write:

$$\begin{aligned}
\overline{H_{\bar{S}}(t)}=&\sum_{n}\left(\varepsilon_{n}(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}(t)\right)^{2}+\left(v_{nu\mathbf{k}}^{\Im}(t)\right)^{2}}{\omega_{u\mathbf{k}}}-\frac{(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^{*})v_{nu\mathbf{k}}^{\Re}(t)+\mathrm{i}\,(g_{nu\mathbf{k}}^{*}-g_{nu\mathbf{k}})v_{nu\mathbf{k}}^{\Im}(t)}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| && (2584)\\
&+\sum_{n\neq m}V_{nm}\left(t\right)|n\rangle\langle m|\prod_{u\mathbf{k}}\mathrm{e}^{\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Re}(t)v_{mu\mathbf{k}}^{\Im}(t)-v_{nu\mathbf{k}}^{\Im}(t)v_{mu\mathbf{k}}^{\Re}(t)\right)}{\omega_{u\mathbf{k}}^{2}}}\prod_{u}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|v_{nu\mathbf{k}}(t)-v_{mu\mathbf{k}}(t)|^{2}}{2\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta_{u}\omega_{u\mathbf{k}}}{2}\right)}, && (2585)\\
R_{n}\left(t\right)=&\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}\left(t\right)|^{2}}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}\left(t\right)}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}\left(t\right)}{\omega_{u\mathbf{k}}}\right)\right) && (2586)\\
=&\sum_{u\mathbf{k}}\left(\frac{\left(v_{nu\mathbf{k}}^{\Re}\left(t\right)\right)^{2}+\left(v_{nu\mathbf{k}}^{\Im}\left(t\right)\right)^{2}-\left(g_{nu\mathbf{k}}+g_{nu\mathbf{k}}^{*}\right)v_{nu\mathbf{k}}^{\Re}\left(t\right)-\mathrm{i}\left(g_{nu\mathbf{k}}^{*}-g_{nu\mathbf{k}}\right)v_{nu\mathbf{k}}^{\Im}\left(t\right)}{\omega_{u\mathbf{k}}}\right) && (2587)
\end{aligned}$$

$$= \sum_{u\mathbf{k}} \left(\frac{\left(v_{nu\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}(t)\right)^2 - 2g_{nu\mathbf{k}}^{\Re} v_{nu\mathbf{k}}^{\Re}(t) - 2g_{nu\mathbf{k}}^{\Im} v_{nu\mathbf{k}}^{\Im}(t)}{\omega_{u\mathbf{k}}} \right), \tag{2588}$$

$$B_{nm}(t) = \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v_{nu\mathbf{k}}^*(t) v_{mu\mathbf{k}}(t) - v_{nu\mathbf{k}}(t) v_{mu\mathbf{k}}^*(t)}{2\omega_{u\mathbf{k}}^2}} \prod_u \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{nu\mathbf{k}}(t) - v_{mu\mathbf{k}}(t)|^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)} \tag{2589}$$

$$= \prod_{u\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Re}(t) v_{mu\mathbf{k}}^{\Im}(t) - v_{nu\mathbf{k}}^{\Im}(t) v_{mu\mathbf{k}}^{\Re}(t)\right)}{\omega_{u\mathbf{k}}^2}} \prod_u \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v_{nu\mathbf{k}}^{\Re}(t) - v_{mu\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}(t) - v_{mu\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}. \tag{2590}$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}(t)}{\partial v_{nu\mathbf{k}}^{\Re}(t)} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Re}(t)} \sum_{u\mathbf{k}} \left(\frac{\left(v_{n'u\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{n'u\mathbf{k}}^{\Im}(t)\right)^2 - 2g_{n'u\mathbf{k}}^{\Re} v_{n'u\mathbf{k}}^{\Re}(t) - 2g_{n'u\mathbf{k}}^{\Im} v_{n'u\mathbf{k}}^{\Im}(t)}{\omega_{u\mathbf{k}}} \right) \tag{2591}$$

$$= \frac{2v_{n'u\mathbf{k}}^{\Re}(t) - 2g_{n'u\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{2592}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Re}(t) - g_{nu\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \delta_{nn'}, \tag{2593}$$

$$\frac{\partial R_{n'}(t)}{\partial v_{nu\mathbf{k}}^{\Im}(t)} = \frac{\partial}{\partial v_{nu\mathbf{k}}^{\Im}(t)} \sum_{u\mathbf{k}} \left(\frac{\left(v_{n'u\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{n'u\mathbf{k}}^{\Im}(t)\right)^2 - 2g_{n'u\mathbf{k}}^{\Re} v_{n'u\mathbf{k}}^{\Re}(t) - 2g_{n'u\mathbf{k}}^{\Im} v_{n'u\mathbf{k}}^{\Im}(t)}{\omega_{u\mathbf{k}}} \right) \tag{2594}$$

$$= \frac{2v_{n'u\mathbf{k}}^{\Im}(t) - 2g_{n'u\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'} \tag{2595}$$

$$= 2\frac{v_{nu\mathbf{k}}^{\Im}(t) - g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}} \delta_{nn'}. \tag{2596}$$

Given that:

$$\varphi_{nm}(t) \equiv \prod_{u\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v_{nu\mathbf{k}}^{\Re}(t) v_{mu\mathbf{k}}^{\Im}(t) - v_{nu\mathbf{k}}^{\Im}(t) v_{mu\mathbf{k}}^{\Re}(t)\right)}{\omega_{u\mathbf{k}}^2}}, \tag{2597}$$

$$B_n(t) \equiv \prod_u \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v_{nu\mathbf{k}}^{\Re}(t)\right)^2 + \left(v_{nu\mathbf{k}}^{\Im}(t)\right)^2}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}, \tag{2598}$$

$$R_{nm}(t) \equiv \prod_u \mathrm{e}^{\sum_{\mathbf{k}} \frac{v_{nu\mathbf{k}}^{\Re}(t) v_{mu\mathbf{k}}^{\Re}(t) + v_{nu\mathbf{k}}^{\Im}(t) v_{mu\mathbf{k}}^{\Im}(t)}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}, \tag{2599}$$

$$R_{nm}(t) = R_{mn}(t), \tag{2600}$$

$$B_{nm}(t) = \varphi_{nm}(t) B_n(t) B_m(t) R_{nm}(t), \tag{2601}$$

$$B_{mn}(t) = \varphi_{nm}^{-1}(t) B_n(t) B_m(t) R_{nm}(t), \frac{\partial \varphi_{n'm'}(t)}{\partial v_{nu\mathbf{k}}^{\Re}(t)} = \frac{\mathrm{i} v_{mu\mathbf{k}}^{\Im}(t)}{\omega_{u\mathbf{k}}^2} \varphi_{nm}(t) \delta_{nn'}, \tag{2602}$$

$$\frac{\partial \varphi_{n'm'}(t)}{\partial v_{nu\mathbf{k}}^{\Im}(t)} = -\frac{\mathrm{i} v_{mu\mathbf{k}}^{\Re}(t)}{\omega_{u\mathbf{k}}^2} \varphi_{nm}(t) \delta_{nn'}, \tag{2603}$$

$$\frac{\partial B_{n'}(t)}{\partial v_{nu\mathbf{k}}^{\Re}(t)} = -\frac{v_{n'u\mathbf{k}}^{\Re}(t)}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) B_{n'}(t) \delta_{nn'}, \tag{2604}$$

$$\frac{\partial B_{n'}(t)}{\partial v_{nu\mathbf{k}}^{\Im}(t)} = -\frac{v_{n'u\mathbf{k}}^{\Im}(t)}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) B_{n'}(t) \delta_{nn'}, \tag{2605}$$

$$\frac{\partial R_{n'm'}(t)}{\partial v_{nu\mathbf{k}}^{\Re}(t)} = \frac{v_{m'u\mathbf{k}}^{\Re}(t)}{\omega_{u\mathbf{k}}^2} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) R_{n'm'}(t) \delta_{nn'}, \tag{2606}$$

$$\frac{\partial R_{n'm'}(t)}{\partial v^{\Im}_{nu\mathbf{k}}(t)} = \frac{v^{\Im}_{m'u\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) R_{n'm'}(t)\,\delta_{nn'}. \tag{2607}$$

Introducing this derivates in the equation and using the chain rule give us:

$$\begin{aligned}
E_{\mathrm{Free,B}}(t) &\equiv E_{\mathrm{Free,B}}\left(R_1(t),\cdots,R_n(t);\varphi_{12}(t),\cdots,\varphi_{n-1,n}(t);R_{12}(t),\cdots,R_{n-1,n}(t);B_1(t),\cdots,B_n(t)\right), && (2608)\\
\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial v^{\Re}_{nu\mathbf{k}}(t)} &= \sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'}(t)}\frac{\partial R_{n'}(t)}{\partial v^{\Re}_{nu\mathbf{k}}(t)} + \sum_{n'<m}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{n'm}(t)}\frac{\partial \varphi_{n'm}(t)}{\partial v^{\Re}_{nu\mathbf{k}}(t)} + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'm}(t)}\frac{\partial R_{n'm}(t)}{\partial v^{\Re}_{nu\mathbf{k}}(t)}\right) + \sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_{n'}(t)}\frac{\partial B_{n'}(t)}{\partial v^{\Re}_{nu\mathbf{k}}(t)} && (2609)\\
&= \sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'}(t)} 2\frac{v^{\Re}_{nu\mathbf{k}}(t) - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\delta_{nn'} + \sum_{n'<m}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{n'm}(t)}\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t)\,\delta_{nn'} + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'm}(t)}\frac{v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\right. && (2610)\\
&\left.\times\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{n'm}(t)\,\delta_{nn'}\right) + \sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_{n'}(t)}\left(-\frac{v^{\Re}_{n'u\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_{n'}(t)\,\delta_{nn'}\right) && (2611)\\
&= \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2\frac{v^{\Re}_{nu\mathbf{k}}(t) - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{nm}(t)}\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t) + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\frac{v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right. && (2612)\\
&\left.\times R_{nm}(t)\right) + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(-\frac{v^{\Re}_{nu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right) && (2613)\\
&= 0 && (2614)
\end{aligned}$$

We can obtain the real part of the variational parameters performing algebra:

$$\begin{aligned}
&\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2\frac{v^{\Re}_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}} + \sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{nm}(t)}\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t) + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\frac{v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right) && (2615)\\
&+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(-\frac{v^{\Re}_{nu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right), && (2616)\\
&\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2\frac{v^{\Re}_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}} - \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(\frac{v^{\Re}_{nu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right) && (2617)\\
&= v^{\Re}_{nu\mathbf{k}}(t)\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2\frac{1}{\omega_{u\mathbf{k}}} - \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(\frac{1}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right)\right) && (2618)\\
&= \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2\frac{g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{nm}(t)}\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t) + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\frac{v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right), && (2619)\\
v^{\Re}_{nu\mathbf{k}}(t) &= \frac{\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2\frac{g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{nm}(t)}\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t) + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\frac{v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right)}{\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2\frac{1}{\omega_{u\mathbf{k}}} - \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(\frac{1}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right)} && (2620)\\
&= \frac{\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} 2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}} - \sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{nm}(t)}\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)\,\varphi_{nm}(t) + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}v^{\Re}_{mu\mathbf{k}}(t)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} - B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}. && (2621)
\end{aligned}$$

Let's consider the imaginary part of the variation parameters:

$$\begin{aligned}
\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial v^{\Im}_{nu\mathbf{k}}(t)} &= \sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'}(t)}\frac{\partial R_{n'}(t)}{\partial v^{\Im}_{nu\mathbf{k}}(t)} + \sum_{n'<m}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{n'm}(t)}\frac{\partial \varphi_{n'm}(t)}{\partial v^{\Im}_{nu\mathbf{k}}(t)} + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'm}(t)}\frac{\partial R_{n'm}(t)}{\partial v^{\Im}_{nu\mathbf{k}}(t)}\right) + \sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_{n'}(t)}\frac{\partial B_{n'}(t)}{\partial v^{\Im}_{nu\mathbf{k}}(t)} && (2622)\\
&= \sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'}(t)} 2\frac{v^{\Im}_{nu\mathbf{k}}(t) - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\delta_{nn'} + \sum_{n'<m}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{n'm}(t)}\left(-\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t)\right)\delta_{nn'} + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{n'm}(t)}\frac{v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\right. && (2623)\\
&\left.\times\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{n'm}(t)\,\delta_{nn'}\right) && (2624)
\end{aligned}$$

$$+\sum_{n'}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_{n'}(t)}\left(-\frac{v^{\Im}_{n'u\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_{n'}(t)\,\delta_{nn'}\right) \tag{2625}$$

$$=\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}2\frac{v^{\Im}_{nu\mathbf{k}}(t)-g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+\sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\left(-\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t)\right)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\frac{v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right) \tag{2626}$$

$$+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(-\frac{v^{\Im}_{nu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right). \tag{2627}$$

Rearranging we obtain:

$$=\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}2\frac{v^{\Im}_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(-\frac{v^{\Im}_{nu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right) \tag{2628}$$

$$=2\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-\sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\left(-\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t)\right)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\frac{v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right), \tag{2629}$$

$$v^{\Im}_{nu\mathbf{k}}(t)=\frac{2\frac{g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-\sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\left(-\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\varphi_{nm}(t)\right)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\frac{v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right)}{\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}2\frac{1}{\omega_{u\mathbf{k}}}-\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\left(\frac{1}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)B_n(t)\right)} \tag{2630}$$

$$=\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-\sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}v^{\Im}_{mu\mathbf{k}}(t)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)-\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\mathrm{i}v^{\Re}_{mu\mathbf{k}}(t)\,\varphi_{nm}(t)\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}. \tag{2631}$$

The variational parameters are:

$$v_{nu\mathbf{k}}(t)=v^{\Re}_{nu\mathbf{k}}(t)+\mathrm{i}v^{\Im}_{nu\mathbf{k}}(t) \tag{2632}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-\sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)\,\varphi_{nm}(t)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}v^{\Re}_{mu\mathbf{k}}(t)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2633}$$

$$+\mathrm{i}\frac{2g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-\sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}v^{\Im}_{mu\mathbf{k}}(t)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)-\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\mathrm{i}v^{\Re}_{mu\mathbf{k}}(t)\,\varphi_{nm}(t)\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2634}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}+2\mathrm{i}g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2635}$$

$$-\sum_{m|m\neq n}\frac{\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)\,\varphi_{nm}(t)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}v^{\Re}_{mu\mathbf{k}}(t)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2636}$$

$$-\sum_{m|m\neq n}\frac{\mathrm{i}\sum_{m|m\neq n}\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}v^{\Im}_{mu\mathbf{k}}(t)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)-\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\mathrm{i}v^{\Re}_{mu\mathbf{k}}(t)\,\varphi_{nm}(t)\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2637}$$

$$=\frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}+2\mathrm{i}g^{\Im}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2638}$$

$$-\sum_{m|m\neq n}\frac{\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\mathrm{i}v^{\Im}_{mu\mathbf{k}}(t)\,\varphi_{nm}(t)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}v^{\Re}_{mu\mathbf{k}}(t)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n(t)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2639}$$

$$-\sum_{m|m\neq n}\frac{\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\mathrm{i}v_{mu\mathbf{k}}^{\Im}\left(t\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}v_{mu\mathbf{k}}^{\Re}\left(t\right)\varphi_{nm}\left(t\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n\left(t\right)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2640}$$

$$=\frac{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}\left(g_{nu\mathbf{k}}^{\Re}+\mathrm{i}g_{nu\mathbf{k}}^{\Im}\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n\left(t\right)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2641}$$

$$-\sum_{m|m\neq n}\frac{\left(v_{mu\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{mu\mathbf{k}}^{\Im}\left(t\right)\right)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}\left(t\right)+\left(v_{mu\mathbf{k}}^{\Re}\left(t\right)+\mathrm{i}v_{mu\mathbf{k}}^{\Im}\left(t\right)\right)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\varphi_{nm}\left(t\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n\left(t\right)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2642}$$

$$=\frac{2\omega_{u\mathbf{k}}g_{nu\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-\sum_{m|m\neq n}v_{mu\mathbf{k}}\left(t\right)\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\varphi_{nm}\left(t\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n\left(t\right)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}. \tag{2643}$$

So we summarize that:

$$v_{nu\mathbf{k}}\left(t\right)=\frac{2\omega_{u\mathbf{k}}g_{nu\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-\sum_{m|m\neq n}v_{mu\mathbf{k}}\left(t\right)\left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)R_{nm}(t)+\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial\varphi_{nm}(t)}\varphi_{nm}\left(t\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)}-B_n\left(t\right)\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}. \tag{2644}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho\left(0\right)=\rho_{\mathrm{S}}\left(0\right)\otimes\rho_B$ with $\rho_{\mathrm{S}}\left(0\right)=\sum_{j,k}\rho_{jk}\left(0\right)|j\rangle\langle k|$ and $\rho_{jk}\left(0\right)=\rho_{kj}^{*}\left(0\right)$. Consider the following notation:

$$\overline{H_{\bar{B}}}=\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} \tag{2645}$$

$$=\sum_{u}\sum_{\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}, \tag{2646}$$

$$\overline{H_{\bar{B}u}}=\sum_{\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}, \tag{2647}$$

$$\overline{H_{\bar{B}}}=\sum_{u}\overline{H_{\bar{B}u}}. \tag{2648}$$

Given that each bath u is independent then the partition function of $\overline{H_{\bar{B}}}$ is equal to:

$$\begin{aligned}\rho_B&=\frac{\mathrm{e}^{-\sum_u\beta_u\overline{H_{\bar{B}u}}}}{\mathrm{Tr}\left(\mathrm{e}^{-\sum_u\beta_u\overline{H_{\bar{B}u}}}\right)}\\&=\frac{\bigotimes_u\mathrm{e}^{-\beta_u\overline{H_{\bar{B}u}}}}{\mathrm{Tr}\left(\bigotimes_u\mathrm{e}^{-\beta_u\overline{H_{\bar{B}u}}}\right)}\text{ (by independence of Hilbert space)}\\&=\frac{\bigotimes_u\mathrm{e}^{-\beta_u\overline{H_{\bar{B}u}}}}{\prod_u\mathrm{Tr}\left(\mathrm{e}^{-\beta_u\overline{H_{\bar{B}u}}}\right)}\text{ (by trace properties)}\\&=\bigotimes_u\left(\frac{\mathrm{e}^{-\beta_u\overline{H_{\bar{B}u}}}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta_u\overline{H_{\bar{B}u}}}\right)}\right)\\&=\bigotimes_u\left(\prod_{\mathbf{k}}\frac{\sum_{j_{u\mathbf{k}}}\mathrm{e}^{-j_{u\mathbf{k}}\beta_u\omega_{u\mathbf{k}}}|j_{u\mathbf{k}}\rangle\langle j_{u\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta_u\omega_{u\mathbf{k}}\right)}\right)\text{ (by (128)).}\end{aligned}$$

The transformation of $\rho(0)$ is:

$$\mathrm{e}^{V(0)}\rho(0)\,\mathrm{e}^{-V(0)} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}(0)\right)\rho(0)\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}(0)\right) \tag{2649}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}(0)\right)\rho_{\mathrm{S}}(0)\otimes\rho_B\left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}(0)\right) \tag{2650}$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}(0)\right)\sum_{j,k}\rho_{jk}(0)\,|j\rangle\langle k|\otimes\rho_B\left(\sum_{n'} |n'\rangle\langle n'| \prod_u B_{n'u-}(0)\right) \tag{2651}$$

$$= \sum_{n,n',j,k} |n\rangle\langle n'|\delta_{nj}\delta_{kn'}\rho_{jk}(0)\otimes\prod_u B_{nu+}(0)\,\rho_B\prod_u B_{n'u-}(0) \tag{2652}$$

$$= \sum_{j,k} |j\rangle\langle k|\rho_{jk}(0)\otimes\prod_u B_{ju+}(0)\,\rho_B\prod_u B_{ku-}(0) \tag{2653}$$

$$= \overline{\rho(0)}. \tag{2654}$$

Recalling that we transform any operator $O(t)$ into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^{\dagger}(t)\,O(t)\,U(t) \tag{2655}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}(t')\right). \tag{2656}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t), \tag{2657}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B(\bar{\rho}(t)) \tag{2658}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n|, \tag{2659}$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right), \tag{2660}$$

$$|n\rangle\langle m| = \frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}, \tag{2661}$$

$$|m\rangle\langle n| = \frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}, \tag{2662}$$

$$J_{nm,x}(t) = \frac{J_{nm}(t) + J_{mn}(t)}{2}, \tag{2663}$$

$$J_{nm,y}(t) = \frac{J_{nm}(t) - J_{mn}(t)}{2\mathrm{i}}, \tag{2664}$$

$$J_{nm}(t) = J_{nm,x}(t) + \mathrm{i}J_{nm,y}(t), \tag{2665}$$

$$J_{mn}(t) = J_{nm,x}(t) - \mathrm{i}J_{nm,y}(t). \tag{2666}$$

We can proof that $J_{nm}(t) = J_{mn}^{\dagger}(t)$ so we can say that the set of matrices $\sigma_{nm,x}, \sigma_{nm,y}, J_{nm,x}(t), J_{nm,y}(t)$ are hermitic. Re-writing the transformed interaction Hamiltonian using the set (2659) give us:

$$\overline{H_{\bar{I}}(t)} = \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m|J_{nm}(t) + \sum_n B_{z,n}(t)\,|n\rangle\langle n| \tag{2667}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)\,|n\rangle\langle m|J_{nm}(t) + V_{mn}(t)\,|m\rangle\langle n|J_{mn}(t)\right) \tag{2668}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \sum_{n<m}\left(V^{\Re}_{nm}(t)\,J_{nm}(t)\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V^{\Im}_{nm}(t)\,J_{nm}(t)\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{2669}$$

$$\left.+V^{\Re}_{nm}(t)\,J_{mn}(t)\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V^{\Im}_{nm}(t)\,J_{mn}(t)\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{2670}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \sum_{n<m}\left(V^{\Re}_{nm}(t)\,J_{nm}(t)\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V^{\Im}_{nm}(t)\,J_{nm}(t)\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{2671}$$

$$\left.+V^{\Re}_{nm}(t)\,J_{mn}(t)\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V^{\Im}_{nm}(t)\,J_{mn}(t)\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{2672}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \sum_{n<m}\left(V^{\Re}_{nm}(t)(J_{nm,x}(t)+\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right) + (J_{nm,x}(t)+\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{2673}$$

$$\left.\times\mathrm{i}V^{\Im}_{nm}(t) + V^{\Re}_{nm}(t)(J_{nm,x}(t)-\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V^{\Im}_{nm}(t)(J_{nm,x}(t)-\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{2674}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \sum_{n<m}\left(V^{\Re}_{nm}(t)\left((J_{nm,x}(t)+\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right) + (J_{nm,x}(t)-\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right)\right)\right. \tag{2675}$$

$$\left.+\mathrm{i}V^{\Im}_{nm}(t)\left((J_{nm,x}(t)+\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}}{2}\right) - (J_{nm,x}(t)-\mathrm{i}J_{nm,y}(t))\left(\frac{\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}}{2}\right)\right)\right) \tag{2676}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \frac{1}{2}\sum_{n<m}\left(V^{\Re}_{nm}(t)((J_{nm,x}(t)+\mathrm{i}J_{nm,y}(t))(\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}) + (J_{nm,x}(t)-\mathrm{i}J_{nm,y}(t))(\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}))\right. \tag{2677}$$

$$\left.+\mathrm{i}V^{\Im}_{nm}(t)\,((J_{nm,x}(t)+\mathrm{i}J_{nm,y}(t))\,(\sigma_{nm,x}-\mathrm{i}\sigma_{nm,y}) - (J_{nm,x}(t)-\mathrm{i}J_{nm,y}(t))\,(\sigma_{nm,x}+\mathrm{i}\sigma_{nm,y}))\right) \tag{2678}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \frac{1}{2}\sum_{n<m}\left(V^{\Re}_{nm}(t)\,(J_{nm,x}(t)\,\sigma_{nm,x} - \mathrm{i}J_{nm,x}(t)\,\sigma_{nm,y} + \mathrm{i}J_{nm,y}(t)\,\sigma_{nm,x} + J_{nm,y}(t)\,\sigma_{nm,y} + J_{nm,x}(t)\right. \tag{2679}$$

$$\times\sigma_{nm,x} + \mathrm{i}J_{nm,x}(t)\,\sigma_{nm,y} - \mathrm{i}J_{nm,y}(t)\,\sigma_{nm,x} + J_{nm,y}(t)\,\sigma_{nm,y}) + \mathrm{i}V^{\Im}_{nm}(t)(J_{nm,x}(t)\,\sigma_{nm,x} - \mathrm{i}J_{nm,x}(t)\,\sigma_{nm,y} + \mathrm{i}J_{nm,y}(t) \tag{2680}$$

$$\left.\times\sigma_{nm,x} + J_{nm,y}(t)\,\sigma_{nm,y} - (J_{nm,x}(t)\,\sigma_{nm,x} + \mathrm{i}J_{nm,x}(t)\,\sigma_{nm,y} - \mathrm{i}J_{nm,y}(t)\,\sigma_{nm,x} + J_{nm,y}(t)\,\sigma_{nm,y}))\right) \tag{2681}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \frac{1}{2}\sum_{n<m}\left(V^{\Re}_{nm}(t)(J_{nm,x}(t)\,\sigma_{nm,x} + J_{nm,y}(t)\,\sigma_{nm,y} + J_{nm,x}(t)\,\sigma_{nm,x} + J_{nm,y}(t)\,\sigma_{nm,y}) + \mathrm{i}V^{\Im}_{nm}(t)\right. \tag{2682}$$

$$\left.\times(-2\mathrm{i}J_{nm,x}(t)\,\sigma_{nm,y} + 2\mathrm{i}J_{nm,y}(t)\,\sigma_{nm,x})\right) \tag{2683}$$

$$= \sum_n B_{z,n}(t)\,|n\rangle\!\langle n| + \sum_{n<m}\left(V^{\Re}_{nm}(t)\,(J_{nm,x}(t)\,\sigma_{nm,x} + J_{nm,y}(t)\,\sigma_{nm,y}) + V^{\Im}_{nm}(t)\,(J_{nm,x}(t)\,\sigma_{nm,y} - J_{nm,y}(t)\,\sigma_{nm,x})\right). \tag{2684}$$

Let's define the set:

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \le n, m \le d-1 \,\wedge\, (n = m \,\text{or}\, n < m)\right\}. \tag{2685}$$

Now consider the following set of operators:

$$A_{1nm} = \sigma_{nm,x}, \tag{2686}$$
$$A_{2nm} = \sigma_{nm,y}, \tag{2687}$$
$$A_{3nm} = |n\rangle\!\langle m|, \tag{2688}$$
$$A_{4nm} = A_{1mn}, \tag{2689}$$
$$A_{5nm} = A_{2nm}, \tag{2690}$$
$$B_{1nm}(t) = J_{nm,x}(t)\,(1-\delta_{mn}), \tag{2691}$$
$$B_{2nm}(t) = J_{nm,y}(t)\,(1-\delta_{mn}), \tag{2692}$$
$$B_{3nm}(t) = B_{z,n}(t)\,\delta_{nm}, \tag{2693}$$
$$B_{4nm}(t) = B_{2nm}(t), \tag{2694}$$
$$B_{5nm}(t) = B_{1nm}(t), \tag{2695}$$
$$C_{1nm}(t) = V^{\Re}_{nm}(t), \tag{2696}$$
$$C_{2nm}(t) = C_{1nm}(t), \tag{2697}$$

$$C_{3nm}\left(t\right)=1, \tag{2698}$$
$$C_{4nm}\left(t\right)=V_{nm}^{\Im}\left(t\right), \tag{2699}$$
$$C_{5nm}\left(t\right)=-C_{4nm}\left(t\right). \tag{2700}$$

The previous notation that denotes the principal elements present in the transformed interaction Hamiltonian allows us to write $\overline{H_I}\left(t\right)$ as:

$$\overline{H_I}\left(t\right)=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(A_{jp}\otimes B_{jp}\left(t\right)\right). \tag{2701}$$

The index shown in the sum are given by $J=\{1,2,3,4,5\}$ and P the set defined in (2685).
We write the interaction Hamiltonian transformed under (2655) as:

$$\widetilde{H_I}\left(t\right)=\sum_{j\in J,p\in P}C_{jp}\left(t\right)\left(\widetilde{A_{jp}}\left(t\right)\otimes\widetilde{B_{jp}}\left(t\right)\right) \tag{2702}$$
$$\widetilde{A_{jp}}\left(t\right)=U^{\dagger}\left(t\right)A_{jp}\left(t\right)U\left(t\right) \tag{2703}$$
$$\widetilde{B_{jp}}\left(t\right)=\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\left(t\right)\mathrm{e}^{-\mathrm{i}H_Bt} \tag{2704}$$

Taking as reference state ρ_B and truncating at second order in $H_I\left(t\right)$, we obtain our master equation in the interaction picture:

$$\frac{\partial\widetilde{\widetilde{\rho_S}}\left(t\right)}{\partial t}=-\int_0^t\mathrm{Tr}_B\left[\widetilde{H_I}\left(t\right),\left[\widetilde{H_I}\left(t'\right),\widetilde{\widetilde{\rho_S}}\left(t\right)\rho_B\right]\right]\mathrm{d}t' \tag{2705}$$

Replacing the equation (2702) in (2705) we can obtain:

$$\frac{\partial\widetilde{\widetilde{\rho_S}}(t)}{\partial t}=-\int_0^t\mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(t'),\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}t' \tag{2706}$$
$$=-\int_0^t\mathrm{Tr}_B\left[\sum_{j\in J,p\in P}C_{jp}(t)\Big(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\Big),\left[\sum_{j'\in J,p'\in P}C_{j'p'}(t')\left(\widetilde{A_{j'p'}}(t')\otimes\widetilde{B_{j'p'}}(t')\right),\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\right]\right]\mathrm{d}t' \tag{2707}$$
$$=-\int_0^t\mathrm{Tr}_B\left[\sum_{j\in J,p\in P}C_{jp}(t)\Big(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\Big),\sum_{j'\in J,p'\in P}C_{j'p'}(t')\Big(\widetilde{A_{j'p'}}(t')\otimes\widetilde{B_{j'p'}}(t')\Big)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B-\widetilde{\widetilde{\rho_S}}(t)\right. \tag{2708}$$
$$\left.\times\rho_B\sum_{j'\in J,p'\in P}C_{j'p'}(t')\left(\widetilde{A_{j'p'}}(t')\otimes\widetilde{B_{j'p'}}(t')\right)\right]\mathrm{d}t' \tag{2709}$$

$$=-\int_0^t \mathrm{Tr}_B\Bigg(\sum_{j\in J,p\in P} C_{jp}(t)\Big(\widetilde{A_{j,p}}(t)\otimes\widetilde{B_{j,p}}(t)\Big)\sum_{j'\in J,p'\in P} C_{j'p'}(t')\Big(\widetilde{A_{j'p'}}(t')\otimes\widetilde{B_{j'p'}}(t')\Big)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B-\sum_{j\in J,p\in P} C_{j,p}(t) \quad (2710)$$

$$\times\Big(\widetilde{A_{j,p}}(t)\otimes\widetilde{B_{j,p}}(t)\Big)\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(t')\Big(\widetilde{A_{j'p'}}(t')\otimes\widetilde{B_{j'p'}}(t')\Big)-\sum_{j'\in J,p'\in P} C_{j'p'}(t')\Big(\widetilde{A_{j'p'}}(t')\otimes\widetilde{B_{j'p'}}(t')\Big) \quad (2711)$$

$$\times\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\sum_{j\in J,p\in P} C_{j,p}(t)\Big(\widetilde{A_{j,p}}(t)\otimes\widetilde{B_{j,p}}(t)\Big)+\widetilde{\widetilde{\rho_S}}(t)\,\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(t')\Big(\widetilde{A_{j'p'}}(t')\otimes\widetilde{B_{j'p'}}(t')\Big)\sum_{j\in J,p\in P} C_{j,p}(t) \quad (2712)$$

$$\times\Big(\widetilde{A_{j,p}}(t)\otimes\widetilde{B_{j,p}}(t)\Big)\Bigg)\,\mathrm{d}t' \quad (2713)$$

$$=-\sum_{j,j'\in J,p,p'\in P}\int_0^t \mathrm{Tr}_B\,\Big(C_{jp}(t)\,C_{j'p'}(t')\,\Big(\widetilde{A_{j,p}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\widetilde{\rho_S}}(t)\otimes\widetilde{B_{j,p}}(t)\,\widetilde{B_{j'p'}}(t')\,\rho_B-\widetilde{A_{j,p}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'p'}}\,(t') \quad (2714)$$

$$\otimes\,\widetilde{B_{j,p}}\,(t)\,\rho_B\widetilde{B_{j'p'}}\,(t')-\widetilde{A_{j'p'}}\,(t')\,\widetilde{\widetilde{\rho_S}}\,(t)\,\widetilde{A_{j,p}}\,(t)\otimes\widetilde{B_{j'p'}}(t')\,\rho_B\widetilde{B_{j,p}}(t)+\widetilde{\widetilde{\rho_S}}\,(t)\,\widetilde{A_{j'p'}}\,(t')\,\widetilde{A_{j,p}}\,(t)\otimes\rho_B\widetilde{B_{j'p'}}\,(t') \quad (2715)$$

$$\times\widetilde{B_{j,p}}(t)\Big)\,\mathrm{d}t'\Big)\,. \quad (2716)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jp,j'p'}\,(t,t')\equiv\mathrm{Tr}_B\,\Big(\widetilde{B_{jp}}\,(t)\,\widetilde{B_{j'p'}}\,(t')\,\rho_B\Big) \quad (2717)$$

$$=\Big\langle\widetilde{B_{jp}}\,(t)\,\widetilde{B_{j'p'}}\,(t')\Big\rangle_B\,. \quad (2718)$$

A property derived from the hermiticity and shown in (415) provides the arguments needed to write:

$$\Lambda^*_{jp,j'p'}\,(t,t')=\mathrm{Tr}_B\,\Big(\widetilde{B_{jp}}\,(t)\,\widetilde{B_{j'p'}}\,(t')\,\rho_B\Big)^\dagger \quad (2719)$$

$$=\mathrm{Tr}_B\,\Big(\rho_B^\dagger\widetilde{B_{j'p'}}^\dagger\,(t')\,\widetilde{B_{jp}}^\dagger\,(t)\Big) \quad (2720)$$

$$=\mathrm{Tr}_B\,\Big(\rho_B\widetilde{B_{j'p'}}\,(t')\,\widetilde{B_{jp}}\,(t)\Big)\ \text{(by hermiticity of the operators)} \quad (2721)$$

$$=\mathrm{Tr}_B\,\Big(\widetilde{B_{j'p'}}\,(t')\,\widetilde{B_{jp}}\,(t)\,\rho_B\Big)\ \text{(by cyclic property of the trace)} \quad (2722)$$

$$=\Lambda_{j'p',jp}\,(t',t)\,. \quad (2723)$$

The correlation functions implied in (2713) are:

$$\mathrm{Tr}_B\,\Big(\widetilde{B_{jp}}\,(t)\,\widetilde{B_{j'p'}}\,(t')\,\rho_B\Big)=\Lambda_{jp,j'p'}\,(t,t')\,, \quad (2724)$$

$$\mathrm{Tr}_B\,\Big(\widetilde{B_{j,p}}\,(t)\,\rho_B\widetilde{B_{j'p'}}\,(t')\Big)=\mathrm{Tr}_B\,\Big(\widetilde{B_{j'p'}}\,(t')\,\widetilde{B_{j,p}}\,(t)\,\rho_B\Big)\ \text{(by cyclic property of the trace)} \quad (2725)$$

$$=\Lambda_{j'p',jp}\,(t',t) \quad (2726)$$

$$=\Lambda^*_{jp,j'p'}\,(t,t')\,, \quad (2727)$$

$$\mathrm{Tr}_B\,\Big(\widetilde{B_{j'p'}}\,(t')\,\rho_B\widetilde{B_{j,p}}\,(t)\Big)=\mathrm{Tr}_B\,\Big(\widetilde{B_{j,p}}\,(t)\,\widetilde{B_{j'p'}}\,(t')\,\rho_B\Big)\ \text{(by cyclic property of the trace)} \quad (2728)$$

$$=\Lambda_{jp,j'p'}\,(t,t') \quad (2729)$$

$$\mathrm{Tr}_B\,\Big(\rho_B\widetilde{B_{j'p'}}\,(t')\,\widetilde{B_{j,p}}\,(t)\Big)=\mathrm{Tr}_B\,\Big(\widetilde{B_{j'p'}}\,(t')\,\widetilde{B_{j,p}}\,(t)\,\rho_B\Big)\ \text{(by cyclic property of the trace)} \quad (2730)$$

$$=\Lambda^*_{jp,j'p'}\,(t,t')\,. \quad (2731)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (2706) and (2713) and using the equations (2724)-(2731) we can re-write:

$$\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t} = -\sum_{j,j'\in J,p,p'\in P}\int_0^t C_{jp}(t)C_{j'p'}(t')\Big(\widetilde{A_{j,p}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,\Lambda_{jp,j'p'}(t,t') - \widetilde{A_{j,p}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,\Lambda^*_{jp,j'p'}(t,t') \tag{2732}$$

$$-\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j,p}}(t)\,\Lambda_{jp,j'p'}(t,t') + \widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{A_{j,p}}(t)\,\Lambda^*_{jp,j'p'}(t,t')\Big)\,\mathrm{d}t' \tag{2733}$$

$$\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t} = -\sum_{j,j'\in J,p,p'\in P}\int_0^t C_{jp}(t)C_{j'p'}(t')\Big(\widetilde{A_{j,p}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\Lambda_{jp,j'p'}(t,t') - \widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j,p}}(t)\Lambda_{jp,j'p'}(t,t') \tag{2734}$$

$$+\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{A_{j,p}}(t)\,\Lambda^*_{jp,j'p'}(t,t') - \widetilde{A_{j,p}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,\Lambda^*_{jp,j'p'}(t,t')\Big)\,\mathrm{d}t' \tag{2735}$$

$$= -\sum_{j,j'\in J,p,p'\in P}\int_0^t C_{jp}(t)\;C_{j'p'}(t')\;\Big(\Lambda_{jp,j'p'}(t,t')\;\Big(\widetilde{A_{j,p}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t) - \widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j,p}}(t)\Big) \tag{2736}$$

$$+\Lambda^*_{jp,j'p'}(t,t')\Big(\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{A_{j,p}}(t) - \widetilde{A_{j,p}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\Big)\Big)\,\mathrm{d}t' \tag{2737}$$

$$= -\sum_{j,j'\in J,p,p'\in P}\int_0^t C_{jp}(t)C_{j'p'}(t')\Big(\Lambda_{jp,j'p'}(t,t')\Big[\widetilde{A_{j,p}}(t)\,,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\Big] + \Lambda^*_{jp,j'p'}(t,t')\Big[\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,, \tag{2738}$$

$$\widetilde{A_{j,p}}(t)\Big]\Big)\,\mathrm{d}t' \tag{2739}$$

$$= -\sum_{j,j'\in J,p,p'\in P}\int_0^t C_{jp}(t)\,C_{j'p'}(t')\Big(\Lambda_{jp,j'p'}(t,t')\,\Big[\widetilde{A_{jp}}(t)\,,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\Big] - \Lambda^*_{jp,j'p'}(t,t')\,\Big[\widetilde{A_{j,p}}(t)\,,\widetilde{\overline{\rho_S}}(t) \tag{2740}$$

$$\times\widetilde{A_{j'p'}}(t')\Big]\Big)\,\mathrm{d}t'. \tag{2741}$$

Returning to the Schrödinger picture we have:

$$U(t)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) = U(t)\,\widetilde{A_{jp}}(t)\,\mathbb{I}\widetilde{A_{j'p'}}(t')\,\mathbb{I}\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) \tag{2742}$$

$$= U(t)\,\widetilde{A_{jp}}(t)\,U^\dagger(t)\,U(t)\,\widetilde{A_{j'p'}}(t')\,U^\dagger(t)\,U(t)\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) \tag{2743}$$

$$= \Big(U(t)\,\widetilde{A_{jp}}(t)\,U^\dagger(t)\Big)\,\Big(U(t)\,\widetilde{A_{j'p'}}(t')\,U^\dagger(t)\Big)\,\Big(U(t)\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t)\Big), \tag{2744}$$

$$U(t)\,\widetilde{A_{jp}}(t')\,U^\dagger(t) \equiv \widetilde{A_{jp}}(t',t), \tag{2745}$$

$$U(t)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) = A_{jp}\widetilde{A_{j'p'}}(t',t)\,\overline{\rho_S}(t) \tag{2746}$$

This procedure applied to the relevant conmutators give us:

$$U(t)\,\Big[\widetilde{A_{jp}}(t)\,,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\Big]\,U^\dagger(t) = U(t)\,\widetilde{A_{jp}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) - U(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{jp}}(t)\,U^\dagger(t) \tag{2747}$$

$$= A_{jp}\widetilde{A_{j'p'}}(t',t)\,\overline{\rho_S}(t) - \widetilde{A_{j'p'}}(t',t)\,\overline{\rho_S}(t)\,A_{jp} \tag{2748}$$

$$= \Big[A_{jp},\widetilde{A_{j'p'}}(t',t)\,\overline{\rho_S}(t)\Big], \tag{2749}$$

$$U(t)\,\Big[\widetilde{A_{j,p}}(t)\,,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\Big]\,U^\dagger(t) = U(t)\,\widetilde{A_{jp}}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,U^\dagger(t) - U(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_{j'p'}}(t')\,\widetilde{A_{jp}}(t)\,U^\dagger(t) \tag{2750}$$

$$= A_{jp}\overline{\rho_S}(t)\,\widetilde{A_{j'p'}}(t',t) - \overline{\rho_S}(t)\,\widetilde{A_{j'p'}}(t',t)\,A_{jp} \tag{2751}$$

$$= \Big[A_{jp},\overline{\rho_S}(t)\,\widetilde{A_{j'p'}}(t',t)\Big]. \tag{2752}$$

Introducing this transformed commutators in the equation (2740) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\Lambda_{jp,j'p'}(t,t')$ as defined before, this equations has the following form:

$$\frac{\partial\overline{\rho_S}(t)}{\partial t}=-\mathrm{i}\left[H_{\overline{S},\mathrm{eff}}(t),\overline{\rho_S}(t)\right]-\sum_{j,j'\in J,p,p'\in P}\int_0^t C_{jp}(t)C_{j'p'}(t')\Big(\Lambda_{jp,j'p'}(t,t')\Big[A_{jp},\widetilde{A_{j'p'}}(t',t)\,\overline{\rho_S}(t)\Big]-\Lambda^*_{jp,j'p'}(t,t') \tag{2753}$$

$$\times\Big[A_{jp},\overline{\rho_S}(t)\,\widetilde{A_{j'p'}}(t',t)\Big]\Big)\,\mathrm{d}t'-\mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right),\overline{\rho_S}(t)\right], \tag{2754}$$

$$t'=t-\tau\ \text{(Change of variables in the integration process)}, \tag{2755}$$

$$\frac{\partial\overline{\rho_S}(t)}{\partial t}=-\mathrm{i}\left[H_{\overline{S},\mathrm{eff}}(t),\overline{\rho_S}(t)\right]-\sum_{j,j'\in J,p,p'\in P}\int_0^t \mathrm{d}\tau C_{jp}(t)\,C_{j'p'}(t-\tau)\,\Big(\Lambda_{jp,j'p'}(t,t-\tau)\,\Big[A_{jp},\widetilde{A_{j'p'}}(t-\tau,t)\,\overline{\rho_S}(t)\Big] \tag{2756}$$

$$-\Lambda^*_{jp,j'p'}(t,t-\tau)\,\Big[A_{jp},\overline{\rho_S}(t)\,\widetilde{A_{j'p'}}(t-\tau,t)\Big]\Big)-\mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{i}t)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t}\right),\overline{\rho_S}(t)\right]. \tag{2757}$$

The equation obtained is a master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian coupled to v-baths and valid at second order in $H_I(t)$. In order to write in a simplified way the equation obtained we define the following notation:

$$\Lambda_{jp,j'p'}(t,t')\equiv\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(t')\,\rho_B\right) \tag{2758}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}(t)\,\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}B_{j'p'}(t')\,\mathrm{e}^{-\mathrm{i}H_Bt'}\rho_B\right) \tag{2759}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}(t)\,\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}B_{j'p'}(t')\,\rho_B\mathrm{e}^{-\mathrm{i}H_Bt'}\right) \tag{2760}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{-\mathrm{i}H_Bt'}\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}(t)\,\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}B_{j'p'}(t')\,\rho_B\right) \tag{2761}$$

$$=\mathrm{Tr}_B\left(\left(\mathrm{e}^{-\mathrm{i}H_Bt'}\mathrm{e}^{\mathrm{i}H_Bt}\right)B_{jp}(t)\left(\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}\right)B_{j'p'}(t')\,\rho_B\right) \tag{2762}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}H_B(t-t')}B_{jp}(t)\,\mathrm{e}^{-\mathrm{i}H_B(t-t')}B_{j'p'}(t')\,\rho_B\right) \tag{2763}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}\tau H_B}B_{jp}(t)\,\mathrm{e}^{-\mathrm{i}\tau H_B}B_{j'p'}(t')\,\rho_B\right) \tag{2764}$$

$$=\mathrm{Tr}_B\left(B_{jp}(t,\tau)\,B_{j'p'}(t',0)\,\rho_B\right). \tag{2765}$$

The correlation functions for crossed terms related to $B_{3nm}(t)$ are given by:

$$\Lambda_{3nn,3mm}(t,t')=\langle B_{z,n}(t,\tau)\,B_{z,m}(t',0)\rangle_B\,, \tag{2766}$$

$$B_{z,n}(t,\tau)=\mathrm{e}^{\mathrm{i}\tau H_B}B_{z,n}(t)\,\mathrm{e}^{-\mathrm{i}\tau H_B} \tag{2767}$$

$$=\mathrm{e}^{\mathrm{i}\tau H_B}\left(\sum_{u\mathbf{k}}\left((g_{nu\mathbf{k}}-v_{nu\mathbf{k}}(t))\,b^\dagger_{u\mathbf{k}}+(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}(t))^*\,b_{u\mathbf{k}}\right)\right)\mathrm{e}^{-\mathrm{i}\tau H_B}, \tag{2768}$$

$$g_{nu\mathbf{k}}-v_{nu\mathbf{k}}(t)\equiv q_{nu\mathbf{k}}(t)\,, \tag{2769}$$

$$B_{z,n}(t,\tau)=\sum_{u\mathbf{k}}\left(q_{nu\mathbf{k}}(t)\,b^\dagger_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^*_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right), \tag{2770}$$

$$\Lambda_{3nn,3mm}(t,t')=\left\langle\sum_{u\mathbf{k}}\left(q_{nu\mathbf{k}}(t)\,b^\dagger_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^*_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\sum_{u'\mathbf{k}'}\left(q_{mu'\mathbf{k}'}(t')\,b^\dagger_{u'\mathbf{k}'}+q^*_{mu'\mathbf{k}'}(t')\,b_{u'\mathbf{k}'}\right)\right\rangle_B \tag{2771}$$

$$=\left\langle\sum_{u\mathbf{k}u'\mathbf{k}'}\left(q_{nu\mathbf{k}}(t)\,b^\dagger_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^*_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu'\mathbf{k}'}(t')\,b^\dagger_{u'\mathbf{k}'}+q^*_{mu'\mathbf{k}'}(t')\,b_{u'\mathbf{k}'}\right)\right\rangle_B \tag{2772}$$

$$=\left\langle\sum_{u\mathbf{k}\mathbf{k}'}\left(q_{nu\mathbf{k}}(t)\,b^\dagger_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^*_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu\mathbf{k}'}(t')\,b^\dagger_{u\mathbf{k}'}+q^*_{mu\mathbf{k}'}(t')\,b_{u\mathbf{k}'}\right)\right\rangle_B \tag{2773}$$

$$+\left\langle\sum_{u\mathbf{k}u'\mathbf{k}'|u\neq u'}\left(q_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu'\mathbf{k}'}(t')\,b_{u'\mathbf{k}'}^{\dagger}+q_{mu'\mathbf{k}'}^{*}(t')\,b_{u'\mathbf{k}'}\right)\right\rangle_{B} \quad (2774)$$

$$=\left\langle\sum_{u\mathbf{k}\mathbf{k}'}\left(q_{nu\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu\mathbf{k}'}\left(t'\right)b_{u\mathbf{k}'}^{\dagger}+q_{mu\mathbf{k}'}^{*}\left(t'\right)b_{u\mathbf{k}'}\right)\right\rangle_{B} \quad (2775)$$

$$+\sum_{u\mathbf{k}u'\mathbf{k}'|u\neq u'}\left(q_{nu\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b_{u\mathbf{k}}^{\dagger}\right\rangle_{B}+q_{nu\mathbf{k}}^{*}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b_{u\mathbf{k}}\right\rangle_{B}\right)\left(q_{mu'\mathbf{k}'}(t')\left\langle b_{u'\mathbf{k}'}^{\dagger}\right\rangle_{B}+q_{mu'\mathbf{k}'}^{*}(t') \quad (2776)$$

$$\times\left\langle b_{u'\mathbf{k}'}\right\rangle_{B}\right) \quad (2777)$$

$$=\left\langle\sum_{u\mathbf{k}\mathbf{k}'}\left(q_{nu\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu\mathbf{k}'}\left(t'\right)b_{u\mathbf{k}'}^{\dagger}+q_{mu\mathbf{k}'}^{*}\left(t'\right)b_{u\mathbf{k}'}\right)\right\rangle_{B} \quad (2778)$$

$$=\sum_{u\mathbf{k}}\left\langle\left(q_{nu\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu\mathbf{k}}\left(t'\right)b_{u\mathbf{k}}^{\dagger}+q_{mu\mathbf{k}}^{*}\left(t'\right)b_{u\mathbf{k}}\right)\right\rangle_{B} \quad (2779)$$

$$+\sum_{u\mathbf{k}\mathbf{k}'|\mathbf{k}\neq\mathbf{k}'}\left\langle\left(q_{nu\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu\mathbf{k}'}\left(t'\right)b_{u\mathbf{k}'}^{\dagger}+q_{mu\mathbf{k}'}^{*}\left(t'\right)b_{u\mathbf{k}'}\right)\right\rangle_{B} \quad (2780)$$

$$=\sum_{u\mathbf{k}}\left\langle\left(q_{nu\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu\mathbf{k}}\left(t'\right)b_{u\mathbf{k}}^{\dagger}+q_{mu\mathbf{k}}^{*}\left(t'\right)b_{u\mathbf{k}}\right)\right\rangle_{B} \quad (2781)$$

$$+\sum_{u\mathbf{k}\mathbf{k}'|\mathbf{k}\neq\mathbf{k}'}\left\langle\left(q_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right\rangle_{B}\left\langle\left(q_{mu\mathbf{k}'}(t')\,b_{u\mathbf{k}'}^{\dagger}+q_{mu\mathbf{k}'}^{*}(t')\,b_{u\mathbf{k}'}\right)\right\rangle_{B} \quad (2782)$$

$$=\sum_{u\mathbf{k}}\left\langle\left(q_{nu\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(q_{mu\mathbf{k}}\left(t'\right)b_{u\mathbf{k}}^{\dagger}+q_{mu\mathbf{k}}^{*}\left(t'\right)b_{u\mathbf{k}}\right)\right\rangle_{B} \quad (2783)$$

$$=\sum_{u\mathbf{k}}\Big\langle q_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}(t')\,b_{u\mathbf{k}}^{\dagger}+q_{nu\mathbf{k}}^{*}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}(t')\,b_{u\mathbf{k}}^{\dagger}+q_{nu\mathbf{k}}(t)\,b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}^{*}(t') \quad (2784)$$

$$\times b_{u\mathbf{k}}+q_{nu\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}^{*}\left(t'\right)b_{u\mathbf{k}}\Big\rangle_{B} \quad (2785)$$

$$=\sum_{u\mathbf{k}}\left(q_{nu\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}(t')\left\langle\left(b_{u\mathbf{k}}^{\dagger}\right)^{2}\right\rangle_{B}+q_{nu\mathbf{k}}^{*}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}(t')\left\langle b_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}\right\rangle_{B}+q_{nu\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right. \quad (2786)$$

$$\left.\times q_{mu\mathbf{k}}^{*}(t')\left\langle b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}\right\rangle_{B}+q_{nu\mathbf{k}}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}^{*}\left(t'\right)\left\langle\left(b_{u\mathbf{k}}\right)^{2}\right\rangle_{B}\right) \quad (2787)$$

$$=\sum_{u\mathbf{k}}\left(q_{nu\mathbf{k}}^{*}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}\left(t'\right)\left\langle b_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}\right\rangle_{B}+q_{nu\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{mu\mathbf{k}}^{*}\left(t'\right)\left\langle b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}\right\rangle_{B}\right) \quad (2788)$$

$$\left\langle b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}\right\rangle_{B}=N_{u\mathbf{k}} \quad (2789)$$

$$\equiv\left(\mathrm{e}^{\beta_{u}\omega_{u\mathbf{k}}}-1\right)^{-1}, \quad (2790)$$

$$\left\langle b_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}\right\rangle_{B}=N_{u\mathbf{k}}+1, \quad (2791)$$

$$\Lambda_{3nn,3mm}(t,t')=\sum_{u\mathbf{k}}\left(q_{nu\mathbf{k}}^{*}\left(t\right)q_{mu\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}\left(t\right)q_{mu\mathbf{k}}^{*}\left(t'\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right), \quad (2792)$$

$$\Lambda_{3n'n',1nm}(t,t')=\left\langle B_{z,n'}\left(t,\tau\right)B_{1,nm}\left(t',0\right)\right\rangle_{B} \quad (2793)$$

$$=\left\langle B_{z,n'}\left(t,\tau\right)J_{nm,x}\left(t'\right)\left(1-\delta_{nm}\right)\right\rangle_{B} \quad (2794)$$

$$=\left(1-\delta_{nm}\right)\left\langle B_{z,n'}\left(t,\tau\right)J_{nm,x}\left(t'\right)\right\rangle_{B} \quad (2795)$$

$$=\left(1-\delta_{nm}\right)\left\langle\left(\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t\right)b_{u\mathbf{k}}^{\dagger}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{n'u\mathbf{k}}^{*}\left(t\right)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\frac{J_{nm}\left(t'\right)+J_{mn}\left(t'\right)}{2}\right\rangle_{B}, \quad (2796)$$

$$J_{nm}\left(t\right)=\prod_{u}\left(B_{nu+}\left(t\right)B_{mu-}\left(t\right)\right)-B_{nm}\left(t\right) \quad (2797)$$

$$=\prod_{u\mathbf{k}}D\left(\alpha_{nu\mathbf{k}}\left(t\right)-\alpha_{mu\mathbf{k}}\left(t\right)\right)\prod_{u\mathbf{k}}\mathrm{e}^{\frac{v_{nu\mathbf{k}}^{*}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}^{*}(t)}{2\omega_{u\mathbf{k}}^{2}}}-B_{nm}\left(t\right), \quad (2798)$$

$$B_{nm}(t) = \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v^*_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}}} \prod_{u} \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)|^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}, \tag{2799}$$

$$\Gamma_{nmu\mathbf{k}}(t) \equiv \mathrm{e}^{\frac{v^*_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}}}, \tag{2800}$$

$$\alpha_{nu\mathbf{k}}(t) = \frac{v_{nu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}, \tag{2801}$$

$$\chi_{nmu}(t) \equiv \sum_{\mathbf{k}} \frac{v^*_{nu\mathbf{k}}(t)\, v_{mu\mathbf{k}}(t) - v_{nu\mathbf{k}}(t)\, v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}} \tag{2802}$$

$$= \mathrm{e}^{\sum_{\mathbf{k}} \Gamma_{nmu\mathbf{k}}(t)}, \tag{2803}$$

$$J_{nm}(t) = \prod_{u} \mathrm{e}^{\chi_{nmu}(t)} \left(\prod_{u\mathbf{k}} D\left(\alpha_{nu\mathbf{k}}(t) - \alpha_{mu\mathbf{k}}(t)\right) - \prod_{u\mathbf{k}} \mathrm{e}^{-\frac{1}{2}\frac{|v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)|^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}\right) \tag{2804}$$

$$\alpha_{(nm)u\mathbf{k}}(t) \equiv \frac{v_{nu\mathbf{k}}(t) - v_{mu\mathbf{k}}(t)}{\omega_{u\mathbf{k}}}, \tag{2805}$$

$$B_{nmu}(t) \equiv \mathrm{e}^{\chi_{nmu}(t)} \prod_{\mathbf{k}} \mathrm{e}^{-\frac{1}{2}\frac{|v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)|^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}, \tag{2806}$$

$$B_{nm}(t) = \prod_{u} B_{nmu}(t), \tag{2807}$$

$$D_{nmu}(t) \equiv \prod_{\mathbf{k}} \mathrm{e}^{-\frac{1}{2}\frac{|v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)|^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}, \tag{2808}$$

$$D_{nm}(t) \equiv \prod_{u} D_{nmu}(t), \tag{2809}$$

$$\chi_{nm}(t) = \sum_{u\mathbf{k}} \frac{v^*_{nu\mathbf{k}}(t)\, v_{mu\mathbf{k}}(t) - v_{nu\mathbf{k}}(t)\, v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}}, \tag{2810}$$

$$\chi_{nm}(t) = -\chi_{mn}(t), \tag{2811}$$

$$J_{nm}(t) = \mathrm{e}^{\chi_{nm}(t)} \left(\prod_{u\mathbf{k}} D\left(\alpha_{(nm)u\mathbf{k}}(t)\right) - D_{nm}(t)\right), \tag{2812}$$

$$J_{nm,x}(t) = \frac{\mathrm{e}^{\chi_{nm}(t)}\left(\prod_{u\mathbf{k}} D\left(\alpha_{(nm)u\mathbf{k}}(t)\right) - D_{nm}(t)\right) + \mathrm{e}^{\chi_{mn}(t)}\left(\prod_{u\mathbf{k}} D\left(\alpha_{(mn)u\mathbf{k}}(t)\right) - D_{mn}(t)\right)}{2}, \tag{2813}$$

$$J_{nm,y}(t) = \frac{\mathrm{e}^{\chi_{nm}(t)}\left(\prod_{u\mathbf{k}} D\left(\alpha_{(nm)u\mathbf{k}}(t)\right) - D_{nm}(t)\right) - \mathrm{e}^{\chi_{mn}(t)}\left(\prod_{u\mathbf{k}} D\left(\alpha_{(mn)u\mathbf{k}}(t)\right) - D_{mn}(t)\right)}{2\mathrm{i}}, \tag{2814}$$

$$\Lambda_{3n'n',1nm}(t,t') = (1-\delta_{nm})\left\langle \sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\, b^{\dagger}_{u\mathbf{k}} \mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} + q^*_{n'u\mathbf{k}}(t)\, b_{u\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(\frac{\mathrm{e}^{\chi_{nm}(t')}\left(\prod_{u'\mathbf{k}'} D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right) - D_{nm}(t')\right)}{2}\right.\right. \tag{2815}$$

$$\left.\left. + \mathrm{e}^{\chi_{mn}(t')} \frac{\left(\prod_{u'\mathbf{k}'} D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right) - D_{mn}(t')\right)}{2}\right)\right\rangle_B \tag{2816}$$

$$= (1-\delta_{nm})\left\langle \sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\, b^{\dagger}_{u\mathbf{k}} \mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} + q^*_{n'u\mathbf{k}}(t)\, b_{u\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(\frac{\mathrm{e}^{\chi_{nm}(t')}\prod_{u'\mathbf{k}'} D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)}{2} + \mathrm{e}^{\chi_{mn}(t')}\right.\right. \tag{2817}$$

$$\left.\left. \times \frac{\prod_{u'\mathbf{k}'} D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)}{2}\right)\right\rangle_B - (1-\delta_{nm})\frac{\mathrm{e}^{\chi_{nm}(t')}D_{nm}(t') + \mathrm{e}^{\chi_{mn}(t')}D_{mn}(t')}{2}\left\langle \sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\, b^{\dagger}_{u\mathbf{k}} \mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right.\right. \tag{2818}$$

$$\left.\left. + q^*_{n'u\mathbf{k}}(t)\, b_{u\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right\rangle_B \tag{2819}$$

$$= (1-\delta_{nm})\left\langle \sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\, b^{\dagger}_{u\mathbf{k}} \mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} + q^*_{n'u\mathbf{k}}(t)\, b_{u\mathbf{k}} \mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\left(\frac{\mathrm{e}^{\chi_{nm}(t')}\prod_{u'\mathbf{k}'} D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)}{2} + \mathrm{e}^{\chi_{mn}(t')}\right.\right. \tag{2820}$$

$$\left.\left. \times \frac{\prod_{u'\mathbf{k}'} D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)}{2}\right)\right\rangle_B \tag{2821}$$

$$= (1-\delta_{nm})\left(\frac{\mathrm{e}^{\chi_{nm}(t')}}{2}\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\, \mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b^{\dagger}_{u\mathbf{k}}\prod_{u'\mathbf{k}'} D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)\right\rangle_B + \left\langle b_{u\mathbf{k}}\prod_{u'\mathbf{k}'} D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)\right\rangle_B\right.\right. \tag{2822}$$

$$\times q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\Big) + \frac{\mathrm{e}^{\chi_{mn}(t')}}{2}\sum_{u\mathbf{k}}\Bigg(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\Bigg\langle b^{\dagger}_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\big(\alpha_{(mn)u'\mathbf{k}'}\left(t'\right)\big)\Bigg\rangle_{B} + q^{*}_{n'u\mathbf{k}}(t) \tag{2823}$$

$$\times\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B}\Bigg)\Bigg) \tag{2824}$$

$$\Bigg\langle b^{\dagger}_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\big(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\big)\Bigg\rangle_{B} = \left\langle b^{\dagger}_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t'\right)\right)\prod_{u'\mathbf{k}'\neq u\mathbf{k}}D\left(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{2825}$$

$$= \left\langle b^{\dagger}_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\prod_{u'\mathbf{k}'\neq u\mathbf{k}}\left\langle D\left(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{2826}$$

$$= -\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)\left\langle D\left(\alpha_{(nm)u\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}N_{u\mathbf{k}}\prod_{u'\mathbf{k}'\neq u\mathbf{k}}\left\langle D\left(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{2827}$$

$$= -\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\prod_{u'\mathbf{k}'}\left\langle D\left(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{2828}$$

$$= -\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}D_{nm}\left(t'\right), \tag{2829}$$

$$\Bigg\langle b_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\big(\alpha_{(mn)u'\mathbf{k}'}\left(t'\right)\big)\Bigg\rangle_{B} = \left\langle b_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t'\right)\right)\prod_{u'\mathbf{k}'\neq u\mathbf{k}}D\left(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{2830}$$

$$= \left\langle b_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}\prod_{u'\mathbf{k}'\neq u\mathbf{k}}\left\langle D\left(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{2831}$$

$$= \alpha_{(nm)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)\prod_{u'\mathbf{k}'}\left\langle D\left(\alpha_{(nm)u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_{B} \tag{2832}$$

$$= \alpha_{(nm)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)D_{nm}\left(t'\right), \tag{2833}$$

$$\Lambda_{3n'n',1nm}\left(t,t'\right) = \left(1-\delta_{nm}\right)\Bigg(\frac{\mathrm{e}^{\chi_{nm}(t')}}{2}\sum_{u\mathbf{k}}\Big(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(-\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}D_{nm}\left(t'\right)\right) + q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} \tag{2834}$$

$$\times\alpha_{(nm)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)D_{nm}\left(t'\right)\Big)+\frac{\mathrm{e}^{\chi_{mn}(t')}}{2}\sum_{u\mathbf{k}}\Big(q_{n'u\mathbf{k}}(t)\Big(-\alpha^{*}_{(mn)u\mathbf{k}}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}N_{u\mathbf{k}}D_{mn}\left(t'\right)\Big) \tag{2835}$$

$$+q^{*}_{n'u\mathbf{k}}\left(t\right)\alpha_{(mn)u\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(N_{u\mathbf{k}}+1\right)D_{mn}\left(t'\right)\Big)\Bigg) \tag{2836}$$

$$= \frac{1-\delta_{nm}}{2}\Bigg(\sum_{u\mathbf{k}}\Big(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(-\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}B_{nm}\left(t'\right)\right) + q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha_{(nm)u\mathbf{k}}\left(t'\right) \tag{2837}$$

$$\times\left(N_{u\mathbf{k}}+1\right)B_{nm}\left(t'\right)\Big)+\sum_{u\mathbf{k}}\Big(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(-\alpha^{*}_{(mn)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}B_{mn}\left(t'\right)\right)+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} \tag{2838}$$

$$\times\alpha_{(mn)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)B_{mn}\left(t'\right)\Big)\Bigg) \tag{2839}$$

$$= \frac{1-\delta_{nm}}{2}\sum_{u\mathbf{k}}\Big(B_{nm}\left(t'\right)\Big(q^{*}_{n'u\mathbf{k}}(t)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha_{(nm)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)-q_{n'u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\Big) \tag{2840}$$

$$+B_{mn}\left(t'\right)\left(q^{*}_{n'u\mathbf{k}}\left(t\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha_{(mn)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)-q_{n'u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^{*}_{(mn)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\right)\Big), \tag{2841}$$

$$\alpha_{(mn)u\mathbf{k}}\left(t\right) = \frac{v_{mu\mathbf{k}}\left(t\right)-v_{nu\mathbf{k}}\left(t\right)}{\omega_{u\mathbf{k}}} \tag{2842}$$

$$= -\alpha_{(nm)u\mathbf{k}}\left(t\right), \tag{2843}$$

$$\Lambda_{3n'n',1nm}\left(t,t'\right) = \frac{1-\delta_{nm}}{2}\sum_{u\mathbf{k}}\Big(B_{nm}\left(t'\right)\Big(q^{*}_{n'u\mathbf{k}}(t)\alpha_{(nm)u\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(N_{u\mathbf{k}}+1\right)-q_{n'u\mathbf{k}}(t)\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} \tag{2844}$$

$$\times N_{u\mathbf{k}}\Big) - B_{mn}\left(t'\right)\Big(q^{*}_{n'u\mathbf{k}}\left(t\right)\alpha_{(nm)u\mathbf{k}}\left(t'\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(N_{u\mathbf{k}}+1\right)-q_{n'u\mathbf{k}}\left(t\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\Big)\Big) \tag{2845}$$

$$= \frac{1-\delta_{nm}}{2}\left(B_{nm}\left(t'\right)-B_{mn}\left(t'\right)\right)\sum_{u\mathbf{k}}\Big(q^{*}_{n'u\mathbf{k}}(t)\,\alpha_{(nm)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{n'u\mathbf{k}}(t) \tag{2846}$$

$$\times\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\Big), \tag{2847}$$

$$\Lambda_{3n'n',2nm}\left(t,t'\right) = \left\langle B_{z,n'}\left(t,\tau\right)B_{2,nm}\left(t',0\right)\right\rangle_{B} \tag{2848}$$

$$= \left\langle B_{z,n'}\left(t,\tau\right)J_{nm,y}\left(t'\right)\left(1-\delta_{nm}\right)\right\rangle_{B} \tag{2849}$$

$$=(1-\delta_{nm})\left\langle B_{z,n'}(t,\tau)\,J_{nm,y}(t')\right\rangle_B \tag{2850}$$

$$=(1-\delta_{nm})\left\langle\left(\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\,b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\frac{J_{nm}(t')-J_{mn}(t')}{2\mathrm{i}}\right\rangle_B \tag{2851}$$

$$=(1-\delta_{nm})\left\langle\left(\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\,b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\left(\frac{\mathrm{e}^{\chi_{nm}(t')}\left(\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)-D_{nm}(t')\right)}{2\mathrm{i}}-\mathrm{e}^{\chi_{mn}(t')}\right. \tag{2852}$$

$$\left.\left.\times\frac{\left(\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)-D_{mn}(t')\right)}{2\mathrm{i}}\right)\right\rangle_B \tag{2853}$$

$$=(1-\delta_{nm})\left\langle\left(\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\,b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\left(\frac{\mathrm{e}^{\chi_{nm}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)-\mathrm{e}^{\chi_{nm}(t')}D_{nm}(t')}{2\mathrm{i}}+\mathrm{e}^{\chi_{mn}(t')}\right. \tag{2854}$$

$$\left.\left.\times\frac{-\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)+D_{mn}(t')}{2\mathrm{i}}\right)\right\rangle_B \tag{2855}$$

$$=(1-\delta_{nm})\left\langle\left(\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\frac{\mathrm{e}^{\chi_{nm}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)-\mathrm{e}^{\chi_{mn}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)}{2\mathrm{i}}\right\rangle_B \tag{2856}$$

$$+(1-\delta_{nm})\left\langle\left(\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)\,b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\frac{\mathrm{e}^{\chi_{mn}(t')}D_{mn}(t')-\mathrm{e}^{\chi_{nm}(t')}D_{nm}(t')}{2\mathrm{i}}\right\rangle_B \tag{2857}$$

$$=(1-\delta_{nm})\left\langle\left(\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t)b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\frac{\mathrm{e}^{\chi_{nm}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)-\mathrm{e}^{\chi_{mn}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)}{2\mathrm{i}}\right\rangle_B \tag{2858}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\sum_{u\mathbf{k}}\left(\mathrm{e}^{\chi_{nm}(t')}\left\langle\left(q_{n'u\mathbf{k}}(t)\,b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)\right\rangle_B\right. \tag{2859}$$

$$\left.-\mathrm{e}^{\chi_{mn}(t')}\left\langle\left(q_{n'u\mathbf{k}}(t)\,b^{\dagger}_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+q^{*}_{n'u\mathbf{k}}(t)\,b_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)\right\rangle_B\right) \tag{2860}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\sum_{u\mathbf{k}}\left(\mathrm{e}^{\chi_{nm}(t')}\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b^{\dagger}_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)\right\rangle_B+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)\right\rangle_B\right)\right. \tag{2861}$$

$$\left.-\mathrm{e}^{\chi_{mn}(t')}\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b^{\dagger}_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)\right\rangle_B+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)\right\rangle_B\right)\right) \tag{2862}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\sum_{u\mathbf{k}}\left(\mathrm{e}^{\chi_{nm}(t')}\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b^{\dagger}_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)\right\rangle_B+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(nm)u'\mathbf{k}'}(t')\right)\right\rangle_B\right)\right. \tag{2863}$$

$$\left.-\mathrm{e}^{\chi_{mn}(t')}\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b^{\dagger}_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)\right\rangle_B+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left\langle b_{u\mathbf{k}}\prod_{u'\mathbf{k}'}D\left(\alpha_{(mn)u'\mathbf{k}'}(t')\right)\right\rangle_B\right)\right) \tag{2864}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\sum_{u\mathbf{k}}\left(\mathrm{e}^{\chi_{nm}(t')}\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(-\alpha^{*}_{(nm)u\mathbf{k}}(t')\,N_{u\mathbf{k}}D_{nm}(t')\right)+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha_{(nm)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,D_{nm}(t')\right)\right. \tag{2865}$$

$$\left.-\mathrm{e}^{\chi_{mn}(t)}\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(-\alpha^{*}_{(mn)u\mathbf{k}}(t')\,N_{u\mathbf{k}}D_{mn}(t')\right)+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(\alpha_{(mn)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,D_{mn}(t')\right)\right)\right) \tag{2866}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\sum_{u\mathbf{k}}\left(\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(-\alpha^{*}_{(nm)u\mathbf{k}}(t')\,N_{u\mathbf{k}}B_{nm}(t')\right)+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha_{(nm)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,B_{nm}(t')\right)\right. \tag{2867}$$

$$\left.-\left(q_{n'u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(-\alpha^{*}_{(mn)u\mathbf{k}}(t')\,N_{u\mathbf{k}}B_{mn}(t')\right)+q^{*}_{n'u\mathbf{k}}(t)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\left(\alpha_{(mn)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,B_{mn}(t')\right)\right)\right) \tag{2868}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\sum_{u\mathbf{k}}\left(B_{nm}(t')\left(q^{*}_{n'u\mathbf{k}}(t)\,\alpha_{(nm)u\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\,(N_{u\mathbf{k}}+1)-q_{n'u\mathbf{k}}(t)\,\alpha^{*}_{(nm)u\mathbf{k}}(t')\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}N_{u\mathbf{k}}\right)\right. \tag{2869}$$

$$\left.-B_{mn}(t')\left(q^{*}_{n'u\mathbf{k}}(t)\,\alpha_{(mn)u\mathbf{k}}(t')\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\,(N_{u\mathbf{k}}+1)-q_{n'u\mathbf{k}}(t)\,\alpha^{*}_{(mn)u\mathbf{k}}(t')\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}N_{u\mathbf{k}}\right)\right) \tag{2870}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\sum_{u\mathbf{k}}\left(B_{nm}(t')\left(q^{*}_{n'u\mathbf{k}}(t)\,\alpha_{(nm)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q_{n'u\mathbf{k}}(t)\,\alpha^{*}_{(nm)u\mathbf{k}}(t')\,N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right. \tag{2871}$$

$$\left.+B_{mn}(t')\left(q^{*}_{n'u\mathbf{k}}(t)\,\alpha_{(nm)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q_{n'u\mathbf{k}}(t)\,\alpha^{*}_{(nm)u\mathbf{k}}(t')\,N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right) \tag{2872}$$

$$=\frac{(1-\delta_{nm})}{2\mathrm{i}}\left(B_{nm}(t')+B_{mn}(t')\right)\sum_{u\mathbf{k}}\left(q^{*}_{n'u\mathbf{k}}(t)\,\alpha_{(nm)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q_{n'u\mathbf{k}}(t)\,\alpha^{*}_{(nm)u\mathbf{k}}(t')\,N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right), \tag{2873}$$

$$\Lambda_{2nm,2n'm'}\left(t,t'\right) = \left\langle B_{2,nm}\left(t,\tau\right) B_{2,n'm'}\left(t',0\right)\right\rangle_B \tag{2874}$$

$$= \left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left\langle \frac{J_{nm}\left(t,\tau\right)-J_{mn}\left(t,\tau\right)}{2\mathrm{i}}\frac{J_{n'm'}\left(t',0\right)-J_{m'n'}\left(t',0\right)}{2\mathrm{i}}\right\rangle_B \tag{2875}$$

$$= -\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4}\left\langle\left(J_{nm}\left(t,\tau\right)-J_{mn}\left(t,\tau\right)\right)\left(J_{n'm'}\left(t',0\right)-J_{m'n'}\left(t',0\right)\right)\right\rangle_B \tag{2876}$$

$$= -\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{4}\left(\left\langle J_{nm}(t,\tau)\, J_{n'm'}\left(t',0\right)\right\rangle_B - \left\langle J_{nm}(t,\tau)\, J_{m'n'}\left(t',0\right)\right\rangle_B - \left\langle J_{mn}(t,\tau)\, J_{n'm'}\left(t',0\right)\right\rangle_B\right. \tag{2877}$$

$$\left. + \left\langle J_{mn}\left(t,\tau\right) J_{m'n'}\left(t',0\right)\right\rangle_B\right), \tag{2878}$$

$$J_{nm}\left(t,\tau\right) = \mathrm{e}^{\mathrm{i}\tau H_B}\left(\prod_{u\mathbf{k}} D\left(\alpha_{nu\mathbf{k}}\left(t\right)-\alpha_{mu\mathbf{k}}\left(t\right)\right)\prod_{u\mathbf{k}}\mathrm{e}^{\frac{v^*_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}}} - B_{nm}\left(t\right)\right)\mathrm{e}^{-\mathrm{i}\tau H_B} \tag{2879}$$

$$= \mathrm{e}^{\mathrm{i}\tau H_B}\left(\prod_{u\mathbf{k}} D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\right)\right)\mathrm{e}^{-\mathrm{i}\tau H_B}\prod_{u\mathbf{k}}\mathrm{e}^{\frac{v^*_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}}} - \mathrm{e}^{\mathrm{i}\tau H_B} B_{nm}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau H_B} \tag{2880}$$

$$= \mathrm{e}^{\mathrm{i}\tau H_B}\left(\prod_{u\mathbf{k}} D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\right)\right)\mathrm{e}^{-\mathrm{i}\tau H_B}\prod_{u\mathbf{k}}\mathrm{e}^{\frac{v^*_{nu\mathbf{k}}(t)v_{mu\mathbf{k}}(t)-v_{nu\mathbf{k}}(t)v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}}} - B_{nm}\left(t\right) \tag{2881}$$

$$= \mathrm{e}^{\chi_{nm}(t)}\prod_{u\mathbf{k}} D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) - B_{nm}\left(t\right), \tag{2882}$$

$$\left\langle J_{nm}(t,\tau)J_{n'm'}\left(t',0\right)\right\rangle_B = \left\langle\left(\mathrm{e}^{\chi_{nm}(t)}\prod_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) - B_{nm}\left(t\right)\right)\left(\mathrm{e}^{\chi_{n'm'}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(n'm')u'\mathbf{k}'}\left(t'\right)\right) - B_{n'm'}\left(t'\right)\right)\right\rangle_B \tag{2883}$$

$$= \left\langle\left(\mathrm{e}^{\chi_{nm}(t)}\prod_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) - B_{nm}\left(t\right)\right)\mathrm{e}^{\chi_{n'm'}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(n'm')u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_B \tag{2884}$$

$$- \left\langle\left(\mathrm{e}^{\chi_{nm}(t)}\prod_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) - B_{nm}\left(t\right)\right)B_{n'm'}\left(t'\right)\right\rangle_B \tag{2885}$$

$$= \left\langle\left(\mathrm{e}^{\chi_{nm}(t)}\prod_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)\left(\mathrm{e}^{\chi_{n'm'}(t')}\prod_{u'\mathbf{k}'}D\left(\alpha_{(n'm')u'\mathbf{k}'}\left(t'\right)\right)\right)\right\rangle_B - B_{nm}\left(t\right)\mathrm{e}^{\chi_{n'm'}(t)} \tag{2886}$$

$$\times\left\langle\prod_{u'\mathbf{k}'}D\left(\alpha_{(n'm')u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_B - B_{n'm'}\left(t'\right)\left(B_{nm}\left(t\right)-B_{nm}\left(t\right)\right) \tag{2887}$$

$$= \mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\left\langle\prod_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\prod_{u'\mathbf{k}'}D\left(\alpha_{(n'm')u'\mathbf{k}'}\left(t'\right)\right)\right\rangle_B - B_{nm}\left(t\right)B_{n'm'}\left(t'\right) \tag{2888}$$

$$= \mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\left\langle\prod_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)D\left(\alpha_{(n'm')u\mathbf{k}}\left(t'\right)\right)\right\rangle_B - B_{nm}\left(t\right)B_{n'm'}\left(t'\right), \tag{2889}$$

We reduce and introduce further notation:

$$\mathrm{e}^{\mathrm{i}\left(\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^*_{(n'm')u\mathbf{k}}\left(t'\right)\right)^{\Im}} = \mathrm{e}^{\frac{\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^*_{(n'm')u\mathbf{k}}\left(t'\right)-\alpha^*_{(nm)u\mathbf{k}}(t)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha_{(n'm')u\mathbf{k}}\left(t'\right)}{2}}, \tag{2890}$$

$$D\left(\alpha_{(nm)u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)D\left(\alpha_{(n'm')u\mathbf{k}}\left(t'\right)\right) = D\left(\alpha_{(nm)u\mathbf{k}}\left(t\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+\alpha_{(n'm')u\mathbf{k}}\left(t'\right)\right)\mathrm{e}^{\mathrm{i}\left(\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^*_{(n'm')u\mathbf{k}}\left(t'\right)\right)^{\Im}}, \tag{2891}$$

$$U_{(nm)(n'm')}\left(t,t'\right) \equiv \prod_{u\mathbf{k}}\mathrm{e}^{\frac{\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^*_{(n'm')u\mathbf{k}}\left(t'\right)-\alpha^*_{(nm)u\mathbf{k}}(t)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha_{(n'm')u\mathbf{k}}\left(t'\right)}{2}} \text{(with } \tau = t-t'\text{)}, \tag{2892}$$

$$\xi^+_{(nm)(n'm')}\left(t,t'\right) \equiv \prod_{u\mathbf{k}}\mathrm{e}^{-\frac{\left|\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+\alpha_{(n'm')u\mathbf{k}}\left(t'\right)\right|^2}{2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}, \tag{2893}$$

$$\xi^-_{(nm)(n'm')}\left(t,t'\right) \equiv \prod_{u\mathbf{k}}\mathrm{e}^{-\frac{\left|\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}-\alpha_{(n'm')u\mathbf{k}}\left(t'\right)\right|^2}{2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{2894}$$

$$= \xi^+_{(mn)(n'm')}\left(t,t'\right), \tag{2895}$$

$$\xi^{+}_{(mn)(m'n')}\left(t,t'\right)=\xi^{+}_{(nm)(n'm')}\left(t,t'\right), \tag{2896}$$

$$\xi^{+}_{(nm)(m'n')}\left(t,t'\right)=\xi^{-}_{(nm)(n'm')}\left(t,t'\right), \tag{2897}$$

$$B_{nm}\left(t\right)=B^{*}_{mn}\left(t\right), \tag{2898}$$

$$U_{(mn)(m'n')}\left(t,t'\right)=\prod_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(\alpha_{(mn)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^{*}_{(m'n')u\mathbf{k}}\left(t'\right)\right)^{\Im}} \tag{2899}$$

$$=\prod_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^{*}_{(n'm')u\mathbf{k}}\left(t'\right)\right)^{\Im}} \tag{2900}$$

$$=U_{(nm)(n'm')}\left(t,t'\right), \tag{2901}$$

$$U_{(nm)(m'n')}\left(t,t'\right)=\prod_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\left(\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^{*}_{(m'n')u\mathbf{k}}\left(t'\right)\right)^{\Im}} \tag{2902}$$

$$=\prod_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\left(\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\alpha^{*}_{(n'm')u\mathbf{k}}\left(t'\right)\right)^{\Im}} \tag{2903}$$

$$=U^{*}_{(nm)(n'm')}\left(t,t'\right), \tag{2904}$$

$$\left\langle J_{nm}(t,\tau)J_{n'm'}\left(t',0\right)\right\rangle_{B}=\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}U_{(nm)(n'm')}\left(t,t'\right)\left\langle\prod_{u\mathbf{k}}D\left(\alpha_{(nm)u\mathbf{k}}(t)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+\alpha_{(n'm')u\mathbf{k}}\left(t'\right)\right)\right\rangle_{B}-B_{nm}(t)B_{n'm'}\left(t'\right) \tag{2905}$$

$$=\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)-B_{nm}\left(t\right)B_{n'm'}\left(t'\right), \tag{2906}$$

$$\chi_{nm}\left(t\right)=\sum_{u\mathbf{k}}\frac{v^{*}_{nu\mathbf{k}}\left(t\right)v_{mu\mathbf{k}}\left(t\right)-v_{nu\mathbf{k}}\left(t\right)v^{*}_{mu\mathbf{k}}\left(t\right)}{2\omega^{2}_{u\mathbf{k}}}, \tag{2907}$$

$$\chi^{*}_{nm}\left(t\right)=-\chi_{nm}\left(t\right), \tag{2908}$$

$$\chi_{m'n'}\left(t'\right)=-\chi_{n'm'}\left(t'\right), \tag{2909}$$

$$\left\langle J_{nm}(t,\tau)J_{m'n'}\left(t',0\right)\right\rangle_{B}=\mathrm{e}^{\chi_{nm}(t)+\chi_{m'n'}\left(t'\right)}U_{(nm)(m'n')}\left(t,t'\right)\xi^{+}_{(nm)(m'n')}\left(t,t'\right)-B_{nm}\left(t\right)B_{m'n'}\left(t'\right) \tag{2910}$$

$$=\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)-B_{nm}\left(t\right)B_{m'n'}\left(t'\right) \tag{2911}$$

$$=\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)-B_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right), \tag{2912}$$

$$\left\langle J_{mn}(t,\tau)J_{n'm'}\left(t',0\right)\right\rangle_{B}=\mathrm{e}^{-\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)-B^{*}_{nm}\left(t\right)B_{n'm'}\left(t'\right), \tag{2913}$$

$$\left\langle J_{mn}(t,\tau)J_{m'n'}\left(t',0\right)\right\rangle_{B}=\mathrm{e}^{-\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)-B^{*}_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right), \tag{2914}$$

$$\Lambda_{2nm,2n'm'}\left(t,t'\right)=-\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4}\left(\left\langle J_{nm}\left(t,\tau\right)J_{n'm'}\left(t',0\right)\right\rangle_{B}-\left\langle J_{nm}\left(t,\tau\right)J_{m'n'}\left(t',0\right)\right\rangle_{B}-\left\langle J_{mn}\left(t,\tau\right)\right.\right. \tag{2915}$$

$$\left.\left.\times J_{n'm'}\left(t',0\right)\right\rangle_{B}+\left\langle J_{mn}(t,\tau)J_{m'n'}\left(t',0\right)\right\rangle_{B}\right) \tag{2916}$$

$$=-\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{4}\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)-B_{nm}(t)\,B_{n'm'}\left(t'\right)\right. \tag{2917}$$

$$-\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)+B_{nm}(t)\,B^{*}_{n'm'}\left(t'\right)-\mathrm{e}^{-\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)} \tag{2918}$$

$$\times U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)+B^{*}_{nm}(t)B_{n'm'}\left(t'\right)+\mathrm{e}^{-\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}U_{(nm)(n'm')}\left(t,t'\right) \tag{2919}$$

$$\left.\times\xi^{+}_{(nm)(n'm')}\left(t,t'\right)-B^{*}_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)\right), \tag{2920}$$

$$=-\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{4}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}+\mathrm{e}^{-\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}\right)\right. \tag{2921}$$

$$-U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(m'n')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}+\mathrm{e}^{-\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}\right)-B_{nm}(t)B_{n'm'}\left(t'\right) \tag{2922}$$

$$\left.+B_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)+B^{*}_{nm}\left(t\right)B_{n'm'}\left(t'\right)-B^{*}_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)\right) \tag{2923}$$

$$=-\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{4}\left(2U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}\right)^{\Re}-2U^{*}_{(nm)(n'm')}\left(t,t'\right)\right. \tag{2924}$$

$$\left.\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}\left(t'\right)}\right)^{\Re}-\left(B_{nm}\left(t\right)-B^{*}_{nm}\left(t\right)\right)\left(B_{n'm'}\left(t'\right)-B^{*}_{n'm'}\left(t'\right)\right)\right) \tag{2925}$$

$$=-\left(1-\delta_{nm}\right)(1-\delta_{n'm'})\left(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}\left(t'\right)}\right)^{\Re}-U^{*}_{(nm)(n'm')}\left(t,t'\right)\right.\right. \tag{2926}$$

$$\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\Big)+B^{\Im}_{nm}\left(t\right)B^{\Im}_{n'm'}\left(t'\right)\Big), \tag{2927}$$

$$\Lambda_{2nm,1n'm'}(t,t')=\left\langle B_{2,nm}\left(t,\tau\right)B_{1,n'm'}\left(t',0\right)\right\rangle_{B} \tag{2928}$$

$$=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left\langle\frac{J_{nm}\left(t,\tau\right)-J_{mn}\left(t,\tau\right)}{2\mathrm{i}}\frac{J_{n'm'}\left(t',0\right)+J_{m'n'}\left(t',0\right)}{2}\right\rangle_{B} \tag{2929}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4\mathrm{i}}\left\langle\left(J_{nm}\left(t,\tau\right)-J_{mn}\left(t,\tau\right)\right)\left(J_{n'm'}\left(t',0\right)+J_{m'n'}\left(t',0\right)\right)\right\rangle_{B} \tag{2930}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4\mathrm{i}}\left\langle J_{nm}\left(t,\tau\right)J_{n'm'}\left(t',0\right)-J_{mn}\left(t,\tau\right)J_{n'm'}\left(t',0\right)+J_{nm}\left(t,\tau\right)J_{m'n'}\left(t',0\right)-J_{mn}\left(t,\tau\right)\right. \tag{2931}$$

$$\left.\times J_{m'n'}\left(t',0\right)\right\rangle_{B} \tag{2932}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4\mathrm{i}}\left(\left\langle J_{nm}\left(t,\tau\right)J_{n'm'}\left(t',0\right)\right\rangle_{B}+\left\langle J_{nm}\left(t,\tau\right)J_{m'n'}\left(t',0\right)\right\rangle_{B}-\left\langle J_{mn}\left(t,\tau\right)J_{n'm'}\left(t',0\right)\right\rangle_{B}\right. \tag{2933}$$

$$\left.-\left\langle J_{mn}\left(t,\tau\right)J_{m'n'}\left(t',0\right)\right\rangle_{B}\right) \tag{2934}$$

$$=\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{4\mathrm{i}}\Big(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}U_{(nm)(n'm')}(t,t')\xi^{+}_{(nm)(n'm')}(t,t')-B_{nm}(t)B_{n'm'}(t')+\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')} \tag{2935}$$

$$\times U^{*}_{(nm)(n'm')}(t,t')\xi^{-}_{(nm)(n'm')}(t,t')-B_{nm}(t)B^{*}_{n'm'}(t')-\mathrm{e}^{-\chi_{nm}(t)+\chi_{n'm'}(t')}U^{*}_{(nm)(m'n')}(t,t')\xi^{-}_{(nm)(m'n')}(t,t') \tag{2936}$$

$$+B^{*}_{nm}\left(t\right)B_{n'm'}\left(t'\right)-\mathrm{e}^{-\chi_{nm}(t)-\chi_{n'm'}(t')}U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)+B^{*}_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)\Big) \tag{2937}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4\mathrm{i}}\Big(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}-\mathrm{e}^{-\chi_{nm}(t)-\chi_{n'm'}(t')}\right) \tag{2938}$$

$$+U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}-\mathrm{e}^{-\chi_{nm}(t)+\chi_{n'm'}(t')}\right)-B_{nm}\left(t\right)B_{n'm'}\left(t'\right) \tag{2939}$$

$$-B_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)+B^{*}_{nm}\left(t\right)B_{n'm'}\left(t'\right)+B^{*}_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)\Big) \tag{2940}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4\mathrm{i}}\Big(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}-\mathrm{e}^{-\chi_{nm}(t)-\chi_{n'm'}(t')}\right) \tag{2941}$$

$$+U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}-\mathrm{e}^{-\chi_{nm}(t)+\chi_{n'm'}(t')}\right)-\left(B_{nm}\left(t\right)-B^{*}_{nm}\left(t\right)\right) \tag{2942}$$

$$\times\left(B_{n'm'}\left(t'\right)+B^{*}_{n'm'}\left(t'\right)\right)\Big) \tag{2943}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4\mathrm{i}}\Big(2\mathrm{i}U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Im}+2\mathrm{i}U^{*}_{(nm)(n'm')}\left(t,t'\right) \tag{2944}$$

$$\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Im}-4\mathrm{i}B^{\Im}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\Big) \tag{2945}$$

$$=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\Big(\frac{1}{2}\Big(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Im}+U^{*}_{(nm)(n'm')}\left(t,t'\right) \tag{2946}$$

$$\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Im}\Big)-B^{\Im}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\Big), \tag{2947}$$

$$\Lambda_{1nm,1n'm'}(t,t')=\left\langle B_{1,nm}\left(t,\tau\right)B_{1,n'm'}\left(t',0\right)\right\rangle_{B} \tag{2948}$$

$$=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left\langle\frac{J_{nm}\left(t,\tau\right)+J_{mn}\left(t,\tau\right)}{2}\frac{J_{n'm'}\left(t',0\right)+J_{m'n'}\left(t',0\right)}{2}\right\rangle_{B} \tag{2949}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4}\left\langle\left(J_{nm}\left(t,\tau\right)+J_{mn}\left(t,\tau\right)\right)\left(J_{n'm'}\left(t',0\right)+J_{m'n'}\left(t',0\right)\right)\right\rangle_{B} \tag{2950}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4}\left\langle J_{nm}\left(t,\tau\right)J_{n'm'}\left(t',0\right)+J_{nm}\left(t,\tau\right)J_{m'n'}\left(t',0\right)+J_{mn}\left(t,\tau\right)J_{n'm'}\left(t',0\right)+J_{mn}\left(t,\tau\right)\right. \tag{2951}$$

$$\left.\times J_{m'n'}\left(t',0\right)\right\rangle_{B} \tag{2952}$$

$$=\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{4}\Big(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}U_{(nm)(n'm')}(t,t')\xi^{+}_{(nm)(n'm')}(t,t')-B_{nm}(t)B_{n'm'}(t')+\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')} \tag{2953}$$

$$\times U^{*}_{(nm)(n'm')}(t,t')\xi^{-}_{(nm)(n'm')}(t,t')-B_{nm}(t)B^{*}_{n'm'}(t')+\mathrm{e}^{-\chi_{nm}(t)+\chi_{n'm'}(t')}U^{*}_{(nm)(m'n')}(t,t')\xi^{-}_{(nm)(n'm')}(t,t') \tag{2954}$$

$$-B^{*}_{nm}\left(t\right)B_{n'm'}\left(t'\right)+\mathrm{e}^{-\chi_{nm}(t)-\chi_{n'm'}(t')}U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)-B^{*}_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)\Big) \tag{2955}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4}\Big(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)2\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re}+2U^{*}_{(nm)(m'n')}\left(t,t'\right) \tag{2956}$$

$$\times\xi^{-}_{(nm)(m'n')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}-B_{nm}\left(t\right)B_{n'm'}\left(t'\right)-B_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)-B^{*}_{nm}\left(t\right)B_{n'm'}\left(t'\right) \tag{2957}$$

$$-B^{*}_{nm}\left(t\right)B^{*}_{n'm'}\left(t'\right)\Big) \tag{2958}$$

$$=\frac{\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)}{4}\Big(2U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re}+2U^{*}_{(nm)(m'n')}\left(t,t'\right) \tag{2959}$$

$$\times\xi^{-}_{(nm)(m'n')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}-4B^{\Re}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\Big) \tag{2960}$$

$$= \left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re}+U^{*}_{(nm)(m'n')}\left(t,t'\right)\right.\right. \tag{2961}$$

$$\left.\left.\times\xi^{-}_{(nm)(m'n')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\right)-B^{\Re}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\right) \tag{2962}$$

$$\Lambda_{jp,j'p'}\left(t,t'\right)=\Lambda^{*}_{j'p',jp}\left(t',t\right), \tag{2963}$$

$$\Lambda_{1nm,2n'm'}\left(t,t'\right)=\Lambda^{*}_{2n'm',1nm}\left(t',t\right) \tag{2964}$$

$$=\left(\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t',t\right)\xi^{+}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')+\chi_{n'm'}(t)}\right)^{\Im}+U^{*}_{(nm)(n'm')}\left(t',t\right)\right.\right.\right. \tag{2965}$$

$$\left.\left.\left.\times\xi^{-}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')-\chi_{n'm'}(t)}\right)^{\Im}\right)-B^{\Im}_{nm}\left(t'\right)B^{\Re}_{n'm'}\left(t\right)\right)^{*} \tag{2966}$$

$$=\left(\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U^{*}_{(nm)(n'm')}\left(t',t\right)\xi^{+}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')+\chi_{n'm'}(t)}\right)^{\Im}+U_{(nm)(n'm')}\left(t',t\right)\right.\right.\right. \tag{2967}$$

$$\left.\left.\left.\times\xi^{-}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')-\chi_{n'm'}(t)}\right)^{\Im}\right)-B^{\Im}_{nm}\left(t'\right)B^{\Re}_{n'm'}\left(t\right)\right) \tag{2968}$$

$$\Lambda_{1nm,3n'm'}\left(t,t'\right)=\Lambda^{*}_{3n'm',1nm}\left(t',t\right) \tag{2969}$$

$$=\left(\frac{1-\delta_{nm}}{2}\left(B_{nm}(t)-B_{mn}(t)\right)\sum_{u\mathbf{k}}\left(q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}(t)(N_{u\mathbf{k}}+1)\,\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}-q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}(t)\,N_{u\mathbf{k}}\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)\right)^{*} \tag{2970}$$

$$=\frac{1-\delta_{nm}}{2}\left(B_{mn}(t)-B_{nm}(t)\right)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}(t)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}(t)\,N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right), \tag{2971}$$

$$\Lambda_{2nm,3n'm'}\left(t,t'\right)=\Lambda^{*}_{3n'm',2nm}\left(t',t\right) \tag{2972}$$

$$=\left(\frac{\left(1-\delta_{nm}\right)}{2\mathrm{i}}\left(B_{nm}\left(t\right)+B_{mn}\left(t\right)\right)\sum_{u\mathbf{k}}\left(q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}\left(t\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}-q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t\right)N_{u\mathbf{k}}\right)\right. \tag{2973}$$

$$\left.\times\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}\right)^{*} \tag{2974}$$

$$=-\frac{\left(1-\delta_{nm}\right)}{2\mathrm{i}}\left(B_{mn}(t)+B_{nm}(t)\right)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}(t)(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}(t)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right). \tag{2975}$$

The correlation functions can be summarized as:

$$\Lambda_{3nn,3mm}\left(t,t'\right)=\sum_{u\mathbf{k}}\left(q^{*}_{nu\mathbf{k}}\left(t\right)q_{mu\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}\left(t\right)q^{*}_{mu\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right), \tag{2976}$$

$$\Lambda_{3n'n',1nm}\left(t,t'\right)=\left(1-\delta_{nm}\right)\mathrm{i}B^{\Im}_{nm}\left(t'\right)\sum_{u\mathbf{k}}\left(q^{*}_{n'u\mathbf{k}}\left(t\right)\alpha_{(nm)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}q_{n'u\mathbf{k}}\left(t\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\right), \tag{2977}$$

$$\Lambda_{3n'n',2nm}\left(t,t'\right)=-\mathrm{i}\left(1-\delta_{nm}\right)B^{\Re}_{nm}\left(t'\right)\sum_{u\mathbf{k}}\left(q^{*}_{n'u\mathbf{k}}\left(t\right)\alpha_{(nm)u\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q_{n'u\mathbf{k}}\left(t\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t'\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right), \tag{2978}$$

$$\Lambda_{2nm,2n'm'}\left(t,t'\right)=-\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re}-U^{*}_{(nm)(m'n')}\left(t,t'\right)\right.\right. \tag{2979}$$

$$\left.\left.\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\right)+B^{\Im}_{nm}\left(t\right)B^{\Im}_{n'm'}\left(t'\right)\right), \tag{2980}$$

$$\Lambda_{2nm,1n'm'}\left(t,t'\right)=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Im}+U^{*}_{(nm)(n'm')}\left(t,t'\right)\right.\right. \tag{2981}$$

$$\left.\left.\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Im}\right)-B^{\Im}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\right), \tag{2982}$$

$$\Lambda_{1nm,1n'm'}\left(t,t'\right)=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re}+U^{*}_{(nm)(m'n')}\left(t,t'\right)\right.\right. \tag{2983}$$

$$\left.\left.\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\right)-B^{\Re}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\right), \tag{2984}$$

$$\Lambda_{1nm,2n'm'}\left(t,t'\right)=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U^{*}_{(nm)(n'm')}\left(t',t\right)\xi^{+}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')+\chi_{n'm'}(t)}\right)^{\Im}+U_{(nm)(n'm')}\left(t',t\right)\right.\right. \tag{2985}$$

$$\left.\left.\times\xi^{-}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')-\chi_{n'm'}(t)}\right)^{\Im}\right)-B^{\Im}_{nm}\left(t'\right)B^{\Re}_{n'm'}\left(t\right)\right), \tag{2986}$$

$$\Lambda_{1nm,3n'm'}\left(t,t'\right)=-\mathrm{i}\left(1-\delta_{nm}\right)B^{\Im}_{nm}\left(t\right)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}\left(t\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right), \tag{2987}$$

$$\Lambda_{2nm,3n'm'}\left(t,t'\right)=\mathrm{i}\left(1-\delta_{nm}\right)B^{\Re}_{nm}\left(t\right)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}\left(t\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right). \tag{2988}$$

Let's consider the following expression related to the sum of coupling constants for all the baths over all their frequences:

$$L_{iu}(\omega) = \sum_{\mathbf{k}} g_{iu\mathbf{k}} \sqrt{\delta(\omega - \omega_{u\mathbf{k}})}. \tag{2989}$$

Using the same argument shown in (842) we can obtain the following approximation:

$$\int_0^\infty f(\omega) L_{iu}(\omega) L_{ju}^*(\omega)\,\mathrm{d}\omega \approx \int_0^\infty f(\omega) \sum_{\mathbf{k}} g_{iu}(\omega_{u\mathbf{k}}) \sqrt{\delta(\omega - \omega_{u\mathbf{k}})} \sum_{\mathbf{k}'} g_{ju}^*(\omega_{u\mathbf{k}'}) \sqrt{\delta(\omega - \omega_{u\mathbf{k}'})}\mathrm{d}\omega \tag{2990}$$

$$= \int_0^\infty f(\omega) \sum_{\mathbf{k},\mathbf{k}'} g_{iu}(\omega_{u\mathbf{k}}) g_{ju}^*(\omega_{u\mathbf{k}'}) \sqrt{\delta(\omega - \omega_{u\mathbf{k}})}\sqrt{\delta(\omega - \omega_{u\mathbf{k}'})}\mathrm{d}\omega \tag{2991}$$

$$= \sum_{\mathbf{k}\neq\mathbf{k}'} \int_0^\infty f(\omega) g_{iu}(\omega_{u\mathbf{k}}) g_{ju}^*(\omega_{u\mathbf{k}'}) \sqrt{\delta(\omega-\omega_{u\mathbf{k}})}\sqrt{\delta(\omega-\omega_{u\mathbf{k}'})}\,\mathrm{d}\omega + \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_{iu}(\omega_{u\mathbf{k}}) \tag{2992}$$

$$\times\, g_{ju}^*(\omega_{u\mathbf{k}})\,\delta(\omega-\omega_{u\mathbf{k}})\mathrm{d}\omega \tag{2993}$$

$$= 0 + \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_{iu}(\omega_{u\mathbf{k}}) g_{ju}^*(\omega_{u\mathbf{k}}) \delta(\omega-\omega_{u\mathbf{k}})\mathrm{d}\omega \tag{2994}$$

$$= \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_{iu}(\omega_{u\mathbf{k}}) g_{ju}^*(\omega_{u\mathbf{k}}) \delta(\omega-\omega_{u\mathbf{k}})\mathrm{d}\omega \tag{2995}$$

$$= \sum_{\mathbf{k}} f(\omega_{u\mathbf{k}}) g_{iu}(\omega_{u\mathbf{k}}) g_{ju}^*(\omega_{u\mathbf{k}}). \tag{2996}$$

if $i = j$ we recover the spectral density defined in the usual way when we integrate for a function $f(\omega)$ that belongs to the set L^2:

$$\sum_{\mathbf{k}} f(\omega_{u\mathbf{k}}) g_{iu}(\omega_{u\mathbf{k}}) g_{iu}^*(\omega_{u\mathbf{k}}) = \sum_{\mathbf{k}} \int_0^\infty f(\omega) g_{iu}(\omega_{u\mathbf{k}}) g_{iu}^*(\omega_{u\mathbf{k}}) \delta(\omega-\omega_{u\mathbf{k}})\mathrm{d}\omega \tag{2997}$$

$$= \int_0^\infty f(\omega) J_{(ii)u}(\omega)\mathrm{d}\omega \tag{2998}$$

$$= \int_0^\infty f(\omega) |L_{iu}(\omega)|^2\,\mathrm{d}\omega. \tag{2999}$$

where

$$J_{(ii)u}(\omega) \equiv \sum_{\mathbf{k}} |g_{iu\mathbf{k}}|^2\,\delta(\omega - \omega_{u\mathbf{k}}) \tag{3000}$$

$$= |L_{iu}(\omega)|^2, \tag{3001}$$

$$v_{iu\mathbf{k}}(\omega_{u\mathbf{k}}, t) \equiv g_{iu\mathbf{k}}(\omega_{u\mathbf{k}}) F_{iu}(\omega_{u\mathbf{k}}, t), \tag{3002}$$

$$v_{iu}(\omega, t) \equiv g_{iu}(\omega) F_{iu}(\omega, t). \tag{3003}$$

In this case $g_{iu}(\omega)$, $v_{iu}(\omega, t)$ and $F_{iu}(\omega, t)$ are the continuous version of $g_{iu}(\omega_{u\mathbf{k}})$, $v_{iu\mathbf{k}}(\omega_{u\mathbf{k}}, t)$ and $F_{iu}(\omega_{u\mathbf{k}}, t)$ respectively. We introduce further notation in order to reduce the length of the expressions of the correlation functions:

$$B_{nm}(t) + B_{mn}(t) = 2B_{nm}^{\Re}(t), \tag{3004}$$

$$B_{nm}(t) - B_{mn}(t) = 2\mathrm{i}B_{nm}^{\Im}(t), \tag{3005}$$

$$P_{nu}(\omega, t) \equiv L_{nu}(\omega)(1 - F_{nu}(\omega, t)), \tag{3006}$$

$$Q_{(nm)u}(\omega, t) \equiv \frac{L_{nu}(\omega) F_{nu}(\omega, t) - L_{mu}(\omega) F_{mu}(\omega, t)}{\omega}. \tag{3007}$$

The integral version of the correlation functions can be obtained as follows:

$$\Lambda_{3nn,3mm}(t,t') = \sum_{u\mathbf{k}} \left(q^*_{nu\mathbf{k}}(t)\, q_{mu\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} + q_{nu\mathbf{k}}(t)\, q^*_{mu\mathbf{k}}(t')\, N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3008}$$

$$= \sum_{u\mathbf{k}} \Big((g^*_{nu\mathbf{k}} - v^*_{nu\mathbf{k}}(t))(g_{mu\mathbf{k}} - v_{mu\mathbf{k}}(t'))(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}}(t))(g^*_{mu\mathbf{k}} - v^*_{mu\mathbf{k}}(t'))N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\Big) \tag{3009}$$

$$= \sum_{u\mathbf{k}} \Big(g^*_{nu\mathbf{k}} g_{mu\mathbf{k}} (1-F^*_{nu}(\omega_{u\mathbf{k}},t))(1-F_{mu}(\omega_{u\mathbf{k}},t'))(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} + g_{nu\mathbf{k}} g^*_{mu\mathbf{k}} (1-F_{nu}(\omega_{u\mathbf{k}},t))(1-F^*_{mu}(\omega_{u\mathbf{k}},t')) \tag{3010}$$

$$\times N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\Big) \tag{3011}$$

$$\approx \sum_{u} \int_0^\infty \Big(L^*_{nu}(\omega) L_{mu}(\omega)(1-F^*_{nu}(\omega,t))(1-F_{mu}(\omega,t'))(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau} + L_{nu}(\omega) L^*_{mu}(\omega)(1-F_{nu}(\omega,t)) \tag{3012}$$

$$\times (1-F^*_{mu}(\omega,t'))\, N_u(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\Big)\,\mathrm{d}\omega \tag{3013}$$

$$= \sum_{u} \int_0^\infty \left(P^*_{nu}(\omega,t)\, P_{mu}(\omega,t')\,(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau} + P_{nu}(\omega,t)\, P^*_{mu}(\omega,t')\, N_u(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega, \tag{3014}$$

$$\chi_{nm}(t) = \sum_{u\mathbf{k}} \frac{v^*_{nu\mathbf{k}}(t)\, v_{mu\mathbf{k}}(t) - v_{nu\mathbf{k}}(t)\, v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}} \tag{3015}$$

$$= \sum_{u} \left(\sum_{\mathbf{k}} \frac{g^*_{nu\mathbf{k}} g_{mu\mathbf{k}} F^*_{nu}(\omega_{u\mathbf{k}},t)\, F_{mu}(\omega_{u\mathbf{k}},t) - g_{nu\mathbf{k}} g^*_{mu\mathbf{k}} F_{nu}(\omega_{u\mathbf{k}},t)\, F^*_{mu}(\omega_{u\mathbf{k}},t)}{2\omega^2_{u\mathbf{k}}}\right) \tag{3016}$$

$$\approx \sum_{u} \int_0^\infty \frac{L^*_{nu}(\omega)\, L_{mu}(\omega)\, F^*_{nu}(\omega,t)\, F_{mu}(\omega,t) - L_{nu}(\omega)\, L^*_{mu}(\omega)\, F_{nu}(\omega,t)\, F^*_{mu}(\omega,t)}{2\omega^2}\mathrm{d}\omega, \tag{3017}$$

$$B_{nm}(t) = \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v^*_{nu\mathbf{k}}(t) v_{mu\mathbf{k}}(t) - v_{nu\mathbf{k}}(t) v^*_{mu\mathbf{k}}(t)}{2\omega^2_{u\mathbf{k}}}} \prod_{u} \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}}(t) - v_{nu\mathbf{k}}(t)|^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}, \tag{3018}$$

$$\approx \mathrm{e}^{\chi_{nm}(t)} \mathrm{e}^{-\sum_u \int_0^\infty \frac{|L_{mu}(\omega) F_{mu}(\omega,t) - L_{nu}(\omega) F_{nu}(\omega,t)|^2}{2\omega^2} \coth\left(\frac{\beta_u \omega}{2}\right)\mathrm{d}\omega} \tag{3019}$$

$$= \mathrm{e}^{\chi_{nm}(t)} \mathrm{e}^{-\sum_u \int_0^\infty \frac{|Q_{(nm)u}(\omega,t)|^2}{2} \coth\left(\frac{\beta_u \omega}{2}\right)\mathrm{d}\omega}, \tag{3020}$$

$$\Lambda_{3n'n',1nm}(t,t') = (1-\delta_{nm})\,\mathrm{i}B^{\Im}_{nm}(t') \sum_{u\mathbf{k}} \left(q^*_{n'u\mathbf{k}}(t)\, \alpha_{(nm)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} - \mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} q_{n'u\mathbf{k}}(t)\, \alpha^*_{(nm)u\mathbf{k}}(t')\, N_{u\mathbf{k}}\right) \tag{3021}$$

$$= (1-\delta_{nm})\,\mathrm{i}B^{\Im}_{nm}(t') \sum_{u\mathbf{k}} \Big((g^*_{n'u\mathbf{k}} - v^*_{n'u\mathbf{k}}(t))\, \frac{v_{nu\mathbf{k}}(t') - v_{mu\mathbf{k}}(t')}{\omega_{u\mathbf{k}}}\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} - \mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\,(g_{n'u\mathbf{k}} - v_{n'u\mathbf{k}}(t)) \tag{3022}$$

$$\times \frac{v^*_{nu\mathbf{k}}(t') - v^*_{mu\mathbf{k}}(t')}{\omega_{u\mathbf{k}}} N_{u\mathbf{k}}\Big) \tag{3023}$$

$$= (1-\delta_{nm})\,\mathrm{i}B^{\Im}_{nm}(t') \sum_{u} \sum_{\mathbf{k}} \Big(g^*_{n'u\mathbf{k}}\,(1-F^*_{n'u}(\omega_{u\mathbf{k}},t))\, \frac{g_{nu\mathbf{k}} F_{nu}(\omega_{u\mathbf{k}},t') - g_{mu\mathbf{k}} F_{mu}(\omega_{u\mathbf{k}},t')}{\omega_{u\mathbf{k}}}\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} \tag{3024}$$

$$-\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} g_{n'u\mathbf{k}}\,(1-F_{n'u}(\omega_{u\mathbf{k}},t))\, \frac{g^*_{nu\mathbf{k}} F^*_{nu}(\omega_{u\mathbf{k}},t') - g^*_{mu\mathbf{k}} F^*_{mu}(\omega_{u\mathbf{k}},t')}{\omega_{u\mathbf{k}}} N_{u\mathbf{k}}\Big) \tag{3025}$$

$$\approx (1-\delta_{nm})\,\mathrm{i}B^{\Im}_{nm}(t') \sum_{u} \int_0^\infty \Big(L^*_{n'u}(\omega)(1-F^*_{n'u}(\omega,t))\, \frac{L_{nu}(\omega) F_{nu}(\omega,t') - L_{mu}(\omega) F_{mu}(\omega,t')}{\omega}\,(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau} \tag{3026}$$

$$-\mathrm{e}^{\mathrm{i}\omega\tau} L_{n'u}(\omega)\,(1-F_{n'u}(\omega,t))\, \frac{L^*_{nu}(\omega)\, F^*_{nu}(\omega,t') - L^*_{mu}(\omega)\, F^*_{mu}(\omega,t')}{\omega} N_u(\omega)\Big)\,\mathrm{d}\omega \tag{3027}$$

$$= (1-\delta_{nm})\,\mathrm{i}B^{\Im}_{nm}(t') \sum_{u} \int_0^\infty \Big(P^*_{n'u}(\omega,t) Q_{(nm)u}(\omega,t')(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau} - \mathrm{e}^{\mathrm{i}\omega\tau} P_{n'u}(\omega,t)\, Q^*_{(nm)u}(\omega,t')\, N_u(\omega)\Big)\mathrm{d}\omega, \tag{3028}$$

$$\Lambda_{3n'n',2nm}(t,t') = -\mathrm{i}\,(1-\delta_{nm})\, B^{\Re}_{nm}(t') \sum_{u\mathbf{k}} \left(q^*_{n'u\mathbf{k}}(t)\, \alpha_{(nm)u\mathbf{k}}(t')\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} - q_{n'u\mathbf{k}}(t)\, \alpha^*_{(nm)u\mathbf{k}}(t')\, N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3029}$$

$$\approx -\mathrm{i}\,(1-\delta_{nm}) B^{\Re}_{nm}(t') \sum_{u} \int_0^\infty \Big(P^*_{n'u}(\omega,t) Q_{(nm)u}(\omega,t')(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau} - \mathrm{e}^{\mathrm{i}\omega\tau} P_{n'u}(\omega,t) Q^*_{(nm)u}(\omega,t') N_u(\omega)\Big)\mathrm{d}\omega, \tag{3030}$$

$$U_{(nm)(n'm')}(t,t') = \prod_{u\mathbf{k}} \mathrm{e}^{\mathrm{i}\left(\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} \alpha^*_{(n'm')u\mathbf{k}}(t')\right)^{\Im}} \text{ (with } \tau = t-t'\text{)}, \tag{3031}$$

$$= \prod_{u} \mathrm{e}^{\mathrm{i}\sum_{\mathbf{k}}\left(\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau} \alpha^*_{(n'm')u\mathbf{k}}(t')\right)^{\Im}} \tag{3032}$$

$$\approx \mathrm{e}^{\mathrm{i}\sum_u \int_0^\infty \left(\frac{L_{nu}(\omega) F_{nu}(\omega,t) - L_{mu}(\omega) F_{mu}(\omega,t)}{\omega} \frac{L^*_{n'u}(\omega) F^*_{n'u}(\omega,t') - L^*_{m'u}(\omega) F^*_{m'u}(\omega,t')}{\omega} \mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im} \mathrm{d}\omega} \tag{3033}$$

$$= \mathrm{e}^{\mathrm{i}\sum_u \int_0^\infty \left(Q_{(nm)u}(\omega,t)Q^*_{(n'm')u}(\omega,t')\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega}, \tag{3034}$$

$$\xi^+_{(nm)(n'm')}(t,t') \equiv \prod_{u\mathbf{k}} \mathrm{e}^{-\frac{\left|\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+\alpha_{(n'm')u\mathbf{k}}(t')\right|^2}{2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3035}$$

$$= \mathrm{e}^{-\sum_u\sum_{\mathbf{k}}\frac{\left|\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}+\alpha_{(n'm')u\mathbf{k}}(t')\right|^2}{2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3036}$$

$$\approx \mathrm{e}^{-\sum_u\int_0^\infty\frac{\left|Q_{(nm)u}(\omega,t)\mathrm{e}^{\mathrm{i}\omega\tau}+Q_{(n'm')u}(\omega,t')\right|^2}{2}\coth\left(\frac{\beta_u\omega}{2}\right)\mathrm{d}\omega}, \tag{3037}$$

$$\xi^-_{(nm)(n'm')}(t,t') = \prod_{u\mathbf{k}} \mathrm{e}^{-\frac{\left|\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}-\alpha_{(n'm')u\mathbf{k}}(t')\right|^2}{2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3038}$$

$$= \mathrm{e}^{-\sum_u\sum_{\mathbf{k}}\frac{\left|\alpha_{(nm)u\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}-\alpha_{(n'm')u\mathbf{k}}(t')\right|^2}{2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3039}$$

$$\approx \mathrm{e}^{-\sum_u\int_0^\infty\frac{\left|Q_{(nm)u}(\omega,t)\mathrm{e}^{\mathrm{i}\omega\tau}-Q_{(n'm')u}(\omega,t')\right|^2}{2}\coth\left(\frac{\beta_u\omega}{2}\right)\mathrm{d}\omega}, \tag{3040}$$

$$\Lambda_{2nm,2n'm'}(t,t') = -(1-\delta_{nm})(1-\delta_{n'm'})\left(\frac{1}{2}\left(U_{(nm)(n'm')}(t,t')\,\xi^+_{(nm)(n'm')}(t,t')\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re} - U^*_{(nm)(m'n')}(t,t')\right.\right. \tag{3041}$$

$$\left.\left.\times\xi^-_{(nm)(n'm')}(t,t')\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\right) + B^{\Im}_{nm}(t)\,B^{\Im}_{n'm'}(t')\right), \tag{3042}$$

$$\Lambda_{2nm,1n'm'}(t,t') = (1-\delta_{nm})(1-\delta_{n'm'})\left(\frac{1}{2}\left(U_{(nm)(n'm')}(t,t')\,\xi^+_{(nm)(n'm')}(t,t')\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Im} + U^*_{(nm)(n'm')}(t,t')\right.\right. \tag{3043}$$

$$\left.\left.\times\xi^-_{(nm)(n'm')}(t,t')\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Im}\right) - B^{\Im}_{nm}(t)\,B^{\Re}_{n'm'}(t')\right), \tag{3044}$$

$$\Lambda_{1nm,1n'm'}(t,t') = (1-\delta_{nm})(1-\delta_{n'm'})\left(\frac{1}{2}\left(U_{(nm)(n'm')}(t,t')\,\xi^+_{(nm)(n'm')}(t,t')\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re} + U^*_{(nm)(m'n')}(t,t')\right.\right. \tag{3045}$$

$$\left.\left.\times\xi^-_{(nm)(n'm')}(t,t')\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\right) - B^{\Re}_{nm}(t)\,B^{\Re}_{n'm'}(t')\right), \tag{3046}$$

$$\Lambda_{1nm,2n'm'}(t,t') = (1-\delta_{nm})(1-\delta_{n'm'})\left(\frac{1}{2}\left(U^*_{(nm)(n'm')}(t',t)\,\xi^+_{(nm)(n'm')}(t',t)\left(\mathrm{e}^{\chi_{nm}(t')+\chi_{n'm'}(t)}\right)^{\Im} + U_{(nm)(n'm')}(t',t)\right.\right. \tag{3047}$$

$$\left.\left.\times\xi^-_{(nm)(n'm')}(t',t)\left(\mathrm{e}^{\chi_{nm}(t')-\chi_{n'm'}(t)}\right)^{\Im}\right) - B^{\Im}_{nm}(t')\,B^{\Re}_{n'm'}(t)\right), \tag{3048}$$

$$\Lambda_{1nm,3n'm'}(t,t') = -\mathrm{i}(1-\delta_{nm})\,B^{\Im}_{nm}(t)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t')\,\alpha^*_{(nm)u\mathbf{k}}(t)\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} - q^*_{n'u\mathbf{k}}(t')\,\alpha_{(nm)u\mathbf{k}}(t)\,N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3049}$$

$$\approx -\mathrm{i}(1-\delta_{nm})\,B^{\Im}_{nm}(t)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t')\,\alpha^*_{(nm)u\mathbf{k}}(t)\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} - q^*_{n'u\mathbf{k}}(t')\,\alpha_{(nm)u\mathbf{k}}(t)\,N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3050}$$

$$\Lambda_{2nm,3n'm'}(t,t') = \mathrm{i}(1-\delta_{nm})\,B^{\Re}_{nm}(t)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}(t')\,\alpha^*_{(nm)u\mathbf{k}}(t)\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau} - q^*_{n'u\mathbf{k}}(t')\,\alpha_{(nm)u\mathbf{k}}(t)\,N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3051}$$

$$\approx \mathrm{i}(1-\delta_{nm})\,B^{\Re}_{nm}(t)\sum_{u\mathbf{k}}\left(P_{n'u}(\omega,t')Q^*_{(nm)u}(\omega,t)(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau} - P^*_{n'u}(\omega,t')Q_{(nm)u}(\omega,t)N_u(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega. \tag{3052}$$

In order to show the explicit form of the matrices present in the RHS of the equation (921) for a general $n \times n$ matrix in a given time let's write the matrix A_i in the base $W(t) = \left\{\left|H_{\bar{S},\mathrm{eff},0}(t)\right\rangle,, \cdots \left|H_{\bar{S},\mathrm{eff},n-1}(t)\right\rangle\right\}$, formed by the time-dependent eigenvectors of $H_{\bar{S},\mathrm{eff}}(t)$ in the following way:

$$A_i = \sum_{j,j'}\left\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)\left|A_i\right|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right\rangle\left|H_{\bar{S},\mathrm{eff},j}(t-\tau)\right\rangle\!\left\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right|. \tag{3053}$$

Let's obtain $U^\dagger(t')A_iU(t')$ in explicit form:

$$U^\dagger(t')A_iU(t') = \sum_{j,j'}\left\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)\left|A_i\right|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right\rangle U^\dagger(t')\left|H_{\bar{S},\mathrm{eff},j}(t-\tau)\right\rangle\!\left\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right|U(t') \tag{3054}$$

$$= \sum_{j,j'}\left\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)\left|A_i\right|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\lambda_j(t-\tau)}\left|H_{\bar{S},\mathrm{eff},j}(t-\tau)\right\rangle\!\left\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right|\mathrm{e}^{\mathrm{i}(t-\tau)\left(-\lambda_{j'}(t-\tau)\right)} \tag{3055}$$

$$=\sum_{j,j'}\left\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)\left|A_i\right|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right\rangle \mathrm{e}^{\mathrm{i}(t-\tau)\left(\lambda_j(t-\tau)-\lambda_{j'}(t-\tau)\right)}\left|H_{\bar{S},\mathrm{eff},j}(t-\tau)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right|, \quad (3056)$$

$$M_{jj'}(t-\tau)=\left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\left|A_i\right|H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\right|, \quad (3057)$$

$$U^{\dagger}(t')A_iU(t')=\sum_{j,j'}M_{jj'}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)\left(\lambda_j(t-\tau)-\lambda_{j'}(t-\tau)\right)}, \quad (3058)$$

$$w_{jj'}(t-\tau)=\lambda_j\left(t-\tau\right)-\lambda_{j'}\left(t-\tau\right), \quad (3059)$$

$$U^{\dagger}(t')A_iU(t')=\sum_{j,j'}M_{jj'}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \quad (3060)$$

$$=\sum_{j}M_{jj}(t-\tau)+\sum_{j\neq j'}M_{jj'}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \quad (3061)$$

$$=\sum_{j}M_{jj}(t-\tau)+\sum_{j\neq j'}M_{j'j}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_{j'j}(t-\tau)}, \quad (3062)$$

$$w_{j'j}(t-\tau)=-w_{jj'}(t-\tau), \quad (3063)$$

$$U^{\dagger}(t')A_iU(t')=\sum_{j}M_{jj}(t-\tau)+\sum_{j\neq j'}M_{j'j}(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \quad (3064)$$

$$=\sum_{j,j'}A_{iw_{jj'}}\left(t-\tau\right)\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \quad (3065)$$

$$=\sum_{j}A_{iw_{jj}}\left(t-\tau\right)\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj}(t-\tau)}+\sum_{j\neq j'}A_{iw_{jj'}}\left(t-\tau\right)\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \quad (3066)$$

$$w_{jj}=0, \quad (3067)$$

$$A_{i0}\left(t-\tau\right)=\sum_{j}\left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\left|A_i\right|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right|, \quad (3068)$$

$$A_{iw_{jj'}}\left(t-\tau\right)=M_{j'j}(t-\tau)\ \ (\text{con } j\neq j') \quad (3069)$$

$$=\left\langle H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\left|A_i\right|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right|. \quad (3070)$$

These matrix have the following property $A_{iw}\left(t-\tau\right)=A^{\dagger}_{i(-w)}\left(t-\tau\right)$. Let $G\left(t-\tau\right)=\{w_{jj'}(t-\tau)\,|j,j'\in\{0,\ldots,n-1\}\}$ and $G^{+}\left(t-\tau\right)=\{x\in G|x>0\}$:

$$U^{\dagger}\left(t'\right)A_iU\left(t'\right)=\sum_{j,j'}M_{jj'}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \quad (3071)$$

$$=\sum_{w_g(t-\tau)\in G(t-\tau)}A_{iw_g}\left(t-\tau\right)\mathrm{e}^{-\mathrm{i}(t-\tau)w_g(t-\tau)} \quad (3072)$$

$$=A_{i0}(t-\tau)+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A_{iw_g}(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w_g(t-\tau)}+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A_{i\left(-w_g\right)}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_g(t-\tau)} \quad (3073)$$

$$\left(U^{\dagger}\left(t'\right)A_iU\left(t'\right)\right)^{\dagger}=\left(A_{i0}(t-\tau)+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A_{iw_g}(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w_g(t-\tau)}+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A_{i\left(-w_g\right)}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_g(t-\tau)}\right)^{\dagger} \quad (3074)$$

$$=A^{\dagger}_{i0}(t-\tau)+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A^{\dagger}_{iw_g}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_g(t-\tau)}+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A^{\dagger}_{i\left(-w_g\right)}(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w_g(t-\tau)} \quad (3075)$$

$$=A_{i0}(t-\tau)+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A_{iw_g}(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w_g(t-\tau)}+\sum_{w_g(t-\tau)\in G^{+}(t-\tau)}A_{i\left(-w_g\right)}(t-\tau)\,\mathrm{e}^{\mathrm{i}(t-\tau)w_g(t-\tau)}, \quad (3076)$$

$$A_{i0}^{\dagger}(t-\tau) = A_{i0}(t-\tau), \tag{3077}$$
$$A_{iw_g}^{\dagger}(t-\tau) = A_{i(-w_g)}(t-\tau), \tag{3078}$$
$$\left(A_{iw_g}^{\dagger}(t-\tau)\right)^{\dagger} = \left(A_{i(-w_g)}(t-\tau)\right)^{\dagger} \tag{3079}$$
$$A_{iw_g}(t-\tau) = A_{i(-w_g)}^{\dagger}(t-\tau). \tag{3080}$$

Now in order to perform the double Fourier decomposition let's recall:

$$\widetilde{A_i}(t,t') \equiv U(t)\,U^{\dagger}(t')\,A_i U(t')\,U^{\dagger}(t). \tag{3081}$$

In this case the decomposition can be written as:

$$\widetilde{A_i}(t,t-\tau) \equiv U(t)\,U^{\dagger}(t-\tau)\,A_i U(t-\tau)\,U^{\dagger}(t) \tag{3082}$$
$$= U(t)\left(\sum_{j,j'} A_{iw_{jj'}}(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)}\right)U^{\dagger}(t). \tag{3083}$$

Now writting $A_{iw_{jj'}}(t-\tau)$ in terms of the eigenstates of $H_{\bar{S},\mathrm{eff}}(t)$ we find:

$$A_{iw_{jj'}}(t-\tau) = \sum_{k,k'} \left\langle H_{\bar{S},\mathrm{eff},k}(t)\right| A_{iw_{jj'}}(t-\tau) \left| H_{\bar{S},\mathrm{eff},k'}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},k}(t)\middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},k'}(t)\right|. \tag{3084}$$

Then the time evolution is given by:

$$\widetilde{A_i}(t,t-\tau) = U(t)\left(\sum_{jj'} A_{iw_{jj'}}(t-\tau)\,\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)}\right)U^{\dagger}(t) \tag{3085}$$
$$= \sum_{jj'} \mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} U(t)\,A_{iw_{jj'}}(t-\tau)\,U^{\dagger}(t) \tag{3086}$$
$$= \sum_{j,j'} \mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} U(t) \sum_{kk'} \left\langle H_{\bar{S},\mathrm{eff},k}(t)\right| A_{iw_{jj'}}(t-\tau) \left| H_{\bar{S},\mathrm{eff},k'}(t)\right\rangle \left|H_{\bar{S},\mathrm{eff},k}(t)\middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},k'}(t)\right| U^{\dagger}(t) \tag{3087}$$
$$= \sum_{jj'} \mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \sum_{kk'} \left\langle H_{\bar{S},\mathrm{eff},k}(t)\right| A_{iw_{jj'}}(t-\tau) \left| H_{\bar{S},\mathrm{eff},k'}(t)\right\rangle U(t) \left|H_{\bar{S},\mathrm{eff},k}(t)\middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},k'}(t)\right| U^{\dagger}(t) \tag{3088}$$
$$= \sum_{jj'} \mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \sum_{kk'} \left\langle H_{\bar{S},\mathrm{eff},k}(t)\right| A_{iw_{jj'}}(t-\tau) \left| H_{\bar{S},\mathrm{eff},k'}(t)\right\rangle \mathrm{e}^{-\mathrm{i}t\lambda_k(t)} \left|H_{\bar{S},\mathrm{eff},k}(t)\middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},k'}(t)\right| \mathrm{e}^{\mathrm{i}t\lambda_{k'}(t)} \tag{3089}$$
$$= \sum_{jj'} \mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)} \sum_{kk'} \left\langle H_{\bar{S},\mathrm{eff},k}(t)\right| A_{iw_{jj'}}(t-\tau) \left| H_{\bar{S},\mathrm{eff},k'}(t)\right\rangle \mathrm{e}^{\mathrm{i}t\left(\lambda_{k'}(t)-\lambda_k(t)\right)} \left|H_{\bar{S},\mathrm{eff},k}(t)\middle\rangle\!\middle\langle H_{\bar{S},\mathrm{eff},k'}(t)\right|, \tag{3090}$$
$$w_{kk'}(t) = \lambda_{k'}(t) - \lambda_k(t) \tag{3091}$$
$$w_{k'k}(t) = -w_{kk'}(t), \tag{3092}$$

$$\widetilde{A_i}(t,t-\tau)=\sum_{jj'}\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)}\sum_{kk'}\left\langle H_{\bar{S},\mathrm{eff},k}\left(t\right)\left|A_{iw_{jj'}}\left(t-\tau\right)\right|H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right\rangle\mathrm{e}^{\mathrm{i}tw_{kk'}(t)}\left|H_{\bar{S},\mathrm{eff},k}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right| \tag{3093}$$

$$=\sum_{jj'kk'}\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)}\left\langle H_{\bar{S},\mathrm{eff},k}\left(t\right)\left|A_{iw_{jj'}}\left(t-\tau\right)\right|H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right\rangle\mathrm{e}^{\mathrm{i}tw_{kk'}(t)}\left|H_{\bar{S},\mathrm{eff},k}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right| \tag{3094}$$

$$=\sum_{jj'kk'}\mathrm{e}^{-\mathrm{i}(t-\tau)w_{jj'}(t-\tau)}\mathrm{e}^{\mathrm{i}tw_{kk'}(t)}\left\langle H_{\bar{S},\mathrm{eff},k}\left(t\right)\left|A_{iw_{jj'}}\left(t-\tau\right)\right|H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right\rangle\left|H_{\bar{S},\mathrm{eff},k}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right| \tag{3095}$$

$$=\sum_{jj'kk'}\mathrm{e}^{\mathrm{i}\tau w_{jj'}(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w_{jj'}(t-\tau)-w_{kk'}(t)\right)}\left\langle H_{\bar{S},\mathrm{eff},k}(t)\left|A_{iw_{jj'}}(t-\tau)\right|H_{\bar{S},\mathrm{eff},k'}(t)\right\rangle\left|H_{\bar{S},\mathrm{eff},k}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}(t)\right| \tag{3096}$$

$$=\sum_{jj'kk'}\mathrm{e}^{\mathrm{i}\tau w_{jj'}(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w_{jj'}(t-\tau)-w_{kk'}(t)\right)}\left\langle H_{\bar{S},\mathrm{eff},k}(t)\left|A_{iw_{jj'}}(t-\tau)\right|H_{\bar{S},\mathrm{eff},k'}(t)\right\rangle\left|H_{\bar{S},\mathrm{eff},k}(t)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}(t)\right| \tag{3097}$$

$$=\sum_{jj'kk'}\mathrm{e}^{\mathrm{i}\tau w_{jj'}(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w_{jj'}(t-\tau)-w_{kk'}(t)\right)}\left\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\left|A_i\right|H_{\bar{S},\mathrm{eff},j}(t-\tau)\right\rangle\left\langle H_{\bar{S},\mathrm{eff},k}(t)\left|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\right.\right\rangle \tag{3098}$$

$$\times\left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\left|H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right.\right\rangle\left|H_{\bar{S},\mathrm{eff},k}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right| \tag{3099}$$

$$=\sum_{jj'kk'}\mathrm{e}^{\mathrm{i}\tau w_{jj'}(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w_{jj'}(t-\tau)-w_{kk'}(t)\right)}A_{iw_{jj'}w_{kk'}}\left(t-\tau,t\right), \tag{3100}$$

$$A_{iw_{jj'}w_{kk'}}(t-\tau,t)\equiv\left\langle H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\left|A_i\right|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right\rangle\left\langle H_{\bar{S},\mathrm{eff},k}\left(t\right)\left|H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\right.\right\rangle\left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\left|H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right.\right\rangle \tag{3101}$$

$$\times\left|H_{\bar{S},\mathrm{eff},k}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right|, \tag{3102}$$

$$A_{iw_{j'j}w_{k'k}}(t-\tau,t)=\left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\left|A_i\right|H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\right\rangle\left\langle H_{\bar{S},\mathrm{eff},k'}\left(t\right)\left|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right.\right\rangle\left\langle H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\left|H_{\bar{S},\mathrm{eff},k}\left(t\right)\right.\right\rangle \tag{3103}$$

$$\times\left|H_{\bar{S},\mathrm{eff},k'}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k}\left(t\right)\right| \tag{3104}$$

$$=\left(\left\langle H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\left|A_i\right|H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\right\rangle\left\langle H_{\bar{S},\mathrm{eff},k}\left(t\right)\left|H_{\bar{S},\mathrm{eff},j'}\left(t-\tau\right)\right.\right\rangle\left\langle H_{\bar{S},\mathrm{eff},j}\left(t-\tau\right)\left|H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right.\right\rangle\right. \tag{3105}$$

$$\left.\times\left|H_{\bar{S},\mathrm{eff},k}\left(t\right)\middle\rangle\middle\langle H_{\bar{S},\mathrm{eff},k'}\left(t\right)\right|\right)^{\dagger} \tag{3106}$$

$$=A^{\dagger}_{iw_{jj'}w_{kk'}}\left(t-\tau,t\right). \tag{3107}$$

Let's prove that $A^{\dagger}_{iw_{j'j}w_{k'k}}\left(t-\tau,t\right)=A_{i\left(-w_{j'j}\right)\left(-w_{k'k}\right)}\left(t-\tau,t\right)=A_{i\left(w_{jj'}\right)\left(w_{kk'}\right)}\left(t-\tau,t\right)$:

$$A^{\dagger}_{iw_{j'j}w_{k'k}}(t-\tau,t)=\left(\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)|A_i|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\rangle\langle H_{\bar{S},\mathrm{eff},k'}(t)\,|H_{\bar{S},\mathrm{eff},j}(t-\tau)\rangle\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\,|H_{\bar{S},\mathrm{eff},k}(t)\rangle|H_{\bar{S},\mathrm{eff},k'}(t)\rangle\langle H_{\bar{S},\mathrm{eff},k}(t)|\right)^{\dagger} \tag{3108}$$

$$=\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)|A_i|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\rangle^{*}\,|H_{\bar{S},\mathrm{eff},k}(t)\rangle\langle H_{\bar{S},\mathrm{eff},k'}(t)|\,\langle H_{\bar{S},\mathrm{eff},k'}(t)\,|H_{\bar{S},\mathrm{eff},j}(t-\tau)\rangle^{*}\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\,|H_{\bar{S},\mathrm{eff},k}(t)\rangle^{*} \tag{3109}$$

$$=\left\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)\middle|A_i^{\dagger}\middle|H_{\bar{S},\mathrm{eff},j}(t-\tau)\right\rangle|H_{\bar{S},\mathrm{eff},k}(t)\rangle\langle H_{\bar{S},\mathrm{eff},k'}(t)|\,\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)\,|H_{\bar{S},\mathrm{eff},k'}(t)\rangle\langle H_{\bar{S},\mathrm{eff},k}(t)\,|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\rangle \tag{3110}$$

$$=\langle H_{\bar{S},\mathrm{eff},j'}(t-\tau)|A_i|H_{\bar{S},\mathrm{eff},j}(t-\tau)\rangle\,|H_{\bar{S},\mathrm{eff},k}(t)\rangle\langle H_{\bar{S},\mathrm{eff},k'}(t)|\,\langle H_{\bar{S},\mathrm{eff},j}(t-\tau)\,|H_{\bar{S},\mathrm{eff},k'}(t)\rangle\langle H_{\bar{S},\mathrm{eff},k}(t)\,|H_{\bar{S},\mathrm{eff},j'}(t-\tau)\rangle \tag{3111}$$

$$=A_{i\left(w_{jj'}\right)\left(w_{kk'}\right)}\left(t-\tau,t\right). \tag{3112}$$

Let $w_{jj'}\rightarrow w$ and $w_{kk'}\rightarrow w'$, it helps to see that the eigenvalues differences can be ordered in a set and also it reduces the length of the terms implied then from the previous property it can be seen that the index $-w$ and $-w'$ change to the functions w and w' by $A^{\dagger}_{iw_{j'j}w_{k'k}}\left(t-\tau,t\right)=A_{i\left(-w_{j'j}\right)\left(-w_{k'k}\right)}\left(t-\tau,t\right)=A_{i\left(w_{jj'}\right)\left(w_{kk'}\right)}\left(t-\tau,t\right)$.

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}\left(t',t\right)=\widetilde{A_i}^{\dagger}\left(t',t\right)$ we can use the equation (921) to write the master equation in the following neater form:

$$U\left(t\right)\frac{\partial\widetilde{\overline{\rho_S}}\left(t\right)}{\partial t}U^{\dagger}\left(t\right)=-\sum_{j,j',p,p',w,w'}\int_0^t\mathrm{d}\tau C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\Big(\Lambda_{jp,j'p'}\left(t,t-\tau\right)\left[A_{jp},\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}(t-\tau,t)\,\overline{\rho_S}\left(t\right)\right] \tag{3113}$$

$$-\Lambda^{*}_{jp,j'p'}\left(t,t-\tau\right)\left[A_{jp},\overline{\rho_S}\left(t\right)\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}\left(t-\tau,t\right)\right]\Big) \tag{3114}$$

$$=-\sum_{j,j',p,p',w,w'}\int_0^t\mathrm{d}\tau C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\Big(\Lambda_{jp,j'p'}\left(t,t-\tau\right)\left[A_{jp},\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \tag{3115}$$

$$-\Lambda^{*}_{jp,j'p'}\left(t,t-\tau\right)\left[A_{jp},\overline{\rho_S}\left(t\right)\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}\left(t-\tau,t\right)\right]\Big). \tag{3116}$$

Given that $A_{j'p'ww'}\left(t-\tau,t\right)=A^{\dagger}_{j'p'(-w)(-w')}\left(t-\tau,t\right)$ and $w\left(t-\tau\right),w'\left(t\right)$ belong to the set of differences of eigenvalues of $H_{\bar{S},\mathrm{eff}}\left(t-\tau\right)$ and $H_{\bar{S},\mathrm{eff}}\left(t\right)$ denoted by J_t and $J_{t-\tau}$ respectively that depends of the time we can take an application where $w\left(t-\tau\right)\rightarrow-w\left(t-\tau\right)$ and $w'\left(t\right)\rightarrow-w'\left(t\right)$ such that the sum:

$$\sum_{ww'} \mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}\left(t-\tau,t\right) = \sum_{ww'} \mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'(-w)(-w')}\left(t-\tau,t\right) \tag{3117}$$

$$= \sum_{ww'} \mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{j'p'ww'}\left(t-\tau,t\right). \tag{3118}$$

is invariant because if $(w\left(t-\tau\right), w'\left(t\right)) \in J_{t-\tau}\times J_t$ then $(-w\left(t-\tau\right), -w'\left(t\right)) \in J_{t-\tau}\times J_t$ where J_t denotes the set of differences of eigenvalues at time t. So the master equation can be written as:

$$U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}(t) = -\sum_{j,j',p,p',w,w'}\int_0^t \mathrm{d}\tau C_{jp}(t)C_{j'p'}(t-\tau)\Big(\Lambda_{jp,j'p'}(t,t-\tau)\Big[A_{jp},\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}(t-\tau,t)\overline{\rho_S}(t)\Big] \tag{3119}$$

$$-\Lambda^{*}_{jp,j'p'}\left(t,t-\tau\right)\Big[A_{jp},\overline{\rho_S}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{j'p'ww'}\left(t-\tau,t\right)\Big]\Big). \tag{3120}$$

With the definition:

$$L_{jpj'p'ww'}\left(t\right) \equiv \int_0^t C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\Lambda_{jp,j'p'}\left(t,t-\tau\right)\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}\left(t-\tau,t\right)\mathrm{d}\tau. \tag{3121}$$

We can show that:

$$L^{\dagger}_{jpj'p'ww'}\left(t\right) = \int_0^t \Big(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\Lambda_{jp,j'p'}\left(t,t-\tau\right)\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}\left(t-\tau,t\right)\mathrm{d}\tau\Big)^{\dagger} \tag{3122}$$

$$= \int_0^t C^{*}_{jp}\left(t\right)C^{*}_{j'p'}\left(t-\tau\right)\Lambda^{*}_{jp,j'p'}\left(t,t-\tau\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{j'p'ww'}\left(t-\tau,t\right)\mathrm{d}\tau \tag{3123}$$

$$= \int_0^t C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\Lambda^{*}_{jp,j'p'}\left(t,t-\tau\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{j'p'ww'}\left(t-\tau,t\right)\mathrm{d}\tau. \tag{3124}$$

So we can write the master equation as:

$$U(t)\,\frac{\partial\widetilde{\overline{\rho_S}}(t)}{\partial t}U^{\dagger}(t) = -\sum_{j,j',p,p',w,w'}\int_0^t \mathrm{d}\tau C_{jp}(t)C_{j'p'}(t-\tau)\Big(\Lambda_{jp,j'p'}(t,t-\tau)\Big[A_{jp},\mathrm{e}^{\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{j'p'ww'}(t-\tau,t) \tag{3125}$$

$$\times\overline{\rho_S}(t)] - \Lambda^{*}_{jp,j'p'}\left(t,t-\tau\right)\Big[A_{jp},\overline{\rho_S}\left(t\right)\mathrm{e}^{-\mathrm{i}\tau w(t-\tau)}\mathrm{e}^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{j'p'ww'}\left(t-\tau,t\right)\Big]\Big). \tag{3126}$$

$$= -\sum_{j,j',p,p',w,w'}\Big(\left[A_{jp},L_{jpj'p'ww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] + \Big[\overline{\rho_S}\left(t\right)L^{\dagger}_{jpj'p'ww'}\left(t\right),A_{jp}\Big]\Big). \tag{3127}$$

If we extend the upper limit of integration to ∞ in the equation (3127) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation. Our master equation in the variational and lab frame are given by:

$$\frac{\partial\overline{\rho_S}(t)}{\partial t} = -\mathrm{i}\Big[H_{\overline{S},\mathrm{eff}}\left(t\right),\overline{\rho_S}\left(t\right)\Big] - \sum_{j,j',p,p',w,w'}\Big(\left[A_{jp}\left(t\right),L_{jpj'p'ww'}\left(t\right)\overline{\rho_S}\left(t\right)\right] - \Big[A_{jp}\left(t\right),\overline{\rho_S}\left(t\right)L^{\dagger}_{jpj'p'ww'}\left(t\right)\Big]\Big) \tag{3128}$$

$$-\,\mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{i}t\right)^k\left(\mathrm{ad}_{H_{\overline{S},\mathrm{eff}}(t)}\right)^k\frac{\partial H_{\overline{S},\mathrm{eff}}\left(t\right)}{\partial t}\right),\overline{\rho_S}\left(t\right)\right], \tag{3129}$$

$$\frac{\partial\rho_S(t)}{\partial t} = -\mathrm{i}\Big[\underline{H_{\overline{S},\mathrm{eff}}\left(t\right)},\rho_S\left(t\right)\Big] - \sum_{j,j',p,p',w,w'}\Big(\Big[\underline{A_{jp}\left(t\right)},\underline{L_{jpj'p'ww'}(t)}\rho_S(t)\Big] - \Big[\underline{A_{jp}\left(t\right)},\rho_S(t)\,\underline{L^{\dagger}_{jpj'p'ww'}(t)}\Big]\Big) \tag{3130}$$

$$-\left[\frac{1-\mathrm{e}^{-\mathrm{ad}_{V(t)}}}{\mathrm{ad}_{V(t)}}\left(\frac{\partial V(t)}{\partial t}\right),\rho_S(t)\right] - \mathrm{i}t\left[\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{i}t\right)^k\left(\mathrm{ad}_{\underline{H_{\overline{S},\mathrm{eff}}(t)}}\right)^k\underline{\frac{\partial H_{\overline{S},\mathrm{eff}}\left(t\right)}{\partial t}}\right),\rho_S\left(t\right)\right]. \tag{3131}$$

VIII. A MULTI-SITE VARIATIONAL TIME-INDEPENDENT MASTER EQUATION

We will focus on proving some special cases of the equation (3131) that are relevant for checking its consistence and consider it for further applications. The main features of this model refers to a N-site model such that each site has it's own environment or phonon bath and there is interaction between sites as well at the same temperature, so taking in (2463) $u=n$, removing the time dependence and dropping the notation $nn\rightarrow n$ then we will obtain:

$$H = H_S + H_I + H_B, \tag{3132}$$
$$H_S = \sum_n \varepsilon_n |n\rangle\langle n| + \sum_{n\neq m} V_{nm}|n\rangle\langle m| \tag{3133}$$
$$H_I = \sum_{nn\mathbf{k}} |n\rangle\langle n| \left(g_{nn\mathbf{k}} b_{n\mathbf{k}}^{\dagger} + g_{nn\mathbf{k}}^{*} b_{n\mathbf{k}}\right) \tag{3134}$$
$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left(g_{n\mathbf{k}} b_{n\mathbf{k}}^{\dagger} + g_{n\mathbf{k}}^{*} b_{n\mathbf{k}}\right), \tag{3135}$$
$$H_B = \sum_{n\mathbf{k}} \omega_{n\mathbf{k}} b_{n\mathbf{k}}^{\dagger} b_{n\mathbf{k}}. \tag{3136}$$

We can separate the Hamiltonian (2537) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}} = \sum_n (\varepsilon_n + R_n)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}|n\rangle\langle m| B_{nm}, \tag{3137}$$
$$\overline{H_{\bar{I}}(t)} = \overline{H_L} + \overline{H_D} \tag{3138}$$
$$= \sum_{n\neq m} V_{nm}|n\rangle\langle m| J_{nm} + \sum_n B_{z,n}|n\rangle\langle n|, \tag{3139}$$
$$\overline{H_L} = \sum_n B_{z,n}|n\rangle\langle n|, \tag{3140}$$
$$\overline{H_D} = \sum_{n\neq m} V_{nm}|n\rangle\langle m| J_{nm}, \tag{3141}$$
$$\overline{H_{\bar{B}}} = \sum_{n\mathbf{k}} \omega_{n\mathbf{k}} b_{n\mathbf{k}}^{\dagger} b_{n\mathbf{k}}, \tag{3142}$$
$$\overline{H} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{3143}$$

Recall that $v_{mu\mathbf{k}} = v_{m\mathbf{k}}\delta_{mu}$ because each site has only one bath associated and setting $v_{n\mathbf{k}} = f_{n\mathbf{k}}$. In our case B_{nm}, R_n, J_{nm} and $B_{z,n}$ are equal to:

$$B_n = \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{n\mathbf{k}}|^2}{\omega_{n\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{n\mathbf{k}}}{2}\right)}, \tag{3144}$$
$$B_{nm} = \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v_{nu\mathbf{k}}^{*}v_{mu\mathbf{k}} - v_{nu\mathbf{k}}v_{mu\mathbf{k}}^{*}}{2\omega_{u\mathbf{k}}^2}} \prod_u \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{mu\mathbf{k}} - v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3145}$$
$$= \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v_{n\mathbf{k}}^{*}\delta_{nu}v_{m\mathbf{k}}\delta_{mu} - v_{n\mathbf{k}}\delta_{nu}v_{m\mathbf{k}}^{*}\delta_{mu}}{2\omega_{u\mathbf{k}}^2}} \prod_u \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{m\mathbf{k}}\delta_{mu} - v_{n\mathbf{k}}\delta_{nu}|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3146}$$
$$= \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{m\mathbf{k}}|^2}{\omega_{m\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{m\mathbf{k}}}{2}\right)} \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \frac{|v_{n\mathbf{k}}|^2}{\omega_{n\mathbf{k}}^2}\coth\left(\frac{\beta\omega_{n\mathbf{k}}}{2}\right)} \tag{3147}$$
$$= B_n B_m, \tag{3148}$$

$$J_{nm} = \prod_{u\mathbf{k}} D\left(\alpha_{nu\mathbf{k}} - \alpha_{mu\mathbf{k}}\right) \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v^*_{nu\mathbf{k}} v_{mu\mathbf{k}} - v_{nu\mathbf{k}} v^*_{mu\mathbf{k}}}{2\omega^2_{u\mathbf{k}}}} - B_{nm} \tag{3149}$$

$$= \prod_{u\mathbf{k}} D\left(\alpha_{nu\mathbf{k}} - \alpha_{mu\mathbf{k}}\right) \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v^*_{nu\mathbf{k}} v_{mu\mathbf{k}} - v_{nu\mathbf{k}} v^*_{mu\mathbf{k}}}{2\omega^2_{u\mathbf{k}}}} - B_n B_m \tag{3150}$$

$$= \prod_{u\mathbf{k}} D\left(\alpha_{n\mathbf{k}}\delta_{nu} - \alpha_{m\mathbf{k}}\delta_{mu}\right) \prod_{u\mathbf{k}} \mathrm{e}^{\frac{v^*_{n\mathbf{k}}\delta_{nu} v_{m\mathbf{k}}\delta_{mu} - v_{n\mathbf{k}}\delta_{nu} v^*_{m\mathbf{k}}\delta_{mu}}{2\omega^2_{u\mathbf{k}}}} - B_n B_m \tag{3151}$$

$$= \prod_{\mathbf{k}} D\left(\alpha_{n\mathbf{k}}\delta_{nn} - \alpha_{m\mathbf{k}}\delta_{mn}\right) \prod_{\mathbf{k}} D\left(\alpha_{n\mathbf{k}}\delta_{nm} - \alpha_{m\mathbf{k}}\delta_{mm}\right) - B_n B_m \tag{3152}$$

$$= \prod_{\mathbf{k}} D\left(\alpha_{n\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{m\mathbf{k}}\right) - B_n B_m, \tag{3153}$$

$$\prod_{\mathbf{k}} D\left(\alpha_{n\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{m\mathbf{k}}\right) \equiv B_{n+} B_{m-}, \tag{3154}$$

$$J_{nm} = B_{n+} B_{m-} - B_n B_m, \tag{3155}$$

$$R_n = \sum_{n\mathbf{k}} \left(\frac{\left|v_{nn\mathbf{k}}\right|^2}{\omega_{n\mathbf{k}}} - \left(g_{nn\mathbf{k}} \frac{v^*_{nn\mathbf{k}}}{\omega_{n\mathbf{k}}} + g^*_{nn\mathbf{k}} \frac{v_{nn\mathbf{k}}}{\omega_{n\mathbf{k}}} \right) \right) \tag{3156}$$

$$= \sum_{n\mathbf{k}} \left(\frac{\left|v_{n\mathbf{k}}\right|^2}{\omega_{n\mathbf{k}}} - \left(g_{n\mathbf{k}} \frac{v^*_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} + g^*_{n\mathbf{k}} \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} \right) \right) \tag{3157}$$

$$= \sum_{n\mathbf{k}} \omega^{-1}_{n\mathbf{k}} \left(\left|v_{n\mathbf{k}}\right|^2 - \left(g_{n\mathbf{k}} v^*_{n\mathbf{k}} + g^*_{n\mathbf{k}} v_{n\mathbf{k}} \right) \right) \tag{3158}$$

$$= \sum_{n\mathbf{k}} \omega^{-1}_{n\mathbf{k}} \left(\left|v_{n\mathbf{k}}\right|^2 - 2\left(g^*_{n\mathbf{k}} v_{n\mathbf{k}} \right)^{\Re} \right), \tag{3159}$$

$$\overline{H_L} = \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\left(g_{nn\mathbf{k}} - v_{nn\mathbf{k}}\right) b^\dagger_{n\mathbf{k}} + \left(g_{nn\mathbf{k}} - v_{nn\mathbf{k}}\right)^* b_{n\mathbf{k}} \right) \tag{3160}$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\right) b^\dagger_{n\mathbf{k}} + \left(g_{n\mathbf{k}} - v_{n\mathbf{k}}\right)^* b_{n\mathbf{k}} \right) \tag{3161}$$

So we can reproduce:

$$\overline{H_{\bar{S}}} = \sum_{n} \left(\varepsilon_n + R_n\right) |n\rangle\langle n| + \sum_{n \neq m} B_n B_m V_{nm} |n\rangle\langle m|, \tag{3162}$$

$$\overline{H_{\bar{I}}} = \overline{H_L} + \overline{H_D} \tag{3163}$$

$$= \sum_{n \neq m} V_{nm} |n\rangle\langle m| J_{nm} + \sum_{n} B_{z,n} |n\rangle\langle n|, \tag{3164}$$

$$\overline{H_L} = \sum_{n} B_{z,n} |n\rangle\langle n|, \tag{3165}$$

$$\overline{H_D} = \sum_{n \neq m} V_{nm} |n\rangle\langle m| J_{nm}, \tag{3166}$$

$$\overline{H_{\bar{B}}} = \sum_{n\mathbf{k}} \omega_{n\mathbf{k}} b^\dagger_{n\mathbf{k}} b_{n\mathbf{k}}, \tag{3167}$$

$$\overline{H} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{3168}$$

The variational parameters are given by:

$$v_{nu\mathbf{k}}\left(t\right) = \frac{2\omega_{u\mathbf{k}} g_{nu\mathbf{k}} \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} - \sum_{m|m\neq n} v_{mu\mathbf{k}}\left(t\right) \left(\frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_{nm}(t)} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right) R_{nm}(t) + \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial \varphi_{nm}(t)} \varphi_{nm}\left(t\right) \right)}{2\omega_{u\mathbf{k}} \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial R_n(t)} - B_n\left(t\right) \frac{\partial E_{\mathrm{Free,B}}(t)}{\partial B_n(t)} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}. \tag{3169}$$

In the model considered here $v_{mu\mathbf{k}} = v_{m\mathbf{k}}\delta_{mu}$, $\beta_u = \beta$ and $u = n$ so:

$$v_{n\mathbf{k}} = \frac{2g_{n\mathbf{k}}\omega_{n\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}}{\partial R_n} - \sum_{m|m\neq n} v_{m\mathbf{k}}\delta_{mn}\left(\frac{\partial E_{\mathrm{Free,B}}}{\partial R_{nm}}\coth\left(\frac{\beta\omega_{n\mathbf{k}}}{2}\right)R_{nm} + \frac{\partial E_{\mathrm{Free,B}}}{\partial \varphi_{nm}}\varphi_{nm}\right)}{2\omega_{n\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}}{\partial R_n} - B_n\frac{\partial E_{\mathrm{Free,B}}}{\partial B_n}\coth\left(\frac{\beta\omega_{n\mathbf{k}}}{2}\right)} \tag{3170}$$

$$= \frac{2g_{n\mathbf{k}}\omega_{n\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}}{\partial R_n}}{2\omega_{n\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}}{\partial R_n} - B_n\frac{\partial E_{\mathrm{Free,B}}}{\partial B_n}\coth\left(\frac{\beta\omega_{n\mathbf{k}}}{2}\right)}. \tag{3171}$$

As we can see $R_{nm}(t) = 1$ and $\varphi_{nm}(t) = 1$ under this model because:

$$\varphi_{nm}(t) = \prod_{u\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v^{\Re}_{nu\mathbf{k}}(t)v^{\Im}_{mu\mathbf{k}}(t) - v^{\Im}_{nu\mathbf{k}}(t)v^{\Re}_{mu\mathbf{k}}(t)\right)}{\omega^2_{u\mathbf{k}}}} \tag{3172}$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v^{\Re}_{nn\mathbf{k}}(t)v^{\Im}_{mn\mathbf{k}}(t) - v^{\Im}_{nn\mathbf{k}}(t)v^{\Re}_{mn\mathbf{k}}(t)\right)}{\omega^2_{u\mathbf{k}}}} \prod_{\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v^{\Re}_{nm\mathbf{k}}(t)v^{\Im}_{mm\mathbf{k}}(t) - v^{\Im}_{nm\mathbf{k}}(t)v^{\Re}_{mm\mathbf{k}}(t)\right)}{\omega^2_{u\mathbf{k}}}} \prod_{u\notin\{n,m\},\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v^{\Re}_{nu\mathbf{k}}(t)v^{\Im}_{mu\mathbf{k}}(t) - v^{\Im}_{nu\mathbf{k}}(t)v^{\Re}_{mu\mathbf{k}}(t)\right)}{\omega^2_{u\mathbf{k}}}} \tag{3173}$$

$$= \prod_{\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v^{\Re}_{nn\mathbf{k}}(t)v^{\Im}_{mn\mathbf{k}}(t) - v^{\Im}_{nn\mathbf{k}}(t)v^{\Re}_{mn\mathbf{k}}(t)\right)\delta_{mn}}{\omega^2_{u\mathbf{k}}}} \prod_{\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v^{\Re}_{nm\mathbf{k}}(t)v^{\Im}_{mm\mathbf{k}}(t) - v^{\Im}_{nm\mathbf{k}}(t)v^{\Re}_{mm\mathbf{k}}(t)\right)\delta_{nm}}{\omega^2_{u\mathbf{k}}}} \prod_{u\notin\{n,m\},\mathbf{k}} \mathrm{e}^{\frac{\mathrm{i}\left(v^{\Re}_{nu\mathbf{k}}(t)v^{\Im}_{mu\mathbf{k}}(t) - v^{\Im}_{nu\mathbf{k}}(t)v^{\Re}_{mu\mathbf{k}}(t)\right)}{\omega^2_{u\mathbf{k}}}} \tag{3174}$$

$$= \mathrm{e}^0 \tag{3175}$$

$$= 1, \tag{3176}$$

$$R_{nm}(t) = \prod_{u} \mathrm{e}^{\sum_{\mathbf{k}} \frac{v^{\Re}_{nu\mathbf{k}}(t)v^{\Re}_{mu\mathbf{k}}(t) + v^{\Im}_{nu\mathbf{k}}(t)v^{\Im}_{mu\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3177}$$

$$= \prod_{u} \mathrm{e}^{\sum_{\mathbf{k}} \frac{v^{\Re}_{n\mathbf{k}}(t)v^{\Re}_{m\mathbf{k}}(t) + v^{\Im}_{n\mathbf{k}}(t)v^{\Im}_{m\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\delta_{nu}\delta_{mu}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3178}$$

$$= \prod_{u} \mathrm{e}^{\sum_{\mathbf{k}} \frac{v^{\Re}_{n\mathbf{k}}(t)v^{\Re}_{m\mathbf{k}}(t) + v^{\Im}_{n\mathbf{k}}(t)v^{\Im}_{m\mathbf{k}}(t)}{\omega^2_{u\mathbf{k}}}\delta_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3179}$$

$$= \mathrm{e}^0 \tag{3180}$$

$$= 1. \tag{3181}$$

If $v_{n\mathbf{k}} = g_{n\mathbf{k}}F_n(\omega_{n\mathbf{k}})$ then we can deduce that $F_n(\omega_{n\mathbf{k}}) \equiv F(\omega_{n\mathbf{k}};\{B_n, R_n\})$ because $F(\omega_{n\mathbf{k}};\{B_n, R_n\})$ depends of $\omega_{n\mathbf{k}}$, B_n and R_n so we arrive to:

$$F(\omega_{n\mathbf{k}};\{B_n, R_n\}) = \frac{2\omega_{n\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}}{\partial R_n}}{2\omega_{n\mathbf{k}}\frac{\partial E_{\mathrm{Free,B}}}{\partial R_n} - B_n\frac{\partial E_{\mathrm{Free,B}}}{\partial B_n}\coth\left(\frac{\beta\omega_{n\mathbf{k}}}{2}\right)}. \tag{3182}$$

The master equation is time-independent and follows the markovian and Born approximation so $\frac{\partial H_{\overline{S},\mathrm{eff}}(t)}{\partial t} = 0$, $H_{\overline{S},\mathrm{eff}}(t) = \overline{H_{\bar{S}}}$ so the equation (3129) can be reduced to:

$$\frac{\partial \overline{\rho_S}(t)}{\partial t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}, \overline{\rho_S}(t)\right] - \sum_{j,j'\in J,p,p'\in P}\int_0^t C_{jp}(t)C_{j'p'}(t')\left(\Lambda_{jp,j'p'}(t,t')\left[A_{jp}, \widetilde{A_{j'p'}}(t',t)\,\overline{\rho_S}(t)\right] - \Lambda^*_{jp,j'p'}(t,t')\right. \tag{3183}$$

$$\left.\times\left[A_{jp}, \overline{\rho_S}(t)\,\widetilde{A_{j'p'}}(t',t)\right]\right)\mathrm{d}t'. \tag{3184}$$

Let $\left(C_{jp}(t)C_{j'p'}(t')\Lambda_{jp,j'p'}(t,t')\left[A_{jp}, \widetilde{A_{j'p'}}(t',t)\,\overline{\rho_S}(t)\right]\right)^\dagger$ equal to:

$$\left(C_{jp}\left(t\right)C_{j'p'}\left(t'\right)\Lambda_{jp,j'p'}\left(t,t'\right)\left[A_{jp},\widetilde{A_{j'p'}}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]\right)^{\dagger}=C_{jp}^{*}\left(t\right)C_{j'p'}^{*}\left(t'\right)\Lambda_{jp,j'p'}^{*}\left(t,t'\right)\left[A_{jp},\widetilde{A_{j'p'}}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]^{\dagger} \tag{3185}$$

$$=C_{jp}\left(t\right)C_{j'p'}\left(t'\right)\Lambda_{jp,j'p'}^{*}\left(t,t'\right)\left[\left(\widetilde{A_{j'p'}}\left(t',t\right)\overline{\rho_S}\left(t\right)\right)^{\dagger},A_{jp}^{\dagger}\right] \tag{3186}$$

$$=C_{jp}\left(t\right)C_{j'p'}\left(t'\right)\Lambda_{jp,j'p'}^{*}\left(t,t'\right)\left[\overline{\rho_S}^{\dagger}\left(t\right)\widetilde{A_{j'p'}}^{\dagger}\left(t',t\right),A_{jp}^{\dagger}\right] \tag{3187}$$

$$=C_{jp}\left(t\right)C_{j'p'}\left(t'\right)\Lambda_{jp,j'p'}^{*}\left(t,t'\right)\left[\overline{\rho_S}\left(t\right)\widetilde{A_{j'p'}}\left(t',t\right),A_{jp}\right] \tag{3188}$$

$$=-C_{jp}\left(t\right)C_{j'p'}\left(t'\right)\Lambda_{jp,j'p'}^{*}\left(t,t'\right)\left[A_{jp},\overline{\rho_S}\left(t\right)\widetilde{A_{j'p'}}\left(t',t\right)\right] \tag{3189}$$

Introducing the notation $A+A^{\dagger}\equiv A+\mathrm{H.c.}$ that denotes hermitian conjugate then we can obtain the master equation as:

$$\frac{\partial\overline{\rho_S}\left(t\right)}{\partial t}=-\mathrm{i}\left[\overline{H_{\bar{S}}},\overline{\rho_S}\left(t\right)\right]-\sum_{j,j'\in J,p,p'\in P}\int_0^t\left(C_{jp}\left(t\right)C_{j'p'}\left(t'\right)\Lambda_{jp,j'p'}\left(t,t'\right)\left[A_{jp},\widetilde{A_{j'p'}}\left(t',t\right)\overline{\rho_S}\left(t\right)\right]+\mathrm{H.c.}\right)\mathrm{d}t'. \tag{3190}$$

We require to study $C_{jp}\left(t\right)$, $\Lambda_{jp,j'p'}\left(t,t'\right)$ and $\widetilde{A_{j'p'}}\left(t',t\right)$ under the hypothesis considered for this model, from the time independence of V_{nm} then $C_{jp}\left(t\right)=C_{jp}$ and:

$$\Lambda_{jp,j'p'}\left(t,t'\right)\equiv\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(t'\right)\rho_B\right) \tag{3191}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}B_{j'p'}\mathrm{e}^{-\mathrm{i}H_Bt'}\rho_B\right) \tag{3192}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}B_{j'p'}\rho_B\mathrm{e}^{-\mathrm{i}H_Bt'}\right) \tag{3193}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{-\mathrm{i}H_Bt'}\mathrm{e}^{\mathrm{i}H_Bt}B_{jp}\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}B_{j'p'}\rho_B\right) \tag{3194}$$

$$=\mathrm{Tr}_B\left(\left(\mathrm{e}^{-\mathrm{i}H_Bt'}\mathrm{e}^{\mathrm{i}H_Bt}\right)B_{jp}\left(\mathrm{e}^{-\mathrm{i}H_Bt}\mathrm{e}^{\mathrm{i}H_Bt'}\right)B_{j'p'}\rho_B\right) \tag{3195}$$

$$=\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}H_B\left(t-t'\right)}B_{jp}\mathrm{e}^{-\mathrm{i}H_B\left(t-t'\right)}B_{j'p'}\rho_B\right) \tag{3196}$$

$$=\mathrm{Tr}_B\left(B_{jp}\mathrm{e}^{-\mathrm{i}\tau H_B}B_{j'p'}\mathrm{e}^{\mathrm{i}\tau H_B}\rho_B\right) \tag{3197}$$

$$=\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(\tau\right)B_{j'p'}\rho_B\right) \tag{3198}$$

$$=\Lambda_{jp,j'p'}\left(\tau,0\right) \tag{3199}$$

$$\equiv\Lambda_{jp,j'p'}\left(\tau\right), \tag{3200}$$

$$t'=t-\tau, \tag{3201}$$

$$\widetilde{A_{jp}}\left(t\right)=U^{\dagger}\left(t\right)A_{jp}\left(t\right)U\left(t\right), \tag{3202}$$

$$\widetilde{A_{jp}}\left(t',t\right)\equiv U\left(t\right)\widetilde{A_{jp}}\left(t'\right)U^{\dagger}\left(t\right) \tag{3203}$$

$$=U\left(t\right)U^{\dagger}\left(t'\right)A_{jp}\left(t'\right)U\left(t'\right)U^{\dagger}\left(t\right) \tag{3204}$$

$$=U\left(t\right)U^{\dagger}\left(t'\right)A_{jp}U\left(t'\right)U^{\dagger}\left(t\right) \tag{3205}$$

$$=U^{\dagger}\left(t'-t\right)A_{jp}U\left(t'-t\right) \tag{3206}$$

$$=U^{\dagger}\left(-\tau\right)A_{jp}U\left(-\tau\right) \tag{3207}$$

$$=\widetilde{A_{jp}}\left(-\tau\right). \tag{3208}$$

We arrive to:

$$\frac{\partial\overline{\rho_S}\left(t\right)}{\partial t}=-\mathrm{i}\left[\overline{H_{\bar{S}}},\overline{\rho_S}\left(t\right)\right]-\sum_{j,j'\in J,p,p'\in P}\int_0^t\left(C_{jp}C_{j'p'}\Lambda_{jp,j'p'}\left(t-t'\right)\left[A_{jp},\widetilde{A_{j'p'}}\left(t'-t\right)\overline{\rho_S}\left(t\right)\right]+\mathrm{H.c.}\right)\mathrm{d}t'. \tag{3209}$$

In order to provide concordance among the notation used in our master equation and the notation of the case considered then we have to divide the transformed interaction hamiltonian as expected in the model studied, this give us as result:

$$
\begin{aligned}
\overline{H_I} &= \sum_{j\in J, p\in P} C_{jp}\left(A_{jp}\otimes B_{jp}\right) && (3210)\\
&= \sum_{(n,m)\in P}\big(\sigma_{nm,x}J_{nm,x}\left(1-\delta_{mn}\right)V_{nm}^{\Re}+\sigma_{nm,y}J_{nm,y}\left(1-\delta_{mn}\right)V_{nm}^{\Re}+|n\rangle\langle m|B_{z,n}\delta_{nm} && (3211)\\
&+\sigma_{nm,x}J_{nm,y}\left(1-\delta_{mn}\right)V_{nm}^{\Im}-V_{nm}^{\Im}\sigma_{nm,y}J_{nm,x}\left(1-\delta_{mn}\right)\big) && (3212)\\
&= \sum_{m,n\in P}\big(\sigma_{nm,x}J_{nm,x}\left(1-\delta_{mn}\right)V_{nm}^{\Re}+\sigma_{nm,y}J_{nm,y}\left(1-\delta_{mn}\right)V_{nm}^{\Re}+|n\rangle\langle m|B_{z,n}\delta_{nm} && (3213)\\
&+\sigma_{nm,x}J_{nm,y}\left(1-\delta_{mn}\right)V_{nm}^{\Im}-V_{nm}^{\Im}\sigma_{nm,y}J_{nm,x}\left(1-\delta_{mn}\right)\big) && (3214)\\
&= \sum_{m,n\in P}\big(\sigma_{nm,x}\left(J_{nm,x}\left(1-\delta_{mn}\right)V_{nm}^{\Re}+J_{nm,y}\left(1-\delta_{mn}\right)V_{nm}^{\Im}\right)+|n\rangle\langle m|B_{z,n}\delta_{nm} && (3215)\\
&+\sigma_{nm,y}\left(J_{nm,y}\left(1-\delta_{mn}\right)V_{nm}^{\Re}-V_{nm}^{\Im}J_{nm,x}\left(1-\delta_{mn}\right)\right)\big) && (3216)\\
&= \sum_{m,n\in P}\sigma_{nm,x}\left(J_{nm,x}V_{nm}^{\Re}+J_{nm,y}V_{nm}^{\Im}\right)\left(1-\delta_{mn}\right)+\sum_{m,n\in P}|n\rangle\langle m|B_{z,n}\delta_{nm} && (3217)\\
&+\sum_{m,n\in P}\sigma_{nm,y}\left(J_{nm,y}V_{nm}^{\Re}-V_{nm}^{\Im}J_{nm,x}\right)\left(1-\delta_{mn}\right). && (3218)
\end{aligned}
$$

The interaction hamiltonian in the model studied is divided as $\overline{H_I}=\sum_{i=1}^{N^2}S_i\otimes E_i$ with:

$$
S_i=\begin{cases}|n\rangle\langle n|=S_n^z, & 1\le i\le N,\\ |n\rangle\langle m|+|m\rangle\langle n|=S_{nm}^x, & N<i\le\frac{N(N+1)}{2},\\ \mathrm{i}\left(|n\rangle\langle m|-|m\rangle\langle n|\right)=S_{nm}^y, & \frac{N(N+1)}{2}<i\le N^2.\end{cases} \tag{3219}
$$

By comparison we deduce that:

$$
E_i=\begin{cases}B_{z,n}=E_n^z, & 1\le i\le N,\\ J_{nm,x}V_{nm}^{\Re}+J_{nm,y}V_{nm}^{\Im}=E_{nm}^x, & N<i\le\frac{N(N+1)}{2},\\ J_{nm,y}V_{nm}^{\Re}-V_{nm}^{\Im}J_{nm,x}=E_{nm}^y, & \frac{N(N+1)}{2}<i\le N^2.\end{cases} \tag{3220}
$$

In terms of the notation of the master equation deduced in the precedent section we write the interaction hamiltonian:

$$
E_i=\begin{cases}C_{3nm}B_{3nm}=E_n^z, & 1\le i\le N,\\ C_{1nm}B_{1nm}+C_{4nm}B_{4nm}=E_{nm}^x, & N<i\le\frac{N(N+1)}{2},\\ C_{2nm}B_{2nm}+C_{5nm}B_{5nm}=E_{nm}^y, & \frac{N(N+1)}{2}<i\le N^2.\end{cases} \tag{3221}
$$

Now we are prepared to show the form of the correlation functions as shown:

$$\Lambda_{ij}\left(t-s\right) = \Lambda_{ij}\left(\tau\right) \tag{3222}$$
$$= \mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}\tau H_B}E_i\mathrm{e}^{-\mathrm{i}\tau H_B}E_j\rho_B\right), \tag{3223}$$
$$\Lambda_{ji}^*\left(\tau\right) = \mathrm{Tr}_B\left(\left(\mathrm{e}^{\mathrm{i}\tau H_B}E_j\mathrm{e}^{-\mathrm{i}\tau H_B}E_i\rho_B\right)^{\dagger}\right) \tag{3224}$$
$$= \mathrm{Tr}_B\left(\rho_B E_i\mathrm{e}^{\mathrm{i}\tau H_B}E_j\mathrm{e}^{-\mathrm{i}\tau H_B}\right) \tag{3225}$$
$$= \mathrm{Tr}_B\left(E_i\mathrm{e}^{\mathrm{i}\tau H_B}E_j\mathrm{e}^{-\mathrm{i}\tau H_B}\rho_B\right) \tag{3226}$$
$$= \mathrm{Tr}_B\left(E_i\mathrm{e}^{\mathrm{i}\tau H_B}E_j\rho_B\mathrm{e}^{-\mathrm{i}\tau H_B}\right) \tag{3227}$$
$$= \mathrm{Tr}_B\left(\mathrm{e}^{-\mathrm{i}\tau H_B}E_i\mathrm{e}^{\mathrm{i}\tau H_B}E_j\rho_B\right) \tag{3228}$$
$$= \Lambda_{ij}\left(-\tau\right), \tag{3229}$$
$$\Lambda_{ji}\left(\tau\right) = \Lambda_{ij}^*\left(-\tau\right). \tag{3230}$$

In these terms we have:

$$\Lambda_{3nn',3mm'}\left(t,t'\right) = \delta_{nn'}\delta_{mm'}\sum_{u\mathbf{k}}\left(q_{nu\mathbf{k}}^*\left(t\right)q_{mu\mathbf{k}}\left(t'\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}+q_{nu\mathbf{k}}\left(t\right)q_{mu\mathbf{k}}^*\left(t'\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3231}$$
$$= \delta_{nn'}\delta_{mm'}\sum_{u\mathbf{k}}\Big(\left(g_{nu\mathbf{k}}^*-v_{nu\mathbf{k}}^*(t)\right)\left(g_{mu\mathbf{k}}-v_{mu\mathbf{k}}\left(t'\right)\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}+\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}(t)\right)\left(g_{mu\mathbf{k}}^*-v_{mu\mathbf{k}}^*\left(t'\right)\right) \tag{3232}$$
$$\times N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\Big) \tag{3233}$$
$$= \delta_{nn'}\delta_{mm'}\sum_{u\mathbf{k}}\Big(g_{nu\mathbf{k}}^*g_{mu\mathbf{k}}\left(1-F_{nu}^*(\omega_{u\mathbf{k}},t)\right)\left(1-F_{mu}\left(\omega_{u\mathbf{k}},t'\right)\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}+\left(1-F_{nu}(\omega_{u\mathbf{k}},t)\right) \tag{3234}$$
$$\times g_{nu\mathbf{k}}g_{mu\mathbf{k}}^*\left(1-F_{mu}^*\left(\omega_{u\mathbf{k}},t'\right)\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\Big) \tag{3235}$$
$$\approx \delta_{nn'}\delta_{mm'}\sum_{u}\int_0^{\infty}\Big(L_{nu}^*\left(\omega\right)L_{mu}\left(\omega\right)\left(1-F_{nu}^*\left(\omega,t\right)\right)\left(1-F_{mu}\left(\omega,t'\right)\right)\left(N_u\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}+L_{nu}\left(\omega\right) \tag{3236}$$
$$\times L_{mu}^*(\omega)\left(1-F_{nu}\left(\omega,t\right)\right)\left(1-F_{mu}^*\left(\omega,t'\right)\right)N_u\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\Big)\mathrm{d}\omega \tag{3237}$$
$$= \delta_{nn'}\delta_{mm'}\sum_{u}\delta_{nu}\delta_{mu}\int_0^{\infty}\Big(L_n^*\left(\omega\right)L_m\left(\omega\right)\left(1-F_n^*\left(\omega,t\right)\right)\left(1-F_m\left(\omega,t'\right)\right)\left(N_u\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}+L_n\left(\omega\right) \tag{3238}$$
$$\times L_m^*(\omega)\left(1-F_n\left(\omega,t\right)\right)\left(1-F_m^*\left(\omega,t'\right)\right)N_u\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\Big)\delta_{nm}\delta_{nu}\mathrm{d}\omega \tag{3239}$$
$$= \delta_{nm}\int_0^{\infty}P_n^*\left(\omega\right)P_n\left(\omega\right)\left(\left(N_n\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}+N_n\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{3240}$$
$$= \delta_{nm}\int_0^{\infty}J_n\left(\omega\right)F_n^2\left(\omega\right)\left(\left(N_n\left(\omega\right)+1\right)\left(\cos\left(\omega\tau\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)+N_n\left(\omega\right)\left(\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right)\right)\mathrm{d}\omega \tag{3241}$$
$$= \delta_{nm}\int_0^{\infty}J_n\left(\omega\right)F_n^2\left(\omega\right)\left(\coth\left(\frac{\beta\omega}{2}\right)\cos\left(\omega\tau\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)\mathrm{d}\omega \tag{3242}$$
$$= \delta_{nm}\Lambda_{nn}^z\left(\tau\right), \tag{3243}$$
$$\chi_{nm} = \sum_{u\mathbf{k}}\frac{v_{nu\mathbf{k}}^*v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v_{mu\mathbf{k}}^*}{2\omega_{u\mathbf{k}}^2} \tag{3244}$$
$$= \sum_{u}\left(\sum_{\mathbf{k}}\frac{g_{nu\mathbf{k}}^*g_{mu\mathbf{k}}F_{nu}^*\left(\omega_{u\mathbf{k}}\right)F_{mu}\left(\omega_{u\mathbf{k}}\right)-g_{nu\mathbf{k}}g_{mu\mathbf{k}}^*F_{nu}\left(\omega_{u\mathbf{k}}\right)F_{mu}^*\left(\omega_{u\mathbf{k}}\right)}{2\omega_{u\mathbf{k}}^2}\right) \tag{3245}$$
$$\approx \sum_{u}\int_0^{\infty}\frac{L_n^*\left(\omega\right)L_m\left(\omega\right)F_n^*\left(\omega\right)F_m\left(\omega\right)-L_n\left(\omega\right)L_m^*\left(\omega\right)F_n\left(\omega\right)F_m^*\left(\omega\right)}{2\omega^2}\delta_{nu}\mathrm{d}\omega \tag{3246}$$
$$= 0, \tag{3247}$$
$$B_{nm} = \prod_{u\mathbf{k}}\mathrm{e}^{\frac{v_{nu\mathbf{k}}^*v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v_{mu\mathbf{k}}^*}{2\omega_{u\mathbf{k}}^2}}\prod_{u}\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}^2}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)} \tag{3248}$$
$$\approx \mathrm{e}^0\mathrm{e}^{-\sum_u\int_0^{\infty}\frac{\left|L_m(\omega)F_m(\omega)-L_n(\omega)F_n(\omega)\right|^2}{2\omega^2}\delta_{nu}\delta_{mu}\coth\left(\frac{\beta_u\omega}{2}\right)\mathrm{d}\omega} \tag{3249}$$
$$= \mathrm{e}^{-\int_0^{\infty}\frac{\left|L_n(\omega)F_n(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta_n\omega}{2}\right)\mathrm{d}\omega}\mathrm{e}^{-\sum_u\int_0^{\infty}\frac{\left|L_m(\omega)F_m(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta_m\omega}{2}\right)\mathrm{d}\omega} \tag{3250}$$

$$=\mathrm{e}^{-\int_0^\infty \frac{J_n(\omega)F_n^2(\omega)}{2\omega^2}\coth\left(\frac{\beta_n\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\int_0^\infty \frac{J_m(\omega)F_m^2(\omega)}{2\omega^2}\coth\left(\frac{\beta_m\omega}{2}\right)\mathrm{d}\omega} \tag{3251}$$

$$=\mathrm{e}^{-\int_0^\infty \frac{J_n(\omega)F_n^2(\omega)}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\int_0^\infty \frac{J_m(\omega)F_m^2(\omega)}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3252}$$

$$=B_nB_m, \tag{3253}$$

$$B_{nm}^{\Im}=0, \tag{3254}$$

$$B_{nm}^{\Re}=B_nB_m, \tag{3255}$$

$$\Lambda_{3n'n',1nm}=(1-\delta_{nm})\,\mathrm{i}B_{nm}^{\Im}\sum_u\int_0^\infty\left(P_{n'u}^*(\omega)\,Q_{(nm)u}(\omega)\,(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}P_{n'u}(\omega)\,Q_{(nm)u}^*(\omega)\,N_u(\omega)\right)\mathrm{d}\omega, \tag{3256}$$

$$=0, \tag{3257}$$

$$\Lambda_{3n'm',2nm}(t,t')=-\mathrm{i}\,(1-\delta_{nm})\,\delta_{n'm'}B_{nm}^{\Re}\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}^*\alpha_{(nm)u\mathbf{k}}\,(N_{u\mathbf{k}}+1)\,\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q_{n'u\mathbf{k}}\alpha_{(nm)u\mathbf{k}}^*N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3258}$$

$$\approx-\mathrm{i}\,(1-\delta_{nm})\,\delta_{n'm'}B_{nm}^{\Re}\sum_u\int_0^\infty\left(P_{n'u}^*(\omega)Q_{(nm)u}(\omega)(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}P_{n'u}(\omega)Q_{(nm)u}^*(\omega)N_u(\omega)\right)\mathrm{d}\omega \tag{3259}$$

$$=-\mathrm{i}\,(1-\delta_{nm})\,\delta_{n'm'}B_nB_m\sum_u\int_0^\infty\left(P_{n'u}^*(\omega)Q_{(nm)u}(\omega)(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}P_{n'u}(\omega)Q_{(nm)u}^*(\omega)N_u(\omega)\right)\mathrm{d}\omega \tag{3260}$$

$$=-\mathrm{i}\,(1-\delta_{nm})\,\delta_{n'm'}B_nB_m\sum_u\int_0^\infty\left(P_{n'}^*(\omega)\,\delta_{n'u}Q_{(nm)u}(\omega)\,(N_u(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}P_{n'}(\omega)\,\delta_{n'u}Q_{(nm)u}^*(\omega)\right. \tag{3261}$$

$$\left.\times N_u(\omega)\,\mathrm{d}\omega\right) \tag{3262}$$

$$=-\mathrm{i}\,(1-\delta_{nm})\,\delta_{n'm'}B_nB_m\int_0^\infty\left(P_{n'}^*(\omega)\,Q_{(nm)n'}(\omega)\,(N_{n'}(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}P_{n'}(\omega)\,Q_{(nm)n'}^*(\omega)\,N_{n'}(\omega)\right)\mathrm{d}\omega \tag{3263}$$

$$=-\mathrm{i}\,(1-\delta_{nm})\,\delta_{n'm'}B_nB_m\int_0^\infty\left(L_{n'}^*(\omega)\,(1-F_{n'}(\omega))\,Q_{(nm)n'}(\omega)\,(N_{n'}(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}L_{n'}(\omega)(1-F_{n'}(\omega))\right. \tag{3264}$$

$$\left.\times Q_{(nm)n'}^*(\omega)\,N_{n'}(\omega)\right)\mathrm{d}\omega \tag{3265}$$

$$=-\mathrm{i}\,(1-\delta_{nm})\,\delta_{n'm'}B_nB_m\int_0^\infty\left(L_{n'}^*(\omega)L_{n'}(\omega)(1-F_{n'}(\omega))F_{n'}(\omega)\,\frac{\delta_{nn'}-\delta_{mn'}}{\omega}\,(N_{n'}(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}L_{n'}(\omega)\right. \tag{3266}$$

$$\left.\times L_{n'}^*(\omega)\,(1-F_{n'}(\omega))\,F_{n'}(\omega)\,\frac{\delta_{nn'}-\delta_{mn'}}{\omega}N_{n'}(\omega)\right)\mathrm{d}\omega \tag{3267}$$

$$=-\mathrm{i}\,(1-\delta_{nm})\,(\delta_{nn'}-\delta_{mn'})\,\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}(\omega)F_{n'}(\omega)(1-F_{n'}(\omega))}{\omega}\left((N_{n'}(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-\mathrm{e}^{\mathrm{i}\omega\tau}N_{n'}(\omega)\right)\mathrm{d}\omega \tag{3268}$$

$$=-\mathrm{i}\,(1-\delta_{nm})\,(\delta_{nn'}-\delta_{mn'})\,\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}(\omega)\,F_{n'}(\omega)\,(1-F_{n'}(\omega))}{\omega}\,((N_{n'}(\omega)+1-N_{n'}(\omega))\cos(\omega\tau) \tag{3269}$$

$$-\mathrm{i}\sin(\omega\tau)\,(2N_{n'}(\omega)+1))\,\mathrm{d}\omega \tag{3270}$$

$$=-\mathrm{i}\,(1-\delta_{nm})(\delta_{nn'}-\delta_{mn'})\,\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}(\omega)F_{n'}(\omega)(1-F_{n'}(\omega))}{\omega}\,(\cos(\omega\tau)-\mathrm{i}\sin(\omega\tau)\,(2N_{n'}(\omega)+1))\,\mathrm{d}\omega \tag{3271}$$

$$=-\,(1-\delta_{nm})(\delta_{nn'}-\delta_{mn'})\,\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}(\omega)F_{n'}(\omega)(1-F_{n'}(\omega))}{\omega}\left(\mathrm{i}\cos(\omega\tau)+\sin(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega, \tag{3272}$$

$$\phi_n^{yz}(\tau)\equiv\int_0^\infty\frac{J_n(\omega)\,F_n(\omega)\,(1-F_n(\omega))}{\omega}\left(\mathrm{i}\cos(\omega\tau)+\sin(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega, \tag{3273}$$

$$\Lambda_{3n'm',2nm}(t,t')=-\,(1-\delta_{nm})\,(\delta_{nn'}-\delta_{mn'})\,\delta_{n'm'}B_nB_m\phi_{n'}^{yz}(\tau), \tag{3274}$$

$$U_{(nm)(n'm')}(t,t')=\mathrm{e}^{\mathrm{i}\sum_u\int_0^\infty\left(Q_{(nm)u}(\omega)Q_{(n'm')u}^*(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega} \tag{3275}$$

$$=\mathrm{e}^{\mathrm{i}\sum_u\int_0^\infty\left(\frac{L_n(\omega)F_n(\omega)\delta_{nu}-L_{mu}(\omega)F_{mu}(\omega)\delta_{mu}}{\omega}\frac{L_{n'u}^*(\omega)F_{n'u}(\omega)\delta_{n'u}-L_{m'u}^*(\omega)F_{m'u}(\omega)\delta_{m'u}}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega} \tag{3276}$$

$$=\mathrm{e}^{\mathrm{i}\int_0^\infty\left(\frac{L_n(\omega)F_n(\omega)}{\omega}\frac{L_{n'}^*(\omega)F_{n'}(\omega)\delta_{n'n}-L_{m'}^*(\omega)F_{m'}(\omega)\delta_{m'n}}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega} \tag{3277}$$

$$\times\,\mathrm{e}^{\mathrm{i}\int_0^\infty\left(-\frac{L_m(\omega)F_m(\omega)}{\omega}\frac{L_{n'}^*(\omega)F_{n'}(\omega)\delta_{n'm}-L_{m'}^*(\omega)F_{m'}(\omega)\delta_{m'm}}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega} \tag{3278}$$

$$=\mathrm{e}^{\mathrm{i}\int_0^\infty\left(\frac{L_n(\omega)F_n(\omega)}{\omega}\frac{L_n^*(\omega)F_n(\omega)\delta_{n'n}-L_n^*(\omega)F_n(\omega)\delta_{m'n}}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega}\,\mathrm{e}^{\mathrm{i}\int_0^\infty\left(-\frac{L_m(\omega)F_m(\omega)}{\omega}\frac{L_m^*(\omega)F_m(\omega)\delta_{n'm}-L_m^*(\omega)F_m(\omega)\delta_{m'm}}{\omega}\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega} \tag{3279}$$

$$=\mathrm{e}^{\mathrm{i}\int_0^\infty\left(\frac{|L_n(\omega)|^2F_n^2(\omega)}{\omega^2}(\delta_{n'n}-\delta_{m'n})\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega}\,\mathrm{e}^{\mathrm{i}\int_0^\infty\left(\frac{|L_m(\omega)|^2F_m^2(\omega)}{\omega^2}(\delta_{m'm}-\delta_{n'm})\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega} \tag{3280}$$

$$=\mathrm{e}^{\mathrm{i}\int_0^\infty\left(\frac{J_n(\omega)F_n^2(\omega)}{\omega^2}(\delta_{n'n}-\delta_{m'n})\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega}\,\mathrm{e}^{\mathrm{i}\int_0^\infty\left(\frac{J_m(\omega)F_m^2(\omega)}{\omega^2}(\delta_{m'm}-\delta_{n'm})\mathrm{e}^{\mathrm{i}\omega\tau}\right)^{\Im}\mathrm{d}\omega} \tag{3281}$$

$$=\mathrm{e}^{\mathrm{i}\int_0^\infty\frac{J_n(\omega)F_n^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega\left(\delta_{n'n}-\delta_{m'n}\right)}\,\mathrm{e}^{\mathrm{i}\int_0^\infty\frac{J_m(\omega)F_m^2(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega\left(\delta_{m'm}-\delta_{n'm}\right)}, \tag{3282}$$

$$\xi^{+}_{(nm)(n'm')}(t,t') \approx \mathrm{e}^{-\sum_u \int_0^\infty \frac{\left|Q_{(nm)u}(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}+Q_{(n'm')u}(\omega)\right|^2}{2}\coth\left(\frac{\beta_u\omega}{2}\right)\mathrm{d}\omega} \tag{3283}$$

$$= \mathrm{e}^{-\sum_u \int_0^\infty \frac{\left|(L_{nu}(\omega)F_{nu}(\omega,t)-L_{mu}(\omega)F_{mu}(\omega,t))\mathrm{e}^{\mathrm{i}\omega\tau}+L_{n'u}(\omega)F_{n'u}(\omega,t)-L_{m'u}(\omega)F_{m'u}(\omega,t)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3284}$$

$$= \mathrm{e}^{-\sum_u \int_0^\infty \frac{\left|(L_n(\omega)F_n(\omega)\delta_{nu}-L_m(\omega)F_m(\omega)\delta_{mu})\mathrm{e}^{\mathrm{i}\omega\tau}+L_{n'}(\omega)F_{n'}(\omega)\delta_{n'u}-L_{m'}(\omega)F_{m'}(\omega)\delta_{m'u}\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3285}$$

$$= \mathrm{e}^{-\sum_u \int_0^\infty \frac{\left|(L_n(\omega)F_n(\omega)\delta_{nu}-L_m(\omega)F_m(\omega)\delta_{mu})\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\sum_u \int_0^\infty \frac{\left|L_{n'}(\omega)F_{n'}(\omega)\delta_{n'u}-L_{m'}(\omega)F_{m'}(\omega)\delta_{m'u}\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3286}$$

$$\times\, \mathrm{e}^{-\sum_u \int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}(L_n(\omega)F_n(\omega)\delta_{nu}-L_m(\omega)F_m(\omega)\delta_{mu})\left(L^*_{n'}(\omega)F_{n'}(\omega)\delta_{n'u}-L^*_{m'}(\omega)F_{m'}(\omega)\delta_{m'u}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3287}$$

$$= \mathrm{e}^{-\sum_u \int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}(L_n(\omega)F_n(\omega)\delta_{nu}-L_m(\omega)F_m(\omega)\delta_{mu})\left(L^*_{n'}(\omega)F_{n'}(\omega)\delta_{n'u}-L^*_{m'}(\omega)F_{m'}(\omega)\delta_{m'u}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3288}$$

$$\times\, \mathrm{e}^{-\int_0^\infty \frac{\left|L_{n'}(\omega)F_{n'}(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\int_0^\infty \frac{\left|L_{m'}(\omega)F_{m'}(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\int_0^\infty \frac{\left|L_m(\omega)F_m(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3289}$$

$$\times\, \mathrm{e}^{-\int_0^\infty \frac{\left|L_n(\omega)F_n(\omega)\right|^2}{2\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3290}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\sum_u \int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}(L_n(\omega)F_n(\omega)\delta_{nu}-L_m(\omega)F_m(\omega)\delta_{mu})\left(L^*_{n'}(\omega)F_{n'}(\omega)\delta_{n'u}-L^*_{m'}(\omega)F_{m'}(\omega)\delta_{m'u}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3291}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\sum_u \int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}(L_n(\omega)F_n(\omega)\delta_{nu}-L_m(\omega)F_m(\omega)\delta_{mu})\left(L^*_{n'}(\omega)F_{n'}(\omega)\delta_{n'u}-L^*_{m'}(\omega)F_{m'}(\omega)\delta_{m'u}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3292}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}(L_n(\omega)F_n(\omega))\left(L^*_{n'}(\omega)F_{n'}(\omega)\delta_{n'n}-L^*_{m'}(\omega)F_{m'}(\omega)\delta_{m'n}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3293}$$

$$\times\, \mathrm{e}^{-\int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}(-L_m(\omega)F_m(\omega))\left(L^*_{n'}(\omega)F_{n'}(\omega)\delta_{n'm}-L^*_{m'}(\omega)F_{m'}(\omega)\delta_{m'm}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3294}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}L_n(\omega)F_n(\omega)\left(L^*_{n}(\omega)F_{n}(\omega)\delta_{n'n}-L^*_{n}(\omega)F_{n}(\omega)\delta_{m'n}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3295}$$

$$\times\, \mathrm{e}^{-\int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}(L_m(\omega)F_m(\omega))\left(L^*_{m}(\omega)F_{m}(\omega)\delta_{m'm}-L^*_{m}(\omega)F_{m}(\omega)\delta_{n'm}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3296}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}J_n(\omega)F^2_n(\omega)\left(\delta_{n'n}-\delta_{m'n}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\int_0^\infty \frac{\left(\mathrm{e}^{\mathrm{i}\omega\tau}J_m(\omega)F^2_m(\omega)\left(\delta_{m'm}-\delta_{n'm}\right)\right)^{\Re}}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3297}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\int_0^\infty \frac{\cos(\omega\tau)J_n(\omega)F^2_n(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\left(\delta_{m'm}-\delta_{n'm}\right)\int_0^\infty \frac{\cos(\omega\tau)J_m(\omega)F^2_m(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}, \tag{3298}$$

$$\xi^{-}_{(nm)(n'm')}(t,t') = B_nB_mB_{n'}B_{m'}\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\int_0^\infty \frac{\cos(\omega\tau)J_n(\omega)F^2_n(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{\left(\delta_{m'm}-\delta_{n'm}\right)\int_0^\infty \frac{\cos(\omega\tau)J_m(\omega)F^2_m(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}. \tag{3299}$$

The composed terms are:

$$U_{(nm)(n'm')}(t,t')\xi^{+}_{(nm)(n'm')}(t,t') = B_nB_mB_{n'}B_{m'}\mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J_n(\omega)F^2_n(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega\left(\delta_{n'n}-\delta_{m'n}\right)}\,\mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J_m(\omega)F^2_m(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega\left(\delta_{m'm}-\delta_{n'm}\right)} \tag{3300}$$

$$\times\, \mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\int_0^\infty \frac{\cos(\omega\tau)J_n(\omega)F^2_n(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega}\,\mathrm{e}^{-\left(\delta_{m'm}-\delta_{n'm}\right)\int_0^\infty \frac{\cos(\omega\tau)J_m(\omega)F^2_m(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3301}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\mathrm{i}\int_0^\infty \frac{J_n(\omega)F^2_n(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\,\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\int_0^\infty \frac{\cos(\omega\tau)J_n(\omega)F^2_n(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3302}$$

$$\times\, \mathrm{e}^{\left(\delta_{m'm}-\delta_{n'm}\right)\mathrm{i}\int_0^\infty \frac{J_m(\omega)F^2_m(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\,\mathrm{e}^{-\left(\delta_{m'm}-\delta_{n'm}\right)\int_0^\infty \frac{\cos(\omega\tau)J_m(\omega)F^2_m(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3303}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\mathrm{i}\int_0^\infty \frac{J_n(\omega)F^2_n(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega}\,\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\int_0^\infty \frac{\cos(\omega\tau)J_n(\omega)F^2_n(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3304}$$

$$\times\, \mathrm{e}^{\mathrm{i}\int_0^\infty \frac{J_m(\omega)F^2_m(\omega)}{\omega^2}\sin(\omega\tau)\mathrm{d}\omega\left(\delta_{m'm}-\delta_{n'm}\right)}\,\mathrm{e}^{-\left(\delta_{m'm}-\delta_{n'm}\right)\int_0^\infty \frac{\cos(\omega\tau)J_m(\omega)F^2_m(\omega)}{\omega^2}\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega} \tag{3305}$$

$$= B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\int_0^\infty \frac{J_n(\omega)F^2_n(\omega)}{\omega^2}\left(-\mathrm{i}\sin(\omega\tau)+\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega} \tag{3306}$$

$$\times\, \mathrm{e}^{-\left(\delta_{m'm}-\delta_{n'm}\right)\int_0^\infty \frac{J_m(\omega)F^2_m(\omega)}{\omega^2}\left(-\mathrm{i}\sin(\omega\tau)+\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega}, \tag{3307}$$

$$\phi^{xy}_n(\tau) \equiv \int_0^\infty \frac{J_n(\omega)F^2_n(\omega)}{\omega^2}\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin(\omega\tau)\right)\mathrm{d}\omega, \tag{3308}$$

$$U_{(nm)(n'm')}(t,t')\xi^{+}_{(nm)(n'm')}(t,t') = B_nB_mB_{n'}B_{m'}\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_n(\tau)-\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_m(\tau)}, \tag{3309}$$

$$U^*_{(nm)(n'm')}(t,t')\xi^{-}_{(nm)(n'm')}(t,t') = B_nB_mB_{n'}B_{m'}\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_n(\tau)+\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_m(\tau)}, \tag{3310}$$

$$\Lambda_{2nm,2n'm'}\left(t,t'\right)=-\left(1-\delta_{nm}\right)(1-\delta_{n'm'})\Bigg(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re}-U^{*}_{(nm)(m'n')}\left(t,t'\right)\right. \tag{3311}$$

$$\left.\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\right)+B^{\Im}_{nm}\left(t\right)B^{\Im}_{n'm'}\left(t'\right)\Bigg) \tag{3312}$$

$$=-\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\left(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)-U^{*}_{(nm)(m'n')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\right)\right) \tag{3313}$$

$$=-\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{2}\left(\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)-\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}-\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)+\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}\right) \tag{3314}$$

$$\times B_nB_mB_{n'}B_{m'} \tag{3315}$$

$$=\frac{(1-\delta_{nm})(1-\delta_{n'm'})}{2}\left(\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)+\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}-\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)-\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}\right) \tag{3316}$$

$$\times B_nB_mB_{n'}B_{m'} \tag{3317}$$

$$\Lambda_{2nm,1n'm'}\left(t,t'\right)=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\Bigg(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Im}+U^{*}_{(nm)(n'm')}\left(t,t'\right)\right. \tag{3318}$$

$$\left.\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Im}\right)-B^{\Im}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\Bigg) \tag{3319}$$

$$=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\Bigg(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\cdot 0+U^{*}_{(nm)(n'm')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\cdot 0\right) \tag{3320}$$

$$-0\cdot B^{\Re}_{n'm'}\left(t'\right)\Bigg) \tag{3321}$$

$$=0, \tag{3322}$$

$$\Lambda_{1nm,1n'm'}\left(t,t'\right)=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\Bigg(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)+\chi_{n'm'}(t')}\right)^{\Re}+U^{*}_{(nm)(m'n')}\left(t,t'\right)\right. \tag{3323}$$

$$\left.\times\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\left(\mathrm{e}^{\chi_{nm}(t)-\chi_{n'm'}(t')}\right)^{\Re}\right)-B^{\Re}_{nm}\left(t\right)B^{\Re}_{n'm'}\left(t'\right)\Bigg) \tag{3324}$$

$$=(1-\delta_{nm})(1-\delta_{n'm'})\Bigg(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)+U^{*}_{(nm)(m'n')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\right)-B_n \tag{3325}$$

$$\times B_mB_{n'}B_{m'}) \tag{3326}$$

$$=(1-\delta_{nm})(1-\delta_{n'm'})\Bigg(\frac{1}{2}\left(U_{(nm)(n'm')}\left(t,t'\right)\xi^{+}_{(nm)(n'm')}\left(t,t'\right)+U^{*}_{(nm)(m'n')}\left(t,t'\right)\xi^{-}_{(nm)(n'm')}\left(t,t'\right)\right)-B_n \tag{3327}$$

$$\times B_mB_{n'}B_{m'}) \tag{3328}$$

$$=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\Bigg(\frac{1}{2}\left(\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)-\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}B_nB_mB_{n'}B_{m'}+\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)}\right. \tag{3329}$$

$$\left.\times\mathrm{e}^{\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}B_nB_mB_{n'}B_{m'}\right)-B_nB_mB_{n'}B_{m'}\Bigg) \tag{3330}$$

$$=(1-\delta_{nm})(1-\delta_{n'm'})\left(\frac{1}{2}\left(\mathrm{e}^{-\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)-\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}+\mathrm{e}^{\left(\delta_{n'n}-\delta_{m'n}\right)\phi^{xy}_{n}(\tau)+\left(\delta_{m'm}-\delta_{n'm}\right)\phi^{xy}_{m}(\tau)}\right)-1\right) \tag{3331}$$

$$\times B_nB_mB_{n'}B_{m'}, \tag{3332}$$

$$\Lambda_{1nm,2n'm'}\left(t,t'\right)=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\Bigg(\frac{1}{2}\left(U^{*}_{(nm)(n'm')}\left(t',t\right)\xi^{+}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')+\chi_{n'm'}(t)}\right)^{\Im}+U_{(nm)(n'm')}\left(t',t\right)\right. \tag{3333}$$

$$\left.\times\xi^{-}_{(nm)(n'm')}\left(t',t\right)\left(\mathrm{e}^{\chi_{nm}(t')-\chi_{n'm'}(t)}\right)^{\Im}\right)-B^{\Im}_{nm}\left(t'\right)B^{\Re}_{n'm'}\left(t\right)\Bigg) \tag{3334}$$

$$=\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right)\Bigg(\frac{1}{2}\left(U^{*}_{(nm)(n'm')}\left(t',t\right)\xi^{+}_{(nm)(n'm')}\left(t',t\right)\cdot 0+U_{(nm)(n'm')}\left(t',t\right)\xi^{-}_{(nm)(n'm')}\left(t',t\right)\cdot 0\right) \tag{3335}$$

$$-0\cdot B^{\Re}_{n'm'}\left(t\right)\Bigg) \tag{3336}$$

$$=0, \tag{3337}$$

$$\Lambda_{1nm,3n'm'}\left(t,t'\right)=-\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B^{\Im}_{nm}\left(t\right)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}\left(t\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3338}$$

$$\approx-\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B^{\Im}_{nm}\left(t\right)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}\left(t\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3339}$$

$$=-\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}\cdot 0\cdot\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}\left(t\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3340}$$

$$=0, \tag{3341}$$

$$\Lambda_{2nm,3n'm'}\left(t,t'\right)=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B^{\Re}_{nm}\left(t\right)\sum_{u\mathbf{k}}\left(q_{n'u\mathbf{k}}\left(t'\right)\alpha^{*}_{(nm)u\mathbf{k}}\left(t\right)\left(N_{u\mathbf{k}}+1\right)\mathrm{e}^{-\mathrm{i}\omega_{u\mathbf{k}}\tau}-q^{*}_{n'u\mathbf{k}}\left(t'\right)\alpha_{(nm)u\mathbf{k}}\left(t\right)N_{u\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega_{u\mathbf{k}}\tau}\right) \tag{3342}$$

$$\approx \mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\sum_u\int_0^\infty\left(P_{n'u}\left(\omega\right)Q^*_{(nm)u}\left(\omega\right)\left(N_u\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P^*_{n'u}\left(\omega\right)Q_{(nm)u}\left(\omega\right)N_u\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{3343}$$
$$=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\left(P_{n'}\left(\omega\right)Q^*_{(nm)n'}\left(\omega\right)\left(N_{n'}\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-P^*_{n'}\left(\omega\right)Q_{(nm)n'}\left(\omega\right)N_{n'}\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{3344}$$
$$=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\left(L_{n'}(\omega)(1-F_{n'}(\omega))\,L^*_{n'}(\omega)F_{n'}(\omega)\,\frac{\delta_{nn'}-\delta_{mn'}}{\omega}\left(N_{n'}(\omega)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-L^*_{n'}(\omega)(1-F_{n'}(\omega))L_{n'}(\omega)\right. \tag{3345}$$
$$\left.\times F_{n'}\left(\omega\right)\frac{\delta_{nn'}-\delta_{mn'}}{\omega}N_{n'}\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{3346}$$
$$=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}\left(\omega\right)(1-F_{n'}\left(\omega\right))\,F_{n'}\left(\omega\right)}{\omega}\left((\delta_{nn'}-\delta_{mn'})(N_{n'}\left(\omega\right)+1)\,\mathrm{e}^{-\mathrm{i}\omega\tau}-(\delta_{nn'}-\delta_{mn'})N_{n'}\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{3347}$$
$$=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}\left(\omega\right)\left(1-F_{n'}\left(\omega\right)\right)F_{n'}\left(\omega\right)}{\omega}\left(\delta_{nn'}-\delta_{mn'}\right)\left(\left(N_{n'}\left(\omega\right)+1\right)\mathrm{e}^{-\mathrm{i}\omega\tau}-N_{n'}\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega\tau}\right)\mathrm{d}\omega \tag{3348}$$
$$=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}\left(\omega\right)(1-F_{n'}\left(\omega\right))F_{n'}\left(\omega\right)}{\omega}\left(\delta_{nn'}-\delta_{mn'}\right)((N_{n'}(\omega)+1)\left(\cos\left(\omega\tau\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)-N_{n'}(\omega)(\cos\left(\omega\tau\right) \tag{3349}$$
$$+\mathrm{i}\sin\left(\omega\tau\right)))\,\mathrm{d}\omega \tag{3350}$$
$$=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}\left(\omega\right)\left(1-F_{n'}\left(\omega\right)\right)F_{n'}\left(\omega\right)}{\omega}\left(\delta_{nn'}-\delta_{mn'}\right)\left(\cos\left(\omega\tau\right)-\mathrm{i}\sin\left(\omega\tau\right)\left(2N_{n'}\left(\omega\right)+1\right)\right)\mathrm{d}\omega \tag{3351}$$
$$=\mathrm{i}\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}\left(\omega\right)\left(1-F_{n'}\left(\omega\right)\right)F_{n'}\left(\omega\right)}{\omega}\left(\delta_{nn'}-\delta_{mn'}\right)\left(\cos\left(\omega\tau\right)-\mathrm{i}\sin\left(\omega\tau\right)\coth\left(\frac{\beta_{n'}\omega}{2}\right)\right)\mathrm{d}\omega \tag{3352}$$
$$=\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\int_0^\infty\frac{J_{n'}\left(\omega\right)\left(1-F_{n'}\left(\omega\right)\right)F_{n'}\left(\omega\right)}{\omega}\left(\delta_{nn'}-\delta_{mn'}\right)\left(\mathrm{i}\cos\left(\omega\tau\right)+\sin\left(\omega\tau\right)\coth\left(\frac{\beta_{n'}\omega}{2}\right)\right)\mathrm{d}\omega \tag{3353}$$
$$=\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\left(\delta_{nn'}-\delta_{mn'}\right)\int_0^\infty\frac{J_{n'}\left(\omega\right)\left(1-F_{n'}\left(\omega\right)\right)F_{n'}\left(\omega\right)}{\omega}\left(\mathrm{i}\cos\left(\omega\tau\right)+\sin\left(\omega\tau\right)\coth\left(\frac{\beta_{n'}\omega}{2}\right)\right)\mathrm{d}\omega \tag{3354}$$
$$=\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\left(\delta_{nn'}-\delta_{mn'}\right)\phi^{yz}_{n'}\left(\tau\right). \tag{3355}$$

We can summarize:

$$\Lambda_{3nn',3mm'}\left(\tau\right)=\delta_{nm}\Lambda^z_{nn}\left(\tau\right), \tag{3356}$$
$$\Lambda_{3n'm',2nm}\left(\tau\right)=-\left(1-\delta_{nm}\right)\left(\delta_{nn'}-\delta_{mn'}\right)\delta_{n'm'}B_nB_m\phi^{yz}_{n'}\left(\tau\right), \tag{3357}$$
$$\Lambda_{3n'n',1nm}\left(\tau\right)=0, \tag{3358}$$
$$\Lambda_{2nm,2n'm'}\left(\tau\right)=\frac{1}{2}\left(\mathrm{e}^{(\delta_{n'n}-\delta_{m'n})\phi^{xy}_n(\tau)+(\delta_{m'm}-\delta_{n'm})\phi^{xy}_m(\tau)}-\mathrm{e}^{-(\delta_{n'n}-\delta_{m'n})\phi^{xy}_n(\tau)-(\delta_{m'm}-\delta_{n'm})\phi^{xy}_m(\tau)}\right)B_nB_m \tag{3359}$$
$$\times B_{n'}B_{m'}\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right), \tag{3360}$$
$$\Lambda_{2nm,1n'm'}\left(\tau\right)=0, \tag{3361}$$
$$\Lambda_{1nm,1n'm'}\left(t,t'\right)=\frac{1}{2}\left(\mathrm{e}^{(\delta_{n'n}-\delta_{m'n})\phi^{xy}_n(\tau)+(\delta_{m'm}-\delta_{n'm})\phi^{xy}_m(\tau)}+\mathrm{e}^{-(\delta_{n'n}-\delta_{m'n})\phi^{xy}_n(\tau)-(\delta_{m'm}-\delta_{n'm})\phi^{xy}_m(\tau)}-2\right)B_n \tag{3362}$$
$$\times B_mB_{n'}B_{m'}\left(1-\delta_{nm}\right)\left(1-\delta_{n'm'}\right), \tag{3363}$$
$$\Lambda_{1nm,2n'm'}\left(t,t'\right)=0, \tag{3364}$$
$$\Lambda_{1nm,3n'm'}\left(t,t'\right)=0, \tag{3365}$$
$$\Lambda_{2nm,3n'm'}\left(t,t'\right)=\left(1-\delta_{nm}\right)\delta_{n'm'}B_nB_m\left(\delta_{nn'}-\delta_{mn'}\right)\phi^{yz}_{n'}\left(\tau\right). \tag{3366}$$

In terms of the notation of the master equation deduced in the precedent section we write the interaction hamiltonian:

$$\overline{H_I}=\sum_{j\in J,p\in P}C_{jp}\left(A_{jp}\otimes B_{jp}\right) \tag{3367}$$
$$=\sum_{m,n\in P}\sigma_{nm,x}\left(J_{nm,x}\left(1-\delta_{mn}\right)V^{\Re}_{nm}+J_{nm,y}\left(1-\delta_{mn}\right)V^{\Im}_{nm}\right)+\sum_{m,n\in P}|n\rangle\langle m|B_{z,n}\delta_{nm}+\sum_{m,n\in P}\sigma_{nm,y}\left(J_{nm,y}\right. \tag{3368}$$
$$\left.\times\left(1-\delta_{mn}\right)V^{\Re}_{nm}-V^{\Im}_{nm}J_{nm,x}\left(1-\delta_{mn}\right)\right) \tag{3369}$$
$$=\sum_{m,n\in P}\sigma_{nm,x}\left(B_{1nm}C_{1nm}+B_{4nm}C_{4nm}\right)+\sum_{m,n\in P}|n\rangle\langle m|B_{3nm}C_{3nm}+\sum_{m,n\in P}\sigma_{nm,y}\left(B_{2nm}C_{2nm}+B_{5nm}C_{5nm}\right). \tag{3370}$$

Then we obtain:

$$E_i = \begin{cases} B_{3nm}C_{3nm} = E^z_n, & 1 \leq i \leq N, \\ B_{1nm}C_{1nm} + B_{4nm}C_{4nm} = E^x_{nm}, & N < i \leq \frac{N(N+1)}{2}, \\ B_{2nm}C_{2nm} + B_{5nm}C_{5nm} = E^y_{nm}, & \frac{N(N+1)}{2} < i \leq N^2. \end{cases} \tag{3371}$$

Now we can find the correlation functions in terms of the notation of the paper studied and recalling that $V^{\Re}_{nm} = V_{nm}$ in [6]:

$$\Lambda_{3nn,3nn}(\tau) = \mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}\tau H_B} B_{3nn}C_{3nn}\mathrm{e}^{-\mathrm{i}\tau H_B} B_{3nn}C_{3nn}\rho_B\right) \tag{3372}$$

$$= \Lambda^z_{nn}(\tau), \tag{3373}$$

$$\Lambda^{xx}_{nmpq}(\tau) = \mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}\tau H_B}\left(C_{1nm}B_{1nm} + C_{4nm}B_{4nm}\right)\mathrm{e}^{-\mathrm{i}\tau H_B}\left(C_{1pq}B_{1pq} + C_{4pq}B_{4pq}\right)\rho_B\right) \tag{3374}$$

$$= V_{nm}V_{pq}\Lambda_{1nm,1pq}(\tau) \tag{3375}$$

$$= (1-\delta_{nm})(1-\delta_{pq})\frac{B_nB_mB_pB_q}{2}\left(\mathrm{e}^{\left(\delta_{pn}-\delta_{qn}\right)\phi^{xy}_n(\tau)+\left(\delta_{qm}-\delta_{pm}\right)\phi^{xy}_m(\tau)} + \mathrm{e}^{-\left(\delta_{pn}-\delta_{qn}\right)\phi^{xy}_n(\tau)-\left(\delta_{qm}-\delta_{pm}\right)\phi^{xy}_m(\tau)} - 2\right) \tag{3376}$$

$$\times V_{nm}V_{pq} \tag{3377}$$

$$= (1-\delta_{nm})(1-\delta_{pq})\frac{1}{2}V_{nm}V_{pq}B_nB_mB_pB_q\left(\mathrm{e}^{\delta_{pn}\phi^{xy}_n(\tau)+\delta_{qm}\phi^{xy}_m(\tau)} + \mathrm{e}^{-\delta_{pn}\phi^{xy}_n(\tau)-\delta_{qm}\phi^{xy}_m(\tau)} - 2\right), \tag{3378}$$

$$\Lambda^{yy}_{nmpq}(\tau) = \mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}\tau H_B}\left(B_{2nm}C_{2nm} + B_{5nm}C_{5nm}\right)\mathrm{e}^{-\mathrm{i}\tau H_B}\left(C_{2pq}B_{2pq} + C_{5pq}B_{5pq}\right)\rho_B\right) \tag{3379}$$

$$= C_{2nm}C_{2pq}\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}\tau H_B}B_{2nm}\mathrm{e}^{-\mathrm{i}\tau H_B}B_{2pq}\rho_B\right) \tag{3380}$$

$$= V_{nm}V_{pq}\Lambda_{2nm,2pq}(\tau) \tag{3381}$$

$$= (1-\delta_{nm})(1-\delta_{pq})\frac{1}{2}B_nB_mB_pB_q\left(\mathrm{e}^{\left(\delta_{pn}-\delta_{qn}\right)\phi^{xy}_n(\tau)+\left(\delta_{qm}-\delta_{pm}\right)\phi^{xy}_m(\tau)} - \mathrm{e}^{-\left(\delta_{pn}-\delta_{qn}\right)\phi^{xy}_n(\tau)-\left(\delta_{qm}-\delta_{pm}\right)\phi^{xy}_m(\tau)}\right) \tag{3382}$$

$$\times V_{nm}V_{pq} \tag{3383}$$

$$= (1-\delta_{nm})(1-\delta_{pq})\frac{1}{2}V_{nm}V_{pq}B_nB_mB_pB_q\left(\mathrm{e}^{\delta_{pn}\phi^{xy}_n(\tau)+\left(\delta_{qm}-\delta_{pm}\right)\phi^{xy}_m(\tau)} - \mathrm{e}^{-\delta_{pn}\phi^{xy}_n(\tau)-\left(\delta_{qm}-\delta_{pm}\right)\phi^{xy}_m(\tau)}\right), \tag{3384}$$

$$\Lambda^{yz}_{nmp}(\tau) = C_{2nm}C_{3pp}\mathrm{Tr}_B\left(\mathrm{e}^{\mathrm{i}\tau H_B}B_{2nm}\mathrm{e}^{-\mathrm{i}\tau H_B}B_{3pp}\rho_B\right) \tag{3385}$$

$$= V_{nm}\Lambda_{2nm,3pp}(\tau) \tag{3386}$$

$$= V_{nm}\Lambda_{2nm,3pp}(\tau) \tag{3387}$$

$$= (1-\delta_{nm})V_{nm}B_nB_m\left(\delta_{np}-\delta_{mp}\right)\phi^{yz}_p(\tau) \tag{3388}$$

$$= (1-\delta_{nm})\delta_{np}V_{nm}B_nB_m\phi^{yz}_n(\tau). \tag{3389}$$

This finally proves that we can obtain the master equation of the cited paper as a limit of the general multibath-multisite time dependent master equation.

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