

# A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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## I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1)$$

$$H_S(t) = \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|, \quad (2)$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (3)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (4)$$

For the states  $|0\rangle, |1\rangle$  we have the orthonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \quad (5)$$

## II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to  $H(t)$ , the unitary transformation defined by  $e^{\pm V}$  where  $V$  is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left( \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right). \quad (6)$$

in terms of the variational scalar parameters  $v_{i\mathbf{k}}$  defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}}(t). \quad (7)$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}} \left( \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \quad (8)$$

$$V = |0\rangle\langle 0| \hat{\varphi}_0 + |1\rangle\langle 1| \hat{\varphi}_1. \quad (9)$$

Here  $*$  denotes the complex conjugate. Expanding  $e^{\pm V}$  using the notation (6) will give us the following result:

$$e^{\pm V} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1)} \quad (10)$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + \dots \quad (11)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + \dots \quad (12)$$

$$= |0\rangle\langle 0| \left( \mathbb{I} \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm \dots \right) + |1\rangle\langle 1| \left( \mathbb{I} \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm \dots \right) \quad (13)$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} \quad (14)$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*b_{\mathbf{k}})} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*b_{\mathbf{k}})} \quad (15)$$

$$= |0\rangle\langle 0|B_0^\pm + |1\rangle\langle 1|B_1^\pm, \quad (16)$$

$$B_i^\pm \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (17)$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} e^{\frac{t^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} e^{-\frac{t^4}{24}([[[X,Y],X],X] + 3[[[X,Y],X],Y] + 3[[[X,Y],Y],Y])} \dots \quad (18)$$

Since  $\left[ \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'} \right] = 0$  for all  $\mathbf{k}', \mathbf{k}$  and  $i, j$  we can show making  $t = 1$  in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right)} = e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right]} \dots \quad (19)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}0} \dots \quad (20)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}}. \quad (21)$$

By induction of this result we can write an expression of  $B_i^\pm$  (shown in equation (17)) as a product of exponentials, which we will call “displacement” operators  $D(\pm v_{i\mathbf{k}})$ :

$$D(\pm v_{i\mathbf{k}}) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, \quad (22)$$

$$B_i^\pm = \prod_{\mathbf{k}} D\left(\pm \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right). \quad (23)$$

this will help us to write operators  $O$  transformed in the variational frame as:

$$\overline{O} \equiv e^V O e^{-V}. \quad (24)$$

We will use the following identities:

$$\overline{|0\chi 0\rangle} = e^V |0\chi 0\rangle e^{-V} \quad (25)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |0\chi 0\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (26)$$

$$= (|0\chi 0\rangle |0\chi 0\rangle B_0^+ + |1\chi 1\rangle |0\chi 0\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (27)$$

$$= |0\chi 0\rangle B_0^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (28)$$

$$= |0\chi 0\rangle |0\chi 0\rangle B_0^+ B_0^- + |0\chi 0\rangle |1\chi 1\rangle B_0^+ B_1^- \quad (29)$$

$$= |0\chi 0\rangle, \quad (30)$$

$$\overline{|1\chi 1\rangle} = (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |1\chi 1\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (31)$$

$$= (|0\chi 0\rangle |1\chi 1\rangle B_0^+ + |1\chi 1\rangle |1\chi 1\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (32)$$

$$= |1\chi 1\rangle B_1^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (33)$$

$$= |1\chi 1\rangle |0\chi 0\rangle B_1^+ B_0^- + B_1^+ |1\chi 1\rangle |1\chi 1\rangle B_1^- \quad (34)$$

$$= B_1^+ |1\chi 1\rangle |1\chi 1\rangle B_1^- \quad (35)$$

$$= |1\chi 1\rangle, \quad (36)$$

$$\overline{|0\chi 1\rangle} = e^V |0\chi 1\rangle e^{-V} \quad (37)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |0\chi 1\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (38)$$

$$= (|0\chi 0\rangle |0\chi 1\rangle B_0^+ + |1\chi 1\rangle |0\chi 1\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (39)$$

$$= (|0\chi 0\rangle |0\chi 1\rangle B_0^+ + |1\chi 1\rangle |0\chi 1\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (40)$$

$$= |0\chi 1\rangle B_0^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (41)$$

$$= |0\chi 1\rangle |0\chi 0\rangle B_0^+ B_0^- + |0\chi 1\rangle |1\chi 1\rangle B_0^+ B_1^- \quad (42)$$

$$= |0\chi 1\rangle B_0^+ B_1^-, \quad (43)$$

$$\overline{|1\chi 0\rangle} = e^V |1\chi 0\rangle e^{-V} \quad (44)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) |1\chi 0\rangle (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (45)$$

$$= (|0\chi 0\rangle |1\chi 0\rangle B_0^+ + |1\chi 1\rangle |1\chi 0\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (46)$$

$$= (|0\chi 0\rangle |1\chi 0\rangle B_0^+ + |1\chi 1\rangle |1\chi 0\rangle B_1^+) (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (47)$$

$$= |1\chi 0\rangle B_1^+ (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (48)$$

$$= |1\chi 0\rangle B_1^+ |0\chi 0\rangle B_0^- + |1\chi 0\rangle B_1^+ |1\chi 1\rangle B_1^- \quad (49)$$

$$= |1\chi 0\rangle |0\chi 0\rangle B_1^+ B_0^- + |1\chi 0\rangle |1\chi 1\rangle B_1^+ B_1^- \quad (50)$$

$$= |1\chi 0\rangle B_1^+ B_0^-, \quad (51)$$

$$\overline{b_{\mathbf{k}}} = e^V b_{\mathbf{k}} e^{-V} \quad (52)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) b_{\mathbf{k}} (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (53)$$

$$= |0\chi 0\rangle B_0^+ b_{\mathbf{k}} B_0^- |0\chi 0\rangle + |0\chi 0\rangle B_0^+ b_{\mathbf{k}} |1\chi 1\rangle B_1^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}} |0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}} B_1^- |1\chi 1\rangle \quad (54)$$

$$= |0\chi 0\rangle |0\chi 0\rangle B_0^+ b_{\mathbf{k}} B_0^- + |0\chi 0\rangle |1\chi 1\rangle B_0^+ b_{\mathbf{k}} B_1^- + |1\chi 1\rangle |0\chi 0\rangle B_1^+ b_{\mathbf{k}} B_0^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}} B_1^- \quad (55)$$

$$= |0\chi 0\rangle \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\chi 1\rangle \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (56)$$

$$= (|0\chi 0\rangle + |1\chi 1\rangle) b_{\mathbf{k}} - |1\chi 1\rangle \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\chi 0\rangle \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (57)$$

$$= b_{\mathbf{k}} - |1\chi 1\rangle \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\chi 0\rangle \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (58)$$

$$\overline{b_{\mathbf{k}}}^\dagger = e^V b_{\mathbf{k}}^\dagger e^{-V} \quad (59)$$

$$= (|0\chi 0\rangle B_0^+ + |1\chi 1\rangle B_1^+) b_{\mathbf{k}}^\dagger (|0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^-) \quad (60)$$

$$= |0\chi 0\rangle B_0^+ b_{\mathbf{k}}^\dagger B_0^- |0\chi 0\rangle + |0\chi 0\rangle B_0^+ b_{\mathbf{k}}^\dagger |1\chi 1\rangle B_1^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}}^\dagger |0\chi 0\rangle B_0^- + |1\chi 1\rangle B_1^+ b_{\mathbf{k}}^\dagger B_1^- |1\chi 1\rangle \quad (61)$$

$$= |0\chi 0\rangle |0\chi 0\rangle B_0^+ b_{\mathbf{k}}^\dagger B_0^- + |0\chi 0\rangle |1\chi 1\rangle B_0^+ b_{\mathbf{k}}^\dagger B_1^- + |1\chi 1\rangle |0\chi 0\rangle B_1^+ b_{\mathbf{k}}^\dagger B_0^- + |1\chi 1\rangle |1\chi 1\rangle B_1^+ b_{\mathbf{k}}^\dagger B_1^- \quad (62)$$

$$= |0\chi 0\rangle \left( b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\chi 1\rangle \left( b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \quad (63)$$

$$= b_{\mathbf{k}}^\dagger - |1\chi 1\rangle \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} - |0\chi 0\rangle \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (64)$$

We have used the following results as well to obtain the transformed  $b_{\mathbf{k}}$  and  $b_{\mathbf{k}}^\dagger$ :

$$B_i^+ b_{\mathbf{k}} B_i^- = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (65)$$

$$B_i^+ b_{\mathbf{k}}^\dagger B_i^- = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (66)$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t) |0\rangle\langle 0|} = \varepsilon_0(t) |0\rangle\langle 0|, \quad (67)$$

$$\overline{\varepsilon_1(t) |1\rangle\langle 1|} = \varepsilon_1(t) |1\rangle\langle 1|, \quad (68)$$

$$\overline{V_{10}(t) |1\rangle\langle 0|} = V_{10}(t) |1\rangle\langle 0| B_1^+ B_0^-, \quad (69)$$

$$\overline{V_{01}(t) |0\rangle\langle 1|} = V_{01}(t) |0\rangle\langle 1| B_0^+ B_1^-, \quad (70)$$

$$\overline{g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}}} = g_{i\mathbf{k}} \left( |0\rangle\langle 0| \left( b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) + g_{i\mathbf{k}}^* \left( |0\rangle\langle 0| \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (71)$$

$$= g_{i\mathbf{k}} \left( (|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left( (|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (72)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| \quad (73)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left( g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |0\rangle\langle 0| - \left( g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |1\rangle\langle 1|, \quad (74)$$

$$\overline{|0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right)} = (|0\rangle\langle 0| B_0^+ + |1\rangle\langle 1| B_1^+) |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_0^- + |1\rangle\langle 1| B_1^-) \quad (75)$$

$$= |0\rangle\langle 0| B_0^+ |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) |0\rangle\langle 0| B_0^- \quad (76)$$

$$= |0\rangle\langle 0| B_0^+ \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) B_0^- \quad (77)$$

$$= |0\rangle\langle 0| \left( g_{0\mathbf{k}} \left( b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (78)$$

$$\overline{|1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right)} = (|0\rangle\langle 0| B_0^+ + |1\rangle\langle 1| B_1^+) |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_0^- + |1\rangle\langle 1| B_1^-) \quad (79)$$

$$= |1\rangle\langle 1| B_1^+ |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) |1\rangle\langle 1| B_1^- \quad (80)$$

$$= |1\rangle\langle 1| B_1^+ \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) B_1^- \quad (81)$$

$$= |1\rangle\langle 1| \left( g_{1\mathbf{k}} \left( b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (82)$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \omega_{\mathbf{k}} (|0\rangle\langle 0| B_0^+ + |1\rangle\langle 1| B_1^+) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} (|0\rangle\langle 0| B_0^- + |1\rangle\langle 1| B_1^-) \quad (83)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left( |0\rangle\langle 0| \prod_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \prod_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) \quad (84)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| B_0^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_0^- + |1\rangle\langle 1| B_1^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_1^- \right) \quad (85)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \Pi_{\mathbf{k}' \neq \mathbf{k}} \left( D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \Pi_{\mathbf{k}' \neq \mathbf{k}} \left( D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right) \right) \right) \quad (86)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \mathbb{I} \right) \quad (87)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (88)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) \right) \quad (89)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left( |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) \quad (92)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \quad (93)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right). \quad (94)$$

So all parts of  $H(t)$  can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)} |0\rangle\langle 0| + \overline{\varepsilon_1(t)} |1\rangle\langle 1| + \overline{V_{10}(t)} |1\rangle\langle 0| + \overline{V_{01}(t)} |0\rangle\langle 1| \quad (95)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_1^+ B_0^- + V_{01}(t) |0\rangle\langle 1| B_0^+ B_1^-, \quad (96)$$

$$\overline{H_I} = \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (97)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (98)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (99)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (100)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (101)$$

$$= \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) \right) \quad (102)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left( |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger}) \right) \right). \quad (103)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} |j\rangle\langle j| \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}) \right). \quad (104)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_S} + \overline{H_I} + \overline{H_B}. \quad (105)$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (106)$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right), \quad (107)$$

$$\chi_{ij} \equiv \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right). \quad (108)$$

$\chi_{ij}$  is an imaginary number so  $e^{\chi_{ij}}$  is the phase associated to  $B_{ij}$ . We can summarize these definitions with other that we will proof and use from now in the following matrix:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_{ij} \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left( \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{\chi_{ij}} \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2i} & \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix}. \quad (109)$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \quad (110)$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \quad (111)$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature  $\beta = \frac{1}{k_B T}$ , considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})}. \quad (112)$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (113)$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (114)$$

So the product of the matrix representation of  $b^\dagger$  and  $b$  with  $-\beta$  is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (115)$$

$$= \sum_{j=0}^{\infty} -j\beta\omega |j\rangle\langle j|, \quad (116)$$

So the density matrix  $\rho_B$  written in the coherence representation can be obtained using the Zassenhaus formula and the fact that  $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$  for all  $i, j$ .

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \quad (117)$$

$$\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \quad (118)$$

The value of  $\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right)$  is:

$$\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (119)$$

$$= \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) \quad (120)$$

$$= \sum_{j_{\mathbf{k}}} \exp(-\beta\omega_{\mathbf{k}})^{j_{\mathbf{k}}} \quad (121)$$

$$= \frac{1}{1 - \exp(-\beta\omega_{\mathbf{k}})} \quad (\text{by geometric series}) \quad (122)$$

$$\equiv f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}), \quad (123)$$

$$\text{Tr}\left(\exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (124)$$

$$= \prod_{\mathbf{k}} \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (125)$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}). \quad (126)$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})} \quad (127)$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (128)$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}. \quad (129)$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle, \quad (130)$$

$$b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle. \quad (131)$$

Then we can prove that  $\langle B_{iz} \rangle_{\overline{H_B}} = 0$  using the following property based on (130)-(131):

$$\langle B_{iz} \rangle_{\overline{H_B}} = \text{Tr}(\rho_B B_{iz}) = \text{Tr}(B_{iz} \rho_B) \quad (132)$$

$$= \text{Tr} \left( \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \right) \rho_B \right) \quad (133)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \rho_B \right) + \sum_{\mathbf{k}} \text{Tr} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \rho_B \right) \quad (134)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left( b_{\mathbf{k}}^{\dagger} \rho_B \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left( b_{\mathbf{k}} \rho_B \right) \quad (135)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} \text{Tr} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) \quad (136)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left( b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left( b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right), \quad (137)$$

$$\text{Tr} \left( b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}^{\dagger}) \quad (138)$$

$$= \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle \langle j_{\mathbf{k}}| \right) \quad (139)$$

$$= 0, \quad (140)$$

$$\text{Tr} \left( b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}) \quad (141)$$

$$= \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle \langle j_{\mathbf{k}}| \right) \quad (142)$$

$$= 0. \quad (143)$$

we therefore find that:

$$\langle B_{iz} \rangle_{\overline{H_B}} = 0. \quad (144)$$

Another important expected value is  $B = \langle B^{\pm} \rangle_{\overline{H_B}}$ , where  $B^{\pm} = e^{\pm \sum_{\mathbf{k}} \left( \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}$  is given by:

$$\langle B^{\pm} \rangle_{H_B} = \text{Tr}(\rho_B B^{\pm}) = \text{Tr}(B^{\pm} \rho_B) \quad (145)$$

$$= \text{Tr} \left( e^{\pm \sum_{\mathbf{k}} \left( \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \rho_B \right) \quad (146)$$

$$= \prod_{\mathbf{k}} \text{Tr}(D(\pm \alpha_{\mathbf{k}}) \rho_B) \quad (147)$$

$$= \prod_{\mathbf{k}} \langle D(\pm \alpha_{\mathbf{k}}) \rangle. \quad (148)$$



Given that we can write a density operator as:

$$\rho = \int P(\alpha) |\alpha\rangle\langle\alpha| d^2\alpha. \quad (149)$$

where  $P(\alpha)$  satisfies  $\int P(\alpha) d^2\alpha = 1$  and describes the state. It follows that the expectation value of an operator  $A$  with respect to the density operator described by  $P(\alpha)$  is given by:

$$\langle A \rangle = \text{Tr}(A\rho) \quad (150)$$

$$= \int P(\alpha) \langle \alpha | A | \alpha \rangle d^2\alpha. \quad (151)$$

We are typically interested in thermal state density operators, for which it can be shown that  $P(\alpha) = \frac{1}{\pi N} \exp\left(-\frac{|\alpha|^2}{N}\right)$  where  $N = (e^{\beta\omega} - 1)^{-1}$  is the average number of excitations in an oscillator of frequency  $\omega$  at inverse temperature  $\beta = 1/k_B T$ .

Using the integral representation (151) we could obtain that the expected value for the displacement operator  $D(h)$  with  $h \in \mathbb{C}$  is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | D(h) | \alpha \rangle d^2\alpha \quad (152)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle d^2\alpha, \quad (153)$$

$$D(h) D(\alpha) = D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)}, \quad (154)$$

$$D(-\alpha) (D(h) D(\alpha)) = D(-\alpha) D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (155)$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (156)$$

$$= D(\alpha) e^{\frac{1}{2}(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2)} e^{\frac{1}{2}(h\alpha^* - h^*\alpha)} \quad (157)$$

$$= D(\alpha) e^{(h\alpha^* - h^*\alpha)}, \quad (158)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(h) \exp(h\alpha^* - h^*\alpha) | 0 \rangle d^2\alpha \quad (159)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | D(h) | 0 \rangle d^2\alpha \quad (160)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | h \rangle d^2\alpha, \quad (161)$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (162)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \langle 0 | \exp\left(-\frac{|h|^2}{2}\right) \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle d^2\alpha \quad (163)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^*\alpha) \exp\left(-\frac{|h|^2}{2}\right) d^2\alpha \quad (164)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N} + h\alpha^* - h^*\alpha\right) d^2\alpha, \quad (165)$$

$$\alpha = x + iy, \quad (166)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 + y^2}{N} + h(x - iy) - h^*(x + iy)\right) dx dy \quad (167)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{N} + hx - h^*x\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N} - ihy - ih^*y\right) dy, \quad (168)$$

$$-\frac{x^2}{N} + hx - h^*x = -\frac{1}{N} (x^2 - Nhx + Nh^*x) \quad (169)$$

$$= -\frac{1}{N} \left( x + \frac{(Nh^* - Nh)}{2} \right)^2 + \frac{N(h^* - h)^2}{4}, \quad (170)$$

$$\frac{y^2}{N} - ihy - ih^*y = -\frac{1}{N} (y^2 + iNhy + iNh^*y) \quad (171)$$

$$= -\frac{1}{N} \left( y^2 + \frac{iN(h + h^*)}{2} \right) - \frac{N(h + h^*)^2}{4}, \quad (172)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{N} \left( x + \frac{(Nh^* - Nh)}{2} \right)^2 - \frac{1}{N} \left( y^2 + \frac{iN(h + h^*)}{2} \right)\right) dx dy, \quad (173)$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx, \quad (174)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x + \frac{(Nh^* - Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{\left(y^2 + \frac{iN(h + h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dy \quad (175)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \quad (176)$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right) \quad (177)$$

$$= \exp\left(-\frac{|h|^2}{2} + \frac{N(h^{*2} - 2hh^* + h^2) - N(h^2 + 2hh^* + h^{*2})}{4}\right) \quad (178)$$

$$= \exp\left(-|h|^2 \left(N + \frac{1}{2}\right)\right) \quad (179)$$

$$= \exp\left(-|h|^2 \left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2}\right)\right) \quad (180)$$

$$= \exp\left(-\frac{|h|^2}{2} \left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1}\right)\right) \quad (181)$$

$$= \exp\left(-\frac{|h|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right). \quad (182)$$

In the last line we used  $\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} = \coth\left(\frac{\beta\omega}{2}\right)$ . So the value of (147) using (182) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \quad (183)$$

We will now force  $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$ . We will also introduce the bath renormalizing driving in  $\overline{H_S}$  to treat it non-perturbatively in the subsequent formalism, we associate the terms related with  $B^+ \sigma^+$  and  $B^- \sigma^-$  with the interaction part of the Hamiltonian  $\overline{H_I}$  and we subtract their expected value in order to satisfy  $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$ .

A final form of the terms of the Hamiltonian  $\overline{H}$  is:

$$\overline{H}(t) = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left( g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (184)$$

$$= \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{jj'} + \sum_j |j\rangle\langle j| B_{jj} + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| (B_j^+ B_{j'}^- - B_{jj'}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (185)$$

$$\equiv \overline{H_S}(t) + \overline{H_I} + \overline{H_B}. \quad (186)$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_i^+ B_j^- \rangle = B_{ij} \quad (187)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (188)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (189)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2} \left( \frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (190)$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2} \left( \frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (191)$$

$$= \prod_{\mathbf{k}} \exp \left( -\frac{1}{2} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) e^{\frac{1}{2} \left( \frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (192)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)}. \quad (193)$$

From the definition  $B_{01} = \langle B_0^+ B_1^- \rangle$  using the displacement operator we have:

$$\langle B_0^+ B_1^- \rangle = B_{01} \quad (194)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (195)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (196)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (197)$$

$$= \prod_{\mathbf{k}} \left( \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (198)$$

$$= \prod_{\mathbf{k}} \left( \exp \left( -\frac{1}{2} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (199)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)}. \quad (200)$$

We can check:

$$\langle B_0^+ B_1^- \rangle = B_{01} \quad (201)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (202)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)^*} \quad (203)$$

$$= \langle B_1^+ B_0^- \rangle^* \quad (204)$$

$$= B_{10}^*. \quad (205)$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}}(t) \equiv (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma^+ + V_{01}(t) B_{01} \sigma^-, \quad (206)$$

$$\overline{H_{\bar{I}}}(t) \equiv V_{10}(t) (B_1^+ B_0^- - B_{10}) \sigma^+ + V_{01}(t) (B_0^+ B_1^- - B_{01}) \sigma^- + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z}, \quad (207)$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (208)$$

$$= H_B. \quad (209)$$

Note that  $\overline{H_{\bar{B}}}$ , which is the bath acting on the effective “system”  $\bar{S}$  in the variational frame, is just the original bath,  $H_B$ , before transforming to the variational frame.

For the Hamiltonian (207) we can verify the condition  $\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = 0$  in the following way:

$$\langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{n\mathbf{k}} \left( (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left( B_j^+ B_{j'}^- - B_{jj'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad (210)$$

$$= \left\langle \sum_{n\mathbf{k}} \left( (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left( B_j^+ B_{j'}^- - B_{jj'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \quad (211)$$

$$= \sum_{n\mathbf{k}} \left( \left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + \left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| \left( \left\langle V_{jj'}(t) B_j^+ B_{j'}^- \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle V_{jj'}(t) B_{jj'} \right\rangle_{\overline{H_{\bar{B}}}} \right) \quad (212)$$

$$= \sum_{n\mathbf{k}} \left( (g_{n\mathbf{k}} - v_{n\mathbf{k}}) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left( \left\langle B_j^+ B_{j'}^- \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_{jj'} \right\rangle_{\overline{H_{\bar{B}}}} \right) \quad (213)$$

$$= \sum_{n\mathbf{k}} \left( (g_{n\mathbf{k}} - v_{n\mathbf{k}}) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_{\bar{B}}}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) (B_{jj'} - B_{jj'}) \quad (214)$$

$$= 0. \quad (215)$$

We used (144) and (193) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_x = B_x^\dagger \quad (216)$$

$$= \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}, \quad (217)$$

$$B_y = B_y^\dagger \quad (218)$$

$$= \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2i}, \quad (219)$$

$$B_{iz} = B_{iz}^\dagger \quad (220)$$

$$= \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \quad (221)$$

Writing the equations (206) and (207) using the previous combinations we obtain that:

$$\overline{H_S}(\hbar) = \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + V_{10}(t) B_{10} \sigma^+ + V_{01}(t) B_{01} \sigma^- \quad (222)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + V_{10}(t) B_{10} \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01} \frac{\sigma_x - i\sigma_y}{2} \quad (223)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + V_{10}(t) (B_{10}^{\Re}(t) + iB_{10}^{\Im}(t)) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) (B_{10}^{\Re}(t) - iB_{10}^{\Im}(t)) \frac{\sigma_x - i\sigma_y}{2} \quad (224)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + B_{10}^{\Re}(t) \left( V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) + iB_{10}^{\Im}(t) \left( V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) \quad (225)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + B_{10}^{\Re}(t) \left( \sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + iB_{10}^{\Im}(t) \left( \sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (226)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j) |j\rangle\langle j| + B_{10}^{\Re}(t) (\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t)) + iB_{10}^{\Im}(t) (i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t)) \quad (227)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + B_{10}^{\Re}(t) (\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t)) + iB_{10}^{\Im}(t) (i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t)) \quad (228)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + (\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t)) - (\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \quad (229)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \quad (230)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \quad (231)$$

$$\overline{H_I} = V_{10}(t) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10}) + V_{01}(t) (\sigma^- B_0^+ B_1^- - \sigma^- B_{01}) + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} \quad (232)$$

$$= |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} + (V_{10}^{\Re}(t) + iV_{10}^{\Im}(t)) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10}) + (V_{10}^{\Re}(t) - iV_{10}^{\Im}(t)) (\sigma^- B_0^+ B_1^- - \sigma^- B_{01}) \quad (233)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10} + \sigma^- B_0^+ B_1^- - \sigma^- B_{01}) + iV_{10}^{\Im}(t) (\sigma^+ B_1^+ B_0^- - \sigma^+ B_{10} - \sigma^- B_0^+ B_1^- + \sigma^- B_{01}) \quad (234)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) \left( \frac{\sigma_x + i\sigma_y}{2} B_1^+ B_0^- - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_0^+ B_1^- - \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (235)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) \left( \frac{\sigma_x + i\sigma_y}{2} B_1^+ B_0^- - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_0^+ B_1^- - \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (236)$$

$$+ iV_{10}^{\Im}(t) \left( \frac{\sigma_x + i\sigma_y}{2} B_1^+ B_0^- - \frac{\sigma_x + i\sigma_y}{2} B_{10} - \frac{\sigma_x - i\sigma_y}{2} B_0^+ B_1^- + \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (237)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) \left( \sigma_x \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} + i\sigma_y \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2} \right) \quad (238)$$

$$+ iV_{10}^{\Im}(t) \left( \sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2} + i\sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (239)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) \left( i\sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2} - \sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (240)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) \left( i^2 \sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (241)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) \left( i^2 \sigma_x \frac{B_1^+ B_0^- - B_0^+ B_1^- - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right) \quad (242)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) (i^2 \sigma_x (-B_y) - \sigma_y B_x) \quad (243)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) (\sigma_x B_y - \sigma_y B_x) \quad (244)$$

### III. FREE-ENERGY MINIMIZATION

The true free energy  $A$  is bounded by the Bogoliubov inequality:

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left( \text{Tr} \left( e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right) + \langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} + O \left( \langle \overline{H_I}^2 \rangle_{\overline{H_S}(t) + \overline{H_B}} \right). \quad (245)$$

We will optimize the set of variational parameters  $\{v_{i\mathbf{k}}\}$  in order to minimize  $A_B$  (i.e. to make it as close to the true free energy  $A$  as possible). Neglecting the higher order terms and using  $\langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} = 0$  we can obtain the following condition to obtain the set  $\{v_{i\mathbf{k}}\}$ :

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = 0. \quad (246)$$

Using this condition and given that  $[\overline{H_S}(t), \overline{H_B}] = 0$ , we have:

$$e^{-\beta(\overline{H_S}(t) + \overline{H_B})} = e^{-\beta\overline{H_S}(t)} e^{-\beta\overline{H_B}}. \quad (247)$$

Then using the fact that  $\overline{H_S}(t)$  and  $\overline{H_B}$  relate to different Hilbert spaces, we obtain:

$$\text{Tr} \left( e^{-\beta\overline{H_S}(t)} e^{-\beta\overline{H_B}} \right) = \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \text{Tr} \left( e^{-\beta\overline{H_B}} \right). \quad (248)$$

So Eq. (246) becomes:

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = -\frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (249)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \text{Tr} \left( e^{-\beta\overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (250)$$

$$= -\frac{1}{\beta} \frac{\partial \left( \ln \left( \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \right) + \ln \left( \text{Tr} \left( e^{-\beta\overline{H_B}} \right) \right) \right)}{\partial v_{i\mathbf{k}}} \quad (251)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta\overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (252)$$

$$= 0 \quad (\text{by Eq. (246)}). \quad (253)$$

But since  $\overline{H_B} = H_B$  which doesn't contain any  $v_{i\mathbf{k}}$ , a derivative of any function of  $H_B$  that does not introduce new  $v_{i\mathbf{k}}$  will be zero. We therefore require the following:

$$\frac{\partial \ln \left( \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} = \frac{1}{e^{-\beta\overline{H_S}(t)}} \frac{\partial \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}} \quad (254)$$

$$= 0. \quad (255)$$

This means we need to impose:

$$\frac{\partial \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}} = 0. \quad (256)$$

First we look at:

$$-\beta \overline{H_{\bar{S}}}(t) = -\beta \left( (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma^+ + V_{01}(t) B_{01} \sigma^- \right). \quad (257)$$

Then the eigenvalues of  $-\beta \overline{H_{\bar{S}}}(t)$  satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr}(-\beta \overline{H_{\bar{S}}}(t)) + \text{Det}(-\beta \overline{H_{\bar{S}}}(t)) = 0. \quad (258)$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr}(\overline{H_{\bar{S}}}(t)), \quad (259)$$

$$\eta \equiv \sqrt{(\text{Tr}(\overline{H_{\bar{S}}}(t)))^2 - 4\text{Det}(\overline{H_{\bar{S}}}(t))}. \quad (260)$$

The solutions of the equation (258) are:

$$\lambda = \beta \frac{-\text{Tr}(\overline{H_{\bar{S}}}(t)) \pm \sqrt{(\text{Tr}(\overline{H_{\bar{S}}}(t)))^2 - 4\text{Det}(\overline{H_{\bar{S}}}(t))}}{2} \quad (261)$$

$$= \beta \frac{-\varepsilon(t) \pm \eta(t)}{2} \quad (262)$$

$$= -\beta \frac{\varepsilon(t) \mp \eta(t)}{2}. \quad (263)$$

The value of  $\text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)})$  can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a  $2 \times 2$  matrix):

$$\text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)}) = \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \exp\left(\frac{\eta(t)\beta}{2}\right) + \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \exp\left(-\frac{\eta(t)\beta}{2}\right) \quad (264)$$

$$= 2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \cosh\left(\frac{\eta(t)\beta}{2}\right). \quad (265)$$

Given that  $v_{ik}$  is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function  $\text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)}) = A(\varepsilon(t), \eta(t))$  to calculate  $\frac{\partial \text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)})}{\partial v_{ik}^{\Re}}$  can lead to:

$$\frac{\partial \text{Tr}(e^{-\beta \overline{H_{\bar{S}}}(t)})}{\partial v_{ik}^{\Re}} = \frac{\partial \left( 2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \cosh\left(\frac{\eta(t)\beta}{2}\right) \right)}{\partial v_{ik}^{\Re}} \quad (266)$$

$$= 2 \left( -\frac{\beta}{2} \frac{\partial \varepsilon(t)}{\partial v_{ik}^{\Re}} \right) \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \cosh\left(\frac{\eta(t)\beta}{2}\right) + 2 \left( \frac{\beta}{2} \frac{\partial \eta(t)}{\partial v_{ik}^{\Re}} \right) \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \sinh\left(\frac{\eta(t)\beta}{2}\right) \quad (267)$$

$$= -\beta \exp\left(-\frac{\varepsilon(t)\beta}{2}\right) \left( \frac{\partial \varepsilon(t)}{\partial v_{ik}^{\Re}} \cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial \eta(t)}{\partial v_{ik}^{\Re}} \sinh\left(\frac{\eta(t)\beta}{2}\right) \right). \quad (268)$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial \varepsilon(t)}{\partial v_{ik}^{\Re}} \cosh\left(\frac{\eta(t)\beta}{2}\right) - \frac{\partial \eta(t)}{\partial v_{ik}^{\Re}} \sinh\left(\frac{\eta(t)\beta}{2}\right) = 0. \quad (269)$$

The derivates included in the expression given are related to:

$$\langle B_0^+ B_1^- \rangle = \left( \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (270)$$

$$= \left( \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right)^* \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (271)$$

$$= \langle B_1^+ B_0^- \rangle^*, \quad (272)$$

$$R_i = \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (273)$$

$$= \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (274)$$

$$\langle B_0^+ B_1^- \rangle = \left( \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (275)$$

$$= \left( \prod_{\mathbf{k}} \exp \left( \frac{1}{2\omega_{\mathbf{k}}^2} (v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*) \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (276)$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = (v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im}) (v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im}) - (v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im}) (v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im}) \quad (277)$$

$$= (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}) - (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} + i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}) \quad (278)$$

$$= 2i (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re}), \quad (279)$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}} - v_{0\mathbf{k}}) (v_{1\mathbf{k}} - v_{0\mathbf{k}})^* \quad (280)$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - (v_{1\mathbf{k}} v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}}) \quad (281)$$

$$= (v_{1\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im})^2 + (v_{0\mathbf{k}}^{\Re})^2 + (v_{0\mathbf{k}}^{\Im})^2 - ((v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im})(v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im}) + (v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im})(v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im})) \quad (282)$$

$$= (v_{1\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im})^2 + (v_{0\mathbf{k}}^{\Re})^2 + (v_{0\mathbf{k}}^{\Im})^2 - 2 (v_{1\mathbf{k}}^{\Re} v_{0\mathbf{k}}^{\Re} + v_{1\mathbf{k}}^{\Im} v_{0\mathbf{k}}^{\Im}) \quad (283)$$

$$= (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2. \quad (284)$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re} - i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re} + i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \right) \right) \quad (285)$$

$$= \sum_{\mathbf{k}} \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}} - g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right), \quad (286)$$

$$\langle B_0^+ B_1^- \rangle = \left( \prod_{\mathbf{k}} \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (287)$$

$$= \left( \prod_{\mathbf{k}} \exp \left( \frac{2i (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re})}{2\omega_{\mathbf{k}}^2} \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (288)$$

$$= \left( \prod_{\mathbf{k}} \exp \left( \frac{i (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (289)$$

Calculating the derivatives respect to  $\alpha_{i\mathbf{k}}^{\Re}$  and  $\alpha_{i\mathbf{k}}^{\Im}$  we have:



$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (290)$$

$$= \frac{\partial \left( \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (291)$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \quad (292)$$

$$\frac{\partial |B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \left( \exp \left( - \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (293)$$

$$= - \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} \exp \left( - \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (294)$$

$$= - \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2, \quad (295)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \sqrt{\left( \text{Tr} \left( \overline{H_{\bar{S}}(t)} \right) \right)^2 - 4 \text{Det} \left( \overline{H_{\bar{S}}(t)} \right)}}{\partial v_{i\mathbf{k}}^{\Re}} \quad (296)$$

$$= \frac{2 \text{Tr} \left( \overline{H_{\bar{S}}(t)} \right) \frac{\partial \text{Tr} \left( \overline{H_{\bar{S}}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}} - 4 \frac{\partial \text{Det} \left( \overline{H_{\bar{S}}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2 \sqrt{\left( \text{Tr} \left( \overline{H_{\bar{S}}(t)} \right) \right)^2 - 4 \text{Det} \left( \overline{H_{\bar{S}}(t)} \right)}} \quad (297)$$

$$= \frac{\varepsilon(t) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1)(\varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta(t)} \quad (298)$$

$$= \frac{\varepsilon(t) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left( (\varepsilon(t) - \varepsilon_i(t) - R_i) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (299)$$

$$= \frac{\varepsilon(t) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left( (\varepsilon(t) - \varepsilon_i(t) - R_i) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{i\mathbf{k}}^{\Re} - v_{i\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (300)$$

$$= \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left( \frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left( - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \varepsilon(t) \right) \quad (301)$$

$$+ 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + 4 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right). \quad (302)$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh \left( \frac{\beta \eta(t)}{2} \right) = \frac{\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \quad (303)$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left( 2 \frac{\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + 2 \frac{(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} + 2 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) - \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \varepsilon(t)}{\eta(t)}}}. \quad (304)$$

Rearrannng this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{(2v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}} - g_{i\mathbf{k}}^*)\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} + g_{i\mathbf{k}}^*)(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (305)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re}(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (306)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 4\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (307)$$

$$= \frac{(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (308)$$

Separating (307) such that the terms with  $v_{i\mathbf{k}}$  are located at one side of the equation permit us to write

$$\frac{(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re})\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Re} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (309)$$

$$v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re} = v_{i\mathbf{k}}^{\Re} \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (310)$$

$$v_{i\mathbf{k}}^{\Re} = \frac{g_{i\mathbf{k}}^{\Re} \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}, \quad (311)$$

$$v_{i\mathbf{k}}^{\Re} = \frac{g_{i\mathbf{k}}^{\Re} \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i\mathbf{k}}^{\Re}}{g_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}. \quad (312)$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (313)$$

$$= \frac{\partial \left( \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (314)$$

$$= 2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (315)$$

$$\frac{\partial |B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial \left( \exp \left( - \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (316)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Im}} \exp\left(-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (317)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Im}} |B_{10}|^2, \quad (318)$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \quad (319)$$

$$= \frac{2\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right) \frac{\partial\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}} - 4\frac{\partial\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2\sqrt{\left(\text{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^2 - 4\text{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \quad (320)$$

$$= \frac{\varepsilon(t) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\frac{\partial((\varepsilon_1(t) + R_1)(\varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta(t)} \quad (321)$$

$$= \frac{\varepsilon(t) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \quad (322)$$

$$= \frac{\varepsilon(t) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2\left(\varepsilon(t) - \varepsilon_i(t) - R_i\right) \left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2(v_{i\mathbf{k}}^{\Im} - v_{i'\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \quad (323)$$

$$= \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \left( \frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left( -i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i) i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right). \quad (324)$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \quad (325)$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \left( \frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left( \left(\frac{g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)^{\Im} \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \left(\frac{g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}\right)^{\Im} + 2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}. \quad (326)$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i\frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - i\left(g_{i\mathbf{k}}^* - g_{i\mathbf{k}}\right) (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (327)$$

$$= \frac{2\left(\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^{\Im}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Im} (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (328)$$

$$= \frac{2\left(\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^{\Im}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (329)$$

$$= \frac{\left(\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^{\Im}\right) \eta(t)}{v_{i\mathbf{k}}^{\Im} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \quad (330)$$

Separating (330) such that the terms with  $v_{i\mathbf{k}}$  are located at one side of the equation permit us to write

$$\frac{(v_{i\mathbf{k}}^{\Im} - g_{i\mathbf{k}}^{\Im})\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Im} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (331)$$

$$v_{i\mathbf{k}}^{\Im} - g_{i\mathbf{k}}^{\Im} = v_{i\mathbf{k}}^{\Im} \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \quad (332)$$

$$+ 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (333)$$

$$v_{i\mathbf{k}}^{\Im} = \frac{g_{i\mathbf{k}}^{\Im} \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}, \quad (334)$$

$$v_{i\mathbf{k}}^{\Im} = \frac{g_{i\mathbf{k}}^{\Im} \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}. \quad (335)$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}) = v_{i\mathbf{k}}^{\Re}(\omega_{\mathbf{k}}) + i v_{i\mathbf{k}}^{\Im}(\omega_{\mathbf{k}}) \quad (336)$$

$$g_{i\mathbf{k}}^{\Re}(\omega_{\mathbf{k}}) \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Re}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \\ = \frac{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}{\quad} \quad (337)$$

$$+ i \frac{g_{i\mathbf{k}}^{\Im}(\omega_{\mathbf{k}}) \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (338)$$

$$= \frac{g_{i\mathbf{k}}(\omega_{\mathbf{k}}) \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}. \quad (339)$$

#### IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is  $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$ , where  $\rho_B^{\text{Thermal}} \equiv \rho_B(0) \equiv \rho_B$ , so the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T(0) e^{-V} \quad (340)$$

$$= (|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+) (\rho_S(0) \otimes \rho_B) (|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-), \quad (341)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: |0\rangle\langle 0|B_0^+|0\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^- \quad (342)$$

$$= |0\rangle\langle 0|B_0^+|0\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^- \quad (343)$$

$$= |0\rangle\langle 0| \otimes B_0^+ \rho_B B_0^-, \quad (344)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: |1\rangle\langle 1|B_1^+|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^- \quad (345)$$

$$= |1\rangle\langle 1|B_1^+ \rho_B B_1^- \quad (346)$$

$$= |1\rangle\langle 1| \otimes B_1^+ \rho_B B_1^-, \quad (347)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1|: |0\rangle\langle 0|B_0^+|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^- \quad (348)$$

$$= |0\rangle\langle 1|B_0^+ \rho_B |1\rangle\langle 1|B_1^- \quad (349)$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_0^+ \rho_B B_1^- \quad (350)$$

$$= |0\rangle\langle 1| \otimes B_0^+ \rho_B B_1^-, \quad (351)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0|: |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^- \quad (352)$$

$$= |1\rangle\langle 0| \otimes B_1^+ \rho_B B_0^-. \quad (353)$$

We transform any operator  $O$  into the interaction picture in the following way:

$$\tilde{O}(t) \equiv U^\dagger(t) O(t) U(t), \quad (354)$$

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right). \quad (355)$$

Here  $\mathcal{T}$  denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (356)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}_T(t)). \quad (357)$$

. In order to separate the Hamiltonian we define the matrix  $\Lambda(t)$  such that  $\Lambda_{1i}(t) = A_i$ ,  $\Lambda_{2i}(t) = B_i$  and  $\Lambda_{3i}(t) = C_i(t)$  written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \quad (358)$$

In this case  $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$  and  $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$  with  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$ .

The previous notation allows us to write the interaction Hamiltonian  $\overline{H_I}(t)$  as pointed in the equation (236):

$$\overline{H_I}(t) = \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) (\sigma_x B_y - \sigma_y B_x) \quad (359)$$

$$= B_{0z} |0\rangle\langle 0| + B_{1z} |1\rangle\langle 1| + V_{10}^{\Re}(t) \sigma_x B_x + V_{10}^{\Re}(t) \sigma_y B_y + V_{10}^{\Im}(t) \sigma_x B_y - V_{10}^{\Im}(t) \sigma_y B_x \quad (360)$$

$$= \sum_i C_i(t) (A_i \otimes B_i(t)). \quad (361)$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \widetilde{\rho_T}(t)]. \quad (362)$$

This equation has the formal solution

$$\widetilde{\rho_T}(t) = \overline{\rho_T}(0) - i \int_0^t [\widetilde{H_I}(s), \widetilde{\rho_T}(s)] ds. \quad (363)$$

Replacing the equation (363) in the equation (362) gives us:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_T}(s)]] ds. \quad (364)$$

This equation allow us to iterate and write in terms of a series expansion with  $\overline{\rho_T}(0)$  the solution as:

$$\widetilde{\rho_T}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \overline{\rho_T}(0)]] \dots]. \quad (365)$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho_S}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \overline{\rho_S}(0) \rho_B]] \dots]. \quad (366)$$

here we have assumed that  $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$ . Consider the following notation:

$$\widetilde{\rho_S}(t) = (1 + W_1(t) + W_2(t) + \dots) \overline{\rho_S}(0) \quad (367)$$

$$= W(t) \overline{\rho_S}(0). \quad (368)$$

in this case

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), (\cdot) \rho_B]] \dots]. \quad (369)$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = (\dot{W}_1(t) + \dot{W}_2(t) + \dots) \overline{\rho_S}(0) \quad (370)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} W(t) \overline{\rho_S}(0) \quad (371)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} \widetilde{\rho_S}(t). \quad (372)$$

where we assumed that  $W(t)$  is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that  $\text{Tr}_B [\widetilde{H_I}(t) \rho_B] = 0$  so  $W_1(t) = 0$ . Thus, to second order and approximating  $W(t) \approx \mathbb{I}$  then the equation (370) becomes:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = \dot{W}_2(t) \widetilde{\rho_S}(t) \quad (373)$$

$$= - \int_0^t dt_1 \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(t_1), \widetilde{\rho_S}(t) \rho_B]]. \quad (374)$$

Replacing  $t_1 \rightarrow t - \tau$

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \int_0^t d\tau \text{Tr}_B [\overline{H_I}(t), [\widetilde{\overline{H_I}}(-\tau), \overline{\rho_S}(t) \rho_B]] . \quad (375)$$

From the interaction picture applied on  $\overline{H_I}(t)$  we find:

$$\widetilde{\overline{H_I}}(t) = U^\dagger(t) e^{iH_B t} \overline{H_I}(t) e^{-iH_B t} U(t) . \quad (376)$$

we use the time-ordering operator  $\mathcal{T}$  because in general  $\overline{H_S}(t)$  doesn't commute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_I}}(t) = \sum_i C_i(t) (\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)) , \quad (377)$$

$$\widetilde{A_i}(t) = U^\dagger(t) e^{iH_B t} A_i e^{-iH_B t} U(t) \quad (378)$$

$$= U^\dagger(t) A_i U(t) e^{iH_B t} e^{-iH_B t} \quad (379)$$

$$= U^\dagger(t) A_i U(t) \mathbb{I} \quad (380)$$

$$= U^\dagger(t) A_i U(t) , \quad (381)$$

$$\widetilde{B_i}(t) = U^\dagger(t) e^{iH_B t} B_i(t) e^{-iH_B t} U(t) \quad (382)$$

$$= U^\dagger(t) U(t) e^{iH_B t} B_i(t) e^{-iH_B t} \quad (383)$$

$$= \mathbb{I} e^{iH_B t} B_i(t) e^{-iH_B t} \quad (384)$$

$$= e^{iH_B t} B_i(t) e^{-iH_B t} . \quad (385)$$

Here we have used the fact that  $[\overline{H_S}(t), H_B] = 0$  because these operators belong to different Hilbert spaces, so  $[U(t), e^{iH_B t}] = 0$ .

Using the expression (377) to replace it in the equation (374)

$$\frac{d\widetilde{\overline{\rho_S}}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{\overline{H_I}}(t), [\widetilde{\overline{H_I}}(s), \widetilde{\overline{\rho_S}}(t) \rho_B]] ds \quad (386)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \left[ \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) , \widetilde{\overline{\rho_S}}(t) \rho_B \right] \right] ds \quad (387)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B - \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \right] ds \quad (388)$$

$$= - \int_0^t \text{Tr}_B \left( \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B - \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \right) \quad (389)$$

$$= - \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) + \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) ds . \quad (390)$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \langle \widetilde{B_j}(t) \widetilde{B_i}(s) \rangle_B \quad (391)$$

$$= \langle \widetilde{B_j}(\tau) \widetilde{B_i}(0) \rangle_B . \quad (392)$$

The correlation functions relevant that appear in the equation (390) are:

$$\text{Tr}_B \left( \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B \right) = \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (393)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (394)$$

$$= \Lambda_{ji}(\tau), \quad (395)$$

$$\text{Tr}_B \left( \widetilde{B}_j(t) \rho_B \widetilde{B}_i(s) \right) = \text{Tr}_B \left( \widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B \right) \quad (396)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (397)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (398)$$

$$= \Lambda_{ij}(-\tau), \quad (399)$$

$$\text{Tr}_B \left( \widetilde{B}_i(s) \rho_B \widetilde{B}_j(t) \right) = \text{Tr}_B \left( \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B \right) \quad (400)$$

$$= \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (401)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (402)$$

$$= \Lambda_{ji}(\tau), \quad (403)$$

$$\text{Tr}_B \left( \rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) \right) = \text{Tr}_B \left( \widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B \right) \quad (404)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (405)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (406)$$

$$= \Lambda_{ij}(-\tau). \quad (407)$$

The cyclic property of the trace was use widely in the development of equations (393) and (407). Replacing in (390)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{ij} (C_i(t) C_j(s) (\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s)) + C_i(t) C_j(s) (\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t))) ds \quad (408)$$

$$= - \int_0^t \sum_{ij} \left( C_i(t) C_j(s) \left( \Lambda_{ij}(\tau) \left[ \widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] + \Lambda_{ji}(-\tau) \left[ \widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t) \right] \right) \right) ds. \quad (409)$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) = \Lambda_{ij}(\tau) \left[ \widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right], \quad (410)$$

$$\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s) = \Lambda_{ji}(-\tau) \left[ \widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t) \right]. \quad (411)$$

Returning to the Schroedinger picture we have:

$$U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) = U(t) \widetilde{A}_i(t) U^\dagger(t) U(t) \widetilde{A}_j(s) U^\dagger(t) U(t) \widetilde{\rho_S}(t) U^\dagger(t), \quad (412)$$

$$= \left( U(t) \widetilde{A}_i(t) U^\dagger(t) \right) \left( U(t) \widetilde{A}_j(s) U^\dagger(t) \right) \left( U(t) \widetilde{\rho_S}(t) U^\dagger(t) \right), \quad (413)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t). \quad (414)$$

This procedure applying to the relevant commutators give us:

$$U(t) \left[ \widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] U^\dagger(t) = \left( U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) - U(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) U^\dagger(t) \right) \quad (415)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) - \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) A_i \quad (416)$$

$$= \left[ A_i, \widetilde{A}_j(t - \tau, t) \widetilde{\rho_S}(t) \right]. \quad (417)$$



Introducing this transformed commutators in the equation (409) allow us to obtain the master equation of the system

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau C_i(t) C_j(t-\tau) \left( \Lambda_{ij}(\tau) \left[ A_i, \widetilde{A}_j(t-\tau, t) \overline{\rho_S}(t) \right] + \Lambda_{ji}(-\tau) \left[ \overline{\rho_S}(t) \widetilde{A}_j(t-\tau, t), A_i \right] \right) \quad (418)$$

where  $i, j \in \{1, 2, 3, 4, 5, 6\}$ .

Here  $\widetilde{A}_j(s, t) = U(t) U^\dagger(s) A_j U(s) U^\dagger(t)$  where  $U(t)$  is given by (355). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in  $H_I(t)$ . The environmental correlation functions can be rewritten as:

$$\Lambda_{ij}(\tau) = \text{Tr}_B \left( \widetilde{B}_i(t) \widetilde{B}_j(s) \rho_B \right) \quad (419)$$

$$= \text{Tr}_B \left( \widetilde{B}_i(\tau) \widetilde{B}_j(0) \rho_B \right). \quad (420)$$

Calculating the correlation functions allow us to obtain:

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left( \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rho_B \right) \quad (421)$$

$$= \int d^2\alpha P(\alpha) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle \quad (422)$$

$$= \frac{1}{\pi N} \int \exp \left( -\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle d^2\alpha \quad (423)$$

$$= \frac{1}{\pi N} \int \exp \left( -\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle d^2\alpha, \quad (424)$$

$$\widetilde{B}_{jz}(\tau) = \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right), \quad (425)$$

$$\widetilde{B}_{jz}(0) = \sum_{\mathbf{k}'} \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right), \quad (426)$$

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left( \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rho_B \right) \quad (427)$$

$$= \text{Tr}_B \left( \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (428)$$

$$= \text{Tr}_B \left( \sum_{\mathbf{k} \neq \mathbf{k}'} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (429)$$

$$+ \text{Tr}_B \left( \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \rho_B \right), \quad (430)$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}} = p_{j\mathbf{k}} \quad (431)$$

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left( \sum_{\mathbf{k} \neq \mathbf{k}'} (p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) (p_{j\mathbf{k}'} b_{\mathbf{k}'}^\dagger + p_{j\mathbf{k}'}^* b_{\mathbf{k}'} ) \rho_B \right) + \text{Tr}_B \left( \sum_{\mathbf{k}} (p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) (p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger + p_{j\mathbf{k}}^* b_{\mathbf{k}} ) \rho_B \right) \quad (432)$$



$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2\alpha_{\mathbf{k}} \right) \quad (457)$$

$$+ \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle d^2\alpha_{\mathbf{k}} \right) \quad (458)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( e^{i\omega_{\mathbf{k}} \tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}} \tau} \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}} \tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} \right) \quad (459)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( (e^{i\omega_{\mathbf{k}}\tau} + e^{-i\omega_{\mathbf{k}}\tau}) \frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} \right), \quad (460)$$

$$\frac{1}{\pi N} \int |\alpha_{\mathbf{k}}|^2 \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) d^2\alpha_{\mathbf{k}} = \frac{1}{\pi N} \int_0^{2\pi} \int_0^\infty r^2 \exp\left(-\frac{r^2}{N}\right) r dr d\theta \quad (461)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 (2 \cos(\omega_{\mathbf{k}}\tau) N) + \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 e^{-i\omega_{\mathbf{k}}\tau} \quad (462)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 (2 \cos(\omega_{\mathbf{k}}\tau) N + e^{-i\omega_{\mathbf{k}}\tau}) \quad (463)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( \frac{2 \cos(\omega_{\mathbf{k}} \tau)}{e^{\beta \omega_{\mathbf{k}}} - 1} + e^{-i \omega_{\mathbf{k}} \tau} \right) \quad (464)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( \frac{2 \cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} + \cos(\omega_{\mathbf{k}}\tau) - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (465)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( \frac{(2 + e^{\beta\omega_{\mathbf{k}}} - 1) \cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (466)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( \frac{(1 + e^{\beta\omega_{\mathbf{k}}}) \cos(\omega_{\mathbf{k}}\tau)}{e^{\beta\omega_{\mathbf{k}}} - 1} - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (467)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( \frac{\left( e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} + e^{\frac{\beta\omega_{\mathbf{k}}}{2}} \right) \cos(\omega_{\mathbf{k}}\tau)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}} - e^{\frac{\beta\omega_{\mathbf{k}}}{2}}} - i \sin(\omega_{\mathbf{k}}\tau) \right) \quad (468)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \cos(\omega_{\mathbf{k}} \tau) - i \sin(\omega_{\mathbf{k}} \tau) \right) \quad (469)$$

$$= \sum_{\mathbf{k}} |g_{j\mathbf{k}} - v_{j\mathbf{k}}|^2 \left( \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \cos(\omega_{\mathbf{k}} \tau) - i \sin(\omega_{\mathbf{k}} \tau) \right), \quad (470)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \rangle_B = \int d^2 \alpha_{\mathbf{k}} P(\alpha_{\mathbf{k}}) \langle \alpha_{\mathbf{k}} | \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) | \alpha_{\mathbf{k}} \rangle \quad (471)$$

$$= \frac{1}{\pi N} \int \exp \left( -\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \left\langle \alpha_{\mathbf{k}} \left| \widetilde{B}_{jz}(\tau) \widetilde{B}_{j'z}(0) \right| \alpha_{\mathbf{k}} \right\rangle d^2 \alpha_{\mathbf{k}} \quad (472)$$

$$= \frac{1}{\pi N} \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \Sigma_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \Sigma_{\mathbf{k}'} \left( (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (473)$$

$$= \frac{1}{\pi N} \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}' \neq \mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}'}^\dagger e^{i\omega_{\mathbf{k}'}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau} \right) \left( (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'})^* b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}} \quad (474)$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}| \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}} \right) |\alpha_{\mathbf{k}}| d^2\alpha_{\mathbf{k}} \quad (475)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}} \right) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (476)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \left| \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \right| \alpha_{\mathbf{k}} \right\rangle d^2\alpha_{\mathbf{k}} \quad (477)$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \left\langle \alpha_{\mathbf{k}} \left| \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \right| \alpha_{\mathbf{k}} \right\rangle d^2\alpha_{\mathbf{k}} \quad (478)$$

$$= \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (479)$$

$$+ \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}}, \quad (480)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (481)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (482)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (483)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}|^2 d^2 \alpha_{\mathbf{k}} \quad (484)$$

$$= N, \quad (485)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (486)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (487)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (488)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (489)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (490)$$

$$= N + 1, \quad (491)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \rangle_B = \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N + 1) \quad (492)$$

$$= \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} + N ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau}), \quad (493)$$

$$D(h') D(h) = \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) D(h' + h), \quad (494)$$

$$\langle D(h') D(h) \rangle_B = \text{Tr}_B \left( \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) D(h' + h) \rho_B \right) \quad (495)$$

$$= \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) \text{Tr}_B (D(h' + h) \rho_B) \quad (496)$$

$$= \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) \frac{1}{\pi N} \int d^2 \alpha P(\alpha) \langle \alpha | D(h' + h) | \alpha \rangle \quad (497)$$

$$= \exp\left(\frac{1}{2} (h' h^* - h'^* h)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \quad (498)$$

$$h' = h \exp(i\omega\tau), \quad (499)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp\left(\frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau))\right) \exp\left(-\frac{|h + h \exp(i\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \quad (500)$$

$$\frac{1}{2} |h|^2 (\exp(i\omega\tau) - \exp(-i\omega\tau)) = \frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau)) \quad (501)$$

$$= \frac{1}{2} |h|^2 (\cos(\omega\tau) + i \sin(\omega\tau) - \cos(\omega\tau) + i \sin(\omega\tau)) \quad (502)$$

$$= \frac{1}{2} |h|^2 (2i \sin(\omega\tau)) \quad (503)$$

$$= i |h|^2 \sin(\omega\tau), \quad (504)$$

$$-\frac{|h + h \exp(i\omega\tau)|^2}{2} = -|h|^2 \frac{|1 + \exp(i\omega\tau)|^2}{2} \quad (505)$$

$$= -|h|^2 \frac{(1 + 2 \cos(\omega\tau) + \cos^2(\omega\tau)) + \sin^2(\omega\tau)}{2} \quad (506)$$



$$= \prod_{\mathbf{k}} \exp \left( - \left| \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( i \sin(\omega_{\mathbf{k}} \tau) - \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (532)$$

$$\left\langle \widetilde{B_0^+ B_1^-(\tau)} \widetilde{B_1^+ B_0^-(0)} \right\rangle_B = \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B \right) \quad (533)$$

$$= \text{Tr}_B \left( \prod_{\mathbf{k}} D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \prod_{\mathbf{k}} D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B \right) \quad (534)$$

$$= \prod_{\mathbf{k}} \text{Tr}_B \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i(\omega_{\mathbf{k}} \tau + \pi)} \right) D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B \right) \quad (535)$$

$$= \prod_{\mathbf{k}} \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau + \pi) + \cos(\omega_{\mathbf{k}} \tau + \pi) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (536)$$

$$= \left\langle \widetilde{B_1^+ B_0^-(\tau)} \widetilde{B_0^+ B_1^-(0)} \right\rangle_B, \quad (537)$$

$$\left\langle \widetilde{B_0^+ B_1^-(\tau)} \widetilde{B_{jz}^-(0)} \right\rangle_B = \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Sigma_{\mathbf{k}'} \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right), \quad (538)$$

$$\langle D(h) b \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h) b | \alpha \rangle \quad (539)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle \alpha | D(-\alpha) D(h) b D(\alpha) | \alpha \rangle \quad (540)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b D(\alpha) | 0 \rangle \quad (541)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \quad (542)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \quad (543)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b + \alpha) | 0 \rangle \quad (544)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (545)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (546)$$

$$= \frac{1}{\pi N} \int \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( - \frac{|h|^2}{2} \right) d^2 \alpha \quad (547)$$

$$= hN \langle D(h) \rangle_B, \quad (548)$$

$$\langle D(h) b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h) b^\dagger | \alpha \rangle \quad (549)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (550)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (551)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) | 0 \rangle \quad (552)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b^\dagger + \alpha^*) | 0 \rangle \quad (553)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b^\dagger + \alpha^*) | 0 \rangle \quad (554)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \quad (555)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 1 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \quad (556)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle -h | 1 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \langle 0 | D(h) | 0 \rangle, \quad (557)$$

$$\langle -h | = \exp \left( -\frac{|-h^*|^2}{2} \right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n |, \quad (558)$$

$$\langle -h | 1 \rangle = \exp \left( -\frac{|-h^*|^2}{2} \right) (-h^*), \quad (559)$$

$$\langle D(h) b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( -\frac{|-h^*|^2}{2} \right) (-h^*) + \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \exp \left( -\frac{|-h^*|^2}{2} \right) \quad (560)$$

$$= -h^* \langle D(h) \rangle_B (N+1), \quad (561)$$

$$\langle b D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \langle \alpha | b D(h) | \alpha \rangle \quad (562)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( -\frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left( -\frac{|h|^2}{2} \right) \quad (563)$$

$$= h \langle D(h) \rangle_B (N+1), \quad (564)$$

$$\langle b^\dagger D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \langle \alpha | b^\dagger D(h) | \alpha \rangle \quad (565)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( -\frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left( -\frac{|h|^2}{2} \right) \quad (566)$$

$$= -h^* \langle D(h) \rangle_B N, \quad (567)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \right\rangle_B = \left\langle \prod_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle_B \quad (568)$$

$$= \prod_{\mathbf{k}} \left( e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \left\langle D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \right\rangle_B \quad (569)$$

$$= \prod_{\mathbf{k}} \left( e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \left\langle D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \right\rangle_B \quad (570)$$

$$= \prod_{\mathbf{k}} \left( \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp \left( -\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (571)$$

$$= B_{10}. \quad (572)$$

The correlation functions can be found readily as:

$$\widetilde{B_1^+ B_0^-}(\tau) = \prod_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right), \quad (573)$$

$$\widetilde{B_0^+ B_1^-}(\tau) = \prod_{\mathbf{k}} \left( D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \exp \left( \frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right), \quad (574)$$

$$\widetilde{B}_x(0) = \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}, \quad (575)$$

$$\widetilde{B}_y(0) = \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2i}, \quad (576)$$

$$B_{10} = \left( \prod_{\mathbf{k}} \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega}{2} \right) \right) \right), \quad (577)$$

$$B_{iz} = \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right), \quad (578)$$

$$\left\langle \widetilde{B}_{iz}(\tau) \widetilde{B}_{jz}(0) \right\rangle_B = \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \right\rangle_B \quad (579)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \quad (580)$$

$$\left\langle \widetilde{B}_x(\tau) \widetilde{B}_x(0) \right\rangle_B = \left\langle \frac{B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01}}{2} \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right\rangle_B \quad (581)$$

$$= \frac{1}{4} \left\langle \left( B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01} \right) \left( B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01} \right) \right\rangle_B \quad (582)$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- \right. \quad (583)$$

$$\left. - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} + B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} + B_{10} B_{01} \right\rangle \quad (584)$$

$$= \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \quad (585)$$

$$\left. + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right\rangle, \quad (586)$$



$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B = \Pi_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (587)$$

$$U = \prod_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right), \quad (588)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (589)$$

$$S = \prod_{\mathbf{k}} \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (590)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B = U \exp(-\phi(\tau)) S, \quad (591)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \right\rangle_B = U^* \exp(-\phi(\tau)) S, \quad (592)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B = \exp(\phi(\tau)) S, \quad (593)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(\tau) \widetilde{B_1^+ B_0^-}(0) \right\rangle_B = \left\langle \widetilde{B_1^+ B_0^-}(\tau) \widetilde{B_0^+ B_1^-}(0) \right\rangle_B, \quad (594)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(\tau) \right\rangle_B = \prod_{\mathbf{k}} \left( \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp \left( - \frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (595)$$

$$= U^{*1/2} S^{1/2}, \quad (596)$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \right\rangle_B = \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \quad (597)$$

$$\left. + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right\rangle, \quad (598)$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \right\rangle_B = \frac{1}{4} \left\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \quad (599)$$

$$\left. - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right\rangle \quad (600)$$

$$= \frac{1}{4} \left( U^* \exp(-\phi(\tau)) S + \exp(\phi(\tau)) S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau)) S + U \exp(-\phi(\tau)) S - B_{10}^{*2} - |B_{10}|^2 \right) \quad (601)$$

$$= \frac{1}{4} \left( 2U^{\Re} \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2(B_{10}^2)^{\Re} - 2|B_{10}|^2 \right) \quad (602)$$

$$= \frac{1}{4} \left( 2U^{\Re} \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2(U^*)^{\Re} S - 2S \right) \quad (603)$$

$$= \frac{S}{2} \left( U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - (U^*)^{\Re} - 1 \right), \quad (604)$$

$$\left\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \right\rangle_B = \left\langle \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \right\rangle_B \quad (605)$$

$$= -\frac{1}{4} \left\langle \left( B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01} \right) \left( B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01} \right) \right\rangle_B \quad (606)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_1^+ B_0^-(\tau) B_1^+ B_0^- \right. \quad (607)$$

$$\left. - B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} + B_{10} B_0^+ B_1^- - B_{10} B_1^+ B_0^- + B_{10} B_{10} - B_{10} B_{01} - B_{01} B_0^+ B_1^- + B_{01} B_1^+ B_0^- - B_{01} B_{10} + B_{01} B_{01} \right\rangle \quad (608)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} \right. \quad (609)$$

$$\left. - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} \right\rangle \quad (610)$$

$$= -\frac{1}{4} \left\langle B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_{01} B_{10} - B_{01} B_{01} - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_{10} B_{10} + B_{10} B_{01} \right\rangle \quad (611)$$

$$= -\frac{1}{4} \left( U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S - \exp(\phi(\tau)) S + U^* \exp(-\phi(\tau)) S + 2S - 2(U^*)^{\Re} S \right) \quad (612)$$

$$= -\frac{S}{4} \left( 2U^{\Re} \exp(-\phi(\tau)) - 2\exp(\phi(\tau)) + 2 - 2U^{\Re} \right) \quad (613)$$

$$= \frac{S}{2} \left( \exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re} \right), \quad (614)$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \right\rangle_B = \left\langle \frac{B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01}}{2} \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \right\rangle_B \quad (615)$$

$$= \frac{1}{4i} \left\langle \left( B_1^+ B_0^-(\tau) + B_0^+ B_1^-(\tau) - B_{10} - B_{01} \right) \left( B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01} \right) \right\rangle_B \quad (616)$$

$$= \frac{1}{4i} \left\langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- \right. \quad (617)$$

$$\left. + B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_{10} B_0^+ B_1^- + B_{10} B_1^+ B_0^- - B_{10} B_{10} + B_{10} B_{01} - B_{01} B_0^+ B_1^- + B_{01} B_1^+ B_0^- - B_{01} B_{10} + B_{01} B_{01} \right\rangle \quad (618)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_{10}^- - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_{10} \rangle \quad (619)$$

$$- B_0^+ B_1^-(\tau) B_{01} \rangle \quad (620)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_{10} B_{10}^- - B_{10} B_{01} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_{01} B_{10} - B_{01} B_{01} \rangle \quad (621)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_{10} B_{10} + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_1^+ B_0^- - B_{01} B_{01} \rangle \quad (622)$$

$$= \frac{1}{4i} (\exp(\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S + U^* S - US) \quad (623)$$

$$= \frac{1}{4i} (-U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S + U^* S - US) \quad (624)$$

$$= \frac{S}{4i} (-U^* \exp(-\phi(\tau)) + U \exp(-\phi(\tau)) + U^* - U) \quad (625)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (626)$$

$$= \frac{2iU^3 S}{4i} (\exp(-\phi(\tau)) - 1) \quad (627)$$

$$= \frac{U^3 S}{2} (\exp(-\phi(\tau)) - 1), \quad (628)$$

$$\langle \widetilde{B}_y(\tau) \widetilde{B}_x(0) \rangle_B = \left\langle \frac{B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01}}{2i} \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2} \right\rangle_B \quad (629)$$

$$= \frac{1}{4i} \left\langle \left( B_0^+ B_1^-(\tau) - B_1^+ B_0^-(\tau) + B_{10} - B_{01} \right) \left( B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01} \right) \right\rangle_B \quad (630)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- \rangle \quad (631)$$

$$+ B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} + B_{10} B_1^+ B_0^- + B_{10} B_0^+ B_1^- - B_{10} B_{10} - B_{10} B_{01} - B_{01} B_1^+ B_0^- - B_{01} B_0^+ B_1^- + B_{01} B_{10} + B_{01} B_{01} \rangle \quad (632)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- \rangle \quad (633)$$

$$+ B_1^+ B_0^-(\tau) B_{10} + B_1^+ B_0^-(\tau) B_{01} \rangle \quad (634)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_{01} B_{10} - B_{01} B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_{10} B_{10} + B_{10} B_{01} \rangle \quad (635)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- - B_{01} B_{01} - B_1^+ B_0^-(\tau) B_1^+ B_0^- - B_1^+ B_0^-(\tau) B_0^+ B_1^- + B_{10} B_{10} \rangle \quad (636)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + B_{10}^2 - B_{10}^{*2}) \quad (637)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U^* S - US) \quad (638)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (639)$$

$$= \frac{2iU^3 S}{4i} (\exp(-\phi(\tau)) - 1) \quad (640)$$

$$= -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}, \quad (641)$$

$$\langle B_1^+ B_0^-(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \Pi_{\mathbf{k}} \exp\left(\frac{1}{2} \left( \frac{v_{1\mathbf{k}'}^* v_{0\mathbf{k}'} - v_{1\mathbf{k}'} v_{0\mathbf{k}'}}{\omega_{\mathbf{k}}^2} \right)\right) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} \langle \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right) \rangle \quad (642)$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}, \quad (643)$$

$$\langle B_0^+ B_1^-(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger \rangle_B = -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01}. \quad (644)$$

$$\langle B_0^+ B_1^-(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}, \quad (645)$$

$$\langle \widetilde{B}_x(\tau) \widetilde{B}_{iz}(0) \rangle_B = \frac{1}{2} \sum_{\mathbf{k}'} \left( -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01} \right. \quad (646)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \quad (647)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left( -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01} \right. \quad (648)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \quad (649)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left( -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) (N_{\mathbf{k}'} + 1) \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{10} + \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{01} \right) \right. \quad (650)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) B_{10} + \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) B_{01} \right) \right) \quad (651)$$





$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( - \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} \right) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( - \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (710)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \quad (711)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} \left( - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (712)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \quad (713)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} \left( (-g_{i\mathbf{k}'} + v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (714)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \quad (715)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + N_{\mathbf{k}'}) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (B_{10} + (N_{\mathbf{k}'} + 1)) \right) \quad (716)$$

$$= \frac{1}{i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) \right) \quad (717)$$

$$= i \sum_{\mathbf{k}'} \left( e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} \right) \quad (718)$$

$$= i \sum_{\mathbf{k}'} \left( e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} \right) \quad (719)$$

$$= i B_{10}^{\Re} \sum_{\mathbf{k}'} \left( e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* N_{\mathbf{k}'} \right). \quad (720)$$

The correlation functions are equal to:

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (721)$$

$$U = \prod_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right), \quad (722)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right), \quad (723)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{|B_{10}|^2}{2} \left( U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^{\Re} - 1 \right), \quad (724)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{|B_{10}|^2}{2} \left( \exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re} \right), \quad (725)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (726)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (727)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (728)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (729)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = i B_{10}^{\Re} \sum_{\mathbf{k}} \left( e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right), \quad (730)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Re} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right). \quad (731)$$

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \quad (732)$$

$$v_{i\mathbf{k}} = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}). \quad (733)$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i(\omega)}{\omega} d\omega. \quad (734)$$

The integral version of the correlation functions are given by:

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}})(g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (735)$$

$$= \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))^* (g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}})) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (736)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) g_{j\mathbf{k}}^* (1 - F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}))^* g_{j\mathbf{k}} (1 - F_j(\omega_{\mathbf{k}})) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (737)$$

$$\approx \int_0^\infty \left( \sqrt{J_i(\omega) J_j^*(\omega)} (1 - F_i(\omega)) (1 - F_j^*(\omega)) e^{i\omega\tau} N(\omega) + \sqrt{J_i^*(\omega) J_j(\omega)} (1 - F_i^*(\omega)) (1 - F_j(\omega)) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega, \quad (738)$$

$$U = \prod_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (739)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \quad (740)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}) g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (741)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}) g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (742)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* g_{1\mathbf{k}} F_0^*(\omega_{\mathbf{k}}) F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} g_{1\mathbf{k}}^* F_0(\omega_{\mathbf{k}}) F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (743)$$

$$\approx \exp \left( \int_0^\infty \frac{\sqrt{J_0^*(\omega) J_1(\omega)} F_0^*(\omega) F_1(\omega) - \sqrt{J_0(\omega) J_1^*(\omega)} F_0(\omega) F_1^*(\omega)}{\omega^2} d\omega \right), \quad (744)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (745)$$

$$= \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (746)$$

$$\approx \int_0^\infty \left| \frac{\sqrt{J_1(\omega) F_1(\omega)} - \sqrt{J_0(\omega) F_0(\omega)}}{\omega} \right|^2 \left( -i \sin(\omega\tau) + \cos(\omega\tau) \coth \left( \frac{\beta\omega}{2} \right) \right) d\omega, \quad (747)$$

$$B_{10} = \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left( \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (748)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left( \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (749)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left( \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \right) \quad (750)$$

$$\approx \exp \left( -\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1(\omega) F_1(\omega)} - \sqrt{J_0(\omega) F_0(\omega)}}{\omega} \right|^2 \coth \left( \frac{\beta\omega}{2} \right) d\omega \right) \exp \left( \int_0^\infty \frac{1}{2} \left( \frac{\sqrt{J_0(\omega) J_1^*(\omega)} F_0(\omega) F_1^*(\omega) - \sqrt{J_0^*(\omega) J_1(\omega)} F_0^*(\omega) F_1(\omega)}{\omega^2} \right) d\omega \right), \quad (751)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{|B_{10}|^2}{2} (U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^{\Re} - 1), \quad (752)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{|B_{10}|^2}{2} (\exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re}), \quad (753)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (754)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1), \quad (755)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (756)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))^* \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (757)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left( g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right)^* - g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}))^* \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (758)$$

$$Q(\omega) = \sqrt{J_i(\omega)} (1 - F_i(\omega)) \left( \frac{\sqrt{J_1(\omega) F_1(\omega)} - \sqrt{J_0(\omega) F_0(\omega)}}{\omega} \right)^*, \quad (759)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B \approx i B_{10}^{\Im} \int_0^\infty (Q(\omega) N(\omega) e^{i\omega\tau} - Q^*(\omega) (N(\omega) + 1) e^{-i\omega\tau}) d\omega, \quad (760)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (761)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left( g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} - g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (762)$$

$$\approx iB_{10}^{\mathfrak{S}} \int_0^\infty \left( Q^*(\omega) N(\omega) e^{i\omega\tau} - Q(\omega) (N(\omega) + 1) e^{-i\omega\tau} \right) d\omega, \quad (763)$$

$$\langle \widetilde{B}_{iz}(\tau) \widetilde{B}_y(0) \rangle_B = iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left( e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (764)$$

$$= iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left( e^{-i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (765)$$

$$\approx iB_{10}^{\mathfrak{R}} \int_0^\infty \left( e^{-i\omega\tau} Q^*(\omega) (N(\omega) + 1) - e^{i\omega\tau} Q(\omega) N(\omega) \right) d\omega, \quad (766)$$

$$\langle \widetilde{B}_y(\tau) \widetilde{B}_{iz}(0) \rangle_B = iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \quad (767)$$

$$= iB_{10}^{\mathfrak{R}} \sum_{\mathbf{k}} \left( g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \quad (768)$$

$$= iB_{10}^{\mathfrak{R}} \int_0^\infty \left( e^{i\omega\tau} Q^*(\omega) N(\omega) - e^{-i\omega\tau} Q(\omega) (N(\omega) + 1) \right) d\omega. \quad (769)$$

The eigenvalues of the Hamiltonian  $\overline{H_{\bar{S}}}$  are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}(\overline{H_{\bar{S}}}) \lambda + \text{Det}(\overline{H_{\bar{S}}}) = 0. \quad (770)$$

The solutions of this equation written in terms of  $\eta$  and  $\xi$  as defined in the previous section are given by  $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$  and they satisfy  $H_S |\pm\rangle = \lambda_{\pm} |\pm\rangle$ . Using this notation is possible to write  $H_{\bar{S}} = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$ .

The time-dependence of the system operators  $\widetilde{A}_i(t)$  may be made explicit using the Fourier decomposition, in the case for time-independent  $\overline{H_{\bar{S}}}$  we will obtain:

$$\widetilde{A}_i(\tau) = e^{i\overline{H_{\bar{S}}}\tau} A_i e^{-i\overline{H_{\bar{S}}}\tau} \quad (771)$$

$$= \sum_w e^{-i w \tau} \mathcal{A}_i(w). \quad (772)$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case  $w \in \{0, \pm\eta\}$ .

In order to use the equation (772) to descompose the equation (355) we need to consider the time ordering operator  $\mathcal{T}$ , it's possible to write using the Dyson series or the expansion of the operator of the form  $U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right)$  like:

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right) \quad (773)$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) \dots H(t_n). \quad (774)$$

Here  $0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t$  is a partition of the set  $[0, t]$ . We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian  $H_E(t)$  such that  $\mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right) \equiv \exp(-it H_E(t))$ . The effective Hamiltonian is expanded in a series of terms of increasing order in time  $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots$  so we can write:

$$U(t) = \exp \left( -it \left( H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots \right) \right). \quad (775)$$

The terms can be found expanding  $\mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_{\bar{S}}}(t') \right)$  and  $U(t)$  then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_S}(t') dt', \quad (776)$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' [\overline{H_S}(t'), \overline{H_S}(t'')], \quad (777)$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' ([\overline{H_S}(t'), \overline{H_S}(t'')], \overline{H_S}(t''')) + [[\overline{H_S}(t'''), \overline{H_S}(t'')], \overline{H_S}(t')]. \quad (778)$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A}_i(t) = U^\dagger(t) A_i(t) U(t) \quad (779)$$

$$= e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \quad (780)$$

$$= \sum_{w(t)} e^{-iw(t)t} \mathcal{A}_i(w(t)). \quad (781)$$

$w(t)$  belongs to the set of differences of eigenvalues of  $H_E(t)$  that depends of the time. As we can see the decomposition matrices are time-dependent as well.

Extending the Fourier decomposition to the matrix  $\widetilde{A}_j(t - \tau, t)$  using the Magnus expansion generates:

$$\widetilde{A}_j(t - \tau, t) = U(t) U^\dagger(t - \tau) A_j(t) U(t - \tau) U^\dagger(t) \quad (782)$$

$$= e^{-itH_E(t)} e^{i(t-\tau)H_E(t-\tau)} A_j(t) e^{-i(t-\tau)H_E(t-\tau)} e^{itH_E(t)} \quad (783)$$

$$= e^{-itH_E(t)} \left( \sum_{w'(t-\tau)} e^{-i(t-\tau)w(t-\tau)} \mathcal{A}_j(w(t-\tau)) \right) e^{itH_E(t)} \quad (784)$$

$$= \sum_{w(t), w'(t-\tau)} e^{iw'(t)t} e^{-i(t-\tau)w(t-\tau)} \mathcal{A}_j(w(t-\tau), w'(t)) \quad (785)$$

$$= \sum_{w(t), w'(t-\tau)} e^{iw'(t)t} e^{-i(t-\tau)w(t-\tau)} \mathcal{A}_j(w(t-\tau), w'(t)) \quad (786)$$

$$= \sum_{w(t), w'(t-\tau)} e^{i\tau w(t-\tau)} e^{-it(w(t-\tau) - w'(t))} \mathcal{A}_j(w(t-\tau), w'(t)) \quad (787)$$

where  $w'(t - \tau)$  and  $w(t)$  belongs to the set of the differences of the eigenvalues of the Hamiltonian  $\overline{H_E}(t - \tau)$  and  $\overline{H_E}(t)$  respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (772) for a general  $2 \times 2$  matrix in a given time let's write the matrix  $A_i$  in the base  $V = \{|+\rangle, |-\rangle\}$  in the following way:

$$A_i = \sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle |\alpha\rangle \langle \beta|. \quad (788)$$

Given that  $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$ , then using the Zassenhaus formula we obtain:

$$e^{i\overline{H_E}\tau} = e^{i(\lambda_+|+\rangle\langle+| + \lambda_-|-\rangle\langle-|)\tau} \quad (789)$$

$$= e^{i\lambda_+|+\rangle\langle+|\tau} e^{i\lambda_-|-\rangle\langle-|\tau} \quad (790)$$

$$= (|-\rangle\langle-| + e^{i\lambda_+\tau}|+\rangle\langle+|) (|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|) \quad (791)$$

$$= e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|. \quad (792)$$

Calculating the transformation (772) directly using the previous relationship we find that:



$$U^\dagger(\tau) A_i(\tau) U(\tau) = (e^{i\lambda_+\tau} |+\rangle\langle+| + e^{i\lambda_-\tau} |-\rangle\langle-|) \left( \sum_{\alpha, \beta \in \mathbb{V}} \langle \alpha | A_i(\tau) | \beta \rangle | \alpha \rangle \langle \beta | \right) (e^{-i\lambda_+\tau} |+\rangle\langle+| + e^{-i\lambda_-\tau} |-\rangle\langle-|) \quad (793)$$

$$= \langle + | A_i(\tau) | + \rangle |+\rangle\langle+| + e^{i\eta\tau} \langle + | A_i(\tau) | - \rangle |+\rangle\langle-| + e^{-i\eta\tau} \langle - | A_i(\tau) | + \rangle |-\rangle\langle+| + \langle - | A_i(\tau) | - \rangle |-\rangle\langle-|. \quad (794)$$

$$= \mathcal{A}_i(0) + \mathcal{A}_i(-w) e^{iw\tau} + \mathcal{A}_i(w) e^{-iw\tau} \quad (795)$$

Here  $w = \lambda_+ - \lambda_-$ . Comparing the RHS of the equations (772) and the explicit expression for  $\widetilde{A}_i(\tau)$  in (780), we obtain the form of the expansion matrices of the Fourier decomposition for a general  $2 \times 2$  matrix:

$$\mathcal{A}_i(0) = \langle + | A_i(\tau) | + \rangle |+\rangle\langle+| + \langle - | A_i(\tau) | - \rangle |-\rangle\langle-|, \quad (796)$$

$$\mathcal{A}_i(-w) = \langle + | A_i(\tau) | - \rangle |+\rangle\langle-|, \quad (797)$$

$$\mathcal{A}_i(w) = \langle - | A_i(\tau) | + \rangle |-\rangle\langle+|. \quad (798)$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e.  $\widetilde{A}_i(\tau) = \widetilde{A}_i^\dagger(\tau)$  and  $\widetilde{B}_i(\tau) = \widetilde{B}_i^\dagger(\tau)$  we can use the equation (772) to write the master equation in the following neater form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau C_i(t) C_j(t-\tau) \left( \Lambda_{ij}(\tau) \left[ A_i, \widetilde{A}_j(t-\tau, t) \overline{\rho_S}(t) \right] + \Lambda_{ji}(-\tau) \left[ \overline{\rho_S}(t) \widetilde{A}_j(t-\tau, t), A_i \right] \right) \quad (799)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau C_i(t) C_j(t-\tau) \left( \Lambda_{ij}(\tau) \left[ A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_j(w(t-\tau), w'(t)) \overline{\rho_S}(t) \right] \right) \quad (800)$$

$$- \Lambda_{ji}(-\tau) \left[ A_i, \overline{\rho_S}(t) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_j(w(t-\tau), w'(t)) \right] \quad (801)$$

Given that  $\mathcal{A}_j(w(t-\tau), w'(t)) = \mathcal{A}_j^\dagger(-w(t-\tau), -w'(t))$  from the Fourier decomposition (772) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\Lambda_{ij}(\tau) = \text{Tr}_B \left( \widetilde{B}_i(t) \widetilde{B}_j(s) \rho_B \right) \quad (802)$$

$$= \text{Tr}_B \left( \widetilde{B}_i(\tau) \widetilde{B}_j(0) \rho_B \right). \quad (803)$$

Let's define:

$$\mathcal{A}_j(w(t-\tau), w'(t)) = \mathcal{A}_{jww'}(t-\tau, t) \quad (804)$$

The master equation can be re-written in the following form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[ A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (805)$$

$$+ \sum_{ijww'} \Lambda_{ji}(-\tau) \left[ A_i, \overline{\rho_S}(t) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (806)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[ A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (807)$$

$$+ \sum_{ijww'} \Lambda_{ji}(-\tau) \left[ A_i, \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (808)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[ A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (809)$$

$$+ \sum_{ijww'} \Lambda_{ji}(-\tau) \left[ A_i, \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (810)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[ A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (811)$$

$$- \Lambda_{ji}(-\tau) \left[ A_i, \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (812)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \text{Tr}_B \left( \left[ A_i, \widetilde{B}_i(\tau) \widetilde{B}_j(0) \rho_B e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \right) \quad (813)$$

$$- \left[ A_i, \widetilde{B}_j(-\tau) \widetilde{B}_i(0) \rho_B \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (814)$$

Given that if we define:

$$D_{ijww'}(t-\tau, t) = C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \quad (815)$$

then

$$D_{ijww'}^\dagger(t-\tau, t) = \left( C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right)^\dagger \quad (816)$$

$$= \Lambda_{ij}^*(\tau) C_i(t) C_j(t-\tau) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}^\dagger(t-\tau, t) \quad (817)$$

We used the fact that  $C_i(t), C_j(t-\tau)$  are real. Now let's consider the following trace recalling that  $\text{Tr}(A)^* = \text{Tr}(A^\dagger)$  so:

$$\text{Tr}_B \left( \widetilde{B}_j(-\tau) \widetilde{B}_i(0) \rho_B \right) = \text{Tr}_B \left( e^{-i\tau H_E(\tau)} B_j e^{i\tau H_E(\tau)} B_i \rho_B \right) \quad (818)$$

$$= \text{Tr}_B \left( B_j e^{i\tau H_E(\tau)} B_i \rho_B e^{-i\tau H_E(\tau)} \right) \text{ (by cyclic permutivity of trace)} \quad (819)$$

$$= \text{Tr}_B \left( B_j e^{i\tau H_E(\tau)} B_i e^{-i\tau H_E(\tau)} \rho_B \right) \text{ (by independence of Hilbert spaces)} \quad (820)$$

$$= \text{Tr}_B \left( B_j \widetilde{B}_i(\tau) \rho_B \right) \text{ (by definition of time evolution)} \quad (821)$$

$$= \text{Tr}_B \left( B_j \widetilde{B}_i(\tau) \rho_B \right) \quad (822)$$

$$= \text{Tr}_B \left( \rho_B B_j \widetilde{B}_i(\tau) \right) \quad (823)$$

$$= \text{Tr}_B \left( \left( \widetilde{B}_i(\tau) B_j \rho_B \right)^\dagger \right) \text{ (by definition of adjoint)} \quad (824)$$

$$= \text{Tr}_B \left( \widetilde{B}_i(\tau) B_j \rho_B \right)^* \quad (825)$$

$$= \Lambda_{ij}^*(\tau) \quad (826)$$

So we can write the master equation like:

$$\frac{d\bar{\rho}_S(t)}{dt} = -i[\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \left( \Lambda_{ij}(\tau) \left[ A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_j(w(t-\tau), w'(t)) \bar{\rho}_S(t) \right] \right. \quad (827)$$

$$\left. - \Lambda_{ij}^*(\tau) \left[ A_i, \bar{\rho}_S(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_j^\dagger(w(t-\tau), w'(t)) \right] \right) \quad (828)$$

$$= -i[\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijww'} \int_0^t d\tau \left( [A_i, D_{ijww'}(t-\tau, t) \bar{\rho}_S(t)] - [A_i, \bar{\rho}_S(t) D_{ijww'}^\dagger(t-\tau, t)] \right) \quad (829)$$

Let's define the response matrix in the following way.

$$\mathcal{D}_{ijww'}(t) = \int_0^t d\tau D_{ijww'}(t-\tau, t) \quad (830)$$

Then the master equation can be written as:

$$\frac{d\bar{\rho}_S(t)}{dt} = -i[\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijww'} \left( [A_i, \mathcal{D}_{ijww'}(t) \bar{\rho}_S(t)] - [A_i, \bar{\rho}_S(t) \mathcal{D}_{ijww'}^\dagger(t)] \right) \quad (831)$$

If we extend the upper limit of integration to  $\infty$  in the equation (830) then the system will be independent of any preparation at  $t = 0$ , so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V} \frac{d\bar{\rho}_S(t)}{dt} e^V = \frac{d(e^{-V} \bar{\rho}_S e^V)}{dt} \quad (832)$$

$$= \frac{d\rho_S}{dt} \quad (833)$$

$$= -ie^{-V} [\bar{H}_S(t), \bar{\rho}_S(t)] e^V - \sum_{ijww'} \int_0^t d\tau \left( e^{-V} [A_i, D_{ijww'}(t-\tau, t) \bar{\rho}_S(t)] e^V - e^{-V} [A_i, \bar{\rho}_S(t) D_{ijww'}^\dagger(t-\tau, t)] e^V \right). \quad (834)$$

For a product we have the following:

$$e^{-V} \overline{AB} e^V = e^{-V} \overline{A} \overline{B} e^V \quad (835)$$

$$= e^{-V} \overline{A} e^V e^{-V} \overline{B} e^V \quad (836)$$

$$= (e^{-V} \overline{A} e^V) (e^{-V} \overline{B} e^V) \quad (837)$$

$$= \overline{AB}. \quad (838)$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V} \overline{[A, B]} e^V = e^{-V} \overline{(AB - BA)} e^V \quad (839)$$

$$= e^{-V} \overline{AB} e^V - e^{-V} \overline{BA} e^V \quad (840)$$

$$= \overline{AB} - \overline{BA} \quad (841)$$

$$= [A, B]. \quad (842)$$

So we will obtain that

$$\frac{d\rho_S}{dt} = -ie^{-V} [\overline{H_S}(t), \overline{\rho_S}(t)] e^V - e^{-V} \sum_{ijww'} \left( [A_i, \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t)] \right) e^V \quad (843)$$

$$= -ie^{-V} [\overline{H_S}(t), \overline{\rho_S}(t)] e^V - \sum_{ijww'} \left( e^{-V} [A_i, \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t)] e^V - e^{-V} [A_i, \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t)] e^V \right) \quad (844)$$

$$= -i [H_S(t), \rho_S(t)] - \sum_{ijww'} \left( [e^{-V} A_i e^V, e^{-V} \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t) e^V] - [e^{-V} A_i e^V, e^{-V} \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t) e^V] \right) \quad (845)$$

$$= -i [H_S(t), \rho_S(t)] - \sum_{ijww'} \left( [e^{-V} A_i e^V, e^{-V} \mathcal{D}_{ijww'}(t) e^V e^{-V} \overline{\rho_S}(t) e^V] - [e^{-V} A_i e^V, e^{-V} \overline{\rho_S}(t) e^V e^{-V} \mathcal{D}_{ijww'}^\dagger(t) e^V] \right) \quad (846)$$

$$= -i [H_S(t), \rho_S(t)] - \sum_{ijww'} \left( [e^{-V} A_i e^V, e^{-V} \mathcal{D}_{ijww'}(t) e^V \rho_S(t)] - [e^{-V} A_i e^V, \rho_S(t) e^{-V} \mathcal{D}_{ijww'}^\dagger(t) e^V] \right). \quad (847)$$

## V. LIMIT CASES

In order to show the plausibility of the master equation (831) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

### A. Time-independent variational quantum master equation

At first let's show that the master equation (831) reproduces the results of the reference [1], for the latter case we have that  $i, j \in \{1, 2, 3\}$  and  $\omega \in (0, \pm\eta)$ . The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left( \delta + \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) |1\rangle\langle 1| + \frac{\Omega}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (848)$$

After performing the transformation (24) on the Hamiltonian (848) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R) |1\rangle\langle 1| + \frac{\Omega_r}{2} \sigma_x, \quad (849)$$

$$\overline{H_I} = B_z |1\rangle\langle 1| + \frac{\Omega}{2} (B_x \sigma_x + B_y \sigma_y), \quad (850)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (851)$$

The Hamiltonian (849) differs from the transformed Hamiltonian  $H_S$  of the reference written like  $H_S = \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z + \frac{\Omega_r}{2} \sigma_x$  by a term proportional to the identity, this can be seen in the following way taking  $\epsilon = \delta + R$

$$(\delta + R) |1\rangle\langle 1| - \frac{\delta}{2} \mathbb{I} = \left( \frac{\delta}{2} + R \right) |1\rangle\langle 1| - \frac{\delta}{2} |0\rangle\langle 0| \quad (852)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\delta + R}{2} \sigma_z \quad (853)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z. \quad (854)$$

In this Hamiltonian we can write  $A_i = \sigma_x$ ,  $A_2 = \sigma_y$  and  $A_3 = \frac{I + \sigma_z}{2} = |1\rangle\langle 1|$  with  $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$ . In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix

$\overline{H_S}$ . Given that  $\overline{H_S} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$  then  $\text{Tr}(\overline{H_S}) = R$  and  $\text{Det}(\overline{H_S}) = \frac{R^2 - \epsilon^2}{4} - \frac{\Omega_r^2}{4}$  then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \quad (855)$$

$$\lambda_{\pm} = \frac{R \pm \sqrt{(-R)^2 - 4 \left( \frac{R^2 - \epsilon^2 - \Omega_r^2}{4} \right)}}{2} \quad (856)$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \quad (857)$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \quad (858)$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \quad (859)$$

$$\lambda_{\pm} = \frac{R \pm \eta}{2}. \quad (860)$$

For  $\lambda_+ = \frac{R+\eta}{2}$  we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \quad (861)$$

so the eigenvector  $|+\rangle = a|0\rangle + b|1\rangle$  satisfies  $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$ , so  $a = \frac{\Omega_r}{\epsilon+\eta}b$  then the normalized eigenvector is  $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|1\rangle$  with  $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$  and  $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$ . The vector is written in reduced way like  $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$ .

For  $\lambda_- = \frac{R-\eta}{2}$  we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \quad (862)$$

so the eigenvector  $|+\rangle = a|0\rangle + b|1\rangle$  satisfies  $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$ , so  $a = -\frac{\epsilon+\eta}{\Omega_r}b$  then the normalized eigenvector is  $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|1\rangle$ . The vector is written in reduced way like  $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$ . Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon + \eta}{2}, \quad (863)$$

$$\lambda_- = \frac{\epsilon - \eta}{2}, \quad (864)$$

$$|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \quad (865)$$

$$|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle, \quad (866)$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}, \quad (867)$$

$$\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}. \quad (868)$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2} \tan^{-1} \left( \frac{\Omega_r}{\epsilon} \right). \quad (869)$$

We can obtain the value of  $\tan(\theta)$  through the following trigonometry identity for  $x = \tan^{-1} \left( \frac{\Omega_r}{\epsilon} \right)$ .

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x) + 1}. \quad (870)$$

So the value of  $\tan(\theta)$  using (870) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}} + 1} \quad (871)$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}}{\frac{\epsilon + \sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}} \quad (872)$$

$$= \frac{\Omega_r}{\epsilon + \eta}. \quad (873)$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (797)-(798) and the fact that  $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$  and  $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$  with  $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$  and  $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$ :

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (874)$$

$$= 2 \sin(\theta) \cos(\theta) \quad (875)$$

$$= \sin(2\theta), \quad (876)$$

$$\langle -|\sigma_x|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (877)$$

$$= -2 \sin(\theta) \cos(\theta) \quad (878)$$

$$= -\sin(2\theta), \quad (879)$$

$$\langle -|\sigma_x|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (880)$$

$$= \cos^2(\theta) - \sin^2(\theta) \quad (881)$$

$$= \cos(2\theta), \quad (882)$$

$$\langle +|\sigma_y|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (883)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (884)$$

$$= 0, \quad (885)$$

$$\langle -|\sigma_y|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (886)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (887)$$

$$= 0, \quad (888)$$

$$\langle -|\sigma_y|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (889)$$

$$= i \cos^2(\theta) + i \sin^2(\theta) \quad (890)$$

$$= i. \quad (891)$$

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (892)$$

$$= \cos(\theta) \cos(\theta) \quad (893)$$

$$= \cos^2(\theta), \quad (894)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (895)$$

$$= \sin(\theta) \sin(\theta) \quad (896)$$

$$= \sin^2(\theta), \quad (897)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (898)$$

$$= -\sin(\theta) \cos(\theta) \quad (899)$$

$$= -\sin(\theta) \cos(\theta). \quad (900)$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta) (|+\rangle\langle+| - |-\rangle\langle-|), \quad (901)$$

$$A_1(\eta) = \cos(2\theta) |-\rangle\langle+|, \quad (902)$$

$$A_2(0) = 0, \quad (903)$$

$$A_2(\eta) = i|-\rangle\langle+|, \quad (904)$$

$$A_3(0) = \cos^2(\theta) |+\rangle\langle+| + \sin^2(\theta) |-\rangle\langle-|, \quad (905)$$

$$A_3(\eta) = -\sin(\theta) \cos(\theta) |-\rangle\langle+|. \quad (906)$$

Now to prove the fact that the model of the “Time-independent variational quantum master equation” is a special case the master equation (834) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (815) is equivalent to:

$$\mathcal{D}_{ijww'}(t) = \int_0^t d\tau D_{ijww'}(t - \tau, t) \quad (907)$$

$$= \int_0^t d\tau C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau) - w'(t))} \mathcal{A}_j(w(t - \tau), w'(t)) \quad (908)$$

$$= \int_0^t d\tau C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) e^{i\tau w} e^{-it(w - w')} \mathcal{A}_j(w, w'). \quad (909)$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by  $\Lambda'_{ij}(\tau)$  relate with the correlation functions defined in the equation (420) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau). \quad (910)$$

So the response matrix can be rewritten as:

$$\mathcal{D}_{ijww'}(t) = \left( \int_0^t d\tau \Lambda'_{ij}(\tau) e^{i\tau w} e^{-it(w - w')} \right) \mathcal{A}_j(w, w') \quad (911)$$

Let's define the response function like:

$$K_{ij}(w, w', t) = \int_0^t C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) e^{iw\tau} e^{-it(w-w')} d\tau \quad (912)$$

$$= \int_0^t \Lambda'_{ij}(\tau) e^{iw\tau} e^{-it(w-w')} d\tau \quad (913)$$

$$= K_{ijww'}(t). \quad (914)$$

Then we have the following equivalence:

$$\mathcal{D}_{ijww'}(t) = K_{ijww'}(t) \mathcal{A}_j(w, w') \quad (915)$$

$$= K_{ijww'}(t) \mathcal{A}_{jww'} \quad (916)$$

We can proof that

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left( [A_i, \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t)] \right) \quad (917)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left( [A_i, K_{ijww'}(t) \mathcal{A}_{jww'} \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) K_{ijww'}^*(t) \mathcal{A}_{jww'}^\dagger] \right) \quad (918)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left( K_{ijww'}(t) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t)] - K_{ijww'}^*(t) [A_i, \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] \right) \quad (919)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left( (K_{ijww'}^{\Re}(t) + iK_{ijww'}^{\Im}(t)) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t)] - (K_{ijww'}^{\Re}(t) - iK_{ijww'}^{\Im}(t)) [A_i, \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] \right) \quad (920)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} K_{ijww'}^{\Re}(t) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t) - \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] - i \sum_{ijww'} K_{ijww'}^{\Im}(t) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t) + \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] \quad (921)$$

Using the notation of the master equation (831), we can say that  $C_1(t) = \frac{\Omega}{2} = C_2(t)$  and  $C_3(t) = 1$ , being  $\Omega$  a constant. Furthermore given that  $\overline{H_S}$  is time-independent then  $B(t) = B$ . Taking the equations(721)-(731) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (420) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left( \widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (922)$$

$$= \frac{\Omega_r^2}{8} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \quad (923)$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left( \widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (924)$$

$$= \frac{\Omega_r^2}{8} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} \right), \quad (925)$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega J(\omega) (1 - F(\omega))^2 G_+(\tau), \quad (926)$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2} \int_0^\infty d\omega \frac{J(\omega)}{\omega} F(\omega) (1 - F(\omega)) iG_-(\tau), \quad (927)$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \quad (928)$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \quad (929)$$

$$= \Lambda'_{13}(\tau) \quad (930)$$

$$= \Lambda'_{31}(\tau) \quad (931)$$

$$= 0. \quad (932)$$



Finally taking the Hamiltonian (848) and given that to reproduce this Hamiltonian we need to impose in (5) that  $V_{10}(t) = \frac{\Omega}{2}$ ,  $\varepsilon_0(t) = 0$  and  $\varepsilon_1(t) = \delta$ , then we obtain that  $\text{Det}(\overline{H_S}) = -\frac{\Omega^2}{4}$ ,  $\text{Tr}(\overline{H_S}) = \epsilon$ . Now  $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$  and using the equation (336) we have that:

$$f_k = \frac{g_k \left( 1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left( \epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k} \right)} \quad (933)$$

$$= \frac{g_k \left( 1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left( 1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k} \right)}. \quad (934)$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (418) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then  $U(t)U^\dagger(t-\tau) = U(\tau)$ . From the equation (831) and using the fact that

$$\widetilde{A}_j(t-\tau, t) = U(\tau) A_j U(-\tau) \quad (935)$$

$$= \sum_w e^{iw\tau} \mathcal{A}_j(-w) \quad (936)$$

$$= \sum_w e^{-iw\tau} \mathcal{A}_j(w). \quad (937)$$

because the matrices  $U(t)$  and  $U(t-\tau)$  commute from the fact that  $H_S(t)$  and  $H_S(t-\tau)$  commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \frac{1}{2} \sum_{ij} \sum_w \gamma_{ij}(w, t) \left[ A_i, \mathcal{A}_j(w) \overline{\rho_S}(t) - \overline{\rho_S}(t) \mathcal{A}_j^\dagger(w) \right] \quad (938)$$

$$- \sum_{ij} \sum_w S_{ij}(w, t) \left[ A_i, \mathcal{A}_j(w) \overline{\rho_S}(t) + \overline{\rho_S}(t) \mathcal{A}_j^\dagger(w) \right]. \quad (939)$$

where  $\mathcal{A}_j^\dagger(w) = \mathcal{A}_j(-w)$ , as we can see the equation (939) contains the rates and energy shifts  $\gamma_{ij}(w, t) = 2K_{ij}^{\Re}(w, t)$  and  $S_{ij}(w, t) = K_{ij}^{\Im}(w, t)$ , respectively, defined in terms of the response functions

$$K_{ij}^{\Im}(w, t) = \int_0^t \Lambda'_{ij}(\tau) e^{iw\tau} d\tau.$$

The fact  $\mathcal{A}_j^\dagger(w) = \mathcal{A}_j(-w)$  can be verified directly for a  $2 \times 2$  matrix. given that  $\overline{H_S}$  is independent of time then we have that:

$$e^{i\overline{H_S}(t-\tau)} = e^{i(\lambda_+|+\rangle\langle+| + \lambda_-|-\rangle\langle-|)(t-\tau)} \quad (940)$$

$$= e^{i\lambda_+|+\rangle\langle+|(t-\tau)} e^{i\lambda_-|-\rangle\langle-|(t-\tau)} \quad (941)$$

$$= \left( |-\rangle\langle-| + e^{i\lambda_+(t-\tau)} |+\rangle\langle+| \right) \left( |+\rangle\langle+| + e^{i\lambda_-(t-\tau)} |-\rangle\langle-| \right) \quad (942)$$

$$= e^{i\lambda_+(t-\tau)} |+\rangle\langle+| + e^{i\lambda_-(t-\tau)} |-\rangle\langle-|. \quad (943)$$

Where  $\lambda_+$ ,  $\lambda_-$  are the eigenvalues associated to the eigenvectors  $|+\rangle\langle+|$ ,  $|-\rangle\langle-|$  of  $\overline{H_S}$ . Calculating the transformation (772) of (796)-(798) directly using the previous relationship we find that:

$$\widetilde{A_i(0)}(t-\tau) = \left( e^{i\lambda_+(t-\tau)}|+\chi| + e^{i\lambda_-(t-\tau)}|-\chi| \right) \left( \langle +|A_i|+ \rangle|+\chi| + \langle -|A_i|- \rangle|-\chi| \right) \left( e^{-i\lambda_+(t-\tau)}|+\chi| + e^{-i\lambda_-(t-\tau)}|-\chi| \right) \quad (944)$$

$$= \langle +|A_i|+ \rangle|+\chi| + \langle -|A_i|- \rangle|-\chi|, \quad (945)$$

$$\widetilde{A_i(w)}(t-\tau) = \left( e^{i\lambda_+(t-\tau)}|+\chi| + e^{i\lambda_-(t-\tau)}|-\chi| \right) \left( \langle +|A_i|- \rangle|+\chi| \right) \left( e^{-i\lambda_+(t-\tau)}|+\chi| + e^{-i\lambda_-(t-\tau)}|-\chi| \right) \quad (946)$$

$$= \langle +|A_i|- \rangle|+\chi| e^{i\omega(t-\tau)}, \quad (947)$$

$$\widetilde{A_i(-w)}(t-\tau) = \left( e^{i\lambda_+(t-\tau)}|+\chi| + e^{i\lambda_-(t-\tau)}|-\chi| \right) \left( \langle -|A_i|+ \rangle|-\chi| \right) \left( e^{-i\lambda_+(t-\tau)}|+\chi| + e^{-i\lambda_-(t-\tau)}|-\chi| \right) \quad (948)$$

$$= \langle -|A_i|+ \rangle|-\chi| e^{-i\omega(t-\tau)}. \quad (949)$$

Here  $w = \lambda_+ - \lambda_-$ . So we can see that for the equation (782) it's possible to deduce for this case of time-independent matrix  $\overline{H_S}$  if  $w \neq w'$  then  $A'_j(w, w') = 0$  so:

$$\widetilde{A_j}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_j(t) U(t-\tau) U^\dagger(t) \quad (950)$$

$$= U(t) \left( \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} A_j(w(t-\tau)) \right) U^\dagger(t) \quad (951)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} U(t) A_j(w(t-\tau)) U^\dagger(t) \quad (952)$$

$$= \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_j(w(t-\tau), w'(t)) \quad (953)$$

$$= \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_{jww'} \quad (954)$$

$$== \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_j(w) \delta_{ww'} \quad (955)$$

$$= \sum_w e^{-i(t-\tau)w} e^{itw} A_j(w) \quad (956)$$

$$= \sum_w e^{i\tau w} A_j(w) \quad (957)$$

$$= U^\dagger(-\tau) A_j U(-\tau) \quad (958)$$

So using now as reference the equation (921) and  $A'_j(w, w') = 0$  we can deduce that:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijw} K_{ij}^{\Re}(w, t) \left[ A_i, A_j(w) \overline{\rho_S}(t) - \overline{\rho_S}(t) A_j^\dagger(w) \right] - i \sum_{ijw} K_{ij}^{\Im}(w, t) \left[ A_i, A_j(w) \overline{\rho_S}(t) + \overline{\rho_S}(t) A_j^\dagger(w) \right] \quad (959)$$

## B. Time-dependent polaron quantum master equation

Following the reference [1], when  $\Omega_k \ll \omega_k$  then  $f_k \approx g_k$  so we recover the full polaron transformation. It means from the equation (107) that  $B_z = 0$ . The Hamiltonian studied is given by:

$$H = \left( \delta + \sum_{\mathbf{k}} \left( g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right) \right) |1\rangle\langle 1| + \frac{\Omega(t)}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (960)$$

If  $v_{\mathbf{k}} \approx g_{\mathbf{k}}$  then  $B(\tau) = B$ , so  $B$  is independent of the time. In order to reproduce the Hamiltonian of the equation (960) using the Hamiltonian of the equation (1) we can say that  $\delta = \varepsilon_1(t)$ ,  $\varepsilon_0(t) = 0$ ,  $V_{10}(t) = \frac{\Omega(t)}{2}$ . Now given that  $v_{\mathbf{k}} \approx g_{\mathbf{k}}$  then, in this case and using the equation (225) and (236) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1) |1\rangle\langle 1| + \frac{B\sigma_x}{2} \Omega(t), \quad (961)$$

$$\overline{H_I} = \frac{\Omega(t)}{2} (B_x \sigma_x + B_y \sigma_y). \quad (962)$$

In this case  $R_1 = \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} g_{\mathbf{k}} \right)$  from (27) and given that  $v_{\mathbf{k}} \approx g_{\mathbf{k}}$  and  $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$  then  $R_1 = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}}^{-1} |g_{\mathbf{k}}|^2) = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}} | \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} |^2)$  as expected, take  $\delta + R_1 = \delta'$ . If  $F(\omega_{\mathbf{k}}) = 1$  and using the equations (922)-(929) we can deduce that the only terms that survive are  $\Lambda_{11}(\tau)$  and  $\Lambda_{22}(\tau)$ . The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} G_+(\tau) d\omega. \quad (963)$$

Writing  $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau)$  so (963) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left( \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau) \right) d\omega. \quad (964)$$

Writing the interaction Hamiltonian (962) in the similar way to the equation (236) allow us to write  $A_1 = \sigma_x$ ,  $A_2 = \sigma_y$ ,  $B_1(t) = B_x$ ,  $B_2(t) = B_y$  and  $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$ . Now taking the equation (225) with  $\delta' |1\rangle\langle 1| = \frac{\delta'}{2} \sigma_z + \frac{\delta'}{2} \mathbb{I}$  help us to reproduce the hamiltonian of the reference [2]. Then  $\overline{H_S}$  is equal to:

$$\overline{H_S} = \frac{\delta'}{2} \sigma_z + \frac{B\sigma_x}{2} \Omega(t). \quad (965)$$

As we can see the function  $B$  is a time-independent function because we consider that  $g_{\mathbf{k}}$  doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \text{Tr}_B \left( \widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (966)$$

$$= \frac{B^2}{2} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \quad (967)$$

$$\Lambda_{22}(\tau) = \text{Tr}_B \left( \widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (968)$$

$$= \frac{B^2}{2} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} \right). \quad (969)$$

These functions match with the equations  $\Lambda_x(\tau)$  and  $\Lambda_y(\tau)$  of the reference [2] and  $\Lambda_i(\tau) = \Lambda_i(-\tau)$  for  $i \in \{x, y\}$  respectively. The master equation for this section based on the equation(418) is:

$$\frac{d\rho_S(t)}{dt} = -i \left[ \frac{\delta'}{2} \sigma_z + \frac{\Omega_r(t) \sigma_x}{2}, \rho_S(t) \right] - \sum_{i=1}^2 \int_0^t d\tau \left( C_i(t) C_i(t-\tau) \Lambda_{ii}(\tau) \left[ A_i, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right] \right. \quad (970)$$

$$\left. + C_i(t) C_i(t-\tau) \Lambda_{ii}(-\tau) \left[ \rho_S(t) \widetilde{A}_i(t-\tau, t), A_i \right] \right). \quad (971)$$

Replacing  $C_i(t) = \frac{\Omega(t)}{2}$  and  $\widetilde{A}_i(t-\tau, t) = \widetilde{\sigma}_i(t-\tau, t)$ , also using the equations (966) and (969) on the equation (971) we obtain that:

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{2} [\delta' \sigma_z + \Omega_r(t) \sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4} \int_0^t d\tau \Omega(t-\tau) ([\sigma_x, \widetilde{\sigma}_x(t-\tau, t) \rho_S(t)] \Lambda_x(\tau) \quad (972)$$

$$+ [\sigma_y, \widetilde{\sigma}_y(t-\tau, t) \rho_S(t)] \Lambda_y(\tau) + [\rho_S(t) \widetilde{\sigma}_x(t-\tau, t), \sigma_x] \Lambda_x(\tau) + [\rho_S(t) \widetilde{\sigma}_y(t-\tau, t), \sigma_y] \Lambda_y(\tau)). \quad (973)$$

As we can see  $\left[A_j, \widetilde{A}_i(t - \tau, t) \rho_S(t)\right]^\dagger = \left[\rho_S(t) \widetilde{A}_i(t - \tau, t), A_j\right]$ ,  $\Lambda_x(\tau) = \Lambda_x(-\tau)$  and  $\Lambda_y(\tau) = \Lambda_y(-\tau)$ , so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

### C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation (S17) of the reference [3] we will impose that  $F(\omega) = 0$ , so there is no transformation in this case. As we can see from the definition (420) the only term that survives is  $\Lambda_{33}(\tau)$ . Taking  $\hbar = 1$  the Hamiltonian of the reference can be written in the form:

$$H = \Delta |1\rangle\langle 1| + \frac{\Omega(t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left( g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right). \quad (974)$$

Using the equation (831), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \frac{1}{2} \sum_w \gamma_{33}(w, t) \left[ A_3, A_3(w) \rho_S(t) - \rho_S(t) A_3^\dagger(w) \right] \quad (975)$$

$$- \sum_w S_{33}(w, t) \left[ A_3, A_3(w) \rho_S(t) + \rho_S(t) A_3^\dagger(w) \right] \Bigg). \quad (976)$$

The correlation functions are relevant if  $F(\omega) = 0$  for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty d\omega J(\omega) G_+(\tau), \quad (977)$$

$$\Lambda_{33}(-\tau) = \int_0^\infty d\omega J(\omega) G_+(-\tau). \quad (978)$$

In our case  $A_3 = \frac{\mathbb{I} + \sigma_z}{2}$ , the equation (976) can be transformed in

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \sum_w (K_{33}(w, t) [A_3, A_3(w) \rho_S(t)] + K_{33}^*(w, t) [\rho_S(t) A_3(w), A_3]). \quad (979)$$

As the paper suggest we will consider that the quantum system is in resonance, so  $\Delta = 0$  and furthermore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to  $\tilde{\Omega}$ . To find the matrices  $A_3(w)$ , we have to remember that  $H_S = \frac{\Omega(t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|)$ , this Hamiltonian using the approximation  $\tilde{\Omega}$  have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\tilde{\Omega}}{2}, \quad (980)$$

$$|+\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |0\rangle), \quad (981)$$

$$\lambda_- = -\frac{\tilde{\Omega}}{2}, \quad (982)$$

$$|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \quad (983)$$

The elements of the decomposition matrices are:

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (984)$$

$$= \frac{1}{2}, \quad (985)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (986)$$

$$= \frac{1}{2}, \quad (987)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (988)$$

$$= -\frac{1}{2}. \quad (989)$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2} |+\rangle\langle +| + \frac{1}{2} |-\rangle\langle -| \quad (990)$$

$$= \frac{\mathbb{I}}{2}, \quad (991)$$

$$A_3(\eta) = -\frac{1}{2} |-\rangle\langle +| \quad (992)$$

$$= \frac{1}{4} (\sigma_z + i\sigma_y), \quad (993)$$

$$A_3(-\eta) = -\frac{1}{2} |+\rangle\langle -| \quad (994)$$

$$= \frac{1}{4} (\sigma_z - i\sigma_y). \quad (995)$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - K_{33}(\tilde{\Omega}, t) \left[ \frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z + i\sigma_y) \rho_S(t) \right] - K_{33}(-\tilde{\Omega}, t) \left[ \frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z - i\sigma_y) \rho_S(t) \right] \quad (996)$$

$$- K_{33}^*(\tilde{\Omega}, t) \left[ \rho_S(t) \frac{1}{4} (\sigma_z + i\sigma_y), \frac{\sigma_z}{2} \right] - K_{33}^*(-\tilde{\Omega}, t) \left[ \rho_S(t) \frac{1}{4} (\sigma_z - i\sigma_y), \frac{\sigma_z}{2} \right]. \quad (997)$$

Calculating the response functions extending the upper limit of  $\tau$  to  $\infty$ , we obtain:

$$K_{33}(\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{i\tilde{\Omega}\tau} d\tau d\omega \quad (998)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (999)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} (n(\omega) + 1) e^{-i\tau\omega} d\tau d\omega \quad (1000)$$

$$= \int_0^\infty \int_0^\infty J(\omega) (n(\omega) + 1) e^{i\tilde{\Omega}\tau - i\tau\omega} d\tau d\omega \quad (1001)$$

$$= \int_0^\infty J(\omega) (n(\omega) + 1) \pi \delta(\tilde{\Omega} - \omega) d\omega \quad (1002)$$

$$= \pi J(\tilde{\Omega}) (n(\tilde{\Omega}) + 1), \quad (1003)$$

$$K_{33}(-\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{-i\tilde{\Omega}\tau} d\tau d\omega \quad (1004)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (1005)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} n(\omega) e^{i\tau\omega} d\tau d\omega \quad (1006)$$

$$= \int_0^\infty \int_0^\infty J(\omega) n(\omega) e^{-i\tilde{\Omega}\tau + i\tau\omega} d\tau d\omega \quad (1007)$$

$$= \int_0^\infty J(\omega) n(\omega) \pi \delta(-\tilde{\Omega} + \omega) d\omega \quad (1008)$$

$$= \pi J(\tilde{\Omega}) n(\tilde{\Omega}). \quad (1009)$$

Here we have used  $\int_0^\infty ds e^{\pm i\epsilon s} = \pi \delta(\epsilon) \pm i \frac{\text{V.P.}}{\epsilon}$ , where V.P. denotes the Cauchy's principal value. These principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix  $A_3(0)$  because the spectral density  $J(\omega)$  is equal to zero when  $\omega = 0$ . Replacing in the equation (996) lead us to obtain:

$$\frac{d\rho_S(t)}{dt} = -i \frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\tilde{\Omega}) \left( (n(\tilde{\Omega}) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\tilde{\Omega}) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (1010)$$

$$- \frac{\pi}{8} J(\tilde{\Omega}) \left( (n(\tilde{\Omega}) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\tilde{\Omega}) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right). \quad (1011)$$

This is the same result than the equation (S17), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{d\rho_S(t)}{dt} = -i \frac{\Omega(t)}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\Omega(t)) \left( (n(\Omega(t)) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\Omega(t)) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (1012)$$

$$- \frac{\pi}{8} J(\Omega(t)) \left( (n(\Omega(t)) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\Omega(t)) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right). \quad (1013)$$

## VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1014)$$

$$H_S(t) = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1015)$$

$$H_I = \sum_{n\mathbf{u}\mathbf{k}} |n\rangle\langle n| \left( g_{n\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger + g_{n\mathbf{u}\mathbf{k}}^* b_{\mathbf{u}\mathbf{k}} \right), \quad (1016)$$

$$H_B = \sum_{\mathbf{u}\mathbf{k}} \omega_{\mathbf{u}\mathbf{k}} b_{\mathbf{u}\mathbf{k}}^\dagger b_{\mathbf{u}\mathbf{k}}. \quad (1017)$$

### A. Variational Transformation

We consider the following operator:

$$V = \sum_{nuk} |n\rangle\langle n| \omega_{uk}^{-1} \left( f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk} \right) \quad (1018)$$

At first let's obtain  $e^{\pm V}$  under the transformation (1018), consider  $\hat{\varphi}_n = \sum_{uk} \omega_{uk}^{-1} \left( f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk} \right)$ , so the equation (1018) can be written as  $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$ , then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \quad (1019)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{(\sum_n |n\rangle\langle n| \hat{\varphi}_n)^2}{2!} + \dots \quad (1020)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1021)$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1022)$$

$$= \sum_n |n\rangle\langle n| \left( \mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots \right) \quad (1023)$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \quad (1024)$$

Given that  $\left[ f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk}, f_{nu'k'} b_{u'k'}^\dagger - f_{nu'k'}^* b_{u'k'} \right] = 0$  for all  $k', k$  and  $u, u'$  then we can proof using the Zassenhaus formula and defining  $D(\pm \alpha_{nuk}) = e^{\pm(\alpha_{nuk} b_{uk}^\dagger - \alpha_{nuk}^* b_{uk})}$  in the same way than (23) with  $\alpha_{nuk} = \frac{f_{nuk}}{\omega_{uk}}$ :

$$e^{\pm \sum_{uk} \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} = \prod_u e^{\pm \sum_k \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} \quad (1025)$$

$$= \prod_u \left( \prod_k e^{\pm \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} \right) \quad (1026)$$

$$= \prod_u \left( \prod_k D(\pm \alpha_{nuk}) \right) \quad (1027)$$

$$= \prod_{uk} D(\pm \alpha_{nuk}) \quad (1028)$$

$$= \prod_u B_{nu\pm} \quad (1029)$$

$$B_{nu\pm} \equiv \prod_k D(\pm \alpha_{nuk}) \quad (1030)$$

As we can see  $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$  and  $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$  this implies that  $e^{-V} e^V = \mathbb{I}$ . This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1031)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1014):

$$\overline{|0\rangle\langle 0|} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |0\rangle\langle 0| \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1032)$$

$$= \prod_u B_{0u+} |0\rangle\langle 0| \prod_u B_{0u-}, \quad (1033)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \quad (1034)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \quad (1035)$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \quad (1036)$$

$$= |0\rangle\langle 0|. \quad (1037)$$

$$\overline{|m\rangle\langle n|} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |m\rangle\langle n| \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1038)$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n| \prod_u B_{nu-}, \quad (1039)$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \quad (1040)$$

$$= |m\rangle\langle n| \prod_u (B_{mu+} B_{nu-}), \quad m \neq n, \quad (1041)$$

$$= |m\rangle\langle n| \prod_u \left( \prod_{\mathbf{k}} D(\alpha_{muk}) \prod_{\mathbf{k}} D(-\alpha_{nuk}) \right), \quad (1042)$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} (D(\alpha_{muk}) D(-\alpha_{nuk})), \quad (1043)$$

$$= |m\rangle\langle n| \prod_{uk} \left( D(\alpha_{muk} - \alpha_{nuk}) \exp \left( \frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (1044)$$

$$\Pi_u(B_{mu+} B_{nu-}) = \prod_{uk} \left( D(\alpha_{muk} - \alpha_{nuk}) \exp \left( \frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (1045)$$

$$\overline{\sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk}} = (\sum_n |n\rangle\langle n| \prod_u B_{nu+}) \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} (\sum_n |n\rangle\langle n| \prod_u B_{nu-}), \quad (1046)$$

$$= (|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \dots) (\sum_n |n\rangle\langle n| \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk}) (|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \dots), \quad (1047)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} + \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} + \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} \prod_u B_{1u-} + \dots, \quad (1048)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} + (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} + (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{1u-} + \dots \quad (1049)$$

$$= |0\rangle\langle 0| (\prod_u B_{0u+} + \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \dots) \quad (1050)$$

$$+ |1\rangle\langle 1| (\prod_u B_{1u+} + \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \dots) + \dots \quad (1051)$$

$$= |0\rangle\langle 0| \left( \sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left( b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) \left( b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left( b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) \left( b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) \quad (1052)$$

$$+ |1\rangle\langle 1| \left( \sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left( b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) \left( b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left( b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) \left( b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) + \dots \quad (1053)$$

$$= |0\rangle\langle 0| \left( \sum_{uk} \omega_{uk} \left( b_{uk}^\dagger - \frac{v_{0uk}}{\omega_{uk}} \right) \left( b_{uk} - \frac{v_{0uk}}{\omega_{uk}} \right) \right) + |1\rangle\langle 1| \left( \sum_{uk} \omega_{uk} \left( b_{uk}^\dagger - \frac{v_{1uk}}{\omega_{uk}} \right) \left( b_{uk} - \frac{v_{1uk}}{\omega_{uk}} \right) \right) + \dots \quad (1054)$$

$$= \sum_{nuk} |n\rangle\langle n| \left( \omega_{uk} \left( b_{uk}^\dagger - \frac{v_{nuk}}{\omega_{uk}} \right) \left( b_{uk} - \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1055)$$

$$= \sum_{nuk} |n\rangle\langle n| \left( \omega_{uk} \left( b_{uk}^\dagger b_{uk} - \frac{v_{nuk}}{\omega_{uk}} b_{uk}^\dagger - \frac{v_{nuk}^*}{\omega_{uk}} b_{uk} + \left| \frac{v_{nuk}}{\omega_{uk}} \right|^2 \right) \right) \quad (1056)$$

$$= \sum_{nuk} |n\rangle\langle n| \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_{nuk} |n\rangle\langle n| \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - v_{nuk} b_{uk}^\dagger - v_{nuk}^* b_{uk} \right) \quad (1057)$$

$$= \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_{nuk} |n\rangle\langle n| \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - (v_{nuk} b_{uk}^\dagger + v_{nuk}^* b_{uk}) \right) \quad (1058)$$

The transformed Hamiltonians of the equations (1015) to (1017) written in terms of (1032) to (1056) are:



$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1059)$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1060)$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) \quad (1061)$$

$$\overline{H_I} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left( \sum_{nuk} |n\rangle\langle n| (g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk}) \right) \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1062)$$

$$= \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left( \sum_{uk} |0\rangle\langle 0| (g_{0uk} b_{uk}^\dagger + g_{0uk}^* b_{uk}) + \dots \right) \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1063)$$

$$= \prod_u B_{0u+} \sum_{uk} |0\rangle\langle 0| (g_{0uk} b_{uk}^\dagger + g_{0uk}^* b_{uk}) \prod_u B_{0u-} + \prod_u B_{1u+} \sum_{uk} |1\rangle\langle 1| (g_{1uk} b_{uk}^\dagger + g_{1uk}^* b_{uk}) \prod_u B_{1u-} + \dots \quad (1064)$$

$$= \sum_{uk} |0\rangle\langle 0| (g_{0uk} \prod_u B_{0u+} b_{uk}^\dagger \prod_u B_{0u-} + g_{0uk}^* \prod_u B_{0u+} b_{uk} \prod_u B_{0u-}) + \sum_{uk} |1\rangle\langle 1| (g_{1uk} \prod_u B_{1u+} b_{uk}^\dagger \prod_u B_{1u-} + g_{1uk}^* \prod_u B_{1u+} b_{uk} \prod_u B_{1u-}) + \dots \quad (1065)$$

$$= \sum_{uk} |0\rangle\langle 0| \left( g_{0uk} \left( b_{uk}^\dagger - \frac{v_{0uk}^*}{\omega_{uk}} \right) + g_{0uk}^* \left( b_{uk} - \frac{v_{0uk}}{\omega_{uk}} \right) \right) + \sum_{uk} |1\rangle\langle 1| \left( g_{1uk} \left( b_{uk}^\dagger - \frac{v_{1uk}^*}{\omega_{uk}} \right) + g_{1uk}^* \left( b_{uk} - \frac{v_{1uk}}{\omega_{uk}} \right) \right) + \dots \quad (1066)$$

$$= \sum_{nuk} |n\rangle\langle n| \left( g_{nuk} \left( b_{uk}^\dagger - \frac{v_{nuk}^*}{\omega_{uk}} \right) + g_{nuk}^* \left( b_{uk} - \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1067)$$

$$= \sum_{nuk} |n\rangle\langle n| \left( g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk} - \left( g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1068)$$

$$\overline{H_B} = \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_{nuk} |n\rangle\langle n| \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - (v_{nuk} b_{uk}^\dagger + v_{nuk}^* b_{uk}) \right) \quad (1069)$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_{nuk} |n\rangle\langle n| \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - (v_{nuk} b_{uk}^\dagger + v_{nuk}^* b_{uk}) \right) \quad (1070)$$

$$+ \sum_{nuk} |n\rangle\langle n| \left( g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk} - \left( g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1071)$$

Let's define the following functions:

$$R_n(t) = \sum_{uk} \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - \left( g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1072)$$

$$B_{z,n}(t) = \sum_{uk} \left( (g_{nuk} - v_{nuk}) b_{uk}^\dagger + (g_{nuk} - v_{nuk})^* b_{uk} \right) \quad (1073)$$

Using the previous functions we have that (1070) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \quad (1074)$$

$$(1075)$$

Now in order to separate the elements of the hamiltonian (1075) let's follow the references of the equations (225) and (236) to separate the hamiltonian, before proceeding to do this we need to consider the term of the form:

$$\langle \Pi_u (B_{mu+} B_{nu-}) \rangle_{\overline{H_0}} = \langle \Pi_{uk} (D(\alpha_{muk} - \alpha_{nuk}) \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk})) \rangle_{\overline{H_0}} \quad (1076)$$

$$= (\Pi_{uk} \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}))) \langle \Pi_{uk} D(\alpha_{muk} - \alpha_{nuk}) \rangle_{\overline{H_0}} \quad (1077)$$

$$= \left( \Pi_{uk} \exp\left(\frac{(v_{muk}^* v_{nuk} - v_{nuk} v_{muk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (1078)$$

$$\equiv B_{nm} \quad (1079)$$

$$\langle \Pi_u (B_{nu+} B_{mu-}) \rangle_{\overline{H_0}} = \left( \Pi_{uk} \exp\left(\frac{(v_{nuk}^* v_{muk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (1080)$$

$$= B_{nm}^* \quad (1081)$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+} B_{nu-}) - B_{nm} \quad (1082)$$

As we can see:

$$J_{nm}^\dagger = \left( \prod_u (B_{mu+} B_{nu-}) - B_{nm} \right)^\dagger \quad (1083)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{nm}^* \quad (1084)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{mn} \quad (1085)$$

$$= J_{mn} \quad (1086)$$

We can separate the Hamiltonian (1075) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}}(t) = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \quad (1087)$$

$$\overline{H_{\bar{I}}}(t) = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1088)$$

$$\overline{H_{\bar{B}}}(t) = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \quad (1089)$$

## B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left( \text{Tr} \left( e^{-\beta(\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}})} \right) \right) + \langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} + O \left( \left\langle \overline{H_{\bar{I}}}^2 \right\rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} \right). \quad (1090)$$

Taking the equations (246)-(254) and given that  $\text{Tr} \left( e^{-\beta \overline{H_{\bar{S}}}(t)} \right) = C(R_0, R_1, \dots, R_{d-1}, B_{01}, \dots, B_{0(d-1)}, \dots, B_{(d-2)(d-1)})$ , where each  $R_i$  and  $B_{kj}$  depend of the set of variational parameters  $\{v_{nuk}\}$ . Given that the numbers  $v_{nuk}$  are complex then we can separate them as  $v_{nuk} = v_{nuk}^{\Re} + i v_{nuk}^{\Im}$ . So our approach will be based on the derivation respect to  $v_{nuk}^{\Re}$  and  $v_{nuk}^{\Im}$ . The Hamiltonian  $\overline{H_{\bar{S}}}(t)$  can be written like:

$$\overline{H_S(t)} = \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{|v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}} - \left( g_{n\mathbf{uk}} \frac{v_{n\mathbf{uk}}^*}{\omega_{\mathbf{uk}}} + g_{n\mathbf{uk}}^* \frac{v_{n\mathbf{uk}}}{\omega_{\mathbf{uk}}} \right) \right) \right) |n\rangle\langle n| \quad (1091)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1092)$$

$$= \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{|v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}} - \frac{g_{n\mathbf{uk}} v_{n\mathbf{uk}}^* + g_{n\mathbf{uk}}^* v_{n\mathbf{uk}}}{\omega_{\mathbf{uk}}} \right) \right) |n\rangle\langle n| \quad (1093)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1094)$$

$$= \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2}{\omega_{\mathbf{uk}}} - \frac{(g_{n\mathbf{uk}} + g_{n\mathbf{uk}}^*) v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im} (g_{n\mathbf{uk}}^* - g_{n\mathbf{uk}})}{\omega_{\mathbf{uk}}} \right) \right) |n\rangle\langle n| \quad (1095)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1096)$$

$$v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^* = (v_{m\mathbf{uk}}^{\Re} - i v_{m\mathbf{uk}}^{\Im})(v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im}) - (v_{m\mathbf{uk}}^{\Re} + i v_{m\mathbf{uk}}^{\Im})(v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im}) \quad (1097)$$

$$= (v_{m\mathbf{uk}}^{\Re} v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - i v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re} + v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Im}) \quad (1098)$$

$$- (v_{m\mathbf{uk}}^{\Re} v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} + i v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re} + v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Im}) \quad (1099)$$

$$= 2i (v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re}) \quad (1100)$$

$$\overline{H_S(t)} = \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2}{\omega_{\mathbf{uk}}} - \frac{(g_{n\mathbf{uk}} + g_{n\mathbf{uk}}^*) v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im} (g_{n\mathbf{uk}}^* - g_{n\mathbf{uk}})}{\omega_{\mathbf{uk}}} \right) \right) |n\rangle\langle n| \quad (1101)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{i(v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re})}{\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1102)$$

$$|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2 = (v_{m\mathbf{uk}} - v_{n\mathbf{uk}})(v_{m\mathbf{uk}} - v_{n\mathbf{uk}})^* \quad (1103)$$

$$= |v_{m\mathbf{uk}}|^2 + |v_{n\mathbf{uk}}|^2 - (v_{n\mathbf{uk}} v_{m\mathbf{uk}}^* + v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*) \quad (1104)$$

$$= (v_{m\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im})^2 + (v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - (v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im})(v_{m\mathbf{uk}}^{\Re} - i v_{m\mathbf{uk}}^{\Im}) \quad (1105)$$

$$- (v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im})(v_{m\mathbf{uk}}^{\Re} + i v_{m\mathbf{uk}}^{\Im}) \quad (1106)$$

$$= (v_{m\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im})^2 + (v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - 2(v_{n\mathbf{uk}}^{\Re} v_{m\mathbf{uk}}^{\Re} + v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Im}) \quad (1107)$$

$$= (v_{m\mathbf{uk}}^{\Re} - v_{n\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im} - v_{n\mathbf{uk}}^{\Im})^2 \quad (1108)$$

$$R_n(t) = \sum_{\mathbf{uk}} \left( \frac{|v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}} - \left( g_{n\mathbf{uk}} \frac{v_{n\mathbf{uk}}^*}{\omega_{\mathbf{uk}}} + g_{n\mathbf{uk}}^* \frac{v_{n\mathbf{uk}}}{\omega_{\mathbf{uk}}} \right) \right) \quad (1109)$$

$$= \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - (g_{n\mathbf{uk}} + g_{n\mathbf{uk}}^*) v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im} (g_{n\mathbf{uk}}^* - g_{n\mathbf{uk}})}{\omega_{\mathbf{uk}}} \right) \quad (1110)$$

$$= \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - 2g_{n\mathbf{uk}}^{\Re} v_{n\mathbf{uk}}^{\Re} - 2g_{n\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Im}}{\omega_{\mathbf{uk}}} \right) \quad (1111)$$

$$B_{mn} = \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1112)$$

$$= \left( \Pi_{\mathbf{uk}} \exp \left( \frac{i(v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re})}{\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{m\mathbf{uk}}^{\Re} - v_{n\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im} - v_{n\mathbf{uk}}^{\Im})^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1113)$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Re}} = \frac{\partial}{\partial v_{nuk}^{\Re}} \sum_{uk} \left( \frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1114)$$

$$= \frac{2v_{nuk}^{\Re} - 2g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1115)$$

$$= 2 \frac{v_{nuk}^{\Re} - g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1116)$$

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Im}} = \frac{\partial}{\partial v_{nuk}^{\Im}} \sum_{uk} \left( \frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1117)$$

$$= \frac{2v_{nuk}^{\Im} - 2g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1118)$$

$$= 2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1119)$$

Given that:

$$\ln B_{mn} = \ln \left( \left( \prod_{uk} \exp \left( \frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \prod_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left( \frac{\beta_u \omega_{uk}}{2} \right) \right) \right) \quad (1120)$$

$$= \sum_{uk} \ln \exp \left( \frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_u \ln \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left( \frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1121)$$

$$= \sum_{uk} \left( \frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_{uk} \left( -\frac{1}{2} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left( \frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1122)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Re}} = \frac{-i v_{muk}^{\Im} - (v_{nuk}^{\Re} - v_{muk}^{\Re}) \coth \left( \frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1123)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Im}} = \frac{i v_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth \left( \frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1124)$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \quad (1125)$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \quad (1126)$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial (B_{nm})^{\dagger}}{\partial a} \quad (1127)$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \quad (1128)$$

$$= B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} - (v_{n\mathbf{k}}^{\Re} - v_{m\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1129)$$

$$= B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1130)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Re}} = \left( \frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \right)^{\dagger} \quad (1131)$$

$$= \left( B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \quad (1132)$$

$$= B_{nm} \left( \frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1133)$$

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \quad (1134)$$

$$= B_{mn} \left( \frac{iv_{m\mathbf{k}}^{\Re} - (v_{n\mathbf{k}}^{\Im} - v_{m\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1135)$$

$$= B_{mn} \left( \frac{iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1136)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Im}} = \left( \frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \right)^{\dagger} \quad (1137)$$

$$= (B_{mn})^{\dagger} \quad (1138)$$

$$= B_{nm} \left( \frac{-iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1139)$$

Introducing this derivatives in the equation (1114) give us:

$$\frac{\partial A_{\mathbf{B}}}{\partial v_{n\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathbf{B}}}{\partial R_n} \left( 2 \frac{v_{n\mathbf{k}}^{\Re} - g_{u\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n < m} \left( \frac{\partial A_{\mathbf{B}}}{\partial B_{nm}} B_{nm} \left( \frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \quad (1140)$$

$$+ \frac{\partial A_{\mathbf{B}}}{\partial B_{mn}} B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1141)$$

$$= 0 \quad (1142)$$

We can obtain the variational parameters:

$$-2 \frac{\partial A_B}{\partial R_n} \frac{v_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1143)$$

$$= -\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \left( \frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left( \frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1144)$$

$$v_{nuk}^{\Re} = \frac{\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} - \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \left( \frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left( \frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right)}{2 \frac{\partial A_B}{\partial R_n} \frac{1}{\omega_{uk}} - \sum_{n \neq m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right)} \quad (1145)$$

$$= \frac{2g_{nuk}^{\Re} \omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} (iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} (-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) \right)}{2\omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n \neq m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right)} \quad (1146)$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_B}{\partial v_{nuk}^{\Im}} = \frac{\partial A_B}{\partial R_n} \left( 2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \right) + \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \left( \frac{-iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1147)$$

$$+ \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left( \frac{iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1148)$$

$$= 0 \quad (1149)$$

Rearranging we obtain



$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1166)$$

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right). \quad (1167)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1168)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\bar{\rho}(t)) \quad (1169)$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1170)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1171)$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \quad (1172)$$

$$B_{nm,y} = \frac{B_{nm} - B_{mn}}{2i} \quad (1173)$$

We can proof that  $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+}B_{n-} - B_m B_n)^\dagger \quad (1174)$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \quad (1175)$$

$$= B_{n+} B_{m-} - B_n B_m \quad (1176)$$

$$= B_{nm} \quad (1177)$$

So we can say that the set of matrices (1170) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1170) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1178)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}) \quad (1179)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left( \Re(V_{nm}(t)) B_{nm} \left( \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) + iV_{nm}^\Im(t) B_{nm} \left( \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) \right) \quad (1180)$$

$$+ \Re(V_{nm}(t)) B_{mn} \left( \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) - iV_{nm}^\Im(t) B_{mn} \left( \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1181)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left( \Re(V_{nm}(t)) \sigma_{nm,x} \left( \frac{B_{nm} + B_{mn}}{2} \right) + \Re(V_{nm}(t)) \sigma_{nm,y} \frac{i(B_{mn} - B_{nm})}{2} \right) \quad (1182)$$

$$+ i\Im(V_{nm}(t)) \sigma_{nm,x} \left( \frac{B_{nm} - B_{mn}}{2} \right) + \Im(V_{nm}(t)) \sigma_{nm,y} \left( \frac{B_{nm} + B_{mn}}{2} \right) \quad (1183)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (\Re(V_{nm}(t)) \sigma_{nm,x} B_{nm,x} - \Im(V_{nm}(t)) \sigma_{nm,x} B_{nm,y} + \Re(V_{nm}(t)) \sigma_{nm,y} B_{nm,y} \quad (1184)$$

$$+ \Im(V_{nm}(t)) \sigma_{nm,y} B_{nm,x}) \quad (1185)$$

Let's define the set

$$P = \{(n, m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\} \quad (1186)$$



Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \quad (1187)$$

$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \quad (1188)$$

$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \quad (1189)$$

$$A_{4,nm}(t) = A_{2,mn}(t) \quad (1190)$$

$$A_{5,nm}(t) = A_{1,nm}(t) \quad (1191)$$

$$B_{1,nm}(t) = B_{nm,x} \quad (1192)$$

$$B_{2,nm}(t) = B_{nm,y} \quad (1193)$$

$$B_{3,nm}(t) = B_{z,n}(t) \quad (1194)$$

$$B_{4,nm}(t) = B_{1,nm}(t) \quad (1195)$$

$$B_{5,nm}(t) = B_{2,nm}(t) \quad (1196)$$

$$C_{1,nm}(t) = \Re(V_{nm}(t)) \quad (1197)$$

$$C_{2,nm}(t) = C_{1,nm}(t) \quad (1198)$$

$$C_{3,nm}(t) = 1 \quad (1199)$$

$$C_{4,nm}(t) = \Im(V_{nm}(t)) \quad (1200)$$

$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \quad (1201)$$

The previous notation allows us to write the interaction Hamiltonian in  $\overline{H_I}(t)$  as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t) (A_{jp} \otimes B_{jp}(t)) \quad (1202)$$

Here  $J = \{1, 2, 3, 4, 5\}$  and  $P$  the set defined in (1186).

We write the interaction Hamiltonian transformed under (1166) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t) (\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)) \quad (1203)$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \quad (1204)$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t) e^{-iH_B t} \quad (1205)$$

Taking as reference state  $\rho_B$  and truncating at second order in  $H_I(t)$ , we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B]] ds \quad (1206)$$

Replacing the equation (1203) in (1206) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H}_I(t), \left[ \widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1207)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \left[ \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1208)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1209)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \Big] ds \quad (1210)$$

$$= - \int_0^t \text{Tr}_B \left( \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right. \quad (1211)$$

$$\left. - \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \right) \quad (1212)$$

$$- \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \quad (1213)$$

$$\left. + \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \right) ds \quad (1214)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B \quad (1215)$$

$$= \left\langle \widetilde{B}_{jp}(\tau) \widetilde{B}_{j'p'}(0) \right\rangle_B \quad (1216)$$

Here  $s \rightarrow t - \tau$  and  $\text{Tr}_B \left( \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B$ . To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators  $\widetilde{B}_{jp}(t)$  and elements related to the system given by  $\widetilde{A}_{jp}(t)$ . The systems considered are in different Hilbert spaces so  $\text{Tr} \left( \widetilde{A}_{jp}(t) \widetilde{B}_{j'p'}(t) \right) = \text{Tr} \left( \widetilde{A}_{jp}(t) \right) \text{Tr} \left( \widetilde{B}_{j'p'}(t) \right)$ . The correlation functions relevant of the master equation (1214) are:

$$\text{Tr}_B \left( \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1217)$$

$$= \left\langle \widetilde{B_{jp}}(0) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1218)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1219)$$

$$\text{Tr}_B \left( \widetilde{B_{jp}}(t) \rho_B \widetilde{B_{j'p'}}(s) \right) = \text{Tr}_B \left( \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1220)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1221)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1222)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1223)$$

$$\text{Tr}_B \left( \widetilde{B_{j'p'}}(s) \rho_B \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) \quad (1224)$$

$$= \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1225)$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1226)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1227)$$

$$\text{Tr}_B \left( \rho_B \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1228)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1229)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1230)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1231)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1207) and (1214) and using the equations (1217)-(1231) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',p,p'} \left( C_{jp}(t) C_{j'p'}(s) \left( \Lambda_{jpj'p'}(\tau) \widetilde{A_{jp}}(t) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau) \widetilde{A_{jp}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \right) \right. \quad (1232)$$

$$\left. + C_{jp}(t) C_{j'p'}(s) \left( \Lambda_{j'p'jp}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jp}}(t) \right) \right) ds \quad (1233)$$

$$= - \int_0^t \sum_{jj'pp'} \left( C_{jp}(t) C_{j'p'}(s) \left( \Lambda_{jpj'p'}(\tau) \left[ \widetilde{A_{jp}}(t), \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[ \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s), \widetilde{A_{jp}}(t) \right] \right) \right) \quad (1234)$$

Rearranging and identifying the commutators allow us to write a more simplified version

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{jj'pp'} \left( C_{jp}(t) C_{j'p'}(t-\tau) \left( \Lambda_{jpj'p'}(\tau) \left[ A_{jp}(t), A_{j'p'}(t-\tau, t) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[ \widetilde{\rho_S}(t) A_{j'p'}(t-\tau, t), A_{jp}(t) \right] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1235)$$

For this case we used that  $A_{jp}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jp}(t) U(t-\tau) U^\dagger(t)$ . This is a non-Markovian equation.

## VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1236)$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1237)$$

$$H_I = \left( \sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left( \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right), \quad (1238)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1239)$$

We will start with a system-bath coupling operator of the form  $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$ .

### A. Variational Transformation

We consider the following operator:

$$V = \left( \sum_{n=1} |n\rangle\langle n| \right) \left( \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}) \right) \quad (1240)$$

At first let's obtain  $e^V$  under the transformation (1240), consider  $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})$ :

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \quad (1241)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{(\sum_{n=1} |n\rangle\langle n| \hat{\varphi})^2}{2!} + \dots \quad (1242)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \quad (1243)$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left( \mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \quad (1244)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \quad (1245)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \quad (1246)$$

Given that  $[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}] = 0$  if  $\mathbf{k}' \neq \mathbf{k}$  then we can proof using the Zassenhaus formula and defining  $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})}$  in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \quad (1247)$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (1248)$$

$$= B_{\pm} \quad (1249)$$

As we can see  $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B$ . because this form imposes that  $e^{-V} e^V = \mathbb{I}$  and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) A \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1250)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1236):

$$\overline{|0\rangle\langle 0|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle 0| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1251)$$

$$= |0\rangle\langle 0|, \quad (1252)$$

$$\overline{|m\rangle\langle n|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle n| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1253)$$

$$= |m\rangle\langle m| B^+ |m\rangle\langle n| n\rangle\langle n| B^-, \quad (1254)$$

$$= |m\rangle\langle n|, \quad m \neq 0, \quad n \neq 0, \quad (1255)$$

$$\overline{|0\rangle\langle m|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle m| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1256)$$

$$= |0\rangle\langle m| B^- \quad m \neq 0, \quad (1257)$$

$$\overline{|m\rangle\langle 0|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle 0| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1258)$$

$$= |0\rangle\langle m| B^+ \quad m \neq 0, \quad (1259)$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1260)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B^- \quad (1261)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B^+ b_{\mathbf{k}}^\dagger B^- \right) \left( B^+ b_{\mathbf{k}} B^- \right) \quad (1262)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left( b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1263)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1264)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \quad (1265)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1266)$$

The transformed Hamiltonians of the equations (1237) to (1239) written in terms of (1251) to (1266) are:

$$\overline{H_S(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1267)$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1268)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1269)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|}) + \sum_{m,n \neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \quad (1270)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) B^- |0\rangle\langle n| + V_{n0}(t) B^+ |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1271)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1272)$$

$$\overline{H_I} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \left( \left( \sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left( \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \right) \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1273)$$

$$= \left( \mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B^+ \right) \left( \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1274)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B^+ (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) B^- \quad (1275)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1276)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1277)$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1278)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1279)$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (1280)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1281)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1282)$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left( g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1283)$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1284)$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \right) \quad (1285)$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left( g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1286)$$

Using the previous functions we have that (1283) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1287)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) \quad (1288)$$

Now in order to separate the elements of the hamiltonian (1288) let's follow the references of the equations (236) and (225) to separate the hamiltonian like:

$$\overline{H_S}(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \quad (1289)$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)), \quad (1290)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (1291)$$

Here B is given by:

$$\begin{aligned} B &= \langle B^+ \rangle \\ &= \langle B^- \rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1292)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1293)$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \quad (1294)$$

$$B_y = \frac{B^- - B^+}{2i} \quad (1295)$$

Using this set of hermitian operators to write the Hamiltonians (1237)-(1239)

$$\overline{H_S(t)} = \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1296)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{0 < m < n} (V_{mn}(t) |m\rangle\langle n| + V_{nm}(t) |n\rangle\langle m|) \quad (1297)$$

$$+ \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1298)$$

$$= \sum_{0 < m < n} ((\Re(V_{mn}(t)) + i\Im(V_{mn}(t))) |m\rangle\langle n| + (\Re(V_{mn}(t)) - i\Im(V_{mn}(t))) |n\rangle\langle m|) + \varepsilon_0(t) |0\rangle\langle 0| \quad (1299)$$

$$+ B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1300)$$

$$= \sum_{0 < m < n} \left( (\Re(V_{nm}(t)) + i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1301)$$

$$+ B \sum_{n=1} \left( V_{0n}(t) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + V_{n0}(t) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) + \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1302)$$

$$= \sum_{0 < m < n} (\Re(V_{nm}(t)) \sigma_{nm,x} + \Im(V_{nm}(t)) \sigma_{nm,y}) + B \sum_{n=1} (\Re(V_{0n}(t)) \sigma_{0n,x} + \Im(V_{mn}(t)) \sigma_{0n,y}) \quad (1303)$$

$$+ \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1304)$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)) \quad (1305)$$

$$= \sum_{n=1} \left( (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) (B^- - B) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) (B^+ - B) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) \quad (1306)$$

$$+ \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1307)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \sum_{n=1} \left( \frac{\sigma_{0n,x}}{2} ((B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) + (B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))) \right) \quad (1308)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) - (B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))) \quad (1309)$$

$$+ \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1310)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \left( \frac{\sigma_{0n,x}}{2} (B^+ + B^- - 2B) \Re(V_{0n}(t)) + i(B^- - B - B^+ + B) \Im(V_{0n}(t)) \right) \quad (1311)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B - B^- + B) \Re(V_{0n}(t)) + i(B - B^- + B - B^+) \Im(V_{0n}(t))) + \sum_{n=1} B_{z,n} |n\rangle\langle n| \quad (1312)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (\sigma_{0n,x} (B_x \Re(V_{0n}(t)) - B_y \Im(V_{0n}(t))) \quad (1313)$$

$$+ \sigma_{0n,y} (B_y \Re(V_{0n}(t)) + B_x \Im(V_{0n}(t)))) \quad (1314)$$

## B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality



$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left( \text{Tr} \left( e^{-\beta(\overline{H}_S + \overline{H}_B)} \right) \right) + \langle \overline{H}_I \rangle_{\overline{H}_S + \overline{H}_B} + O \left( \langle \overline{H}_I^2 \rangle_{\overline{H}_S + \overline{H}_B} \right). \quad (1315)$$

Taking the equations (246)-(254) and given that  $\text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right) = C(R_1, R_2, \dots, R_{d-1}, B)$ , where each  $R_i$  and  $B$  depend of the set of variational parameters  $\{v_{\mathbf{k}}\}$ . From (254) and using the chain rule we obtain that:

$$\frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \frac{\partial R_n}{\partial v_{\mathbf{k}}}, \quad (1316)$$

$$= 0 \quad (1317)$$

Let's recall the equations (1284) and (1286), we can write them in terms of the variational parameters

$$B = \exp \left( - (1/2) \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) \right) \quad (1318)$$

$$R_n = \sum_{\mathbf{k}} \omega_{\mathbf{k}}^{-1} (v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}} v_{\mathbf{k}}) \quad (1319)$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \quad (1320)$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1321)$$

Introducing this derivates in the equation (1316) give us:

$$\frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \left( -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \right) + \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1322)$$

$$= v_{\mathbf{k}} \left( \frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t) \quad (1323)$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1324)$$

$$= \frac{2g_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1325)$$

Now taking  $v_{\mathbf{k}} = g_{\mathbf{k}} v_{\mathbf{k}}$  then we can obtain  $v_{\mathbf{k}}$  like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B}}. \quad (1326)$$

### C. Master Equation

Let's consider that the initial state of the system is given by  $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$ , as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) (|0\rangle\langle 0| \otimes \rho_B) \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1327)$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \quad (1328)$$

$$0 = \rho(0) \quad (1329)$$

We transform any operator  $O$  into the interaction picture in the following way:

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1330)$$

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right). \quad (1331)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1332)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}(t)) \quad (1333)$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I}(t) = B_{z,0} |0\rangle\langle 0| + \sum_{n=1} (\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n} |n\rangle\langle n|) \quad (1334)$$

$$+ \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x} \quad (1335)$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1336)$$

$$B_{z,n} = \sum_{\mathbf{k}} \left( g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \text{ if } n \neq 0 \quad (1337)$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \quad (1338)$$

$$A_{2n}(t) = \sigma_{0n,y} \quad (1339)$$

$$A_{3n}(t) = |n\rangle\langle n| \quad (1340)$$

$$A_{4n}(t) = A_{2n}(t) \quad (1341)$$

$$A_{5n}(t) = A_{1n}(t) \quad (1342)$$

$$B_{1n}(t) = B_x \quad (1343)$$

$$B_{2n}(t) = B_y \quad (1344)$$

$$B_{3n}(t) = B_{z,n} \quad (1345)$$

$$B_{4n}(t) = B_{1n}(t) \quad (1346)$$

$$B_{5n}(t) = B_{2n}(t) \quad (1347)$$

$$C_{10}(t) = 0 \quad (1348)$$

$$C_{20}(t) = 0 \quad (1349)$$

$$C_{40}(t) = 0 \quad (1350)$$

$$C_{50}(t) = 0 \quad (1351)$$

$$C_{30}(t) = 1 \quad (1352)$$

$$C_{1n}(t) = \Re(V_{0n}(t)) \quad (1353)$$

$$C_{2n}(t) = C_{1n}(t) \quad (1354)$$

$$C_{3n}(t) = 1 \quad (1355)$$

$$C_{4n}(t) = \Im(V_{0n}(t)) \quad (1356)$$

$$C_{5n}(t) = -\Im(V_{0n}(t)) \quad (1357)$$

The previous notation allows us to write the interaction Hamiltonian in  $\overline{H}_I(t)$  as:

$$\overline{H}_I = \sum_{j \in J} \sum_{n=1} C_{jn}(t) (A_{jn} \otimes B_{jn}(t)) \quad (1358)$$

Here  $J = \{1, 2, 3, 4, 5\}$ .

We write the interaction Hamiltonian transformed under (1330) as:

$$\widetilde{H}_I(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1359)$$

$$\widetilde{A}_i(t) = U^\dagger(t) A_i U(t) \quad (1360)$$

$$\widetilde{B}_i(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \quad (1361)$$

Taking as reference state  $\rho_B$  and truncating at second order in  $H_I(t)$ , we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho}_S(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H}_I(t), \left[ \widetilde{H}_I(s), \widetilde{\rho}_S(t) \rho_B \right] \right] ds \quad (1362)$$

Replacing the equation (1359) in (1362) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H}_I(t), \left[ \widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1363)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \left[ \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1364)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1365)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \Big] ds \quad (1366)$$

$$= - \int_0^t \text{Tr}_B \left( \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right) \quad (1367)$$

$$- \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \quad (1368)$$

$$- \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1369)$$

$$+ \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \Big) ds \quad (1370)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jn j'n'}(\tau) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B \quad (1371)$$

$$= \left\langle \widetilde{B}_{jn}(\tau) \widetilde{B}_{j'n'}(0) \right\rangle_B \quad (1372)$$

Here  $s \rightarrow t - \tau$  and  $\text{Tr}_B \left( \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \rho_B \right) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B$ . To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators  $\widetilde{B}_{jn}(t)$  and elements related to the system given by  $\widetilde{A}_{jn}(t)$ . The systems considered are in different Hilbert spaces so  $\text{Tr} \left( \widetilde{A}_{jn}(t) \widetilde{B}_{j'n'}(t) \right) = \text{Tr} \left( \widetilde{A}_{jn}(t) \right) \text{Tr} \left( \widetilde{B}_{j'n'}(t) \right)$ . The correlation functions relevant of the master equation (1370) are:

$$\text{Tr}_B \left( \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1373)$$

$$= \left\langle \widetilde{B_{jn}}(0) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1374)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1375)$$

$$\text{Tr}_B \left( \widetilde{B_{jn}}(t) \rho_B \widetilde{B_{j'n'}}(s) \right) = \text{Tr}_B \left( \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1376)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1377)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1378)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1379)$$

$$\text{Tr}_B \left( \widetilde{B_{j'n'}}(s) \rho_B \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) \quad (1380)$$

$$= \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1381)$$

$$= \left\langle \widetilde{B_{jn}}(\tau) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1382)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1383)$$

$$\text{Tr}_B \left( \rho_B \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1384)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1385)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1386)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1387)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1363) and (1370) and using the equations (1373)-(1387) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left( C_{jn}(t) C_{j'n'}(s) \left( \Lambda_{jn j'n'}(\tau) \widetilde{A_{jn}}(t) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'n' jn}(-\tau) \widetilde{A_{jn}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \right) \right. \quad (1388)$$

$$\left. + C_{jn}(t) C_{j'n'}(s) \left( \Lambda_{j'n' jn}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \widetilde{A_{jn}}(t) - \Lambda_{jn j'n'}(\tau) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jn}}(t) \right) \right) ds \quad (1389)$$

$$= - \int_0^t \sum_{j,j',n,n'} \left( C_{jn}(t) C_{j'n'}(s) \left( \Lambda_{jn j'n'}(\tau) \left[ \widetilde{A_{jn}}(t), \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'n' jn}(-\tau) \left[ \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s), \widetilde{A_{jn}}(t) \right] \right) \right) \quad (1390)$$

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left( C_{jn}(t) C_{j'n'}(t-\tau) \left( \Lambda_{jn j'n'}(\tau) [A_{jn}(t), A_{j'n'}(t-\tau, t) \widetilde{\rho_S}(t)] + \Lambda_{j'n' jn}(-\tau) [\widetilde{\rho_S}(t) A_{j'n'}(t-\tau, t), A_{jn}(t)] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1391)$$

For this case we used that  $A_{jn}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jn}(t) U(t-\tau) U^\dagger(t)$ . This is a non-Markovian equation and if we take  $n = 2$  (two sites),  $\mu_0(t) = 0$ ,  $\mu_1(t) = 1$  then we can reproduce a similar expression to (418) as expected.

## VIII. BIBLIOGRAPHY

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