Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving a better bound for the free energy of the system using the variational parameters optimization we consider the generalization in [1] of the Bogoliubov inequality. Let´s define the partition functions of $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ given by $Z\left(t\right)$ and $Z_0\left(t\right)$ respectively as: $\int_0^1 x^x \mathrm{d}x$

$$Z\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}(t)}\right), \tag{1}$$

$$Z_0\left(t\right) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}_0(t)}\right). \tag{2}$$

where the transformed hamiltonians $\overline{H}\left(t\right)$ and $\overline{H_0}\left(t\right)$ are defined as:

$$\overline{H}\left(t\right) \equiv \overline{H_{\bar{I}}}\left(t\right) + \overline{H_0}\left(t\right), \tag{3}$$

$$\overline{H_0}\left(t\right) \equiv \overline{H_{\bar{S}}}\left(t\right) + \overline{H_{\bar{B}}}. \tag{4}$$

For any operator $A\left(t\right)$ we define the expected value respect to $\overline{H_0}\left(t\right)$ as:

$$\left\langle A\left(t\right)\right\rangle_{\overline{H_0}(t)} \equiv \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}. \tag{5}$$

The terms $\overline{H_{\bar{S}}}\left(t\right)$, $\overline{H_{\bar{B}}}$ and $\overline{H_{\bar{I}}}\left(t\right)$ are related to the variational transformation performed in [1,2], this transformation allowed us to construct $\overline{H_{\bar{I}}}\left(t\right)$ such that $\left\langle\overline{H_{\bar{I}}}\left(t\right)\right\rangle_{\overline{H_0}(t)} = 0$. The diagonalization of $\overline{H_0}\left(t\right)$ in terms of it's eigenstates and eigenvalues such that $\overline{H_0}\left(t\right)|n\rangle = E_{0,n}\left(t\right)|n\rangle$, being $|n\rangle$ an eigenstate of $\overline{H_0}\left(t\right)$ with eigenvalue $E_{0,n}\left(t\right)$ is $\overline{H_0}\left(t\right) = \sum_n E_{0,n}\left(t\right)|n\rangle\langle n|$, with $\langle n|n'\rangle = \delta_{nn'}$, so a simple form of $\mathrm{e}^{-\beta\overline{H_0}(t)}$ can be found as follows:

$$\mathrm{e}^{r(X+Y)} = \mathrm{e}^{rX}\mathrm{e}^{rY}\mathrm{e}^{-\frac{r^2}{2}[X,Y]}\mathrm{e}^{\frac{r^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\cdots \text{ (Zassenhaus formula)}, \tag{6}$$

$$\mathrm{e}^{X+Y} = \mathrm{e}^{X}\mathrm{e}^{Y}\mathrm{e}^{-\frac{r^2}{2}0}\mathrm{e}^{\frac{r^3}{6}(2[Y,0]+[X,0])}\cdots \text{ (setting } r=1 \text{ and } [X,Y]=0 \text{ in (6))} \tag{7}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}\mathbb{I} \tag{8}$$

$$= \mathrm{e}^{X}\mathrm{e}^{Y}, \tag{9}$$

$$\mathrm{e}^{-\beta\overline{H_0}(t)} = \mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by the diagonalization of } \overline{H_0}\left(t\right)) \tag{10}$$

$$\mathrm{e}^{-\sum_n \beta E_{0,n}(t)|n\rangle\langle n|} = \prod_n \mathrm{e}^{-\beta E_{0,n}(t)|n\rangle\langle n|} \text{ (by (9) and } [|n\rangle\langle n|, |n'\rangle\langle n'|] = 0) \tag{11}$$

$$= \prod_n \sum_{j=0}^{\infty} \frac{(-\beta E_{0,n}(t)\,|n\rangle\langle n|)^j}{j!} \text{ (by the exponential formula)} \tag{12}$$

$$= \prod_n \left(\mathbb{I} + \sum_{j=1}^{\infty} \frac{(-\beta E_{0,n}(t))^j\,|n\rangle\langle n|}{j!}\right) \text{ (using } (aA)^j = a^j A^j \text{ and } (|n\rangle\langle n|)^2 = |n\rangle\langle n|) \tag{13}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| + \sum_{j=1}^{\infty} \frac{(-\beta E_{0,n}(t))^j\,|n\rangle\langle n|}{j!}\right) \tag{14}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + |n\rangle\langle n| \left(\sum_{j=0}^{\infty} \frac{(-\beta E_{0,n}(t))^j}{j!}\right)\right) \tag{15}$$

$$= \prod_n \left(\mathbb{I} - |n\rangle\langle n| + \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \text{ (by the exponential formula)} \tag{16}$$

$$= \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right). \tag{17}$$

We will prove by induction a neat form for (17), we will show that:

$$\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \mathbb{I} + \sum_{j=1}^{n} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \tag{18}$$

For $n = 1$ the formula is trivial, in the case $n = 2$ we obtain that:

$$\prod_{j=1}^{2} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2|\right) \tag{19}$$

$$= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,1}(t)} - 1\right)|1\rangle\langle 1| + \left(\mathrm{e}^{-\beta E_{0,2}(t)} - 1\right)|2\rangle\langle 2| \text{ (by } \langle i|j\rangle = \delta_{ij}) \tag{20}$$

$$= \mathbb{I} + \sum_{j=1}^{2} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \tag{21}$$

It is our case base, our induction step is (18), in the case $n + 1$ we will have:

$$\prod_{j=1}^{n+1} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right) = \left(\prod_{j=1}^{n} \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|\right)\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) \tag{22}$$

$$= \left(\mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right)\left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1|\right) \text{ (by induction step)} \tag{23}$$

$$= \mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n+1}(t)} - 1\right)|n+1\rangle\langle n+1| + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| \text{ (by } \langle i|j\rangle = \delta_{ij}) \tag{24}$$

$$= \mathbb{I} + \sum_{j=1}^{n+1} \left(\mathrm{e}^{-\beta E_{0,j}(t)} - 1\right)|j\rangle\langle j|. \tag{25}$$

By mathematical induction we proved that (18) is true for all $n \in \mathbb{N}$. Given that the resolution of the identity is $\mathbb{I} = \sum_n |n\rangle\langle n|$ so we find that:

$$\mathrm{e}^{-\beta\overline{H_0}(t)} = \prod_n \left(\mathbb{I} + \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n|\right) \tag{26}$$

$$= \mathbb{I} + \sum_n \left(\mathrm{e}^{-\beta E_{0,n}(t)} - 1\right)|n\rangle\langle n| \tag{27}$$

$$= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \sum_n |n\rangle\langle n| \tag{28}$$

$$= \mathbb{I} + \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n| - \mathbb{I} \text{ (by the resolution of the identity)} \tag{29}$$

$$= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|. \tag{30}$$

The partition function $Z_0\left(t\right)$ is equal to:

$$Z_0\left(t\right) = \mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right) \tag{31}$$

$$= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(|n\rangle\langle n|\right) \tag{32}$$

$$= \sum_n \mathrm{e}^{-\beta E_{0,n}(t)}. \tag{33}$$

The explicit form of the average value $\langle A\left(t\right)\rangle_{\overline{H_0}(t)}$ can be found from the partition function $Z_0\left(t\right)$:

$$\langle A\left(t\right)\rangle_{\overline{H_0}(t)} = \frac{\mathrm{Tr}\left(A\left(t\right)\mathrm{e}^{-\beta\overline{H_0}(t)}\right)}{Z_0\left(t\right)} \tag{34}$$

$$= \frac{\mathrm{Tr}\left(\sum_n A\left(t\right)\mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta\overline{H_0}(t)}\right)} \tag{35}$$

$$= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}|n\rangle\langle n|\right)} \tag{36}$$

$$= \frac{\mathrm{Tr}\left(\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}} \tag{37}$$

$$= \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}\mathrm{Tr}\left(A\left(t\right)|n\rangle\langle n|\right)}{\sum_n \mathrm{e}^{-\beta E_{0,n}(t)}}. \tag{38}$$

At first we show a double sequence of inequalities of order M, N which generalizes the quantum Bogoliubov inequality to any order as shown in [3]:

$$Z\left(t\right) \geq Z_0\left(t\right)\mathrm{e}^{-\left\langle\overline{H_I}(t)\right\rangle_{\overline{H_0}(t)}}\left(1 + F_M\left(\overrightarrow{u}\left(t\right);\alpha\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right);\alpha\right)\right). \tag{39}$$

where the funcion $F_N\left(\overrightarrow{u}\left(t\right);\alpha\right)$ is defined as:

$$F_N\left(\overrightarrow{w}\left(t\right);\alpha\right) \equiv \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k\frac{w_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}. \tag{40}$$

In this case α is a parameter that can be optimized, $\beta \equiv \frac{1}{\mathrm{k_B}T}$, $\overrightarrow{w}\left(t\right)$ is a vector such that $\overrightarrow{w}\left(t\right) = \left(w_1, w_2, ...\right)$ and $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ are two vectors of average values that we will define below. For this objective we define the diagonalized hamiltonian $\overline{H_{\overline{I}D}}\left(t\right)$ respect to the basis of eigenstates of $\overline{H_0}\left(t\right)$ as:

$$\overline{H_{\overline{I}D}}\,(t) \equiv \sum_n \left\langle n \left| \overline{H_{\overline{I}}}\,(t) \right| n \right\rangle |n\rangle\langle n|. \tag{41}$$

We will prove an important property related to $\overline{H_{\overline{I}D}}\,(t)$, which is a Hamiltonian written as a linear combination of a set of ortonormal operators. Let's consider a vector space R with two operations $+$ and $\cdot$, if there exist $a, b \in R$ such that $a \cdot b = 0$ and $b \cdot a = 0$ then for any $k \in \mathbb{N}$ we have $(a+b)^k = a^k + b^k$ where $a^k = a^{k-1} \cdot a$ is a recursive definition of the power of an element written in terms of $\cdot$. At first we prove that this result yields for any $k \in \mathbb{N}$ by induction, the case $k = 1$ is trivial so we will focus on the case $k = 2$, we have that:

$$\begin{aligned}
(a+b)^2 &= (a+b) \cdot (a+b) && (42)\\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b && (43)\\
&= a^2 + a \cdot b + b \cdot a + b^2 && (44)\\
&= a^2 + 0 + 0 + b^2 \text{ (because } a \cdot b = b \cdot a = 0) && (45)\\
&= a^2 + b^2. && (46)
\end{aligned}$$

This is the base case. By induction step we will consider that $(a+b)^k = a^k + b^k$ with $k \geq 2$, now for $k+1$ we will have that:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)^k \cdot (a+b) && (47)\\
&= \left(a^k + b^k\right) \cdot (a+b) \text{ (by induction step)} && (48)\\
&= a^k \cdot a + a^k \cdot b + b^k \cdot a + b^k \cdot b && (49)\\
&= a^{k+1} + a^{k-1} \cdot a \cdot b + b^{k-1} \cdot b \cdot a + b^{k+1} \text{ (by recursive definition of } a^k) && (50)\\
&= a^{k+1} + a^{k-1} \cdot (a \cdot b) + b^{k-1} \cdot (b \cdot a) + b^{k+1} \text{ (by associativity on } R) && (51)\\
&= a^{k+1} + a^{k-1} \cdot (0) + b^{k-1} \cdot (0) + b^{k+1} \text{ (because } a \cdot b = b \cdot a = 0) && (52)\\
&= a^{k+1} + b^{k+1}. && (53)
\end{aligned}$$

By the principle of mathematical induction we can conclude that the proposition is true for all $k \in \mathbb{N}$. Now we will extend the result, let $a_1, ..., a_n \in R$ such that $a_i \cdot a_j = 0$ for all $i \neq j$ then $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$. The case $n = 1$ is trivial as well so we will focus on $n = 2$, this case was proved in the precedent lines so it will be our base case. By induction step we will consider that $(a_1 + ... + a_n)^k = a_1^k + ... + a_n^k$ with $n \geq 2$, now for $n+1$ we will have that:

$$\begin{aligned}
a_{n+1} \cdot (a_1 + ... + a_n) &= a_{n+1} \cdot a_1 + ... + a_{n+1} \cdot a_n && (54)\\
&= 0 \text{ (because } a_i \cdot a_j = 0 \text{ for all } i \neq j \text{)}, && (55)\\
(a_1 + ... + a_n + a_{n+1})^k &= ((a_1 + ... + a_n) + a_{n+1})^k && (56)\\
&= (a_1 + ... + a_n)^k + a_{n+1}^k \text{ (by (47) and (55))} && (57)\\
&= a_1^k + ... + a_n^k + a_{n+1}^k \text{ (by inductive step)}. && (58)
\end{aligned}$$

So we can conclude by mathematical induction that the proposition is true for all $n \in \mathbb{N}$. We can prove the following property for $\left(\overline{H_{\overline{I}D}}\,(t)\right)^k$:

$$\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle |n\rangle\langle n|\left\langle n'\left|\overline{H_{I}}\left(t\right)\right|n'\right\rangle |n'\rangle\langle n'| = \left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle \left\langle n'\left|\overline{H_{I}}\left(t\right)\right|n'\right\rangle |n\rangle\langle n|n'\rangle\langle n'| \tag{59}$$

$$= \left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle \left\langle n'\left|\overline{H_{I}}\left(t\right)\right|n'\right\rangle |n\rangle\langle n'|\delta_{nn'}, \tag{60}$$

$$\left(\overline{H_{I}}_{D}\left(t\right)\right)^{k} = \left(\sum_{n}\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle |n\rangle\langle n|\right)^{k} \text{ (by (41))} \tag{61}$$

$$= \sum_{n}\left(\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle |n\rangle\langle n|\right)^{k} \text{ (by (58) and (60))}, \tag{62}$$

$$\left(aA\right)^{k} = a^{k}A^{k} \text{ (by the property of the power of a matrix)}, \tag{63}$$

$$\left(|n\rangle\langle n|\right)^{k} = |n\rangle\langle n| \text{ (because } |n\rangle\langle n| \text{ is a projector and } k\in\mathbb{N}^{*}\text{)}, \tag{64}$$

$$\left(\overline{H_{I}}_{D}\left(t\right)\right)^{k} = \sum_{n}\left(\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle\right)^{k} |n\rangle\langle n| \text{ (by (63) and (64))}. \tag{65}$$

The vectors $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ are defined as $\overrightarrow{u}\left(t\right)\equiv\left(u_{1},u_{2},...\right)$ and $\overrightarrow{v}\left(t\right)\equiv\left(v_{1},v_{2},...\right)$. We can define the elements of $\overrightarrow{u}\left(t\right)$ and $\overrightarrow{v}\left(t\right)$ in terms of the matrix $\overline{H_{I}}_{D}\left(t\right)$:

$$u_{k}\left(t\right)\equiv\left\langle\left(\overline{H_{I}}_{D}\left(t\right)-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}\right\rangle_{\overline{H_{0}}\left(t\right)} \tag{66}$$

$$=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\mathrm{Tr}\left(\left(\sum_{n}\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle |n\rangle\langle n|-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}|n\rangle\langle n|\right)}{Z_{0}\left(t\right)} \text{ (by (38))}, \tag{67}$$

$$\left(\sum_{n}\langle n|\overline{H_{I}}\left(t\right)|n\rangle |n\rangle\langle n|-\langle\overline{H_{I}}(t)\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}=\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\left(\sum_{n}\langle n|\overline{H_{I}}\left(t\right)|n\rangle |n\rangle\langle n|\right)^{j}\left(\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k-j} \text{ (by binomial theorem)} \tag{68}$$

$$=\sum_{j=0}^{k}\left(-1\right)^{j}\binom{k}{j}\left(\sum_{n}\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle^{j}|n\rangle\langle n|\right)\left(\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k-j} \text{ (by (65))} \tag{69}$$

$$=\sum_{n}\left(\sum_{j=0}^{k}\left(-1\right)^{j}\binom{k}{j}\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle^{j}\left(\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k-j}\right)|n\rangle\langle n| \tag{70}$$

$$=\sum_{n}\left(\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}|n\rangle\langle n|, \tag{71}$$

$$=\sum_{n}\left(\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}|n\rangle\langle n|, \tag{72}$$

$$u_{k}\left(t\right)=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\mathrm{Tr}\left(\sum_{n'}\left(\left\langle n'\left|\overline{H_{I}}\left(t\right)\right|n'\right\rangle-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}|n'\rangle\langle n'|n\rangle\langle n|\right)}{Z_{0}\left(t\right)} \tag{73}$$

$$=\frac{\sum_{nn'}\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\mathrm{Tr}\left(\left(\left\langle n'\left|\overline{H_{I}}\left(t\right)\right|n'\right\rangle-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}|n'\rangle\langle n|\delta_{nn'}\right)}{Z_{0}\left(t\right)} \tag{74}$$

$$=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}\mathrm{Tr}\left(|n\rangle\langle n|\right)}{Z_{0}\left(t\right)} \tag{75}$$

$$=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}1}{Z_{0}\left(t\right)} \tag{76}$$

$$=\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|\overline{H_{I}}\left(t\right)\right|n\right\rangle-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}}{Z_{0}\left(t\right)}, \tag{77}$$

$$v_{k}\left(t\right)\equiv\frac{\sum_{n}\mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\left(\overline{H_{0}}\left(t\right)-E_{0,n}\left(t\right)+\overline{H_{I}}\left(t\right)-\left\langle\overline{H_{I}}\left(t\right)\right\rangle_{\overline{H_{0}}\left(t\right)}\right)^{k}\right|n\right\rangle}{Z_{0}\left(t\right)}. \tag{78}$$

By construction $\left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} = 0$, so we summarize the double inequality that generalizes the Bogoliubov inequality and it's coefficients as:

$$Z\left(t\right) \geq Z_0\left(t\right)\left(1 + F_M\left(\overrightarrow{u}\left(t\right)\right) + F_N\left(\overrightarrow{v}\left(t\right) - \overrightarrow{u}\left(t\right)\right)\right), \tag{79}$$

$$Z\left(t\right) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H}\left(t\right)}\right), \tag{80}$$

$$Z_0\left(t\right) = \sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}, \tag{81}$$

$$F_N\left(\overrightarrow{u}\left(t\right)\right) = \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}\left(-\beta\right)^k \frac{u_k\left(t\right)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}, \tag{82}$$

$$u_k\left(t\right) = \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle^k}{Z_0\left(t\right)}, \tag{83}$$

$$v_k\left(t\right) = \frac{\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right)^k\right|n\right\rangle}{Z_0\left(t\right)}. \tag{84}$$

As we can see the expression (83) was written in shorter terms, we want to do the same for (84) in order to write that expressions in a similar format. The expressions that we will show will appear widely in the obtention of a formula for $v_k\left(t\right)$:

$$\begin{aligned}
\left(\overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right)\left|n\right\rangle &= \overline{H_0}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (85)\\
&= E_{0,n}\left(t\right)\left|n\right\rangle - E_{0,n}\left(t\right)\left|n\right\rangle && (86)\\
&= 0, && (87)\\
\left\langle n\right|\left(\overline{H_0}\left(t\right) - E_{0,n}\right) &= \left\langle n\right|\overline{H_0}\left(t\right) - \left\langle n\right| E_{0,n}\left(t\right) && (88)\\
&= \left\langle n\right| E_{0,n}\left(t\right) - \left\langle n\right| E_{0,n}\left(t\right) && (89)\\
&= 0. && (90)
\end{aligned}$$

At first we calculated $v_1\left(t\right)$ using the definition (84) :

$$\begin{aligned}
v_1\left(t\right) &= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_0}\left(t\right) - E_{0,n}\left(t\right) + \overline{H_I}\left(t\right)\right|n\right\rangle && (91)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_0}\left(t\right) - E_{0,n}\left(t\right)\right|n\right\rangle + \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left\langle n\left|\overline{H_I}\left(t\right)\right|n\right\rangle && (92)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|\overline{H_0}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} && (93)\\
&= \frac{1}{Z_0\left(t\right)}\sum_n \mathrm{e}^{-\beta E_{0,n}\left(t\right)}\left(\left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle - \left\langle n\left|E_{0,n}\left(t\right)\right|n\right\rangle\right) + \left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} && (94)\\
&= 0 + \left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} \quad \text{(by construction } \left\langle \overline{H_I}\left(t\right)\right\rangle_{\overline{H_0}\left(t\right)} = 0\text{)} && (95)\\
&= 0. && (96)
\end{aligned}$$

For $k \geq 2$ and $k \in \mathbb{N}$ we calculated:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^k \right| n \right\rangle \tag{97}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{98}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{99}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \left(E_{0,n}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right) \right| n \right\rangle \tag{100}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle . \tag{101}$$

In general we can write a formula for $v_k(t)$ that implies an expected value of a dependent expression of $\overline{H_{\overline{I}}}(t)$ and $\overline{H_0}(t)$:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) - E_{0,n}(t) + \overline{H_{\overline{I}}}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{102}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H_0}(t) + \overline{H_{\overline{I}}}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{103}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\overline{H}(t) - E_{0,n}(t)\right)^{k-2} \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{104}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t) \left(\sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j}(t)\, E_{0,n}^j(t) \right) \overline{H_{\overline{I}}}(t) \right| n \right\rangle \tag{105}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, E_{0,n}^j(t) \right| n \right\rangle \tag{106}$$

$$= \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{107}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n \mathrm{e}^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_{\overline{I}}}(t)\, \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right| n \right\rangle \tag{108}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t)\; \overline{H}^{k-2-j}(t)\, \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{109}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_{\overline{I}}}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} . \tag{110}$$

The formula (110) is well defined taking as example $k = 2, 3$.

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^{2-2-j} \overline{H_{\overline{I}}}(t)\; \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \tag{111}$$

$$= (-1)^0 \left\langle \overline{H_{\overline{I}}}(t) \left(\overline{H_{\overline{I}}}(t) + \overline{H_0}(t)\right)^0 \overline{H_{\overline{I}}}(t)\; \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \tag{112}$$

$$= \left\langle \overline{H_{\overline{I}}}^2(t) \right\rangle_{\overline{H_0}(t)} , \tag{113}$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{3-2-j} \overline{H_I}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{114}$$

$$= \left\langle \sum_{j=0}^{1} (-1)^j \binom{1}{j} \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{1-j} \overline{H_I}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{115}$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_I}(t) \left(\overline{H_I}(t)+\overline{H_0}(t)\right)^1 \overline{H_I}(t)\ \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_I}(t) \left(\overline{H_I}(t)+\overline{H_0}(t)\right)^0 \overline{H_I}(t)\ \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \tag{116}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\ \mathbb{I} - \overline{H_I}(t)\ \mathbb{I} \overline{H_I}(t)\ \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{117}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t) - \overline{H_I}(t)\ \overline{H_I}(t)\ \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{118}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t)\ \overline{H_0}(t)\ \overline{H_I}(t) - \overline{H_I}(t)\ \overline{H_I}(t)\ \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \tag{119}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) \left(\overline{H_0}(t)\ \overline{H_I}(t) - \overline{H_I}(t)\ \overline{H_0}(t)\right) \right\rangle_{\overline{H_0}(t)} \tag{120}$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) \left[\overline{H_0}(t), \overline{H_I}(t)\right] \right\rangle_{\overline{H_0}(t)} \text{ (because } \left[\overline{H_0}(t), \overline{H_I}(t)\right] = \overline{H_0}(t)\overline{H_I}(t) - \overline{H_I}(t)\overline{H_0}(t)). \tag{121}$$

So we summarize:

$$\overline{H_I}_D(t) = \sum_n \left\langle n \left| \overline{H_I}(t) \right| n \right\rangle |n\rangle\langle n|, \tag{122}$$

$$u_k(t) = \left\langle \left(\overline{H_I}_D(t)\right)^k \right\rangle_{\overline{H_0}(t)}, \tag{123}$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{k-2-j} \overline{H_I}(t)\ \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \tag{124}$$

The free energy is defined as:

$$E_{\text{free}}(t) \equiv -\frac{1}{\beta} \ln(Z(t)). \tag{125}$$

We define the free energy at first order as:

$$E_{\text{free},1}(t) \equiv -\frac{1}{\beta} \ln(Z_0(t)). \tag{126}$$

It is well-known that the function $f(x) = -\ln(x)$ is a decreasing function so we can transform (39):

$$E_{\text{free}}(t) \leq -\frac{1}{\beta} \ln\left(Z_0(t)\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right)\right) \tag{127}$$

$$= -\frac{1}{\beta} \ln(Z_0(t)) - \frac{1}{\beta} \ln\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) \tag{128}$$

$$= E_{\text{free},1}(t) - \frac{1}{\beta} \ln\left(1 + F_M\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) \tag{129}$$

$$\equiv E_{\text{free,MN}}(t). \tag{130}$$

here $E_{\text{free,MN}}(t)$ is the free energy associate to the strong version of the Quantum Bogoliubov inequality of M, N order. In our approach we will set $N = M$, so the inequality (130) of N, N order is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln\left(1 + F_N\left(\overrightarrow{u}(t)\right) + F_N\left(\overrightarrow{v}(t) - \overrightarrow{u}(t)\right)\right) \tag{131}$$

$$= E_{\text{free,NN}}(t). \tag{132}$$

A weaker form of the inequality (132) is obtained making $\overrightarrow{u}(t)=0$ as suggest [3]:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{v}(t)\right)\right) \tag{133}$$
$$\equiv E_{\text{free,N}}(t). \tag{134}$$

The algebraic equation associated with $\alpha_{\text{opt}}(t)$ such that $E_{\text{free,N}}(t)$ is closer to $E_{\text{free}}(t)$ follows from the fact that in the optimal parameter $\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha}|_{\alpha_{\text{opt}(t)}}=0$, calculating this derivate we have:

$$\frac{\partial E_{\text{free,N}}(t)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(E_{\text{free},1}(t) - \frac{1}{\beta}\ln\left(1+F_N\left(\overrightarrow{v}(t)\right)\right)\right) \tag{135}$$
$$= -\frac{1}{\beta}\frac{\frac{\partial}{\partial\alpha}\left(F_N\left(\overrightarrow{v}(t)\right)\right)}{1+F_N\left(\overrightarrow{v}(t)\right)} \tag{136}$$
$$= 0. \tag{137}$$

The precedent equation is equivalent to make the numerator equal to 0:

$$\frac{\partial F_N\left(\overrightarrow{v}(t)\right)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left(\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{138}$$
$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\partial}{\partial\alpha}\frac{\alpha^i}{i!}\ \text{(by product rule)} \tag{139}$$
$$= -\mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!} + \mathrm{e}^{-\alpha}\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} \tag{140}$$
$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=1}^{2N-1-k}\frac{\alpha^{i-1}}{(i-1)!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right) \tag{141}$$
$$= \mathrm{e}^{-\alpha}\left(\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{j=0}^{2N-2-k}\frac{\alpha^j}{j!} - \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\sum_{i=0}^{2N-1-k}\frac{\alpha^i}{i!}\right)\ \text{(setting } j=i-1\text{)} \tag{142}$$
$$= \mathrm{e}^{-\alpha}\left(-\sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha^{2N-1-k}}{(2N-1-k)!}\right)\ \text{(performing the difference)} \tag{143}$$
$$= 0. \tag{144}$$

Then the optimal value $\alpha_{\text{opt}}(t)$ will sastisfy the following equation:

$$G\left(\alpha_{\text{opt}}(t)\right) \equiv \sum_{k=2}^{2N-1}(-\beta)^k\frac{u_k(t)}{k!}\frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \tag{145}$$
$$= 0. \tag{146}$$

The elements presented are the required to find variational parameters of the system using the inequality (134) and the self consistent equation (SCE) (145) to a particular order required.

II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify $v_2(t), v_3(t), v_4(t), v_5(t)$ using the (124), we have already $v_2(t), v_3(t)$ and the form of $v_4(t)$ and $v_5(t)$ is given by:

$$v_4(t) = \sum_{j=0}^{4-2} (-1)^j \binom{4-2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{4-2-j} \overline{H_I}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{147}$$

$$= \sum_{j=0}^{2} (-1)^j \binom{2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{2-j} \overline{H_I}(t)\, \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \tag{148}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t)\, \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^1 \overline{H_I}(t)\, \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \langle \overline{H_I}(t) (\overline{H_I}(t) \tag{149}$$

$$+ \overline{H_0}(t))^0\, \overline{H_I}(t)\, \overline{H_0}^2(t) \rangle_{\overline{H_0}(t)} \tag{150}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t)\, \mathbb{I} \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^1 \overline{H_I}(t)\, \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t)\, \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{151}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t)\, \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{152}$$

$$= \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t) - 2\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \overline{H_0}(t) + \overline{H_I}^2(t)\, \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \tag{153}$$

$$= \langle \overline{H_I}(t) \left(\overline{H_I}^2(t) + \overline{H_I}(t)\, \overline{H_0}(t) + \overline{H_0}(t)\, \overline{H_I}(t) + \overline{H_0}^2(t)\right) \overline{H_I}(t) - 2\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \overline{H_0}(t) + \overline{H_I}^2(t) \tag{154}$$

$$\times \overline{H_0}^2(t) \rangle_{\overline{H_0}(t)} \tag{155}$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t)\, \overline{H_0}(t)\, \overline{H_I}(t) + \overline{H_I}(t)\, \overline{H_0}(t)\, \overline{H_I}^2(t) + \overline{H_I}(t)\, \overline{H_0}^2(t)\, \overline{H_I}(t) - 2\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \overline{H_0}(t) \tag{156}$$

$$+ \overline{H_I}^2(t)\, \overline{H_0}^2(t) \rangle_{\overline{H_0}(t)} \tag{157}$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t)\overline{H_0}(t)\, \overline{H_I}(t) + \overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t) + \overline{H_I}(t)\overline{H_0}^2(t)\, \overline{H_I}(t) - 2\overline{H_I}^3(t)\overline{H_0}(t) + \overline{H_I}^2(t)\overline{H_0}^2(t) - 2\overline{H_I}(t) \tag{158}$$

$$\times \overline{H_0}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \tag{159}$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t)\overline{H_0}(t)\, \overline{H_I}(t) + \overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t) + \overline{H_I}(t)\overline{H_0}^2(t)\, \overline{H_I}(t) - \overline{H_I}^3(t)\overline{H_0}(t) - \overline{H_I}^3(t)\overline{H_0}(t) + \overline{H_I}^2(t) \tag{160}$$

$$\times \overline{H_0}^2(t) - \overline{H_I}(t)\, \overline{H_0}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) - \overline{H_I}(t)\, \overline{H_0}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \tag{161}$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t) - \overline{H_I}^3(t)\overline{H_0}(t) + \overline{H_I}(t)\overline{H_0}(t)\, \overline{H_I}^2(t) - \overline{H_I}^3(t)\overline{H_0}(t) + \overline{H_I}(t)\overline{H_0}^2(t)\, \overline{H_I}(t) - \overline{H_I}(t) \tag{162}$$

$$\times \overline{H_0}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) + \overline{H_I}^2(t)\, \overline{H_0}^2(t) - \overline{H_I}(t)\, \overline{H_0}(t)\, \overline{H_I}(t)\, \overline{H_0}(t) \rangle_{\overline{H_0}(t)} \tag{163}$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}(t) \Big(\big((\overline{H_I}(t)\overline{H_0}(t))\overline{H_I}(t) - \overline{H_I}(t)(\overline{H_I}(t)\overline{H_0}(t)) \big) + \left(\overline{H_0}(t)\overline{H_I}^2(t) - \overline{H_I}^2(t)\overline{H_0}(t)\right) + \big(\overline{H_0}(t)(\overline{H_0}(t) \tag{164}$$

$$\times \overline{H_I}(t)) - (\overline{H_0}(t)\, \overline{H_I}(t))\, \overline{H_0}(t)\big) + \big((\overline{H_I}(t)\, \overline{H_0}(t))\, \overline{H_0}(t) - \overline{H_0}(t)\, (\overline{H_I}(t)\, \overline{H_0}(t))\big)\Big) \rangle_{\overline{H_0}(t)} \tag{165}$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \left(\left[\overline{H_I}(t)\overline{H_0}(t), \overline{H_I}(t) \right] + \left[\overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[\overline{H_0}(t), \overline{H_0}(t)\overline{H_I}(t) \right] + \left[\overline{H_I}(t)\overline{H_0}(t), \overline{H_0}(t) \right] \right) \right\rangle_{\overline{H_0}(t)}, \tag{166}$$

$$v_5(t) = \sum_{j=0}^{5-2} (-1)^j \binom{5-2}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{5-2-j} \overline{H_I}(t)\, \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \tag{167}$$

$$= \sum_{j=0}^{3} (-1)^j \binom{3}{j} \left\langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^{3-j} \overline{H_I}(t)\, \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \tag{168}$$

$$= \langle \overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t)\overline{H_0}^0(t) - 3\overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t)\overline{H_0}(t) - \overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)^0 \overline{H_I}(t)\, \overline{H_0}^3(t) \tag{169}$$

$$+ 3\overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \overline{H_0}^2(t) \rangle_{\overline{H_0}(t)} \tag{170}$$

$$= \langle \overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t) - 3\overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t)\overline{H_0}(t) + 3\overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)\overline{H_I}(t)\overline{H_0}^2(t) - \overline{H_I}(t) \tag{171}$$

$$\times \overline{H_I}(t)\, \overline{H_0}^3(t) \rangle_{\overline{H_0}(t)} \tag{172}$$

$$= \langle \overline{H_I}(t) \left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t) - 3\overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)^2 \overline{H_I}(t)\, \overline{H_0}(t) + 3\overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)\overline{H_I}(t)\, \overline{H_0}^2(t) \tag{173}$$

$$- \overline{H_I}(t)\, \overline{H_I}(t)\, \overline{H_0}^3(t) \rangle_{\overline{H_0}(t)} \tag{174}$$

$$= \langle \overline{H_I}(t)\left(\overline{H_I}(t) + \overline{H_0}(t)\right)^3 \overline{H_I}(t) - 3\overline{H_I}(t)\left(\overline{H_I}^2(t) + \overline{H_I}(t)\overline{H_0}(t) + \overline{H_0}(t)\overline{H_I}(t) + \overline{H_0}^2(t)\right)\overline{H_I}(t)\overline{H_0}(t) + 3\overline{H_I}(t) \tag{175}$$

$$\times \left(\overline{H_I}(t) + \overline{H_0}(t)\right) \overline{H_I}(t)\, \overline{H_0}^2(t) - \overline{H_I}(t)\, \overline{H_I}(t)\, \overline{H_0}^3(t) \rangle_{\overline{H_0}(t)} \tag{176}$$

$$=\left\langle \overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}^{3}(t)+\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)+\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{2}(t)+\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)+\overline{H_{\overline{I}}}(t)\right.\right. \tag{177}$$

$$\times\overline{H_0}^{2}(t)+\overline{H_0}^{3}(t)\Big)\,\overline{H_{\overline{I}}}(t)-3\overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}^{2}(t)+\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)+\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_0}^{2}(t)\right)\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)+3\overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}(t)\right. \tag{178}$$

$$\left.\left.+\overline{H_0}(t)\right)\overline{H_{\overline{I}}}(t)\,\overline{H_0}^{2}(t)-\overline{H_{\overline{I}}}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}^{3}(t)\right\rangle_{\overline{H_0(t)}} \tag{179}$$

$$=\left\langle \overline{H_{\overline{I}}}^{5}(t)+\overline{H_{\overline{I}}}^{3}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{2}(t)+\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{3}(t)+\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_{\overline{I}}}(t)\right. \tag{180}$$

$$\times\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}^{2}(t)+\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_{\overline{I}}}(t)\,\overline{H_0}^{3}(t)\,\overline{H_{\overline{I}}}(t)-3\overline{H_{\overline{I}}}^{4}(t)\,\overline{H_0}(t)-3\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)-3\overline{H_{\overline{I}}}(t) \tag{181}$$

$$\left.\times\overline{H_0}(t)\overline{H_{\overline{I}}}^{2}(t)\overline{H_0}(t)-3\overline{H_{\overline{I}}}(t)\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}(t)\overline{H_0}(t)+3\overline{H_{\overline{I}}}^{3}(t)\overline{H_0}^{2}(t)+3\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\overline{H_0}^{2}(t)-\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}^{3}(t)\right\rangle_{\overline{H_0(t)}} \tag{182}$$

$$=\left\langle \overline{H_{\overline{I}}}^{5}(t)+\overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{2}(t)+\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{3}(t)+\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}^{2}(t)+\overline{H_{\overline{I}}}(t)\,\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}(t)\right.\right. \tag{183}$$

$$+\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)+\overline{H_0}^{3}(t)\,\overline{H_{\overline{I}}}(t)-3\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\overline{H_0}(t)-3\overline{H_{\overline{I}}}^{3}(t)\overline{H_0}(t)-3\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}(t)\overline{H_0}(t)-\overline{H_{\overline{I}}}(t) \tag{184}$$

$$\left.\left.\times\overline{H_0}^{3}(t)+3\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}^{2}(t)+3\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}^{2}(t)-3\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\right)\right\rangle_{\overline{H_0(t)}} \tag{185}$$

$$=\left\langle \overline{H_{\overline{I}}}^{5}(t)+\overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}^{2}(t)\overline{H_0}(t)\overline{H_{\overline{I}}}(t)-\overline{H_{\overline{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\overline{H_{\overline{I}}}^{2}(t)-\overline{H_{\overline{I}}}^{3}(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_{\overline{I}}}^{3}(t)-\overline{H_0}(t)\overline{H_{\overline{I}}}^{2}(t)\right.\right. \tag{186}$$

$$\times\overline{H_0}(t)+\overline{H_0}(t)\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\overline{H_{\overline{I}}}(t)-\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\overline{H_{\overline{I}}}(t)\overline{H_0}(t)+\overline{H_{\overline{I}}}(t)\overline{H_0}^{2}(t)\overline{H_{\overline{I}}}(t)-\overline{H_0}(t)\overline{H_{\overline{I}}}^{2}(t)\overline{H_0}(t)+\overline{H_0}^{3}(t)\,\overline{H_{\overline{I}}}(t) \tag{187}$$

$$-\overline{H_{\overline{I}}}(t)\,\overline{H_0}^{3}(t)+\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}^{2}(t)-\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)+2\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}^{2}(t)-2\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)+3\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t) \tag{188}$$

$$\left.\left.\times\overline{H_0}^{2}(t)-3\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)-\overline{H_{\overline{I}}}^{3}(t)\,\overline{H_0}(t)+\overline{H_0}^{2}(t)\,\overline{H_{\overline{I}}}^{2}(t)\right)\right\rangle_{\overline{H_0(t)}} \tag{189}$$

$$=\left\langle \overline{H_{\overline{I}}}^{5}(t)+\overline{H_{\overline{I}}}(t)\left(\left(\left(\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\right)\overline{H_{\overline{I}}}(t)-\overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\right)\right)+\left(\left(\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\right)\overline{H_{\overline{I}}}^{2}(t)-\overline{H_{\overline{I}}}^{2}(t)\left(\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\right)\right)\right.\right. \tag{190}$$

$$+\left(\overline{H_0}(t)\overline{H_{\overline{I}}}^{3}(t)-\overline{H_{\overline{I}}}^{3}(t)\,\overline{H_0}(t)\right)+\left(\overline{H_0}(t)\left(\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\overline{H_{\overline{I}}}(t)\right)-\left(\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\overline{H_{\overline{I}}}(t)\right)\overline{H_0}(t)\right)+\left(\overline{H_0}(t)\left(\overline{H_0}(t)\overline{H_{\overline{I}}}^{2}(t)\right)\right. \tag{191}$$

$$\left.-\left(\overline{H_0}(t)\overline{H_{\overline{I}}}^{2}(t)\right)\overline{H_0}(t)\right)+\left(\overline{H_0}^{3}(t)\overline{H_{\overline{I}}}(t)-\overline{H_{\overline{I}}}(t)\overline{H_0}^{3}(t)\right)+\left(\left(\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\right)\left(\overline{H_0}(t)\overline{H_{\overline{I}}}(t)\right)-\left(\overline{H_0}(t)\overline{H_{\overline{I}}}(t)\right)\left(\overline{H_{\overline{I}}}(t)\overline{H_0}(t)\right)\right) \tag{192}$$

$$+3\overline{H_0}(t)\left(\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\right)\overline{H_0}(t)+2\overline{H_{\overline{I}}}(t)\left(\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)-\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\right)\overline{H_0}(t)+\left(\left(\overline{H_{\overline{I}}}^{2}(t)\overline{H_0}(t)\right)\left(\overline{H_0}(t)\right)\right. \tag{193}$$

$$\left.\left.\left.-\left(\overline{H_0}(t)\right)\left(\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\right)\right)\right)\right\rangle_{\overline{H_0(t)}} \tag{194}$$

$$=\left\langle \overline{H_{\overline{I}}}^{5}(t)+\overline{H_{\overline{I}}}(t)\left(\left[\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\,,\overline{H_{\overline{I}}}(t)\right]+\left[\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,,\overline{H_{\overline{I}}}^{2}(t)\right]+\left[\overline{H_0}(t)\,,\overline{H_{\overline{I}}}^{3}(t)\right]+\left[\overline{H_0}(t)\,,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\right]\right.\right. \tag{195}$$

$$+\left[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{2}(t)\right]+\left[\overline{H_0}^{3}(t)\,,\overline{H_{\overline{I}}}(t)\right]+\left[\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\right]+3\overline{H_0}(t)\left[\overline{H_{\overline{I}}}(t)\,,\overline{H_0}(t)\right]\overline{H_0}(t)+2\overline{H_{\overline{I}}}(t) \tag{196}$$

$$\left.\left.\times\left[\overline{H_{\overline{I}}}(t)\,,\overline{H_0}(t)\right]\overline{H_0}(t)+\left[\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\right]\right)\right\rangle_{\overline{H_0(t)}}. \tag{197}$$

Summarizing we have that:

$$v_2(t)=\left\langle \overline{H_{\overline{I}}}^{2}(t)\right\rangle_{\overline{H_0(t)}}, \tag{198}$$

$$v_3(t)=\left\langle \overline{H_{\overline{I}}}^{3}(t)+\overline{H_{\overline{I}}}(t)\left[\overline{H_0}(t)\,,\overline{H_{\overline{I}}}(t)\right]\right\rangle_{\overline{H_0(t)}}, \tag{199}$$

$$v_4(t)=\left\langle \overline{H_{\overline{I}}}^{4}(t)+\overline{H_{\overline{I}}}(t)\left(\left[\overline{H_{\overline{I}}}(t)\overline{H_0}(t),\overline{H_{\overline{I}}}(t)\right]+\left[\overline{H_0}(t),\overline{H_{\overline{I}}}^{2}(t)\right]+\left[\overline{H_0}(t),\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\right]+\left[\overline{H_{\overline{I}}}(t)\overline{H_0}(t),\overline{H_0}(t)\right]\right)\right\rangle_{\overline{H_0(t)}}, \tag{200}$$

$$v_5(t)=\left\langle \overline{H_{\overline{I}}}^{5}(t)+\overline{H_{\overline{I}}}(t)\left(\left[\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\,,\overline{H_{\overline{I}}}(t)\right]+\left[\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,,\overline{H_{\overline{I}}}^{2}(t)\right]+\left[\overline{H_0}(t)\,,\overline{H_{\overline{I}}}^{3}(t)\right]+\left[\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_{\overline{I}}}^{2}(t)\right]\right.\right. \tag{201}$$

$$+\left[\overline{H_0}(t)\,,\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\right]+\left[\overline{H_0}^{3}(t)\,,\overline{H_{\overline{I}}}(t)\right]+\left[\overline{H_{\overline{I}}}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\,\overline{H_{\overline{I}}}(t)\right]+3\overline{H_0}(t)\left[\overline{H_{\overline{I}}}(t)\,,\overline{H_0}(t)\right]\overline{H_0}(t) \tag{202}$$

$$\left.\left.+2\overline{H_{\overline{I}}}(t)\left[\overline{H_{\overline{I}}}(t)\,,\overline{H_0}(t)\right]\overline{H_0}(t)+\left[\overline{H_{\overline{I}}}^{2}(t)\,\overline{H_0}(t)\,,\overline{H_0}(t)\right]\right)\right\rangle_{\overline{H_0(t)}}. \tag{203}$$

Now we will obtain the expected values related to $v_2(t)$, $v_3(t)$, $v_4(t)$ and $v_5(t)$. Recall the hamiltonian of interest for the system studied in [2]:

$$\overline{H_{\bar{S}}}\,(t) \equiv (\varepsilon_0(t) + R_0(t))\,|0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t))\,|1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t) - B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\right) - \sigma_y \left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) + B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right), \quad (204)$$

$$\overline{H_{\bar{I}}}\,(t) \equiv \sum_i B_{iz}\,(t)\,|i\rangle\langle i| + V_{10}^{\Re}\,(t)\,(\sigma_x B_x\,(t) + \sigma_y B_y\,(t)) + V_{10}^{\Im}\,(t)\,(\sigma_x B_y\,(t) - \sigma_y B_x\,(t))\,, \quad (205)$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (206)$$

$$= H_B. \quad (207)$$

In this case $\varepsilon_j\,(t)$, $R_j\,(t)$ for $j \in \{0,1\}$, $B_{10}^{\Re}(t)$, $B_{10}^{\Im}(t)$, $V_{10}^{\Re}(t)$ and $V_{10}^{\Im}(t)$ are scalars and the other operators are:

$$\sigma_x \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \quad (208)$$

$$\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (209)$$

$$\sigma_y \equiv -\mathrm{i}|1\rangle\langle 0| + \mathrm{i}|0\rangle\langle 1| \quad (210)$$

$$\equiv \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad (211)$$

$$\sigma_z \equiv |1\rangle\langle 1| - |0\rangle\langle 0| \quad (212)$$

$$\equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (213)$$

$$\begin{pmatrix} B_{iz}\,(t) & B_i^{\pm}\,(t) \\ B_x\,(t) & B_i\,(t) \\ B_y\,(t) & B_{ij}\,(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}\,(t))^*\, b_{\mathbf{k}} \right) & \mathrm{e}^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \\ \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} & \mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \mathrm{e}^{\chi_{ij}(t)} \end{pmatrix}, \quad (214)$$

$$\chi_{ij}\,(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*\,(t)\, v_{j\mathbf{k}}\,(t) - v_{i\mathbf{k}}\,(t)\, v_{j\mathbf{k}}^*\,(t)}{\omega_{\mathbf{k}}^2} \right), \quad (215)$$

$$B_i^+\,(t)\, B_j^-\,(t) = \mathrm{e}^{\chi_{ij}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}\,(t)}{\omega_{\mathbf{k}}} \right), \quad (216)$$

$$D\,(\pm v_{\mathbf{k}}\,(t)) \equiv \mathrm{e}^{\pm\left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}. \quad (217)$$

As we can see they verify the relationship $\sigma_x\sigma_y = \mathrm{i}\sigma_z$. The explicit form of $\overline{H_{\bar{I}}}^2\,(t)$ is:

$$\overline{H_{\bar{I}}}^2\,(t) = \sum_i B_{iz}^2(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i B_{iz}(t)|i\rangle\langle i|(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)\sum_i B_{iz}(t)\,|i\rangle\langle i|\,(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Re}(t) \quad (218)$$

$$\times\,(\sigma_x B_x\,(t) + \sigma_y B_y\,(t)) \sum_i B_{iz}\,(t)\,|i\rangle\langle i| + \left(V_{10}^{\Re}(t)\right)^2 (\sigma_x B_x\,(t) + \sigma_y B_y\,(t))^2 + V_{10}^{\Re}(t)V_{10}^{\Im}(t)\,(\sigma_x B_x\,(t) + \sigma_y B_y\,(t)) \quad (219)$$

$$\times\,(\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))(\sigma_x B_x(t) \quad (220)$$

$$+\sigma_y B_y\,(t)) + \left(V_{10}^{\Im}\,(t)\right)^2 (\sigma_x B_y\,(t) - \sigma_y B_x\,(t))^2 \quad (221)$$

$$= \sum_i B_{iz}^2(t)\,|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i (B_{iz}(t)\,B_x(t)\,|i\rangle\langle i|\sigma_x + B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i (B_{iz}(t)\,B_y(t)\,|i\rangle\langle i|\sigma_x - B_{iz}(t) \quad (222)$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i (\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2 \left(\sigma_x^2 B_x^2(t) + \sigma_x\sigma_y B_x(t)B_y(t) + \sigma_y\right. \quad (223)$$

$$\left.\times\sigma_x B_y(t)B_x(t) + \sigma_y^2 B_y^2(t)\right) + V_{10}^{\Im}(t)\sum_i (\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 \left(\sigma_x^2 B_y^2(t) + \sigma_y^2 B_x^2(t)\right. \quad (224)$$

$$\left.-\sigma_x\sigma_y B_y(t)B_x(t) - \sigma_y\sigma_x B_x(t)B_y(t)\right) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)\left(\sigma_x^2 B_y(t)B_x(t) + \sigma_x\sigma_y B_y^2(t) - \sigma_y\sigma_x B_x^2(t) - \sigma_y^2 B_x(t)B_y(t)\right. \quad (225)$$

$$\left.+\sigma_x^2 B_x(t)B_y(t) - \sigma_x\sigma_y B_x^2(t) + \sigma_y\sigma_x B_y^2(t) - \sigma_y^2 B_y(t)B_x(t)\right), \quad (226)$$

$$\sigma_x\sigma_y = \mathrm{i}\sigma_z \text{ (by Pauli matrices properties)}, \tag{227}$$

$$\overline{H_I}^2(t) = \sum_i B_{iz}^2(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i (B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x + B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i (B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x - B_{iz}(t) \tag{228}$$

$$\times B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i (\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2 (B_x^2(t) + \mathrm{i}\sigma_z B_x(t)B_y(t) - \mathrm{i}\sigma_z \tag{229}$$

$$\times B_y(t)B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t)\sum_i (\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2 (B_y^2(t) + B_x^2(t) - \mathrm{i}\sigma_z \tag{230}$$

$$\times B_y(t)B_x(t) + \mathrm{i}\sigma_z B_x(t)B_y(t)). \tag{231}$$

To introduce the direct calculation of the expected values recall that the hamiltonian $\overline{H_0}(t)$ is a direct sum of the hamiltonians of two Hilbert spaces given by $\overline{H_{\bar{S}}}(t)$ and $\overline{H_{\bar{B}}}$, so we can write in general the hamiltonian $\overline{H_0}(t)$ as:

$$\overline{H_0}(t) = \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} + \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}. \tag{232}$$

where $\mathbb{I}_{\bar{B}}$ and $\mathbb{I}_{\bar{S}}$ are the identity of the systems $\bar{B}$ and $\bar{S}$ respectively.
We can show that:

$$\begin{aligned}
\left[\overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}}, \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}}\right] &= \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} \cdot \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}} \otimes \overline{H_{\bar{B}}} \cdot \overline{H_{\bar{S}}}(t) \otimes \mathbb{I}_{\bar{B}} &(233)\\
&= \overline{H_{\bar{S}}}(t)\,\mathbb{I}_{\bar{S}} \otimes \mathbb{I}_{\bar{B}}\overline{H_{\bar{B}}} - \mathbb{I}_{\bar{S}}\overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}}\mathbb{I}_{\bar{B}} &(234)\\
&= \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} - \overline{H_{\bar{S}}}(t) \otimes \overline{H_{\bar{B}}} \text{ (by definition of identity operator)} &(235)\\
&= 0. &(236)
\end{aligned}$$

Let's introduce the following partition functions $Z_{\bar{S}}(t)$ and $Z_{\bar{B}}$ related to the systems $\bar{S}$ and $\bar{B}$ respectively.:

$$Z_{\bar{S}}(t) \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right), \tag{237}$$

$$Z_{\bar{B}} \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \tag{238}$$

Using (9), (233) and $\mathrm{Tr}(\mathrm{A}\otimes\mathrm{B}) = \mathrm{Tr}(\mathrm{A})\,\mathrm{Tr}(\mathrm{B})$ we can infer that the partition function $Z_0(t)$ can be factorized as:

$$\begin{aligned}
Z_0(t) &= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0(t)}}\right). &(239)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right) \text{ (by (4))}, &(240)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by (9))} &(241)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)} \otimes \mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (because } \bar{S} \text{ and } \bar{B} \text{ are disjoint Hilbert spaces)} &(242)\\
&= \mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right) \text{ (by Tr}(A\otimes B)\text{=Tr}(A)\text{Tr}(B)), &(243)\\
&= Z_{\bar{S}}(t)\,Z_{\bar{B}} \text{ (by (237) and (238)))}. &(244)
\end{aligned}$$

For an operator $J(t)$ that can be factorized as $J(t) = S(t) \otimes B(t)$ with $S(t) \in \mathrm{gen}\left(\overline{H_{\bar{S}}}(t)\right)$ and $B(t) \in \mathrm{gen}\left(\overline{H_{\bar{B}}}\right)$, being $\mathrm{gen}\,(\mathrm{A})$ the vectorial space generated by the eigenvectors of the operator A, we calculate it's expected value respect to $\overline{H_0}(t)$ using a simple way as follows:

$$\langle J(t)\rangle_{\overline{H_0(t)}} = \frac{\mathrm{Tr}\left(J(t)\,\mathrm{e}^{-\beta\overline{H_0(t)}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_0(t)}}\right)} \text{ (by (5))} \tag{245}$$

$$= \frac{\mathrm{Tr}\left((S(t)\otimes B(t))\left(\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\otimes \mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by } J(t)=S(t)\otimes B(t) \text{ and } \mathrm{e}^{-\beta\overline{H_0(t)}} = \mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\otimes \mathrm{e}^{-\beta\overline{H_{\bar B}}}) \tag{246}$$

$$= \frac{\mathrm{Tr}\left(\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\right)\otimes\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (rearranging and factorizing)} \tag{247}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\right)\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\right)\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \text{ (by Tr}(A\otimes B)\text{=Tr}(A)\text{Tr}(B)) \tag{248}$$

$$= \frac{\mathrm{Tr}\left(S(t)\,\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar S}(t)}}\right)}\frac{\mathrm{Tr}\left(B(t)\,\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar B}}}\right)} \tag{249}$$

$$= \langle S(t)\rangle_{\overline{H_{\bar S}(t)}}\langle B(t)\rangle_{\overline{H_{\bar B}}} \text{ (by (5)).} \tag{250}$$

The factorization of $\left\langle \overline{H_I}^2(t)\right\rangle_{\overline{H_0(t)}}$ in terms of expected values of elements from $\mathrm{gen}\left(\overline{H_{\bar S}}(t)\right)$ and $\mathrm{gen}\left(\overline{H_{\bar B}}\right)$ is:

$$\left\langle \overline{H_I}^2(t)\right\rangle_{\overline{H_0(t)}} = \sum_i \langle |i\rangle\langle i|\rangle_{\overline{H_{\bar S}(t)}}\langle B_{iz}^2(t)\rangle_{\overline{H_{\bar B}}} + V_{10}^{\Re}(t)\sum_i\left(\langle |i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}(t)}}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \langle |i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}(t)}}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}}\right) \tag{251}$$

$$+V_{10}^{\Im}(t)\sum_i\left(\langle |i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar S}(t)}}\langle B_{iz}(t)B_y(t)\rangle_{\overline{H_{\bar B}}} - \langle |i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar S}(t)}}\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar B}}}\right) + V_{10}^{\Re}(t)\sum_i\left(\langle \sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}(t)}}\right. \tag{252}$$

$$\times\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} + \left.\langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}(t)}}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\right) + \left(V_{10}^{\Re}(t)\right)^2\left(\langle B_x^2(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}(t)}}\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar B}}}\right. \tag{253}$$

$$\left.-\mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}(t)}}\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar B}}} + \left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar B}}}\right) + V_{10}^{\Im}(t)\sum_i\left(\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar S}(t)}}\langle B_y(t)B_{iz}(t)\rangle_{\overline{H_{\bar B}}} - \langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar S}(t)}}\right. \tag{254}$$

$$\left.\times\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}\right) + \left(V_{10}^{\Im}(t)\right)^2\left(\langle B_y^2(t)\rangle_{\overline{H_{\bar B}}} + \left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar B}}} - \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}(t)}}\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}} + \mathrm{i}\,\langle\sigma_z\rangle_{\overline{H_{\bar S}(t)}}\right. \tag{255}$$

$$\left.\times\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}\right). \tag{256}$$

In order to obtain the expected values of $\left\langle \overline{H_I}^2(t)\right\rangle_{\overline{H_0(t)}}$ respect to the part related to the bath we need to calculate the following expected values that appear in the equation (231) and can be obtained using the factorization of (251). The expected values relevant for calculations are $\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_{iz}(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\langle B_y(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar B}}}$, $\left\langle B_x^2(t)\right\rangle_{\overline{H_{\bar B}}}$, $\left\langle B_y^2(t)\right\rangle_{\overline{H_{\bar B}}}$, $\langle B_x(t)\,B_y(t)\rangle_{\overline{H_{\bar B}}}$ and $\langle B_y(t)\,B_x(t)\rangle_{\overline{H_{\bar B}}}$. Recalling the form of the hamiltonian $\overline{H_{\bar B}} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$ we can extend the result (244), introducing the notation:

$$A_1\otimes\cdots\otimes A_n \equiv \bigotimes_k A_k, \tag{257}$$

$$Z_{\mathbf{k}} \equiv \mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) \tag{258}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{-1} \tag{259}$$

$$= f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{260}$$

with the creation $b_{\mathbf{k}}$ and annihilation $b_{\mathbf{k}}^\dagger$ operators defned in terms of their actions as:

$$b_{\mathbf{k}}\mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{261}$$

$$b_{\mathbf{k}}^\dagger\mid j_{\mathbf{k}}\rangle \equiv \sqrt{j_{\mathbf{k}}+1}\mid j_{\mathbf{k}}+1\rangle. \tag{262}$$

being $|j_{\mathbf{k}}\rangle$ an eigenstate of $H_{\mathbf{k}} \equiv \omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}$. With this notation we can write the partition function as:

$$Z_{\bar{B}} = \mathrm{Tr}\left(\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right), \tag{263}$$

$$\mathrm{e}^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} = \bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}, \tag{264}$$

$$Z_{\bar{B}} = \mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right) \text{ (by (264))} \tag{265}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right) \text{ (by } \mathrm{Tr}\left(A\otimes B\right)=\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right)) \tag{266}$$

$$= \prod_{\mathbf{k}} Z_{\mathbf{k}} \text{ (by (264))}. \tag{267}$$

For a function $f\left(t\right)$ which can be factorized as:

$$f\left(t\right) \equiv \prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right). \tag{268}$$

with $f_{\mathbf{k}}\left(t\right) \in \mathrm{gen}\left(\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)$, it's expected value is given by:

$$\left\langle f\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(f\left(t\right)\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\overline{H_{\bar{B}}}}\right)} \tag{269}$$

$$= \frac{\mathrm{Tr}\left(\prod_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \text{ (by (264) and (268))} \tag{270}$$

$$= \frac{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}} f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\bigotimes_{\mathbf{k}}\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{271}$$

$$= \frac{\prod_{\mathbf{k}}\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\prod_{\mathbf{k}}\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{272}$$

$$= \prod_{\mathbf{k}}\frac{\mathrm{Tr}\left(f_{\mathbf{k}}\left(t\right)\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}\right)} \tag{273}$$

$$= \prod_{\mathbf{k}}\left\langle f_{\mathbf{k}}\left(t\right)\right\rangle_{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}}. \tag{274}$$

It means that for an operator that can be factorized in terms of functions generated by $\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}$ for each $\mathbf{k}$ we only require to calculate the expected value respect to the Hilbert space where the operator belongs. This process lead us to the following explicit forms of the expected values relevant for our calculations:

$$\left\langle B_{iz}^{2}\left(t\right)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\left(t\right)\right)^{*}b_{\mathbf{k}}\right)\right)^{2}\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (214))}, \tag{275}$$

$$= \left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)^{2}+\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(\left(g_{i\mathbf{k}'}\right.\right.\right. \tag{276}$$

$$\left.\left.\left.-v_{i\mathbf{k}'}(t)\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\left(t\right)\right)^{*}b_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by square expansion properties)}, \tag{277}$$

$$= \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}\neq\mathbf{k}'} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{278}$$

$$\times \left\langle \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) b_{\mathbf{k}'}^{\dagger} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^{*} b_{\mathbf{k}'} \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (274))}, \tag{279}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{280}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{281}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} |j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (262))}, \tag{282}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \mathrm{Tr}\left(|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{283}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+1)} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by trace properties)}, \tag{284}$$

$$= 0, \tag{285}$$

$$\left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{286}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{287}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} |j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (261))}, \tag{288}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \mathrm{Tr}\left(|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{289}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}})} \cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by trace properties)}, \tag{290}$$

$$= 0, \tag{291}$$

$$\left\langle B_{iz}^{2}(t) \right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}\neq\mathbf{k}'} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left\langle b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{292}$$

$$\times \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) \left\langle b_{\mathbf{k}'}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^{*} \left\langle b_{\mathbf{k}'} \right\rangle_{\overline{H_{\bar{B}}}} \right) \tag{293}$$

$$= \sum_{\mathbf{k}} \left\langle \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}\neq\mathbf{k}'} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \cdot 0 + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \cdot 0 \right) \left(\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right) \cdot 0 \right. \tag{294}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)\right)^{*} \cdot 0 \right) \text{ (by (280) and (286))} \tag{295}$$

$$= \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{296}$$

$$= \sum_{\mathbf{k}} \left\langle \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{2} \left(b_{\mathbf{k}}^{\dagger}\right)^{2} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(\left(g_{i\mathbf{k}} \right.\right. \tag{297}$$

$$\left.\left. - v_{i\mathbf{k}}(t)\right)^{*}\right)^{2} b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{298}$$

$$= \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{2} \left\langle \left(b_{\mathbf{k}}^{\dagger}\right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left|g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right|^{2} \left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} + \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right)^{2} \left\langle b_{\mathbf{k}}^{2} \right\rangle_{\overline{H_{\bar{B}}}}, \tag{299}$$

$$\left\langle \left(b_{\mathbf{k}}^{\dagger}\right)^{2} \right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger}\right)^{2} \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{300}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger}\right)^2 |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{301}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)}|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (262) applied twice)} \tag{302}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)}\mathrm{Tr}\left(|j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{303}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{\left(j_{\mathbf{k}}+2\right)\left(j_{\mathbf{k}}+1\right)}\cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{304}$$

$$= 0, \tag{305}$$

$$\left\langle b_{\mathbf{k}}^2\right\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} b_{\mathbf{k}}^2 |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{306}$$

$$= \frac{\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)}|j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by (261) applied twice)} \tag{307}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)}\mathrm{Tr}\left(|j_{\mathbf{k}}-2\rangle\langle j_{\mathbf{k}}|\right)}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{308}$$

$$= \frac{\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}}\left(j_{\mathbf{k}}-1\right)}\cdot 0}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \text{ (by properties of the trace)} \tag{309}$$

$$= 0, \tag{310}$$

$$\left\langle b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} + b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\right)\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{311}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{312}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \text{ (now (261) and (262))} \tag{313}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}\sqrt{j_{\mathbf{k}}}|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}\sqrt{j_{\mathbf{k}}+1}|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{314}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\sqrt{j_{\mathbf{k}}}\sqrt{j_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}b_{\mathbf{k}}\sqrt{j_{\mathbf{k}}+1}\sqrt{j_{\mathbf{k}}+1}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{315}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}j_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| + \sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(j_{\mathbf{k}}+1\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{316}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right)|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{317}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right)\mathrm{Tr}\left(|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \tag{318}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\sum_{j_{\mathbf{k}}} \mathrm{e}^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right) \text{ (by properties of trace)} \tag{319}$$

$$= \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\sum_{j_{\mathbf{k}}=0}^{\infty} \left(\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)^{j_{\mathbf{k}}}\left(2j_{\mathbf{k}}+1\right), \tag{320}$$

$$\sum_{j_{\mathbf{k}}=0}^{\infty} x^{j_{\mathbf{k}}}(2j_{\mathbf{k}}+1) = \frac{1+x}{(1-x)^2}, \tag{321}$$

$$\left\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right\rangle_{\overline{H_{\bar{B}}}} = \left(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) \frac{\mathrm{e}^{-\beta\omega_{\mathbf{k}}}+1}{(1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}})^2} \text{ (setting } x=\mathrm{e}^{-\beta\omega_{\mathbf{k}}} \text{ in (321) and by (311))}, \tag{322}$$

$$= \frac{1+\mathrm{e}^{-\beta\omega_{\mathbf{k}}}}{1-\mathrm{e}^{-\beta\omega_{\mathbf{k}}}} \tag{323}$$

$$= \frac{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}} \frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}+\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}}{\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}} \frac{\mathrm{e}^{\frac{\beta\omega_{\mathbf{k}}}{2}}-\mathrm{e}^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}{2}} \tag{324}$$

$$= \frac{\cosh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\sinh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{325}$$

$$= \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \tag{326}$$

$$\left\langle B_{iz}^2(t)\right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \text{ (by (300), (306) and (326))}, \tag{327}$$

$$\left\langle B_{iz}(t)\, B_x(t)\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}} \Big((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}}\Big) \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_{\bar{B}}}} \tag{328}$$

$$= \frac{1}{2}\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}}\right) \left(\mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + \mathrm{e}^{\chi_{01}(t)} \right.\right. \tag{329}$$

$$\left.\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t)+B_{01}(t)}{2} \left\langle \sum_{\mathbf{k}} \Big((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}}\Big)\right\rangle_{\overline{H_{\bar{B}}}} \tag{330}$$

$$= \frac{1}{2}\left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \, b_{\mathbf{k}}\right) \left(\mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + \mathrm{e}^{\chi_{01}(t)} \right.\right. \tag{331}$$

$$\left.\left. \times \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \text{ (by (280) and (286))}, \tag{332}$$

$$\left\langle F(h)\right\rangle_{\overline{H_{\bar{B}}}} \equiv \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle \alpha\right| F(h) \left|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (using the coherent representation with } N=\left(\mathrm{e}^{\beta\omega}-1\right)^{-1}\text{)}, \tag{333}$$

$$D(\alpha_{\mathbf{k}}) \equiv \mathrm{e}^{\left(\frac{\alpha_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{\alpha_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}}\right)} \text{ (displacement operator definition)}, \tag{334}$$

$$|\alpha\rangle \equiv D(\alpha)\,|0\rangle \text{ (displacement operator properties)}, \tag{335}$$

$$\langle\alpha| \equiv \langle 0| D(-\alpha), \tag{336}$$

$$D(-\alpha)D(h)D(\alpha) \equiv D(h)\, \mathrm{e}^{h\alpha^* - h^*\alpha} \text{ (displacement operator properties)}, \tag{337}$$

$$D(0) \equiv \mathbb{I} \text{ (identity written in terms of the displacement operator)}, \tag{338}$$

$$D(-\alpha)D(0)D(\alpha) = D(0)\, \mathrm{e}^{0\cdot\alpha^* - 0^*\cdot\alpha} \tag{339}$$

$$= D(0) \tag{340}$$

$$= \mathbb{I}, \tag{341}$$

$$D(-\alpha)\, b^{\dagger} D(\alpha) = b^{\dagger} + \alpha^*, \tag{342}$$

$$D(-\alpha)\, b D(\alpha) = b + \alpha, \tag{343}$$

$$\left\langle D(h)\right\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}, \tag{344}$$

$$\left\langle b^{\dagger} D(h)\right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle \alpha \left| b^{\dagger} D(h) \right| \alpha \right\rangle \mathrm{d}^2\alpha \text{ (definition of expected value)} \tag{345}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| D(-\alpha)\, b^{\dagger} D(h)\, D(\alpha) \right| 0 \right\rangle \mathrm{d}^2\alpha \text{ (by (335) and (336))} \tag{346}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| D(-\alpha)\, b^{\dagger} \mathbb{I} D(h)\, D(\alpha) \right| 0 \right\rangle \mathrm{d}^2\alpha \text{ (inserting identity operator)} \tag{347}$$

$$= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| \Big(D(-\alpha)\, b^{\dagger} D(\alpha)\Big) \left(D(-\alpha)\, D(h)\, D(\alpha)\right) \right| 0 \right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} \tag{348}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| \left(b^\dagger + \alpha^*\right) D(h) e^{h\alpha^* - h^*\alpha} \right| 0 \right\rangle d^2\alpha \text{ (by (342) and (337))} \tag{349}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| b^\dagger D(h) e^{h\alpha^* - h^*\alpha} \right| 0 \right\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| \alpha^* D(h) e^{h\alpha^* - h^*\alpha} \right| 0 \right\rangle d^2\alpha \tag{350}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} 0 D(h) e^{h\alpha^* - h^*\alpha} |0\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| \alpha^* D(h) e^{h\alpha^* - h^*\alpha} \right| 0 \right\rangle d^2\alpha \tag{351}$$

$$= \frac{1}{\pi N}\int 0 d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 |D(h)| 0\rangle d^2\alpha \tag{352}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 |D(h)| 0\rangle d^2\alpha \tag{353}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha^* \langle 0 | h\rangle d^2\alpha \text{ (by (335))} \tag{354}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha^* e^{-\frac{|h|^2}{2}} d^2\alpha \text{ (because } \langle 0|h\rangle = e^{-\frac{|h|^2}{2}}\text{)}, \tag{355}$$

$$x = \alpha^{\Re} \in \mathbb{R}, \tag{356}$$

$$y = \alpha^{\Im} \in \mathbb{R}, \tag{357}$$

$$\alpha = x + iy, \tag{358}$$

$$\left\langle b^\dagger D(h) \right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha^* e^{-\frac{|h|^2}{2}} d^2\alpha \tag{359}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)/N} e^{h(x-iy) - h^*(x+iy)} (x - iy)\, dx dy \text{ (by (356) and (357))} \tag{360}$$

$$= -h^* e^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)} N \tag{361}$$

$$= -h^* \langle D(h)\rangle_{\overline{H_{\bar{B}}}} N, \tag{362}$$

$$|h\rangle = e^{-\frac{|h|^2}{2}} \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle, \tag{363}$$

$$\langle b D(h)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \langle \alpha |b D(h)| \alpha\rangle d^2\alpha \text{ (definition of expected value)} \tag{364}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \langle 0 |D(-\alpha) b \mathbb{I} D(h) D(\alpha)| 0\rangle d^2\alpha \text{ (by (335) and (336))} \tag{365}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \langle 0 |(D(-\alpha) b D(\alpha)) (D(-\alpha) D(h) D(\alpha))| 0\rangle d^2\alpha \text{ (by associative property)} \tag{366}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| (b + \alpha) D(h) e^{h\alpha^* - h^*\alpha} \right| 0 \right\rangle d^2\alpha \text{ (by (343) and (337))} \tag{367}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| b D(h) e^{h\alpha^* - h^*\alpha} \right| 0 \right\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} \left\langle 0 \left| \alpha D(h) e^{h\alpha^* - h^*\alpha} \right| 0 \right\rangle d^2\alpha \tag{368}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \langle 0 |b| h\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha \langle 0|h\rangle d^2\alpha \tag{369}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \langle 0| b e^{-\frac{|h|^2}{2}} \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha \langle 0|h\rangle d^2\alpha \text{ (by (363))} \tag{370}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \langle 0| e^{-\frac{|h|^2}{2}} \sum_{n=1}^{\infty} \frac{h^n}{\sqrt{n!}} \sqrt{n} |n-1\rangle d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha \langle 0|h\rangle d^2\alpha \text{ (by (261))} \tag{371}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} e^{-\frac{|h|^2}{2}} \sum_{n=1}^{\infty} \frac{h^n}{\sqrt{n!}} \sqrt{n} \delta_{0,n-1} d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha \langle 0|h\rangle d^2\alpha \text{ (by } \langle n|n'\rangle = \delta_{nn'}\text{)} \tag{372}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} e^{-\frac{|h|^2}{2}} \frac{h^1}{\sqrt{1!}} \sqrt{1} d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha \langle 0|h\rangle d^2\alpha \tag{373}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} e^{-\frac{|h|^2}{2}} h d^2\alpha + \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} \alpha e^{-\frac{|h|^2}{2}} d^2\alpha \text{ (because } \langle 0|h\rangle = e^{-\frac{|h|^2}{2}}\text{)} \tag{374}$$

$$= \frac{1}{\pi N}\int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^*\alpha} e^{-\frac{|h|^2}{2}} (\alpha + h) d^2\alpha \tag{375}$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)/N} e^{h(x-iy) - h^*(x+iy)} (x + iy + h)\, dx dy \tag{376}$$

$$
\begin{aligned}
&= h\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)}(N+1) && (377)\\
&= h\left\langle D(h)\right\rangle_{\overline{H_{\bar{B}}}}(N+1)\,, && (378)\\
\left\langle D(h)\,b\right\rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|D(h)\,b\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (definition of expected value)} && (379)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,\mathbb{I}D(h)\,bD(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (335) and (336))} && (380)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(D(-\alpha)\,D(h)\,D(\alpha)\right)\left(D(-\alpha)\,bD(\alpha)\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} && (381)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}(b+\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (343) and (337))} && (382)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}b\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\right|0\right\rangle \mathrm{d}^2\alpha && (383)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\left\langle 0|h\right\rangle \mathrm{d}^2\alpha && (384)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\right|D(h)\,0\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (261))} && (385)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha && (386)\\
&= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}(x+\mathrm{i}y)\,\mathrm{d}x\mathrm{d}y && (387)\\
&= hN\mathrm{e}^{-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)} && (388)\\
&= hN\left\langle D(h)\right\rangle_B\,, && (389)\\
\left\langle D(h)\,b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle \alpha\left|D(h)\,b^\dagger\right|\alpha\right\rangle \mathrm{d}^2\alpha \text{ (definition of expected value)} && (390)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(-\alpha)\,D(h)\,\mathbb{I}b^\dagger D(\alpha)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (335) and (336))} && (391)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\left(D(-\alpha)\,D(h)\,D(\alpha)\right)\left(D(-\alpha)\,b^\dagger D(\alpha)\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by associative property)} && (392)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \mathrm{d}^2\alpha \text{ (by (343) and (337))} && (393)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\left\langle 0\left|\alpha^* D(h)\,\mathrm{e}^{h\alpha^*-h^*\alpha}\right|0\right\rangle \mathrm{d}^2\alpha && (394)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha && (395)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle 0\left|D(h)\,b^\dagger\right|0\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha && (396)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h\left|\sqrt{0+1}\right|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\left\langle 0|h\right\rangle \mathrm{d}^2\alpha && (397)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\left\langle -h\left|\sqrt{0+1}\right|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (336))}\,, && (398)\\
\left\langle h\right| &= \mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(h^*)^n}{\sqrt{n!}}\left\langle n\right|\,, && (399)\\
\left\langle D(h)\,b^\dagger\right\rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\sum_{n=0}^{\infty}\frac{(-h^*)^n}{\sqrt{n!}}\left\langle n|1\right\rangle \mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by (399))} && (400)\\
&= \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\frac{(-h^*)^1}{\sqrt{1!}}\mathrm{d}^2\alpha + \frac{1}{\pi N}\int \mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\alpha^*\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha \text{ (by } \left\langle n|n'\right\rangle=\delta_{nn'}) && (401)\\
&= \frac{1}{\pi N}\int (\alpha^*-h^*)\,\mathrm{e}^{-\frac{|\alpha|^2}{N}}\mathrm{e}^{h\alpha^*-h^*\alpha}\mathrm{e}^{-\frac{|h|^2}{2}}\mathrm{d}^2\alpha && (402)\\
&= \frac{\mathrm{e}^{-\frac{|h|^2}{2}}}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(x^2+y^2\right)/N}\mathrm{e}^{h(x-\mathrm{i}y)-h^*(x+\mathrm{i}y)}(x-\mathrm{i}y-h^*)\,\mathrm{d}x\mathrm{d}y && (403)\\
&= -h^*\left\langle D(h)\right\rangle_B(N+1)\,, && (404)
\end{aligned}
$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{2}\left\langle \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right.\right. \quad (405)$$

$$\left.\left.+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad \text{(replacing the definitions in (214))} \quad (406)$$

$$= \frac{1}{2}\left\langle \mathrm{e}^{\chi_{10}(t)}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)+\mathrm{e}^{\chi_{01}(t)}\right. \quad (407)$$

$$\left.\times\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \quad (408)$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\langle b_{\mathbf{k}}\right. \quad (409)$$

$$\left.\times\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\Bigg)+\frac{\mathrm{e}^{\chi_{01}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right.\right.\right. \quad (410)$$

$$\left.\left.\left.-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}\right), \quad (411)$$

$$\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad \text{(by (344))}, \quad (412)$$

$$N_{\mathbf{k}} = \left(\mathrm{e}^{\beta\omega_{\mathbf{k}}}-1\right)^{-1}, \quad (413)$$

$$\langle b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} = \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (378))}, \quad (414)$$

$$\left\langle b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right\rangle_{\overline{H_{\bar{B}}}} = -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (362))}, \quad (415)$$

$$\prod_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} = \mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad \text{(by (412) and (274))}, \quad (416)$$

$$\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} = \left\langle b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (274))} \quad (417)$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D(\alpha_{\mathbf{k}'})\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (274))} \quad (418)$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\prod_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \quad (419)$$

$$= -\alpha_{\mathbf{k}}^{*}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad \text{(by (412))}, \quad (420)$$

$$\left\langle b_{\mathbf{k}}\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} = \langle b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D(\alpha_{\mathbf{k}'})\right\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (274))} \quad (421)$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\langle D(\alpha_{\mathbf{k}'})\rangle_{\overline{H_{\bar{B}}}} \quad \text{(by (414))} \quad (422)$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\prod_{\mathbf{k}}\langle D(\alpha_{\mathbf{k}})\rangle_{\overline{H_{\bar{B}}}} \quad (423)$$

$$= \alpha_{\mathbf{k}}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad \text{(by (412))}, \quad (424)$$

$$\langle B_{iz}(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \frac{\mathrm{e}^{\chi_{10}(t)}}{2}\left(\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left\langle b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}}+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\langle b_{\mathbf{k}}\right. \quad (425)$$

$$\times \prod_{\mathbf{k}'} \left(D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \Bigg\rangle_{\overline{H_{\bar{B}}}} \Bigg) + \frac{\mathrm{e}^{\chi_{01}(t)}}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \right. \tag{426}$$

$$\left. \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} \left(D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}} \right) \text{ (by (214))}, \tag{427}$$

$$= \frac{\mathrm{e}^{\chi_{10}(t)}}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \tag{428}$$

$$\left. \times \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \frac{\mathrm{e}^{\chi_{01}(t)}}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-N_{\mathbf{k}} \right. \right. \tag{429}$$

$$\left. \times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \tag{430}$$

$$\left. \times \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) \tag{431}$$

$$= \frac{1}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \tag{432}$$

$$\left. \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \frac{1}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-N_{\mathbf{k}} \right. \right. \tag{433}$$

$$\left. \times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \mathrm{e}^{\chi_{01}(t)} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \tag{434}$$

$$\left. \times (N_{\mathbf{k}} + 1) \mathrm{e}^{\chi_{01}(t)} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right) \tag{435}$$

$$= \frac{1}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} B_{10}(t) \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right. \tag{436}$$

$$\left. \times (N_{\mathbf{k}} + 1) B_{10}(t) \right) + \frac{1}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} B_{01}(t) \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \tag{437}$$

$$\left. \times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) \right) \text{ (by (214) using the defintion } B_{ij}(t)\text{)} \tag{438}$$

$$= \frac{B_{10}(t)}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \tag{439}$$

$$+ \frac{B_{01}(t)}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \tag{440}$$

$$= \frac{B_{10}(t)}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \tag{441}$$

$$- \frac{B_{01}(t)}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \tag{442}$$

$$= \frac{B_{10}(t) - B_{01}(t)}{2} \left(\sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \tag{443}$$

$$\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_{\bar{B}}}} = \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))\, b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \right\rangle_{\overline{H_{\bar{B}}}} \tag{444}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t)\right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} + \frac{\left(B_{10}(t) - B_{01}(t)\right)}{2\mathrm{i}} \tag{445}$$

$$\times \left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \text{(by the properties of expected value)}, \tag{446}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t)\right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} + \frac{\left(B_{10}(t) - B_{01}(t)\right)}{2\mathrm{i}} \cdot 0 \tag{447}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \left(B_{0}^{+}(t) B_{1}^{-}(t) - B_{1}^{+}(t) B_{0}^{-}(t)\right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \tag{448}$$

$$= \frac{\left\langle \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \left(\mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_{\overline{H_{\bar{B}}}}}{2\mathrm{i}} \tag{449}$$

$$= \frac{1}{2\mathrm{i}} \left\langle \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right. \tag{450}$$

$$\left. + \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \mathrm{e}^{\chi_{01}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \mathrm{e}^{\chi_{10}(t)} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{451}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \mathrm{e}^{\chi_{01}(t)} \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} - \mathrm{e}^{\chi_{10}(t)} \left\langle b_{\mathbf{k}}^{\dagger} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \right. \tag{452}$$

$$\times \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \mathrm{e}^{\chi_{01}(t)} \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} - \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{453}$$

$$\left. \times \mathrm{e}^{\chi_{10}(t)} \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right) \tag{454}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \mathrm{e}^{\chi_{01}(t)} \left(- \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \right) - \mathrm{e}^{\chi_{10}(t)} \right. \tag{455}$$

$$\times \left(- \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \right) \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \mathrm{e}^{\chi_{01}(t)} \tag{456}$$

$$\times \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \tag{457}$$

$$\left. \times \mathrm{e}^{\chi_{10}(t)} \mathrm{e}^{-\sum_{\mathbf{k}} \frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^{2}}{2} \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)} \right) \tag{458}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left(- \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{01}(t) \right) - \left(- \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{10}(t) \right) \right. \tag{459}$$

$$\times \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \tag{460}$$

$$\left. \times \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \right) \tag{461}$$

$$= \frac{1}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{01}(t) + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} B_{10}(t) \right. \tag{462}$$

$$\left. - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{01}(t) - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) B_{10}(t) \right) \tag{463}$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^{*} N_{\mathbf{k}} - \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right) \tag{464}$$

$$\langle B_{x}(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} = \left\langle \frac{B_{1}^{+}(t) B_{0}^{-}(t) + B_{0}^{+}(t) B_{1}^{-}(t) - B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right) b_{\mathbf{k}}^{\dagger} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)\right)^{*} b_{\mathbf{k}} \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{465}$$

$$=\left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}-\frac{B_{10}(t)+B_{01}(t)}{2} \tag{466}$$

$$\times\left\langle \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{467}$$

$$=\left\langle \frac{B_1^+(t)B_0^-(t)+B_0^+(t)B_1^-(t)}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}-\frac{B_{10}(t)+B_{01}(t)}{2}\cdot 0 \tag{468}$$

$$=\frac{1}{2}\left\langle\left(B_1^+(t)\,B_0^-(t)+B_0^+(t)\,B_1^-(t)\right)\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{469}$$

$$=\frac{1}{2}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)+\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}\right.\right. \tag{470}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{471}$$

$$\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}=\alpha_{\mathbf{k}}N_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{472}$$

$$\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}=-\alpha_{\mathbf{k}}^{*}\left(N_{\mathbf{k}}+1\right)\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}, \tag{473}$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{474}$$

$$=-\alpha_{\mathbf{k}}^{*}\left(N_{\mathbf{k}}+1\right)\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{475}$$

$$=-\alpha_{\mathbf{k}}^{*}\left(N_{\mathbf{k}}+1\right)\prod_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{476}$$

$$=-\alpha_{\mathbf{k}}^{*}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{477}$$

$$\left\langle\left(\prod_{\mathbf{k}'}D(\alpha_{\mathbf{k}'})\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}=\left\langle D\left(\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\left\langle\prod_{\mathbf{k}'\neq\mathbf{k}}D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{478}$$

$$=\alpha_{\mathbf{k}}N_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}\prod_{\mathbf{k}'\neq\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}'}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{479}$$

$$=\alpha_{\mathbf{k}}N_{\mathbf{k}}\prod_{\mathbf{k}}\left\langle D\left(\alpha_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{480}$$

$$=\alpha_{\mathbf{k}}N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}, \tag{481}$$

$$\langle B_x(t)B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\right. \tag{482}$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{483}$$

$$\left.\times\mathrm{e}^{\chi_{10}(t)}+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{484}$$

$$=\frac{1}{2}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\mathrm{e}^{\chi_{10}(t)}\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right. \tag{485}$$

$$+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{10}(t)} \tag{486}$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*} \tag{487}$$

$$\times\left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right) \tag{488}$$

$$=\frac{1}{2}\sum_{\mathbf{k}}\left(-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)B_{10}(t)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\right. \tag{489}$$

$$\times(N_{\mathbf{k}}+1)B_{01}(t)+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{10}(t)+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \tag{490}$$

$$\left.\times N_{\mathbf{k}}B_{01}(t)\right) \tag{491}$$

$$=\frac{B_{10}(t)-B_{01}(t)}{2}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}(N_{\mathbf{k}}+1)\right) \tag{492}$$

$$\langle B_{y}(t)\,B_{iz}(t)\rangle_{\overline{H_{\bar{B}}}}=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)+B_{10}(t)-B_{01}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{493}$$

$$=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}}+\frac{B_{10}(t)-B_{01}(t)}{2\mathrm{i}} \tag{494}$$

$$\times\left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{495}$$

$$=\left\langle\frac{B_{0}^{+}(t)\,B_{1}^{-}(t)-B_{1}^{+}(t)\,B_{0}^{-}(t)}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{496}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left\langle\left(\mathrm{e}^{\chi_{01}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)-\mathrm{e}^{\chi_{10}(t)}\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,b_{\mathbf{k}}^{\dagger}\right.\right. \tag{497}$$

$$\left.\left.+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}b_{\mathbf{k}}\right)\right\rangle_{\overline{H_{\bar{B}}}} \tag{498}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,\mathrm{e}^{\chi_{01}(t)}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\right. \tag{499}$$

$$\times\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}\right\rangle_{\overline{H_{\bar{B}}}}+(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{500}$$

$$\left.\times\mathrm{e}^{\chi_{01}(t)}-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left\langle\left(\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{501}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))\,\mathrm{e}^{\chi_{01}(t)}\left(-\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right. \tag{502}$$

$$-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(-\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}(N_{\mathbf{k}}+1)\,\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)+\mathrm{e}^{\chi_{01}(t)} \tag{503}$$

$$\times\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*}\left(\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)-\mathrm{e}^{\chi_{10}(t)}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)^{*} \tag{504}$$

$$\left.\times\left(\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}\mathrm{e}^{-\sum_{\mathbf{k}}\frac{\left|\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right|^{2}}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}\right)\right) \tag{505}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\left(N_{\mathbf{k}}+1\right)B_{01}(t)+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t)\right)\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)^{*}\right. \tag{506}$$

$$\times(N_{\mathbf{k}}+1)B_{10}(t)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right)N_{\mathbf{k}}B_{01}(t)-(g_{i\mathbf{k}}-v_{i\mathbf{k}}(t))^{*}\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \tag{507}$$

$$\left.\times N_{\mathbf{k}}B_{10}(t)\right) \tag{508}$$

$$
\begin{aligned}
&= \frac{B_{01}(t)+B_{10}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}}+1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} \right) && (509)\\
\mathrm{Var}_{\overline{H_{\bar{B}}}}(A) &\equiv \left\langle A^2 \right\rangle_{\overline{H_{\bar{B}}}} - \langle A \rangle^2_{\overline{H_{\bar{B}}}}, && (510)\\
\mathrm{Var}(aX+b) &= a^2 \mathrm{Var}(X), && (511)\\
\langle B_x(t) \rangle_{\overline{H_{\bar{B}}}} &= 0, && (512)\\
\langle B_y(t) \rangle_{\overline{H_{\bar{B}}}} &= 0, && (513)\\
\left\langle B_x^2(t) \right\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Var}_{\overline{H_{\bar{B}}}}(B_x(t)) + \langle B_x(t) \rangle^2_{\overline{H_{\bar{B}}}} && (514)\\
&= \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(\frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) && (515)\\
&= \frac{1}{4} \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t) \right) && (516)\\
&= \frac{1}{4} \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) \right) && (517)\\
&= \frac{1}{4} \left(\left\langle \left(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t) + B_{01}(t))^2 \right) && (518)\\
&= \frac{1}{4} \left(\left\langle \left(B_1^+(t)B_0^-(t) \right)^2 + B_1^+(t)B_0^-(t)B_0^+(t)B_1^-(t) + B_0^+(t)B_1^-(t)B_1^+(t)B_0^-(t) + \left(B_0^+(t)B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} \right. && (519)\\
&\left. - (B_{10}(t) + B_{01}(t))^2 \right) && (520)\\
&= \frac{1}{4} \left(\left\langle \left(B_1^+(t)B_0^-(t) \right)^2 + 2\mathbb{I} + \left(B_0^+(t)B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t) + B_{01}(t))^2 \right), && (521)\\
(D(h))^2 &= D(h)D(h) && (522)\\
&= D(h+h)\, \mathrm{e}^{\frac{1}{2}\left(\frac{h^*h - hh^*}{\omega^2} \right)} && (523)\\
&= D(2h), && (524)\\
\left\langle \left(B_i^+(t)B_j^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} &= \left\langle \left(\prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \mathrm{e}^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} && (525)\\
&= \left\langle \prod_{\mathbf{k}} D\left(2\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} \right\rangle_{\overline{H_{\bar{B}}}} && (526)\\
&= \prod_{\mathbf{k}} \mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} \mathrm{e}^{-2\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} && (527)\\
&= \left(\prod_{\mathbf{k}} \mathrm{e}^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}} \mathrm{e}^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right)^2 \left(\prod_{\mathbf{k}} \mathrm{e}^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right)} \right)^2 && (528)\\
&= B_{ij}^2(t) \left| B_{ij}(t) \right|^2, && (529)\\
\left\langle B_x^2(t) \right\rangle_{\overline{H_{\bar{B}}}} &= \frac{1}{4} \left(\left\langle \left(B_1^+(t)B_0^-(t) \right)^2 + 2\mathbb{I} + \left(B_0^+(t)B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t) + B_{01}(t))^2 \right) && (530)\\
&= \frac{1}{4} \left(\left\langle \left(B_1^+(t)B_0^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + 2\langle \mathbb{I} \rangle_{\overline{H_{\bar{B}}}} + \left\langle \left(B_0^+(t)B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t) + B_{01}(t))^2 \right) && (531)\\
&= \frac{1}{4} \left(\left\langle \left(B_1^+(t)B_0^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + 2 + \left\langle \left(B_0^+(t)B_1^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} - (B_{10}(t) + B_{01}(t))^2 \right) && (532)\\
&= \frac{1}{4} \left(B_{10}^2(t) \left| B_{10}^2(t) \right| + 2 + B_{01}^2(t) \left| B_{01}^2(t) \right| - \left(B_{10}^2(t) + 2B_{10}(t)B_{01}(t) + B_{01}^2(t) \right) \right) && (533)\\
&= \frac{1}{4} \left(B_{10}^2(t) \left| B_{10}^2(t) \right| + 2 + B_{01}^2(t) \left| B_{10}^2(t) \right| - \left(B_{10}^2(t) + 2\left| B_{10}^2(t) \right| + B_{01}^2(t) \right) \right) && (534)\\
&= \frac{1}{4} \left(B_{10}^2(t) + B_{01}^2(t) - 2 \right) \left(\left| B_{10}^2(t) \right| - 1 \right) && (535)\\
\left\langle B_y^2(t) \right\rangle_{\overline{H_{\bar{B}}}} &= \mathrm{Var}_{\overline{H_{\bar{B}}}}(B_y(t)) + \langle B_y(t) \rangle^2_{\overline{H_{\bar{B}}}} && (536)\\
&= \mathrm{Var}_{\overline{H_{\bar{B}}}} \left(\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \right) && (537)
\end{aligned}
$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t) + B_{10}(t) - B_{01}(t)\right) \tag{538}$$

$$= -\frac{1}{4}\mathrm{Var}_{\overline{H_{\bar{B}}}}\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right) \tag{539}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{540}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - 2\mathbb{I} + \left(B_1^+(t)\,B_0^-(t)\right)^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}}\right) \tag{541}$$

$$= -\frac{1}{4}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - 2\left\langle\mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)^2\right) \tag{542}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\left|B_{01}(t)\right|^2 - 2 + B_{10}^2(t)\left|B_{10}(t)\right|^2 - \left(B_{01}(t) - B_{10}(t)\right)^2\right) \tag{543}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\left|B_{01}(t)\right|^2 - 2 + B_{10}^2(t)\left|B_{10}(t)\right|^2 - B_{01}^2(t) + 2B_{01}(t)\,B_{10}(t) - B_{10}^2(t)\right) \tag{544}$$

$$= -\frac{1}{4}\left(B_{01}^2(t)\left|B_{10}(t)\right|^2 - 2 + B_{10}^2(t)\left|B_{10}(t)\right|^2 - B_{01}^2(t) + 2\left|B_{10}(t)\right|^2 - B_{10}^2(t)\right) \tag{545}$$

$$= -\frac{1}{4}\left(B_{10}^2(t) + B_{01}^2(t) + 2\right)\left(\left|B_{10}(t)\right|^2 - 1\right), \tag{546}$$

$$\langle B_x(t)B_y(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_x(t)\,\frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{547}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\left\langle B_x(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{548}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}(t) - B_{01}(t)}{2\mathrm{i}}\cdot 0 \tag{549}$$

$$= \left\langle B_x(t)\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{550}$$

$$= \left\langle \frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\,\frac{B_0^+(t)\,B_1^-(t) - B_1^+(t)\,B_0^-(t)}{2\mathrm{i}}\right\rangle_{\overline{H_{\bar{B}}}} \tag{551}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle\big(B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t)\big)\big(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\big)\big\rangle_{\overline{H_{\bar{B}}}} - \big\langle\big(B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t)\big)\big\rangle_{\overline{H_{\bar{B}}}} \tag{552}$$

$$\times\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{553}$$

$$= \frac{1}{4\mathrm{i}}\Big(\big\langle B_1^+(t)\,B_0^-(t)\,B_0^+(t)B_1^-(t) - B_1^+(t)\,B_0^-(t)\,B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)\,B_0^+(t)\,B_1^-(t) - B_0^+(t)\,B_1^-(t) \tag{554}$$

$$\times B_1^+(t)B_0^-(t)\big\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\Big) \tag{555}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\mathbb{I} - \left(B_1^+(t)\,B_0^-(t)\right)^2 + \left(B_0^+(t)\,B_1^-(t)\right)^2 - \mathbb{I}\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{556}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{10}(t) + B_{01}(t)\right)\right) \tag{557}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2 - \left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}(t) - B_{10}(t)\right)\left(B_{01}(t) + B_{10}(t)\right)\right) \tag{558}$$

$$= \frac{1}{4\mathrm{i}}\left(\left\langle\left(B_0^+(t)\,B_1^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle\left(B_1^+(t)\,B_0^-(t)\right)^2\right\rangle_{\overline{H_{\bar{B}}}} - \left(B_{01}^2(t) - B_{10}^2(t)\right)\right) \tag{559}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t)\left|B_{10}(t)\right|^2 - B_{10}^2(t)\left|B_{10}(t)\right|^2 - B_{01}^2(t) + B_{10}^2(t)\right) \tag{560}$$

$$= \frac{1}{4\mathrm{i}}\left(B_{01}^2(t) - B_{10}^2(t)\right)\left(\left|B_{10}(t)\right|^2 - 1\right), \tag{561}$$

$$\langle B_y(t)B_x(t)\rangle_{\overline{H_{\bar{B}}}} = \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{562}$$

$$= \left\langle B_y(t)\,\frac{B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} - \left\langle B_y(t)\,\frac{B_{10}(t) + B_{01}(t)}{2}\right\rangle_{\overline{H_{\bar{B}}}} \tag{563}$$

$$= \frac{1}{2}\left\langle B_y(t)\left(B_1^+(t)\,B_0^-(t) + B_0^+(t)\,B_1^-(t)\right)\right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t) + B_{01}(t)}{2}\left\langle B_y(t)\right\rangle_{\overline{H_{\bar{B}}}} \tag{564}$$

$$= \frac{1}{2} \left\langle B_y(t) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} - \frac{B_{10}(t) + B_{01}(t)}{2} \cdot 0 \tag{565}$$

$$= \frac{1}{2} \left\langle B_y(t) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{566}$$

$$= \frac{1}{2} \left\langle \frac{B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{567}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) \right) \left(B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} + \frac{(B_{10}(t) - B_{01}(t))}{4\mathrm{i}} \left\langle \left(B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} \tag{568}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t) \right) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} + \frac{(B_{10}(t) - B_{01}(t))\,(B_{10}(t) + B_{01}(t))}{4\mathrm{i}} \tag{569}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t) \right) \left(B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t) \right) \right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} \tag{570}$$

$$= \frac{1}{4\mathrm{i}} \left\langle B_0^+(t)\, B_1^-(t)\, B_1^+(t)\, B_0^-(t) + B_0^+(t)\, B_1^-(t)\, B_0^+(t)\, B_1^-(t) - B_1^+(t)\, B_0^-(t)\, B_1^+(t)\, B_0^-(t) - B_1^+(t)\, B_0^-(t)\, B_0^+(t)\, B_1^-(t) \right\rangle_{\overline{H_{\bar{B}}}} \tag{571}$$

$$+ \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} \tag{572}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \mathbb{I} + \left(B_0^+(t)\, B_1^-(t) \right)^2 - \left(B_1^+(t)\, B_0^-(t) \right)^2 - \mathbb{I} \right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} \tag{573}$$

$$= \frac{1}{4\mathrm{i}} \left\langle \left(B_0^+(t)\, B_1^-(t) \right)^2 - \left(B_1^+(t)\, B_0^-(t) \right)^2 \right\rangle_{\overline{H_{\bar{B}}}} + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} \tag{574}$$

$$= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t)\, |B_{10}(t)|^2 - B_{10}^2(t)\, |B_{10}(t)|^2 \right) + \frac{B_{10}^2(t) - B_{01}^2(t)}{4\mathrm{i}} \tag{575}$$

$$= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t)\, |B_{10}(t)|^2 - B_{10}^2(t)\, |B_{10}(t)|^2 + B_{10}^2(t) - B_{01}^2(t) \right) \tag{576}$$

$$= \frac{1}{4\mathrm{i}} \left(B_{01}^2(t) - B_{10}^2(t) \right) \left(|B_{10}(t)|^2 - 1 \right). \tag{577}$$

Summarizing the expected values obtained in the precedent lines we have:

$$\left\langle B_{iz}^2(t) \right\rangle_{\overline{H_{\bar{B}}}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right), \tag{578}$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_{\bar{B}}}} = \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right), \tag{579}$$

$$\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_{\bar{B}}}} = \frac{B_{10}(t) + B_{01}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right), \tag{580}$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} = \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) \right), \tag{581}$$

$$\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_{\bar{B}}}} = \frac{B_{01}(t) + B_{10}(t)}{2\mathrm{i}} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* (N_{\mathbf{k}} + 1) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) N_{\mathbf{k}} \right), \tag{582}$$

$$\left\langle B_x^2(t) \right\rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4} \left(B_{10}^2(t) + B_{01}^2(t) - 2 \right) \left(\left| B_{10}^2(t) \right| - 1 \right), \tag{583}$$

$$\left\langle B_y^2(t) \right\rangle_{\overline{H_{\bar{B}}}} = -\frac{1}{4} \left(B_{10}^2(t) + B_{01}^2(t) + 2 \right) \left(|B_{10}(t)|^2 - 1 \right), \tag{584}$$

$$\langle B_x(t) B_y(t) \rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4\mathrm{i}} \left(B_{01}^2(t) - B_{10}^2(t) \right) \left(|B_{10}(t)|^2 - 1 \right), \tag{585}$$

$$\langle B_y(t) B_x(t) \rangle_{\overline{H_{\bar{B}}}} = \frac{1}{4\mathrm{i}} \left(B_{01}^2(t) - B_{10}^2(t) \right) \left(|B_{10}(t)|^2 - 1 \right). \tag{586}$$

The density matrix associated to $\rho_{\overline{S}} = \frac{\mathrm{e}^{-\beta \overline{H_{\bar{S}}}(t)}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta \overline{H_{\bar{S}}}(t)}\right)}$ follows the form:

$$\rho_{\overline{S},00}=\frac{1}{2}-\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{587}$$

$$\rho_{\overline{S},01}=-\frac{B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{588}$$

$$\rho_{\overline{S},10}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{589}$$

$$\rho_{\overline{S},11}=\frac{1}{2}+\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}. \tag{590}$$

The expected values respect to the system $\overline{S}$ of relevance for calculating $\left\langle\overline{H_I}^2\left(t\right)\right\rangle_{\overline{H_{\bar{S}}(t)}}$ are $\langle|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle|i\rangle\langle i|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle|i\rangle\langle i|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle\sigma_x|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$, $\langle\sigma_y|i\rangle\langle i|\rangle_{\overline{H_{\bar{S}}(t)}}$ and $\langle\sigma_z\rangle_{\overline{H_{\bar{S}}(t)}}$, we took account that $\sigma_x\sigma_y=\mathrm{i}\sigma_z$ and $\sigma_y\sigma_x=-\mathrm{i}\sigma_z$. The values needed for our calculation are:

$$\langle|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}-\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{591}$$

$$\langle|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}}=\frac{1}{2}+\frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{2\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{592}$$

$$\langle|0\rangle\langle 0|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{593}$$

$$\langle|1\rangle\langle 1|\sigma_x\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{B_{10}^*\left(t\right)V_{10}^*\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{594}$$

$$\langle|0\rangle\langle 0|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}}=-\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{595}$$

$$\langle |1\rangle\langle 1|\sigma_y\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B^*_{10}\left(t\right)V^*_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)\right|^2\left|V_{10}\left(t\right)\right|^2}}, \tag{596}$$

$$\langle \sigma_x|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B^*_{10}\left(t\right)V^*_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{597}$$

$$\langle \sigma_x|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{598}$$

$$\langle \sigma_y|0\rangle\langle 0|\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\mathrm{i}B^*_{10}\left(t\right)V^*_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{599}$$

$$\langle \sigma_y|1\rangle\langle 1|\rangle_{\overline{H_{\bar{S}}(t)}} = -\frac{\mathrm{i}B_{10}\left(t\right)V_{10}\left(t\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}, \tag{600}$$

$$\langle \sigma_z\rangle_{\overline{H_{\bar{S}}(t)}} = \frac{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)\tanh\left(\frac{\beta}{2}\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}\right)}{\sqrt{\left(\sum_i\left(-1\right)^i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)\right)^2+4\left|B_{10}\left(t\right)V_{10}\left(t\right)\right|^2}}. \tag{601}$$

Our next step is to find $v_3\left(t\right)$, the commutator $\left[\overline{H_0}\left(t\right),\overline{H_{\bar{I}}}\left(t\right)\right]$ is a central point for our calculations and it is equal to:

$$\left[\,\overline{H_0}(t)\,,\overline{H_{\bar{I}}}(t)\right] = \Big[(\varepsilon_0(t)+R_0(t))\,|0\rangle\langle 0| + (\varepsilon_1(t)+R_1(t))\,|1\rangle\langle 1| + \sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t) - B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right) - \sigma_y\Big(B^{\Re}_{10}(t)V^{\Im}_{10}(t) \tag{602}$$

$$+B^{\Im}_{10}(t)V^{\Re}_{10}(t)\Big) + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}, \sum_i B_{iz}(t)\,|i\rangle\langle i| + V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t)) + V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\Big] \tag{603}$$

$$= \Big[\sum_i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i| + \sigma_x\left(B^{\Re}_{10}\left(t\right)V^{\Re}_{10}\left(t\right) - B^{\Im}_{10}\left(t\right)V^{\Im}_{10}\left(t\right)\right) - \sigma_y\left(B^{\Re}_{10}\left(t\right)V^{\Im}_{10}\left(t\right) + B^{\Im}_{10}\left(t\right)V^{\Re}_{10}\left(t\right)\right) \tag{604}$$

$$+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b^\dagger_{\mathbf{k}}b_{\mathbf{k}}, \sum_i B_{iz}\left(t\right)|i\rangle\langle i| + V^{\Re}_{10}\left(t\right)\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right) + V^{\Im}_{10}\left(t\right)\left(\sigma_x B_y\left(t\right)-\sigma_y B_x\left(t\right)\right)\Big] \tag{605}$$

$$= \sum_i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V^{\Re}_{10}\left(t\right)\left(\sigma_x B_x\left(t\right)+\sigma_y B_y\left(t\right)\right) + \sum_i\left(\varepsilon_i\left(t\right)+R_i\left(t\right)\right)|i\rangle\langle i|V^{\Im}_{10}\left(t\right)\left(\sigma_x B_y\left(t\right)-\sigma_y B_x\left(t\right)\right) + \sigma_x \tag{606}$$

$$\times\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t) - B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i| + \sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t) - B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Re}_{10}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{607}$$

$$+\sigma_x\left(B^{\Re}_{10}(t)V^{\Re}_{10}(t) - B^{\Im}_{10}(t)V^{\Im}_{10}(t)\right)V^{\Im}_{10}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t)) - \sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t) + B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)\sum_i B_{iz}(t)\,|i\rangle\langle i| \tag{608}$$

$$-\sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t) + B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Re}_{10}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) - \sigma_y\left(B^{\Re}_{10}(t)V^{\Im}_{10}(t) + B^{\Im}_{10}(t)V^{\Re}_{10}(t)\right)V^{\Im}_{10}(t)(\sigma_x B_y(t) \tag{609}$$

$$-\sigma_y B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (610)$$

$$-\sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t) |i\rangle\langle i| \quad (611)$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \quad (612)$$

$$+ V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (613)$$

$$\times \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \quad (614)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \quad (615)$$

$$= \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \quad (616)$$

$$\times \sum_i B_{iz}(t) |i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t) (B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Im}(t) \quad (617)$$

$$\times (B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) |i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Re}(t) (-\mathrm{i}\sigma_z B_x(t) + B_y(t)) \quad (618)$$

$$-\left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t) (-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (619)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t)\right. \quad (620)$$

$$\left. + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t) (B_x(t) - \mathrm{i}\sigma_z B_y(t)) \quad (621)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + V_{10}^{\Re}(t) (\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (622)$$

$$\times \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \quad (623)$$

$$+ V_{10}^{\Im}(t) (\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (624)$$

The term $\overline{H_{\bar{I}}}(t) \left[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)\right]$ is given by:

$$\overline{H_{\bar{I}}}(t)[\overline{H_0}(t), \overline{H_{\bar{I}}}(t)] = \left(\sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t))\right) \left(\sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Re}(t)\right. \quad (625)$$

$$\times (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t)(\sigma_x B_y(t) - \sigma_y B_x(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \quad (626)$$

$$\times \sum_i B_{iz}(t) |i\rangle\langle i| + \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) V_{10}^{\Re}(t)(B_x(t) + \mathrm{i}\sigma_z B_y(t)) + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \quad (627)$$

$$\times V_{10}^{\Im}(t)(B_y(t) - \mathrm{i}\sigma_z B_x(t)) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \sum_i B_{iz}(t) |i\rangle\langle i| - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) \quad (628)$$

$$\times V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t) + B_y(t)) - \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t) - B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t) \quad (629)$$

$$\times |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \quad (630)$$

$$\times \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (631)$$

$$- V_{10}^{\Re}(t)(\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t) (B_x(t) - \mathrm{i}\sigma_z B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \quad (632)$$

$$+ V_{10}^{\Re}(t) (\mathrm{i}\sigma_z B_x(t) + B_y(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (633)$$

$$- V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t)(B_y(t) + \mathrm{i}\sigma_z B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)\right) \quad (634)$$

$$\left. + V_{10}^{\Im}(t) (\mathrm{i}\sigma_z B_y(t) - B_x(t)) \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)\right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\right) \quad (635)$$

$$= \sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{636}$$

$$+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\,V_{10}^{\Re}(t)\,(B_x(t) \tag{637}$$

$$+\mathrm{i}\sigma_z B_y(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\,V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) \tag{638}$$

$$+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|-\sum_i B_{iz}(t)|i\rangle\langle i|\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\,V_{10}^{\Re}(t)\,(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-\sum_i B_{iz}(t)|i\rangle\langle i| \tag{639}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)V_{10}^{\Im}(t)(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)|i\rangle\langle i|+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{640}$$

$$\times V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x \tag{641}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)+\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)-\sum_i B_{iz}(t)|i\rangle\langle i|\sum_i B_{iz}(t)|i\rangle\langle i| \tag{642}$$

$$\times\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{643}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t) \tag{644}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t) \tag{645}$$

$$\times(B_y(t)+\mathrm{i}\sigma_z B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)+\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big) \tag{646}$$

$$-\sum_i B_{iz}(t)|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{647}$$

$$+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x \tag{648}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t)) \tag{649}$$

$$+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_x\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\,V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,\sigma_y \tag{650}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t)) \tag{651}$$

$$-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\,V_{10}^{\Im}(t)\,(-\mathrm{i}\sigma_z B_y(t)-B_x(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{652}$$

$$\times\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{653}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_x\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t) \tag{654}$$

$$+\sigma_y B_y(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)-V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t) \tag{655}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Re}(t)\,(B_x(t)-\mathrm{i}\sigma_z B_y(t)) \tag{656}$$

$$\times\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)+V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\mathrm{i}\sigma_z B_x(t)+B_y(t))\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)-V_{10}^{\Re}(t) \tag{657}$$

$$\times(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\,V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t) \tag{658}$$

$$+R_i(t))|i\rangle\langle i|-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(B_y(t)+\mathrm{i}\sigma_z B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t)) \tag{659}$$

$$\times V_{10}^{\Im}(t)\,(\mathrm{i}\sigma_z B_y(t)-B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)-V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{660}$$

$$+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i|V_{10}^{\Re}(t)\,(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i(\varepsilon_i(t)+R_i(t))|i\rangle\langle i| \tag{661}$$

$$\times V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))+V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|+V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{662}$$

$$-\sigma_y B_x(t))\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\,V_{10}^{\Re}(t)(B_x(t)+\mathrm{i}\sigma_z B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_x\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) \tag{663}$$

$$\times V_{10}^{\Im}(t)\,(B_y(t)-\mathrm{i}\sigma_z B_x(t))-V_{10}^{\Im}(t)\,(\sigma_x B_y(t)-\sigma_y B_x(t))\,\sigma_y\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)\sum_i B_{iz}(t)|i\rangle\langle i|-V_{10}^{\Im}(t)\,(\sigma_x B_y(t) \tag{664}$$

$$-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)(-\mathrm{i}\sigma_z B_x(t)+B_y(t))-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{665}$$

$$\times V_{10}^{\Im}(t)\left(-\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)+V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{666}$$

$$\times V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t)) \tag{667}$$

$$\times\sum_i B_{iz}(t)\,|i\rangle\langle i|\sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{668}$$

$$-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\sum_i B_{iz}(t)|i\rangle\langle i|\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))\sum_i(\varepsilon_i(t)+R_i(t)) \tag{669}$$

$$\times|i\rangle\langle i|-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))V_{10}^{\Re}(t)\left(B_x(t)-\mathrm{i}\sigma_z B_y(t)\right)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Re}(t) \tag{670}$$

$$\times\left(\mathrm{i}\sigma_z B_x(t)+B_y(t)\right)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)-V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V_{10}^{\Re}(t)\left(\sigma_x B_x(t)+\sigma_y B_y(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{671}$$

$$-V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)|i\rangle\langle i|-V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)V_{10}^{\Im}(t)\left(B_y(t)\right. \tag{672}$$

$$\left.+\mathrm{i}\sigma_z B_x(t)\right)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_z B_y(t)-B_x(t)\right)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right) \tag{673}$$

$$-V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\,V_{10}^{\Im}(t)\left(\sigma_x B_y(t)-\sigma_y B_x(t)\right)\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} \tag{674}$$

$$=V_{10}^{\Re}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\right)+V_{10}^{\Im}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y\right. \tag{675}$$

$$\left.\times B_{iz}(t)\,B_x(t)\right)+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_{i\neq i'}B_{iz}(t)\,B_{i'z}(t)\,|i\rangle\langle i'|+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Re}(t)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\right. \tag{676}$$

$$\left.\times B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)\right)+\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\sum_i\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_x(t)\right)-\left(B_{10}^{\Re}(t)\right. \tag{677}$$

$$\left.\times V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_{i\neq i'}B_{iz}(t)B_{i'z}(t)\,|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|-\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\sum_i\left(-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+B_{iz}(t)\right. \tag{678}$$

$$\left.\times B_y(t)\,|i\rangle\langle i|\right)+\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)+|i\rangle\langle i|B_{iz}(t)\,B_x(t)\right)+\sum_{i,\mathbf{k}}|i\rangle\langle i|B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t) \tag{679}$$

$$+V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+|i\rangle\langle i|\sigma_y B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\right)+V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-|i\rangle\langle i|\sigma_y B_{iz}(t)\right. \tag{680}$$

$$\left.\times\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\right)-\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i B_{iz}^2(t)\,|i\rangle\langle i|\sigma_x+\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i B_{iz}^2(t)\,|i\rangle\langle i|\sigma_y-\sum_{i,\mathbf{k}}|i\rangle\langle i| \tag{681}$$

$$\times B_{iz}^2(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-V_{10}^{\Re}(t)\sum_{i,i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|B_{iz}(t)B_x(t)+|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)B_y(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\right. \tag{682}$$

$$\left.\times V_{10}^{\Im}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\,B_x(t)-\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_x(t)+|i\rangle\langle i|\right. \tag{683}$$

$$\left.\times B_{iz}(t)\,B_y(t)\right)-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|i\rangle\langle i|\sigma_y B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)-V_{10}^{\Im}(t)\sum_{i\neq i'}\left(\varepsilon_{i'}(t)+R_{i'}(t)\right)\left(|i\rangle\langle i|\sigma_x|i'\rangle\langle i'|\right. \tag{684}$$

$$\left.\times B_{iz}(t)\,B_y(t)-|i\rangle\langle i|\sigma_y|i'\rangle\langle i'|B_{iz}(t)\,B_x(t)\right)-V_{10}^{\Im}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i\left(|i\rangle\langle i|B_{iz}(t)\,B_y(t)+\mathrm{i}|i\rangle\langle i|\sigma_z B_x(t)\right)+V_{10}^{\Im}(t) \tag{685}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)\sum_i\left(\mathrm{i}|i\rangle\langle i|\sigma_z B_{iz}(t)\,B_y(t)-|i\rangle\langle i|B_{iz}(t)\,B_x(t)\right)-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\left(|i\rangle\langle i|\sigma_x B_{iz}(t)\,B_y(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-|i\rangle\langle i|\sigma_y\right. \tag{686}$$

$$\left.\times B_{iz}(t)\,B_x(t)\,\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B_x^2(t)+\sigma_x|i\rangle\langle i|\sigma_y B_x(t)\,B_y(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)\,B_x(t)+\sigma_y|i\rangle\langle i|\right. \tag{687}$$

$$\left.\times\sigma_y B_y^2(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_i\left(\varepsilon_i(t)+R_i(t)\right)\left(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B_x^2(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_y B_y(t)\,B_x(t)\right) \tag{688}$$

$$+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)\sum_i\left(|i\rangle\langle i|B_x(t)\,B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) \tag{689}$$

$$\times\left(\sigma_x B_x^2(t)+\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)-\sigma_x B_y^2(t)\right)+V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right)V_{10}^{\Im}(t)\left(B_x(t)\,B_y(t)-\mathrm{i}\sigma_z B_y^2(t)\right. \tag{690}$$

$$\left.-\mathrm{i}\sigma_z B_x^2(t)-B_y(t)\,B_x(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\right)\sum_i\left(\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_{iz}(t)+|i\rangle\langle i|B_y(t)\,B_{iz}(t)\right)-V_{10}^{\Re}(t) \tag{691}$$

$$\times\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right)V_{10}^{\Re}(t)\left(-\sigma_y B_x^2(t)+\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B_y^2(t)\right)-V_{10}^{\Re}(t)\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\right. \tag{692}$$

$$\left.\times V_{10}^{\Re}(t)\right)V_{10}^{\Im}(t)\left(-\sigma_y B_x(t)B_y(t)+\sigma_x B_y^2(t)-\sigma_x B_x^2(t)-\sigma_y B_y(t)B_x(t)\right)+V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\left(\sigma_x|i\rangle\langle i|B_x(t)+\sigma_y|i\rangle\langle i|B_y(t)\right)\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t) \tag{693}$$

$$+\Big(V_{10}^{\Re}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\Big)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}} \tag{694}$$

$$\times\Big(B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)-B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\Big)-V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big) \tag{695}$$

$$\times\sum_i(\sigma_x|i\rangle\langle i|\sigma_x B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i|\sigma_x B_y(t)\,B_{iz}(t))+V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i(\sigma_x|i\rangle\langle i|\sigma_y B_x(t)\,B_{iz}(t)+\sigma_y|i\rangle\langle i| \tag{696}$$

$$\times\sigma_y B_y(t)\,B_{iz}(t))-V_{10}^{\Re}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\Big(\sigma_x|i\rangle\langle i|B_x(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\sigma_y|i\rangle\langle i|B_y(t)\,B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big)-\Big(V_{10}^{\Re}(t)\Big)^2\sum_i(\varepsilon_i(t)+R_i(t))(|i\rangle\langle i|B_x^2(t) \tag{697}$$

$$-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)+|i\rangle\langle i|B_y^2(t))-\Big(V_{10}^{\Re}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)(\sigma_x B_x^2(t)+\sigma_y B_y(t)\,B_x(t) \tag{698}$$

$$-\sigma_y B_x(t)\,B_y(t)+\sigma_x B_y^2(t))+\Big(V_{10}^{\Re}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)(\sigma_y B_x^2(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_x B_x(t)\,B_y(t)+\sigma_y B_y^2(t)) \tag{699}$$

$$-\Big(V_{10}^{\Re}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big)-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_i(\varepsilon_i(t)+R_i(t)) \tag{700}$$

$$\times(|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)-|i\rangle\langle i|B_y(t)\,B_x(t))-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)(\sigma_x B_x(t) \tag{701}$$

$$\times B_y(t)+\sigma_y B_y^2(t)+\sigma_y B_x^2(t)-\sigma_x B_y(t)B_x(t))+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)(\sigma_y B_x(t)B_y(t)-\sigma_x B_y^2(t)-\sigma_x B_x^2(t) \tag{702}$$

$$-\sigma_y B_y(t)\,B_x(t))-V_{10}^{\Re}(t)\,V_{10}^{\Im}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_x(t)\,B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_x^2(t)-B_y(t)\,B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_i(\varepsilon_i(t) \tag{703}$$

$$+R_i(t))(\sigma_x|i\rangle\langle i|\sigma_x B_y(t)\,B_x(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x^2(t)+\sigma_x|i\rangle\langle i|\sigma_y B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_y B_x(t)\,B_y(t))+\Big(V_{10}^{\Im}(t)\Big)^2\sum_i(\varepsilon_i(t)+R_i(t))(\sigma_x|i\rangle\langle i| \tag{704}$$

$$\times\sigma_x B_y^2(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x(t)\,B_y(t)-\sigma_x|i\rangle\langle i|\sigma_y B_y(t)\,B_x(t)+\sigma_y|i\rangle\langle i|\sigma_y B_x^2(t))+V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)\sum_i(|i\rangle\langle i|B_y(t) \tag{705}$$

$$\times B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_{iz}(t))+V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)V_{10}^{\Re}(t)(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)+\sigma_y B_y^2(t)+\sigma_x B_x(t)B_y(t)) \tag{706}$$

$$+\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t)-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)(B_y^2(t)+\mathrm{i}\sigma_z B_x(t)B_y(t)-\mathrm{i}\sigma_z B_y(t)B_x(t)+B_x^2(t))-V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big) \tag{707}$$

$$\times\sum_i(\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)B_{iz}(t)-|i\rangle\langle i|B_x(t)B_{iz}(t))-V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)V_{10}^{\Re}(t)(-\sigma_y B_y(t)\,B_x(t)-\sigma_x B_x^2(t)+\sigma_x B_y^2(t) \tag{708}$$

$$-\sigma_y B_x(t)\,B_y(t))-\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)(-\sigma_y B_y^2(t)-\sigma_x B_x(t)\,B_y(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_y B_x^2(t))+V_{10}^{\Im}(t) \tag{709}$$

$$\times\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\Big(\sigma_x|i\rangle\langle i|B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_{iz}(t)\Big)+V_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+\mathrm{i}\sigma_z \tag{710}$$

$$\times B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)\Big)+\Big(V_{10}^{\Im}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)+\mathrm{i}\sigma_z B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_y(t)-\mathrm{i}\sigma_z B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)+B_x(t) \tag{711}$$

$$\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_x(t)\Big)-V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Re}(t)-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)\sum_i(\sigma_x|i\rangle\langle i|\sigma_x B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_x B_x(t)\,B_{iz}(t))+V_{10}^{\Im}(t)\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t) \tag{712}$$

$$+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)\sum_i(\sigma_x|i\rangle\langle i|\sigma_y B_y(t)B_{iz}(t)-\sigma_y|i\rangle\langle i|\sigma_y B_x(t)B_{iz}(t))-V_{10}^{\Im}(t)\sum_{i,\mathbf{k}}\omega_{\mathbf{k}}\Big(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big) \tag{713}$$

$$-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\sum_i(\varepsilon_i(t)+R_i(t))(|i\rangle\langle i|B_y(t)B_x(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)-|i\rangle\langle i|B_x(t)B_y(t))-V_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) \tag{714}$$

$$-B_{10}^{\Im}(t)V_{10}^{\Im}(t)\Big)(\sigma_x B_y(t)B_x(t)-\sigma_y B_x^2(t)-\sigma_y B_y^2(t)-\sigma_x B_x(t)B_y(t))+V_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big(B_{10}^{\Re}(t)V_{10}^{\Im}(t)+B_{10}^{\Im}(t)V_{10}^{\Re}(t)\Big)(\sigma_y B_y(t)B_x(t) \tag{715}$$

$$+\sigma_x B_x^2(t)+\sigma_x B_y^2(t)-\sigma_y B_x(t)B_y(t))-\sum_{\mathbf{k}}V_{10}^{\Im}(t)V_{10}^{\Re}(t)\,\omega_{\mathbf{k}}\Big(B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big) \tag{716}$$

$$-\Big(V_{10}^{\Im}(t)\Big)^2\sum_i(\varepsilon_i(t)+R_i(t))(|i\rangle\langle i|B_y^2(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)+|i\rangle\langle i|B_x^2(t))-\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)V_{10}^{\Re}(t) \tag{717}$$

$$-B_{10}^{\Im}(t)\,V_{10}^{\Im}(t)\Big)(\sigma_x B_y^2(t)-\sigma_y B_x(t)\,B_y(t)+\sigma_y B_y(t)\,B_x(t)+\sigma_x B_x^2(t))+\Big(V_{10}^{\Im}(t)\Big)^2\Big(B_{10}^{\Re}(t)\,V_{10}^{\Im}(t)+B_{10}^{\Im}(t)\,V_{10}^{\Re}(t)\Big)(\sigma_y B_y^2(t) \tag{718}$$

$$+\sigma_x B_x(t)\,B_y(t)-\sigma_x B_y(t)\,B_x(t)+\sigma_y B_x^2(t))-\Big(V_{10}^{\Im}(t)\Big)^2\sum_{\mathbf{k}}\omega_{\mathbf{k}}\Big(B_y^2(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\mathrm{i}\sigma_z B_x(t)B_y(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\mathrm{i}\sigma_z B_y(t)B_x(t)\,b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+B_x^2(t) \tag{719}$$

$$\times b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\Big), \tag{720}$$

Now let's obtain the form of $\overline{H_{\bar{I}}}^3(t)$:

$$\overline{H_{\bar{I}}}^3(t)=\Big(\sum_i B_{iz}(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)(\sigma_x B_x(t)+\sigma_y B_y(t))+V_{10}^{\Im}(t)(\sigma_x B_y(t)-\sigma_y B_x(t))\Big)\Big(\sum_i B_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x \tag{721}$$

$$+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) \tag{722}$$

$$+\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+\left(V_{10}^{\Im}(t)\right)^2 \tag{723}$$

$$\times\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right)) \tag{724}$$

$$=\sum_iB_{iz}\,(t)\,|i\rangle\langle i|\sum_iB_{iz}^2\,(t)\,|i\rangle\langle i|+\sum_iB_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+\sum_iB_{iz}\,(t)\,|i\rangle\langle i|V_{10}^{\Im}\,(t) \tag{725}$$

$$\times\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+\sum_iB_{iz}(t)|i\rangle\langle i|V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+\sum_iB_{iz}(t)|i\rangle\langle i| \tag{726}$$

$$\times\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+\sum_iB_{iz}(t)\,|i\rangle\langle i|V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) \tag{727}$$

$$+\sum_iB_{iz}(t)\,|i\rangle\langle i|\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\sum_iB_{iz}^2(t)\,|i\rangle\langle i|+V_{10}^{\Re}(t) \tag{728}$$

$$\times(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}\,(t)\sum_i(B_{iz}(t)B_x(t)\,|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}\,(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t) \tag{729}$$

$$\times|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t) \tag{730}$$

$$+\sigma_yB_y(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\,V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t) \tag{731}$$

$$-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Re}(t)(\sigma_xB_x(t)+\sigma_yB_y(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t) \tag{732}$$

$$-\sigma_yB_x(t))\sum_iB_{iz}^2(t)|i\rangle\langle i|+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x+B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_y \tag{733}$$

$$\times B_x(t))V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)\,|i\rangle\langle i|\sigma_x-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y)+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\,V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t)+\sigma_y|i\rangle\langle i| \tag{734}$$

$$\times B_y(t)B_{iz}(t))+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Re}(t)\right)^2\left(B_x^2(t)+\mathrm{i}\sigma_zB_x(t)B_y(t)-\mathrm{i}\sigma_zB_y(t)B_x(t)+B_y^2(t)\right)+V_{10}^{\Im}(t)(\sigma_xB_y(t)-\sigma_yB_x(t)) \tag{735}$$

$$\times V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}(t)-\sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t))+V_{10}^{\Im}(t)\,(\sigma_xB_y(t)-\sigma_yB_x(t))\left(V_{10}^{\Im}(t)\right)^2\left(B_y^2(t)+B_x^2(t)-\mathrm{i}\sigma_zB_y\,(t)\,B_x\,(t)\right. \tag{736}$$

$$\left.+\mathrm{i}\sigma_zB_x\,(t)\,B_y\,(t)\right) \tag{737}$$

$$=\sum_iB_{iz}^3\,(t)\,|i\rangle\langle i|+V_{10}^{\Re}\,(t)\sum_i\left(B_{iz}^2\,(t)\,B_x\,(t)\,|i\rangle\langle i|\sigma_x+B_{iz}^2\,(t)\,B_y\,(t)\,|i\rangle\langle i|\sigma_y\right)+V_{10}^{\Im}\,(t)\sum_i\left(B_{iz}^2(t)B_y(t)|i\rangle\langle i|\sigma_x-B_{iz}^2(t)B_x(t)|i\rangle\langle i|\sigma_y\right) \tag{738}$$

$$+V_{10}^{\Re}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_x(t)B_{iz}(t)+|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_x^2(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)\right. \tag{739}$$

$$\left.\times B_x(t)B_y(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+|i\rangle\langle i|B_{iz}(t)B_y^2(t)\right)+V_{10}^{\Im}(t)\sum_{i\neq i'}\left(|i'\rangle\langle i'|\sigma_x|i\rangle\langle i|B_{i'z}(t)B_y(t)B_{iz}(t)-|i'\rangle\langle i'|\sigma_y|i\rangle\langle i|B_{i'z}(t)\right. \tag{740}$$

$$\left.B_x(t)B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_{iz}(t)B_y^2(t)+|i\rangle\langle i|B_{iz}(t)B_x^2(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_y(t)B_x(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_{iz}(t)B_x(t)B_y(t)\right)+V_{10}^{\Re}(t) \tag{741}$$

$$\times\sum_i\left(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}^2(t)+\sigma_y|i\rangle\langle i|B_y(t)B_{iz}^2(t)\right)+\left(V_{10}^{\Re}(t)\right)^2\sum_i(B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_x+B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_y+B_y(t) \tag{742}$$

$$\times B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_x+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i(B_x(t)B_{iz}(t)B_y(t)\,\sigma_x|i\rangle\langle i|\sigma_x-B_x(t)B_{iz}(t)B_x(t)\,\sigma_x|i\rangle\langle i|\sigma_y \tag{743}$$

$$+B_y(t)B_{iz}(t)B_y(t)\,\sigma_y|i\rangle\langle i|\sigma_x-B_y(t)B_{iz}(t)B_x(t)\,\sigma_y|i\rangle\langle i|\sigma_y)+\left(V_{10}^{\Re}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_x^2(t)B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|\right. \tag{744}$$

$$\left.\times B_y(t)B_x(t)B_{iz}(t)+|i\rangle\langle i|B_y^2(t)B_{iz}(t)\right)+\left(V_{10}^{\Re}(t)\right)^3\left(\sigma_xB_x^3(t)+\sigma_yB_x^2(t)B_y(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_xB_x(t)B_y^2(t)+\sigma_yB_y(t)B_x^2(t)\right. \tag{745}$$

$$\left.-\sigma_xB_y(t)B_x(t)B_y(t)+\sigma_xB_y^2(t)B_x(t)+\sigma_yB_y^3(t)\right)+V_{10}^{\Re}(t)V_{10}^{\Im}(t)\sum_i\left(|i\rangle\langle i|B_x(t)B_y(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)B_{iz}(t)-\mathrm{i}|i\rangle\langle i|\sigma_zB_y^2(t)\right. \tag{746}$$

$$\left.\times B_{iz}(t)+\mathrm{i}|i\rangle\langle i|\sigma_zB_y(t)B_x(t)B_{iz}(t)\right)+V_{10}^{\Re}(t)\left(V_{10}^{\Im}(t)\right)^2\left(\sigma_xB_x(t)B_y^2(t)+\sigma_xB_x^3(t)-\sigma_yB_x(t)B_y(t)B_x(t)+\sigma_yB_x^2(t)B_y(t)+\sigma_yB_y^3(t)\right. \tag{747}$$

$$\left.+\sigma_yB_y(t)\,B_x^2(t)+\sigma_xB_y^2(t)\,B_x(t)-\sigma_xB_y(t)\,B_x(t)\,B_y(t)\right)+V_{10}^{\Im}(t)\sum_i\left(\sigma_x|i\rangle\langle i|B_y(t)\,B_{iz}^2(t)-\sigma_y|i\rangle\langle i|B_x(t)\,B_{iz}^2(t)\right)+V_{10}^{\Re}(t)\,V_{10}^{\Im}(t) \tag{748}$$

$$\times(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_x(t)+\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t)-\sigma_y|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_y(t))+\left(V_{10}^{\Im}(t)\right)^2 \tag{749}$$

$$\times(\sigma_x|i\rangle\langle i|\sigma_xB_y(t)B_{iz}(t)B_y(t)-\sigma_x|i\rangle\langle i|\sigma_yB_y(t)B_{iz}(t)B_x(t)-\sigma_y|i\rangle\langle i|\sigma_xB_x(t)B_{iz}(t)B_y(t)+\sigma_y|i\rangle\langle i|\sigma_yB_x(t)B_{iz}(t)B_x(t))+V_{10}^{\Re}(t) \tag{750}$$

$$\times V_{10}^{\Im}(t)\sum_i\left(|i\rangle\langle i|B_y(t)\,B_x(t)\,B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_y^2(t)\,B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x^2(t)\,B_{iz}(t)-|i\rangle\langle i|B_x(t)\,B_y(t)\,B_{iz}(t)\right)+V_{10}^{\Im}(t)\left(V_{10}^{\Re}(t)\right)^2 \tag{751}$$

$$\times\left(\sigma_x B_y(t)B_x^2(t)+\sigma_y B_y(t)B_x(t)B_y(t)-\sigma_y B_y^2(t)B_x(t)+\sigma_x B_y^3(t)-\sigma_y B_x^3(t)+\sigma_x B_x^2(t)B_y(t)-\sigma_x B_x(t)B_y(t)B_x(t)-\sigma_y B_x(t)B_y^2(t)\right) \tag{752}$$

$$+\left(V_{10}^{\Im}(t)\right)^2\sum_i\left(|i\rangle\langle i|B_y^2(t)B_{iz}(t)-\mathrm{i}\sigma_z|i\rangle\langle i|B_y(t)\,B_x(t)\,B_{iz}(t)+\mathrm{i}\sigma_z|i\rangle\langle i|B_x(t)\,B_y(t)\,B_{iz}(t)+|i\rangle\langle i|B_x^2(t)\,B_{iz}(t)\right)+\left(V_{10}^{\Im}(t)\right)^3\left(\sigma_x B_y^3(t)\right. \tag{753}$$

$$\left.+\sigma_x B_y(t)\,B_x^2(t)-\sigma_y B_y^2(t)\,B_x(t)+\sigma_y B_y(t)\,B_x(t)\,B_y(t)-\sigma_y B_x(t)\,B_y^2(t)-\sigma_y B_x^3(t)-\sigma_x B_x(t)\,B_y(t)\,B_x(t)+\sigma_x B_x^2(t)\,B_y(t)\right). \tag{754}$$

III. BIBLIOGRAPHY

[1] McCutcheon D P S, Dattani N S, Gauger E M, Lovett B W and Nazir A 2011 Phys. Rev. B 84 081305.
[2] Dattani N S, Chaparro E C, A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between.
[3] A. Decoster, J. Phys. A 37, 9051 2004.
[4] Dara P S McCutcheon and Ahsan Nazir 2010 New J. Phys. 12 113042.
[5] Supplement: Theoretical model of phonon induced dephasing. A.J. Ramsay ey al 2009.
[6] Felix A Pollock et al 2013 New J. Phys. 15 075018.

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