

# Generalized Bogoliubov inequality for a general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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## I. GENERAL ELEMENTS FOR FREE ENERGY MINIMIZATION WITH NTH BOGOLIUBOV INEQUALITY

In order to provide a general approach for achieving the free energy of the system using the variational parameters optimization we consider the generalization in [5] of the Bogoliubov inequality. We consider the partition functions of  $\bar{H}(t)$  and  $\bar{H}_0(t)$  respect to  $\bar{H}_0(t)$  as:

$$Z(t) = \left\langle e^{-\beta \bar{H}(t)} \right\rangle_{\bar{H}_0(t)}, \quad (1)$$

$$\bar{H}(t) = \bar{H}_I(t) + \bar{H}_0(t), \quad (2)$$

$$\bar{H}_0(t) = \bar{H}_S(t) + \bar{H}_B, \quad (3)$$

$$Z_0(t) = e^{-\beta \langle \bar{H}(t) \rangle_{\bar{H}_0(t)}} \quad (4)$$

$$= e^{-\beta \langle \bar{H}_I(t) + \bar{H}_0(t) \rangle_{\bar{H}_0(t)}} \quad (5)$$

$$= e^{-\beta \langle \bar{H}_I(t) \rangle_{\bar{H}_0(t)}} e^{-\beta \langle \bar{H}_0(t) \rangle_{\bar{H}_0(t)}} \quad (6)$$

$$= e^0 e^{-\beta \langle \bar{H}_0(t) \rangle_{\bar{H}_0(t)}} \quad (7)$$

$$= e^{-\beta \langle \bar{H}_0(t) \rangle_{\bar{H}_0(t)}} \quad (8)$$

Here  $\bar{H}_0(t) = \bar{H}_S(t) + \bar{H}_B$ , also we used  $\langle \bar{H}_I(t) \rangle_{\bar{H}_0(t)} = 0$ . Taking the Quantum Bogoliubov inequality from [5]:

$$Z(t) \geq Z_0(t) e^{-\langle \bar{H}_I(t) \rangle_{\bar{H}_0(t)}} (1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))), \quad (9)$$

$$F_N(\vec{v}(t)) = e^{-\alpha} \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \sum_{i=0}^{2N-1-k} \frac{\alpha^i}{i!}. \quad (10)$$

where

$$\bar{H}_{ID}(t) = \sum_n \langle n | \bar{H}_I(t) | n \rangle | n \rangle \langle n | \quad (\text{with } | n \rangle \text{ is an eigenstate of } \bar{H}_0(t)), \quad (11)$$

$$\bar{H}_0(t) | n \rangle = E_{0,n}(t) | n \rangle, \quad (12)$$

$$Z_0(t) = \sum_n e^{-\beta E_{0,n}(t)}, \quad (13)$$

$$u_k(t) = \left\langle \left( \bar{H}_{ID}(t) - \langle \bar{H}_I(t) \rangle_{\bar{H}_0(t)} \right)^k \right\rangle_{\bar{H}_0(t)} \quad (14)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left( \langle n | \bar{H}_I(t) | n \rangle - \langle \bar{H}_I(t) \rangle_{\bar{H}_0(t)} \right)^k, \quad (15)$$

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \left( \bar{H}_0(t) - E_{0,n}(t) + \bar{H}_I(t) - \langle \bar{H}_I(t) \rangle_{\bar{H}_0(t)} \right)^k \right| n \right\rangle. \quad (16)$$

By construction  $\langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} = 0$ , so we arrive to:

$$Z(t) \geq Z_0(t) (1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))), \quad (17)$$

$$u_k(t) = \left\langle (\overline{H_{ID}}(t))^k \right\rangle_{\overline{H_0}(t)}, \quad (18)$$

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^k \right| n \right\rangle. \quad (19)$$

As we can see the expression (17) was written in terms of the expected value of an operator, we want to do the same for (19) in order to write that expressions in a short form, following this we obtained:

$$(\overline{H_0}(t) - E_{0,n}(t)) |n\rangle = \overline{H_0}(t) |n\rangle - E_{0,n}(t) |n\rangle \quad (20)$$

$$= E_{0,n}(t) |n\rangle - E_{0,n}(t) |n\rangle \quad (21)$$

$$= 0, \quad (22)$$

$$\langle n | (\overline{H_0}(t) - E_{0,n}(t)) = \langle n | \overline{H_0}(t) - \langle n | E_{0,n}(t) \quad (23)$$

$$= \langle n | E_{0,n}(t) - \langle n | E_{0,n}(t) \quad (24)$$

$$= 0. \quad (25)$$

At first we calculated  $v_1(t)$  like:

$$v_1(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t) | n \rangle \quad (26)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_0}(t) - E_{0,n}(t) | n \rangle + \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) | n \rangle \quad (27)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | E_{0,n}(t) - E_{0,n}(t) | n \rangle + \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) | n \rangle \quad (28)$$

$$= 0 + \langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} \quad (29)$$

$$= 0. \quad (30)$$

For  $k \geq 2$  and  $k \in N$  we calculated:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^k \right| n \right\rangle \quad (31)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (32)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (33)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| (E_{0,n}(t) - E_{0,n}(t) + \overline{H_I}(t)) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} (E_{0,n}(t) - E_{0,n}(t) + \overline{H_I}(t)) \right| n \right\rangle \quad (34)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} \overline{H_I}(t) \right| n \right\rangle. \quad (35)$$

We will obtain the explicit form of  $v_2(t)$  and  $v_3(t)$  using (35):

$$v_2(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{2-2} \overline{H_I}(t) \right| n \right\rangle \quad (36)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^0 \overline{H_I}(t) \right| n \right\rangle \quad (37)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) \mathbb{I} \overline{H_I}(t) | n \rangle \quad (38)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}^2(t) | n \rangle \quad (39)$$

$$= \langle \overline{H_I}(t)^2 \rangle_{\overline{H_0}(t)}, \quad (40)$$

$$v_3(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{3-2} \overline{H_I}(t) | n \rangle \quad (41)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^1 \overline{H_I}(t) | n \rangle \quad (42)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \overline{H_I}(t) | n \rangle, \quad (43)$$

$$\overline{H_0}(t) | n \rangle = E_{0,n}(t) | n \rangle, \quad (44)$$

$$\langle n | \overline{H_0}(t) = \langle n | E_{0,n}(t), \quad (45)$$

$$v_3(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t)) \overline{H_I}(t) | n \rangle \quad (46)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) E_{0,n}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_I}(t) \overline{H_I}(t) | n \rangle \quad (47)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}^3(t) - \overline{H_I}(t) E_{0,n}(t) \overline{H_I}(t) | n \rangle \quad (48)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}^3(t) - \overline{H_I}(t) \overline{H_I}(t) E_{0,n}(t) | n \rangle \quad (49)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}^3(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}(t) | n \rangle \quad (50)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}^3(t) + \overline{H_I}(t) (\overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t)) | n \rangle \quad (51)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}^3(t) + \overline{H_I}(t) [\overline{H_0}(t), \overline{H_I}(t)] | n \rangle \quad (52)$$

$$= \langle \overline{H_I}^3(t) + \overline{H_I}(t) [\overline{H_0}(t), \overline{H_I}(t)] \rangle_{\overline{H_0}(t)}. \quad (53)$$

In general we have:

$$v_k(t) = \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) (\overline{H_0}(t) - E_{0,n}(t) + \overline{H_I}(t))^{k-2} \overline{H_I}(t) | n \rangle \quad (54)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) (\overline{H_0}(t) + \overline{H_I}(t) - E_{0,n}(t))^{k-2} \overline{H_I}(t) | n \rangle \quad (55)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \langle n | \overline{H_I}(t) (\overline{H}(t) - E_{0,n}(t))^{k-2} \overline{H_I}(t) | n \rangle \quad (56)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) \left( \sum_j^{k-2} (-1)^j \binom{k-2}{j} \overline{H}^{k-2-j}(t) E_{0,n}^j(t) \right) \overline{H_I}(t) \right| n \right\rangle \quad (57)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \langle n | \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) E_{0,n}^j(t) | n \rangle \quad (58)$$

$$= \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right| n \right\rangle \quad (59)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \frac{1}{Z_0(t)} \sum_n e^{-\beta E_{0,n}(t)} \left\langle n \left| \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right| n \right\rangle \quad (60)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) \overline{H}^{k-2-j}(t) \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (61)$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{k-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \quad (62)$$

The formula (62) is well defined taking as example  $k = 2, 3$ .

$$v_2(t) = \left\langle \sum_{j=0}^{2-2} (-1)^j \binom{2-2}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{2-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (63)$$

$$= (-1)^0 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^0 \overline{H_I}(t) \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} \quad (64)$$

$$= \left\langle \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)}, \quad (65)$$

$$v_3(t) = \left\langle \sum_{j=0}^{3-2} (-1)^j \binom{3-2}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{3-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (66)$$

$$= \left\langle \sum_{j=0}^1 (-1)^j \binom{1}{j} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{1-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (67)$$

$$= \left\langle (-1)^0 \binom{1}{0} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^1 \overline{H_I}(t) \overline{H_0}^0(t) + (-1)^1 \binom{1}{1} \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^0 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} \quad (68)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \mathbb{I} - \overline{H_I}(t) \mathbb{I} \overline{H_I}(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \quad (69)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \quad (70)$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} \quad (71)$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) (\overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t)) \right\rangle_{\overline{H_0}(t)} \quad (72)$$

$$= \left\langle \overline{H_I}(t)^3 + \overline{H_I}(t) [\overline{H_0}(t), \overline{H_I}(t)] \right\rangle_{\overline{H_0}(t)}. \quad (73)$$

So we summarize:

$$\overline{H_{ID}}(t) = \sum_n \langle n | \overline{H_I}(t) | n \rangle |n\rangle \langle n|, \quad (74)$$

$$u_k(t) = \left\langle (\overline{H_{ID}}(t))^k \right\rangle_{\overline{H_0}(t)}, \quad (75)$$

$$v_k(t) = \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{k-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)}. \quad (76)$$

Then we obtained finally:

$$Z(t) \geq Z_0(t) (1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))), \quad (77)$$

The free energy is defined as:

$$E_{\text{free}}(t) = -\frac{1}{\beta} \ln(Z(t)). \quad (78)$$

It is well known that the function  $f(x) = \ln(x)$  is monotonic and increasing so we can transform (77):

$$E_{\text{free},1}(t) = -\frac{1}{\beta} \ln(Z_0(t)), \quad (79)$$

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_M(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (80)$$

$$\equiv E_{\text{free,MN}}(t). \quad (81)$$

here  $E_{\text{free,MN}}(t)$  is the free energy associate to the strong version of the Quantum Bogoliubov inequality of  $MN$  order. In our approach we will set  $N = M$ , so our quantum Bogoliubov inequality of  $N$  order is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{u}(t)) + F_N(\vec{v}(t) - \vec{u}(t))) \quad (82)$$

$$= E_{\text{free,NN}}(t). \quad (83)$$

A weaker form of the inequality (83) making  $\vec{u}(t) = 0$  is:

$$E_{\text{free}}(t) \leq E_{\text{free},1}(t) - \frac{1}{\beta} \ln(1 + F_N(\vec{v}(t))) \quad (84)$$

$$\equiv E_{\text{free,N}}(t). \quad (85)$$

The algebraic equation associated with  $\alpha_{\text{opt}}(t)$  such that  $E_{\text{free,N}}(t)$  is closer to  $E_{\text{free}}(t)$  is given by the following expression:

$$G(\alpha_{\text{opt}}(t)) = \sum_{k=2}^{2N-1} (-\beta)^k \frac{u_k(t)}{k!} \frac{\alpha_{\text{opt}}^{2N-1-k}}{(2N-1-k)!} \quad (86)$$

$$= 0. \quad (87)$$

The elements presented are the required to find variational parameters through the determination of the SCE (self consistent equations) of the system in particular to the order expected.

## II. SCE FROM 3RD QUANTUM BOGOLIUBOV INEQUALITY

Our first approach is to obtain the SCE for the 3rd order, for this we need to identify  $v_2(t)$ ,  $v_3(t)$ ,  $v_4(t)$ ,  $v_5(t)$  using the (76), we have already  $v_2(t)$ ,  $v_3(t)$  and the form of  $v_4(t)$  and  $v_5(t)$  is given by:

$$v_4(t) = \sum_{j=0}^{4-2} (-1)^j \binom{4-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{4-2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (88)$$

$$= \sum_{j=0}^2 (-1)^j \binom{2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{2-j} \overline{H_I}(t) \overline{H_0}^j(t) \right\rangle_{\overline{H_0}(t)} \quad (89)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \overline{H_0}^0(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^1 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}(t) (\overline{H_I}(t) \right. \quad (90)$$

$$\left. + \overline{H_0}(t))^0 \overline{H_I}(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (91)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \mathbb{I} \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^1 \overline{H_I}(t) \overline{H_0}^1(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (92)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \right\rangle_{\overline{H_0}(t)} - 2 \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) \right\rangle_{\overline{H_0}(t)} + \left\langle \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (93)$$

$$= \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) - 2\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) \right\rangle_{\overline{H_0}(t)} \quad (94)$$

$$= \langle \overline{H_I}(t) \left( \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^2(t) \right) \overline{H_I}(t) - 2\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) \rangle \quad (95)$$

$$+\overline{H_I}^2(t)\overline{H_0}^2(t)\Big\rangle_{\overline{H_0}(t)} \quad (96)$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - 2\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}(t) \quad (97)$$

$$+\overline{H_I}^2(t)\overline{H_0}^2(t)\Big\rangle_{\overline{H_0}(t)} \quad (98)$$

$$= \langle \overline{H}_I^4(t) + \overline{H}_I^2(t) \overline{H}_0(t) \overline{H}_I(t) + \overline{H}_I(t) \overline{H}_0(t) \overline{H}_I^2(t) + \overline{H}_I(t) \overline{H}_0^2(t) \overline{H}_I(t) - 2\overline{H}_I^3(t) \overline{H}_0(t) + \overline{H}_I^2(t) \overline{H}_0^2(t) \rangle \quad (99)$$

$$-2\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\rangle_{\overline{H_0}(t)} \quad (100)$$

$$= \langle \overline{H}_I^4(t) + \overline{H}_I^2(t) \overline{H}_0(t) \overline{H}_I(t) + \overline{H}_I(t) \overline{H}_0(t) \overline{H}_I^2(t) + \overline{H}_I(t) \overline{H}_0^2(t) \overline{H}_I(t) - \overline{H}_I^3(t) \overline{H}_0(t) - \overline{H}_I^3(t) \overline{H}_0(t) \rangle \quad (101)$$

$$+\overline{H_I}^{-2}(t)\overline{H_0}^{-2}(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\Big\rangle_{\overline{H_0}(t)} \quad (102)$$

$$= \langle \overline{H_I}^4(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) \rangle \quad (103)$$

$$-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)+\overline{H_I}^2(t)\overline{H_0}^2(t)-\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t)\Big\rangle_{\overline{H_0}(t)} \quad (104)$$

$$= \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \left( [\overline{H_I}(t) \overline{H_0}(t), \overline{H_I}(t)] + [\overline{H_0}(t), \overline{H_I}^2(t)] + [\overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t)] + [\overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t)] \right) \right\rangle_{\overline{H_0}(t)}, \quad (105)$$

$$v_5(t) = \sum_{j=0}^{5-2} (-1)^j \binom{5-2}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{5-2-j} \overline{H_I}(t) \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \quad (106)$$

$$= \sum_{j=0}^3 (-1)^j \binom{3}{j} \left\langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^{3-j} \overline{H_I}(t) \overline{H_0}(t)^j \right\rangle_{\overline{H_0}(t)} \quad (107)$$

$$= \langle \overline{H}_I(t) (\overline{H}_I(t) + \overline{H}_0(t))^3 \overline{H}_I(t) \overline{H}_0^0(t) - 3 \overline{H}_I(t) (\overline{H}_I(t) + \overline{H}_0(t))^2 \overline{H}_I(t) \overline{H}_0(t) - \overline{H}_I(t) (\overline{H}_I(t) + \overline{H}_0(t))^0 \overline{H}_I(t) \overline{H}_0^3(t) \rangle \quad (108)$$

$$+3\overline{H_I}(t) \left( \overline{H_I}(t) + \overline{H_0}(t) \right) \overline{H_I}(t) \overline{H_0}^2(t) \Big\rangle_{\overline{H_0}(t)} \quad (109)$$

$$= \langle \overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^3 \overline{H_I}(t) - 3\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t))^2 \overline{H_I}(t) \overline{H_0}(t) + 3\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}^2(t) \rangle \quad (110)$$

$$-\overline{H_I}(t)\overline{H_I}(t)\overline{H_0^3}(t)\Big\rangle_{\overline{H_0}(t)} \quad (111)$$

$$= \langle \overline{H}_I(t) (\overline{H}_I(t) + \overline{H}_0(t))^3 \overline{H}_I(t) - 3\overline{H}_I(t) (\overline{H}_I(t) + \overline{H}_0(t))^2 \overline{H}_I(t) \overline{H}_0(t) + 3\overline{H}_I(t) (\overline{H}_I(t) + \overline{H}_0(t)) \overline{H}_I(t) \overline{H}_0^2(t) \rangle \quad (112)$$

$$-\overline{H_I}(t)\overline{H_I}(t)\overline{H_0^3}(t)\rangle_{\overline{H_0}(t)} \quad (113)$$

$$= \langle \overline{H}_I(t) (\overline{H}_I(t) + \overline{H}_0(t))^3 \overline{H}_I(t) - 3\overline{H}_I(t) (\overline{H}_I^2(t) + \overline{H}_I(t) \overline{H}_0(t) + \overline{H}_0(t) \overline{H}_I(t) + \overline{H}_0^2(t)) \overline{H}_I(t) \overline{H}_0(t) \rangle \quad (114)$$

$$+3\overline{H_I}(t) (\overline{H_I}(t) + \overline{H_0}(t)) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}^3(t) \Big\rangle_{\overline{H_0}(t)} \quad (115)$$

$$= \langle \overline{H_I}(t) \left( \overline{H_I}^3(t) + \overline{H_I}^2(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_0}^2(t) \overline{H_I}(t) + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \right) \rangle \quad (116)$$

$$+\overline{H_I}(t)\overline{H_0}^2(t)+\overline{H_0}^3(t)\Big)\overline{H_I}(t)-3\overline{H_I}(t)\Big(\overline{H_I}^2(t)+\overline{H_I}(t)\overline{H_0}(t)+\overline{H_0}(t)\overline{H_I}(t)+\overline{H_0}^2(t)\Big)\overline{H_I}(t)\overline{H_0}(t) \quad (117)$$

$$+3\overline{H_I}(t) \left( \overline{H_I}(t) + \overline{H_0}(t) \right) \overline{H_I}(t) \overline{H_0}^2(t) - \overline{H_I}(t) \overline{H_I}(t) \overline{H_0}^3(t) \Big\rangle_{\overline{H_0}(t)} \quad (118)$$

$$= \langle \overline{H_I}^5(t) + \overline{H_I}^3(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}^2(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^3(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \rangle \quad (119)$$

$$+\overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}^2(t)+\overline{H_I}^2(t)\overline{H_0}^2(t)\overline{H_I}(t)+\overline{H_I}(t)\overline{H_0}^3(t)\overline{H_I}(t)-3\overline{H_I}^4(t)\overline{H_0}(t)-3\overline{H_I}^2(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}(t) \quad (120)$$

$$-3\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}^2(t)\overline{H_0}(t) - 3\overline{H_I}(t)\overline{H_0}^2(t)\overline{H_I}(t)\overline{H_0}(t) + 3\overline{H_I}^3(t)\overline{H_0}^2(t) + 3\overline{H_I}(t)\overline{H_0}(t)\overline{H_I}(t)\overline{H_0}^2(t) \quad (121)$$

$$-\overline{H_I}^2(t)\overline{H_0}^3(t)\big\rangle_{\overline{H_0}(t)} \quad (122)$$

$$= \langle \overline{H_I^5}(t) + \overline{H_I}(t) \left( \overline{H_I^2}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I^2}(t) + \overline{H_0}(t) \overline{H_I^3}(t) + \overline{H_0^2}(t) \overline{H_I^2}(t) + \overline{H_I}(t) \overline{H_0^2}(t) \overline{H_I}(t) \right) \quad (123)$$

$$+ \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) + \overline{H_0}^3(t) \overline{H_I}(t) - 3\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) - 3\overline{H_I}^3(t) \overline{H_0}(t) - 3\overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) \quad (124)$$

$$- \overline{H_I}(t) \overline{H_0}^3(t) + 3\overline{H_I}^2(t) \overline{H_0}^2(t) + 3\overline{H_0}(t) \overline{H_I}(t) \overline{H_0}^2(t) - 3\overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) \Big) \Big\rangle_{\overline{H_0}(t)} \quad (125)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \overline{H_I}^2(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}^2(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}^3(t) \right. \right. \quad (126)$$

$$\left. - \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) + \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) + \overline{H_I}(t) \overline{H_0}^2(t) \overline{H_I}(t) - \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) \right. \quad (127)$$

$$\left. + \overline{H_0}^3(t) \overline{H_I}(t) - \overline{H_I}(t) \overline{H_0}^3(t) + \overline{H_I}^2(t) \overline{H_0}^2(t) - \overline{H_0}(t) \overline{H_I}^2(t) \overline{H_0}(t) + 2\overline{H_I}^2(t) \overline{H_0}^2(t) - 2\overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \overline{H_0}(t) \right. \quad (128)$$

$$\left. + 3\overline{H_0}(t) \overline{H_I}(t) \overline{H_0}^2(t) - 3\overline{H_0}^2(t) \overline{H_I}(t) \overline{H_0}(t) - \overline{H_I}^3(t) \overline{H_0}(t) + \overline{H_0}^2(t) \overline{H_I}^2(t) \right) \Big\rangle_{\overline{H_0}(t)} \quad (129)$$

$$= \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[ \overline{H_0}(t), \overline{H_I}^3(t) \right] + \left[ \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}^2(t) \right] \right. \quad (130)$$

$$\left. + \left[ \overline{H_0}(t), \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right] + \left[ \overline{H_0}^3(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + 3\overline{H_0}(t) \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) \right. \quad (131)$$

$$\left. + 2\overline{H_I}(t) \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right) \Big\rangle_{\overline{H_0}(t)}. \quad (132)$$

Summarizing we have that:

$$v_2(t) = \left\langle \overline{H_I}^2(t) \right\rangle_{\overline{H_0}(t)}, \quad (133)$$

$$v_3(t) = \left\langle \overline{H_I}^3(t) + \overline{H_I}(t) \left[ \overline{H_0}(t), \overline{H_I}(t) \right] \right\rangle_{\overline{H_0}(t)}, \quad (134)$$

$$v_4(t) = \left\langle \overline{H_I}^4(t) + \overline{H_I}(t) \left( \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[ \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[ \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right) \right\rangle_{\overline{H_0}(t)}, \quad (135)$$

$$v_5(t) = \left\langle \overline{H_I}^5(t) + \overline{H_I}(t) \left( \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_I}^2(t) \right] + \left[ \overline{H_0}(t), \overline{H_I}^3(t) \right] + \left[ \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}^2(t) \right] \right. \quad (136)$$

$$\left. + \left[ \overline{H_0}(t), \overline{H_I}(t) \overline{H_0}(t) \overline{H_I}(t) \right] + \left[ \overline{H_0}^3(t), \overline{H_I}(t) \right] + \left[ \overline{H_I}(t) \overline{H_0}(t), \overline{H_0}(t) \overline{H_I}(t) \right] + 3\overline{H_0}(t) \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) \right. \quad (137)$$

$$\left. + 2\overline{H_I}(t) \left[ \overline{H_I}(t), \overline{H_0}(t) \right] \overline{H_0}(t) + \left[ \overline{H_I}^2(t) \overline{H_0}(t), \overline{H_0}(t) \right] \right) \Big\rangle_{\overline{H_0}(t)}. \quad (138)$$

Now we will obtain the expected values related to  $v_2(t)$ ,  $v_3(t)$ ,  $v_4(t)$  and  $v_5(t)$ . We recall that the expected value can be calculated as:

$$\langle \overline{H_I}(t) \rangle_{\overline{H_0}(t)} = \frac{\text{Tr} \left( \overline{H_I}(t) e^{-\beta \overline{H_0}(t)} \right)}{\text{Tr} \left( e^{-\beta \overline{H_0}(t)} \right)}. \quad (139)$$

Recall the hamiltonian of  $\overline{H_I}(t)$  then the explicit form of  $\overline{H_I}^2(t)$  is:

$$\overline{H_I}^2(t) = \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i B_{iz}(t) |i\rangle\langle i| (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \sum_i B_{iz}(t) |i\rangle\langle i| (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (140)$$

$$+ V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i B_{iz}(t) |i\rangle\langle i| + \left( V_{10}^{\Re}(t) \right)^2 (\sigma_x B_x(t) + \sigma_y B_y(t))^2 + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x B_x(t) \quad (141)$$

$$+ \sigma_y B_y(t)) (\sigma_x B_y(t) - \sigma_y B_x(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) V_{10}^{\Im}(t) (\sigma_x B_y(t) \quad (142)$$

$$- \sigma_y B_x(t)) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \left( V_{10}^{\Im}(t) \right)^2 (\sigma_x B_y(t) - \sigma_y B_x(t))^2 \quad (143)$$

$$= \sum_i B_{iz}^2(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \sum_i (B_{iz}(t) B_x(t) |i\rangle\langle i| \sigma_x + B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_y) + V_{10}^{\Im}(t) \sum_i (B_{iz}(t) B_y(t) |i\rangle\langle i| \sigma_x \quad (144)$$

$$-B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y + V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2(\sigma_x^2B_x^2(t) + \sigma_x\sigma_yB_x(t)B_y(t)) \quad (145)$$

$$+ \sigma_y\sigma_xB_y(t)B_x(t) + \sigma_y^2B_y^2(t)) + V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2(\sigma_x^2B_y^2(t) + \sigma_y^2B_x^2(t)) \quad (146)$$

$$- \sigma_x\sigma_yB_y(t)B_x(t) - \sigma_y\sigma_xB_x(t)B_y(t)) + V_{10}^{\Re}(t)V_{10}^{\Im}(t)(\sigma_x^2B_y(t)B_x(t) + \sigma_x\sigma_yB_y^2(t) - \sigma_y\sigma_xB_x^2(t) - \sigma_y^2B_x(t)B_y(t)) \quad (147)$$

$$+ \sigma_x^2B_x(t)B_y(t) - \sigma_x\sigma_yB_x^2(t) + \sigma_y\sigma_xB_y^2(t) - \sigma_y^2B_y(t)B_x(t)), \quad (148)$$

$$\sigma_x\sigma_y = -i\sigma_z, \quad (149)$$

$$\overline{H_I}^2(t) = \sum_i B_{iz}^2(t)|i\rangle\langle i| + V_{10}^{\Re}(t)\sum_i(B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_x + B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Im}(t)\sum_i(B_{iz}(t)B_y(t)|i\rangle\langle i|\sigma_x \quad (150)$$

$$- B_{iz}(t)B_x(t)|i\rangle\langle i|\sigma_y) + V_{10}^{\Re}(t)\sum_i(\sigma_x|i\rangle\langle i|B_x(t)B_{iz}(t) + \sigma_y|i\rangle\langle i|B_y(t)B_{iz}(t)) + \left(V_{10}^{\Re}(t)\right)^2(B_x^2(t) - i\sigma_zB_x(t)B_y(t)) \quad (151)$$

$$+ i\sigma_zB_y(t)B_x(t) + B_y^2(t)) + V_{10}^{\Im}(t)\sum_i(\sigma_x|i\rangle\langle i|B_y(t)B_{iz}(t) - \sigma_y|i\rangle\langle i|B_x(t)B_{iz}(t)) + \left(V_{10}^{\Im}(t)\right)^2(B_y^2(t) + B_x^2(t)) \quad (152)$$

$$+ i\sigma_zB_y(t)B_x(t) - i\sigma_zB_x(t)B_y(t)). \quad (153)$$

In order to obtain the expected values of  $\langle \overline{H_I}^2(t) \rangle_{\overline{H_0}(t)}$  respect to the part related to the bath we need to calculate the following expected values  $\langle B_{iz}^2(t) \rangle_{\overline{H_B}}, \langle B_{iz}(t)B_x(t) \rangle_{\overline{H_B}}, \langle B_{iz}(t)B_y(t) \rangle_{\overline{H_B}}, \langle B_x(t)B_{iz}(t) \rangle_{\overline{H_B}}, \langle B_y(t)B_{iz}(t) \rangle_{\overline{H_B}}, \langle B_x^2(t) \rangle_{\overline{H_B}}, \langle B_x(t)B_y(t) \rangle_{\overline{H_B}}, \langle B_y(t)B_x(t) \rangle_{\overline{H_B}}, \langle B_y^2(t) \rangle_{\overline{H_B}}$ :

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \left\langle \left( \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}) \right)^2 \right\rangle_{\overline{H_B}} \quad (154)$$

$$= \left\langle \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*b_{\mathbf{k}})^2 + \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}) \sum_{\mathbf{k}'} ((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^*b_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (155)$$

$$- v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^*b_{\mathbf{k}'})) \right\rangle_{\overline{H_B}} \quad (156)$$

$$= \left\langle \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*b_{\mathbf{k}})^2 \right\rangle_{\overline{H_B}} + \left\langle \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*b_{\mathbf{k}}) \right\rangle_{\overline{H_B}} \quad (157)$$

$$\times \left\langle \sum_{\mathbf{k}'} ((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^*b_{\mathbf{k}'}) \right\rangle_{\overline{H_B}} \quad (158)$$

$$= \left\langle \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*b_{\mathbf{k}})^2 \right\rangle_{\overline{H_B}} \quad (159)$$

$$= \sum_{\mathbf{k}} \left\langle (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 (b_{\mathbf{k}}^\dagger)^2 + |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 (b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger) + ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*)^2 b_{\mathbf{k}}^2 \right\rangle_{\overline{H_B}} \quad (160)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^2 \left\langle (b_{\mathbf{k}}^\dagger)^2 \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \left\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} \left\langle ((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^*)^2 b_{\mathbf{k}}^2 \right\rangle_{\overline{H_B}} \quad (161)$$

$$\left\langle (b_{\mathbf{k}}^\dagger)^2 \right\rangle_{\overline{H_B}} = \frac{\text{Tr} \left( (b_{\mathbf{k}}^\dagger)^2 \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (162)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger)^2 |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (163)$$

$$= \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}} \sqrt{(j_{\mathbf{k}}+2)(j_{\mathbf{k}}+1)} |j_{\mathbf{k}}+2\rangle\langle j_{\mathbf{k}}| \right)}{f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (164)$$



$$= 0, \quad (165)$$

$$\langle b_{\mathbf{k}}^2 \rangle_{\overline{H_B}} = \frac{\text{Tr} \left( \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \sqrt{j_{\mathbf{k}} (j_{\mathbf{k}} - 1)} |j_{\mathbf{k}} - 2\rangle \langle j_{\mathbf{k}}| \right)}{f^{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \quad (166)$$

$$= 0, \quad (167)$$

$$\langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} = (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( (b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger) \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (168)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (169)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \text{Tr} \left( \sum_{j_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}}\rangle \langle j_{\mathbf{k}}| \right) \quad (170)$$

$$= (1 - e^{-\beta \omega_{\mathbf{k}}}) \sum_{j_{\mathbf{k}}} (2j_{\mathbf{k}} + 1) e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \quad (171)$$

$$= \frac{1 + e^{-\beta \omega_{\mathbf{k}}}}{1 - e^{-\beta \omega_{\mathbf{k}}}} \quad (172)$$

$$= \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right), \quad (173)$$

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right), \quad (174)$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} = \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_B}} \quad (175)$$

$$= \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left( e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) + e^{\chi_{01}(t)} \right. \right. \quad (176)$$

$$\left. \times \prod_{\mathbf{k}'} D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}}, \quad (177)$$

$$\langle b^\dagger D(h) \rangle_{\overline{H_B}} = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | b^\dagger D(h) | \alpha \rangle d^2 \alpha \quad (178)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) b^\dagger D(\alpha) D(-\alpha) D(h) D(\alpha) | 0 \rangle d^2 \alpha \quad (179)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) b^\dagger D(\alpha) D(h) e^{h\alpha^* - h^* \alpha} | 0 \rangle d^2 \alpha \quad (180)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | (b^\dagger + \alpha^*) D(h) e^{h\alpha^* - h^* \alpha} | 0 \rangle d^2 \alpha \quad (181)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \langle 0 | (b^\dagger + \alpha^*) | h \rangle d^2 \alpha \quad (182)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \langle 0 | (b^\dagger + \alpha^*) | h \rangle d^2 \alpha, \quad (183)$$

$$| \alpha \rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle, \quad (184)$$

$$\langle b^\dagger D(h) \rangle_{\overline{H_B}} = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \left( \langle 0 | b^\dagger | h \rangle + \alpha^* \langle 0 | h \rangle \right) d^2 \alpha \quad (185)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \left( \langle 0 | b^\dagger e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle + \alpha^* \langle 0 | h \rangle \right) d^2 \alpha \quad (186)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \left( \langle 0 | e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1} | n+1 \rangle + \alpha^* \langle 0 | h \rangle \right) d^2 \alpha \quad (187)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \alpha^* \langle 0 | h \rangle d^2 \alpha \quad (188)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \alpha^* e^{-\frac{|h|^2}{2}} d^2 \alpha \quad (189)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{N}} e^{h(x-iy) - h^*(x+iy)} (x-iy) dx dy \quad (190)$$

$$= -h^* N \left( \langle D(h) \rangle_{\overline{H_B}} \right)^2 \quad (191)$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} = \frac{1}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \left( e^{\chi_{10}(t)} \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right. \right. \quad (192)$$

$$\left. + e^{\chi_{01}(t)} \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right) \right\rangle_{\overline{H_B}} \quad (193)$$

$$= \frac{1}{2} \left\langle e^{\chi_{10}(t)} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right. \quad (194)$$

$$\left. + e^{\chi_{01}(t)} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right) \right\rangle_{\overline{H_B}} \quad (195)$$

$$= \frac{e^{\chi_{10}(t)}}{2} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \quad (196)$$

$$\left. \times \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \right) \quad (197)$$

$$+ \frac{e^{\chi_{01}(t)}}{2} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \quad (198)$$

$$\left. \times \left\langle b_{\mathbf{k}} \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \right\rangle_{\overline{H_B}} \right) \quad (199)$$

$$= \frac{B_{10}(t)}{2} \left( - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right. \quad (200)$$

$$\left. + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (201)$$

$$+ \frac{B_{01}(t)}{2} \left( - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right. \quad (202)$$

$$\left. + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (203)$$

$$= \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left( - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right. \quad (204)$$

$$\left. + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\frac{\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (205)$$

$$\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} = \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right\rangle_{\overline{H_B}} \quad (206)$$

$$= \frac{1}{2i} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \right\rangle_{\overline{H_B}} \quad (207)$$

$$= \frac{B_{10}(t)}{2i} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \sum_{\mathbf{k}} (g_{i\mathbf{k}} \right. \quad (208)$$

$$\left. - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (209)$$

$$+ \frac{B_{01}(t)}{2i} \left( - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right. \quad (210)$$

$$\left. + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (211)$$

$$= \frac{B_{10}(t) + B_{01}(t)}{2i} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right. \quad (212)$$

$$\left. - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right), \quad (213)$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} = \langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} \quad (214)$$

$$= \left\langle \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (215)$$

$$= \frac{1}{2} \left\langle (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_B}} \quad (216)$$

$$= \frac{1}{2} \left\langle e^{\chi_{10}(t)} \prod_{\mathbf{k}'} D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_B}} \quad (217)$$

$$+ \frac{1}{2} \left\langle e^{\chi_{01}(t)} \prod_{\mathbf{k}'} D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_B}}, \quad (218)$$

$$\langle D(h) b \rangle_{\overline{H_B}} = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | D(h) b | \alpha \rangle d^2 \alpha \quad (219)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle d^2 \alpha \quad (220)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) e^{h\alpha^* - h^* \alpha} (b + \alpha) | 0 \rangle d^2 \alpha \quad (221)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) e^{h\alpha^* - h^* \alpha} \alpha | 0 \rangle d^2 \alpha \quad (222)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \alpha \langle 0 | D(h) | 0 \rangle d^2 \alpha \quad (223)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \alpha \langle 0 | h \rangle d^2 \alpha \quad (224)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int \alpha e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} d^2 \alpha \quad (225)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{N}} e^{h(x-iy) - h^*(x+iy)} (x+iy) dx dy \quad (226)$$

$$= N h e^{-|h|^2 \coth(\frac{\beta\omega}{2})} \quad (227)$$

$$= N h \langle D(h) \rangle_{\overline{H_B}}^2, \quad (228)$$

$$\langle D(h) b^\dagger \rangle_{\overline{H_B}} = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | D(h) b^\dagger | \alpha \rangle d^2 \alpha \quad (229)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) | 0 \rangle d^2 \alpha \quad (230)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) e^{h\alpha^* - h^* \alpha} (b^\dagger + \alpha^*) | 0 \rangle d^2 \alpha \quad (231)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \langle 0 | D(h) b^\dagger | 0 \rangle d^2 \alpha + \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \alpha^* e^{h\alpha^* - h^* \alpha} \langle 0 | D(h) | 0 \rangle d^2 \alpha \quad (232)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} \langle -h | 1 \rangle d^2 \alpha + \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \alpha^* e^{h\alpha^* - h^* \alpha} d^2 \alpha, \quad (233)$$

$$\langle \alpha | = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \langle n |, \quad (234)$$

$$\langle D(h) b^\dagger \rangle_{\overline{H_B}} = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} e^{-\frac{|h|^2}{2}} (-h^*) d^2 \alpha + \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \alpha^* e^{h\alpha^* - h^* \alpha} d^2 \alpha \quad (235)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{h\alpha^* - h^* \alpha} (-h^* + \alpha^*) d^2 \alpha \quad (236)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{N}} e^{h(x-iy) - h^*(x+iy)} (-h^* + x - iy) dx dy \quad (237)$$

$$= -(N+1) h^* e^{-|h|^2 \coth(\frac{\beta\omega}{2})}, \quad (238)$$

$$= -(N+1) h^* \langle D(h) \rangle_{\overline{H_B}}^2, \quad (239)$$

$$\langle D(h) \rangle_{\overline{H_B}} = e^{-\frac{|h|^2}{2} \coth(\frac{\beta\omega}{2})}, \quad (240)$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} = \frac{e^{\chi_{10}(t)}}{2} \left\langle \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_B}} \quad (241)$$

$$+ \frac{e^{\chi_{01}(t)}}{2} \left\langle \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) \left( \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \right\rangle_{\overline{H_B}} \quad (242)$$

$$= \frac{e^{\chi_{10}(t)}}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right. \right. \right. \quad (243)$$

$$\left. \left. - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} + \frac{e^{\chi_{01}(t)}}{2} \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \right. \right. \quad (244)$$

$$\left. \left. \times \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right) \right\rangle_{\overline{H_B}} \quad (245)$$

$$= \frac{e^{\chi_{10}(t)}}{2} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \prod_{\mathbf{k}'} \left( D \left( \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right. \right. \quad (246)$$

$$\left. \left. - \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right) + \frac{e^{\chi_{01}(t)}}{2} \left( \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left\langle \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} \right. \quad (247)$$

$$\left. \left. + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left\langle \prod_{\mathbf{k}'} \left( D \left( \frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} - \frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right) \right) \quad (248)$$

$$= \frac{B_{10}(t)}{2} \left( \sum_{\mathbf{k}} e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}^2} \coth(\frac{\beta\omega_{\mathbf{k}}}{2})} \left( -(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (N_{\mathbf{k}} + 1) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right. \right. \quad (249)$$



$$= \text{Var}_{\overline{H_B}} \left( \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (272)$$

$$= \frac{1}{4} \text{Var}_{\overline{H_B}} (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)) \quad (273)$$

$$= \frac{1}{4} \text{Var}_{\overline{H_B}} (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t)) \quad (274)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (275)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 + B_1^+(t) B_0^-(t) B_0^+(t) B_1^-(t) + B_0^+(t) B_1^-(t) B_1^+(t) B_0^-(t) + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} \right. \quad (276)$$

$$\left. - (B_{10}(t) + B_{01}(t))^2 \right) \quad (277)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 + 2\mathbb{I} + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right), \quad (278)$$

$$(D(h))^2 = D(h) D(h) \quad (279)$$

$$= D(h+h) e^{\frac{1}{2} \left( \frac{h^* h - h h^*}{\omega^2} \right)} \quad (280)$$

$$= D(2h), \quad (281)$$

$$\left\langle (B_i^+(t) B_j^-(t))^2 \right\rangle_{\overline{H_B}} = \left\langle \left( \prod_{\mathbf{k}} D \left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right)^2 \right\rangle_{\overline{H_B}} \quad (282)$$

$$= \left\langle \prod_{\mathbf{k}} D \left( 2 \left( \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} \right\rangle_{\overline{H_B}} \quad (283)$$

$$= \prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} e^{-2 \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (284)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 + 2\mathbb{I} + (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (285)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} + 2 + \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (286)$$

$$= \frac{1}{4} \left( \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} + 2 + \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{10}(t) + B_{01}(t))^2 \right) \quad (287)$$

$$= \frac{1}{4} \left( e^{2\chi_{10}(t)} \prod_{\mathbf{k}} e^{-2 \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)} + 2 + e^{2\chi_{01}(t)} \prod_{\mathbf{k}} e^{-2 \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)} \right. \quad (288)$$

$$\left. - (B_{10}(t) + B_{01}(t))^2 \right) \quad (289)$$

$$= \frac{1}{4} (B_{10}^2(t) |B_{10}^2(t)| + 2 + B_{01}^2(t) |B_{01}^2(t)| - (B_{10}^2(t) + 2B_{10}(t) B_{01}(t) + B_{01}^2(t))) \quad (290)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = \text{Var}_{\overline{H_B}} (B_y(t)) + \langle B_y(t) \rangle_{\overline{H_B}}^2 \quad (291)$$

$$= \text{Var}_{\overline{H_B}} \left( \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right) \quad (292)$$

$$= -\frac{1}{4} \text{Var}_{\overline{H_B}} (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)) \quad (293)$$

$$= -\frac{1}{4} \text{Var}_{\overline{H_B}} (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t)) \quad (294)$$

$$= -\frac{1}{4} \left( \left\langle (B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} - (B_{01}(t) - B_{10}(t))^2 \right) \quad (295)$$

$$= -\frac{1}{4} \left( \left\langle (B_0^+(t) B_1^-(t))^2 - 2\mathbb{I} + (B_1^+(t) B_0^-(t))^2 - (B_{01}(t) - B_{10}(t))^2 \right\rangle_{\overline{H_B}} \right) \quad (296)$$

$$= -\frac{1}{4} \left( \left\langle (B_0^+(t) B_1^-(t))^2 \right\rangle_{\overline{H_B}} + \left\langle (B_1^+(t) B_0^-(t))^2 \right\rangle_{\overline{H_B}} - 2 - (B_{01}(t) - B_{10}(t))^2 \right), \quad (297)$$

$$\left\langle (B_i^+(t) B_j^-(t))^2 \right\rangle_{\overline{H_B}} = \prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2}} e^{-2 \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (298)$$

$$= \left( \prod_{\mathbf{k}} e^{\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2}} e^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^2 \left( \prod_{\mathbf{k}} e^{-\frac{\left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \right)^2 \quad (299)$$

$$= B_{ij}^2(t) |B_{ij}(t)|^2, \quad (300)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = -\frac{1}{4} (B_{01}^2(t) |B_{10}(t)|^2 - 2 + B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}(t) - B_{10}(t))^2), \quad (301)$$

$$\langle B_x(t) B_y(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \right\rangle_{\overline{H_B}} \quad (302)$$

$$= \frac{1}{4i} \langle (B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)) (B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)) \rangle_{\overline{H_B}} \quad (303)$$

$$= \frac{1}{4i} \langle \mathbb{I} - (B_1^+(t)B_0^-(t))^2 + B_{10}^2(t) - B_{10}(t)B_{01}(t) + (B_0^+(t)B_1^-(t))^2 - \mathbb{I} + B_{10}(t)B_{01}(t) - B_{01}^2(t) \rangle_{\overline{H_B}} \quad (304)$$

$$= \frac{1}{4i} \langle (B_0^+(t)B_1^-(t))^2 - (B_1^+(t)B_0^-(t))^2 - (B_{01}^2(t) - B_{10}^2(t)) \rangle_{\overline{H_B}} \quad (305)$$

$$= \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}^2(t) - B_{10}^2(t))), \quad (306)$$

$$\langle B_y(t) B_x(t) \rangle_{\overline{H_B}} = \left\langle \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right\rangle_{\overline{H_B}} \quad (307)$$

$$= \frac{1}{4i} \langle (B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)) (B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)) \rangle_{\overline{H_B}} \quad (308)$$

$$= \frac{1}{4i} \langle \mathbb{I} + (B_0^+(t)B_1^-(t))^2 - B_{10}(t)B_{01}(t) - B_{01}^2(t) - (B_1^+(t)B_0^-(t))^2 - \mathbb{I} + B_{10}^2(t) + B_{10}(t)B_{01}(t) \rangle_{\overline{H_B}} \quad (309)$$

$$= \frac{1}{4i} \langle (B_0^+(t)B_1^-(t))^2 - B_{01}^2(t) - (B_1^+(t)B_0^-(t))^2 + B_{10}^2(t) \rangle_{\overline{H_B}} \quad (310)$$

$$= \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{01}^2(t) - (B_{10}^2(t) |B_{10}(t)|^2 - B_{10}^2(t))). \quad (311)$$

The density matrix associated to  $\rho_{\overline{S}} = \frac{e^{-\beta \overline{H_0}(t)}}{\text{Tr}(e^{-\beta \overline{H_0}(t)})}$  follows the form:

$$\rho_{\overline{S},00} = \frac{1}{2} + \frac{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{2 \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}} \quad (312)$$

$$\rho_{\overline{S},01} = - \frac{B_{10}^*(t) V_{10}^*(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}} \quad (313)$$

$$\rho_{\overline{S},10} = - \frac{B_{10}(t) V_{10}(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}} \quad (314)$$

$$\rho_{\overline{S},11} = \frac{1}{2} - \frac{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{2 \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}} \quad (315)$$

The expected values respect to the system  $\overline{S}$  of relevance for calculating  $\langle \overline{H_I}^2(t) \rangle_{H_{\overline{S}}}$  are  $\langle |i\rangle\langle i| \rangle_{H_{\overline{S}}}$ ,  $\langle |i\rangle\langle i| \sigma_x \rangle_{H_{\overline{S}}}$ ,  $\langle |i\rangle\langle i| \sigma_y \rangle_{H_{\overline{S}}}$ ,  $\langle \sigma_x |i\rangle\langle i| \rangle_{H_{\overline{S}}}$ ,  $\langle \sigma_y |i\rangle\langle i| \rangle_{H_{\overline{S}}}$  and  $\langle \sigma_z \rangle_{H_{\overline{S}}}$ , we took account that  $\sigma_x \sigma_y = -i \sigma_z$  and  $\sigma_y \sigma_x = i \sigma_z$ . The values needed for our calculation are:

$$\langle |0\rangle\langle 0| \rangle_{\overline{H_S(t)}} = \frac{1}{2} - \frac{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{2 \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (316)$$

$$\langle |1\rangle\langle 1| \rangle_{\overline{H_S(t)}} = \frac{1}{2} + \frac{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{2 \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (317)$$

$$\langle |0\rangle\langle 0| \sigma_x \rangle_{\overline{H_S(t)}} = - \frac{B_{10}(t) V_{10}(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (318)$$

$$\langle |1\rangle\langle 1| \sigma_x \rangle_{\overline{H_S(t)}} = - \frac{B_{10}^*(t) V_{10}^*(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (319)$$

$$\langle |0\rangle\langle 0| \sigma_y \rangle_{\overline{H_S(t)}} = - \frac{i B_{10}(t) V_{10}(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (320)$$

$$\langle |1\rangle\langle 1| \sigma_y \rangle_{\overline{H_S(t)}} = - \frac{i B_{10}^*(t) V_{10}^*(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t)|^2 |V_{10}(t)|^2}}, \quad (321)$$

$$\langle \sigma_x | 0\rangle\langle 0| \rangle_{\overline{H_S(t)}} = - \frac{B_{10}^*(t) V_{10}^*(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (322)$$

$$\langle \sigma_x | 1\rangle\langle 1| \rangle_{\overline{H_S(t)}} = - \frac{B_{10}(t) V_{10}(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (323)$$

$$\langle \sigma_y | 0\rangle\langle 0| \rangle_{\overline{H_S(t)}} = - \frac{i B_{10}^*(t) V_{10}^*(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (324)$$

$$\langle \sigma_y | 1\rangle\langle 1| \rangle_{\overline{H_S(t)}} = - \frac{i B_{10}(t) V_{10}(t) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}, \quad (325)$$

$$\langle \sigma_z \rangle_{\overline{H_S(t)}} = - \frac{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right) \tanh \left( \frac{\beta}{2} \sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2} \right)}{\sqrt{\left( \sum_i (-1)^i (\varepsilon_i(t) + R_i(t)) \right)^2 + 4 |B_{10}(t) V_{10}(t)|^2}}. \quad (326)$$



Summarizing the expected values of the bath we have:

$$\langle B_{iz}^2(t) \rangle_{\overline{H_B}} = \sum_{\mathbf{k}} |g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (327)$$

$$\langle B_{iz}(t) B_x(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) - B_{01}(t)}{2} \sum_{\mathbf{k}} \left( e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left( -(g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right. \right. \quad (328)$$

$$\left. + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right), \quad (329)$$

$$\langle B_{iz}(t) B_y(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} \left( e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right. \right. \quad (330)$$

$$\left. - \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) \right), \quad (331)$$

$$\langle B_x(t) B_{iz}(t) \rangle_{\overline{H_B}} = \frac{B_{01}(t) - B_{10}(t)}{2} \sum_{\mathbf{k}} \left( e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (N_{\mathbf{k}} + 1) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right. \right. \quad (332)$$

$$\left. - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (333)$$

$$\langle B_y(t) B_{iz}(t) \rangle_{\overline{H_B}} = \frac{B_{10}(t) + B_{01}(t)}{2i} \sum_{\mathbf{k}} \left( e^{-\frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}}} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (N_{\mathbf{k}} + 1) \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right. \right. \quad (334)$$

$$\left. - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* N_{\mathbf{k}} \left( \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (335)$$

$$\langle B_x^2(t) \rangle_{\overline{H_B}} = \frac{1}{4} (B_{10}^2(t) |B_{10}^2(t)| + 2 + B_{01}^2(t) |B_{01}^2(t)| - (B_{10}(t) + B_{01}(t))^2), \quad (336)$$

$$\langle B_y^2(t) \rangle_{\overline{H_B}} = -\frac{1}{4} (B_{01}^2(t) |B_{10}(t)|^2 - 2 + B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}(t) - B_{10}(t))^2), \quad (337)$$

$$\langle B_x(t) B_y(t) \rangle_{\overline{H_B}} = \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}^2(t) - B_{10}^2(t))), \quad (338)$$

$$\langle B_y(t) B_x(t) \rangle_{\overline{H_B}} = \frac{1}{4i} (B_{01}^2(t) |B_{10}(t)|^2 - B_{10}^2(t) |B_{10}(t)|^2 - (B_{01}^2(t) - B_{10}^2(t))). \quad (339)$$

Our next step is to find  $v_3(t)$ , the commutator  $[\overline{H_0}(t), \overline{H_I}(t)]$  is a central point for our calculations and it is equal to:

$$[\overline{H_0}(t), \overline{H_I}(t)] = [(\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) \quad (340)$$

$$+ B_{10}^{\Im}(t) V_{10}^{\Re}(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (341)$$

$$= \left[ \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \right. \quad (342)$$

$$\left. + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \right] \quad (343)$$

$$= \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (344)$$

$$+ \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) \sum_i B_{iz}(t) |i\rangle\langle i| + \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \quad (345)$$

$$+ \sigma_x (B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t)) V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Re}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)) \sum_i B_{iz}(t) |i\rangle\langle i| \quad (346)$$

$$- \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Re}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t)) V_{10}^{\Im}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) - \sigma_y (B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Im}(t)) V_{10}^{\Re}(t) (\sigma_x B_y(t) \quad (347)$$

$$-\sigma_y B_x(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (348)$$

$$- \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (349)$$

$$- \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t) (\sigma_x B_x(t) \quad (350)$$

$$+ \sigma_y B_y(t)) \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Re}(t) (\sigma_x B_x(t) \quad (351)$$

$$+ \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) \quad (352)$$

$$- B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \quad (353)$$

$$= \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (354)$$

$$+ \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| + \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Re}(t) (B_x(t) - i\sigma_z B_y(t)) \quad (355)$$

$$+ \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) V_{10}^{\Im}(t) (B_y(t) + i\sigma_z B_x(t)) - \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \sum_i B_{iz}(t) |i\rangle\langle i| \quad (356)$$

$$- \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Re}(t) (i\sigma_z B_x(t) + B_y(t)) - \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) V_{10}^{\Im}(t) (i\sigma_z B_y(t) - B_x(t)) \quad (357)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \sum_i B_{iz}(t) |i\rangle\langle i| + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (358)$$

$$- \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + \sum_i B_{iz}(t) |i\rangle\langle i| \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (359)$$

$$- \sum_i B_{iz}(t) |i\rangle\langle i| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Re}(t) (B_x(t) + i\sigma_z B_y(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) \quad (360)$$

$$- B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) + V_{10}^{\Re}(t) (-i\sigma_z B_x(t) + B_y(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (361)$$

$$- V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_i (\varepsilon_i(t) + R_i(t)) |i\rangle\langle i| - V_{10}^{\Im}(t) (B_y(t) - i\sigma_z B_x(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) \quad (362)$$

$$+ V_{10}^{\Im}(t) (-i\sigma_z B_y(t) - B_x(t)) \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) - V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (363)$$

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