The Mother of all Master Equations

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H_T(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \sum_i \varepsilon_i(t) |i\rangle\langle i| + \sum_{i\neq j} V_{ij}(t) |i\rangle\langle j|, \tag{2}$$

$$H_I = \sum_i |i\rangle\langle i| \sum_{\mathbf{k}} \left(g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} \right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \tag{4}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V(t)}$, where $V(t)$ is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv \sum_i |i\rangle\langle i| \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) \tag{5}$$

in terms of the variational scalar parameters $\{v_{\mathbf{k}}\}$, which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. Operators $O(t)$ in the variational frame will be written as:

$$\overline{O}(t) \equiv \mathrm{e}^{V(t)} O(t) \mathrm{e}^{-V(t)}. \tag{6}$$

We assume that the bath starts equilibrium with inverse temperature $\beta = 1/k_{\mathrm{B}}T$:

$$\rho_B \equiv \rho_B(0) \tag{7}$$

$$= \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)}. \tag{8}$$

With the following definitions:

$$\begin{pmatrix} B_{iz}(t) & B_i^{\pm}(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} & e^{-\frac{1}{2}\sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth(\beta\omega_{\mathbf{k}}/2)} \\ \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2\mathrm{i}} & e^{-\frac{1}{2}\sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\chi_{ij}(t)} \end{pmatrix}, \tag{9}$$

$$R_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \tag{10}$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2} \right), \tag{11}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{12}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{13}$$

we may write the transformed Hamiltonian as a sum of the form:

$$\overline{H_T}(t) \equiv \overline{H_{\bar{S}}}(t) + \overline{H_{\bar{I}}}(t) + \overline{H_{\bar{B}}}, \tag{14}$$

$$\overline{H_{\bar{S}}}(t) \equiv \sum_i (\varepsilon_i(t) + R_i(t))|i\rangle\langle i| + \sigma_x\left(B_{10}^{\Re}(t)V_{10}^{\Re}(t) - B_{10}^{\Im}(t)V_{10}^{\Im}(t)\right) - \sigma_y\left(B_{10}^{\Re}(t)V_{10}^{\Im}(t) + B_{10}^{\Im}(t)V_{10}^{\Re}(t)\right), \tag{15}$$

$$\overline{H_{\bar{I}}}(t) \equiv \sum_i B_{iz}(t)|i\rangle\langle i| + V_{10}^{\Re}(t)(B_x(t)\sigma_x + B_y(t)\sigma_y) - V_{10}^{\Im}(t)(B_x(t)\sigma_y - B_y(t)\sigma_x), \tag{16}$$

$$\overline{H_{\bar{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \tag{17}$$

$$= H_B. \tag{18}$$

III. FREE-ENERGY MINIMIZATION

The true free energy $E_{\mathrm{Free}}(t)$ is bounded by the Bogoliubov inequality:

$$E_{\mathrm{Free}}(t) \leq E_{\mathrm{Free,B}}(t) \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t) + H_{\bar{B}}}}\right)\right) + \left\langle \overline{H_{\bar{I}}}(t) \right\rangle_{\overline{H_{\bar{S}}(t) + H_{\bar{B}}}} + O\left(\left\langle \overline{H_{\bar{I}}}^2(t) \right\rangle_{\overline{H_{\bar{S}}(t) + H_{\bar{B}}}}\right). \tag{19}$$

We will optimize the set of variational parameters $\{v_{\mathbf{k}}(t)\}$ in order to minimize $E_{\mathrm{Free,B}}(t)$ (i.e. to make it as close to the true free energy $E_{\mathrm{Free}}(t)$ as possible). Neglecting the higher order terms and using $\left\langle \overline{H_{\bar{I}}}(t) \right\rangle_{\overline{H_{\bar{S}}(t) + H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{\mathbf{k}}(t)\}$:

$$\frac{\partial A_{\mathrm{B}}(\{v_{\mathbf{k}}(t)\}; t)}{\partial v_{i\mathbf{k}}(t)} = 0. \tag{20}$$

This leads us to:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}, t) = \frac{g_i(\omega_{\mathbf{k}})\left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t))\right) + 2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}}|B_{10}(t)|^2|V_{10}(t)|^2\coth(\beta\omega_{\mathbf{k}}/2)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|B_{10}(t)|^2|V_{10}(t)|^2\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)}, \tag{21}$$

if $i = 1$ then $i' = 0$ and viceversa. Also we have the following definitions:

$$\eta(t) \equiv \sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right)\right)^2 - 4\mathrm{Det}\left(\overline{H_{\bar{S}}}(t)\right)}, \tag{22}$$

$$\varepsilon(t) \equiv \mathrm{Tr}\left(\overline{H_{\bar{S}}}(t)\right). \tag{23}$$

IV. MASTER EQUATION

We transform any operator $O(t)$ into the interaction picture in the following way:

$$\widetilde{O}(t) \equiv U^{\dagger}(t)\, O(t)\, U(t)\,, \tag{24}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_T}(t')\right) \tag{25}$$

$$= \exp\left(-\mathrm{i}\overline{H_{T,\mathrm{eff}}}(t)\right), \text{ where} \tag{26}$$

$$H_{X,\mathrm{eff}}(t) \equiv \frac{1}{t}\int_0^t H_X(t')\,\mathrm{d}t' - \frac{\mathrm{i}}{2t}\int_0^t\int_0^{t'} \left[H_X(t'), H_X(t'')\right]\mathrm{d}t'\mathrm{d}t'' \tag{27}$$

here we used a perturbative expansion of $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_T}(t')\right)$.
Therefore:

$$\widetilde{\overline{\rho_{\bar{S}}}}(t) = U^{\dagger}(t)\,\overline{\rho_{\bar{S}}}(t)\,U(t)\,. \tag{28}$$

We will initialize the density operator as: $\rho_{\mathrm{Total}}(0) = \rho_{\bar{S}}(0)\otimes\rho_B(0)$, where $\rho_B(0) \equiv \rho_B^{\mathrm{Thermal}} \equiv \rho_B$. Taking as reference state ρ_B and truncating at second order in $\overline{H_{\bar{I}}}(t)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_{\bar{S}}}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(t'), \widetilde{\overline{\rho_{\bar{S}}}}(t)\,\rho_B\right]\right]\mathrm{d}t'. \tag{29}$$

To simplify this we define the following matrix related to describe $\overline{H_{\bar{I}}}(t)$:

$$\begin{pmatrix} A \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}, \tag{30}$$

$$\overline{H_{\bar{I}}}(t) = \sum_i C_i(t)\left(A_i \otimes B_i(t)\right), \tag{31}$$

$$\widetilde{\overline{H_{\bar{I}}}}(t) = \sum_i C_i(t)\left(\widetilde{A_i}(t)\otimes\widetilde{B_i}(t)\right). \tag{32}$$

Taking the master equation (29) and expanding the commutators yields:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_{\bar{S}}}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\Bigg(\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_{\bar{S}}}}(t)\rho_B - \sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\widetilde{\overline{\rho_{\bar{S}}}}(t)\rho_B\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right) \tag{33}$$

$$-\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\widetilde{\overline{\rho_{\bar{S}}}}(t)\rho_B\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right) + \widetilde{\overline{\rho_{\bar{S}}}}(t)\rho_B\sum_i C_i(t')\left(\widetilde{A_i}(t')\otimes\widetilde{B_i}(t')\right)\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right)\Bigg)\,\mathrm{d}t'. \tag{34}$$

We can keep the A and $C(t)$ as they are when tracing over the bath degrees of freedom, but we will replace the expected value of the $B(t)$ operators, known as correlation functions, by $\mathscr{B}(t,t')$ such that:

$$\mathscr{B}_{ij}(t,t') \equiv \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(t')\,\rho_B\right). \tag{35}$$

This allows us to remove the trace over the bath and write down a more tangible master equation:

$$\frac{\mathrm{d}\widetilde{\overline{\rho_{\bar{S}}}}(t)}{\mathrm{d}t} = -\int_0^t\left(\sum_{ij} C_i(t)\,C_j(t')\left(\mathscr{B}_{ij}(t,t')\left[\widetilde{A_i}(t), \widetilde{A_j}(t')\widetilde{\overline{\rho_{\bar{S}}}}(t)\right] - \mathscr{B}_{ij}^{*}(t,t')\left[\widetilde{A_i}(t), \widetilde{\overline{\rho_{\bar{S}}}}(t)\,\widetilde{A_j}(t')\right]\right)\right)\mathrm{d}t'. \tag{36}$$

here we considered the following notation:

$$\widetilde{A_j}\left(t',t\right)=U\left(t\right)U^{\dagger}\left(t'\right)A_jU\left(t'\right)U^{\dagger}\left(t\right). \tag{37}$$

Given that $t'=t-\tau$ then we can perform the change of variables in the integral of the equation (36), also doing the reverse of the transformation to interaction picture we get:

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ij}\int_0^t\mathrm{d}\tau C_i(t)\,C_j(t-\tau)\left(\mathscr{B}_{ij}(t,t-\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_{\bar{S}}}\left(t\right)\Big]+\mathscr{B}^*_{ij}(t,t-\tau)\Big[\overline{\rho_{\bar{S}}}\left(t\right)\widetilde{A_j}(t-\tau,t)\,,A_i\Big]\right). \tag{38}$$

The Fourier decomposition of the operators $\widetilde{A_i}\left(t\right)$ and $\widetilde{A_j}\left(t-\tau,t\right)$ using the expansion $\overline{H_{\bar{S},\mathrm{eff}}}\left(t\right)$ is:

$$\widetilde{A_i}\left(t\right)=\sum_{w(t)}e^{-\mathrm{i}tw(t)}A_i\left(w\left(t\right)\right). \tag{39}$$

$$\widetilde{A_j}\left(t-\tau,t\right)=\sum_{w(t-\tau),w'(t)}e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_j\left(w\left(t-\tau\right),w'\left(t\right)\right), \tag{40}$$

where $w\left(t\right)$ belongs to the set of differences of eigenvalues of $\overline{H_{\bar{S},\mathrm{eff}}}\left(t\right)$.
Replacing (40) in (38) we deduce that:

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t)\,,\overline{\rho_{\bar{S}}}(t)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(t,t-\tau)\Big[A_i,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}(t-\tau,t)\,\overline{\rho_{\bar{S}}}\left(t\right)\Big] \tag{41}$$

$$+\mathscr{B}^*_{ij}(t,t-\tau)\left[\overline{\rho_{\bar{S}}}(t)\,e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A^{\dagger}_{jww'}(t-\tau,t)\,,A_i\right]\Big). \tag{42}$$

Let's define the operator:

$$D_{ijww'}\left(t-\tau,t\right)\equiv C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(t,t-\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}A_{jww'}\left(t-\tau,t\right). \tag{43}$$

With this notation applied to (42) we arrive to the following master equation:

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijww'}\int_0^t\mathrm{d}\tau\left(\left[A_i,D_{ijww'}\left(t-\tau,t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]-\left[\overline{\rho_{\bar{S}}}\left(t\right)D^{\dagger}_{ijww'}\left(t-\tau,t\right),A_i\right]\right). \tag{44}$$

We define a response matrix $\mathscr{D}_{ijww'}\left(t\right)$ as:

$$\mathscr{D}_{ijww'}\left(t\right)=\int_0^t D_{ijww'}\left(t-\tau,t\right)\mathrm{d}\tau. \tag{45}$$

Finally we end up with our final master equation in the variationally optimized

$$\frac{\mathrm{d}\overline{\rho_{\bar{S}}}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}\left(t\right),\overline{\rho_{\bar{S}}}\left(t\right)\right]-\sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\overline{\rho_{\bar{S}}}\left(t\right)\right]-\left[\overline{\rho_{\bar{S}}}\left(t\right)\mathscr{D}^{\dagger}_{ijww'}\left(t\right),A_i\right]\right) \tag{46}$$

$$\dot{\rho}=-\mathrm{i}\left[H_{\bar{S}}\left(t\right),\rho\right]-\sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}\left(t\right)\rho\right]-\left[\rho\mathscr{D}^{\dagger}_{ijww'}\left(t\right),A_i\right]\right) \tag{47}$$

We will now show that many useful master equations can be derived as special cases of the above "mother" of all master equations.

V. LIMITING CASES

Many limiting cases can be derived from the "mother" of all master equations. We can set $g^{\Im}_{i\mathbf{k}}=0$, or $V^{\Im}_{10}=0$, $g_{1\mathbf{k}}=g_{0\mathbf{k}}$, for example. Let us look at some particular cases.

A. Time-independent VPQME of 2011

The hamiltonian associated to this system is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega}{2}\sigma_{x}+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{48}$$

It's possible to summarize this hamiltonian in terms of the hamiltonian (1) using the following matrix:

$$\begin{pmatrix}V_{10}^{\Im}(t) & g_{0\mathbf{k}}^{\Re} & v_{0\mathbf{k}}(t) & B_{10}(t)\\ V_{10}^{\Re}(t) & g_{0\mathbf{k}}^{\Im} & v_{1\mathbf{k}}(t) & \Omega_{r}\\ \varepsilon_{0}(t) & g_{1\mathbf{k}}^{\Re} & B_{0z}(t) & R_{0}(t)\\ \varepsilon_{1}(t) & g_{1\mathbf{k}}^{\Im} & B_{1z}(t) & R_{1}(t)\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & B\\ \frac{\Omega}{2} & 0 & v_{\mathbf{k}} & B\Omega\\ 0 & g_{\mathbf{k}} & 0 & 0\\ \delta & 0 & B_{z} & R\end{pmatrix}. \tag{49}$$

We now have the corresponding set of hamiltonians that satisfy the separation shown in (14)-(18):

$$\overline{H_{S}}=(\delta+R)|1\rangle\langle 1|+\frac{\Omega_{r}}{2}\sigma_{x}, \tag{50}$$

$$\overline{H_{I}}=B_{z}|1\rangle\langle 1|+\frac{\Omega}{2}\left(B_{x}\sigma_{x}+B_{y}\sigma_{y}\right), \tag{51}$$

$$H_{B}=\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{52}$$

Let's look now at $v_{\mathbf{k}}$:

$$v_{\mathbf{k}}=\frac{g_{i}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}}\right)} \tag{53}$$

$$=\frac{g_{\mathbf{k}}\left(1-\frac{\varepsilon(t)}{\eta}\tanh\left(\frac{\beta\eta}{2}\right)\right)}{1-\frac{\varepsilon(t)}{\eta}\tanh\left(\frac{\beta\eta}{2}\right)\left(1-\frac{\Omega_{r}^{2}}{2\varepsilon(t)\omega_{\mathbf{k}}}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right)}. \tag{54}$$

The bath and system-bath interaction operators become:

$$\begin{pmatrix}B_{z}(t) & B^{\pm}(t)\\ B_{x}(t) & B(t)\\ B_{y}(t) & R(t)\end{pmatrix}\equiv\begin{pmatrix}\sum_{\mathbf{k}}\left(g_{\mathbf{k}}-v_{\mathbf{k}}(t)\right)\left(b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\right) & e^{\pm\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}\right)}\\ \frac{B^{+}+B^{-}-2B}{2} & e^{-(1/2)\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{2}\coth(\beta\omega_{\mathbf{k}}/2)}\\ \frac{B^{-}-B^{+}}{2\mathrm{i}} & \sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}^{2}}{\omega_{\mathbf{k}}}-2g_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\end{pmatrix}, \tag{55}$$

$$\begin{pmatrix}A\\ B(t)\\ C(t)\end{pmatrix}=\begin{pmatrix}\sigma_{x} & \sigma_{y} & \frac{I-\sigma_{z}}{2} & \sigma_{x} & \sigma_{y} & \frac{I+\sigma_{z}}{2}\\ B_{x} & B_{y} & B_{z} & B_{y} & B_{x} & 0\\ \frac{\Omega}{2} & \frac{\Omega}{2} & 1 & 0 & 0 & 1\end{pmatrix}. \tag{56}$$

Therefore $C(t)$ is no longer time-dependent. Defining:

$$\Lambda_{ij}(\tau)\equiv C_{i}C_{j}\mathscr{B}_{ij}(\tau). \tag{57}$$

The response function is given by:

$$K_{ijw}(t)=\int_{0}^{t}C_{i}C_{j}\mathscr{B}_{ij}(\tau)\,e^{\mathrm{i}w\tau}\mathrm{d}\tau \tag{58}$$

Defining $A_j(w) \equiv A_{jw}$ then we can write the master equation as:

$$\frac{\mathrm{d}\overline{\rho}_{\bar{S}}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H}_{\bar{S}}(t), \overline{\rho}_S(t)\right] - \sum_{ijw}\Big(K^{\Re}_{ijw}(t)\Big[A_i, A_{jw}\overline{\rho}_S(t) - \overline{\rho}_S(t)\,A^{\dagger}_{jw}\Big] + \mathrm{i}K^{\Im}_{ijw}(t)\Big[A_i, A_{jw}\overline{\rho}_S(t) + \overline{\rho}_S(t)\,A^{\dagger}_{jw}\Big]\Big). \tag{59}$$

B. Time-dependent polaron master equation

Following the reference [1], if $v_{\mathbf{k}} \to g_{\mathbf{k}}$ then we recover the full polaron transformation. It means from the equation (9) that $B_z = 0$. The Hamiltonian studied in this case is given by:

$$H = \left(\delta + \sum_{\mathbf{k}}\left(g_{\mathbf{k}}b^{\dagger}_{\mathbf{k}} + g^{*}_{\mathbf{k}}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega(t)}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{60}$$

If $v_{\mathbf{k}} \to g_{\mathbf{k}}$ then $B(\tau) = B$ from the equation (9), so B is independent of the time. It's possible to summarize (60) in terms of the hamiltonian (1) using the following matrix:

$$\begin{pmatrix} V^{\Im}_{10}(t) & g^{\Re}_{0\mathbf{k}} & v_{0\mathbf{k}}(t) & B_{10}(t) \\ V^{\Re}_{10}(t) & g^{\Im}_{0\mathbf{k}} & v_{1\mathbf{k}}(t) & \Omega_r(t) \\ \varepsilon_0(t) & g^{\Re}_{1\mathbf{k}} & B_{0z}(t) & R_0(t) \\ \varepsilon_1(t) & g^{\Im}_{1\mathbf{k}} & B_{1z}(t) & R_1(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & B \\ \frac{\Omega(t)}{2} & 0 & g_{\mathbf{k}} & B\Omega(t) \\ 0 & g^{\Re}_{\mathbf{k}} & 0 & 0 \\ \delta & g^{\Im}_{\mathbf{k}} & 0 & -\sum_{\mathbf{k}}\omega^{-1}_{\mathbf{k}}|g_{\mathbf{k}}|^2 \end{pmatrix}. \tag{61}$$

Using the equation (15) and (16) we obtain the following transformed Hamiltonians:

$$\overline{H_{\bar{S}}}(t) = (\delta + R_1)|1\rangle\langle 1| + \frac{B\sigma_x}{2}\Omega(t), \tag{62}$$

$$\overline{H_{\bar{I}}}(t) = \frac{\Omega(t)}{2}(B_x\sigma_x + B_y\sigma_y). \tag{63}$$

For the time-dependent polaron master equation we have $F(\omega_{\mathbf{k}}) = 1$ so it's continuous form is $F(\omega) = 1$, we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$:

$$\Lambda_{11}(\tau) = \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{64}$$

$$\Lambda_{22}(\tau) = \frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right). \tag{65}$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A_i}(t-\tau, t) = \widetilde{\sigma_i}(t-\tau, t)$ we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)\right] - \frac{\Omega(t)}{4}\int_0^t \mathrm{d}\tau\Omega(t-\tau)\left(\left[\sigma_x, \widetilde{\sigma_x}(t-\tau, t)\rho_S(t)\right]\Lambda_x(\tau)\right. \tag{66}$$

$$\left. + \left[\sigma_y, \widetilde{\sigma_y}(t-\tau, t)\rho_S(t)\right]\Lambda_y(\tau) + \left[\rho_S(t)\widetilde{\sigma_x}(t-\tau, t), \sigma_x\right]\Lambda_x(\tau) + \left[\rho_S(t)\widetilde{\sigma_y}(t-\tau, t), \sigma_y\right]\Lambda_y(\tau)\right). \tag{67}$$

As we can see $\left[A_j, \widetilde{A_i}(t-\tau, t)\rho_S(t)\right]^{\dagger} = \left[\rho_S(t)\widetilde{A_i}(t-\tau, t), A_j\right]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extending the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. Taking $\bar{h} = 1$ the Hamiltonian of the reference can be written as:

$$H = \Delta\left|1\middle\rangle\!\middle\langle 1\right| + \frac{\Omega\left(t\right)}{2}\left(\left|1\middle\rangle\!\middle\langle 0\right| + \left|0\middle\rangle\!\middle\langle 1\right|\right) + \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \left|1\middle\rangle\!\middle\langle 1\right| \sum_{\mathbf{k}}\left(g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^{*} b_{\mathbf{k}}\right). \tag{68}$$

Given that $F\left(\omega\right) = 0$ then for the weak-coupling approximation we have:

$$\Lambda_{33}\left(\tau\right) = \int_{0}^{\infty} \mathrm{d}\omega J\left(\omega\right) G_{+}\left(\tau\right). \tag{69}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the master equation in this case is:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)\right)] - \sum_{w}\left(K_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right)\right] + K_{33}^{*}\left(w,t\right)\left[\rho_S\left(t\right) A_3^{\dagger}\left(w\right), A_3\right]\right). \tag{70}$$

Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x, \rho_S(t))\right] - \frac{\pi}{8} J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right) + 1\right)[\sigma_z, (\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)] + n\left(\Omega(t)\right)[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)]\right) \tag{71}$$

$$- \frac{\pi}{8} J\left(\Omega\left(t\right)\right)\left(\left(n\left(\Omega\left(t\right)\right) + 1\right)\left[\rho_S\left(t\right)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\Omega\left(t\right)\right)\left[\rho_S\left(t\right)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right). \tag{72}$$

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