

# A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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## I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1)$$

$$H_S(t) = \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|, \quad (2)$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (3)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (4)$$

For the states  $|0\rangle, |1\rangle$  we have the orthonormal condition:

$$\langle i|j\rangle = \delta_{ij} \quad (5)$$

## II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to  $H(t)$ , the unitary transformation defined by  $e^{\pm V}$  where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left( \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) \quad (6)$$

in terms of the variational scalar parameters  $v_{i\mathbf{k}}$  defined as:

$$v_{i\mathbf{k}} = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}} \quad (7)$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}} \left( \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \quad (8)$$

$$V = |0\rangle\langle 0| \hat{\varphi}_0 + |1\rangle\langle 1| \hat{\varphi}_1. \quad (9)$$

Here  $*$  denotes the complex conjugate. Expanding  $e^{\pm V}$  using the notation (6) will give us the following result:

$$e^{\pm V} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1)} \quad (10)$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1))^2}{2!} + \dots \quad (11)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!} + \dots \quad (12)$$

$$= |0\rangle\langle 0| \left(1 \pm \hat{\varphi}_0 + \frac{\hat{\varphi}_0^2}{2!} \pm \dots\right) + |1\rangle\langle 1| \left(1 \pm \hat{\varphi}_1 + \frac{\hat{\varphi}_1^2}{2!} \pm \dots\right) \quad (13)$$

$$= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0} + |1\rangle\langle 1|e^{\pm\hat{\varphi}_1} \quad (14)$$

$$= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*b_{\mathbf{k}})} + |1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*b_{\mathbf{k}})} \quad (15)$$

$$= |0\rangle\langle 0|B_{0\pm} + |1\rangle\langle 1|B_{1\pm}, \quad (16)$$

$$B_{i\pm} \equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (17)$$

Let's recall the Zassenhaus formula:

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} e^{\frac{t^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} e^{-\frac{t^4}{24}([[[X,Y],X],X] + 3[[[X,Y],X],Y] + 3[[[X,Y],Y],Y])} \dots \quad (18)$$

Since  $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right] = 0$  for all  $\mathbf{k}', \mathbf{k}$  and  $i, j$  we can show making  $t = 1$  in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + \left(\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right)} = e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right]} \dots \quad (19)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} e^{-\frac{1}{2}0} \dots \quad (20)$$

$$= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}} \quad (21)$$

By induction of this result we can write expression of  $B_{i\pm}$  as a product of exponentials, which we will call "displacement" operators  $D(\pm v_{i\mathbf{k}})$ :

$$B_{i\pm} = \prod_{\mathbf{k}} D\left(\pm \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right), \quad (22)$$

$$D(\pm v_{i\mathbf{k}}) \equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. \quad (23)$$

this will help us to write operators  $O$  in the variational frame :

$$\overline{O} \equiv e^V O e^{-V}. \quad (24)$$

We use the following identities:

$$\overline{|0\rangle\langle 0|} = e^V |0\rangle\langle 0| e^{-V} \quad (25)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |0\rangle\langle 0| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (26)$$

$$= (|0\rangle\langle 0|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (27)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (28)$$

$$= |0\rangle\langle 0|B_{0+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (29)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}B_{1-} \quad (30)$$

$$= |0\rangle\langle 0|, \quad (31)$$

$$\overline{|1\rangle\langle 1|} = (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |1\rangle\langle 1| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (32)$$

$$= (|0\rangle\langle 0|1\rangle\langle 1|B_{0+} + |1\rangle\langle 1|1\rangle\langle 1|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (33)$$

$$= |1\rangle\langle 1|B_{1+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (34)$$

$$= |1\rangle\langle 1|0\rangle\langle 0|B_{1+}B_{0-} + B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} \quad (35)$$

$$= B_{1+}|1\rangle\langle 1|1\rangle\langle 1|B_{1-} \quad (36)$$

$$= |1\rangle\langle 1|B_{1+}B_{1-} \quad (37)$$

$$= |1\rangle\langle 1|, \quad (38)$$

$$\overline{|0\rangle\langle 1|} = e^V |0\rangle\langle 1| e^{-V} \quad (39)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |0\rangle\langle 1| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (40)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_{0+} + |1\rangle\langle 1|B_{1+}|0\rangle\langle 1|) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (41)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_{0+} + |1\rangle\langle 1|0\rangle\langle 1|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (42)$$

$$= |0\rangle\langle 1|B_{0+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (43)$$

$$= |0\rangle\langle 1|0\rangle\langle 0|B_{0+}B_{0-} + |0\rangle\langle 1|1\rangle\langle 1|B_{0+}B_{1-} \quad (44)$$

$$= |0\rangle\langle 1|B_{0+}B_{1-}, \quad (45)$$

$$\overline{|1\rangle\langle 0|} = e^V |1\rangle\langle 0| e^{-V} \quad (46)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) |1\rangle\langle 0| (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (47)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (48)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_{0+} + |1\rangle\langle 1|1\rangle\langle 0|B_{1+}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (49)$$

$$= |1\rangle\langle 0|B_{1+} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (50)$$

$$= |1\rangle\langle 0|B_{1+}|0\rangle\langle 0|B_{0-} + |1\rangle\langle 0|B_{1+}|1\rangle\langle 1|B_{1-} \quad (51)$$

$$= |1\rangle\langle 0|B_{1+}B_{0-} + |1\rangle\langle 0|1\rangle\langle 1|B_{1+}B_{1-} \quad (52)$$

$$= |1\rangle\langle 0|B_{1+}B_{0-}, \quad (53)$$

$$\overline{b_{\mathbf{k}}} = e^V b_{\mathbf{k}} e^{-V} \quad (54)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) b_{\mathbf{k}} (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (55)$$

$$= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-}|0\rangle\langle 0| + |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}|1\rangle\langle 1|B_{1-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-}|1\rangle\langle 1| \quad (56)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}B_{1-} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}B_{1-} \quad (57)$$

$$= |0\rangle\langle 0| \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (58)$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - |1\rangle\langle 1| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (59)$$

$$= b_{\mathbf{k}} - |1\rangle\langle 1| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (60)$$

$$\overline{b_{\mathbf{k}}^\dagger} = e^V b_{\mathbf{k}}^\dagger e^{-V} \quad (61)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) b_{\mathbf{k}}^\dagger (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (62)$$

$$= |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^\dagger B_{0-}|0\rangle\langle 0| + |0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^\dagger|1\rangle\langle 1|B_{1-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^\dagger|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^\dagger B_{1-}|1\rangle\langle 1| \quad (63)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_{0+}b_{\mathbf{k}}^\dagger B_{0-} + |0\rangle\langle 0|1\rangle\langle 1|B_{0+}b_{\mathbf{k}}^\dagger B_{1-} + |1\rangle\langle 1|0\rangle\langle 0|B_{1+}b_{\mathbf{k}}^\dagger B_{0-} + |1\rangle\langle 1|1\rangle\langle 1|B_{1+}b_{\mathbf{k}}^\dagger B_{1-} \quad (64)$$

$$= |0\rangle\langle 0| \left( b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \quad (65)$$

$$= b_{\mathbf{k}}^\dagger - |1\rangle\langle 1| \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} - |0\rangle\langle 0| \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (66)$$

We have used the following:

$$B_{i+} b_{\mathbf{k}} B_{i-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (67)$$

$$B_{i+} b_{\mathbf{k}}^{\dagger} B_{i-} = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}. \quad (68)$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t) |0\rangle\langle 0|} = \varepsilon_0(t) |0\rangle\langle 0|, \quad (69)$$

$$\overline{\varepsilon_1(t) |1\rangle\langle 1|} = \varepsilon_1(t) |1\rangle\langle 1|, \quad (70)$$

$$\overline{V_{10}(t) |1\rangle\langle 0|} = V_{10}(t) |1\rangle\langle 0| B_{1+} B_{0-}, \quad (71)$$

$$\overline{V_{01}(t) |0\rangle\langle 1|} = V_{01}(t) |0\rangle\langle 1| B_{0+} B_{1-}, \quad (72)$$

$$\overline{g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}}} = g_{i\mathbf{k}} \left( |0\rangle\langle 0| \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \right) + g_{i\mathbf{k}}^* \left( |0\rangle\langle 0| \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (73)$$

$$= g_{i\mathbf{k}} \left( (|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left( (|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (74)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} |1\rangle\langle 1| \quad (75)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left( g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |0\rangle\langle 0| - \left( g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) |1\rangle\langle 1|, \quad (76)$$

$$\overline{|0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}})} = (|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}) |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}) \quad (77)$$

$$= |0\rangle\langle 0| B_{0+} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) |0\rangle\langle 0| B_{0-} \quad (78)$$

$$= |0\rangle\langle 0| B_{0+} \left( g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) B_{0-} \quad (79)$$

$$= |0\rangle\langle 0| \left( g_{0\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (80)$$

$$\overline{|1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}})} = (|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}) |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) (|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}) \quad (81)$$

$$= |1\rangle\langle 1| B_{1+} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) |1\rangle\langle 1| B_{1-} \quad (82)$$

$$= |1\rangle\langle 1| B_{1+} \left( g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) B_{1-} \quad (83)$$

$$= |1\rangle\langle 1| \left( g_{1\mathbf{k}} \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (84)$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} = \omega_{\mathbf{k}} (|0\rangle\langle 0| B_{0+} + |1\rangle\langle 1| B_{1+}) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} (|0\rangle\langle 0| B_{0-} + |1\rangle\langle 1| B_{1-}) \quad (85)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| B_{0+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{0-} + |1\rangle\langle 1| B_{1+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B_{1-} \right) \quad (86)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| \Pi_{\mathbf{k}'} D \left( \frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D \left( \frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \left( |0\rangle\langle 0| \Pi_{\mathbf{k}'} D \left( -\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D \left( -\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right) \quad (87)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| D \left( \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left( -\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \Pi_{\mathbf{k}' \neq \mathbf{k}} D \left( \frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) D \left( -\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) + |1\rangle\langle 1| D \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left( -\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \Pi_{\mathbf{k}' \neq \mathbf{k}} D \left( \frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) D \left( -\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) \right) \quad (88)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| D \left( \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left( -\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \mathbb{I} + |1\rangle\langle 1| D \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} D \left( -\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \mathbb{I} \right) \quad (89)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) + |1\rangle\langle 1| \left( b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} \left( |0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (92)$$

$$= \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (93)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left( \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left( \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (94)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^* b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^* b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger \right) \quad (95)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger) \right). \quad (96)$$

So all parts of  $H(t)$  can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)} |0\rangle\langle 0| + \overline{\varepsilon_1(t)} |1\rangle\langle 1| + \overline{V_{10}(t)} |1\rangle\langle 0| + \overline{V_{01}(t)} |0\rangle\langle 1| \quad (97)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_{1+} B_{0-} + V_{01}(t) |0\rangle\langle 1| B_{0+} B_{1-}, \quad (98)$$

$$\overline{H_I} = \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (99)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (100)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} \left( b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} \left( b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left( b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (101)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) + \sum_{\mathbf{k}} |1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left( g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^* \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left( g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^* \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (102)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (103)$$

$$= \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) \right) \quad (104)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left( |1\rangle\langle 1| \left( \frac{|v_{1\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{1\mathbf{k}}^* b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) + |0\rangle\langle 0| \left( \frac{|v_{0\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (v_{0\mathbf{k}}^* b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^\dagger) \right) \right). \quad (105)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{j+} B_{j'-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} |j\rangle\langle j| \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - (g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}) \right) \quad (106)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_S} + \overline{H_I} + \overline{H_B} \quad (107)$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \quad (108)$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \quad (109)$$

With the following definitions:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_i \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{-(1/2) \sum_{\mathbf{k}} \left( \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^2 \coth(\beta \omega_{\mathbf{k}}/2)} \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2i} & \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix} \quad (110)$$

$$(\cdot)^{\Re} \equiv \Re(\cdot) \quad (111)$$

$$(\cdot)^{\Im} \equiv \Im(\cdot) \quad (112)$$

We assume that the bath is at equilibrium with inverse temperature  $\beta = 1/k_B T$ , considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})} \quad (113)$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (114)$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (115)$$

So the product of the matrix representation of  $b^\dagger$  and  $b$  is:

$$-\beta \omega b^\dagger b = -\beta \omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (116)$$

$$= \sum_{j=0}^{\infty} -j \beta \omega |j\rangle \langle j|, \quad (117)$$

So the density matrix  $\rho_B$  written in the coherence representation can be obtained using the Zassenhaus formula and the fact that  $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$  for all  $i, j$ .

$$\exp\left(-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \quad (118)$$

$$\exp\left(-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \quad (119)$$

The value of  $\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)\right)$  is:

$$\text{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (120)$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \quad (121)$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \quad (122)$$

$$= \frac{1}{1 - \exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad (\text{by geometric series}) \quad (123)$$

$$\equiv f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \quad (124)$$

$$\text{Tr}\left(\exp\left(-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)\right) = \text{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (125)$$

$$= \prod_{\mathbf{k}} \text{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (126)$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \quad (127)$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})} \quad (128)$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \quad (129)$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \quad (130)$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1\rangle, \quad (131)$$

$$b_{\mathbf{k}}^{\dagger} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1\rangle. \quad (132)$$

Then we can prove that  $\langle B_{iz} \rangle_{\overline{H_B}} = 0$  using the following property based on (131)-(132):

$$\langle B_{iz} \rangle_{\overline{H_B}} = \text{Tr}(\rho_B B_{iz}) = \text{Tr}(B_{iz} \rho_B) \quad (133)$$

$$= \text{Tr}\left(\left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}}\right)\right) \rho_B\right) \quad (134)$$

$$= \sum_{\mathbf{k}} \text{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^{\dagger} \rho_B\right) + \sum_{\mathbf{k}} \text{Tr}\left((g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \rho_B\right) \quad (135)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left( b_{\mathbf{k}}^\dagger \rho_B \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left( b_{\mathbf{k}} \rho_B \right) \quad (136)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} \text{Tr} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) \quad (137)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \text{Tr} \left( b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \text{Tr} \left( b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right), \quad (138)$$

$$\text{Tr} \left( b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}}^\dagger |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}^\dagger) \quad (139)$$

$$= \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (140)$$

$$= 0, \quad (141)$$

$$\text{Tr} \left( b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) b_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}) \quad (142)$$

$$= \text{Tr} \left( \left( \sum_{j_{\mathbf{k}}} \exp(-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}) \right) \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (143)$$

$$= 0. \quad (144)$$

we therefore find that:

$$\langle B_{iz} \rangle_{\overline{H_B}} = 0 \quad (145)$$

Another important expected value is  $B = \langle B_{\pm} \rangle_{\overline{H_B}}$ , where  $B_{\pm} = e^{\pm \sum_{\mathbf{k}} \left( \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}$  is given by:

$$\langle B_{\pm} \rangle_{H_B} = \text{Tr} (\rho_B B_{\pm}) = \text{Tr} (B_{\pm} \rho_B) \quad (146)$$

$$= \text{Tr} \left( e^{\pm \sum_{\mathbf{k}} \left( \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \rho_B \right) \quad (147)$$

$$= \prod_{\mathbf{k}} \text{Tr} (D(\pm \alpha_{\mathbf{k}}) \rho_B) \quad (148)$$

$$= \prod_{\mathbf{k}} \langle D(\pm \alpha_{\mathbf{k}}) \rangle. \quad (149)$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha) |\alpha \rangle \langle \alpha| d^2 \alpha \quad (150)$$

where  $P(\alpha)$  satisfies  $\int P(\alpha) d^2 \alpha = 1$  and describes the state. It follows that the expectation value of an operator  $A$  with respect to the density operator described by  $P(\alpha)$  is given by:

$$\langle A \rangle = \text{Tr} (A \rho) \quad (151)$$

$$= \int P(\alpha) \langle \alpha | A | \alpha \rangle d^2 \alpha \quad (152)$$

We are typically interested in thermal state density operators, for which it can be shown that  $P(\alpha) = \frac{1}{\pi N} \exp \left( -\frac{|\alpha|^2}{N} \right)$  where  $N = (e^{\beta \omega} - 1)^{-1}$  is the average number of excitations in an oscillator of frequency  $\omega$  at inverse temperature  $\beta = 1/k_B T$ .

Using the integral representation (152) we could obtain that the expected value for the displacement operator  $D(h)$  with  $h \in \mathbb{C}$  is equal to:



$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | D(h) | \alpha \rangle d^2 \alpha \quad (153)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle d^2 \alpha \quad (154)$$

$$D(h) D(\alpha) = D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (155)$$

$$D(-\alpha) (D(h) D(\alpha)) = D(-\alpha) D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (156)$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (157)$$

$$= D(\alpha) e^{\frac{1}{2}(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2)} e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (158)$$

$$= D(\alpha) e^{(h\alpha^* - h^* \alpha)}, \quad (159)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle 0 | D(h) \exp(h\alpha^* - h^* \alpha) | 0 \rangle d^2 \alpha \quad (160)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle d^2 \alpha \quad (161)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \langle 0 | h \rangle d^2 \alpha \quad (162)$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (163)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \langle 0 | \exp\left(-\frac{|h|^2}{2}\right) \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle d^2 \alpha \quad (164)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \exp(h\alpha^* - h^* \alpha) \exp\left(-\frac{|h|^2}{2}\right) d^2 \alpha \quad (165)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N} + h\alpha^* - h^* \alpha\right) d^2 \alpha, \quad (166)$$

$$\alpha = x + iy, \quad (167)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 + y^2}{N} + h(x - iy) - h^*(x + iy)\right) dx dy \quad (168)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{N} + hx - h^* x\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N} - ihy - ih^* y\right) dy, \quad (169)$$

$$-\frac{x^2}{N} + hx - h^* x = -\frac{1}{N} (x^2 - Nhx + Nh^* x) \quad (170)$$

$$= -\frac{1}{N} \left( x + \frac{(Nh^* - Nh)}{2} \right)^2 + \frac{N(h^* - h)^2}{4}, \quad (171)$$

$$-\frac{y^2}{N} - ihy - ih^* y = -\frac{1}{N} (y^2 + iNhy + iNh^* y) \quad (172)$$

$$= -\frac{1}{N} \left( y^2 + \frac{iN(h + h^*)}{2} \right) - \frac{N(h + h^*)^2}{4}, \quad (173)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{N} \left( x + \frac{(Nh^* - Nh)}{2} \right)^2 - \frac{1}{N} \left( y^2 + \frac{iN(h + h^*)}{2} \right)\right) dx dy, \quad (174)$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx, \quad (175)$$

$$\langle D(h) \rangle = \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x + \frac{(Nh^* - Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{\left(y^2 + \frac{iN(h + h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right) dy \quad (176)$$

$$= \frac{\exp\left(-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}\right)}{\pi N} \left( \sqrt{2\pi} \sqrt{\frac{N}{2}} \right)^2 \quad (177)$$

$$= \exp \left( -\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4} \right) \quad (178)$$

$$= \exp \left( -\frac{|h|^2}{2} + \frac{N(h^{*2} - 2hh^* + h^2) - N(h^2 + 2hh^* + h^{*2})}{4} \right) \quad (179)$$

$$= \exp \left( -|h|^2 \left( N + \frac{1}{2} \right) \right) \quad (180)$$

$$= \exp \left( -|h|^2 \left( \frac{1}{e^{\beta\omega} - 1} + \frac{1}{2} \right) \right) \quad (181)$$

$$= \exp \left( -\frac{|h|^2}{2} \left( \frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} \right) \right) \quad (182)$$

$$= \exp \left( -\frac{|h|^2}{2} \coth \left( \frac{\beta\omega}{2} \right) \right). \quad (183)$$

In the last line we used  $\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} = \coth \left( \frac{\beta\omega}{2} \right)$ . So the value of (148) using (183) is given by:

$$B = \exp \left( -\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (184)$$

We will now force  $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$ . We will also introduce the bath renormalizing driving in  $\overline{H_S}$  to treat it non-perturbatively in the subsequent formalism, we associate the terms related with  $B_+ \sigma_+$  and  $B_- \sigma_-$  with the interaction part of the Hamiltonian  $\overline{H_I}$  and we subtract their expected value in order to satisfy  $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$ .

A final form of the terms of the Hamiltonian  $\overline{H}$  is:

$$\overline{H}(t) = \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| |B_{jj'} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle \langle j| \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} + v_{j\mathbf{k}}) b_{\mathbf{k}} \right) \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} \left( g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (185)$$

$$= \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| |B_{jj'} + \sum_{j\mathbf{k}} |j\rangle \langle j| B_{j\mathbf{k}} + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_{jj'} + B_{j\mathbf{k}} - B_{jj'}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (186)$$

$$\equiv \overline{H_S}(t) + \overline{H_I} + \overline{H_B}. \quad (187)$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_{1+} B_{0-} \rangle = B_{10} \quad (188)$$

$$= \left\langle \prod_{\mathbf{k}} D \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D \left( -\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle \quad (189)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) D \left( -\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \quad (190)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (191)$$

$$= \prod_{\mathbf{k}} \left\langle D \left( \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (192)$$

$$= \prod_{\mathbf{k}} \exp \left( -\frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \quad (193)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)}. \quad (194)$$

From the definition  $B_{01} = \langle B_{0+} B_{1-} \rangle$  using the displacement operator we have:

$$\langle B_{0+} B_{1-} \rangle = B_{01} \quad (195)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (196)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (197)$$

$$= \left\langle \prod_{\mathbf{k}} \left( D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \right\rangle \quad (198)$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (199)$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (200)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (201)$$

This can be checked in the following way:

$$\langle B_{0+} B_{1-} \rangle = B_{01} \quad (202)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \quad (203)$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)^*} \quad (204)$$

$$= \langle B_{1+} B_{0-} \rangle^* \quad (205)$$

$$= B_{10}^*. \quad (206)$$

The parts of the Hamiltonian splitted are:

$$\overline{H_{\overline{S}}}(t) \equiv (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma_+ + V_{01}(t) B_{01} \sigma_-, \quad (207)$$

$$\overline{H_{\overline{I}}} \equiv V_{10}(t) (B_{1+} B_{0-} - B_{10}) \sigma_+ + V_{01}(t) (B_{0+} B_{1-} - B_{01}) \sigma_- + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z}, \quad (208)$$

$$\overline{H_{\overline{B}}} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (209)$$

$$= H_B. \quad (210)$$

Note that  $\overline{H_{\overline{B}}}$ , which is the bath acting on the effective “system”  $\overline{S}$  in the variational frame, is just the original bath,  $H_B$ , before transforming to the variational frame.

For the Hamiltonian (208) we can verify the condition  $\langle \overline{H_{\overline{I}}} \rangle_{\overline{H_{\overline{B}}}} = 0$  in the following way:

$$\langle \overline{H_I} \rangle_{\overline{H_B}} = \langle \sum_{n\mathbf{k}} ((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}) |n\rangle \langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_{jj} + B_{jj'} - B_{jj'}) \rangle_{\overline{H_B}} \quad (211)$$

$$= \langle \sum_{n\mathbf{k}} ((g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}}) |n\rangle \langle n| \rangle_{\overline{H_B}} + \langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_{jj} + B_{jj'} - B_{jj'}) \rangle_{\overline{H_B}} \quad (212)$$

$$= \sum_{n\mathbf{k}} \left( \langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}) b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + \langle (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle \langle n| + \sum_{j \neq j'} |j\rangle \langle j'| \left( \langle V_{jj'}(t) B_{jj} + B_{jj'} \rangle_{\overline{H_B}} - \langle V_{jj'}(t) B_{jj'} \rangle_{\overline{H_B}} \right) \quad (213)$$

$$= \sum_{n\mathbf{k}} \left( (g_{n\mathbf{k}} - v_{n\mathbf{k}}) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle \langle n| + \sum_{j \neq j'} |j\rangle \langle j'| V_{jj'}(t) \left( \langle B_{jj} + B_{jj'} \rangle_{\overline{H_B}} - \langle B_{jj'} \rangle_{\overline{H_B}} \right) \quad (214)$$

$$= \sum_{n\mathbf{k}} \left( (g_{n\mathbf{k}} - v_{n\mathbf{k}}) \langle b_{\mathbf{k}}^\dagger \rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}})^* \langle b_{\mathbf{k}} \rangle_{\overline{H_B}} \right) |n\rangle \langle n| + \sum_{j \neq j'} |j\rangle \langle j'| V_{jj'}(t) (B_{jj} - B_{jj'}) \cdot \quad (215)$$

$$= 0. \quad (216)$$

We used (145) and (194) to evaluate the expected values.

Let's consider the following Hermitian combinations:

$$B_x = B_x^\dagger \quad (217)$$

$$= \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2}, \quad (218)$$

$$B_y = B_y^\dagger \quad (219)$$

$$= \frac{B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10} - B_{01}}{2i}, \quad (220)$$

$$B_{iz} = B_{iz}^\dagger \quad (221)$$

$$= \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right). \quad (222)$$

Writing the equations (207) and (208) using the previous combinations we obtain that:

$$\overline{H_S}(t) = (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + V_{10}(t) B_{10} \sigma_+ + V_{01}(t) B_{01} \sigma_- \quad (223)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + V_{10}(t) B_{10} \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01} \frac{\sigma_x - i\sigma_y}{2} \quad (224)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + V_{10}(t) \left( B_{10}^{\Re}(t) + i B_{10}^{\Im}(t) \right) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \left( B_{01}^{\Re}(t) - i B_{01}^{\Im}(t) \right) \frac{\sigma_x - i\sigma_y}{2} \quad (225)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + B_{10}^{\Re}(t) \left( V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) + i B_{10}^{\Im}(t) \left( V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) \quad (226)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + B_{10}^{\Re}(t) \left( \sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i \sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + i B_{10}^{\Im}(t) \left( \sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i \sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (227)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + B_{10}^{\Re}(t) \left( \sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i \sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + i B_{10}^{\Im}(t) \left( \sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i \sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (228)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + B_{10}^{\Re}(t) \left( \sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + i B_{10}^{\Im}(t) \left( i \sigma_x V_{10}^{\Im}(t) + i \sigma_y V_{10}^{\Re}(t) \right) \quad (229)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \left( \sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) - \left( \sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (230)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (231)$$

$$= (\varepsilon_0(t) + R_0) |0\rangle \langle 0| + (\varepsilon_1(t) + R_1) |1\rangle \langle 1| + \sigma_x \left( B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left( B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right). \quad (232)$$

$$\overline{H_I} V_{10}(t)(\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}) + V_{01}(t)(\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) + |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} \quad (233)$$

$$\Rightarrow |0\rangle\langle 0| B_{0z} + |1\rangle\langle 1| B_{1z} + \left( V_{10}^{\mathcal{R}}(t) + i V_{10}^{\mathcal{I}}(t) \right) (\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10}) + \left( V_{10}^{\mathcal{R}}(t) - i V_{10}^{\mathcal{I}}(t) \right) (\sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) \quad (234)$$

$$\Rightarrow \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) (\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} + \sigma_- B_{0+} B_{1-} - \sigma_- B_{01}) + i V_{10}^{\mathcal{I}}(t) (\sigma_+ B_{1+} B_{0-} - \sigma_+ B_{10} - \sigma_- B_{0+} B_{1-} + \sigma_- B_{01}) \quad (235)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) \left( \frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (236)$$

$$\Rightarrow \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) \left( \frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} + \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} - \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) + i V_{10}^{\mathcal{I}}(t) \left( \frac{\sigma_x + i\sigma_y}{2} B_{1+} B_{0-} - \frac{\sigma_x + i\sigma_y}{2} B_{10} - \frac{\sigma_x - i\sigma_y}{2} B_{0+} B_{1-} + \frac{\sigma_x - i\sigma_y}{2} B_{01} \right) \quad (237)$$

$$+ \frac{\sigma_x - i\sigma_y}{2} B_{01} \quad (238)$$

$$= \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) \left( \sigma_x \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} + i\sigma_y \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} \right) + i V_{10}^{\mathcal{I}}(t) \left( \sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} \right. \quad (239)$$

$$\left. + i\sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} \right) \quad (240)$$

$$\Rightarrow \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) (\sigma_x B_{x+} \sigma_y B_y) + V_{10}^{\mathcal{I}}(t) \left( i\sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} \right) \quad (241)$$

$$\Rightarrow \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) (\sigma_x B_{x+} \sigma_y B_y) + V_{10}^{\mathcal{I}}(t) \left( i^2 \sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} \right) \quad (242)$$

$$\Rightarrow \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) (\sigma_x B_{x+} \sigma_y B_y) + V_{10}^{\mathcal{I}}(t) \left( i^2 \sigma_x \frac{B_{1+} B_{0-} - B_{0+} B_{1-} - B_{10} + B_{01}}{2i} - \sigma_y \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} \right) \quad (243)$$

$$\Rightarrow \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) (\sigma_x B_{x+} \sigma_y B_y) + V_{10}^{\mathcal{I}}(t) (i^2 \sigma_x (-B_y) - \sigma_y B_x) \quad (244)$$

$$\Rightarrow \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\mathcal{R}}(t) (\sigma_x B_{x+} \sigma_y B_y) + V_{10}^{\mathcal{I}}(t) (\sigma_x B_y - \sigma_y B_x). \quad (245)$$

### III. FREE-ENERGY MINIMIZATION

The true free energy  $A$  is bounded by the Bogoliubov inequality:

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left( \text{Tr} \left( e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right) + \langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} + O \left( \langle \overline{H_I}^2 \rangle_{\overline{H_S}(t) + \overline{H_B}} \right). \quad (246)$$

We will optimize the set of variational parameters  $\{v_{i\mathbf{k}}\}$  in order to minimize  $A_B$  (i.e. to make it as close to the true free energy  $A$  as possible). Neglecting the higher order terms and using  $\langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} = 0$  we can obtain the following condition to obtain the set  $\{v_{i\mathbf{k}}\}$ :

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = 0. \quad (247)$$

Using this condition and given that  $[\overline{H_S}(t), \overline{H_B}] = 0$ , we have:

$$e^{-\beta(\overline{H_S}(t) + \overline{H_B})} = e^{-\beta\overline{H_S}(t)} e^{-\beta\overline{H_B}}. \quad (248)$$

Then using the fact that  $\overline{H_S}(t)$  and  $\overline{H_B}$  relate to different Hilbert spaces, we obtain:

$$\text{Tr} \left( e^{-\beta\overline{H_S}(t)} e^{-\beta\overline{H_B}} \right) = \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \text{Tr} \left( e^{-\beta\overline{H_B}} \right). \quad (249)$$

So Eq. (247) becomes:

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}} = -\frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (250)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \text{Tr} \left( e^{-\beta\overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (251)$$

$$= -\frac{1}{\beta} \frac{\partial \left( \ln \left( \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \right) + \ln \left( \text{Tr} \left( e^{-\beta\overline{H_B}} \right) \right) \right)}{\partial v_{i\mathbf{k}}} \quad (252)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta\overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta} \frac{\partial \ln \left( \text{Tr} \left( e^{-\beta\overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}} \quad (253)$$

$$= 0 \quad (\text{by Eq. (247)}). \quad (254)$$

But since  $\bar{H}_B = H_B$  which doesn't contain any  $v_{ik}$ , a derivative of any function of  $H_B$  that does not introduce new  $v_{ik}$  will be zero. We therefore require the following:

$$\frac{\partial \ln \left( \text{Tr} \left( e^{-\beta \bar{H}_S(t)} \right) \right)}{\partial v_{ik}} = \frac{1}{e^{-\beta \bar{H}_S(t)}} \frac{\partial \text{Tr} \left( e^{-\beta \bar{H}_S(t)} \right)}{\partial v_{ik}} \quad (255)$$

$$= 0. \quad (256)$$

This means we need to impose:

$$\frac{\partial \text{Tr} \left( e^{-\beta \bar{H}_S(t)} \right)}{\partial v_{ik}} = 0. \quad (257)$$

First we look at:

$$-\beta \bar{H}_S(t) = -\beta \left( (\varepsilon_0(t) + R_0) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1) |1\rangle\langle 1| + V_{10}(t) B_{10} \sigma_+ + V_{01}(t) B_{01} \sigma_- \right). \quad (258)$$

Then the eigenvalues of  $-\beta \bar{H}_S(t)$  satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr} \left( -\beta \bar{H}_S(t) \right) + \text{Det} \left( -\beta \bar{H}_S(t) \right) = 0. \quad (259)$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr} \left( \bar{H}_S(t) \right), \quad (260)$$

$$\eta \equiv \sqrt{\left( \text{Tr} \left( \bar{H}_S(t) \right) \right)^2 - 4 \text{Det} \left( \bar{H}_S(t) \right)}. \quad (261)$$

The solutions of the equation (259) are:

$$\lambda = \beta \frac{-\text{Tr} \left( \bar{H}_S(t) \right) \pm \sqrt{\left( \text{Tr} \left( \bar{H}_S(t) \right) \right)^2 - 4 \text{Det} \left( \bar{H}_S(t) \right)}}{2} \quad (262)$$

$$= \beta \frac{-\varepsilon(t) \pm \eta(t)}{2} \quad (263)$$

$$= -\beta \frac{\varepsilon(t) \mp \eta(t)}{2}. \quad (264)$$

The value of  $\text{Tr} \left( e^{-\beta \bar{H}_S(t)} \right)$  can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a  $2 \times 2$  matrix):

$$\text{Tr} \left( e^{-\beta \bar{H}_S(t)} \right) = \exp \left( -\frac{\varepsilon(t) \beta}{2} \right) \exp \left( \frac{\eta(t) \beta}{2} \right) + \exp \left( -\frac{\varepsilon(t) \beta}{2} \right) \exp \left( -\frac{\eta(t) \beta}{2} \right) \quad (265)$$

$$= 2 \exp \left( -\frac{\varepsilon(t) \beta}{2} \right) \cosh \left( \frac{\eta(t) \beta}{2} \right). \quad (266)$$

Given that  $v_{ik}$  is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function  $\text{Tr} \left( e^{-\beta \bar{H}_S(t)} \right) = A(\varepsilon(t), \eta(t))$  to calculate  $\frac{\partial \text{Tr} \left( e^{-\beta \bar{H}_S(t)} \right)}{\partial v_{ik}^{\Re}}$  can lead to:

$$\frac{\partial \text{Tr} \left( e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \left( 2 \exp \left( -\frac{\varepsilon(t)\beta}{2} \right) \cosh \left( \frac{\eta(t)\beta}{2} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (267)$$

$$= 2 \left( -\frac{\beta}{2} \frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} \right) \exp \left( -\frac{\varepsilon(t)\beta}{2} \right) \cosh \left( \frac{\eta(t)\beta}{2} \right) + 2 \left( \frac{\beta}{2} \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} \right) \exp \left( -\frac{\varepsilon(t)\beta}{2} \right) \sinh \left( \frac{\eta(t)\beta}{2} \right) \quad (268)$$

$$= -\beta \exp \left( -\frac{\varepsilon(t)\beta}{2} \right) \left( \frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} \cosh \left( \frac{\eta(t)\beta}{2} \right) - \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} \sinh \left( \frac{\eta(t)\beta}{2} \right) \right). \quad (269)$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} \cosh \left( \frac{\eta(t)\beta}{2} \right) - \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} \sinh \left( \frac{\eta(t)\beta}{2} \right) = 0. \quad (270)$$

The derivates included in the expression given are related to:

$$\langle B_{0+} B_{1-} \rangle = \left( \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (271)$$

$$= \left( \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right)^* \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (272)$$

$$= \langle B_{1+} B_{0-} \rangle^*, \quad (273)$$

$$R_i = \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \quad (274)$$

$$= \sum_{\mathbf{k}} \left( \frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (275)$$

$$\langle B_{0+} B_{1-} \rangle = \left( \prod_{\mathbf{k}} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)} \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (276)$$

$$= \left( \prod_{\mathbf{k}} \exp \left( \frac{1}{2\omega_{\mathbf{k}}^2} (v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*) \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (277)$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = (v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im}) (v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im}) - (v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im}) (v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im}) \quad (278)$$

$$= (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}) - (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} + i v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im}) \quad (279)$$

$$= 2i (v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re}), \quad (280)$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = (v_{1\mathbf{k}} - v_{0\mathbf{k}})(v_{1\mathbf{k}} - v_{0\mathbf{k}})^* \quad (281)$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - (v_{1\mathbf{k}} v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}}) \quad (282)$$

$$= (v_{1\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im})^2 + (v_{0\mathbf{k}}^{\Re})^2 + (v_{0\mathbf{k}}^{\Im})^2 - ((v_{1\mathbf{k}}^{\Re} + i v_{1\mathbf{k}}^{\Im})(v_{0\mathbf{k}}^{\Re} - i v_{0\mathbf{k}}^{\Im}) + (v_{1\mathbf{k}}^{\Re} - i v_{1\mathbf{k}}^{\Im})(v_{0\mathbf{k}}^{\Re} + i v_{0\mathbf{k}}^{\Im})) \quad (283)$$

$$= (v_{1\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im})^2 + (v_{0\mathbf{k}}^{\Re})^2 + (v_{0\mathbf{k}}^{\Im})^2 - 2(v_{1\mathbf{k}}^{\Re} v_{0\mathbf{k}}^{\Re} + v_{1\mathbf{k}}^{\Im} v_{0\mathbf{k}}^{\Im}) \quad (284)$$

$$= (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2. \quad (285)$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - \left( g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re} - i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re} + i v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \right) \right) \quad (286)$$

$$= \sum_{\mathbf{k}} \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (287)$$

$$\langle B_{0+} B_{1-} \rangle = \left( \prod_{\mathbf{k}} \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (288)$$

$$= \left( \prod_{\mathbf{k}} \exp \left( \frac{2i(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re})}{2\omega_{\mathbf{k}}^2} \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (289)$$

$$= \left( \prod_{\mathbf{k}} \exp \left( \frac{i(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (290)$$

Calculating the derivates respect to  $\alpha_{i\mathbf{k}}^{\Re}$  and  $\alpha_{i\mathbf{k}}^{\Im}$  we have:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (291)$$

$$= \frac{\partial \left( \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (292)$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \quad (293)$$

$$\frac{\partial |B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \left( \exp \left( -\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}} \quad (294)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} \exp \left( -\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (295)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2, \quad (296)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial \sqrt{(\text{Tr}(\overline{H_S(t)}))^2 - 4 \text{Det}(\overline{H_S(t)})}}{\partial v_{i\mathbf{k}}^{\Re}} \quad (297)$$

$$= \frac{2 \text{Tr}(\overline{H_S(t)}) \frac{\partial \text{Tr}(\overline{H_S(t)})}{\partial v_{i\mathbf{k}}^{\Re}} - 4 \frac{\partial \text{Det}(\overline{H_S(t)})}{\partial v_{i\mathbf{k}}^{\Re}}}{2 \sqrt{(\text{Tr}(\overline{H_S(t)}))^2 - 4 \text{Det}(\overline{H_S(t)})}} \quad (298)$$

$$= \frac{\varepsilon(t) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0) - |V_{10}(t)|^2 |B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta(t)} \quad (299)$$

$$= \frac{\varepsilon(t) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left( (\varepsilon(t) - \varepsilon_i(t) - R_i) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\partial v_{i\mathbf{k}}^{\Re}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (300)$$

$$= \frac{\varepsilon(t) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left( (\varepsilon(t) - \varepsilon_i(t) - R_i) \left( \frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (301)$$

$$= \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left( \frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left( -\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \quad (302)$$

$$+ 4 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \quad (303)$$

From the equation (270) and replacing the derivates obtained we have:



$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \quad (304)$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \left( 2 \frac{\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + 2 \frac{(\varepsilon(t) - \varepsilon_i(t) - R_i) \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} + 2 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\eta(t)}}} \quad (305)$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{(2v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}}^{\Re} + g_{i\mathbf{k}}^{\Re}) (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (306)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re} (\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i)) + 4 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (307)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re} - 2g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( 2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{4|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 4 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (308)$$

$$= \frac{(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re})\eta(t)}{v_{i\mathbf{k}}^{\Re} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (309)$$

Separating (308) such that the terms with  $v_{i\mathbf{k}}$  are located at one side of the equation permit us to write

$$\frac{(v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re})\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Re} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth(\beta\omega_{\mathbf{k}}/2)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (310)$$

$$v_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re} = v_{i\mathbf{k}}^{\Re} \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\Re} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \quad (311)$$

$$+ 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (312)$$

$$g_{i\mathbf{k}}^{\Re} \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (313)$$

$$v_{i\mathbf{k}}^{\Re} = \frac{\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2 |B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (314)$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial(\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (315)$$

$$= \frac{\partial \left( \left( \frac{(v_{i\mathbf{k}}^{\Re})^2 + (v_{i\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}}^{\Re} + g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (316)$$

$$= 2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^{\Re} - g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}} \quad (317)$$

$$\frac{\partial |B_{10}|^2}{\partial v_{i\mathbf{k}}^{\Im}} = \frac{\partial \left( \exp \left( - \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re})^2 + (v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}} \quad (318)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Im}} \exp\left(-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re})^2+(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im})^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (319)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Im}} |B_{10}|^2 \quad (320)$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{(\text{Tr}(H_{\overline{S}}(t)))^2-4\text{Det}(H_{\overline{S}}(t))}}{\partial v_{i\mathbf{k}}^{\Re}} \quad (321)$$

$$= \frac{2\text{Tr}(H_{\overline{S}}(t)) \frac{\partial\text{Tr}(H_{\overline{S}}(t))}{\partial v_{i\mathbf{k}}^{\Re}} - 4 \frac{\partial\text{Det}(H_{\overline{S}}(t))}{\partial v_{i\mathbf{k}}^{\Re}}}{2\sqrt{(\text{Tr}(H_{\overline{S}}(t)))^2-4\text{Det}(H_{\overline{S}}(t))}} \quad (322)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2 \frac{\partial((\varepsilon_1(t)+R_1)(\varepsilon_0(t)+R_0)-|V_{10}(t)|^2|B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Im}}}{\eta(t)} \quad (323)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2 \left( (\varepsilon(t)-\varepsilon_i(t)-R_i) \left(2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im})}{\partial v_{i\mathbf{k}}^{\Im}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (324)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) - 2 \left( (\varepsilon(t)-\varepsilon_i(t)-R_i) \left(2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \frac{2(v_{i\mathbf{k}}^{\Im}-v_{i\mathbf{k}}^{\Im})}{\omega_{\mathbf{k}}^2} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{\eta(t)} \quad (325)$$

$$= \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \left( \frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left( -i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t)-\varepsilon_i(t)-R_i) i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (326)$$

$$+ 4 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (327)$$

From the equation (270) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}}} \quad (328)$$

$$= \frac{2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \left( \frac{2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{4}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{2}{\eta(t)} \left( \left( \frac{g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)^{\Im} \varepsilon(t) - 2(\varepsilon(t)-\varepsilon_i(t)-R_i) \left( \frac{g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right)^{\Im} + 2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)} \quad (329)$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{(2v_{i\mathbf{k}}^{\Im}-i(g_{i\mathbf{k}}^*-g_{i\mathbf{k}}))\eta(t)}{v_{i\mathbf{k}}^{\Im} \left( 2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i) - \frac{4|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - i(g_{i\mathbf{k}}^*-g_{i\mathbf{k}}) \left( \varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i) \right) + 4 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (330)$$

$$= \frac{2(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im})\eta(t)}{v_{i\mathbf{k}}^{\Im} \left( 2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i) - \frac{4|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Im}(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)) + 4 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (331)$$

$$= \frac{2(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im})\eta(t)}{v_{i\mathbf{k}}^{\Im} \left( 2\varepsilon(t)-4(\varepsilon(t)-\varepsilon_i(t)-R_i) - \frac{4|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Im}(2\varepsilon_i(t)+2R_i-\varepsilon(t)) + 4 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (332)$$

$$= \frac{(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im})\eta(t)}{v_{i\mathbf{k}}^{\Im} \left( \varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Im}(2\varepsilon_i(t)+2R_i-\varepsilon(t)) + 2 \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (333)$$

Separating (333) such that the terms with  $v_{i\mathbf{k}}$  are located at one side of the equation permit us to write

$$\frac{(v_{i\mathbf{k}}^{\Im} - g_{i\mathbf{k}}^{\Im})\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Im} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (334)$$

$$v_{i\mathbf{k}}^{\Im} - g_{i\mathbf{k}}^{\Im} = v_{i\mathbf{k}}^{\Im} \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) \quad (335)$$

$$- \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\Im} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (336)$$

$$v_{i\mathbf{k}}^{\Im} = \frac{g_{i\mathbf{k}}^{\Im} \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (337)$$

$$v_{i\mathbf{k}}^{\Im} = \frac{g_{i\mathbf{k}}^{\Im} \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (338)$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}}) = v_{i\mathbf{k}}^{\Re}(\omega_{\mathbf{k}}) + i v_{i\mathbf{k}}^{\Im}(\omega_{\mathbf{k}}) \quad (339)$$

$$= \frac{g_{i\mathbf{k}}^{\Re}(\omega_{\mathbf{k}}) \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Re}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (340)$$

$$+ i \frac{g_{i\mathbf{k}}^{\Im}(\omega_{\mathbf{k}}) \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Im}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (341)$$

$$= \frac{g_{i\mathbf{k}}(\omega_{\mathbf{k}}) \left( 1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i - \varepsilon(t)) \right) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} |B_{10}|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left( \varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i) - \frac{2|V_{10}(t)|^2|B_{10}|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (342)$$

#### IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is  $\rho_T(0) = \rho_S(0) \otimes \rho_B^{\text{Thermal}}$ , the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^V \rho_T(0) e^{-V} \quad (343)$$

$$= (|0\rangle\langle 0|B_{0+} + |1\rangle\langle 1|B_{1+}) (\rho_S(0) \otimes \rho_B^{\text{Thermal}}) (|0\rangle\langle 0|B_{0-} + |1\rangle\langle 1|B_{1-}) \quad (344)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0|: |0\rangle\langle 0|B_{0+}|0\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \quad (345)$$

$$= |0\rangle\langle 0|B_{0+}|0\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \quad (346)$$

$$= |0\rangle\langle 0| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{0-} \quad (347)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1|: |1\rangle\langle 1|B_{1+}|1\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \quad (348)$$

$$= |1\rangle\langle 1|B_{1+}\rho_B^{\text{Thermal}}B_{1-} \quad (349)$$

$$= |1\rangle\langle 1| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{1-} \quad (350)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1|: |0\rangle\langle 0|B_{0+}|0\rangle\langle 1|\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \quad (351)$$

$$= |0\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}|1\rangle\langle 1|B_{1-} \quad (352)$$

$$= |0\rangle\langle 1|1\rangle\langle 1|B_{0+}\rho_B^{\text{Thermal}}B_{1-} \quad (353)$$

$$= |0\rangle\langle 1| \otimes B_{0+}\rho_B^{\text{Thermal}}B_{1-} \quad (354)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0|: |1\rangle\langle 1|B_{1+}|1\rangle\langle 0|\rho_B^{\text{Thermal}}|0\rangle\langle 0|B_{0-} \quad (355)$$

$$= |1\rangle\langle 0| \otimes B_{1+}\rho_B^{\text{Thermal}}B_{0-} \quad (356)$$

We transform any operator  $O$  into the interaction picture in the following way:

$$\tilde{O}(t) \equiv U^\dagger(t) O(t) U(t) \quad (357)$$

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right). \quad (358)$$

Here  $\mathcal{T}$  denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (359)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\overline{\rho_T}(t)) \quad (360)$$

. In order to separate the Hamiltonian we define the matrix  $\Lambda(t)$  such that  $\Lambda_{1i}(t) = A_i$ ,  $\Lambda_{2i}(t) = B_i$  and  $\Lambda_{3i}(t) = C_i(t)$  written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix} \quad (361)$$

In this case  $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$  and  $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$  with  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$ .

The previous notation allows us to write the interaction Hamiltonian  $\overline{H_I}(t)$  as pointed in the equation (245):

$$\overline{H_I}(t) = \sum_i B_{iz} |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x + \sigma_y B_y) + V_{10}^{\Im}(t) (\sigma_x B_y - \sigma_y B_x) \quad (362)$$

$$= B_{0z} |0\rangle\langle 0| + B_{1z} |1\rangle\langle 1| + V_{10}^{\Re}(t) \sigma_x B_x + V_{10}^{\Re}(t) \sigma_y B_y + V_{10}^{\Im}(t) \sigma_x B_y - V_{10}^{\Im}(t) \sigma_y B_x \quad (363)$$

$$= \sum_i C_i(t) (A_i \otimes B_i(t)) \quad (364)$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{d\widetilde{\rho}_T(t)}{dt} = -i[\widetilde{H}_I(t), \widetilde{\rho}_T(t)]. \quad (365)$$

This equation has the formal solution

$$\widetilde{\rho}_T(t) = \rho(0) - i \int_0^t [\widetilde{H}_I(s), \widetilde{\rho}_T(s)] ds. \quad (366)$$

Replacing the equation (366) in the equation (365) give us:

$$\frac{d\widetilde{\rho}_T(t)}{dt} = -i[\widetilde{H}_I(t), \rho_T(0)] - \int_0^t [\widetilde{H}_I(t), [\widetilde{H}_I(s), \widetilde{\rho}_T(s)]] ds. \quad (367)$$

This equation allow us to iterate and write in terms of a series expansion with  $\rho_T(0)$  the solution as:

$$\widetilde{\rho}_T(t) = \rho_T(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \dots [\widetilde{H}_I(t_n), \rho_T(0)]] \dots] \quad (368)$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho}_{\overline{S}}(t) = \rho_{\overline{S}}(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \dots [\widetilde{H}_I(t_n), \rho_{\overline{S}}(0) \rho_B^{\text{Thermal}}]] \dots] \quad (369)$$

here we have assumed that  $\rho_T(0) = \rho_{\overline{S}}(0) \otimes \rho_B^{\text{Thermal}}$ . Consider the following notation:

$$\widetilde{\rho}_{\overline{S}}(t) = (1 + W_1(t) + W_2(t) + \dots) \rho_{\overline{S}}(0) \quad (370)$$

$$= W(t) \rho_{\overline{S}}(0) \quad (371)$$

in this case

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H}_I(t_1), [\widetilde{H}_I(t_2), \dots [\widetilde{H}_I(t_n), (\cdot) \rho_B^{\text{Thermal}}]] \dots] \quad (372)$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{d\widetilde{\rho}_{\overline{S}}(t)}{dt} = (\dot{W}_1(t) + \dot{W}_2(t) + \dots) \rho_{\overline{S}}(0) \quad (373)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} W(t) \rho_{\overline{S}}(0) \quad (374)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} \widetilde{\rho}_{\overline{S}}(t) \quad (375)$$

where we assumed that  $W(t)$  is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that  $\text{Tr}_B [\widetilde{H}_I(t) \rho_B(0)] = 0$  so  $W_1(t) = 0$ . Thus, to second order and taking  $W(t) \approx \mathbb{I}$  then the equation (373) becomes:

$$\frac{d\widetilde{\rho}_{\overline{S}}(t)}{dt} = -i[H_{\overline{S}}, \rho_{\overline{S}}(t)] - \int_0^t d\tau [H_I, [\widetilde{H}_I(-\tau), \rho_{\overline{S}}(t) \rho_B^{\text{Thermal}}]] \quad (376)$$

Replacing  $t_1 \rightarrow t - \tau$

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\tilde{H}_I(t_1), [\tilde{H}_I(t_2), \dots [\tilde{H}_I(t_n), (\cdot) \rho_B^{\text{Thermal}}]] \dots] \quad (377)$$

Taking as reference state  $\rho_B^{\text{Thermal}}$  and truncating at second order in  $\overline{H_I}(t)$ , we obtain our master equation in the interaction picture in the transformed frame:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}}] \right] ds \quad (378)$$

From the interaction picture applied on  $\overline{H_I}(t)$  we find:

$$\widetilde{H_I}(t) = U^\dagger(t) e^{iH_B t} \overline{H_I}(t) e^{-iH_B t} U(t) \quad (379)$$

we use the time-ordering operator  $\mathcal{T}$  because in general  $\overline{H_S}(t)$  doesn't commute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{H_I}(t) = \sum_i C_i(t) \left( \widetilde{A}_i(t) \otimes \widetilde{B}_i(t) \right) \quad (380)$$

$$\widetilde{A}_i(t) = U^\dagger(t) e^{iH_B t} A_i e^{-iH_B t} U(t) \quad (381)$$

$$= U^\dagger(t) A_i U(t) e^{iH_B t} e^{-iH_B t} \quad (382)$$

$$= U^\dagger(t) A_i U(t) \mathbb{I} \quad (383)$$

$$= U^\dagger(t) A_i U(t) \quad (384)$$

$$\widetilde{B}_i(t) = U^\dagger(t) e^{iH_B t} B_i(t) e^{-iH_B t} U(t) \quad (385)$$

$$= U^\dagger(t) U(t) e^{iH_B t} B_i(t) e^{-iH_B t} \quad (386)$$

$$= \mathbb{I} e^{iH_B t} B_i(t) e^{-iH_B t} \quad (387)$$

$$= e^{iH_B t} B_i(t) e^{-iH_B t} \quad (388)$$

Here we have used the fact that  $[\overline{H_S}(t), H_B] = 0$  because these operators belong to different Hilbert spaces, so  $[U(t), e^{iH_B t}] = 0$ .

Using the expression (380) to replace it in the equation (378)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}}] \right] ds \quad (389)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)), [\sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)), \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}}] \right] ds \quad (390)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)), \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} - \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \right] ds \quad (391)$$

$$= - \int_0^t \text{Tr}_B \left( \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} - \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \right) \quad (392)$$

$$= - \sum_{i \in J} C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) + \widetilde{\rho_S}(t) \rho_B^{\text{Thermal}} \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \quad (393)$$

In order to calculate the correlation functions we define:

$$\Lambda_{ji}(\tau) = \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (394)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (395)$$

The correlation functions relevant that appear in the equation (393) are:

$$\text{Tr}_B \left( \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B^{\text{Thermal}} \right) = \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (396)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (397)$$

$$= \Lambda_{ji}(\tau) \quad (398)$$

$$\text{Tr}_B \left( \widetilde{B}_j(t) \rho_B^{\text{Thermal}} \widetilde{B}_i(s) \right) = \text{Tr}_B \left( \widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B^{\text{Thermal}} \right) \quad (399)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (400)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (401)$$

$$= \Lambda_{ij}(-\tau) \quad (402)$$

$$\text{Tr}_B \left( \widetilde{B}_i(s) \rho_B^{\text{Thermal}} \widetilde{B}_j(t) \right) = \text{Tr}_B \left( \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B^{\text{Thermal}} \right) \quad (403)$$

$$= \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (404)$$

$$= \left\langle \widetilde{B}_j(\tau) \widetilde{B}_i(0) \right\rangle_B \quad (405)$$

$$= \Lambda_{ji}(\tau) \quad (406)$$

$$\text{Tr}_B \left( \rho_B^{\text{Thermal}} \widetilde{B}_i(s) \widetilde{B}_j(t) \right) = \text{Tr}_B \left( \widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B^{\text{Thermal}} \right) \quad (407)$$

$$= \left\langle \widetilde{B}_i(s) \widetilde{B}_j(t) \right\rangle_B \quad (408)$$

$$= \left\langle \widetilde{B}_i(-\tau) \widetilde{B}_j(0) \right\rangle_B \quad (409)$$

$$= \Lambda_{ij}(-\tau) \quad (410)$$

The cyclic property of the trace was use widely in the development of equations (396) and (410). Replacing in (393)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{ij} (C_i(t) C_j(s) (\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s)) + C_i(t) C_j(s) (\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t))) ds \quad (411)$$

$$= - \int_0^t \sum_{ij} (C_i(t) C_j(s) (\Lambda_{ij}(\tau) [\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t)] + \Lambda_{ji}(-\tau) [\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t)])) ds \quad (412)$$

We could identify the following commutators in the equation deduced:

$$\Lambda_{ij}(\tau) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \Lambda_{ij}(\tau) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) = \Lambda_{ij}(\tau) [\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t)] \quad (413)$$

$$\Lambda_{ji}(-\tau) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \Lambda_{ji}(-\tau) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s) = \Lambda_{ji}(-\tau) [\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t)] \quad (414)$$

Returning to the Schroedinger picture we have:

$$U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) = U(t) \widetilde{A}_i(t) U^\dagger(t) U(t) \widetilde{A}_j(s) U^\dagger(t) U(t) \widetilde{\rho_S}(t) U^\dagger(t) \quad (415)$$

$$= \left( U(t) \widetilde{A}_i(t) U^\dagger(t) \right) \left( U(t) \widetilde{A}_j(s) U^\dagger(t) \right) \left( U(t) \widetilde{\rho_S}(t) U^\dagger(t) \right) \quad (416)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) \quad (417)$$

This procedure applying to the relevant commutators give us:

$$U(t) [\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t)] U^\dagger(t) = \left( U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) - U(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) U^\dagger(t) \right) \quad (418)$$

$$= A_i \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) - \widetilde{A}_j(s, t) \widetilde{\rho_S}(t) A_i \quad (419)$$

$$= [A_i, \widetilde{A}_j(t - \tau, t) \widetilde{\rho_S}(t)] \quad (420)$$

Introducing this transformed commutators in the equation (412) allow us to obtain the master equation of the system

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau \left( C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) \left[ A_i, \widetilde{A}_j(t-\tau, t) \overline{\rho_S}(t) \right] \right. \quad (421)$$

$$\left. + C_j(t) C_i(t-\tau) \Lambda_{ji}(-\tau) \left[ \overline{\rho_S}(t) \widetilde{A}_j(t-\tau, t), A_i \right] \right) \quad (422)$$

where  $i, j \in \{1, 2, 3, 4, 5, 6\}$ .

Here  $\widetilde{A}_j(s, t) = U(t) U^\dagger(s) A_j U(s) U^\dagger(t)$  where  $U(t)$  is given by (358). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in  $H_I(t)$ . The environmental correlation functions are given by:

$$\Lambda_{ij}(\tau) = \text{Tr}_B \left( \widetilde{B}_i(t) \widetilde{B}_j(s) \rho_B^{\text{Thermal}} \right) \quad (423)$$

$$= \text{Tr}_B \left( \widetilde{B}_i(\tau) \widetilde{B}_j(0) \rho_B^{\text{Thermal}} \right) \quad (424)$$

Calculating the correlation functions allow us to obtain:

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left( \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rho_B^{\text{Thermal}} \right) \quad (425)$$

$$= \int d^2\alpha P(\alpha) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle \quad (426)$$

$$= \frac{1}{\pi N} \int \exp \left( -\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle d^2\alpha \quad (427)$$

$$= \frac{1}{\pi N} \int \exp \left( -\frac{|\alpha|^2}{N} \right) \langle \alpha | \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) | \alpha \rangle d^2\alpha \quad (428)$$

$$\widetilde{B}_{jz}(\tau) = \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \quad (429)$$

$$\widetilde{B}_{jz}(0) = \sum_{\mathbf{k}'} \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \quad (430)$$

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left( \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rho_B \right) \quad (431)$$

$$= \text{Tr}_B \left( \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (432)$$

$$= \text{Tr}_B \left( \sum_{\mathbf{k} \neq \mathbf{k}'} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (433)$$

$$+ \text{Tr}_B \left( \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \rho_B \right) \quad (434)$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}} = p_{j\mathbf{k}} \quad (435)$$

$$\langle \widetilde{B}_{jz}(\tau) \widetilde{B}_{jz}(0) \rangle_B = \text{Tr}_B \left( \sum_{\mathbf{k} \neq \mathbf{k}'} \left( p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( p_{j\mathbf{k}'} b_{\mathbf{k}'}^\dagger + p_{j\mathbf{k}'}^* b_{\mathbf{k}'} \right) \rho_B \right) + \text{Tr}_B \left( \sum_{\mathbf{k}} \left( p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger + p_{j\mathbf{k}}^* b_{\mathbf{k}} \right) \rho_B \right) \quad (436)$$

$$= 0 + \text{Tr}_B \left( \sum_{\mathbf{k}} \left( p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left( p_{j\mathbf{k}} b_{\mathbf{k}}^\dagger + p_{j\mathbf{k}}^* b_{\mathbf{k}} \right) \rho_B \right) \quad (437)$$

$$= \text{Tr}_B \left( \sum_{\mathbf{k}} \left( p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} + |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} + p_{j\mathbf{k}}^* p_{j\mathbf{k}} b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \rho_B \right) \quad (438)$$

$$= \text{Tr}_B \left( \sum_{\mathbf{k}} p_{j\mathbf{k}}^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left( \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left( \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left( \sum_{\mathbf{k}} p_{j\mathbf{k}}^* p_{j\mathbf{k}} b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) \quad (439)$$

$$= \text{Tr}_B \left( \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left( \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) \quad (440)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( e^{i\omega_{\mathbf{k}}\tau} \text{Tr}_B \left( b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rho_B \right) + e^{-i\omega_{\mathbf{k}}\tau} \text{Tr}_B \left( b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rho_B \right) \right) \quad (441)$$

$$= \sum_{\mathbf{k}} |p_{j\mathbf{k}}|^2 \left( e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp \left( -\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp \left( -\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}} \right) \quad (442)$$





$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}'} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) \sum_{\mathbf{k}''} ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}') b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}')^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (474)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k} \neq \mathbf{k}'} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}') b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}')^* b_{\mathbf{k}'} e^{-i\omega_{\mathbf{k}'}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (475)$$

$$+ \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (476)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) ((g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau}) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (477)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (478)$$

$$= \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (479)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \mathbb{q} | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | \mathbb{p} \rangle d^2 \alpha_{\mathbf{k}} \quad (480)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (481)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (482)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) |\alpha_{\mathbf{k}}|^2 d^2 \alpha_{\mathbf{k}} \quad (483)$$

$$= N \quad (484)$$

$$\frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \mathbb{q} | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) | \mathbb{p} \rangle d^2 \alpha_{\mathbf{k}} \quad (485)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (486)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (487)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (488)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle 0 | |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \mathbb{q} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \mathbb{p} \rangle d^2 \alpha_{\mathbf{k}} \quad (489)$$

$$= N + 1 \quad (490)$$

$$\langle \widetilde{B_{jz}}(\tau) \widetilde{B_{j'z}}(0) \rangle_B = \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N+1) \quad (491)$$

$$= \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} + N ((g_{j\mathbf{k}} - v_{j\mathbf{k}}) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau})) \quad (492)$$

$$D(h') D(h) = \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) D(h' + h) \quad (493)$$

$$\langle D(h') D(h) \rangle_B = \text{Tr}_B \left( \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) D(h' + h) \rho_B^{\text{Thermal}} \right) \quad (494)$$

$$= \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) \text{Tr}_B \left( D(h' + h) \rho_B^{\text{Thermal}} \right) \quad (495)$$

$$= \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) \frac{1}{\pi N} \int d^2 \alpha P(\alpha) \langle \alpha | D(h' + h) | \alpha \rangle \quad (496)$$

$$= \exp\left(\frac{1}{2} (h'h^* - h'^*h)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (497)$$

$$h' = h \exp(i\omega\tau) \quad (498)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp\left(\frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau))\right) \exp\left(-\frac{|h + h \exp(i\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (499)$$

$$\frac{1}{2} |h|^2 (\exp(i\omega\tau) - \exp(-i\omega\tau)) = \frac{1}{2} (h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau)) \quad (500)$$

$$= \frac{1}{2} |h|^2 (\cos(\omega\tau) + i \sin(\omega\tau) - \cos(\omega\tau) + i \sin(\omega\tau)) \quad (501)$$

$$= \frac{1}{2} |h|^2 (2i \sin(\omega\tau)) \quad (502)$$

$$= i |h|^2 \sin(\omega\tau) \quad (503)$$

$$-\frac{|h + h \exp(i\omega\tau)|^2}{2} = -|h|^2 \frac{|1 + \exp(i\omega\tau)|^2}{2} \quad (504)$$

$$= -|h|^2 \frac{|1 + \cos(\omega\tau) + i \sin(\omega\tau)|^2}{2} \quad (505)$$

$$= -|h|^2 \frac{(1 + \cos(\omega\tau))^2 + \sin^2(\omega\tau)}{2} \quad (506)$$

$$= -|h|^2 \frac{(1 + 2 \cos(\omega\tau) + \cos^2(\omega\tau)) + \sin^2(\omega\tau)}{2} \quad (507)$$

$$= -|h|^2 \frac{2 + 2 \cos(\omega\tau)}{2} \quad (508)$$

$$= -|h|^2 (1 + \cos(\omega\tau)) \quad (509)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp(i|h|^2 \sin(\omega\tau)) \exp\left(-|h|^2(1+\cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (510)$$

$$= \exp \left( i |h|^2 \sin(\omega\tau) - |h|^2 (1 + \cos(\omega\tau)) \coth \left( \frac{\beta\omega}{2} \right) \right) \quad (511)$$

$$= \exp \left( -|h|^2 \left( -i \sin(\omega\tau) + \cos(\omega\tau) \coth \left( \frac{\beta\omega}{2} \right) \right) \right) \exp \left( -|h|^2 \coth \left( \frac{\beta\omega}{2} \right) \right) \quad (512)$$

$$= \langle D(h) \rangle_B \exp(-\phi(\tau)) \quad (513)$$

$$\exp(-\phi(\tau)) = \exp\left(-|h|^2\left(\cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right) - i\sin(\omega\tau)\right)\right) \quad (514)$$

$$\phi(\tau) = |h|^2 \left( \cos(\omega\tau) \coth\left(\frac{\beta\omega}{2}\right) - i \sin(\omega\tau) \right) \quad (515)$$

$$\langle D(h') D(h) \rangle_B = \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \exp\left(-\frac{|h+h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right) \quad (516)$$

$$h' = v \exp(i\omega\tau) \quad (517)$$

$$\langle \widetilde{B_{1+} B_{0-}}(\tau) \widetilde{B_{1+} B_{0-}}(0) \rangle_B = \text{Tr}_B \left( \widetilde{B_{1+} B_{0-}}(\tau) \widetilde{B_{1+} B_{0-}}(0) \rho_B^{\text{Thermal}} \right) \quad (518)$$

$$= \text{Tr}_B \left( \widetilde{B_{1+} B_{0-}}(\tau) \widetilde{B_{1+} B_{0-}}(0) \rho_B^{\text{Thermal}} \right) \quad (519)$$

$$= \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (520)$$

$$= \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (521)$$

$$= \Pi_{\mathbf{k}} \left( \exp \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \rho_B^{\text{Thermal}} \right) \quad (522)$$

$$= \Pi_{\mathbf{k}} \left( \exp \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \Pi_{\mathbf{k}} \left( \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (523)$$

$$= \Pi_{\mathbf{k}} \left( \exp \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (524)$$

$$\langle \widetilde{B_+ B_-} \tau \widetilde{B_+ B_-}(0) \rangle_B = \Pi_{\mathbf{k}} \left( \exp \left( \frac{v_0 \mathbf{k} \cdot \mathbf{v}_1 \mathbf{k} - v_0 \mathbf{k} \cdot \mathbf{v}_1^* \mathbf{k}}{\omega_{\mathbf{k}}^2} \right) \exp \left( - \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (525)$$

$$\langle \widetilde{B_{+1} + \bar{B}_{0-}(\tau)} \widetilde{B_{0+} + \bar{B}_{1-}(0)} \rangle_B = \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left( D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}} \omega_{\mathbf{k}}} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (526)$$

$$= \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} \left( D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \rho_B^{\text{Thermal}} \right) \quad (527)$$

$$= \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \Pi_{\mathbf{k}} D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) \Pi_{\mathbf{k}} D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \quad (528)$$

$$= \text{Tr}_B \left( \Pi_{\mathbf{k}} D \left( \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) \Pi_{\mathbf{k}} D \left( \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \quad (529)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left( \left( D \left( \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i\omega\tau} \right) D \left( \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \right) \rho_B^{\text{Thermal}} \right) \quad (530)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left( \left( D \left( \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i(\omega \tau + \pi)} \right) D \left( \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \right) \rho_B^{\text{Thermal}} \right) \quad (531)$$

$$= \Pi_{\mathbf{k}} \exp \left( - \left| \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega \tau + \pi) + \cos(\omega \tau + \pi) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (532)$$

$$= \Pi_{\mathbf{k}} \exp\left(-\left|\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}}\right|^2 \left(i \sin(\omega \tau) - \cos(\omega \tau) \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)\right)\right) \exp\left(-\left|\frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2}\right)\right) \quad (533)$$

$$\langle \widetilde{B_{0+} B_{1-}(\tau)} \widetilde{B_{1+} B_{0-}(0)} \rangle_B = \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \rho_B^{\text{Thermal}} \right) \right) \quad (534)$$

$$= \text{Tr}_B \left( \Pi_{\mathbf{k}} D \left( \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i \omega_{\mathbf{k}} \tau} \right) \Pi_{\mathbf{k}} D \left( \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \quad (535)$$

$$= \Pi_{\mathbf{k}} \text{Tr}_B \left( D \left( \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i(\omega_{\mathbf{k}} \tau + \pi)} \right) D \left( \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right) \rho_B^{\text{Thermal}} \right) \quad (536)$$

$$= \Pi_{\mathbf{k}} \exp \left( - \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau + \pi) + \cos(\omega_{\mathbf{k}} \tau + \pi) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_1 \mathbf{k} - v_0 \mathbf{k}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (537)$$

$$= \langle |B_{1+} \widehat{B}_{0-}(\tau) B_{0+} \widehat{B}_{1-}(0) \rangle_B \quad (538)$$

$$\langle \widehat{B}_{0+} \widehat{B}_{1-}(\tau) \widehat{B}_{jz}(0) \rangle_B = \text{Tr}_B \left( \Pi_{\mathbf{k}} \left( D \left( \frac{v_0 \mathbf{k} - v_1 \mathbf{k}}{\omega_{\mathbf{k}}} e^{i \omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left( \frac{v_0^* \mathbf{k} v_1 \mathbf{k} - v_0 \mathbf{k} v_1^* \mathbf{k}}{\omega_{\mathbf{k}}^2} \right)} \right) \Sigma_{\mathbf{k}'} \left( (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'})^* b_{\mathbf{k}'} \right) \rho_B^{\text{Thermal}} \right) \quad (539)$$

$$\langle D(h)b \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h)b | \alpha \rangle \quad (540)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle \alpha | D(-\alpha) D(h) b D(\alpha) | \alpha \rangle \quad (541)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b D(\alpha) | 0 \rangle \quad (542)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \quad (543)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \quad (544)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b + \alpha) | 0 \rangle \quad (545)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (546)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (547)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( - \frac{|h|^2}{2} \right) d^2 \alpha \quad (548)$$

$$= hN \langle D(h) \rangle_B \quad (549)$$

$$\langle D(h) b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h) b^\dagger | \alpha \rangle \quad (550)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (551)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (552)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) | 0 \rangle \quad (553)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b^\dagger + \alpha^*) | 0 \rangle \quad (554)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b^\dagger + \alpha^*) | 0 \rangle \quad (555)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \quad (556)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle \quad (557)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle -h | 1 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 0 \rangle \quad (558)$$

$$\langle -h | = \exp \left( - \frac{|-h^*|^2}{2} \right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n | \quad (559)$$

$$\langle -h | 1 \rangle = \exp \left( - \frac{|-h^*|^2}{2} \right) (-h^*) \quad (560)$$

$$\langle D(h) b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( - \frac{|-h^*|^2}{2} \right) (-h^*) + \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \exp \left( - \frac{|-h^*|^2}{2} \right) \quad (561)$$

$$= -h^* \langle D(h) \rangle_B (N+1) \quad (562)$$

$$\langle b D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \langle \alpha | b D(h) | \alpha \rangle \quad (563)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( - \frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left( - \frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left( - \frac{|h|^2}{2} \right) \quad (564)$$

$$= h \langle D(h) \rangle_B (N+1) \quad (565)$$

$$\langle b^\dagger D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \langle \alpha | b^\dagger D(h) | \alpha \rangle \quad (566)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left( -\frac{|h|^2}{2} \right) h + \frac{1}{\pi N} \int d^2 \alpha \exp \left( -\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha \exp \left( -\frac{|h|^2}{2} \right) \quad (567)$$

$$= -h^* \langle D(h) \rangle_B N \quad (568)$$

$$\langle \widetilde{B_{1+} B_{0-}}(\tau) \rangle_B = \left\langle \prod_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle_B \quad (569)$$

$$= \prod_{\mathbf{k}} \left( e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \langle D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \rangle_B \quad (570)$$

$$= \prod_{\mathbf{k}} \left( e^{\frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \langle D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \rangle_B \quad (571)$$

$$= \prod_{\mathbf{k}} \left( \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp \left( -\frac{1}{2} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (572)$$

$$= B_{10} \quad (573)$$

The correlation functions can be found readily as:

$$\widetilde{B_{1+} B_{0-}}(\tau) = \prod_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \quad (574)$$

$$\widetilde{B_{0+} B_{1-}}(\tau) = \prod_{\mathbf{k}} \left( D \left( \frac{v_{0\mathbf{k}} - v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \exp \left( \frac{1}{2} \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \quad (575)$$

$$\widetilde{B_x}(0) = \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} \quad (576)$$

$$\widetilde{B_y}(0) = \frac{B_{0+} B_{1-} - B_{1+} B_{0-} + B_{10} - B_{01}}{2i} \quad (577)$$

$$B_{10} = \left( \prod_{\mathbf{k}} \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left( \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (578)$$

$$B_{iz} = \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} \right) \quad (579)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \left\langle \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} \tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} \tau} \right) \sum_{\mathbf{k}} \left( (g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} \right) \right\rangle_B \quad (580)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}} \tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}} \tau} (N_{\mathbf{k}} + 1) \quad (581)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \left\langle \frac{B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{01}}{2} \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}}{2} \right\rangle_B \quad (582)$$

$$= \frac{1}{4} \langle (B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{01}) (B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10} - B_{01}) \rangle_B \quad (583)$$

$$= \frac{1}{4} \langle B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10} - B_{1+} B_{0-}(\tau) B_{01} + B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10} - B_{0+} B_{1-}(\tau) B_{01} + B_{10} B_{0+} B_{1-} + B_{10} B_{0+} B_{1-} - B_{01} B_{1+} B_{0-} - B_{01} B_{0+} B_{1-} + B_{01} B_{10} + B_{01} B_{01} \rangle \quad (584)$$

$$- B_{0+} B_{1-}(\tau) B_{10} - B_{0+} B_{1-}(\tau) B_{01} + B_{10} B_{0+} B_{1-} + B_{10} B_{0+} B_{1-} - B_{01} B_{1+} B_{0-} - B_{01} B_{0+} B_{1-} + B_{01} B_{10} + B_{01} B_{01} \rangle \quad (585)$$

$$= \frac{1}{4} \langle B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10} - B_{1+} B_{0-}(\tau) B_{01} + B_{0+} B_{1-}(\tau) B_{1+} B_{0-} - B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10} - B_{0+} B_{1-}(\tau) B_{01} \rangle \quad (586)$$

$$+ B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{10} - B_{0+} B_{1-}(\tau) B_{01} \rangle \quad (587)$$

$$\langle \widetilde{B_{0+}B_{1-}}(\tau) \widetilde{B_{0+}B_{1-}}(0) \rangle_B = \prod_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (588)$$

$$U = \prod_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (589)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}} \tau) + \cos(\omega_{\mathbf{k}} \tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (590)$$

$$S = \prod_{\mathbf{k}} \exp \left( - \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (591)$$

$$\langle \widetilde{B_{0+}B_{1-}}(\tau) \widetilde{B_{0+}B_{1-}}(0) \rangle_B = U \exp(-\phi(\tau)) S \quad (592)$$

$$\langle \widetilde{B_{1+}B_{0-}}(\tau) \widetilde{B_{1+}B_{0-}}(0) \rangle_B = U^* \exp(-\phi(\tau)) S \quad (593)$$

$$\langle \widetilde{B_{1+}B_{0-}}(\tau) \widetilde{B_{0+}B_{1-}}(0) \rangle_B = \exp(\phi(\tau)) S \quad (594)$$

$$\langle \widetilde{B_{0+}B_{1-}}(\tau) \widetilde{B_{1+}B_{0-}}(0) \rangle_B = \langle \widetilde{B_{1+}B_{0-}}(\tau) \widetilde{B_{0+}B_{1-}}(0) \rangle_B \quad (595)$$

$$\langle \widetilde{B_{1+}B_{0-}}(\tau) \rangle_B = \prod_{\mathbf{k}} \left( \exp \left( \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \exp \left( - \frac{1}{2} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (596)$$

$$= U^{*1/2} S^{1/2} \quad (597)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{1}{4} \langle B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{10-}B_{1+}B_{0-}(\tau) B_{01+}B_{0+}B_{1-}(\tau) B_{1+}B_{0-} - \quad (598)$$

$$+ B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{10-}B_{0+}B_{1-}(\tau) B_{01} \rangle \quad (599)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{1}{4} \langle B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{10-}B_{1+}B_{0-}(\tau) B_{01+}B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{10-}B_{0+}B_{1-}(\tau) B_{01} \rangle \quad (600)$$

$$= \frac{1}{4} \left( U^* \exp(-\phi(\tau)) S + \exp(\phi(\tau)) S - B_{10}^2 - |B_{10}|^2 + \exp(\phi(\tau)) S + U \exp(-\phi(\tau)) S - B_{10}^{*2} - |B_{10}|^2 \right) \quad (602)$$

$$= \frac{1}{4} \left( 2U^{\Re} \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2(B_{10}^2)^{\Re} - 2|B_{10}|^2 \right) \quad (603)$$

$$= \frac{1}{4} \left( 2U^{\Re} \exp(-\phi(\tau)) S + 2\exp(\phi(\tau)) S - 2(U^*)^{\Re} S - 2S \right) \quad (604)$$

$$= \frac{S}{2} \left( U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - (U^*)^{\Re} - 1 \right) \quad (605)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \left\langle \frac{B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10-}B_{01}}{2i} \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10-}B_{01}}{2i} \right\rangle_B \quad (606)$$

$$= -\frac{1}{4} \langle (B_{0+}B_{1-}(\tau) - B_{1+}B_{0-}(\tau) + B_{10-}B_{01})(B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10-}B_{01}) \rangle_B \quad (607)$$

$$= -\frac{1}{4} \langle B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{10-}B_{01} - B_{1+}B_{0-}(\tau) B_{0+}B_{1-} + B_{1+}B_{0-}(\tau) B_{1+}B_{0-} - \quad (608)$$

$$- B_{1+}B_{0-}(\tau) B_{10+}B_{1+}B_{0-}(\tau) B_{01} + B_{10-}B_{01} B_{0+}B_{1-} - B_{10-}B_{01} B_{1+}B_{0-} - B_{10-}B_{01} B_{10-}B_{01} B_{0+}B_{1-} + B_{01} B_{10+}B_{01} B_{01} \rangle \quad (609)$$

$$= -\frac{1}{4} \langle B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{10-}B_{01} - B_{1+}B_{0-}(\tau) B_{0+}B_{1-} + B_{1+}B_{0-}(\tau) B_{10+}B_{1+}B_{0-}(\tau) B_{01} \rangle \quad (610)$$

$$= -\frac{1}{4} \langle B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{0+}B_{1-}(\tau) B_{10+}B_{1+}B_{0-}(\tau) B_{01} \rangle \quad (611)$$

$$= -\frac{1}{4} \langle B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{1+}B_{0-} + B_{01} B_{10-}B_{01} B_{01} - B_{1+}B_{0-}(\tau) B_{0+}B_{1-} + B_{1+}B_{0-}(\tau) B_{10+}B_{1+}B_{0-}(\tau) B_{01} \rangle \quad (612)$$

$$= -\frac{1}{4} \left( U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S - \exp(\phi(\tau)) S + U^* \exp(-\phi(\tau)) S + 2S - 2(U^*)^{\Re} S \right) \quad (613)$$

$$= -\frac{S}{4} \left( 2U^{\Re} \exp(-\phi(\tau)) - 2\exp(\phi(\tau)) + 2 - 2U^{\Re} \right) \quad (614)$$

$$= \frac{S}{2} \left( \exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re} \right) \quad (615)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \left\langle \frac{B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10-}B_{01}}{2} \frac{B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10-}B_{01}}{2i} \right\rangle_B \quad (616)$$

$$= \frac{1}{4i} \langle (B_{1+}B_{0-}(\tau) + B_{0+}B_{1-}(\tau) - B_{10-}B_{01})(B_{0+}B_{1-} - B_{1+}B_{0-} + B_{10-}B_{01}) \rangle_B \quad (617)$$

$$= \frac{1}{4i} \langle B_{1+}B_{0-}(\tau) B_{0+}B_{1-} - B_{1+}B_{0-}(\tau) B_{1+}B_{0-} + B_{1+}B_{0-}(\tau) B_{10-}B_{01} + B_{0+}B_{1-}(\tau) B_{0+}B_{1-} - B_{0+}B_{1-}(\tau) B_{1+}B_{0-} - \quad (618)$$

$$+ B_{0+}B_{1-}(\tau) B_{10-}B_{01} - B_{0+}B_{1-}(\tau) B_{10+}B_{1+}B_{0-}(\tau) B_{01} + B_{10-}B_{01} B_{0+}B_{1-} - B_{10-}B_{01} B_{1+}B_{0-} - B_{10-}B_{01} B_{10+}B_{01} B_{01} \rangle \quad (619)$$

$$= \frac{1}{4i} \langle B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{1+} B_{0-}(\tau) B_{10-} B_{1+} B_{0-}(\tau) B_{01+} B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{1+} B_{0-}(\tau) B_{01+} B_{0+} B_{1-}(\tau) B_{0+} B_{1-} \rangle \quad (620)$$

$$- B_{0+} B_{1-}(\tau) B_{01+} \rangle \quad (621)$$

$$= \frac{1}{4i} \langle B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{10-} B_{10-} B_{01+} B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{01+} B_{10-} B_{01+} B_{0+} B_{1-} \rangle \quad (622)$$

$$= \frac{1}{4i} \langle B_{1+} B_{0-}(\tau) B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} + B_{10-} B_{10-} B_{0+} B_{1-}(\tau) B_{0+} B_{1-} - B_{0+} B_{1-}(\tau) B_{1+} B_{0-} - B_{01+} B_{01+} \rangle \quad (623)$$

$$= \frac{1}{4i} (\exp(\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S - \exp(\phi(\tau)) S + U^* S - US) \quad (624)$$

$$= \frac{1}{4i} (-U^* \exp(-\phi(\tau)) S + U \exp(-\phi(\tau)) S + U^* S - US) \quad (625)$$

$$= \frac{S}{4i} (-U^* \exp(-\phi(\tau)) + U \exp(-\phi(\tau)) + U^* - U) \quad (626)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (627)$$

$$= \frac{2iU^3 S}{4i} (\exp(-\phi(\tau)) - 1) \quad (628)$$

$$= \frac{U^3 S}{2} (\exp(-\phi(\tau)) - 1) \quad (629)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \left\langle \frac{B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10-} B_{01+}}{2i} \frac{B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10-} B_{01+}}{2} \right\rangle_B \quad (630)$$

$$= \frac{1}{4i} \langle (B_{0+} B_{1-}(\tau) - B_{1+} B_{0-}(\tau) + B_{10-} B_{01+}) (B_{1+} B_{0-} + B_{0+} B_{1-} - B_{10-} B_{01+}) \rangle_B \quad (631)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{01+} - B_{1+} B_{0-}(\tau) B_{01+} B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10-} B_{01+} B_{0+} B_{1-} \rangle \quad (632)$$

$$+ B_{1+} B_{0-}(\tau) B_{10-} B_{1+} B_{0-}(\tau) B_{01+} B_{10-} B_{01+} B_{0+} B_{1-} - B_{10-} B_{10-} B_{01+} B_{0+} B_{1-} - B_{01+} B_{0+} B_{1-} + B_{01+} B_{10-} + B_{01+} B_{01+} \rangle \quad (633)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{01+} - B_{1+} B_{0-}(\tau) B_{01+} B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{10-} B_{01+} B_{0+} B_{1-} \rangle \quad (634)$$

$$+ B_{1+} B_{0-}(\tau) B_{10-} B_{1+} B_{0-}(\tau) B_{01+} \rangle \quad (635)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{01+} - B_{01+} B_{10-} B_{01+} B_{0+} B_{1-} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} - B_{1+} B_{0-}(\tau) B_{01+} B_{10-} + B_{10-} B_{01+} \rangle \quad (636)$$

$$= \frac{1}{4i} \langle B_{0+} B_{1-}(\tau) B_{1+} B_{0-} + B_{0+} B_{1-}(\tau) B_{10-} B_{01+} - B_{01+} B_{01+} - B_{1+} B_{0-}(\tau) B_{1+} B_{0-} - B_{1+} B_{0-}(\tau) B_{01+} B_{10-} \rangle \quad (637)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + B_{10}^2 - B_{10}^{*2}) \quad (638)$$

$$= \frac{1}{4i} (U \exp(-\phi(\tau)) S - U^* \exp(-\phi(\tau)) S + U^* S - US) \quad (639)$$

$$= \frac{S(U - U^*)}{4i} (\exp(-\phi(\tau)) - 1) \quad (640)$$

$$= \frac{2iU^3 S}{4i} (\exp(-\phi(\tau)) - 1) \quad (641)$$

$$= -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} \quad (642)$$

$$\langle B_{1+} B_{0-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'}^\dagger \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \Pi_{\mathbf{k}} \exp\left(\frac{1}{2} \left( \frac{v_{1\mathbf{k}'}^* v_{0\mathbf{k}'} - v_{1\mathbf{k}'} v_{0\mathbf{k}'}}{\omega_{\mathbf{k}}^2} \right)\right) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{N_{\mathbf{k}'}} \left\langle \Pi_{\mathbf{k}} \left( D \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \right) \right\rangle \quad (643)$$

$$= (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{N_{\mathbf{k}'}} B_{10} \quad (644)$$

$$\langle B_{0+} B_{1-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger \rangle_B = -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01} \quad (645)$$

$$\langle B_{0+} B_{1-}(\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{N_{\mathbf{k}'}} B_{01} \quad (646)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = \frac{1}{2} \sum_{\mathbf{k}'} \left( -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01} \right. \quad (647)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{N_{\mathbf{k}'}} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{N_{\mathbf{k}'}} B_{01} \right) \quad (648)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left( -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01} \right. \quad (649)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{N_{\mathbf{k}'}} B_{10} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{N_{\mathbf{k}'}} B_{01} \right) \quad (650)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left( -(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10} + \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^*_{B_{10}} + \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^*_{B_{01}} \right) \quad (651)$$

$$\left. + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*_{N_{\mathbf{k}'}} \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{B_{10}} + \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)_{B_{01}} \right) \right) \quad (652)$$







$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( - \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} \right) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( - \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (711)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \quad (712)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} \left( - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (713)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \quad (714)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{01} N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10} N_{\mathbf{k}'} \right) \right. \quad (715)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} \left( (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10} \right) \right) \right) \quad (716)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* (B_{10} + N_{\mathbf{k}'}) - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (B_{10} + (N_{\mathbf{k}'} + 1)) \right) \quad (717)$$

$$= \frac{1}{i} \sum_{\mathbf{k}'} \left( e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) \right) \quad (718)$$

$$= i \sum_{\mathbf{k}'} \left( e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} \right) \quad (719)$$

$$= i \sum_{\mathbf{k}'} \left( e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re} (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re} N_{\mathbf{k}'} \right) \quad (720)$$

$$= i B_{10}^{\Re} \sum_{\mathbf{k}'} \left( e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) \left( \frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* N_{\mathbf{k}'} \right) \quad (721)$$

The correlation functions are equal to:

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (722)$$

$$U = \prod_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (723)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left( \frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (724)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{|B_{10}|^2}{2} \left( U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^{\Re} - 1 \right) \quad (725)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{|B_{10}|^2}{2} \left( \exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re} \right) \quad (726)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1) \quad (727)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1) \quad (728)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (729)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (730)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_y}(0) \rangle_B = i B_{10}^{\Re} \sum_{\mathbf{k}} \left( e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (731)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Re} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \quad (732)$$

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}) \quad (733)$$

$$v_{i\mathbf{k}} = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}) \quad (734)$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i(\omega)}{\omega} d\omega \quad (735)$$

The integral version of the correlation functions are given by:

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_{jz}}(0) \rangle_B = \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - v_{i\mathbf{k}})(g_{j\mathbf{k}} - v_{j\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (736)$$

$$= \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))^* (g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}})) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (737)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) g_{j\mathbf{k}}^* (1 - F_j(\omega_{\mathbf{k}}))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}))^* g_{j\mathbf{k}} (1 - F_j(\omega_{\mathbf{k}})) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (738)$$

$$\approx \int_0^\infty (\sqrt{J_i(\omega) J_j^*(\omega)} (1 - F_i(\omega)) (1 - F_j^*(\omega)) e^{i\omega\tau} N(\omega) + \sqrt{J_i^*(\omega) J_j(\omega)} (1 - F_i^*(\omega)) (1 - F_j(\omega)) e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (739)$$

$$U = \prod_{\mathbf{k}} \left( \exp \left( \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (740)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \quad (741)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}) g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (742)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}) g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (743)$$

$$= \exp \left( \sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* g_{1\mathbf{k}} F_0^*(\omega_{\mathbf{k}}) F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} g_{1\mathbf{k}}^* F_0(\omega_{\mathbf{k}}) F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \quad (744)$$

$$\approx \exp \left( \int_0^\infty \frac{\sqrt{J_0^*(\omega) J_1(\omega)} F_0^*(\omega) F_1(\omega) - \sqrt{J_0(\omega) J_1^*(\omega)} F_0(\omega) F_1^*(\omega)}{\omega^2} d\omega \right) \quad (745)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (746)$$

$$= \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \left( -i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (747)$$

$$\approx \int_0^\infty \left| \frac{\sqrt{J_1(\omega) F_1(\omega)} - \sqrt{J_0(\omega) F_0(\omega)}}{\omega} \right|^2 \left( -i \sin(\omega\tau) + \cos(\omega\tau) \coth \left( \frac{\beta\omega}{2} \right) \right) d\omega \quad (748)$$

$$B_{10} = \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left( \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (749)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left( \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \quad (750)$$

$$= \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right|^2 \coth \left( \frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left( \sum_{\mathbf{k}} \frac{1}{2} \left( \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}) g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2} \right) \right) \quad (751)$$

$$\approx \exp \left( -\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1(\omega) F_1(\omega)} - \sqrt{J_0(\omega) F_0(\omega)}}{\omega} \right|^2 \coth \left( \frac{\beta\omega}{2} \right) d\omega \right) \exp \left( \int_0^\infty \frac{1}{2} \left( \frac{\sqrt{J_0(\omega) J_1^*(\omega)} F_0(\omega) F_1^*(\omega) - \sqrt{J_0^*(\omega) J_1(\omega)} F_0^*(\omega) F_1(\omega)}{\omega^2} \right) d\omega \right) \quad (752)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{|B_{10}|^2}{2} (U^{\Re} \exp(-\phi(\tau)) + \exp(\phi(\tau)) - U^{\Re} - 1) \quad (753)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{|B_{10}|^2}{2} (\exp(\phi(\tau)) - U^{\Re} \exp(-\phi(\tau)) - 1 + U^{\Re}) \quad (754)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_y}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1) \quad (755)$$

$$\langle \widetilde{B_y}(\tau) \widetilde{B_x}(0) \rangle_B = \frac{U^{\Im} |B_{10}|^2}{2} (\exp(-\phi(\tau)) - 1) \quad (756)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}}) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (757)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}))^* \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (758)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left( g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \right)^* - g_{i\mathbf{k}}^* (1 - F_i(\omega_{\mathbf{k}}))^* \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (759)$$

$$Q(\omega) = \sqrt{J_i(\omega)} (1 - F_i(\omega)) \left( \frac{\sqrt{J_1(\omega)} F_1(\omega) - \sqrt{J_0(\omega)} F_0(\omega)}{\omega} \right)^* \quad (760)$$

$$\langle \widetilde{B_{iz}}(\tau) \widetilde{B_x}(0) \rangle_B \approx i B_{10}^{\Im} \int_0^\infty (Q(\omega) N(\omega) e^{i\omega\tau} - Q^*(\omega) (N(\omega) + 1) e^{-i\omega\tau}) d\omega \quad (761)$$

$$\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \rangle_B = i B_{10}^{\Im} \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (762)$$

$$= i B_{10}^{\Im} \sum_{\mathbf{k}} \left( g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} - g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (763)$$

$$\approx iB_{10}^3 \int_0^\infty \left( Q^*(\omega) N(\omega) e^{i\omega\tau} - Q(\omega) (N(\omega) + 1) e^{-i\omega\tau} \right) d\omega \quad (764)$$

$$\langle \widetilde{B_{iz}(\tau)} \widetilde{B_{y(0)}} \rangle_B = iB_{10}^3 \sum_{\mathbf{k}} \left( e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (765)$$

$$= iB_{10}^3 \sum_{\mathbf{k}} \left( e^{-i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (766)$$

$$\approx iB_{10}^3 \int_0^\infty \left( e^{-i\omega\tau} Q^*(\omega) (N(\omega) + 1) - e^{i\omega\tau} Q(\omega) N(\omega) \right) d\omega \quad (767)$$

$$\langle \widetilde{B_y(\tau)} \widetilde{B_{iz}(0)} \rangle_B = iB_{10}^3 \sum_{\mathbf{k}} \left( (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \quad (768)$$

$$= iB_{10}^3 \sum_{\mathbf{k}} \left( g_{i\mathbf{k}}^* (1 - F_i^*(\omega_{\mathbf{k}})) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}} (1 - F_i(\omega_{\mathbf{k}})) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left( \frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^* \right) \quad (769)$$

$$= iB_{10}^3 \int_0^\infty \left( e^{i\omega\tau} Q^*(\omega) N(\omega) - e^{-i\omega\tau} Q(\omega) (N(\omega) + 1) \right) d\omega \quad (770)$$

The eigenvalues of the Hamiltonian  $\overline{H_S}$  are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}(\overline{H_S}) \lambda + \text{Det}(\overline{H_S}) = 0 \quad (771)$$

The solutions of this equation written in terms of  $\eta$  and  $\xi$  as defined in the previous section are given by  $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$  and they satisfy  $H_S |\pm\rangle = \lambda_{\pm} |\pm\rangle$ . Using this notation is possible to write  $H_S = \lambda_+ |+\rangle \langle +| + \lambda_- |-\rangle \langle -|$ .

The time-dependence of the system operators  $\widetilde{A_i}(t)$  may be made explicit using the Fourier decomposition:

$$\widetilde{A_i}(\tau) = e^{i\overline{H_S}\tau} A_i e^{-i\overline{H_S}\tau} \quad (772)$$

$$= \sum_w e^{-i w \tau} A_i(w) \quad (773)$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case  $w \in \{0, \pm\eta\}$ .

In order to use the equation (773) to descompose the equation (358) we need to consider the time ordering operator  $\mathcal{T}$ , it's possible to write using the Dyson series or the expansion of the operator of the form  $U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right)$  like:

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right) \quad (774)$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) \dots H(t_n) \quad (775)$$

Here  $0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t$  is a partition of the set  $[0, t]$ . We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian  $H_E(t)$  such that  $\mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right) \equiv \exp(-it H_E(t))$ . The effective Hamiltonian is expanded in a series of terms of increasing order in time  $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots$  so we can write:

$$U(t) = \exp \left( -it \left( H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots \right) \right) \quad (776)$$

The terms can be found expanding  $\mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right)$  and  $U(t)$  then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_S}(t') dt' \quad (777)$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' [\overline{H_S}(t'), \overline{H_S}(t'')] \quad (778)$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' ([\overline{H_S}(t'), \overline{H_S}(t'')], \overline{H_S}(t''')) + [[\overline{H_S}(t'''), \overline{H_S}(t'')], \overline{H_S}(t')] \quad (779)$$

In this case the Fourier decomposition using the Magnus expansion is

$$\widetilde{A}_i(t) = e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \quad (780)$$

$$= \sum_{w(t)} e^{-iw(t)t} A_i(w(t)) \quad (781)$$

$w(t)$  belongs to the set of differences of eigenvalues that depends of the time. As we can see the eigenvectors are time dependent as well.

Extending the Fourier decomposition to the matrix  $\widetilde{A}_j(t - \tau, t)$  using the Magnus expansion generates:

$$\widetilde{A}_j(t - \tau, t) = U(t - \tau) U^\dagger(t) A_j(t) U(t) U^\dagger(t - \tau) \quad (782)$$

$$= e^{-i(t-\tau)H_E(t-\tau)} e^{iH_E(t)t} A_j(t) e^{-iH_E(t)t} e^{i(t-\tau)H_E(t-\tau)} \quad (783)$$

$$= e^{-i(t-\tau)H_E(t-\tau)} \sum_{w(t)} e^{-iw(t)t} A_j(w(t)) e^{i(t-\tau)H_E(t-\tau)} \quad (784)$$

$$= \sum_{w(t), w'(t-\tau)} e^{-iw(t)t} e^{iw'(t-\tau)} A_j(w(t), w'(t-\tau)) \quad (785)$$

where  $w'(t - \tau)$  and  $w(t)$  belongs to the set of the differences of the eigenvalues of the Hamiltonian  $H_S(t - \tau)$  and  $H_S(t)$  respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (773) for a general  $2 \times 2$  matrix in a given time let's write the matrix  $A_i$  in the base  $V = \{|+\rangle, |-\rangle\}$  in the following way:

$$A_i = \sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle | \alpha \rangle \langle \beta | \quad (786)$$

Given that  $[|+\rangle \langle +|, |-\rangle \langle -|] = 0$ , then using the Zassenhaus formula we obtain:

$$e^{i\overline{H_S}\tau} = e^{i(\lambda_+|+\rangle \langle +| + \lambda_-|-\rangle \langle -|)\tau} \quad (787)$$

$$= e^{i\lambda_+|+\rangle \langle +|\tau} e^{i\lambda_-|-\rangle \langle -|\tau} \quad (788)$$

$$= (|-\rangle \langle -| + e^{i\lambda_+\tau} |+\rangle \langle +|) (|+\rangle \langle +| + e^{i\lambda_-\tau} |-\rangle \langle -|) \quad (789)$$

$$= e^{i\lambda_+\tau} |+\rangle \langle +| + e^{i\lambda_-\tau} |-\rangle \langle -| \quad (790)$$

Calculating the transformation (773) directly using the previous relationship we find that:

$$\widetilde{A}_i(\tau) = (e^{i\lambda_+\tau} |+\rangle \langle +| + e^{i\lambda_-\tau} |-\rangle \langle -|) \left( \sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle | \alpha \rangle \langle \beta | \right) (e^{-i\lambda_+\tau} |+\rangle \langle +| + e^{-i\lambda_-\tau} |-\rangle \langle -|) \quad (791)$$

$$= \langle + | A_i | + \rangle | + \rangle \langle + | + e^{i\eta\tau} \langle + | A_i | - \rangle | + \rangle \langle - | + e^{-i\eta\tau} \langle - | A_i | + \rangle | - \rangle \langle + | + \langle - | A_i | - \rangle | - \rangle \langle - | \quad (792)$$

Here  $\eta = \lambda_+ - \lambda_-$ . Comparing the RHS of the equations (773) and the explicit expression for  $\widetilde{A}_i(\tau)$  and we obtain the form of the expansion matrices of the Fourier decomposition for a general  $2 \times 2$  matrix:

$$A_i(0) = \langle + | A_i | + \rangle | + \rangle \langle + | + \langle - | A_i | - \rangle | - \rangle \langle - | \quad (793)$$

$$A_i(w) = \langle + | A_i | - \rangle | + \rangle \langle - | \quad (794)$$

$$A_i(-w) = \langle - | A_i | + \rangle | - \rangle \langle + | \quad (795)$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e.  $\widetilde{A}_i(\tau) = \widetilde{A}_i^\dagger(\tau)$  and  $\widetilde{B}_i(\tau) = \widetilde{B}_i^\dagger(\tau)$  we can use the equation (773) to write the master equation in the following neater form:

$$\frac{d\bar{\rho}_S}{dt} = -i[\bar{H}_S(t), \bar{\rho}_S(t)] - \frac{1}{2} \sum_{ij} \sum_{w, w'} \gamma_{ij}(w, w', t) [A_i, A_j(w, w') \bar{\rho}_S(t) - \bar{\rho}_S(t) A_j^\dagger(w, w')] - i \sum_{ij} \sum_w S_{ij}(w, w', t) [A_i, A_j(w, w') \bar{\rho}_S(t) + \bar{\rho}_S(t) A_j^\dagger(w, w')] \quad (796)$$

where  $A_j^\dagger(w) = A(-w)$  as expected from the equations (794) and (795). As we can see the equation shown contains the rates and energy shifts  $\gamma_{ij}(w, w', t) = 2K_{ij}^{\Re}(w, w', t)$  and  $S_{ij}(w, w', t) = K_{ij}^{\Im}(w, w', t)$ , respectively, defined in terms of the response functions

$$K_{ij}(w, w', t) = \int_0^t C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) e^{i\omega\tau} e^{-it(w-w')} d\tau \quad (797)$$

$$= K_{ijww'}(t) \quad (798)$$

If we extend the upper limit of integration to  $\infty$  in the equation (797) then the system will be independent of any preparation at  $t = 0$ , so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

We are interested in recover the density matrix in the lab frame from the density matrix of the transformed frame. At first let's recall the transformation using the master equation:

$$\frac{d\bar{\rho}_S}{dt} = -i[\bar{H}_S(t), \bar{\rho}_S(t)] - \sum_{ijww'} K_{ijww'}(t) [A_i, A_{jww'} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jww'}^\dagger] \quad (799)$$

Applying the inverse transformation we will obtain that:

$$e^{-V} \frac{d\bar{\rho}_S}{dt} e^V = \frac{d(e^{-V} \bar{\rho}_S e^V)}{dt} \quad (800)$$

$$= \frac{d\rho_S}{dt} \quad (801)$$

$$= -ie^{-V} [\bar{H}_S(t), \bar{\rho}_S(t)] e^V - \sum_{ijww'} K_{ijww'}(t) e^{-V} [A_i, A_{jww'} \bar{\rho}_S(t) - \bar{\rho}_S(t) A_{jww'}^\dagger] e^V \quad (802)$$

For a product we have the following:

$$e^{-V} \overline{AB} e^V = e^{-V} \overline{A} \overline{B} e^V \quad (803)$$

$$= e^{-V} \overline{A} e^V e^{-V} \overline{B} e^V \quad (804)$$

$$= (e^{-V} \overline{A} e^V) (e^{-V} \overline{B} e^V) \quad (805)$$

$$= \overline{AB} \quad (806)$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V} \overline{[A, B]} e^V = e^{-V} \overline{(AB - BA)} e^V \quad (807)$$

$$= e^{-V} \overline{AB} e^V - e^{-V} \overline{BA} e^V \quad (808)$$

$$= \overline{AB} - \overline{BA} \quad (809)$$

$$= [A, B] \quad (810)$$

So we will obtain that

$$\frac{d\rho_{\overline{S}}}{dt} = -i e^{-V} [\overline{H}_{\overline{S}}(t), \overline{\rho}_{\overline{S}}(t)] e^V - \sum_{ijww'} K_{ijww'}(t) e^{-V} \left[ A_i, A_{jww'} \overline{\rho}_{\overline{S}}(t) - \overline{\rho}_{\overline{S}}(t) A_{jww'}^\dagger \right] e^V \quad (811)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \sum_{ijww'} K_{ijww'}(t) \left[ e^{-V} A_i e^V, e^{-V} A_{jww'} \overline{\rho}_{\overline{S}}(t) e^V - e^{-V} \overline{\rho}_{\overline{S}}(t) A_{jww'}^\dagger e^V \right] \quad (812)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \sum_{ijww'} K_{ijww'}(t) \left[ e^{-V} A_i e^V, e^{-V} A_{jww'} e^V e^{-V} \overline{\rho}_{\overline{S}}(t) e^V - e^{-V} \overline{\rho}_{\overline{S}}(t) e^V e^{-V} A_{jww'}^\dagger e^V \right] \quad (813)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \sum_{ijww'} K_{ijww'}(t) \left[ e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_{\overline{S}}(t) - \rho_{\overline{S}}(t) e^{-V} A_{jww'}^\dagger e^V \right] \quad (814)$$

$$= -i [H_{\overline{S}}(t), \rho_{\overline{S}}(t)] - \left( \sum_{ijww'} K_{ijww'}(t) \left( [e^{-V} A_i e^V, e^{-V} A_{jww'} e^V \rho_{\overline{S}}(t)] - [e^{-V} A_i e^V, \rho_{\overline{S}}(t) e^{-V} A_{jww'}^\dagger e^V] \right) \right) \quad (815)$$

## V. LIMIT CASES

In order to show the plausibility of the master equation (796) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

### A. Time-independent variational quantum master equation

At first let's show that the master equation (796) reproduces the results of the reference [1], for the latter case we have that  $i, j \in \{1, 2, 3\}$  and  $\omega \in (0, \pm\eta)$ . The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left( \delta + \sum_j g_j (b_k^\dagger + b_k) \right) |1\rangle\langle 1| + \frac{\Omega}{2} \sigma_x + \sum_k \omega_k b_k^\dagger b_k \quad (816)$$

After performing the transformation (24) on the Hamiltonian (816) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H}_S = (\delta + R) |1\rangle\langle 1| + \frac{\Omega_r}{2} \sigma_x \quad (817)$$

$$\overline{H}_I = B_z |1\rangle\langle 1| + \frac{\Omega}{2} (B_x \sigma_x + B_y \sigma_y) \quad (818)$$

$$H_B = \sum_k \omega_k b_k^\dagger b_k \quad (819)$$

The Hamiltonian (817) differs from the transformed Hamiltonian  $H_S$  of the reference written like  $H_S = \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z + \frac{\Omega_r}{2} \sigma_x$  by a term proportional to the identity, this can be seen in the following way taking  $\epsilon = \delta + R$

$$(\delta + R) |1\rangle\langle 1| - \frac{\delta}{2} \mathbb{I} = \left( \frac{\delta}{2} + R \right) |1\rangle\langle 1| - \frac{\delta}{2} |0\rangle\langle 0| \quad (820)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\delta + R}{2} \sigma_z \quad (821)$$

$$= \frac{R}{2} \mathbb{I} + \frac{\epsilon}{2} \sigma_z \quad (822)$$

In this Hamiltonian we can write  $A_i = \sigma_x$ ,  $A_2 = \sigma_y$  and  $A_3 = \frac{I+\sigma_z}{2}$ . In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix  $\overline{H_S}$ .

$$\lambda_+ = \frac{\epsilon + \eta}{2} \quad (823)$$

$$\lambda_- = \frac{\epsilon - \eta}{2} \quad (824)$$

$$|+\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} \begin{pmatrix} \epsilon + \eta \\ \Omega_r \end{pmatrix} \quad (825)$$

$$|-\rangle = \frac{1}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} \begin{pmatrix} -\Omega_r \\ \epsilon + \eta \end{pmatrix} \quad (826)$$

Using this basis we can find the decomposition matrices using the equations (794)-(795) and the fact that  $|+\rangle = \cos(\theta)|1\rangle + \sin(\theta)|0\rangle$  and  $|-\rangle = -\sin(\theta)|1\rangle + \cos(\theta)|0\rangle$  with  $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$  and  $\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$ :

$$\langle + | \sigma_x | + \rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (827)$$

$$= 2 \sin(\theta) \cos(\theta) \quad (828)$$

$$= \sin(2\theta) \quad (829)$$

$$\langle - | \sigma_x | - \rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (830)$$

$$= -2 \sin(\theta) \cos(\theta) \quad (831)$$

$$= -\sin(2\theta) \quad (832)$$

$$\langle - | \sigma_x | + \rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (833)$$

$$= \cos^2(\theta) - \sin^2(\theta) \quad (834)$$

$$= \cos(2\theta) \quad (835)$$

$$\langle + | \sigma_y | + \rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (836)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (837)$$

$$= 0 \quad (838)$$

$$\langle - | \sigma_y | - \rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (839)$$

$$= -i \sin(\theta) \cos(\theta) + i \sin(\theta) \cos(\theta) \quad (840)$$

$$= 0 \quad (841)$$

$$\langle - | \sigma_y | + \rangle = \begin{pmatrix} -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (842)$$

$$= i \cos^2(\theta) + i \sin^2(\theta) \quad (843)$$

$$= i \quad (844)$$

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (845)$$

$$= \cos(\theta) \cos(\theta) \quad (846)$$

$$= \cos^2(\theta) \quad (847)$$



$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = (-\sin(\theta) \cos(\theta)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (848)$$

$$= \sin(\theta) \sin(\theta) \quad (849)$$

$$= \sin^2(\theta) \quad (850)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = (-\sin(\theta) \cos(\theta)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (851)$$

$$= -\sin(\theta) \cos(\theta) \quad (852)$$

$$= -\sin(\theta) \cos(\theta) \quad (853)$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta) (|+\rangle \langle +| - |-\rangle \langle -|) \quad (854)$$

$$A_1(\eta) = \cos(2\theta) |-\rangle \langle +| \quad (855)$$

$$A_2(0) = 0 \quad (856)$$

$$A_2(\eta) = i |-\rangle \langle +| \quad (857)$$

$$A_3(0) = \cos^2(\theta) |+\rangle \langle +| + \sin^2(\theta) |-\rangle \langle -| \quad (858)$$

$$A_3(\eta) = -\sin(\theta) \cos(\theta) |-\rangle \langle +| \quad (859)$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by  $\Lambda'_{ij}(\tau)$  relate with the correlation functions defined in the equation (424) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t) C_j(t - \tau) \Lambda_{ij}(\tau) \quad (860)$$

Using the notation of the master equation (796), we can say that  $C_1(t) = \frac{\Omega}{2} = C_2(t)$  and  $C_3(t) = 1$ , being  $\Omega$  a constant. Furthermore given that  $\overline{H_S}$  is time-independent then  $B(t) = B$ . Taking the equations(722)-(732) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (424) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left( \widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (861)$$

$$= \frac{\Omega_r^2}{8} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right) \quad (862)$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left( \widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (863)$$

$$= \frac{\Omega_r^2}{8} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} \right) \quad (864)$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega J(\omega) (1 - F(\omega))^2 G_+(\tau) \quad (865)$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2} \int_0^\infty d\omega \frac{J(\omega)}{\omega} F(\omega) (1 - F(\omega)) iG_-(\tau) \quad (866)$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau) \quad (867)$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) = \Lambda'_{13}(\tau) = \Lambda'_{31}(\tau) = 0 \quad (868)$$

Finally taking the Hamiltonian (816) and given that to reproduce this Hamiltonian we need to impose in (5) that  $V_{10}(t) = \frac{\Omega}{2}$ ,  $\varepsilon_0(t) = 0$  and  $\varepsilon_1(t) = \delta$ , then we obtain that  $\text{Det}(\overline{H_S}) = -\frac{\Omega_r^2}{4}$ ,  $\text{Tr}(\overline{H_S}) = \epsilon$ . Now  $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$  and using the equation (339) we have that:

$$f_k = \frac{g_k \left( 1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left( \epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k} \right)} \quad (869)$$

$$= \frac{g_k \left( 1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left( 1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k} \right)} \quad (870)$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (422) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then  $U(t)U^\dagger(t-\tau) = U(\tau)$ . From the equation (796) and using the fact that

$$\widetilde{A}_j(t-\tau, t) = U(\tau) A_j U(-\tau) \quad (871)$$

$$= \sum_w e^{iw\tau} A_i(-w) \quad (872)$$

$$= \sum_w e^{-iw\tau} A_i(w) \quad (873)$$

because the matrices  $U(t)$  and  $U(t-\tau)$  commute from the fact that  $H_S(t)$  and  $H_S(t-\tau)$  commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[H_S(t), \overline{\rho_S}(t)] - \frac{1}{2} \sum_{ij} \sum_w \gamma_{ij}(w, t) \left[ A_i, A_j(w) \overline{\rho_S}(t) - \overline{\rho_S}(t) A_j^\dagger(w) \right] \quad (874)$$

$$- \sum_{ij} \sum_w S_{ij}(w, t) \left[ A_i, A_j(w) \overline{\rho_S}(t) + \overline{\rho_S}(t) A_j^\dagger(w) \right] \quad (875)$$

where  $A_j^\dagger(w) = A(-w)$ , as we can see the equation (875) contains the rates and energy shifts  $\gamma_{ij}(w, t) = 2K_{ij}^{\Re}(w, t)$  and  $S_{ij}(w, t) = K_{ij}^{\Im}(w, t)$ , respectively, defined in terms of the response functions

$$K_{ij}(w, t) = \int_0^t \Lambda'_{ij}(\tau) e^{iw\tau} d\tau \quad (876)$$

## B. Time-dependent polaron quantum master equation

Following the reference [1], when  $\Omega_k \ll \omega_k$  then  $f_k \approx g_k$  so we recover the full polaron transformation. It means from the equation (109) that  $B_z = 0$ . The Hamiltonian studied is given by:

$$H = \left( \delta + \sum_{\mathbf{k}} \left( g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right) \right) |1\rangle\langle 1| + \frac{\Omega(t)}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (877)$$

If  $v_{\mathbf{k}} \approx g_{\mathbf{k}}$  then  $B(\tau) = B$ , so  $B$  is independent of the time. In order to reproduce the Hamiltonian of the equation (877) using the Hamiltonian of the equation (1) we can say that  $\delta = \varepsilon_1(t)$ ,  $\varepsilon_0(t) = 0$ ,  $V_{10}(t) = \frac{\Omega(t)}{2}$ . Now given that  $v_{\mathbf{k}} \approx g_{\mathbf{k}}$  then, in this case and using the equation (226) and (245) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1) |1\rangle\langle 1| + \frac{B\sigma_x}{2} \Omega(t) \quad (878)$$

$$\overline{H_1} = \frac{\Omega(t)}{2} (B_x \sigma_x + B_y \sigma_y) \quad (879)$$

In this case  $R_1 = \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} g_{\mathbf{k}} \right)$  from (27) and given that  $v_{\mathbf{k}} \approx g_{\mathbf{k}}$  and  $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$  then  $R_1 = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}}^{-1} |g_{\mathbf{k}}|^2) = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}} | \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} |^2)$  as expected, take  $\delta + R_1 = \delta'$ . If  $F(\omega_{\mathbf{k}}) = 1$  and using the equations (861)-(868) we can deduce that the only terms that survive are  $\Lambda_{11}(\tau)$  and  $\Lambda_{22}(\tau)$ . The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} G_+(\tau) d\omega \quad (880)$$

Writing  $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau)$  so (880) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left( \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau) \right) d\omega \quad (881)$$

Writing the interaction Hamiltonian (879) in the similar way to the equation (245) allow us to write  $A_1 = \sigma_x$ ,  $A_2 = \sigma_y$ ,  $B_1(t) = B_x$ ,  $B_2(t) = B_y$  and  $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$ . Now taking the equation (226) with  $\delta' |1\rangle\langle 1| = \frac{\delta'}{2} \sigma_z + \frac{\delta'}{2} \mathbb{I}$  help us to reproduce the hamiltonian of the reference [2]. Then  $\overline{H_S}$  is equal to:

$$\overline{H_S} = \frac{\delta'}{2} \sigma_z + \frac{B\sigma_x}{2} \Omega(t) \quad (882)$$

As we can see the function  $B$  is a time-independent function because we consider that  $g_{\mathbf{k}}$  doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \text{Tr}_B \left( \widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (883)$$

$$= \frac{B^2}{2} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right) \quad (884)$$

$$\Lambda_{22}(\tau) = \text{Tr}_B \left( \widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (885)$$

$$= \frac{B^2}{2} \left( e^{\phi(\tau)} + e^{-\phi(\tau)} \right) \quad (886)$$

These functions match with the equations  $\Lambda_x(\tau)$  and  $\Lambda_y(\tau)$  of the reference [2] and  $\Lambda_i(\tau) = \Lambda_i(-\tau)$  for  $i \in \{x, y\}$  respectively. The master equation for this section based on the equation(422) is:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i \left[ \frac{\delta'}{2} \sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t) \right] - \sum_{i=1}^2 \int_0^t d\tau \left( C_i(t) C_i(t-\tau) \Lambda_{ii}(\tau) \left[ A_i, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right] \right. \quad (887)$$

$$\left. + C_i(t) C_i(t-\tau) \Lambda_{ii}(-\tau) \left[ \rho_S(t) \widetilde{A}_i(t-\tau, t), A_i \right] \right) \quad (888)$$

Replacing  $C_i(t) = \frac{\Omega(t)}{2}$  and  $\widetilde{A}_i(t-\tau, t) = \widetilde{\sigma}_i(t-\tau, t)$ , also using the equations (883) and (886) on the equation (888) we obtain that:

$$\frac{d\overline{\rho_S}(t)}{dt} = -\frac{i}{2} [\delta' \sigma_z + \Omega_r(t) \sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4} \int_0^t d\tau \Omega(t-\tau) ([\sigma_x, \widetilde{\sigma}_x(t-\tau, t) \rho_S(t)] \Lambda_x(\tau) \quad (889)$$

$$+ [\sigma_y, \widetilde{\sigma}_y(t-\tau, t) \rho_S(t)] \Lambda_y(\tau) + [\rho_S(t) \widetilde{\sigma}_x(t-\tau, t), \sigma_x] \Lambda_x(\tau) + [\rho_S(t) \widetilde{\sigma}_y(t-\tau, t), \sigma_y] \Lambda_y(\tau)) \quad (890)$$

As we can see  $[A_j, \widetilde{A}_i(t-\tau, t) \rho_S(t)]^\dagger = [\rho_S(t) \widetilde{A}_i(t-\tau, t), A_j]$ ,  $\Lambda_x(\tau) = \Lambda_x(-\tau)$  and  $\Lambda_y(\tau) = \Lambda_y(-\tau)$ , so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

### C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation (S17) of the reference [3] we will impose that  $F(\omega) = 0$ , so there is no transformation in this case. As we can see from the definition (424) the only term that survives is  $\Lambda_{33}(\tau)$ . Taking  $\hbar = 1$  the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} (g_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{\mathbf{k}}^* b_{\mathbf{k}}) \quad (891)$$

Using the equation (796), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \frac{1}{2} \sum_w \gamma_{33}(w, t) [A_3, A_3(w) \rho_S(t) - \rho_S(t) A_3^{\dagger}(w)] \quad (892)$$

$$- \sum_w S_{33}(w, t) [A_3, A_3(w) \rho_S(t) + \rho_S(t) A_3^{\dagger}(w)] \quad (893)$$

The correlation functions are relevant if  $F(\omega) = 0$  for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^{\infty} d\omega J(\omega) G_+(\tau) \quad (894)$$

$$\Lambda_{33}(-\tau) = \int_0^{\infty} d\omega J(\omega) G_+(-\tau) \quad (895)$$

In our case  $A_3 = \frac{\mathbb{I} + \sigma_z}{2}$ , the equation (893) can be transformed in

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S(t)] - \sum_w (K_{33}(w, t) [A_3, A_3(w) \rho_S(t)] + K_{33}^*(w, t) [\rho_S(t) A_3(w), A_3]) \quad (896)$$

As the paper suggest we will consider that the quantum system is in resonance, so  $\Delta = 0$  and furthermore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to  $\tilde{\Omega}$ . To find the matrices  $A_3(w)$ , we have to remember that  $H_S = \frac{\Omega(t)}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)$ , this Hamiltonian have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\tilde{\Omega}}{2} \quad (897)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \quad (898)$$

$$\lambda_- = -\frac{\tilde{\Omega}}{2} \quad (899)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |0\rangle) \quad (900)$$

The elements of the decomposition matrices are:

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (901)$$

$$= \frac{1}{2} \quad (902)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = \frac{1}{2} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (903)$$

$$= \frac{1}{2} \quad (904)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (905)$$

$$= -\frac{1}{2} \quad (906)$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -| \quad (907)$$

$$= \frac{\mathbb{I}}{2} \quad (908)$$

$$A_3(\eta) = -\frac{1}{2} |-\rangle \langle +| \quad (909)$$

$$= \frac{1}{4} (\sigma_z + i\sigma_y) \quad (910)$$

$$A_3(-\eta) = -\frac{1}{2} |+\rangle \langle -| \quad (911)$$

$$= \frac{1}{4} (\sigma_z - i\sigma_y) \quad (912)$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - K_{33}(\tilde{\Omega}, t) \left[ \frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z + i\sigma_y) \rho_S(t) \right] - K_{33}(-\tilde{\Omega}, t) \left[ \frac{\sigma_z}{2}, \frac{1}{4} (\sigma_z - i\sigma_y) \rho_S(t) \right] \quad (913)$$

$$- K_{33}^*(\tilde{\Omega}, t) \left[ \rho_S(t) \frac{1}{4} (\sigma_z + i\sigma_y), \frac{\sigma_z}{2} \right] - K_{33}^*(-\tilde{\Omega}, t) \left[ \rho_S(t) \frac{1}{4} (\sigma_z - i\sigma_y), \frac{\sigma_z}{2} \right] \quad (914)$$

Calculating the response functions extending the upper limit of  $\tau$  to  $\infty$ , we obtain:

$$K_{33}(\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{i\tilde{\Omega}\tau} d\tau d\omega \quad (915)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (916)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} (n(\omega) + 1) e^{-i\tau\omega} d\tau d\omega \quad (917)$$

$$= \int_0^\infty \int_0^\infty J(\omega) (n(\omega) + 1) e^{i\tilde{\Omega}\tau - i\tau\omega} d\tau d\omega \quad (918)$$

$$= \int_0^\infty J(\omega) (n(\omega) + 1) \pi \delta(\tilde{\Omega} - \omega) d\omega \quad (919)$$

$$= \pi J(\tilde{\Omega}) (n(\tilde{\Omega}) + 1) \quad (920)$$

$$K_{33}(-\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{-i\tilde{\Omega}\tau} d\tau d\omega \quad (921)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} ((n(\omega) + 1) e^{-i\tau\omega} + n(\omega) e^{i\tau\omega}) d\tau d\omega \quad (922)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} n(\omega) e^{i\tau\omega} d\tau d\omega \quad (923)$$

$$= \int_0^\infty \int_0^\infty J(\omega) n(\omega) e^{-i\tilde{\Omega}\tau + i\tau\omega} d\tau d\omega \quad (924)$$

$$= \int_0^\infty J(\omega) n(\omega) \pi \delta(-\tilde{\Omega} + \omega) d\omega \quad (925)$$

$$= \pi J(\tilde{\Omega}) n(\tilde{\Omega}) \quad (926)$$

Here we have used  $\int_0^\infty ds e^{\pm i\epsilon s} = \pi \delta(\epsilon) \pm i \frac{\text{V.P.}}{\epsilon}$ , where V.P. denotes the Cauchy's principal value. These principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix  $A_3(0)$  because the spectral density  $J(\omega)$  is equal to zero when  $\omega = 0$ . Replacing in the equation (913) lead us to obtain:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\tilde{\Omega}) \left( (n(\tilde{\Omega}) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\tilde{\Omega}) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (927)$$

$$- \frac{\pi}{8} J(\tilde{\Omega}) \left( (n(\tilde{\Omega}) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\tilde{\Omega}) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right) \quad (928)$$

This is the same result than the equation (S17), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{d\rho_S(t)}{dt} = -i\frac{\Omega(t)}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\Omega(t)) \left( (n(\Omega(t)) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\Omega(t)) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (929)$$

$$- \frac{\pi}{8} J(\Omega(t)) \left( (n(\Omega(t)) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\Omega(t)) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right) \quad (930)$$

## VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (931)$$

$$H_S(t) = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (932)$$

$$H_I = \sum_{n\mathbf{k}} |n\rangle\langle n| \left( g_{n\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{n\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (933)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (934)$$

### A. Variational Transformation

We consider the following operator:

$$V = \sum_{nuk} |n\rangle\langle n| \omega_{uk}^{-1} \left( f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk} \right) \quad (935)$$

At first let's obtain  $e^{\pm V}$  under the transformation (935), consider  $\hat{\varphi}_n = \sum_{uk} \omega_{uk}^{-1} \left( f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk} \right)$ , so the equation (935) can be written as  $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$ , then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \quad (936)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{(\sum_n |n\rangle\langle n| \hat{\varphi}_n)^2}{2!} + \dots \quad (937)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (938)$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (939)$$

$$= \sum_n |n\rangle\langle n| \left( \mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots \right) \quad (940)$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \quad (941)$$

Given that  $\left[ f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk}, f_{nu'k'} b_{u'k'}^\dagger - f_{nu'k'}^* b_{u'k'} \right] = 0$  for all  $k', k$  and  $u, u'$  then we can proof using the Zassenhaus formula and defining  $D(\pm \alpha_{nuk}) = e^{\pm (\alpha_{nuk} b_{uk}^\dagger - \alpha_{nuk}^* b_{uk})}$  in the same way than (23) with  $\alpha_{nuk} = \frac{f_{nuk}}{\omega_{uk}}$ :

$$e^{\pm \sum_{uk} \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} = \prod_u e^{\pm \sum_k \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} \quad (942)$$

$$= \prod_u \left( \prod_k e^{\pm \omega_{uk}^{-1} (f_{nuk} b_{uk}^\dagger - f_{nuk}^* b_{uk})} \right) \quad (943)$$

$$= \prod_u \left( \prod_k D(\pm \alpha_{nuk}) \right) \quad (944)$$

$$= \prod_{uk} D(\pm \alpha_{nuk}) \quad (945)$$

$$= \prod_u B_{nu\pm} \quad (946)$$

$$B_{nu\pm} \equiv \prod_k D(\pm \alpha_{nuk}) \quad (947)$$

As we can see  $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$  and  $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$  this implies that  $e^{-V} e^V = \mathbb{I}$ . This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (948)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (931):

$$\overline{|0\rangle\langle 0|} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |0\rangle\langle 0| \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (949)$$

$$= \prod_u B_{0u+} |0\rangle\langle 0| |0\rangle\langle 0| \prod_u B_{0u-}, \quad (950)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \quad (951)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \quad (952)$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \quad (953)$$

$$= |0\rangle\langle 0|. \quad (954)$$

$$\overline{|m\rangle\langle n|} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |m\rangle\langle n| \left( \sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (955)$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n| |n\rangle\langle n| \prod_u B_{nu-}, \quad (956)$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \quad (957)$$

$$= |m\rangle\langle n| \prod_u (B_{mu+} B_{nu-}), \quad m \neq n, \quad (958)$$

$$= |m\rangle\langle n| \prod_u \left( \prod_{\mathbf{k}} D(\alpha_{mu\mathbf{k}}) \prod_{\mathbf{k}} D(-\alpha_{nu\mathbf{k}}) \right), \quad (959)$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} (D(\alpha_{mu\mathbf{k}}) D(-\alpha_{nu\mathbf{k}})), \quad (960)$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left( D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp \left( \frac{1}{2} (-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}}) \right) \right). \quad (961)$$

$$\Pi_u(B_{mu+} B_{nu-}) = \prod_{u\mathbf{k}} \left( D(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}) \exp \left( \frac{1}{2} (-\alpha_{mu\mathbf{k}} \alpha_{nu\mathbf{k}}^* + \alpha_{mu\mathbf{k}}^* \alpha_{nu\mathbf{k}}) \right) \right). \quad (962)$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = (\sum_n |n\rangle\langle n| \prod_u B_{nu+}) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} (\sum_n |n\rangle\langle n| \prod_u B_{nu-}), \quad (963)$$

$$= (|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \dots) (\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}) (|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \dots), \quad (964)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + \dots, \quad (965)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{1u-} + \dots \quad (966)$$

$$= |0\rangle\langle 0| (\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \dots) \quad (967)$$

$$+ |1\rangle\langle 1| (\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \dots) + \dots \quad (968)$$

$$= |0\rangle\langle 0| \left( \sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left( b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left( b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left( b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left( b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) \quad (969)$$

$$+ |1\rangle\langle 1| \left( \sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left( b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}^*}{\omega_{0\mathbf{k}}} \right) \left( b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left( b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}^*}{\omega_{1\mathbf{k}}} \right) \left( b_{1\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) + \dots \quad (970)$$

$$= |0\rangle\langle 0| \left( \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left( b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) \left( b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + |1\rangle\langle 1| \left( \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left( b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) \left( b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \dots \quad (971)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left( \omega_{n\mathbf{k}} \left( b_{n\mathbf{k}}^\dagger - \frac{v_{nn\mathbf{k}}^*}{\omega_{n\mathbf{k}}} \right) \left( b_{n\mathbf{k}} - \frac{v_{nn\mathbf{k}}}{\omega_{n\mathbf{k}}} \right) \right) \quad (972)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left( \omega_{n\mathbf{k}} \left( b_{n\mathbf{k}}^\dagger b_{n\mathbf{k}} - \frac{v_{nn\mathbf{k}}}{\omega_{n\mathbf{k}}} b_{n\mathbf{k}}^\dagger - \frac{v_{nn\mathbf{k}}^*}{\omega_{n\mathbf{k}}} b_{n\mathbf{k}} + \left| \frac{v_{nn\mathbf{k}}}{\omega_{n\mathbf{k}}} \right|^2 \right) \right) \quad (973)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \omega_{n\mathbf{k}} b_{n\mathbf{k}}^\dagger b_{n\mathbf{k}} + \sum_{n\mathbf{k}} |n\rangle\langle n| \left( \frac{|v_{nn\mathbf{k}}|^2}{\omega_{n\mathbf{k}}} - v_{nn\mathbf{k}} b_{n\mathbf{k}}^\dagger - v_{nn\mathbf{k}}^* b_{n\mathbf{k}} \right) \quad (974)$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{n\mathbf{k}} |n\rangle\langle n| \left( \frac{|v_{nn\mathbf{k}}|^2}{\omega_{n\mathbf{k}}} - (v_{nn\mathbf{k}} b_{n\mathbf{k}}^\dagger + v_{nn\mathbf{k}}^* b_{n\mathbf{k}}) \right) \quad (975)$$

The transformed Hamiltonians of the equations (932) to (934) written in terms of (949) to (973) are:



$$\overline{H_S}(t) = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (976)$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (977)$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu} + B_{nu}) \quad (978)$$

$$\overline{H_I} = \left( \sum_n |n\rangle\langle n| \prod_u B_{nu} \right) \left( \sum_{n \neq m} |n\rangle\langle m| (g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk}) \right) \left( \sum_n |n\rangle\langle n| \prod_u B_{nu} \right) \quad (979)$$

$$= \left( \sum_n |n\rangle\langle n| \prod_u B_{nu} \right) \left( \sum_{u \neq v} |0\rangle\langle 0| (g_{0uk} b_{uk}^\dagger + g_{0uk}^* b_{uk}) + \dots \right) \left( \sum_n |n\rangle\langle n| \prod_u B_{nu} \right) \quad (980)$$

$$= \prod_u B_{0u} + \sum_{u \neq v} |0\rangle\langle 0| (g_{0uk} b_{uk}^\dagger + g_{0uk}^* b_{uk}) \prod_u B_{0u} + \dots \quad (981)$$

$$= \sum_{u \neq v} |0\rangle\langle 0| (g_{0uk} \prod_u B_{0u} + b_{uk}^\dagger \prod_u B_{0u} + g_{0uk}^* \prod_u B_{0u} + b_{uk} \prod_u B_{0u}) + \dots \quad (982)$$

$$= \sum_{u \neq v} |0\rangle\langle 0| \left( g_{0uk} \left( b_{uk}^\dagger - \frac{v_{0uk}^*}{\omega_{uk}} \right) + g_{0uk}^* \left( b_{uk} - \frac{v_{0uk}}{\omega_{uk}} \right) \right) + \dots \quad (983)$$

$$= \sum_{n \neq m} |n\rangle\langle n| \left( g_{nuk} \left( b_{uk}^\dagger - \frac{v_{nuk}^*}{\omega_{uk}} \right) + g_{nuk}^* \left( b_{uk} - \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (984)$$

$$= \sum_{n \neq m} |n\rangle\langle n| \left( g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk} - \left( g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (985)$$

$$\overline{H_B} = \sum_{u \neq v} \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_{n \neq m} |n\rangle\langle n| \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - (v_{nuk} b_{uk}^\dagger + v_{nuk}^* b_{uk}) \right) \quad (986)$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu} + B_{nu}) + \sum_{u \neq v} \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_{n \neq m} |n\rangle\langle n| \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - (v_{nuk} b_{uk}^\dagger + v_{nuk}^* b_{uk}) \right) \quad (987)$$

$$+ \sum_{n \neq m} |n\rangle\langle n| \left( g_{nuk} b_{uk}^\dagger + g_{nuk}^* b_{uk} - \left( g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (988)$$

Let's define the following functions:

$$R_n(t) = \sum_{u \neq v} \left( \frac{|v_{nuk}|^2}{\omega_{uk}} - \left( g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (989)$$

$$B_{z,n}(t) = \sum_{u \neq v} \left( (g_{nuk} - v_{nuk}) b_{uk}^\dagger + (g_{nuk} - v_{nuk})^* b_{uk} \right) \quad (990)$$

Using the previous functions we have that (987) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu} + B_{nu}) + \sum_{u \neq v} \omega_{uk} b_{uk}^\dagger b_{uk} + \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \quad (991)$$

$$(992)$$

Now in order to separate the elements of the hamiltonian (992) let's follow the references of the equations (226) and (245) to separate the hamiltonian, before proceeding to do this we need to consider the term of the form:

$$\langle \Pi_u (B_{mu+} B_{nu-}) \rangle_{\overline{H_0}} = \langle \Pi_{uk} (D(\alpha_{muk} - \alpha_{nuk}) \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk})) \rangle_{\overline{H_0}} \quad (993)$$

$$= (\Pi_{uk} \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}))) \langle \Pi_{uk} D(\alpha_{muk} - \alpha_{nuk}) \rangle_{\overline{H_0}} \quad (994)$$

$$= \left( \Pi_{uk} \exp\left(\frac{(v_{muk}^* v_{nuk} - v_{nuk} v_{muk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (995)$$

$$\equiv B_{nm} \quad (996)$$

$$\langle \Pi_u (B_{nu+} B_{mu-}) \rangle_{\overline{H_0}} = \left( \Pi_{uk} \exp\left(\frac{(v_{nuk}^* v_{muk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (997)$$

$$= B_{nm}^* \quad (998)$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+} B_{nu-}) - B_{nm} \quad (999)$$

As we can see:

$$J_{nm}^\dagger = \left( \prod_u (B_{mu+} B_{nu-}) - B_{nm} \right)^\dagger \quad (1000)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{nm}^* \quad (1001)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{mn} \quad (1002)$$

$$= J_{mn} \quad (1003)$$

We can separate the Hamiltonian (992) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\overline{S}}}(t) = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \quad (1004)$$

$$\overline{H_{\overline{I}}}(t) = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1005)$$

$$\overline{H_{\overline{B}}}(t) = \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} \quad (1006)$$

## B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left( \text{Tr} \left( e^{-\beta(\overline{H_{\overline{S}}}(t) + \overline{H_{\overline{B}}})} \right) \right) + \langle \overline{H_{\overline{I}}} \rangle_{\overline{H_{\overline{S}}}(t) + \overline{H_{\overline{B}}}} + O \left( \left\langle \overline{H_{\overline{I}}}^2 \right\rangle_{\overline{H_{\overline{S}}}(t) + \overline{H_{\overline{B}}}} \right). \quad (1007)$$

Taking the equations (247)-(255) and given that  $\text{Tr} \left( e^{-\beta \overline{H_{\overline{S}}}(t)} \right) = C(R_0, R_1, \dots, R_{d-1}, B_{01}, \dots, B_{0(d-1)}, \dots, B_{(d-2)(d-1)})$ , where each  $R_i$  and  $B_{kj}$  depend of the set of variational parameters  $\{v_{nuk}\}$ . Given that the numbers  $v_{nuk}$  are complex then we can separate them as  $v_{nuk} = v_{nuk}^{\Re} + i v_{nuk}^{\Im}$ . So our approach will be based on the derivation respect to  $v_{nuk}^{\Re}$  and  $v_{nuk}^{\Im}$ . The Hamiltonian  $\overline{H_{\overline{S}}}(t)$  can be written like:

$$\overline{H_S(t)} = \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{|v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}} - \left( g_{n\mathbf{uk}} \frac{v_{n\mathbf{uk}}^*}{\omega_{\mathbf{uk}}} + g_{n\mathbf{uk}}^* \frac{v_{n\mathbf{uk}}}{\omega_{\mathbf{uk}}} \right) \right) \right) |n\rangle\langle n| \quad (1008)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1009)$$

$$= \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{|v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}} - \frac{g_{n\mathbf{uk}} v_{n\mathbf{uk}}^* + g_{n\mathbf{uk}}^* v_{n\mathbf{uk}}}{\omega_{\mathbf{uk}}} \right) \right) |n\rangle\langle n| \quad (1010)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1011)$$

$$= \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2}{\omega_{\mathbf{uk}}} - \frac{(g_{n\mathbf{uk}} + g_{n\mathbf{uk}}^*) v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im} (g_{n\mathbf{uk}}^* - g_{n\mathbf{uk}})}{\omega_{\mathbf{uk}}} \right) \right) |n\rangle\langle n| \quad (1012)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1013)$$

$$v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^* = (v_{m\mathbf{uk}}^{\Re} - i v_{m\mathbf{uk}}^{\Im})(v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im}) - (v_{m\mathbf{uk}}^{\Re} + i v_{m\mathbf{uk}}^{\Im})(v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im}) \quad (1014)$$

$$= (v_{m\mathbf{uk}}^{\Re} v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - i v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re} + v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Im}) \quad (1015)$$

$$- (v_{m\mathbf{uk}}^{\Re} v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} + i v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re} + v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Im}) \quad (1016)$$

$$= 2i (v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re}) \quad (1017)$$

$$\overline{H_S(t)} = \sum_n \left( \varepsilon_n(t) + \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2}{\omega_{\mathbf{uk}}} - \frac{(g_{n\mathbf{uk}} + g_{n\mathbf{uk}}^*) v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im} (g_{n\mathbf{uk}}^* - g_{n\mathbf{uk}})}{\omega_{\mathbf{uk}}} \right) \right) |n\rangle\langle n| \quad (1018)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left( \Pi_{\mathbf{uk}} \exp \left( \frac{i(v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re})}{\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1019)$$

$$|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2 = (v_{m\mathbf{uk}} - v_{n\mathbf{uk}})(v_{m\mathbf{uk}} - v_{n\mathbf{uk}})^* \quad (1020)$$

$$= |v_{m\mathbf{uk}}|^2 + |v_{n\mathbf{uk}}|^2 - (v_{n\mathbf{uk}} v_{m\mathbf{uk}}^* + v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*) \quad (1021)$$

$$= (v_{m\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im})^2 + (v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - (v_{n\mathbf{uk}}^{\Re} + i v_{n\mathbf{uk}}^{\Im})(v_{m\mathbf{uk}}^{\Re} - i v_{m\mathbf{uk}}^{\Im}) \quad (1022)$$

$$- (v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im})(v_{m\mathbf{uk}}^{\Re} + i v_{m\mathbf{uk}}^{\Im}) \quad (1023)$$

$$= (v_{m\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im})^2 + (v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - 2(v_{n\mathbf{uk}}^{\Re} v_{m\mathbf{uk}}^{\Re} + v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Im}) \quad (1024)$$

$$= (v_{m\mathbf{uk}}^{\Re} - v_{n\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im} - v_{n\mathbf{uk}}^{\Im})^2 \quad (1025)$$

$$R_n(t) = \sum_{\mathbf{uk}} \left( \frac{|v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}} - \left( g_{n\mathbf{uk}} \frac{v_{n\mathbf{uk}}^*}{\omega_{\mathbf{uk}}} + g_{n\mathbf{uk}}^* \frac{v_{n\mathbf{uk}}}{\omega_{\mathbf{uk}}} \right) \right) \quad (1026)$$

$$= \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - (g_{n\mathbf{uk}} + g_{n\mathbf{uk}}^*) v_{n\mathbf{uk}}^{\Re} - i v_{n\mathbf{uk}}^{\Im} (g_{n\mathbf{uk}}^* - g_{n\mathbf{uk}})}{\omega_{\mathbf{uk}}} \right) \quad (1027)$$

$$= \sum_{\mathbf{uk}} \left( \frac{(v_{n\mathbf{uk}}^{\Re})^2 + (v_{n\mathbf{uk}}^{\Im})^2 - 2g_{n\mathbf{uk}}^{\Re} v_{n\mathbf{uk}}^{\Re} - 2g_{n\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Im}}{\omega_{\mathbf{uk}}} \right) \quad (1028)$$

$$B_{mn} = \left( \Pi_{\mathbf{uk}} \exp \left( \frac{(v_{m\mathbf{uk}}^* v_{n\mathbf{uk}} - v_{m\mathbf{uk}} v_{n\mathbf{uk}}^*)}{2\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{m\mathbf{uk}} - v_{n\mathbf{uk}}|^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1029)$$

$$= \left( \Pi_{\mathbf{uk}} \exp \left( \frac{i(v_{n\mathbf{uk}}^{\Im} v_{m\mathbf{uk}}^{\Re} - v_{m\mathbf{uk}}^{\Im} v_{n\mathbf{uk}}^{\Re})}{\omega_{\mathbf{uk}}^2} \right) \right) \Pi_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{m\mathbf{uk}}^{\Re} - v_{n\mathbf{uk}}^{\Re})^2 + (v_{m\mathbf{uk}}^{\Im} - v_{n\mathbf{uk}}^{\Im})^2}{\omega_{\mathbf{uk}}^2} \coth \left( \frac{\beta_u \omega_{\mathbf{uk}}}{2} \right) \right) \quad (1030)$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Re}} = \frac{\partial}{\partial v_{nuk}^{\Re}} \sum_{uk} \left( \frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1031)$$

$$= \frac{2v_{nuk}^{\Re} - 2g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1032)$$

$$= 2 \frac{v_{nuk}^{\Re} - g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1033)$$

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Im}} = \frac{\partial}{\partial v_{nuk}^{\Im}} \sum_{uk} \left( \frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1034)$$

$$= \frac{2v_{nuk}^{\Im} - 2g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1035)$$

$$= 2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1036)$$

Given that:

$$\ln B_{mn} = \ln \left( \left( \prod_{uk} \exp \left( \frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \prod_u \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left( \frac{\beta_u \omega_{uk}}{2} \right) \right) \right) \quad (1037)$$

$$= \sum_{uk} \ln \exp \left( \frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_u \ln \exp \left( -\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left( \frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1038)$$

$$= \sum_{uk} \left( \frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_{uk} \left( -\frac{1}{2} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left( \frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1039)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Re}} = \frac{-i v_{muk}^{\Im} - (v_{nuk}^{\Re} - v_{muk}^{\Re}) \coth \left( \frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1040)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Im}} = \frac{i v_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth \left( \frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1041)$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \quad (1042)$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \quad (1043)$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial (B_{nm})^{\dagger}}{\partial a} \quad (1044)$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \quad (1045)$$

$$= B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} - (v_{n\mathbf{k}}^{\Re} - v_{m\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1046)$$

$$= B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1047)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Re}} = \left( \frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \right)^{\dagger} \quad (1048)$$

$$= \left( B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \quad (1049)$$

$$= B_{nm} \left( \frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1050)$$

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \quad (1051)$$

$$= B_{mn} \left( \frac{iv_{m\mathbf{k}}^{\Re} - (v_{n\mathbf{k}}^{\Im} - v_{m\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1052)$$

$$= B_{mn} \left( \frac{iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1053)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Im}} = \left( \frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \right)^{\dagger} \quad (1054)$$

$$= (B_{mn})^{\dagger} \quad (1055)$$

$$= B_{nm} \left( \frac{-iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1056)$$

Introducing this derivatives in the equation (1031) give us:

$$\frac{\partial A_{\mathbf{B}}}{\partial v_{n\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathbf{B}}}{\partial R_n} \left( 2 \frac{v_{n\mathbf{k}}^{\Re} - g_{u\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n < m} \left( \frac{\partial A_{\mathbf{B}}}{\partial B_{nm}} B_{nm} \left( \frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \quad (1057)$$

$$+ \frac{\partial A_{\mathbf{B}}}{\partial B_{mn}} B_{mn} \left( \frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1058)$$

$$= 0 \quad (1059)$$

We can obtain the variational parameters:

$$-2 \frac{\partial A_B}{\partial R_n} \frac{v_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1060)$$

$$= -\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \left( \frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left( \frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1061)$$

$$v_{nuk}^{\Re} = \frac{\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} - \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \left( \frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left( \frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right)}{2 \frac{\partial A_B}{\partial R_n} \frac{1}{\omega_{uk}} - \sum_{n \neq m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right)} \quad (1062)$$

$$= \frac{2g_{nuk}^{\Re} \omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} (iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} (-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) \right)}{2\omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n \neq m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right)} \quad (1063)$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_B}{\partial v_{nuk}^{\Im}} = \frac{\partial A_B}{\partial R_n} \left( 2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \right) + \sum_{n < m} \left( \frac{\partial A_B}{\partial B_{nm}} B_{nm} \left( \frac{-iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1064)$$

$$+ \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left( \frac{iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1065)$$

$$= 0 \quad (1066)$$

Rearranging we obtain



$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1083)$$

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right). \quad (1084)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1085)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\bar{\rho}(t)) \quad (1086)$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1087)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1088)$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \quad (1089)$$

$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2i} \quad (1090)$$

We can proof that  $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+} B_{n-} - B_m B_n)^\dagger \quad (1091)$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \quad (1092)$$

$$= B_{n+} B_{m-} - B_n B_m \quad (1093)$$

$$= B_{nm} \quad (1094)$$

So we can say that the set of matrices (1087) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1087) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1095)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}) \quad (1096)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left( \Re(V_{nm}(t)) B_{nm} \left( \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) + iV_{nm}^\Im(t) B_{nm} \left( \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) \right) \quad (1097)$$

$$+ \Re(V_{nm}(t)) B_{mn} \left( \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) - iV_{nm}^\Im(t) B_{mn} \left( \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1098)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left( \Re(V_{nm}(t)) \sigma_{nm,x} \left( \frac{B_{nm} + B_{mn}}{2} \right) + \Re(V_{nm}(t)) \sigma_{nm,y} \frac{i(B_{mn} - B_{nm})}{2} \right) \quad (1099)$$

$$+ i\Im(V_{nm}(t)) \sigma_{nm,x} \left( \frac{B_{nm} - B_{mn}}{2} \right) + \Im(V_{nm}(t)) \sigma_{nm,y} \left( \frac{B_{nm} + B_{mn}}{2} \right) \quad (1100)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (\Re(V_{nm}(t)) \sigma_{nm,x} B_{nm,x} - \Im(V_{nm}(t)) \sigma_{nm,x} B_{nm,y} + \Re(V_{nm}(t)) \sigma_{nm,y} B_{nm,y} \quad (1101)$$

$$+ \Im(V_{nm}(t)) \sigma_{nm,y} B_{nm,x}) \quad (1102)$$

Let's define the set

$$P = \{(n, m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\} \quad (1103)$$



Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \quad (1104)$$

$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \quad (1105)$$

$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \quad (1106)$$

$$A_{4,nm}(t) = A_{2,mn}(t) \quad (1107)$$

$$A_{5,nm}(t) = A_{1,nm}(t) \quad (1108)$$

$$B_{1,nm}(t) = B_{nm,x} \quad (1109)$$

$$B_{2,nm}(t) = B_{nm,y} \quad (1110)$$

$$B_{3,nm}(t) = B_{z,n}(t) \quad (1111)$$

$$B_{4,nm}(t) = B_{1,nm}(t) \quad (1112)$$

$$B_{5,nm}(t) = B_{2,nm}(t) \quad (1113)$$

$$C_{1,nm}(t) = \Re(V_{nm}(t)) \quad (1114)$$

$$C_{2,nm}(t) = C_{1,nm}(t) \quad (1115)$$

$$C_{3,nm}(t) = 1 \quad (1116)$$

$$C_{4,nm}(t) = \Im(V_{nm}(t)) \quad (1117)$$

$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \quad (1118)$$

The previous notation allows us to write the interaction Hamiltonian in  $\overline{H_I}(t)$  as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t) (A_{jp} \otimes B_{jp}(t)) \quad (1119)$$

Here  $J = \{1, 2, 3, 4, 5\}$  and  $P$  the set defined in (1103).

We write the interaction Hamiltonian transformed under (1083) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t) \right) \quad (1120)$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \quad (1121)$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t) e^{-iH_B t} \quad (1122)$$

Taking as reference state  $\rho_B$  and truncating at second order in  $H_I(t)$ , we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H_I}(t), \left[ \widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1123)$$

Replacing the equation (1120) in (1123) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H}_I(t), \left[ \widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1124)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \left[ \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1125)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1126)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \Big] ds \quad (1127)$$

$$= - \int_0^t \text{Tr}_B \left( \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right. \quad (1128)$$

$$\left. - \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \right. \quad (1129)$$

$$\left. - \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \right. \quad (1130)$$

$$\left. + \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left( \widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \sum_{j \in J, p \in P} C_{jp}(t) \left( \widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \right) ds \quad (1131)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B \quad (1132)$$

$$= \left\langle \widetilde{B}_{jp}(\tau) \widetilde{B}_{j'p'}(0) \right\rangle_B \quad (1133)$$

Here  $s \rightarrow t - \tau$  and  $\text{Tr}_B \left( \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B$ . To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators  $\widetilde{B}_{jp}(t)$  and elements related to the system given by  $\widetilde{A}_{jp}(t)$ . The systems considered are in different Hilbert spaces so  $\text{Tr} \left( \widetilde{A}_{jp}(t) \widetilde{B}_{j'p'}(t) \right) = \text{Tr} \left( \widetilde{A}_{jp}(t) \right) \text{Tr} \left( \widetilde{B}_{j'p'}(t) \right)$ . The correlation functions relevant of the master equation (1131) are:

$$\text{Tr}_B \left( \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1134)$$

$$= \left\langle \widetilde{B_{jp}}(0) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1135)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1136)$$

$$\text{Tr}_B \left( \widetilde{B_{jp}}(t) \rho_B \widetilde{B_{j'p'}}(s) \right) = \text{Tr}_B \left( \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1137)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1138)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1139)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1140)$$

$$\text{Tr}_B \left( \widetilde{B_{j'p'}}(s) \rho_B \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) \quad (1141)$$

$$= \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1142)$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1143)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1144)$$

$$\text{Tr}_B \left( \rho_B \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1145)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1146)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1147)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1148)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1124) and (1131) and using the equations (1134)-(1148) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',p,p'} \left( C_{jp}(t) C_{j'p'}(s) \left( \Lambda_{jpj'p'}(\tau) \widetilde{A_{jp}}(t) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau) \widetilde{A_{jp}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \right) \right. \quad (1149)$$

$$\left. + C_{jp}(t) C_{j'p'}(s) \left( \Lambda_{j'p'jp}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jp}}(t) \right) \right) ds \quad (1150)$$

$$= - \int_0^t \sum_{jj'pp'} \left( C_{jp}(t) C_{j'p'}(s) \left( \Lambda_{jpj'p'}(\tau) \left[ \widetilde{A_{jp}}(t), \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[ \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s), \widetilde{A_{jp}}(t) \right] \right) \right) \quad (1151)$$

Rearranging and identifying the commutators allow us to write a more simplified version

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{jj'pp'} \left( C_{jp}(t) C_{j'p'}(t-\tau) \left( \Lambda_{jpj'p'}(\tau) \left[ A_{jp}(t), A_{j'p'}(t-\tau, t) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[ \widetilde{\rho_S}(t) A_{j'p'}(t-\tau, t), A_{jp}(t) \right] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1152)$$

For this case we used that  $A_{jp}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jp}(t) U(t-\tau) U^\dagger(t)$ . This is a non-Markovian equation.

## VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1153)$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1154)$$

$$H_I = \left( \sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left( \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right), \quad (1155)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1156)$$

We will start with a system-bath coupling operator of the form  $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$ .

### A. Variational Transformation

We consider the following operator:

$$V = \left( \sum_{n=1} |n\rangle\langle n| \right) \left( \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}) \right) \quad (1157)$$

At first let's obtain  $e^V$  under the transformation (1157), consider  $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})$ :

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \quad (1158)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{(\sum_{n=1} |n\rangle\langle n| \hat{\varphi})^2}{2!} + \dots \quad (1159)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \quad (1160)$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left( \mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \quad (1161)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \quad (1162)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \quad (1163)$$

Given that  $[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}] = 0$  if  $\mathbf{k}' \neq \mathbf{k}$  then we can proof using the Zassenhaus formula and defining  $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})}$  in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \quad (1164)$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (1165)$$

$$= B_{\pm} \quad (1166)$$

As we can see  $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_-$  because this form imposes that  $e^{-V} e^V = \mathbb{I}$  and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) A \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (1167)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1153):

$$\overline{|0\rangle\langle 0|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle 0| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \quad (1168)$$

$$= |0\rangle\langle 0|, \quad (1169)$$

$$\overline{|m\rangle\langle n|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle n| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \quad (1170)$$

$$= |m\rangle\langle m| B_+ |m\rangle\langle n| n\rangle\langle n| B_-, \quad (1171)$$

$$= |m\rangle\langle n|, \quad m \neq 0, \quad n \neq 0, \quad (1172)$$

$$\overline{|0\rangle\langle m|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |0\rangle\langle m| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right), \quad (1173)$$

$$= |0\rangle\langle m| B_- \quad m \neq 0, \quad (1174)$$

$$\overline{|m\rangle\langle 0|} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) |m\rangle\langle 0| \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (1175)$$

$$= |0\rangle\langle m| B_+ \quad m \neq 0, \quad (1176)$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (1177)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B_+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_- \quad (1178)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( B_+ b_{\mathbf{k}}^\dagger B_- \right) (B_+ b_{\mathbf{k}} B_-) \quad (1179)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left( b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1180)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1181)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \quad (1182)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1183)$$

The transformed Hamiltonians of the equations (1154) to (1156) written in terms of (1168) to (1183) are:

$$\overline{H_S(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1184)$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1185)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \overline{\sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|)} + \sum_{m,n \neq 0} \overline{V_{mn}(t) |m\rangle\langle n|} \quad (1186)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left( \overline{V_{0n}(t) |0\rangle\langle n|} + \overline{V_{n0}(t) |n\rangle\langle 0|} \right) + \sum_{m,n \neq 0} \overline{V_{mn}(t) |m\rangle\langle n|} \quad (1187)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) B_- |0\rangle\langle n| + V_{n0}(t) B_+ |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1188)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1189)$$

$$\overline{H_I} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) \left( \left( \sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left( \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \right) \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (1190)$$

$$= \left( \mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B_+ \right) \left( \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (1191)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B_+ (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) B_- \quad (1192)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1193)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1194)$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1195)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1196)$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (1197)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1198)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1199)$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left( g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1200)$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1201)$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left( \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \right) \quad (1202)$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left( g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1203)$$

Using the previous functions we have that (1200) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B_- + V_{n0}(t) |n\rangle\langle 0| B_+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1204)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) \quad (1205)$$

Now in order to separate the elements of the hamiltonian (1205) let's follow the references of the equations (245) and (226) to separate the hamiltonian like:

$$\overline{H_S}(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \quad (1206)$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B_- - B) + V_{n0}(t) |n\rangle\langle 0| (B_+ - B)), \quad (1207)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (1208)$$

Here B is given by:

$$\begin{aligned} B &= \langle B_+ \rangle \\ &= \langle B_- \rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1209)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1210)$$

$$B_x = \frac{B_+ + B_- - 2B}{2} \quad (1211)$$

$$B_y = \frac{B_- - B_+}{2i} \quad (1212)$$

Using this set of hermitian operators to write the Hamiltonians (1154)-(1156)

$$\overline{H_S(t)} = \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1213)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{0 < m < n} (V_{mn}(t) |m\rangle\langle n| + V_{nm}(t) |n\rangle\langle m|) \quad (1214)$$

$$+ \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1215)$$

$$= \sum_{0 < m < n} ((\Re(V_{mn}(t)) + i\Im(V_{mn}(t))) |m\rangle\langle n| + (\Re(V_{mn}(t)) - i\Im(V_{mn}(t))) |n\rangle\langle m|) + \varepsilon_0(t) |0\rangle\langle 0| \quad (1216)$$

$$+ B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1217)$$

$$= \sum_{0 < m < n} \left( (\Re(V_{nm}(t)) + i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1218)$$

$$+ B \sum_{n=1} \left( V_{0n}(t) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + V_{n0}(t) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) + \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1219)$$

$$= \sum_{0 < m < n} (\Re(V_{nm}(t)) \sigma_{nm,x} + \Im(V_{nm}(t)) \sigma_{nm,y}) + B \sum_{n=1} (\Re(V_{0n}(t)) \sigma_{0n,x} + \Im(V_{mn}(t)) \sigma_{0n,y}) \quad (1220)$$

$$+ \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1221)$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B_- - B) + V_{n0}(t) |n\rangle\langle 0| (B_+ - B)) \quad (1222)$$

$$= \sum_{n=1} \left( (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) (B_- - B) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) (B_+ - B) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) \quad (1223)$$

$$+ \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1224)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \sum_{n=1} \left( \frac{\sigma_{0n,x}}{2} ((B_- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) + (B_+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))) \right) \quad (1225)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B_+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) - (B_- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))) \quad (1226)$$

$$+ \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1227)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \left( \frac{\sigma_{0n,x}}{2} (B_+ + B_- - 2B) \Re(V_{0n}(t)) + i(B_- - B - B_+ + B) \Im(V_{0n}(t)) \right) \quad (1228)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B_+ - B - B_- + B) \Re(V_{0n}(t)) + i(B - B_- + B - B_+) \Im(V_{0n}(t))) + \sum_{n=1} B_{z,n} |n\rangle\langle n| \quad (1229)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (\sigma_{0n,x} (B_x \Re(V_{0n}(t)) - B_y \Im(V_{0n}(t))) \quad (1230)$$

$$+ \sigma_{0n,y} (B_y \Re(V_{0n}(t)) + B_x \Im(V_{0n}(t)))) \quad (1231)$$

## B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality



$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left( \text{Tr} \left( e^{-\beta(\overline{H}_S + \overline{H}_B)} \right) \right) + \langle \overline{H}_I \rangle_{\overline{H}_S + \overline{H}_B} + O \left( \langle \overline{H}_I^2 \rangle_{\overline{H}_S + \overline{H}_B} \right). \quad (1232)$$

Taking the equations (247)-(255) and given that  $\text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right) = C(R_1, R_2, \dots, R_{d-1}, B)$ , where each  $R_i$  and  $B$  depend of the set of variational parameters  $\{v_{\mathbf{k}}\}$ . From (255) and using the chain rule we obtain that:

$$\frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \frac{\partial R_n}{\partial v_{\mathbf{k}}}, \quad (1233)$$

$$= 0 \quad (1234)$$

Let's recall the equations (1201) and (1203), we can write them in terms of the variational parameters

$$B = \exp \left( - (1/2) \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) \right) \quad (1235)$$

$$R_n = \sum_{\mathbf{k}} \omega_{\mathbf{k}}^{-1} (v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}} v_{\mathbf{k}}) \quad (1236)$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \quad (1237)$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1238)$$

Introducing this derivates in the equation (1233) give us:

$$\frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \left( -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \right) + \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1239)$$

$$= v_{\mathbf{k}} \left( \frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t) \quad (1240)$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1241)$$

$$= \frac{2g_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1242)$$

Now taking  $v_{\mathbf{k}} = g_{\mathbf{k}} v_{\mathbf{k}}$  then we can obtain  $v_{\mathbf{k}}$  like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left( e^{-\beta \overline{H}_S(t)} \right)}{\partial B}}. \quad (1243)$$

### C. Master Equation

Let's consider that the initial state of the system is given by  $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$ , as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_+ \right) (|0\rangle\langle 0| \otimes \rho_B) \left( |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B_- \right) \quad (1244)$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \quad (1245)$$

$$0 = \rho(0) \quad (1246)$$

We transform any operator  $O$  into the interaction picture in the following way:

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1247)$$

$$U(t) \equiv \mathcal{T} \exp \left( -i \int_0^t dt' \overline{H_S}(t') \right). \quad (1248)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1249)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}(t)) \quad (1250)$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I}(t) = B_{z,0} |0\rangle\langle 0| + \sum_{n=1} (\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n} |n\rangle\langle n|) \quad (1251)$$

$$+ \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x} \quad (1252)$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1253)$$

$$B_{z,n} = \sum_{\mathbf{k}} \left( g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \text{ if } n \neq 0 \quad (1254)$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \quad (1255)$$

$$A_{2n}(t) = \sigma_{0n,y} \quad (1256)$$

$$A_{3n}(t) = |n\rangle\langle n| \quad (1257)$$

$$A_{4n}(t) = A_{2n}(t) \quad (1258)$$

$$A_{5n}(t) = A_{1n}(t) \quad (1259)$$

$$B_{1n}(t) = B_x \quad (1260)$$

$$B_{2n}(t) = B_y \quad (1261)$$

$$B_{3n}(t) = B_{z,n} \quad (1262)$$

$$B_{4n}(t) = B_{1n}(t) \quad (1263)$$

$$B_{5n}(t) = B_{2n}(t) \quad (1264)$$

$$C_{10}(t) = 0 \quad (1265)$$

$$C_{20}(t) = 0 \quad (1266)$$

$$C_{40}(t) = 0 \quad (1267)$$

$$C_{50}(t) = 0 \quad (1268)$$

$$C_{30}(t) = 1 \quad (1269)$$

$$C_{1n}(t) = \Re(V_{0n}(t)) \quad (1270)$$

$$C_{2n}(t) = C_{1n}(t) \quad (1271)$$

$$C_{3n}(t) = 1 \quad (1272)$$

$$C_{4n}(t) = \Im(V_{0n}(t)) \quad (1273)$$

$$C_{5n}(t) = -\Im(V_{0n}(t)) \quad (1274)$$

The previous notation allows us to write the interaction Hamiltonian in  $\overline{H}_I(t)$  as:

$$\overline{H}_I = \sum_{j \in J} \sum_{n=1} C_{jn}(t) (A_{jn} \otimes B_{jn}(t)) \quad (1275)$$

Here  $J = \{1, 2, 3, 4, 5\}$ .

We write the interaction Hamiltonian transformed under (1247) as:

$$\widetilde{H}_I(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1276)$$

$$\widetilde{A}_i(t) = U^\dagger(t) A_i U(t) \quad (1277)$$

$$\widetilde{B}_i(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \quad (1278)$$

Taking as reference state  $\rho_B$  and truncating at second order in  $H_I(t)$ , we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho}_S(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H}_I(t), \left[ \widetilde{H}_I(s), \widetilde{\rho}_S(s) \rho_B \right] \right] ds \quad (1279)$$

Replacing the equation (1276) in (1279) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[ \widetilde{H}_I(t), \left[ \widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1280)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \left[ \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1281)$$

$$= - \int_0^t \text{Tr}_B \left[ \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] \quad (1282)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \right] ds \quad (1283)$$

$$= - \int_0^t \text{Tr}_B \left( \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right) \quad (1284)$$

$$- \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \quad (1285)$$

$$- \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1286)$$

$$+ \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left( \widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left( \widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \right] ds \quad (1287)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jn j'n'}(\tau) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B \quad (1288)$$

$$= \left\langle \widetilde{B}_{jn}(\tau) \widetilde{B}_{j'n'}(0) \right\rangle_B \quad (1289)$$

Here  $s \rightarrow t - \tau$  and  $\text{Tr}_B \left( \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \rho_B \right) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B$ . To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators  $\widetilde{B}_{jn}(t)$  and elements related to the system given by  $\widetilde{A}_{jn}(t)$ . The systems considered are in different Hilbert spaces so  $\text{Tr} \left( \widetilde{A}_{jn}(t) \widetilde{B}_{j'n'}(t) \right) = \text{Tr} \left( \widetilde{A}_{jn}(t) \right) \text{Tr} \left( \widetilde{B}_{j'n'}(t) \right)$ . The correlation functions relevant of the master equation (1287) are:

$$\text{Tr}_B \left( \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1290)$$

$$= \left\langle \widetilde{B_{jn}}(0) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1291)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1292)$$

$$\text{Tr}_B \left( \widetilde{B_{jn}}(t) \rho_B \widetilde{B_{j'n'}}(s) \right) = \text{Tr}_B \left( \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1293)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1294)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1295)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1296)$$

$$\text{Tr}_B \left( \widetilde{B_{j'n'}}(s) \rho_B \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) \quad (1297)$$

$$= \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1298)$$

$$= \left\langle \widetilde{B_{jn}}(\tau) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1299)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1300)$$

$$\text{Tr}_B \left( \rho_B \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left( \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1301)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1302)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1303)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1304)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1280) and (1287) and using the equations (1290)-(1304) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left( C_{jn}(t) C_{j'n'}(s) \left( \Lambda_{jn j'n'}(\tau) \widetilde{A_{jn}}(t) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'n' jn}(-\tau) \widetilde{A_{jn}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \right) \right. \quad (1305)$$

$$\left. + C_{jn}(t) C_{j'n'}(s) \left( \Lambda_{j'n' jn}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \widetilde{A_{jn}}(t) - \Lambda_{jn j'n'}(\tau) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jn}}(t) \right) \right) ds \quad (1306)$$

$$= - \int_0^t \sum_{j,j',n,n'} \left( C_{jn}(t) C_{j'n'}(s) \left( \Lambda_{jn j'n'}(\tau) \left[ \widetilde{A_{jn}}(t), \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'n' jn}(-\tau) \left[ \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s), \widetilde{A_{jn}}(t) \right] \right) \right) \quad (1307)$$

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left( C_{jn}(t) C_{j'n'}(t-\tau) \left( \Lambda_{jn j'n'}(\tau) \left[ A_{jn}(t), A_{j'n'}(t-\tau, t) \widetilde{\rho_S}(t) \right] + \Lambda_{j'n' jn}(-\tau) \left[ \widetilde{\rho_S}(t) A_{j'n'}(t-\tau, t), A_{jn}(t) \right] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1308)$$

For this case we used that  $A_{jn}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jn}(t) U(t-\tau) U^\dagger(t)$ . This is a non-Markovian equation and if we take  $n = 2$  (two sites),  $\mu_0(t) = 0$ ,  $\mu_1(t) = 1$  then we can reproduce a similar expression to (422) as expected.

## VIII. BIBLIOGRAPHY

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