A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1}$$

$$H_S(t) = \varepsilon_0(t)|0\rangle\langle 0| + \varepsilon_1(t)|1\rangle\langle 1| + V_{10}(t)|1\rangle\langle 0| + V_{01}(t)|0\rangle\langle 1|, \tag{2}$$

$$H_I = |0\rangle\langle 0|\sum_{\mathbf{k}}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right), \tag{3}$$

$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{4}$$

For the states $|0\rangle, |1\rangle$ we have the ortonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \tag{5}$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V}$ where is the variationally optimizable anti-Hermitian operator:

$$V \equiv |0\rangle\langle 0|\sum_{\mathbf{k}}\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right) + |1\rangle\langle 1|\sum_{\mathbf{k}}\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right). \tag{6}$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}}\alpha_{i\mathbf{k}}(t). \tag{7}$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i \equiv \sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right), \tag{8}$$

$$V = |0\rangle\langle 0|\hat{\varphi}_0 + |1\rangle\langle 1|\hat{\varphi}_1. \tag{9}$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V}$ using the notation (6) will give us the following result:

$$
\begin{aligned}
e^{\pm V} &= e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)} && (10)\\
&= \mathbb{I}\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)+\frac{\left(\pm\left(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1\right)\right)^2}{2!}+\dots && (11)\\
&= |0\rangle\langle 0|+|1\rangle\langle 1|\pm(|0\rangle\langle 0|\hat{\varphi}_0+|1\rangle\langle 1|\hat{\varphi}_1)+\frac{|0\rangle\langle 0|\hat{\varphi}_0^2}{2!}+\frac{|1\rangle\langle 1|\hat{\varphi}_1^2}{2!}+\dots && (12)\\
&= |0\rangle\langle 0|\left(\mathbb{I}\pm\hat{\varphi}_0+\frac{\hat{\varphi}_0^2}{2!}\pm\dots\right)+|1\rangle\langle 1|\left(\mathbb{I}\pm\hat{\varphi}_1+\frac{\hat{\varphi}_1^2}{2!}\pm\dots\right) && (13)\\
&= |0\rangle\langle 0|e^{\pm\hat{\varphi}_0}+|1\rangle\langle 1|e^{\pm\hat{\varphi}_1} && (14)\\
&= |0\rangle\langle 0|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{0\mathbf{k}}b_{\mathbf{k}}^\dagger-\alpha_{0\mathbf{k}}^*b_{\mathbf{k}}\right)}+|1\rangle\langle 1|e^{\pm\sum_{\mathbf{k}}\left(\alpha_{1\mathbf{k}}b_{\mathbf{k}}^\dagger-\alpha_{1\mathbf{k}}^*b_{\mathbf{k}}\right)} && (15)\\
&= |0\rangle\langle 0|B_0^{\pm}+|1\rangle\langle 1|B_1^{\pm}, && (16)\\
B_i^{\pm} &\equiv e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}. && (17)
\end{aligned}
$$

Let's recall the Zassenhaus formula:

$$
e^{t(X+Y)}=e^{tX}\,e^{tY}\,e^{-\frac{t^2}{2}[X,Y]}\,e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\,e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])}\cdots. \tag{18}
$$

Since $\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}'}}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}^\dagger-\frac{v_{j\mathbf{k}'}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}'}\right]=0$ for all $\mathbf{k}^{'},\mathbf{k}$ and i,j we can show making $t=1$ in (18) the following result:

$$
\begin{aligned}
e^{\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)+\left(\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)} &= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}\left[\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}},\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right]}\dots && (19)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{-\frac{1}{2}0}\dots && (20)\\
&= e^{\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}e^{\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}}. && (21)
\end{aligned}
$$

By induction of this result we can write an expresion of $B_i^{\pm}$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D\left(\pm v_{i\mathbf{k}}\right)$:

$$
\begin{aligned}
D\left(\pm v_{i\mathbf{k}}\right) &\equiv e^{\pm\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^\dagger-\frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}, && (22)\\
B_i^{\pm} &= \prod_{\mathbf{k}}D\left(\pm\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right). && (23)
\end{aligned}
$$

this will help us to write operators O transformed in the variational frame as:

$$
\overline{O}\equiv \mathrm{e}^{V}O\mathrm{e}^{-V}. \tag{24}
$$

We will use the following identities:

$$
\begin{aligned}
\overline{|0\rangle\langle 0|} &= \mathrm{e}^{V}|0\rangle\langle 0|\mathrm{e}^{-V} && (25)\\
&= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (26)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|0\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (27)\\
&= |0\rangle\langle 0|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (28)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+B_1^- && (29)\\
&= |0\rangle\langle 0|, && (30)\\
\overline{|1\rangle\langle 1|} &= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (31)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 1|B_0^+ + |1\rangle\langle 1|1\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (32)\\
&= |1\rangle\langle 1|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (33)\\
&= |1\rangle\langle 1|0\rangle\langle 0|B_1^+B_0^- + B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- && (34)\\
&= B_1^+|1\rangle\langle 1|1\rangle\langle 1|B_1^- && (35)\\
&= |1\rangle\langle 1|, && (36)\\
\overline{|0\rangle\langle 1|} &= \mathrm{e}^{V}|0\rangle\langle 1|\mathrm{e}^{-V} && (37)\\
&= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|0\rangle\langle 1|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (38)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|B_1^+|0\rangle\langle 1|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (39)\\
&= \left(|0\rangle\langle 0|0\rangle\langle 1|B_0^+ + |1\rangle\langle 1|0\rangle\langle 1|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (40)\\
&= |0\rangle\langle 1|B_0^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (41)\\
&= |0\rangle\langle 1|0\rangle\langle 0|B_0^+B_0^- + |0\rangle\langle 1|1\rangle\langle 1|B_0^+B_1^- && (42)\\
&= |0\rangle\langle 1|B_0^+B_1^-, && (43)\\
\overline{|1\rangle\langle 0|} &= \mathrm{e}^{V}|1\rangle\langle 0|\mathrm{e}^{-V} && (44)\\
&= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)|1\rangle\langle 0|\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (45)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (46)\\
&= \left(|0\rangle\langle 0|1\rangle\langle 0|B_0^+ + |1\rangle\langle 1|1\rangle\langle 0|B_1^+\right)\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (47)\\
&= |1\rangle\langle 0|B_1^+\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (48)\\
&= |1\rangle\langle 0|B_1^+|0\rangle\langle 0|B_0^- + |1\rangle\langle 0|B_1^+|1\rangle\langle 1|B_1^- && (49)\\
&= |1\rangle\langle 0|0\rangle\langle 0|B_1^+B_0^- + |1\rangle\langle 0|1\rangle\langle 1|B_1^+B_1^- && (50)\\
&= |1\rangle\langle 0|B_1^+B_0^-, && (51)\\
\overline{b_{\mathbf{k}}} &= \mathrm{e}^{V}b_{\mathbf{k}}\mathrm{e}^{-V} && (52)\\
&= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (53)\\
&= |0\rangle\langle 0|B_0^+b_{\mathbf{k}}B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+b_{\mathbf{k}}|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}B_1^-|1\rangle\langle 1| && (54)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+b_{\mathbf{k}}B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+b_{\mathbf{k}}B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+b_{\mathbf{k}}B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}B_1^- && (55)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (56)\\
&= \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} && (57)\\
&= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, && (58)\\
\overline{b_{\mathbf{k}}}^{\dagger} &= \mathrm{e}^{V}b_{\mathbf{k}}^{\dagger}\mathrm{e}^{-V} && (59)\\
&= \left(|0\rangle\langle 0|B_0^+ + |1\rangle\langle 1|B_1^+\right)b_{\mathbf{k}}^{\dagger}\left(|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^-\right) && (60)\\
&= |0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}B_0^-|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}|1\rangle\langle 1|B_1^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}|0\rangle\langle 0|B_0^- + |1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}B_1^-|1\rangle\langle 1| && (61)\\
&= |0\rangle\langle 0|0\rangle\langle 0|B_0^+b_{\mathbf{k}}^{\dagger}B_0^- + |0\rangle\langle 0|1\rangle\langle 1|B_0^+b_{\mathbf{k}}^{\dagger}B_1^- + |1\rangle\langle 1|0\rangle\langle 0|B_1^+b_{\mathbf{k}}^{\dagger}B_0^- + |1\rangle\langle 1|1\rangle\langle 1|B_1^+b_{\mathbf{k}}^{\dagger}B_1^- && (62)\\
&= |0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right) && (63)\\
&= b_{\mathbf{k}}^{\dagger} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. && (64)
\end{aligned}
$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$:

$$B_i^{+} b_{\mathbf{k}} B_i^{-} = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{65}$$

$$B_i^{+} b_{\mathbf{k}}^{\dagger} B_i^{-} = b_{\mathbf{k}}^{\dagger} - \frac{v_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}. \tag{66}$$

We therefore have the following relationships:

$$\begin{aligned}
\overline{\varepsilon_0\left(t\right)|0\rangle\langle 0|} &= \varepsilon_0\left(t\right)|0\rangle\langle 0|, &(67)\\
\overline{\varepsilon_1\left(t\right)|1\rangle\langle 1|} &= \varepsilon_1\left(t\right)|1\rangle\langle 1|, &(68)\\
\overline{V_{10}\left(t\right)|1\rangle\langle 0|} &= V_{10}\left(t\right)|1\rangle\langle 0|B_1^{+}B_0^{-}, &(69)\\
\overline{V_{01}\left(t\right)|0\rangle\langle 1|} &= V_{01}\left(t\right)|0\rangle\langle 1|B_0^{+}B_1^{-}, &(70)\\
\overline{g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}} &= g_{i\mathbf{k}}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)\right)+g_{\mathbf{ik}}^{*}\left(|0\rangle\langle 0|\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+|1\rangle\langle 1|\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right) &(71)\\
&= g_{i\mathbf{k}}\left(\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right)+g_{i\mathbf{k}}^{*}\left(\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|\right) &(72)\\
&= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}-g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|-g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}|0\rangle\langle 0|-g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}|1\rangle\langle 1|-g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}|1\rangle\langle 1| &(73)\\
&= g_{i\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{i\mathbf{k}}^{*}b_{\mathbf{k}}-\left(g_{i\mathbf{k}}\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|0\rangle\langle 0|-\left(g_{i\mathbf{k}}\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}+g_{i\mathbf{k}}^{*}\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)|1\rangle\langle 1|, &(74)\\
\overline{|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)} &= \left(|0\rangle\langle 0|B_0^{+}+|1\rangle\langle 1|B_1^{+}\right)|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}+|1\rangle\langle 1|B_1^{-}\right) &(75)\\
&= |0\rangle\langle 0|B_0^{+}|0\rangle\langle 0|\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)|0\rangle\langle 0|B_0^{-} &(76)\\
&= |0\rangle\langle 0|B_0^{+}\left(g_{0\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{0\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_0^{-} &(77)\\
&= |0\rangle\langle 0|\left(g_{0\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+g_{0\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), &(78)\\
\overline{|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)} &= \left(|0\rangle\langle 0|B_0^{+}+|1\rangle\langle 1|B_1^{+}\right)|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)\left(|0\rangle\langle 0|B_0^{-}+|1\rangle\langle 1|B_1^{-}\right) &(79)\\
&= |1\rangle\langle 1|B_1^{+}|1\rangle\langle 1|\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1k}^{*}b_{\mathbf{k}}\right)|1\rangle\langle 1|B_1^{-} &(80)\\
&= |1\rangle\langle 1|B_1^{+}\left(g_{1\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{1\mathbf{k}}^{*}b_{\mathbf{k}}\right)B_1^{-} &(81)\\
&= |1\rangle\langle 1|\left(g_{1\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger}-\frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+g_{1\mathbf{k}}^{*}\left(b_{\mathbf{k}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right), &(82)\\
\overline{\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}} &= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}+|1\rangle\langle 1|B_1^{+}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{-}+|1\rangle\langle 1|B_1^{-}\right) &(83)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left(|0\rangle\langle 0|\prod_{\mathbf{k}'}D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)+|1\rangle\langle 1|\prod_{\mathbf{k}'}D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right) &(84)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|B_0^{+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_0^{-}+|1\rangle\langle 1|B_1^{+}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}B_1^{-}\right) &(85)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\prod_{\mathbf{k}'\neq\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)D\left(-\frac{v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right)\right) &(86)\\
&= \omega_{\mathbf{k}}\left(|0\rangle\langle 0|D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}+|1\rangle\langle 1|D\left(\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\mathbb{I}\right) &(87)
\end{aligned}$$

$$
\begin{aligned}
&= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) && (88) \\
&= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \tfrac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} + \left| \tfrac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} \right) \right) && (89) \\
&= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) && (90) \\
&= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) && (91) \\
&= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^{2} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^{\dagger} \right) \right) && (92) \\
&= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^{*} b_{\mathbf{k}} - v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^{*} b_{\mathbf{k}} - v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) && (93) \\
&= \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*} b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*} b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right). && (94)
\end{aligned}
$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$
\begin{aligned}
\overline{H_S(t)} &= \overline{\varepsilon_0(t)\,|0\rangle\langle 0|} + \overline{\varepsilon_1(t)\,|1\rangle\langle 1|} + \overline{V_{10}(t)\,|1\rangle\langle 0|} + \overline{V_{01}(t)\,|0\rangle\langle 1|} && (95) \\
&= \varepsilon_0(t)\,|0\rangle\langle 0| + \varepsilon_1(t)\,|1\rangle\langle 1| + V_{10}(t)\,|1\rangle\langle 0| B_1^{+} B_0^{-} + V_{01}(t)\,|0\rangle\langle 1| B_0^{+} B_1^{-}, && (96) \\
\overline{H_I} &= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}} \right)} && (97) \\
&= \overline{\sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}} \right)} + \overline{\sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}} \right)} && (98) \\
&= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^{*} \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^{*} \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) && (99) \\
&= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{0\mathbf{k}}^{*} b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} + g_{1\mathbf{k}}^{*} b_{\mathbf{k}} \right) - \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \frac{v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{0\mathbf{k}}^{*} \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \frac{v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{1\mathbf{k}}^{*} \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right), && (100) \\
\overline{H_B} &= \overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} && (101) \\
&= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*} b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*} b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right) && (102) \\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^{*} b_{\mathbf{k}} + v_{1\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^{*} b_{\mathbf{k}} + v_{0\mathbf{k}} b_{\mathbf{k}}^{\dagger} \right) \right) \right). && (103)
\end{aligned}
$$

Finally merging these expressions gives the transformed Hamiltonian:

$$
\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^{+} B_{j'}^{-} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^{\dagger} + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^{*} b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^{2}}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^{*}}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^{*} \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right). \tag{104}
$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$
\overline{H(t)} = \overline{H_{\bar{S}}} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{105}
$$

Let's define:

$$R_i \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right), \tag{106}$$

$$B_{iz} \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \, b_{\mathbf{k}} \right), \tag{107}$$

$$\chi_{ij} \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right). \tag{108}$$

χ_{ij} is an imaginary number so $e^{\chi_{ij}}$ is the phase associated to B_{ij}. Wwe can summarize these definitions with other that we will proof and use from now in the following matrix:

$$\begin{pmatrix} B_{iz} & B_{i\pm} \\ B_x & B_{ij} \\ B_y & R_i \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}})\, b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}})^* \, b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{10}^*}{2} & e^{\chi_{ij}} \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \\ \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{10}^*}{2\mathrm{i}} & \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \end{pmatrix}. \tag{109}$$

$$(\cdot)^{\Re} \equiv \Re(\cdot), \tag{110}$$

$$(\cdot)^{\Im} \equiv \Im(\cdot). \tag{111}$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_{\mathrm{B}} T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B} \right)}. \tag{112}$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{113}$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{114}$$

So the product of the matrix representation of $b^\dagger$ and b with $-\beta$ is:

$$-\beta\omega b^{\dagger}b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{115}$$

$$= \sum_{j=0}^{\infty} -j\beta\omega \left|j\right\rangle\!\left\langle j\right|, \tag{116}$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\!\langle j|, |i\rangle\!\langle i|] = 0$ for all i, j.

$$\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) = \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|, \tag{117}$$

$$\exp\left(-\beta\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right) = \prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|. \tag{118}$$

The value of $\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right)\right)$ is:

$$\mathrm{Tr}\left(\exp\left(-\beta\omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|\right) \tag{119}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) \tag{120}$$

$$= \sum_{j_{\mathbf{k}}} \exp\left(-\beta\omega_{\mathbf{k}}\right)^{j_{\mathbf{k}}} \tag{121}$$

$$= \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} \quad \text{(by geometric series)} \tag{122}$$

$$\equiv f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right), \tag{123}$$

$$\mathrm{Tr}\left(\exp\left(-\beta\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}\right)\right) = \mathrm{Tr}\left(\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|\right) \tag{124}$$

$$= \prod_{\mathbf{k}} \mathrm{Tr}\left(\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|\right) \tag{125}$$

$$= \prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right). \tag{126}$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} \tag{127}$$

$$= \frac{\prod_{\mathbf{k}}\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)} \tag{128}$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} \exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right) |j_{\mathbf{k}}\rangle\!\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}. \tag{129}$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle, \tag{130}$$
$$b_{\mathbf{k}}^{\dagger} \mid j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1} \mid j_{\mathbf{k}}+1\rangle. \tag{131}$$

Then we can prove that $\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0$ using the following property based on (130)-(131):

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = \mathrm{Tr}\left(\rho_B B_{iz}\right) = \mathrm{Tr}\left(B_{iz}\rho_B\right) \tag{132}$$
$$= \mathrm{Tr}\left(\left(\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}\right)\right)\rho_B\right) \tag{133}$$
$$= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}\rho_B\right) \tag{134}$$
$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\rho_B\right)+\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\rho_B\right) \tag{135}$$
$$= \sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\mathrm{Tr}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right) \tag{136}$$
$$= \sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}\right)\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right)+\sum_{\mathbf{k}}\left(\mathrm{g}_{i\mathbf{k}}-\mathrm{v}_{i\mathbf{k}}\right)^{*}\mathrm{Tr}\left(b_{\mathbf{k}}\prod_{\mathbf{k}}\frac{\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\mathrm{Bose-Einstein}}\left(-\beta\omega_{\mathbf{k}}\right)}\right), \tag{137}$$

$$\mathrm{Tr}\left(b_{\mathbf{k}}^{\dagger}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})\right)b_{\mathbf{k}}^{\dagger}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\ \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}^{\dagger}\text{)} \tag{138}$$
$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}}+1}\,|j_{\mathbf{k}}+1\rangle\langle j_{\mathbf{k}}|\right) \tag{139}$$
$$= 0, \tag{140}$$
$$\mathrm{Tr}\left(b_{\mathbf{k}}\sum_{j_{\mathbf{k}}}\exp(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}})|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) = \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)b_{\mathbf{k}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right)\ \text{(by cyclic permutivity of trace, move } b_{\mathrm{k}}\text{)} \tag{141}$$
$$= \mathrm{Tr}\left(\left(\sum_{j_{\mathbf{k}}}\exp\left(-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}\right)\right)\sqrt{j_{\mathbf{k}}}\,|j_{\mathbf{k}}-1\rangle\langle j_{\mathbf{k}}|\right) \tag{142}$$
$$= 0. \tag{143}$$

we therefore find that:

$$\langle B_{iz}\rangle_{\overline{H_{\bar{B}}}} = 0. \tag{144}$$

Another important expected value is $B = \langle B^{\pm}\rangle_{\overline{H_{\bar{B}}}}$, where $B^{\pm} = e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}$ is given by:

$$\left\langle B^{\pm}\right\rangle_{H_B} = \mathrm{Tr}\left(\rho_B B_{\pm}\right) = \mathrm{Tr}\left(B_{\pm}\rho_B\right) \tag{145}$$
$$= \mathrm{Tr}\left(e^{\pm\sum_{\mathbf{k}}\left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}^{\dagger}-\frac{v_{\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}b_{\mathbf{k}}\right)}\rho_B\right) \tag{146}$$
$$= \prod_{\mathbf{k}}\mathrm{Tr}\left(D\left(\pm\alpha_{\mathbf{k}}\right)\rho_B\right) \tag{147}$$
$$= \prod_{\mathbf{k}}\left\langle D\left(\pm\alpha_{\mathbf{k}}\right)\right\rangle. \tag{148}$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha)\, |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha. \tag{149}$$

where $P(\alpha)$ satisfies $\int P(\alpha)\,\mathrm{d}^2\alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by$P(\alpha)$ is given by:

$$\begin{aligned} \langle A\rangle =& \mathrm{Tr}\,(A\rho) && (150)\\ =& \int P(\alpha)\,\langle\alpha\,|A|\,\alpha\rangle \mathrm{d}^2\alpha. && (151)\end{aligned}$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N}\exp\left(-\frac{|\alpha|^2}{N}\right)$ where $N = \left(e^{\beta\omega}-1\right)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = 1/k_B T$.

Using the integral representation (151) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\begin{aligned}
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle\alpha|D(h)\,|\alpha\rangle \mathrm{d}^2\alpha && (152)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(-\alpha)\,D(h)\,D(\alpha)\,|0\rangle \mathrm{d}^2\alpha, && (153)\\
D(h)\,D(\alpha) &= D(h+\alpha)\,e^{\frac{1}{2}(h\alpha^*-h^*\alpha)}, && (154)\\
D(-\alpha)\,(D(h)\,D(\alpha)) &= D(-\alpha)\,D(h+\alpha)\,e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (155)\\
&= D(h)\,e^{\frac{1}{2}(-\alpha(h+\alpha)^*+\alpha^*(h+\alpha))}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (156)\\
&= D(\alpha)\,e^{\frac{1}{2}\left(-\alpha h^*-|\alpha|^2+\alpha^* h+|\alpha|^2\right)}e^{\frac{1}{2}(h\alpha^*-h^*\alpha)} && (157)\\
&= D(\alpha)\,e^{(h\alpha^*-h^*\alpha)}, && (158)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\langle 0|D(h)\exp(h\alpha^*-h^*\alpha)\,|0\rangle \mathrm{d}^2\alpha && (159)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\,\langle 0|D(h)\,|0\rangle \mathrm{d}^2\alpha && (160)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\,\langle 0|h\rangle \mathrm{d}^2\alpha, && (161)\\
|\alpha\rangle &= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, && (162)\\
\langle D(h)\rangle &= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\,\langle 0|\exp\left(-\frac{|h|^2}{2}\right)\sum_{n=0}^{\infty}\frac{h^n}{\sqrt{n!}}|n\rangle \mathrm{d}^2\alpha && (163)\\
&= \frac{1}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha && (164)\\
&= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int \exp\left(-\frac{|\alpha|^2}{N}+h\alpha^*-h^*\alpha\right)\mathrm{d}^2\alpha, && (165)
\end{aligned}$$

$$\alpha = x + \mathrm{i}y, \tag{166}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{N}+h\left(x-\mathrm{i}y\right)-h^*\left(x+\mathrm{i}y\right)\right)\mathrm{d}x\mathrm{d}y \tag{167}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{N}+hx-h^*x\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y\right)\mathrm{d}y, \tag{168}$$

$$-\frac{x^2}{N}+hx-h^*x = -\frac{1}{N}\left(x^2-Nhx+Nh^*x\right) \tag{169}$$

$$= -\frac{1}{N}\left(x+\frac{\left(Nh^*-Nh\right)}{2}\right)^2+\frac{N\left(h^*-h\right)^2}{4}, \tag{170}$$

$$-\frac{y^2}{N}-\mathrm{i}hy-\mathrm{i}h^*y = -\frac{1}{N}\left(y^2+\mathrm{i}Nhy+\mathrm{i}Nh^*y\right) \tag{171}$$

$$= -\frac{1}{N}\left(y^2+\frac{\mathrm{i}N\left(h+h^*\right)}{2}\right)-\frac{N\left(h+h^*\right)^2}{4}, \tag{172}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{N}\left(x+\frac{(Nh^*-Nh)}{2}\right)^2-\frac{1}{N}\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)\right)\mathrm{d}x\mathrm{d}y, \tag{173}$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty}\exp\left(-\frac{\left(x-\mu\right)^2}{2\sigma^2}\right)\mathrm{d}x, \tag{174}$$

$$\langle D\left(h\right)\rangle = \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(x+\frac{(Nh^*-Nh)}{2}\right)^2}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}x\int_{-\infty}^{\infty}\exp\left(-\frac{\left(y^2+\frac{\mathrm{i}N(h+h^*)}{2}\right)}{2\left(\sqrt{\frac{N}{2}}\right)^2}\right)\mathrm{d}y \tag{175}$$

$$= \frac{\exp\left(-\frac{|h|^2}{2}+\frac{N(h^*-h)^2}{4}-\frac{N(h+h^*)^2}{4}\right)}{\pi N}\left(\sqrt{2\pi}\sqrt{\frac{N}{2}}\right)^2 \tag{176}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N\left(h^*-h\right)^2}{4}-\frac{N\left(h+h^*\right)^2}{4}\right) \tag{177}$$

$$= \exp\left(-\frac{|h|^2}{2}+\frac{N\left(h^{*2}-2hh^*+h^2\right)-N\left(h^2+2hh^*+h^{*2}\right)}{4}\right) \tag{178}$$

$$= \exp\left(-|h|^2\left(N+\frac{1}{2}\right)\right) \tag{179}$$

$$= \exp\left(-|h|^2\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)\right) \tag{180}$$

$$= \exp\left(-\frac{|h|^2}{2}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right)\right) \tag{181}$$

$$= \exp\left(-\frac{|h|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right). \tag{182}$$

In the last line we used $\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (147) using (182) is given by:

$$B = \exp\left(-\sum_{\mathbf{k}}\frac{|\alpha_{\mathbf{k}}|^2}{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{183}$$

We will now force $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B^+\sigma^+$ and $B^-\sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}$ and we subtract their expected value in order to satisfy $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}$ is:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+ B_{j'}^- + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}})^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{184}$$

$$= \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| B_{jj'} + \sum_j |j\rangle\langle j| B_{jz} + \sum_{j\neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+ B_{j'}^- - B_{jj'} \right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \tag{185}$$

$$\equiv \overline{H_{\bar{S}}(t)} + \overline{H_{\bar{I}}} + \overline{H_{\bar{B}}}. \tag{186}$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_i^+ B_j^- \rangle = B_{ij} \tag{187}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \tag{188}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \tag{189}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle \tag{190}$$

$$= \prod_{\mathbf{k}} \left\langle D\left(\frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{191}$$

$$= \prod_{\mathbf{k}} \exp\left(-\frac{1}{2} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \tag{192}$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{i\mathbf{k}}^* v_{j\mathbf{k}} - v_{i\mathbf{k}} v_{j\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}. \tag{193}$$

From the definition $B_{01} = \langle B_0^+ B_1^- \rangle$ using the displacement operator we have:

$$\langle B_0^+ B_1^- \rangle = B_{01} \tag{194}$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle \tag{195}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \tag{196}$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \right\rangle \tag{197}$$

$$= \prod_{\mathbf{k}} \left(\left\langle D\left(\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \right\rangle e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \tag{198}$$

$$= \prod_{\mathbf{k}} \left(\exp\left(-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \tag{199}$$

$$= \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}. \tag{200}$$

We can check:

$$
\begin{aligned}
\langle B_0^+ B_1^- \rangle &= B_{01} && (201)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} && (202)\\
&= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)^*} && (203)\\
&= \langle B_1^+ B_0^- \rangle^* && (204)\\
&= B_{10}^*. && (205)
\end{aligned}
$$

The parts of the splitted Hamiltonian are:

$$
\begin{aligned}
\overline{H_{\bar{S}}\left(t\right)} &\equiv \left(\varepsilon_0\left(t\right)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1| + V_{10}\left(t\right)B_{10}\sigma^+ + V_{01}\left(t\right)B_{01}\sigma^-, && (206)\\
\overline{H_{\bar{I}}} &\equiv V_{10}\left(t\right)\left(B_1^+B_0^- - B_{10}\right)\sigma^+ + V_{01}\left(t\right)\left(B_0^+B_1^- - B_{01}\right)\sigma^- + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z}, && (207)\\
\overline{H_{\bar{B}}} &\equiv \sum_{\mathbf{k}}\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} && (208)\\
&= H_B. && (209)
\end{aligned}
$$

Note that $\overline{H_{\bar{B}}}$, which is the bath acting on the effective "system" $\bar{S}$ in the variational frame, is just the original bath, H_B, before transforming to the variational frame.

For the Hamiltonian (207) we can verify the condition $\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} = 0$ in the following way:

$$
\begin{aligned}
\langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{B}}}} &= \left\langle \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)b_{\mathbf{k}}^\dagger + \left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^* b_{\mathbf{k}}\right)|n\rangle\langle n| + \sum_{j\neq j'} V_{jj'}\left(t\right)|j\rangle\langle j'|\left(B_j^+B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} && (210)\\
&= \left\langle \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)b_{\mathbf{k}}^\dagger + \left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^* b_{\mathbf{k}}\right)|n\rangle\langle n|\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle \sum_{j\neq j'} V_{jj'}\left(t\right)|j\rangle\langle j'|\left(B_j^+B_{j'}^- - B_{jj'}\right)\right\rangle_{\overline{H_{\bar{B}}}} && (211)\\
&= \sum_{n\mathbf{k}}\left(\left\langle\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + \left\langle\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^* b_{\mathbf{k}}\right\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|\left(\left\langle V_{jj'}\left(t\right)B_j^+B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}} - \langle V_{jj'}\left(t\right)B_{jj'}\rangle_{\overline{H_{\bar{B}}}}\right) && (212)\\
&= \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}\left(t\right)\left(\left\langle B_j^+B_{j'}^-\right\rangle_{\overline{H_{\bar{B}}}} - \langle B_{jj'}\rangle_{\overline{H_{\bar{B}}}}\right) && (213)\\
&= \sum_{n\mathbf{k}}\left(\left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)\left\langle b_{\mathbf{k}}^\dagger\right\rangle_{\overline{H_{\bar{B}}}} + \left(g_{n\mathbf{k}}-v_{n\mathbf{k}}\right)^*\langle b_{\mathbf{k}}\rangle_{\overline{H_{\bar{B}}}}\right)|n\rangle\langle n| + \sum_{j\neq j'}|j\rangle\langle j'|V_{jj'}\left(t\right)\left(B_{jj'} - B_{jj'}\right) && (214)\\
&= 0. && (215)
\end{aligned}
$$

We used (144) and (193) to evaluate the expression. Let's consider the following Hermitian combinations:

$$
\begin{aligned}
B_x &= B_x^\dagger && (216)\\
&= \frac{B_1^+B_0^- + B_0^+B_1^- - B_{10} - B_{01}}{2}, && (217)\\
B_y &= B_y^\dagger && (218)\\
&= \frac{B_0^+B_1^- - B_1^+B_0^- + B_{10} - B_{01}}{2\mathrm{i}}, && (219)\\
B_{iz} &= B_{iz}^\dagger && (220)\\
&= \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^* b_{\mathbf{k}}\right). && (221)
\end{aligned}
$$

Writing the equations (206) and (207) using the previous combinations we obtain that:

$$
\begin{aligned}
\overline{H_{\bar{S}}}(t) &= \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j| + V_{10}\left(t\right)B_{10}\sigma^{+} + V_{01}\left(t\right)B_{01}\sigma^{-} && (222)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j| + V_{10}\left(t\right)B_{10}\frac{\sigma_x+\mathrm{i}\sigma_y}{2} + V_{01}\left(t\right)B_{01}\frac{\sigma_x-\mathrm{i}\sigma_y}{2} && (223)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j| + V_{10}\left(t\right)\left(B_{10}^{\Re}\left(t\right)+\mathrm{i}B_{10}^{\Im}\left(t\right)\right)\frac{\sigma_x+\mathrm{i}\sigma_y}{2} + V_{01}(t)\left(B_{10}^{\Re}\left(t\right)-\mathrm{i}B_{10}^{\Im}\left(t\right)\right)\frac{\sigma_x-\mathrm{i}\sigma_y}{2} && (224)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j\left(t\right)+R_j\right)|j\rangle\langle j| + B_{10}^{\Re}(t)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}+V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right) + \mathrm{i}B_{10}^{\Im}(t)\left(V_{10}(t)\frac{\sigma_x+\mathrm{i}\sigma_y}{2}-V_{01}(t)\frac{\sigma_x-\mathrm{i}\sigma_y}{2}\right) && (225)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t)+R_j\right)|j\rangle\langle j| + B_{10}^{\Re}(t)\left(\sigma_x\frac{V_{10}(t)+V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)-V_{01}(t)}{2}\right) + \mathrm{i}B_{10}^{\Im}(t)\left(\sigma_x\frac{V_{10}(t)-V_{01}(t)}{2}+\mathrm{i}\sigma_y\frac{V_{10}(t)+V_{01}(t)}{2}\right) && (226)\\
&= \sum_{j\in\{0,1\}} \left(\varepsilon_j(t)+R_j\right)|j\rangle\langle j| + B_{10}^{\Re}(t)\left(\sigma_x V_{10}^{\Re}\left(t\right)-\sigma_y V_{10}^{\Im}\left(t\right)\right) + \mathrm{i}B_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x V_{10}^{\Im}(t)+\mathrm{i}\sigma_y V_{10}^{\Re}(t)\right) && (227)\\
&= \left(\varepsilon_0(t)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1(t)+R_1\right)|1\rangle\langle 1| + B_{10}^{\Re}(t)\left(\sigma_x V_{10}^{\Re}\left(t\right)-\sigma_y V_{10}^{\Im}\left(t\right)\right) + \mathrm{i}B_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x V_{10}^{\Im}(t)+\mathrm{i}\sigma_y V_{10}^{\Re}(t)\right) && (228)\\
&= \left(\varepsilon_0(t)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1(t)+R_1\right)|1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-\sigma_y B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)\right) - \left(\sigma_x B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)+\sigma_y B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right) && (229)\\
&= \left(\varepsilon_0\left(t\right)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right) - \sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right) && (230)\\
&= \left(\varepsilon_0(t)+R_0\right)|0\rangle\langle 0| + \left(\varepsilon_1\left(t\right)+R_1\right)|1\rangle\langle 1| + \sigma_x\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Re}\left(t\right)-B_{10}^{\Im}\left(t\right)V_{10}^{\Im}\left(t\right)\right) - \sigma_y\left(B_{10}^{\Re}\left(t\right)V_{10}^{\Im}\left(t\right)+B_{10}^{\Im}\left(t\right)V_{10}^{\Re}\left(t\right)\right). && (231)\\
\overline{H_{\bar{I}}} &= V_{10}\left(t\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}\right) + V_{01}\left(t\right)\left(\sigma^{-}B_0^{+}B_1^{-}-\sigma^{-}B_{01}\right) + |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} && (232)\\
&= |0\rangle\langle 0|B_{0z} + |1\rangle\langle 1|B_{1z} + \left(V_{10}^{\Re}\left(t\right)+\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}\right) + \left(V_{10}^{\Re}\left(t\right)-\mathrm{i}V_{10}^{\Im}\left(t\right)\right)\left(\sigma^{-}B_0^{+}B_1^{-}-\sigma^{-}B_{01}\right) && (233)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}+\sigma^{-}B_0^{+}B_1^{-}-\sigma^{-}B_{01}\right) + \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma^{+}B_1^{+}B_0^{-}-\sigma^{+}B_{10}-\sigma^{-}B_0^{+}B_1^{-}+\sigma^{-}B_{01}\right) && (234)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}B_0^{-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}B_1^{-}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) && (235)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}B_0^{-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}B_1^{-}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) && (236)\\
&+ \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_1^{+}B_0^{-}-\frac{\sigma_x+\mathrm{i}\sigma_y}{2}B_{10}-\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_0^{+}B_1^{-}+\frac{\sigma_x-\mathrm{i}\sigma_y}{2}B_{01}\right) && (237)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}+\mathrm{i}\sigma_y\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2}\right) && (238)\\
&+ \mathrm{i}V_{10}^{\Im}\left(t\right)\left(\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2}+\mathrm{i}\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) && (239)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2}-\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) && (240)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) && (241)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}(t)\left(\sigma_x B_x+\sigma_y B_y\right) + V_{10}^{\Im}(t)\left(\mathrm{i}^2\sigma_x\frac{B_1^{+}B_0^{-}-B_0^{+}B_1^{-}-B_{10}+B_{01}}{2\mathrm{i}}-\sigma_y\frac{B_1^{+}B_0^{-}+B_0^{+}B_1^{-}-B_{10}-B_{01}}{2}\right) && (242)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x+\sigma_y B_y\right) + V_{10}^{\Im}\left(t\right)\left(\mathrm{i}^2\sigma_x\left(-B_y\right)-\sigma_y B_x\right) && (243)\\
&= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x+\sigma_y B_y\right) + V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y-\sigma_y B_x\right). && (244)
\end{aligned}
$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right) + \left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}}^2\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}}\right). \tag{245}$$

We will optimize the set of variational parameters $\{v_{i\mathbf{k}}\}$ in order to minimize A_{B} (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\left\langle\overline{H_{\bar{I}}}\right\rangle_{\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}} = 0$ we can obtain the following condition to obtain the set $\{v_{i\mathbf{k}}\}$:

$$\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} = 0. \tag{246}$$

Using this condition and given that $\left[\overline{H_{\bar{S}}}\left(t\right), \overline{H_{\bar{B}}}\right] = 0$, we have:

$$e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)} = e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}. \tag{247}$$

Then using the fact that $\overline{H_{\bar{S}}}\left(t\right)$ and $\overline{H_{\bar{B}}}$ relate to different Hilbert spaces, we obtain:

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}e^{-\beta\overline{H_{\bar{B}}}}\right) = \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right). \tag{248}$$

So Eq. (246) becomes:

$$\begin{aligned}
\frac{\partial A_{\mathrm{B}}}{\partial v_{i\mathbf{k}}} &= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}}(t)+\overline{H_{\bar{B}}}\right)}\right)\right)}{\partial v_{i\mathbf{k}}} && (249)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (250)\\
&= -\frac{1}{\beta}\frac{\partial\left(\ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right) + \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)\right)}{\partial v_{i\mathbf{k}}} && (251)\\
&= -\frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} - \frac{1}{\beta}\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{B}}}}\right)\right)}{\partial v_{i\mathbf{k}}} && (252)\\
&= 0 \quad (\text{by Eq. (246)}). && (253)
\end{aligned}$$

But since $\bar{H}_{\bar{B}} = H_B$ which doesn't contain any $v_{i\mathbf{k}}$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}$ will be zero. We therefore require the following:

$$\begin{aligned}
\frac{\partial \ln\left(\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)\right)}{\partial v_{i\mathbf{k}}} &= \frac{1}{e^{-\beta\overline{H_{\bar{S}}}(t)}}\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} && (254)\\
&= 0. && (255)
\end{aligned}$$

This means we need to impose:

$$\frac{\partial \mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{i\mathbf{k}}} = 0. \tag{256}$$

First we look at:

$$-\beta\overline{H_{\bar{S}}}\left(t\right)=-\beta\left(\left(\varepsilon_{0}\left(t\right)+R_{0}\right)|0\rangle\langle 0|+\left(\varepsilon_{1}\left(t\right)+R_{1}\right)|1\rangle\langle 1|+V_{10}\left(t\right)B_{10}\sigma^{+}+V_{01}\left(t\right)B_{01}\sigma^{-}\right). \tag{257}$$

Then the eigenvalues of $-\beta\overline{H_{\bar{S}}}\left(t\right)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^{2}-\mathrm{Tr}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)+\mathrm{Det}\left(-\beta\overline{H_{\bar{S}}}\left(t\right)\right)=0. \tag{258}$$

Let's define:

$$\varepsilon\left(t\right)\equiv\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right), \tag{259}$$

$$\eta\equiv\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}. \tag{260}$$

The solutions of the equation (258) are:

$$\lambda=\beta\frac{-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\pm\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}}\left(t\right)\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}}\left(t\right)\right)}}{2} \tag{261}$$

$$=\beta\frac{-\varepsilon\left(t\right)\pm\eta\left(t\right)}{2} \tag{262}$$

$$=-\beta\frac{\varepsilon\left(t\right)\mp\eta\left(t\right)}{2}. \tag{263}$$

The value of $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)=\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(\frac{\eta\left(t\right)\beta}{2}\right)+\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\exp\left(-\frac{\eta\left(t\right)\beta}{2}\right) \tag{264}$$

$$=2\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right). \tag{265}$$

Given that $v_{\mathbf{ik}}$ is a complex numnber then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)=A\left(\varepsilon\left(t\right),\eta\left(t\right)\right)$ to calculate $\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{\mathbf{ik}}^{\Re}}$ can lead to:

$$\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}}(t)}\right)}{\partial v_{\mathbf{ik}}^{\Re}}=\frac{\partial\left(2\exp\left(-\frac{\varepsilon(t)\beta}{2}\right)\cosh\left(\frac{\eta(t)\beta}{2}\right)\right)}{\partial v_{\mathbf{ik}}^{\Re}} \tag{266}$$

$$=2\left(-\frac{\beta}{2}\frac{\partial\varepsilon\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)+2\left(\frac{\beta}{2}\frac{\partial\eta\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\right)\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right) \tag{267}$$

$$=-\beta\exp\left(-\frac{\varepsilon\left(t\right)\beta}{2}\right)\left(\frac{\partial\varepsilon\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)\right). \tag{268}$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\cosh\left(\frac{\eta\left(t\right)\beta}{2}\right)-\frac{\partial\eta\left(t\right)}{\partial v_{\mathbf{ik}}^{\Re}}\sinh\left(\frac{\eta\left(t\right)\beta}{2}\right)=0. \tag{269}$$

The derivates included in the expression given are related to:

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{270}$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right)^* \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{271}$$

$$= \langle B_1^+ B_0^- \rangle^*, \tag{272}$$

$$R_i = \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right) \tag{273}$$

$$= \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{274}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)} \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{275}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2\omega_{\mathbf{k}}^2} \left(v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* \right) \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right), \tag{276}$$

$$v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^* = \left(v_{0\mathbf{k}}^{\Re} - \mathrm{i} v_{0\mathbf{k}}^{\Im} \right) \left(v_{1\mathbf{k}}^{\Re} + \mathrm{i} v_{1\mathbf{k}}^{\Im} \right) - \left(v_{0\mathbf{k}}^{\Re} + \mathrm{i} v_{0\mathbf{k}}^{\Im} \right) \left(v_{1\mathbf{k}}^{\Re} - \mathrm{i} v_{1\mathbf{k}}^{\Im} \right) \tag{277}$$

$$= \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} + \mathrm{i} v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - \mathrm{i} v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im} \right) - \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Re} - \mathrm{i} v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} + \mathrm{i} v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} + v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Im} \right) \tag{278}$$

$$= 2\mathrm{i} \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} \right), \tag{279}$$

$$|v_{1\mathbf{k}} - v_{0\mathbf{k}}|^2 = \left(v_{1\mathbf{k}} - v_{0\mathbf{k}} \right) \left(v_{1\mathbf{k}} - v_{0\mathbf{k}} \right)^* \tag{280}$$

$$= |v_{1\mathbf{k}}|^2 + |v_{0\mathbf{k}}|^2 - \left(v_{1\mathbf{k}} v_{0\mathbf{k}}^* + v_{1\mathbf{k}}^* v_{0\mathbf{k}} \right) \tag{281}$$

$$= \left(v_{1\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} \right)^2 + \left(v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{0\mathbf{k}}^{\Im} \right)^2 - \left(\left(v_{1\mathbf{k}}^{\Re} + \mathrm{i} v_{1\mathbf{k}}^{\Im} \right) \left(v_{0\mathbf{k}}^{\Re} - \mathrm{i} v_{0\mathbf{k}}^{\Im} \right) + \left(v_{1\mathbf{k}}^{\Re} - \mathrm{i} v_{1\mathbf{k}}^{\Im} \right) \left(v_{0\mathbf{k}}^{\Re} + \mathrm{i} v_{0\mathbf{k}}^{\Im} \right) \right) \tag{282}$$

$$= \left(v_{1\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} \right)^2 + \left(v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{0\mathbf{k}}^{\Im} \right)^2 - 2 \left(v_{1\mathbf{k}}^{\Re} v_{0\mathbf{k}}^{\Re} + v_{1\mathbf{k}}^{\Im} v_{0\mathbf{k}}^{\Im} \right) \tag{283}$$

$$= \left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} \right)^2. \tag{284}$$

Rewriting in terms of real and imaginary parts.

$$R_i = \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re} \right)^2 + \left(v_{i\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re} - \mathrm{i} v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re} + \mathrm{i} v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}} \right) \right) \tag{285}$$

$$= \sum_{\mathbf{k}} \left(\frac{\left(v_{i\mathbf{k}}^{\Re} \right)^2 + \left(v_{i\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re} \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - \mathrm{i} v_{i\mathbf{k}}^{\Im} \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \tag{286}$$

$$\langle B_0^+ B_1^- \rangle = \left(\prod_{\mathbf{k}} \exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{287}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{2\mathrm{i} \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} \right)}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \tag{288}$$

$$= \left(\prod_{\mathbf{k}} \exp\left(\frac{\mathrm{i} \left(v_{0\mathbf{k}}^{\Re} v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} v_{1\mathbf{k}}^{\Re} \right)}{\omega_{\mathbf{k}}^2} \right) \right) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{\left(v_{1\mathbf{k}}^{\Re} - v_{0\mathbf{k}}^{\Re} \right)^2 + \left(v_{1\mathbf{k}}^{\Im} - v_{0\mathbf{k}}^{\Im} \right)^2}{\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right), \tag{289}$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}=\frac{\partial\left(\varepsilon_{1}\left(t\right)+R_{1}+\varepsilon_{0}\left(t\right)+R_{0}\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{290}$$

$$=\frac{\partial\left(\left(\frac{\left(v_{i\mathbf{k}}^{\Re}\right)^{2}+\left(v_{i\mathbf{k}}^{\Im}\right)^{2}}{\omega_{\mathbf{k}}}-v_{i\mathbf{k}}^{\Re}\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}-\mathrm{i}v_{i\mathbf{k}}^{\Im}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{291}$$

$$=\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}, \tag{292}$$

$$\frac{\partial|B_{10}|^{2}}{\partial v_{i\mathbf{k}}^{\Re}}=\frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v_{i\mathbf{k}}^{\Re}} \tag{293}$$

$$=-\frac{2\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{294}$$

$$=-\frac{2\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}\left|B_{10}\right|^{2}, \tag{295}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}=\frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{296}$$

$$=\frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}\left(t\right)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}\left(t\right)}\right)}} \tag{297}$$

$$=\frac{\varepsilon\left(t\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}\right)\left(\varepsilon_{0}(t)+R_{0}\right)-|V_{10}(t)|^{2}|B_{10}(t)|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Re}}}{\eta\left(t\right)} \tag{298}$$

$$=\frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)}{\partial v_{i\mathbf{k}}^{\Re}}|B_{10}|^{2}|V_{10}\left(t\right)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{299}$$

$$=\frac{\varepsilon(t)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Re}-v_{i'\mathbf{k}}^{\Re}\right)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta\left(t\right)} \tag{300}$$

$$=\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon\left(t\right)-4\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta\left(t\right)}\right)+\frac{1}{\eta\left(t\right)}\left(-\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}\varepsilon\left(t\right)\right. \tag{301}$$

$$\left.+2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)\frac{g_{i\mathbf{k}}+g_{i\mathbf{k}}^{*}}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^{2}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{302}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta\left(t\right)}{2}\right)=\frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}}} \tag{303}$$

$$=\frac{\frac{2v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}-\frac{2g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\left(2\frac{\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})-\frac{2}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+2\frac{(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}^{2}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)-\frac{g_{i\mathbf{k}}^{\Re}}{\omega_{\mathbf{k}}}\varepsilon(t)}{\eta(t)}}. \tag{304}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right)=\frac{\left(2v^{\Re}_{i\mathbf{k}}-g_{i\mathbf{k}}-g^{*}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{305}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}B_{10}^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{306}$$

$$=\frac{\left(2v^{\Re}_{i\mathbf{k}}-2g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+4\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{307}$$

$$=\frac{\left(v^{\Re}_{i\mathbf{k}}-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{v^{\Re}_{i\mathbf{k}}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+2\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{308}$$

Separating (307) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{\left(v^{\Re}_{i\mathbf{k}}-g^{\Re}_{i\mathbf{k}}\right)\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Re}_{i\mathbf{k}}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Re}_{i\mathbf{k}}(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t))+2\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{309}$$

$$v^{\Re}_{i\mathbf{k}}-g^{\Re}_{i\mathbf{k}}=v^{\Re}_{i\mathbf{k}}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_{i}(t)-R_{i})-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Re}_{i\mathbf{k}}(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t))+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Re}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^{2}|V_{10}(t)|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \tag{310}$$

$$v^{\Re}_{i\mathbf{k}}=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}}{g^{\Re}_{i\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \tag{311}$$

$$v^{\Re}_{i\mathbf{k}}=\frac{g^{\Re}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(2\varepsilon_{i}\left(t\right)+2R_{i}-\varepsilon\left(t\right)\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}}\frac{v^{\Re}_{i'\mathbf{k}}}{g^{\Re}_{i\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}\left(t\right)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon\left(t\right)-2\left(\varepsilon\left(t\right)-\varepsilon_{i}\left(t\right)-R_{i}\right)-\frac{2|V_{10}(t)|^{2}|B_{10}|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \tag{312}$$

The imaginary part can be found in the following way:

$$\frac{\partial\varepsilon\left(t\right)}{\partial v^{\Im}_{i\mathbf{k}}}=\frac{\partial\left(\varepsilon_{1}\left(t\right)+R_{1}+\varepsilon_{0}\left(t\right)+R_{0}\right)}{\partial v^{\Im}_{i\mathbf{k}}} \tag{313}$$

$$=\frac{\partial\left(\left(\frac{\left(v^{\Re}_{i\mathbf{k}}\right)^{2}+\left(v^{\Im}_{i\mathbf{k}}\right)^{2}}{\omega_{\mathbf{k}}}-v^{\Re}_{i\mathbf{k}}\frac{g_{i\mathbf{k}}+g^{*}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}v^{\Im}_{i\mathbf{k}}\frac{g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}} \tag{314}$$

$$=2\frac{v^{\Im}_{i\mathbf{k}}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g^{*}_{i\mathbf{k}}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \tag{315}$$

$$\frac{\partial\left|B_{10}\right|^{2}}{\partial v^{\Im}_{i\mathbf{k}}}=\frac{\partial\left(\exp\left(-\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{1\mathbf{k}}-v^{\Re}_{0\mathbf{k}}\right)^{2}+\left(v^{\Im}_{1\mathbf{k}}-v^{\Im}_{0\mathbf{k}}\right)^{2}}{\omega^{2}_{\mathbf{k}}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)}{\partial v^{\Im}_{i\mathbf{k}}} \tag{316}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}\exp\left(-\sum_{\mathbf{k}}\frac{\left(v_{1\mathbf{k}}^{\Re}-v_{0\mathbf{k}}^{\Re}\right)^{2}+\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)^{2}}{\omega_{\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{317}$$

$$= -\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}\right|^{2}, \tag{318}$$

$$\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}} = \frac{\partial\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}}{\partial v_{i\mathbf{k}}^{\Re}} \tag{319}$$

$$= \frac{2\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\frac{\partial\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}}-4\frac{\partial\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{2\sqrt{\left(\mathrm{Tr}\left(\overline{H_{\bar{S}}(t)}\right)\right)^{2}-4\mathrm{Det}\left(\overline{H_{\bar{S}}(t)}\right)}} \tag{320}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\frac{\partial\left(\left(\varepsilon_{1}(t)+R_{1}\right)\left(\varepsilon_{0}(t)+R_{0}\right)-\left|V_{10}(t)\right|^{2}\left|B_{10}(t)\right|^{2}\right)}{\partial v_{i\mathbf{k}}^{\Im}}}{\eta(t)} \tag{321}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}\frac{\partial\left(v_{1\mathbf{k}}^{\Im}-v_{0\mathbf{k}}^{\Im}\right)}{\partial v_{i\mathbf{k}}^{\Im}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{322}$$

$$= \frac{\varepsilon(t)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-2\left(\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\left(2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\right)+\frac{2\left(v_{i\mathbf{k}}^{\Im}-v_{i'\mathbf{k}}^{\Im}\right)}{\omega_{\mathbf{k}}^{2}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}{\eta(t)} \tag{323}$$

$$= \frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{1}{\eta(t)}\left(-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}\varepsilon(t)+2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^{2}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right). \tag{324}$$

From the equation (269) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Im}}} \tag{325}$$

$$= \frac{2\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}-\mathrm{i}\frac{g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left(\frac{2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)}\right)+\frac{2}{\eta(t)}\left(\frac{\left(g_{i\mathbf{k}}^{*}\right)^{\Im}}{\omega_{\mathbf{k}}}\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\frac{\left(g_{i\mathbf{k}}^{*}\right)^{\Im}}{\omega_{\mathbf{k}}}+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}^{2}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)}. \tag{326}$$

Rearranging this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\left(2v_{i\mathbf{k}}^{\Im}-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4\left|V_{10}(t)\right|^{2}\left|B_{10}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\mathrm{i}\left(g_{i\mathbf{k}}^{*}-g_{i\mathbf{k}}\right)\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{327}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4\left|V_{10}(t)\right|^{2}B_{10}^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{328}$$

$$= \frac{2\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(2\varepsilon(t)-4\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{4\left|V_{10}(t)\right|^{2}\left|B_{10}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-2g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+4\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \tag{329}$$

$$= \frac{\left(v_{i\mathbf{k}}^{\Im}-g_{i\mathbf{k}}^{\Im}\right)\eta(t)}{v_{i\mathbf{k}}^{\Im}\left(\varepsilon(t)-2\left(\varepsilon(t)-\varepsilon_{i}(t)-R_{i}\right)-\frac{2\left|V_{10}(t)\right|^{2}\left|B_{10}\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}^{\Im}\left(2\varepsilon_{i}(t)+2R_{i}-\varepsilon(t)\right)+2\frac{v_{i'\mathbf{k}}^{\Im}}{\omega_{\mathbf{k}}}\left|B_{10}\right|^{2}\left|V_{10}(t)\right|^{2}\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \tag{330}$$

Separating (330) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}})\eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)}=v^{\Im}_{i\mathbf{k}}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-g^{\Im}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i-\varepsilon(t))+2\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (331)$$

$$v^{\Im}_{i\mathbf{k}}-g^{\Im}_{i\mathbf{k}}=v^{\Im}_{i\mathbf{k}}\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}g^{\Im}_{i\mathbf{k}}(2\varepsilon_i(t)+2R_i-\varepsilon(t)) \quad (332)$$

$$+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (333)$$

$$v^{\Im}_{i\mathbf{k}}=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}, \quad (334)$$

$$v^{\Im}_{i\mathbf{k}}=\frac{g^{\Im}_{i\mathbf{k}}\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \quad (335)$$

The variational parameters are:

$$v_{i\mathbf{k}}(\omega_{\mathbf{k}})=v^{\Re}_{i\mathbf{k}}(\omega_{\mathbf{k}})+\mathrm{i}v^{\Im}_{i\mathbf{k}}(\omega_{\mathbf{k}}) \quad (336)$$

$$=\frac{g^{\Re}_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)v^{\Re}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\eta(t)\,\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (337)$$

$$+\,\mathrm{i}\frac{g^{\Im}_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v^{\Im}_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)} \quad (338)$$

$$=\frac{g_{i\mathbf{k}}(\omega_{\mathbf{k}})\left(1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}(2\varepsilon_i(t)+2R_i-\varepsilon(t))\right)+2\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\frac{v_{i'\mathbf{k}}(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}|B_{10}|^2|V_{10}(t)|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{1-\frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)}\left(\varepsilon(t)-2(\varepsilon(t)-\varepsilon_i(t)-R_i)-\frac{2|V_{10}(t)|^2|B_{10}|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}}\right)}. \quad (339)$$

IV. MASTER EQUATION

In order to describe the dynamics of the QD under the influence of the phonon environment, we use the time-convolutionless projection operator technique. The initial density operator is $\rho_T(0) = \rho_S(0)\otimes\rho_B^{\mathrm{Thermal}}$, where $\rho_B^{\mathrm{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$
\begin{aligned}
\overline{\rho_T(0)} &\equiv e^{V}\rho_T\left(0\right)e^{-V} && (340)\\
&= \left(|0\rangle\langle 0|B_0^{+} + |1\rangle\langle 1|B_1^{+}\right)\left(\rho_S(0)\otimes\rho_B\right)\left(|0\rangle\langle 0|B_0^{-} + |1\rangle\langle 1|B_1^{-}\right), && (341)\\
\text{for } \rho_S(0) = |0\rangle\langle 0|\text{: } & |0\rangle\langle 0|0\rangle B_0^{+}\langle 0|\rho_B|0\rangle\langle 0|B_0^{-} && (342)\\
&= |0\rangle B_0^{+}\langle 0|\rho_B|0\rangle\langle 0|B_0^{-} && (343)\\
&= |0\rangle\langle 0|\otimes B_0^{+}\rho_B B_0^{-}, && (344)\\
\text{for } \rho_S(0) = |1\rangle\langle 1|\text{: } & |1\rangle\langle 1|B_1^{+}|1\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^{-} && (345)\\
&= |1\rangle\langle 1|B_1^{+}\rho_B B_1^{-} && (346)\\
&= |1\rangle\langle 1|\otimes B_1^{+}\rho_B B_1^{-}, && (347)\\
\text{for } \rho_S(0) = |0\rangle\langle 1| : & |0\rangle\langle 0|B_0^{+}|0\rangle\langle 1|\rho_B|1\rangle\langle 1|B_1^{-} && (348)\\
&= |0\rangle\langle 1|B_0^{+}\rho_B|1\rangle\langle 1|B_1^{-} && (349)\\
&= |0\rangle\langle 1|1\rangle\langle 1|B_0^{+}\rho_B B_1^{-} && (350)\\
&= |0\rangle\langle 1|\otimes B_0^{+}\rho_B B_1^{-}, && (351)\\
\text{for } \rho_S(0) = |1\rangle\langle 0| : & |1\rangle\langle 1|B_1^{+}|1\rangle\langle 0|\rho_B|0\rangle\langle 0|B_0^{-} && (352)\\
&= |1\rangle\langle 0|\otimes B_1^{+}\rho_B B_0^{-}. && (353)
\end{aligned}
$$

We transform any operator O into the interaction picture in the following way:

$$
\begin{aligned}
\widetilde{O}\left(t\right) &\equiv U^{\dagger}\left(t\right)O\left(t\right)U\left(t\right), && (354)\\
U\left(t\right) &\equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}\left(t'\right)\right). && (355)
\end{aligned}
$$

Here $\mathcal{T}$ denotes a time ordering operator. Therefore:

$$
\begin{aligned}
\widetilde{\overline{\rho_S}}\left(t\right) &= U^{\dagger}\left(t\right)\overline{\rho_S}\left(t\right)U\left(t\right), \text{ where} && (356)\\
\overline{\rho_S}\left(t\right) &= \mathrm{Tr}_B\left(\bar{\rho}_T\left(t\right)\right). && (357)
\end{aligned}
$$

. In order to separate the Hamiltonian we define the matrix $\Lambda\left(t\right)$ such that $\Lambda_{1i}\left(t\right) = A_i$, $\Lambda_{2i}\left(t\right) = B_i$ and $\Lambda_{3i}\left(t\right) = C_i\left(t\right)$ written as:

$$
\begin{pmatrix} A\left(t\right) \\ B\left(t\right) \\ C\left(t\right) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x & B_y & B_{1z} & B_y & B_x & B_{0z} \\ V_{10}^{\Re}\left(t\right) & V_{10}^{\Re}\left(t\right) & 1 & V_{10}^{\Im}\left(t\right) & -V_{10}^{\Im}\left(t\right) & 1 \end{pmatrix}. \quad (358)
$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_{\bar{I}}}\left(t\right)$ as pointed in the equation (236):

$$
\begin{aligned}
\overline{H_{\bar{I}}}\left(t\right) &= \sum_i B_{iz}|i\rangle\langle i| + V_{10}^{\Re}\left(t\right)\left(\sigma_x B_x + \sigma_y B_y\right) + V_{10}^{\Im}\left(t\right)\left(\sigma_x B_y - \sigma_y B_x\right) && (359)\\
&= B_{0z}|0\rangle\langle 0| + B_{1z}|1\rangle\langle 1| + V_{10}^{\Re}\left(t\right)\sigma_x B_x + V_{10}^{\Re}\left(t\right)\sigma_y B_y + V_{10}^{\Im}\left(t\right)\sigma_x B_y - V_{10}^{\Im}\left(t\right)\sigma_y B_x && (360)\\
&= \sum_i C_i\left(t\right)\left(A_i \otimes B_i\left(t\right)\right). && (361)
\end{aligned}
$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_T}}(t)}{\mathrm{d}t} = \text{-i}[\widetilde{\widetilde{H_{\bar{I}}}}(t), \widetilde{\widetilde{\rho_T}}(t)]. \tag{362}$$

This equation has the formal solution

$$\widetilde{\widetilde{\rho_T}}(t) = \overline{\rho_T}(0) \text{ -i} \int_0^t [\widetilde{\widetilde{H_{\bar{I}}}}(s), \widetilde{\widetilde{\rho_T}}(s)]\mathrm{d}s. \tag{363}$$

Replacing the equation (363) in the equation (362) gives us:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_T}}(t)}{\mathrm{d}t} = -\mathrm{i}[\widetilde{\widetilde{H_{\bar{I}}}}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{\widetilde{H_{\bar{I}}}}(t), [\widetilde{\widetilde{H_{\bar{I}}}}(s), \widetilde{\widetilde{\rho_T}}(s)]]\mathrm{d}s. \tag{364}$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\widetilde{\rho_T}}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n [\widetilde{\widetilde{H_{\bar{I}}}}(t_1), [\widetilde{\widetilde{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\widetilde{H_{\bar{I}}}}(t_n), \overline{\rho_T}(0)]] \cdots]. \tag{365}$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\widetilde{\rho_S}}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\widetilde{H_{\bar{I}}}}(t_1), [\widetilde{\widetilde{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\widetilde{H_{\bar{I}}}}(t_n), \overline{\rho_S}(0)\rho_B]] \cdots]. \tag{366}$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\widetilde{\rho_S}}(t) = (1 + W_1(t) + W_2(t) + ...)\,\overline{\rho_S}(0) \tag{367}$$
$$= W(t)\,\overline{\rho_S}(0). \tag{368}$$

in this case

$$W_n(t) = (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \mathrm{Tr}_B [\widetilde{\widetilde{H_{\bar{I}}}}(t_1), [\widetilde{\widetilde{H_{\bar{I}}}}(t_2), \cdots [\widetilde{\widetilde{H_{\bar{I}}}}(t_n), (\cdot)\rho_B]] \cdots]. \tag{369}$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) \overline{\rho_S}(0) \tag{370}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} W(t)\,\overline{\rho_S}(0) \tag{371}$$
$$= \left(\dot{W}_1(t) + \dot{W}_2(t) + ...\right) W(t)^{-1} \widetilde{\widetilde{\rho_S}}(t). \tag{372}$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\mathrm{Tr}_B[\widetilde{\widetilde{H_{\bar{I}}}}(t)\rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (370) becomes:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = \dot{W}_2(t)\,\widetilde{\widetilde{\rho_S}}(t) \tag{373}$$
$$= -\int_0^t \mathrm{d}t_1 \mathrm{Tr}_B \left[\widetilde{\widetilde{H_{\bar{I}}}}(t), \left[\widetilde{\widetilde{H_{\bar{I}}}}(t_1), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]. \tag{374}$$

Replacing $t_1 \to t - \tau$

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_B \left[\overline{H_{\bar{I}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(-\tau), \overline{\rho_S}(t)\rho_B\right]\right]. \tag{375}$$

From the interaction picture applied on $\overline{H_{\bar{I}}}(t)$ we find:

$$\widetilde{\overline{H_{\bar{I}}}}(t) = U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}\overline{H_{\bar{I}}}(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t). \tag{376}$$

we use the time-ordering operator $\mathcal{T}$ because in general $\overline{H_{\bar{S}}}(t)$ doesn't conmute with itself at two different times. We write the interaction Hamiltonian as:

$$\begin{aligned}
\widetilde{\overline{H_{\bar{I}}}}(t) &= \sum_i C_i(t)\left(\widetilde{A_i}(t)\otimes\widetilde{B_i}(t)\right), && (377)\\
\widetilde{A_i}(t) &= U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}A_i\mathrm{e}^{-\mathrm{i}H_B t}U(t) && (378)\\
&= U^\dagger(t)\,A_i U(t)\,\mathrm{e}^{\mathrm{i}H_B t}\mathrm{e}^{-\mathrm{i}H_B t} && (379)\\
&= U^\dagger(t)\,A_i U(t)\,\mathbb{I} && (380)\\
&= U^\dagger(t)\,A_i U(t), && (381)\\
\widetilde{B_i}(t) &= U^\dagger(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}U(t) && (382)\\
&= U^\dagger(t)\,U(t)\,\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} && (383)\\
&= \mathbb{I}\mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t} && (384)\\
&= \mathrm{e}^{\mathrm{i}H_B t}B_i(t)\,\mathrm{e}^{-\mathrm{i}H_B t}. && (385)
\end{aligned}$$

Here we have used the fact that $\left[\overline{H_{\bar{S}}}(t), H_B\right] = 0$ because these operators belong to different Hilbert spaces, so $\left[U(t), \mathrm{e}^{\mathrm{i}H_B t}\right] = 0$.
Using the expression (377) to replace it in the equation (374)

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \mathrm{Tr}_B\left[\widetilde{\overline{H_{\bar{I}}}}(t), \left[\widetilde{\overline{H_{\bar{I}}}}(s), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s && (386)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \left[\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right), \widetilde{\overline{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s && (387)\\
&= -\int_0^t \mathrm{Tr}_B\left[\sum_j C_j(t)\left(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\right), \sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\widetilde{\overline{\rho_S}}(t)\rho_B - \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(s)\left(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\right)\right]\mathrm{d}s && (388)\\
&= -\textstyle\int_0^t \mathrm{Tr}_B\big(\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B - \sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big) && (389)\\
&\textstyle-\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\widetilde{\overline{\rho_S}}(t)\rho_B\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big) + \widetilde{\overline{\rho_S}}(t)\rho_B\sum_i C_i(s)\big(\widetilde{A_i}(s)\otimes\widetilde{B_i}(s)\big)\sum_j C_j(t)\big(\widetilde{A_j}(t)\otimes\widetilde{B_j}(t)\big)\big)\mathrm{d}s. && (390)
\end{aligned}$$

In order to calculate the correlation functions we define:

$$\begin{aligned}
\mathscr{B}_{ij}(\tau) &= \mathrm{Tr}_B\left(\widetilde{B_i}(t)\,\widetilde{B_j}(s)\,\rho_B\right) && (391)\\
&= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)\,\widetilde{B_j}(0)\,\rho_B\right). && (392)
\end{aligned}$$

The correlation functions relevant that appear in the equation (390) are:

$$
\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right) &= \left\langle \widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B && (393)\\
&= \left\langle \widetilde{B_j}(\tau)\,\widetilde{B_i}(0)\right\rangle_B && (394)\\
&= \mathscr{B}_{ji}(\tau)\,, && (395)\\
\mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\rho_B\widetilde{B_i}(s)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) && (396)\\
&= \left\langle \widetilde{B_i}(s)\,\widetilde{B_j}(t)\right\rangle_B && (397)\\
&= \left\langle \widetilde{B_i}(-\tau)\,\widetilde{B_j}(0)\right\rangle_B && (398)\\
&= \mathscr{B}_{ij}(-\tau)\,, && (399)\\
\mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\rho_B\widetilde{B_j}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_j}(t)\,\widetilde{B_i}(s)\,\rho_B\right) && (400)\\
&= \left\langle \widetilde{B_j}(t)\,\widetilde{B_i}(s)\right\rangle_B && (401)\\
&= \left\langle \widetilde{B_j}(\tau)\,\widetilde{B_i}(0)\right\rangle_B && (402)\\
&= \mathscr{B}_{ji}(\tau)\,, && (403)\\
\mathrm{Tr}_B\left(\rho_B\widetilde{B_i}(s)\,\widetilde{B_j}(t)\right) &= \mathrm{Tr}_B\left(\widetilde{B_i}(s)\,\widetilde{B_j}(t)\,\rho_B\right) && (404)\\
&= \left\langle \widetilde{B_i}(s)\,\widetilde{B_j}(t)\right\rangle_B && (405)\\
&= \left\langle \widetilde{B_i}(-\tau)\,\widetilde{B_j}(0)\right\rangle_B && (406)\\
&= \mathscr{B}_{ij}(-\tau)\,. && (407)
\end{aligned}
$$

The cyclic property of the trace was use widely in the development of equations (393) and (407). Replacing in (390)

$$
\begin{aligned}
\frac{\mathrm{d}\widetilde{\overline{\rho_S}}(t)}{\mathrm{d}t} &= -\int_0^t \sum_{ij}\Big(C_i(t)\,C_j(s)\left(\mathscr{B}_{ij}(\tau)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t) - \mathscr{B}_{ji}(-\tau)\,\widetilde{A_i}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_j}(s)\right) && (408)\\
&+C_i(t)\,C_j(s)\left(\mathscr{B}_{ji}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_j}(s)\,\widetilde{A_i}(t) - \mathscr{B}_{ij}(\tau)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_i}(t)\right)\Big)\,\mathrm{d}s && (409)\\
&= -\int_0^t \sum_{ij}\left(C_i(t)\,C_j(s)\left(\mathscr{B}_{ij}(\tau)\left[\widetilde{A_i}(t)\,,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t)\right] + \mathscr{B}_{ji}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_j}(s)\,,\widetilde{A_i}(t)\right]\right)\right)\mathrm{d}s. && (410)
\end{aligned}
$$

We could identify the following commutators in the equation deduced:

$$
\begin{aligned}
\mathscr{B}_{ij}(\tau)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t) - \mathscr{B}_{ij}(\tau)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_i}(t) &= \mathscr{B}_{ij}(\tau)\left[\widetilde{A_i}(t)\,,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_s}}(t)\right], && (411)\\
\mathscr{B}_{ji}(-\tau)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_j}(s)\,\widetilde{A_i}(t) - \mathscr{B}_{ji}(-\tau)\,\widetilde{A_i}(t)\,\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_j}(s) &= \mathscr{B}_{ji}(-\tau)\left[\widetilde{\overline{\rho_S}}(t)\,\widetilde{A_j},\widetilde{A_i}(t)\right]. && (412)
\end{aligned}
$$

Returning to the Schroedinger picture we have:

$$
\begin{aligned}
U(t)\,\widetilde{A_i}(t)\,\widetilde{A_j}(s)\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t) &= U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\,U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\,U(t)\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t)\,, && (413)\\
&= \left(U(t)\,\widetilde{A_i}(t)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{A_j}(s)\,U^\dagger(t)\right)\left(U(t)\,\widetilde{\overline{\rho_S}}(t)\,U^\dagger(t)\right), && (414)\\
&= A_i\widetilde{A_j}(s,t)\,\overline{\rho_S}(t)\,. && (415)
\end{aligned}
$$

This procedure applying to the relevant commutators give us:

$$U\left(t\right)\left[\widetilde{A_{i}}\left(t\right),\widetilde{A_{j}}\left(s\right)\widetilde{\overline{\rho_{S}}}\left(t\right)\right]U^{\dagger}\left(t\right)=\Big(U\left(t\right)\widetilde{A_{i}}\left(t\right)\widetilde{A_{j}}\left(s\right)\widetilde{\overline{\rho_{S}}}\left(t\right)U^{\dagger}\left(t\right)-U\left(t\right)\widetilde{A_{j}}\left(s\right)\widetilde{\overline{\rho_{S}}}\left(t\right)\widetilde{A_{i}}\left(t\right)U^{\dagger}\left(t\right)\Big) \tag{416}$$

$$=A_{i}\widetilde{A_{j}}\left(s,t\right)\overline{\rho_{S}}\left(t\right)-\widetilde{A_{j}}\left(s,t\right)\overline{\rho_{S}}\left(t\right)A_{i} \tag{417}$$

$$=\left[A_{i},\widetilde{A_{j}}\left(t-\tau,t\right)\overline{\rho_{S}}\left(t\right)\right]. \tag{418}$$

Introducing this transformed commutators in the equation (410) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathscr{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{\mathrm{d}\overline{\rho_{S}}(t)}{\mathrm{d}t}=-\mathrm{i}[H_{S}(t),\overline{\rho_{S}}(t)]-\sum_{ij}\int_{0}^{t}\mathrm{d}\tau C_{i}(t)C_{j}(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_{i},\widetilde{A_{j}}(t-\tau,t)\,\overline{\rho_{S}}\,(t)\Big]+\mathscr{B}_{ji}(-\tau)\Big[\overline{\rho_{S}}\,(t)\widetilde{A_{j}}(t-\tau,t),A_{i}\Big]\Big) \tag{419}$$

where $i,j\in\{1,2,3,4,5.6\}$.
Here $\widetilde{A_{j}}\left(s,t\right)=U\left(t\right)U^{\dagger}\left(s\right)A_{j}U\left(s\right)U^{\dagger}\left(t\right)$ where $U\left(t\right)$ is given by (355). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_{I}\left(t\right)$.

Calculating the correlation functions allow us to obtain:

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_{B}=\mathrm{Tr}_{B}\left(\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\rho_{B}\right) \tag{420}$$

$$=\int\mathrm{d}^{2}\alpha P\left(\alpha\right)\left\langle\alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle \tag{421}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha\right|^{2}}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle\mathrm{d}^{2}\alpha \tag{422}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha\right|^{2}}{N}\right)\left\langle\alpha\left|\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\right|\alpha\right\rangle\mathrm{d}^{2}\alpha, \tag{423}$$

$$\widetilde{B_{jz}}\left(\tau\right)=\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right), \tag{424}$$

$$\widetilde{B_{jz}}\left(0\right)=\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\right), \tag{425}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_{B}=\mathrm{Tr}_{B}\left(\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{jz}}\left(0\right)\rho_{B}\right) \tag{426}$$

$$=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\Big((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)\sum_{\mathbf{k}'}\Big((g_{j\mathbf{k}'}-v_{j\mathbf{k}'})b_{\mathbf{k}'}^{\dagger}+(g_{j\mathbf{k}'}-v_{j\mathbf{k}'})^{*}b_{\mathbf{k}'}\Big)\rho_{B}\right) \tag{427}$$

$$=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\Big((g_{j\mathbf{k}}-v_{j\mathbf{k}})\,b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)\Big(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}\Big)\rho_{B}\right) \tag{428}$$

$$+\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\Big(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)\Big(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}b_{\mathbf{k}}\Big)\rho_{B}\right), \tag{429}$$

$$g_{j\mathbf{k}}-v_{j\mathbf{k}}=p_{j\mathbf{k}} \tag{430}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_{B}=\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}\neq\mathbf{k}'}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}'}b_{\mathbf{k}'}^{\dagger}+p_{j\mathbf{k}'}^{*}b_{\mathbf{k}'}\right)\rho_{B}\right) \tag{431}$$

$$+\mathrm{Tr}_{B}\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}\right)\rho_{B}\right) \tag{432}$$

$$=0+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(p_{j\mathbf{k}}b_{\mathbf{k}}^{\dagger}+p_{j\mathbf{k}}^{*}b_{\mathbf{k}}\right)\rho_B\right) \tag{433}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left(p_{j\mathbf{k}}^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left|p_{j\mathbf{k}}\right|^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left|p_{j\mathbf{k}}\right|^2 b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\rho_B\right) \tag{434}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2 b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}} p_{j\mathbf{k}}^{*2}b_{\mathbf{k}}b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{435}$$

$$=\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2 b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right)+\mathrm{Tr}_B\left(\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2 b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\rho_B\right) \tag{436}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\rho_B\right)+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\mathrm{Tr}_B\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\rho_B\right)\right) \tag{437}$$

$$=\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{438}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{439}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{440}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{441}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{442}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{443}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{444}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{445}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{446}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{447}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+\left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+\left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{448}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)\right. \tag{449}$$

$$\left.+\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)\right) \tag{450}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+\left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{451}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|\left|\alpha_{\mathbf{k}}\right|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{452}$$

$$+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{453}$$

$$1=\frac{1}{\pi N}\int\exp\left(-\frac{\left|\alpha_{\mathbf{k}}\right|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{454}$$

$$b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\left|0\right\rangle=0, \tag{455}$$

$$b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\left|0\right\rangle=\left|0\right\rangle, \tag{456}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}|p_{j\mathbf{k}}|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle||\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{457}$$

$$+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{458}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right) \tag{459}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\left(e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}}\right), \tag{460}$$

$$\frac{1}{\pi N}\int|\alpha_{\mathbf{k}}|^2\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int_0^{2\pi}\int_0^{\infty}r^2\exp\left(-\frac{r^2}{N}\right)r\mathrm{d}r\mathrm{d}\theta \tag{461}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N\right)+\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2e^{-\mathrm{i}\omega_{\mathbf{k}}\tau} \tag{462}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(2\cos\left(\omega_{\mathbf{k}}\tau\right)N+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{463}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}+e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right) \tag{464}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{2\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}+\cos\left(\omega_{\mathbf{k}}\tau\right)-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{465}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(2+e^{\beta\omega_{\mathbf{k}}}-1\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{466}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(1+e^{\beta\omega_{\mathbf{k}}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{\beta\omega_{\mathbf{k}}}-1}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{467}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\frac{\left(e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}+e^{\frac{\beta\omega_{\mathbf{k}}}{2}}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)}{e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}-e^{-\frac{\beta\omega_{\mathbf{k}}}{2}}}-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{468}$$

$$=\sum_{\mathbf{k}}\left|p_{j\mathbf{k}}\right|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right) \tag{469}$$

$$=\sum_{\mathbf{k}}\left|g_{j\mathbf{k}}-v_{j\mathbf{k}}\right|^2\left(\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\cos\left(\omega_{\mathbf{k}}\tau\right)-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right)\right), \tag{470}$$

$$\left\langle\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\right\rangle_B = \int\mathrm{d}^2\alpha_{\mathbf{k}}P(\alpha_{\mathbf{k}})\left\langle\alpha_{\mathbf{k}}\middle|\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\middle|\alpha_{\mathbf{k}}\right\rangle \tag{471}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|\widetilde{B_{jz}}(\tau)\widetilde{B_{j'z}}(0)\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{472}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}'}\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{473}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}\neq\mathbf{k}'}\left(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}+\left(g_{j'\mathbf{k}'}-v_{j'\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{474}$$

$$+\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{475}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\langle\alpha_{\mathbf{k}}|\sum_{\mathbf{k}}\left((g_{j\mathbf{k}}-v_{j\mathbf{k}})b_{\mathbf{k}}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+(g_{j\mathbf{k}}-v_{j\mathbf{k}})^*b_{\mathbf{k}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\left(\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}+\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}\right)|\alpha_{\mathbf{k}}\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{476}$$

$$=\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{477}$$

$$+\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{478}$$

$$=\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^*e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{479}$$

$$+\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^*\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}, \tag{480}$$

$$\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{481}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}^{\dagger}D\left(\alpha_{\mathbf{k}}\right)D\left(-\alpha_{\mathbf{k}}\right)b_{\mathbf{k}}D\left(\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{482}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{483}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)|\alpha_{\mathbf{k}}|^2\,\mathrm{d}^2\alpha_{\mathbf{k}} \tag{484}$$

$$= N, \tag{485}$$

$$\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle\alpha_{\mathbf{k}}\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|\alpha_{\mathbf{k}}\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} = \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}D(\alpha_{\mathbf{k}})D(-\alpha_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}D(\alpha_{\mathbf{k}})\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{486}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left|\left(b_{\mathbf{k}}+\alpha_{\mathbf{k}}\right)\left(b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}^{*}\right)\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{487}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+\alpha_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}}\alpha_{\mathbf{k}}^{*}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{488}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+|\alpha_{\mathbf{k}}|^2\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{489}$$

$$= \frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\left||\alpha_{\mathbf{k}}|^2\right|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}}+\frac{1}{\pi N}\int\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right)\left\langle 0\middle|b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}\middle|0\right\rangle\mathrm{d}^2\alpha_{\mathbf{k}} \tag{490}$$

$$= N+1, \tag{491}$$

$$\left\langle\widetilde{B_{jz}}\left(\tau\right)\widetilde{B_{j'z}}\left(0\right)\right\rangle_B = \sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N+\sum_{\mathbf{k}}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N+1\right) \tag{492}$$

$$= \sum_{\mathbf{k}}\Big(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}+N\Big(\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}+\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}\left(g_{j'\mathbf{k}}-v_{j'\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\Big)\Big), \tag{493}$$

$$D\left(h'\right)D\left(h\right) = \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)D\left(h'+h\right), \tag{494}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \mathrm{Tr}_B\left(\exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)D\left(h'+h\right)\rho_B\right) \tag{495}$$

$$= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\mathrm{Tr}_B\left(D\left(h'+h\right)\rho_B\right) \tag{496}$$

$$= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\frac{1}{\pi N}\int\mathrm{d}^2\alpha P\left(\alpha\right)\left\langle\alpha\left|D\left(h'+h\right)\right|\alpha\right\rangle \tag{497}$$

$$= \exp\left(\frac{1}{2}\left(h'h^{*}-h'^{*}h\right)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{498}$$

$$h' = h\exp\left(\mathrm{i}\omega\tau\right), \tag{499}$$

$$\left\langle D\left(h\exp\left(\mathrm{i}\omega\tau\right)\right)D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(hh^{*}\exp\left(\mathrm{i}\omega\tau\right)-h^{*}h\exp\left(-\mathrm{i}\omega\tau\right)\right)\right)\exp\left(-\frac{|h+h\exp\left(\mathrm{i}\omega\tau\right)|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{500}$$

$$\frac{1}{2}|h|^2\left(\exp(\mathrm{i}\omega\tau)-\exp(-\mathrm{i}\omega\tau)\right) = \frac{1}{2}\left(hh^{*}\exp\left(\mathrm{i}\omega\tau\right)-h^{*}h\exp\left(-\mathrm{i}\omega\tau\right)\right) \tag{501}$$

$$= \frac{1}{2}|h|^2\left(\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)-\cos\left(\omega\tau\right)+\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{502}$$

$$= \frac{1}{2}|h|^2\left(2\mathrm{i}\sin\left(\omega\tau\right)\right) \tag{503}$$

$$= \mathrm{i}\,|h|^2\sin\left(\omega\tau\right), \tag{504}$$

$$-\frac{|h+h\exp\left(\mathrm{i}\omega\tau\right)|^2}{2} = -|h|^2\frac{|1+\exp\left(\mathrm{i}\omega\tau\right)|^2}{2} \tag{505}$$

$$= -|h|^2\frac{\left(1+2\cos\left(\omega\tau\right)+\cos^2\left(\omega\tau\right)\right)+\sin^2\left(\omega\tau\right)}{2} \tag{506}$$

$$= -\left|h\right|^2 \frac{2+2\cos\left(\omega\tau\right)}{2} \tag{507}$$

$$= -\left|h\right|^2 \left(1+\cos\left(\omega\tau\right)\right), \tag{508}$$

$$\left\langle D(h\exp(\mathrm{i}\omega\tau))D(h)\right\rangle_B = \exp\left(\mathrm{i}|h|^2\sin(\omega\tau)\right)\exp\left(-|h|^2(1+\cos(\omega\tau))\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{509}$$

$$= \exp\left(\mathrm{i}\left|h\right|^2\sin\left(\omega\tau\right) - \left|h\right|^2\left(1+\cos\left(\omega\tau\right)\right)\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{510}$$

$$= \exp\left(-\left|h\right|^2\left(-\mathrm{i}\sin\left(\omega\tau\right)+\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)\right)\right)\exp\left(-\left|h\right|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \tag{511}$$

$$= \left\langle D\left(h\right)\right\rangle_B \exp\left(-\phi\left(\tau\right)\right), \tag{512}$$

$$\exp\left(-\phi\left(\tau\right)\right) = \exp\left(-\left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right)\right), \tag{513}$$

$$\phi\left(\tau\right) = \left|h\right|^2\left(\cos\left(\omega\tau\right)\coth\left(\frac{\beta\omega}{2}\right)-\mathrm{i}\sin\left(\omega\tau\right)\right), \tag{514}$$

$$\left\langle D\left(h'\right)D\left(h\right)\right\rangle_B = \exp\left(\frac{1}{2}\left(h'h^*-h'^*h\right)\right)\exp\left(-\frac{|h+h'|^2}{2}\coth\left(\frac{\beta\omega}{2}\right)\right), \tag{515}$$

$$h' = v\exp\left(i\omega\tau\right), \tag{516}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\widetilde{B_1^+B_0^-}\left(\tau\right)\widetilde{B_1^+B_0^-}\left(0\right)\rho_B\right) \tag{517}$$

$$= \mathrm{Tr}_B\left(\widetilde{B_1^+B_0^-}\left(\tau\right)\widetilde{B_1^+B_0^-}\left(0\right)\rho_B\right) \tag{518}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{519}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{520}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{521}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\prod_{\mathbf{k}}\left(\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right) \tag{522}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{523}$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{524}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\widetilde{B_0^+B_1^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{525}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \tag{526}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{527}$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}}D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \tag{528}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{529}$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\rho_B\right) \tag{530}$$

$$= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau+\pi)+\cos(\omega_{\mathbf{k}}\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{531}$$

$$= \prod_{\mathbf{k}} \exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)-\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \quad (532)$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_1^+B_0^-}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\rho_B\right) \quad (533)$$

$$= \mathrm{Tr}_B\left(\prod_{\mathbf{k}} D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\prod_{\mathbf{k}} D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \quad (534)$$

$$= \prod_{\mathbf{k}}\mathrm{Tr}_B\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}(\omega_{\mathbf{k}}\tau+\pi)}\right)D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\rho_B\right) \quad (535)$$

$$= \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau+\pi)+\cos(\omega_{\mathbf{k}}\tau+\pi)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \quad (536)$$

$$= \left\langle |\widetilde{B_1^+B_0^-}(\tau)\,\widetilde{B_0^+B_1^-}(0)\right\rangle_B, \quad (537)$$

$$\left\langle \widetilde{B_0^+B_1^-}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \mathrm{Tr}_B\left(\prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\sum_{\mathbf{k}'}\left(\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{j\mathbf{k}'}-v_{j\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right)\rho_B\right), \quad (538)$$

$$\langle D(h)\,b\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\,|D(h)\,b|\,\alpha\rangle \quad (539)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,\alpha\rangle \quad (540)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\,|D(-\alpha)\,D(h)\,bD(\alpha)|\,0\rangle \quad (541)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,bD(\alpha)|\,0\rangle \quad (542)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\langle 0\,|D(-\alpha)\,D(h)\,D(\alpha)\,(b+\alpha)|\,0\rangle \quad (543)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\,|D(h)\,(b+\alpha)|\,0\rangle \quad (544)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)b|0\rangle+\frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha|0\rangle \quad (545)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\langle 0\,|D(h)\,\alpha|\,0\rangle \quad (546)$$

$$= \frac{1}{\pi N}\int \alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\exp\left(-\frac{|h|^2}{2}\right)\mathrm{d}^2\alpha \quad (547)$$

$$= hN\,\langle D(h)\rangle_B\,, \quad (548)$$

$$\left\langle D(h)\,b^\dagger\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle\alpha\,\middle|D(h)\,b^\dagger\middle|\,\alpha\right\rangle \quad (549)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\,\middle|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \quad (550)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\,\middle|D(-\alpha)\,D(h)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \quad (551)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\,\middle|D(-\alpha)\,D(h)\,D(\alpha)\,D(-\alpha)\,b^\dagger D(\alpha)\middle|\,0\right\rangle \quad (552)$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle 0\left|D(-\alpha)D(h)D(\alpha)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{553}$$

$$= \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(h\alpha^*-h^*\alpha\right)\left\langle 0\left|D(h)\left(b^\dagger+\alpha^*\right)\right|0\right\rangle \tag{554}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\left\langle 0\left|D(h)b^\dagger\right|0\right\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)\alpha^*|0\rangle \tag{555}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle 0|D(h)|1\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle \tag{556}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\langle -h|1\rangle+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\langle 0|D(h)|0\rangle, \tag{557}$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right)\sum_n \frac{(-h^*)^n}{\sqrt{n!}}\langle n|, \tag{558}$$

$$\langle -h|1\rangle = \exp\left(-\frac{|-h^*|^2}{2}\right)(-h^*), \tag{559}$$

$$\left\langle D(h)b^\dagger\right\rangle_B = \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|-h^*|^2}{2}\right)(-h^*)+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha^*\exp\left(-\tfrac{|-h^*|^2}{2}\right) \tag{560}$$

$$= -h^*\langle D(h)\rangle_B(N+1), \tag{561}$$

$$\langle bD(h)\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|bD(h)|\alpha\rangle \tag{562}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{563}$$

$$= h\langle D(h)\rangle_B(N+1), \tag{564}$$

$$\left\langle b^\dagger D(h)\right\rangle_B = \frac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\left\langle\alpha\left|b^\dagger D(h)\right|\alpha\right\rangle \tag{565}$$

$$= \tfrac{1}{\pi N}\int \mathrm{d}^2\alpha \exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\exp\left(-\tfrac{|h|^2}{2}\right)h+\tfrac{1}{\pi N}\int \mathrm{d}^2\alpha\exp\left(-\tfrac{|\alpha|^2}{2}\right)\exp(h\alpha^*-h^*\alpha)\alpha\exp\left(-\tfrac{|h|^2}{2}\right) \tag{566}$$

$$= -h^*\langle D(h)\rangle_B N, \tag{567}$$

$$\left\langle \widetilde{B_1^+B_0^-}(\tau)\right\rangle_B = \left\langle \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\right\rangle_B \tag{568}$$

$$= \prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle_B \tag{569}$$

$$= \prod_{\mathbf{k}}\left(e^{\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)}\right)\prod_{\mathbf{k}}\left\langle D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\right\rangle_B \tag{570}$$

$$= \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}-\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{571}$$

$$= B_{10}. \tag{572}$$

The correlation functions can be found readily as:

$$\widetilde{B_1^+B_0^-}(\tau) = \prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^*v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right), \tag{573}$$

$$\widetilde{B_0^+B_1^-}(\tau) = \prod_{\mathbf{k}}\left(D\left(\frac{v_{0\mathbf{k}}-v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\exp\left(\frac{1}{2}\left(\frac{v_{0\mathbf{k}}^*v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right), \tag{574}$$

$$\widetilde{B_x}\left(0\right) = \frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}, \tag{575}$$

$$\widetilde{B_y}\left(0\right) = \frac{B_0^+ B_1^- - B_1^+ B_0^- + B_{10} - B_{01}}{2\mathrm{i}}, \tag{576}$$

$$B_{10} = \left(\prod_{\mathbf{k}} \exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\left(\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{577}$$

$$B_{iz} = \sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* b_{\mathbf{k}}\right), \tag{578}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \left\langle\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right) b_{\mathbf{k}}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}}\tau} + \left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^* b_{\mathbf{k}} e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\right)\sum_{\mathbf{k}}\left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}} - v_{jk}\right)^* b_{\mathbf{k}}\right)\right\rangle_B \tag{579}$$

$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right)^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}}\left(g_{i\mathbf{k}} - v_{i\mathbf{k}}\right)^*\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}\right) e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}} + 1\right), \tag{580}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \left\langle\frac{B_1^+ B_0^-\left(\tau\right) + B_0^+ B_1^-\left(\tau\right) - B_{10} - B_{01}}{2}\frac{B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}}{2}\right\rangle_B \tag{581}$$

$$= \frac{1}{4}\left\langle\left(B_1^+ B_0^-\left(\tau\right) + B_0^+ B_1^-\left(\tau\right) - B_{10} - B_{01}\right)\left(B_1^+ B_0^- + B_0^+ B_1^- - B_{10} - B_{01}\right)\right\rangle_B \tag{582}$$

$$= \tfrac{1}{4}\Big\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- + B_0^+ B_1^-(\tau) B_0^+ B_1^- \tag{583}$$

$$- B_0^+ B_1^-(\tau) B_{10} - B_0^+ B_1^-(\tau) B_{01} - B_{10} B_1^+ B_0^- - B_{10} B_0^+ B_1^- + B_{10} B_{10} + B_{10} B_{01} - B_{01} B_1^+ B_0^- - B_{01} B_0^+ B_1^- + B_{01} B_{10} + B_{01} B_{01}\Big\rangle \tag{584}$$

$$= \frac{1}{4}\Big\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \tag{585}$$

$$+ B_0^+ B_1^-\left(\tau\right) B_0^+ B_1^- - B_0^+ B_1^-\left(\tau\right) B_{10} - B_0^+ B_1^-\left(\tau\right) B_{01}\Big\rangle, \tag{586}$$

$$\left\langle\widetilde{B_0^+ B_1^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B = \Pi_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right) + \cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right)\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\right), \tag{587}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right), \tag{588}$$

$$\phi\left(\tau\right) = \sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\left(-\mathrm{i}\sin\left(\omega_{\mathbf{k}}\tau\right) + \cos\left(\omega_{\mathbf{k}}\tau\right)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{589}$$

$$S = \prod_{\mathbf{k}}\exp\left(-\left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{590}$$

$$\left\langle\widetilde{B_0^+ B_1^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B = U\exp(-\phi(\tau))S, \tag{591}$$

$$\left\langle\widetilde{B_1^+ B_0^-}(\tau)\widetilde{B_1^+ B_0^-}(0)\right\rangle_B = U^*\exp(-\phi(\tau))S, \tag{592}$$

$$\left\langle\widetilde{B_1^+ B_0^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B = \exp(\phi(\tau))S, \tag{593}$$

$$\left\langle\widetilde{B_0^+ B_1^-}(\tau)\widetilde{B_1^+ B_0^-}(0)\right\rangle_B = \left\langle\widetilde{B_1^+ B_0^-}(\tau)\widetilde{B_0^+ B_1^-}(0)\right\rangle_B, \tag{594}$$

$$\left\langle\widetilde{B_1^+ B_0^-}\left(\tau\right)\right\rangle_B = \prod_{\mathbf{k}}\left(\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right)\right)\prod_{\mathbf{k}}\exp\left(-\frac{1}{2}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{595}$$

$$= U^{*1/2} S^{1/2}, \tag{596}$$

$$\left\langle\widetilde{B_x}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B = \frac{1}{4}\Big\langle B_1^+ B_0^-(\tau) B_1^+ B_0^- + B_1^+ B_0^-(\tau) B_0^+ B_1^- - B_1^+ B_0^-(\tau) B_{10} - B_1^+ B_0^-(\tau) B_{01} + B_0^+ B_1^-(\tau) B_1^+ B_0^- \tag{597}$$

$$+ B_0^+ B_1^-\left(\tau\right) B_0^+ B_1^- - B_0^+ B_1^-\left(\tau\right) B_{10} - B_0^+ B_1^-\left(\tau\right) B_{01}\Big\rangle, \tag{598}$$

$$\left\langle\widetilde{B_x}\left(\tau\right)\widetilde{B_x}\left(0\right)\right\rangle_B = \frac{1}{4}\Big\langle B_1^+ B_0^-\left(\tau\right) B_1^+ B_0^- + B_1^+ B_0^-\left(\tau\right) B_0^+ B_1^- - B_1^+ B_0^-\left(\tau\right) B_{10} - B_1^+ B_0^-\left(\tau\right) B_{01} \tag{599}$$

$$+ B_0^+ B_1^-\left(\tau\right) B_1^+ B_0^- + B_0^+ B_1^-\left(\tau\right) B_0^+ B_1^- - B_0^+ B_1^-\left(\tau\right) B_{10} - B_0^+ B_1^-\left(\tau\right) B_{01}\Big\rangle \tag{600}$$

$$= \frac{1}{4}\Big(U^*\exp\left(-\phi\left(\tau\right)\right)S + \exp(\phi\left(\tau\right))S - B_{10}^2 - \left|B_{10}\right|^2 + \exp(\phi\left(\tau\right))S + U\exp(-\phi\left(\tau\right))S - B_{10}^{*2} - \left|B_{10}\right|^2\Big) \tag{601}$$

$$= \frac{1}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right)S + 2\exp\left(\phi\left(\tau\right)\right)S - 2\left(B_{10}^{2}\right)^{\Re} - 2\left|B_{10}\right|^{2}\right) \tag{602}$$

$$= \frac{1}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right)S + 2\exp\left(\phi\left(\tau\right)\right)S - 2\left(U^{*}\right)^{\Re}S - 2S\right) \tag{603}$$

$$= \frac{S}{2}\left(U^{\Re}\exp\left(-\phi\left(\tau\right)\right) + \exp\left(\phi\left(\tau\right)\right) - \left(U^{*}\right)^{\Re} - 1\right), \tag{604}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_0^{+}B_1^{-}\left(\tau\right) - B_1^{+}B_0^{-}\left(\tau\right) + B_{10} - B_{01}}{2\mathrm{i}}\frac{B_0^{+}B_1^{-} - B_1^{+}B_0^{-} + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B \tag{605}$$

$$= -\frac{1}{4}\left\langle\left(B_0^{+}B_1^{-}\left(\tau\right) - B_1^{+}B_0^{-}\left(\tau\right) + B_{10} - B_{01}\right)\left(B_0^{+}B_1^{-} - B_1^{+}B_0^{-} + B_{10} - B_{01}\right)\right\rangle_B \tag{606}$$

$$= -\tfrac{1}{4}\langle B_0^{+}B_1^{-}(\tau)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}(\tau)B_1^{+}B_0^{-} + B_0^{+}B_1^{-}(\tau)B_{10} - B_0^{+}B_1^{-}(\tau)B_{01} - B_1^{+}B_0^{-}(\tau)B_0^{+}B_1^{-} + B_1^{+}B_0^{-}(\tau)B_1^{+}B_0^{-} \tag{607}$$

$$- B_1^{+}B_0^{-}(\tau)B_{10} + B_1^{+}B_0^{-}(\tau)B_{01} + B_{10}B_0^{+}B_1^{-} - B_{10}B_1^{+}B_0^{-} + B_{10}B_{10} - B_{10}B_{01} - B_{01}B_0^{+}B_1^{-} + B_{01}B_1^{+}B_0^{-} - B_{01}B_{10} + B_{01}B_{01}\rangle \tag{608}$$

$$= -\frac{1}{4}\langle B_0^{+}B_1^{-}\left(\tau\right)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}\left(\tau\right)B_1^{+}B_0^{-} + B_0^{+}B_1^{-}\left(\tau\right)B_{10} - B_0^{+}B_1^{-}\left(\tau\right)B_{01} \tag{609}$$

$$- B_1^{+}B_0^{-}\left(\tau\right)B_0^{+}B_1^{-} + B_1^{+}B_0^{-}\left(\tau\right)B_1^{+}B_0^{-} - B_1^{+}B_0^{-}\left(\tau\right)B_{10} + B_1^{+}B_0^{-}\left(\tau\right)B_{01}\rangle \tag{610}$$

$$= -\tfrac{1}{4}\langle B_0^{+}B_1^{-}(\tau)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}(\tau)B_1^{+}B_0^{-} + B_{01}B_{10} - B_{01}B_{01} - B_1^{+}B_0^{-}(\tau)B_0^{+}B_1^{-} + B_1^{+}B_0^{-}(\tau)B_1^{+}B_0^{-} - B_{10}B_{10} + B_{10}B_{01}\rangle \tag{611}$$

$$= -\frac{1}{4}\left(U\exp(-\phi\left(\tau\right))S - \exp\left(\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^{*}\exp\left(-\phi\left(\tau\right)\right)S + 2S - 2\left(U^{*}\right)^{\Re}S\right) \tag{612}$$

$$= -\frac{S}{4}\left(2U^{\Re}\exp\left(-\phi\left(\tau\right)\right) - 2\exp\left(\phi\left(\tau\right)\right) + 2 - 2U^{\Re}\right) \tag{613}$$

$$= \frac{S}{2}\left(\exp\left(\phi\left(\tau\right)\right) - U^{\Re}\exp\left(-\phi\left(\tau\right)\right) - 1 + U^{\Re}\right), \tag{614}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\frac{B_1^{+}B_0^{-}\left(\tau\right) + B_0^{+}B_1^{-}\left(\tau\right) - B_{10} - B_{01}}{2}\frac{B_0^{+}B_1^{-} - B_1^{+}B_0^{-} + B_{10} - B_{01}}{2\mathrm{i}}\right\rangle_B \tag{615}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_1^{+}B_0^{-}\left(\tau\right) + B_0^{+}B_1^{-}\left(\tau\right) - B_{10} - B_{01}\right)\left(B_0^{+}B_1^{-} - B_1^{+}B_0^{-} + B_{10} - B_{01}\right)\right\rangle_B \tag{616}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_1^{+}B_0^{-}(\tau)B_0^{+}B_1^{-} - B_1^{+}B_0^{-}(\tau)B_1^{+}B_0^{-} + B_1^{+}B_0^{-}(\tau)B_{10} - B_1^{+}B_0^{-}(\tau)B_{01} + B_0^{+}B_1^{-}(\tau)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}(\tau)B_1^{+}B_0^{-} + B_0^{+}B_1^{-}(\tau)B_{10} \tag{617}$$

$$- B_0^{+}B_1^{-}(\tau)B_{01} - B_{10}B_0^{+}B_1^{-} + B_{10}B_1^{+}B_0^{-} - B_{10}B_{10} + B_{10}B_{01} - B_{01}B_0^{+}B_1^{-} + B_{01}B_1^{+}B_0^{-} - B_{01}B_{10} + B_{01}B_{01}\rangle \tag{618}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_1^{+}B_0^{-}(\tau)B_0^{+}B_1^{-} - B_1^{+}B_0^{-}(\tau)B_1^{+}B_0^{-} + B_1^{+}B_0^{-}(\tau)B_{10} - B_1^{+}B_0^{-}(\tau)B_{01} + B_0^{+}B_1^{-}(\tau)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}(\tau)B_1^{+}B_0^{-} + B_0^{+}B_1^{-}(\tau)B_{10} \tag{619}$$

$$- B_0^{+}B_1^{-}(\tau)B_{01}\rangle \tag{620}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_1^{+}B_0^{-}(\tau)B_0^{+}B_1^{-} - B_1^{+}B_0^{-}(\tau)B_1^{+}B_0^{-} + B_{10}B_{10} - B_{10}B_{01} + B_0^{+}B_1^{-}(\tau)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}(\tau)B_1^{+}B_0^{-} + B_{01}B_{10} - B_{01}B_{01}\rangle \tag{621}$$

$$= \frac{1}{4\mathrm{i}}\langle B_1^{+}B_0^{-}(\tau)B_0^{+}B_1^{-} - B_1^{+}B_0^{-}(\tau)B_1^{+}B_0^{-} + B_{10}B_{10} + B_0^{+}B_1^{-}(\tau)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}(\tau)B_1^{+}B_0^{-} - B_{01}B_{01}\rangle \tag{622}$$

$$= \frac{1}{4\mathrm{i}}\left(\exp\left(\phi\left(\tau\right)\right)S - U^{*}\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S - \exp\left(\phi\left(\tau\right)\right)S + U^{*}S - US\right) \tag{623}$$

$$= \frac{1}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi\left(\tau\right)\right)S + U\exp\left(-\phi\left(\tau\right)\right)S + U^{*}S - US\right) \tag{624}$$

$$= \frac{S}{4\mathrm{i}}\left(-U^{*}\exp\left(-\phi\left(\tau\right)\right) + U\exp\left(-\phi\left(\tau\right)\right) + U^{*} - U\right) \tag{625}$$

$$= \frac{S\left(U - U^{*}\right)}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{626}$$

$$= \frac{2\mathrm{i}U^{\Im}S}{4\mathrm{i}}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right) \tag{627}$$

$$= \frac{U^{\Im}S}{2}\left(\exp\left(-\phi\left(\tau\right)\right) - 1\right), \tag{628}$$

$$\left\langle\widetilde{B_y}(\tau)\,\widetilde{B_x}(0)\right\rangle_B = \left\langle\frac{B_0^{+}B_1^{-}\left(\tau\right) - B_1^{+}B_0^{-}\left(\tau\right) + B_{10} - B_{01}}{2\mathrm{i}}\frac{B_1^{+}B_0^{-} + B_0^{+}B_1^{-} - B_{10} - B_{01}}{2}\right\rangle_B \tag{629}$$

$$= \frac{1}{4\mathrm{i}}\left\langle\left(B_0^{+}B_1^{-}\left(\tau\right) - B_1^{+}B_0^{-}\left(\tau\right) + B_{10} - B_{01}\right)\left(B_1^{+}B_0^{-} + B_0^{+}B_1^{-} - B_{10} - B_{01}\right)\right\rangle_B \tag{630}$$

$$= \tfrac{1}{4\mathrm{i}}\langle B_0^{+}B_1^{-}(\tau)B_1^{+}B_0^{-} + B_0^{+}B_1^{-}(\tau)B_0^{+}B_1^{-} - B_0^{+}B_1^{-}(\tau)B_{10} - B_0^{+}B_1^{-}(\tau)B_{01} - B_1^{+}B_0^{-}(\tau)B_1^{+}B_0^{-} - B_1^{+}B_0^{-}(\tau)B_0^{+}B_1^{-} \tag{631}$$

$$+ B_1^{+}B_0^{-}(\tau)B_{10} + B_1^{+}B_0^{-}(\tau)B_{01} + B_{10}B_1^{+}B_0^{-} + B_{10}B_0^{+}B_1^{-} - B_{10}B_{10} - B_{10}B_{01} - B_{01}B_1^{+}B_0^{-} - B_{01}B_0^{+}B_1^{-} + B_{01}B_{10} + B_{01}B_{01}\rangle \tag{632}$$

$$= \frac{1}{4\mathrm{i}} \left\langle B_0^+ B_1^- (\tau) B_1^+ B_0^- + B_0^+ B_1^- (\tau) B_0^+ B_1^- - B_0^+ B_1^- (\tau) B_{10} - B_0^+ B_1^- (\tau) B_{01} \right. \tag{633}$$

$$\left. - B_1^+ B_0^- (\tau) B_1^+ B_0^- - B_1^+ B_0^- (\tau) B_0^+ B_1^- + B_1^+ B_0^- (\tau) B_{10} + B_1^+ B_0^- (\tau) B_{01} \right\rangle \tag{634}$$

$$= \frac{1}{4\mathrm{i}} \left\langle B_0^+ B_1^- (\tau) B_1^+ B_0^- + B_0^+ B_1^- (\tau) B_0^+ B_1^- - B_{01} B_{10} - B_{01} B_{01} - B_1^+ B_0^- (\tau) B_1^+ B_0^- \right. \tag{635}$$

$$\left. - B_1^+ B_0^- (\tau) B_0^+ B_1^- + B_{10} B_{10} + B_{10} B_{01} \right\rangle \tag{636}$$

$$= \frac{1}{4\mathrm{i}} \left\langle B_0^+ B_1^- (\tau) B_1^+ B_0^- + B_0^+ B_1^- (\tau) B_0^+ B_1^- - B_{01} B_{10} - B_{01} B_{01} - B_1^+ B_0^- (\tau) B_1^+ B_0^- \right. \tag{637}$$

$$\left. - B_1^+ B_0^- (\tau) B_0^+ B_1^- + B_{10} B_{10} + B_{10} B_{01} \right\rangle \tag{638}$$

$$= \tfrac{1}{4\mathrm{i}} \left\langle B_0^+ B_1^- (\tau) B_1^+ B_0^- + B_0^+ B_1^- (\tau) B_0^+ B_1^- - B_{01} B_{01} - B_1^+ B_0^- (\tau) B_1^+ B_0^- - B_1^+ B_0^- (\tau) B_0^+ B_1^- + B_{10} B_{10} \right\rangle \tag{639}$$

$$= \frac{1}{4\mathrm{i}} \left(U \exp\left(-\phi\left(\tau\right)\right) S - U^* \exp\left(-\phi\left(\tau\right)\right) S + B_{10}^2 - B_{10}^{*2} \right) \tag{640}$$

$$= \frac{1}{4\mathrm{i}} \left(U \exp\left(-\phi\left(\tau\right)\right) S - U^* \exp\left(-\phi\left(\tau\right)\right) S + U^* S - U S \right) \tag{641}$$

$$= \frac{S\left(U - U^*\right)}{4\mathrm{i}} \left(\exp\left(-\phi\left(\tau\right)\right) - 1 \right) \tag{642}$$

$$= \frac{2\mathrm{i} U^{\Im} S}{4\mathrm{i}} \left(\exp\left(-\phi\left(\tau\right)\right) - 1 \right) \tag{643}$$

$$= -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10}, \tag{644}$$

$$\left\langle B_1^+ B_0^- (\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \prod_{\mathbf{k}} \exp\left(\tfrac{1}{2} \left(\tfrac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2} \right) \right) \left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} \left\langle \prod_{\mathbf{k}} \left(D\left(\tfrac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} e^{\mathrm{i}\omega_{\mathbf{k}}\tau} \right) \right) \right\rangle \tag{645}$$

$$= \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}, \tag{646}$$

$$\left\langle B_0^+ B_1^- (\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger \right\rangle_B = -\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{01}. \tag{647}$$

$$\left\langle B_0^+ B_1^- (\tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right\rangle_B = \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}, \tag{648}$$

$$\left\langle \widetilde{B_x}(\tau) \widetilde{B_{iz}}(0) \right\rangle_B = \tfrac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{01} \right. \tag{649}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \tag{650}$$

$$= \tfrac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{10} - \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \left(N_{\mathbf{k}'} + 1\right) B_{01} \right. \tag{651}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10} + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* \left(\tfrac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01} \right) \tag{652}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right) \left(N_{\mathbf{k}'} + 1\right) \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* B_{01} \right) \right. \tag{653}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) B_{10} + \left(\frac{v_{0\mathbf{k}'} - v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) B_{01} \right) \right) \tag{654}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-\left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'} + 1) \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* B_{01} \right) \right. \tag{655}$$

$$\left. + \left(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}\right)^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) B_{10} - \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) B_{01} \right) \right) \tag{656}$$

$$= \tfrac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* (B_{10} - B_{01}) + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\tfrac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) (B_{10} - B_{01}) \right) \tag{657}$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} 2\mathrm{i} B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \right) \tag{658}$$

$$= \mathrm{i} \sum_{\mathbf{k}'} B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \right)^* \right) \tag{659}$$

$$= \mathrm{i} \sum_{\mathbf{k}'} B_{10}^{\Im} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* N_{\mathbf{k}'} \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right) e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})(N_{\mathbf{k}'} + 1) \left(\frac{v_{1\mathbf{k}'} - v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}} \right)^* e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau} \right), \tag{660}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \left\langle \sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_{B} \tag{661}$$

$$= \sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}}{2}\right)\right\rangle_{B} \tag{662}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}-B_{10}-B_{01}\right)\right\rangle_{B} \tag{663}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_{1}^{+}B_{0}^{-}+B_{0}^{+}B_{1}^{-}\right)\right\rangle_{B} \tag{664}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\Big\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-} \tag{665}$$

$$+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\Big\rangle, \tag{666}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B} = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B} \tag{667}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{668}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}'}^{*}v_{0\mathbf{k}'}-v_{1\mathbf{k}'}v_{0\mathbf{k}'}^{*}}{\omega_{\mathbf{k}'}^{2}}\right)\right)\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{669}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{670}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B} \tag{671}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left\langle b_{\mathbf{k}'}^{\dagger}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{672}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}N_{\mathbf{k}'}\right)e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{673}$$

$$= -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle\prod_{\mathbf{k}}D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right\rangle_{B}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau} \tag{674}$$

$$= -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{675}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^{\dagger}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0}^{+}B_{1}^{-}\right\rangle_{B} = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{01}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}, \tag{676}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{1}^{+}B_{0}^{-}\right\rangle_{B} = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}B_{1}^{+}B_{0}^{-}\right\rangle_{B} \tag{677}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left\langle b_{\mathbf{k}'}\prod_{\mathbf{k}}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\right)\right\rangle_{B} \tag{678}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\left\langle b_{\mathbf{k}'}D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{679}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{680}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\prod_{\mathbf{k}}\exp\left(\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^{*}v_{0\mathbf{k}}-v_{1\mathbf{k}}v_{0\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right)\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)\left\langle D\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\right\rangle_{B}\left\langle\prod_{\mathbf{k}\neq\mathbf{k}'}\left(D\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\right)\right\rangle_{B} \tag{681}$$

$$= \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{10}, \tag{682}$$

$$\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_{0+}B_{1}^{-}\right\rangle_{B} = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{01}, \tag{683}$$

$$\left\langle \widetilde{B_{iz}}\left(\tau\right)\widetilde{B_{x}}\left(0\right)\right\rangle_{B} = \frac{1}{2}\sum_{\mathbf{k}'}\left(-\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{10}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}-\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)B_{01}N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right. \tag{684}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{10}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\left(N_{\mathbf{k}'}+1\right)B_{01}\right) \tag{685}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{01}-B_{10}\right)+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\left(B_{10}-B_{01}\right)\right) \tag{686}$$

$$= \frac{1}{2}\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{01}-B_{10}\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\left(B_{01}-B_{10}\right)\right) \tag{687}$$

$$= \mathrm{i}\sum_{\mathbf{k}'}B_{10}^{\Im}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(N_{\mathbf{k}'}+1\right)\right), \tag{688}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \left\langle\left(\frac{B_0^+B_1^-(\tau)-B_1^+B_0^-(\tau)+B_{10}-B_{01}}{2\mathrm{i}}\right)\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{689}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-(\tau)-B_1^+B_0^-(\tau)+B_{10}-B_{01}\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{690}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(B_0^+B_1^-(\tau)-B_1^+B_0^-(\tau)\right)\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^* b_{\mathbf{k}'}\right)\right\rangle_B \tag{691}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left\langle B_0^+B_1^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger-B_1^+B_0^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger+B_0^+B_1^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}-B_1^+B_0^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right\rangle, \tag{692}$$

$$\left\langle B_0^+B_1^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger\right\rangle_B = -\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}, \tag{693}$$

$$\left\langle B_0^+B_1^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}, \tag{694}$$

$$\left\langle B_1^+B_0^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger\right\rangle_B = -\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}, \tag{695}$$

$$\left\langle B_1^+B_0^-(\tau)\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}\right\rangle_B = \left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}, \tag{696}$$

$$\left\langle \widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right. \tag{697}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{698}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{01}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)^*(N_{\mathbf{k}'}+1)B_{10}\right. \tag{699}$$

$$\left.+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}\right) \tag{700}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{01})+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(-B_{10}-B_{01})\right) \tag{701}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)(N_{\mathbf{k}'}+1)e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*(B_{10}+B_{01})-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*N_{\mathbf{k}'}e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(B_{10}+B_{01})\right), \tag{702}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(\frac{B_0^+B_1^--B_1^+B_0^-+B_{10}-B_{01}}{2\mathrm{i}}\right)\right\rangle_B \tag{703}$$

$$= \frac{1}{2\mathrm{i}}\left\langle\sum_{\mathbf{k}'}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^--B_1^+B_0^-+B_{10}-B_{01}\right)\right\rangle_B \tag{704}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\right)\left(B_0^+B_1^--B_1^+B_0^-\right)\right\rangle_B \tag{705}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left\langle\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^--\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)b_{\mathbf{k}'}^\dagger e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_0^+B_1^-\right. \tag{706}$$

$$\left.-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*b_{\mathbf{k}'}e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}B_1^+B_0^-\right\rangle \tag{707}$$

$$= \frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle\right. \tag{708}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right\rangle \tag{709}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left\langle e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle\right. \tag{710}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right\rangle \tag{711}$$

$$= \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle\right. \tag{712}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^*\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle\right) \tag{713}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_1^+B_0^-\right\rangle_B = -\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{10}N_{\mathbf{k}'}, \tag{714}$$

$$\left\langle b_{\mathbf{k}'}^\dagger B_0^+B_1^-\right\rangle_B = -\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^* B_{01}N_{\mathbf{k}'}, \tag{715}$$

$$\left\langle b_{\mathbf{k}'}B_1^+B_0^-\right\rangle_B = \left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{10}, \tag{716}$$

$$\left\langle b_{\mathbf{k}'}B_0^+B_1^-\right\rangle_B = \left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(N_{\mathbf{k}'}+1\right)B_{01}, \tag{717}$$

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \tfrac{1}{2\mathrm{i}}\textstyle\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(-\left(\tfrac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}^*N_{\mathbf{k}'}\right)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(-\left(\tfrac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^*B_{10}N_{\mathbf{k}'}\right)\right. \tag{718}$$

$$+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{10}^{*}\right)-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{10}\right)\right) \tag{719}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{*}N_{\mathbf{k}'}+\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}N_{\mathbf{k}'}\right)\right. \tag{720}$$

$$\left.+e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{0\mathbf{k}'}-v_{1\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{10}^{*}\right)\right)-\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)\,B_{10}\right)\right) \tag{721}$$

$$=\frac{1}{2\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}\left(B_{10}+B_{10}^{*}\right)N_{\mathbf{k}'}\right. \tag{722}$$

$$\left.-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)\left(B_{10}+B_{10}^{*}\right)(N_{\mathbf{k}'}+1)\right) \tag{723}$$

$$=\frac{1}{\mathrm{i}}\sum_{\mathbf{k}'}\left(e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}-e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(N_{\mathbf{k}'}+1)\right) \tag{724}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}\right) \tag{725}$$

$$=\mathrm{i}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)B_{10}^{\Re}(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}B_{10}^{\Re}N_{\mathbf{k}'}\right) \tag{726}$$

$$=\mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}'}\left(e^{-\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)^{*}\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)-e^{\mathrm{i}\omega_{\mathbf{k}'}\tau}\left(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}\right)\left(\frac{v_{1\mathbf{k}'}-v_{0\mathbf{k}'}}{\omega_{\mathbf{k}'}}\right)^{*}N_{\mathbf{k}'}\right) \tag{727}$$

The correlation functions are equal to:

$$\left\langle\widetilde{B_{iz}}(\tau)\,\widetilde{B_{jz}}(0)\right\rangle_{B}=\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)^{*}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(g_{j\mathbf{k}}-v_{j\mathbf{k}}\right)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{728}$$

$$U=\prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^{*}v_{1\mathbf{k}}-v_{0\mathbf{k}}v_{1\mathbf{k}}^{*}}{\omega_{\mathbf{k}}^{2}}\right)\right), \tag{729}$$

$$\phi(\tau)=\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^{2}\left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau)+\cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right), \tag{730}$$

$$\left\langle\widetilde{B_{x}}(\tau)\,\widetilde{B_{x}}(0)\right\rangle_{B}=\frac{|B_{10}|^{2}}{2}\left(U^{\Re}\exp(-\phi(\tau))+\exp(\phi(\tau))-U^{\Re}-1\right), \tag{731}$$

$$\left\langle\widetilde{B_{y}}(\tau)\,\widetilde{B_{y}}(0)\right\rangle_{B}=\frac{|B_{10}|^{2}}{2}\left(\exp(\phi(\tau))-U^{\Re}\exp(-\phi(\tau))-1+U^{\Re}\right), \tag{732}$$

$$\left\langle\widetilde{B_{x}}(\tau)\,\widetilde{B_{y}}(0)\right\rangle_{B}=\frac{U^{\Im}|B_{10}|^{2}}{2}\left(\exp(-\phi(\tau))-1\right), \tag{733}$$

$$\left\langle\widetilde{B_{y}}(\tau)\,\widetilde{B_{x}}(0)\right\rangle_{B}=\frac{U^{\Im}|B_{10}|^{2}}{2}\left(\exp(-\phi(\tau))-1\right), \tag{734}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\,\widetilde{B_{x}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{735}$$

$$\left\langle\widetilde{B_{x}}(\tau)\,\widetilde{B_{iz}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{736}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\,\widetilde{B_{y}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}N_{\mathbf{k}}\right), \tag{737}$$

$$\left\langle\widetilde{B_{y}}(\tau)\,\widetilde{B_{iz}}(0)\right\rangle_{B}=\mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^{*}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)(N_{\mathbf{k}}+1)\,e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^{*}\right). \tag{738}$$

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \tag{739}$$

$$v_{i\mathbf{k}} = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}). \tag{740}$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i(\omega)}{\omega} \mathrm{d}\omega. \tag{741}$$

The integral version of the correlation functions are given by:

$$\left\langle \widetilde{B_{iz}}(\tau)\widetilde{B_{jz}}(0)\right\rangle_B = \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-v_{i\mathbf{k}})(g_{j\mathbf{k}}-v_{j\mathbf{k}})^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+(g_{i\mathbf{k}}-v_{i\mathbf{k}})^*(g_{j\mathbf{k}}-v_{j\mathbf{k}})e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{742}$$

$$= \sum_{\mathbf{k}}\left((g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}}))^*(g_{j\mathbf{k}}-g_{j\mathbf{k}}F_j(\omega_{\mathbf{k}}))e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{743}$$

$$= \sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))\, g_{j\mathbf{k}}^*(1-F_j(\omega_{\mathbf{k}}))^* e^{\mathrm{i}\omega_{\mathbf{k}}\tau}N_{\mathbf{k}}+g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^* g_{j\mathbf{k}}(1-F_j(\omega_{\mathbf{k}}))e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{744}$$

$$\approx \int_0^\infty \left(\sqrt{J_i(\omega)J_j^*(\omega)}(1-F_i(\omega))\left(1-F_j^*(\omega)\right)e^{\mathrm{i}\omega\tau}N(\omega)+\sqrt{J_i^*(\omega)J_j(\omega)}(1-F_i^*(\omega))(1-F_j(\omega))e^{-\mathrm{i}\omega\tau}(N(\omega)+1)\right)\mathrm{d}\omega, \tag{745}$$

$$U = \prod_{\mathbf{k}}\left(\exp\left(\frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{746}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{v_{0\mathbf{k}}^* v_{1\mathbf{k}} - v_{0\mathbf{k}} v_{1\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right) \tag{747}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})\, g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})\, g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{748}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}})\, g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})\, g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{749}$$

$$= \exp\left(\sum_{\mathbf{k}} \frac{g_{0\mathbf{k}}^* g_{1\mathbf{k}} F_0^*(\omega_{\mathbf{k}})\, F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} g_{1\mathbf{k}}^* F_0(\omega_{\mathbf{k}})\, F_1^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right) \tag{750}$$

$$\approx \exp\left(\int_0^\infty \frac{\sqrt{J_0^*(\omega)\, J_1(\omega)} F_0^*(\omega)\, F_1(\omega) - \sqrt{J_0(\omega)\, J_1^*(\omega)} F_0(\omega)\, F_1^*(\omega)}{\omega^2} \mathrm{d}\omega\right), \tag{751}$$

$$\phi(\tau) = \sum_{\mathbf{k}} \left|\frac{v_{1\mathbf{k}} - v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{752}$$

$$= \sum_{\mathbf{k}} \left|\frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2 \left(-\mathrm{i}\sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau)\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \tag{753}$$

$$\approx \int_0^\infty \left|\frac{\sqrt{J_1(\omega)} F_1(\omega) - \sqrt{J_0(\omega)} F_0(\omega)}{\omega}\right|^2 \left(-\mathrm{i}\sin(\omega\tau) + \cos(\omega\tau)\coth\left(\frac{\beta\omega}{2}\right)\right)\mathrm{d}\omega, \tag{754}$$

$$B_{10} = \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{755}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} - \frac{g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{v_{1\mathbf{k}}^* v_{0\mathbf{k}} - v_{1\mathbf{k}} v_{0\mathbf{k}}^*}{\omega_{\mathbf{k}}^2}\right)\right) \tag{756}$$

$$= \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\left|\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}-\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right|^2\coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right)\exp\left(\sum_{\mathbf{k}}\frac{1}{2}\left(\frac{g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})g_{1\mathbf{k}}^*F_1^*(\omega_{\mathbf{k}})-g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})g_{0\mathbf{k}}^*F_0^*(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2}\right)\right) \tag{757}$$

$$\approx \exp\left(-\frac{1}{2}\int_0^\infty\left|\frac{\sqrt{J_1(\omega)}F_1(\omega)-\sqrt{J_0(\omega)}F_0(\omega)}{\omega}\right|^2\coth\left(\frac{\beta\omega}{2}\right)\mathrm{d}\omega\right)\exp\left(\int_0^\infty\frac{1}{2}\left(\frac{\sqrt{J_0(\omega)J_1^*(\omega)}F_0(\omega)F_1^*(\omega)-\sqrt{J_0^*(\omega)J_1(\omega)}F_0^*(\omega)F_1(\omega)}{\omega^2}\right)\mathrm{d}\omega\right), \tag{758}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(U^{\Re}\exp\left(-\phi\left(\tau\right)\right)+\exp\left(\phi\left(\tau\right)\right)-U^{\Re}-1\right), \tag{759}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{|B_{10}|^2}{2}\left(\exp\left(\phi\left(\tau\right)\right)-U^{\Re}\exp\left(-\phi\left(\tau\right)\right)-1+U^{\Re}\right), \tag{760}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_y}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right), \tag{761}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_x}(0)\right\rangle_B = \frac{U^{\Im}|B_{10}|^2}{2}\left(\exp\left(-\phi\left(\tau\right)\right)-1\right), \tag{762}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{763}$$

$$= \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}})\right)N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-\left(g_{i\mathbf{k}}-g_{i\mathbf{k}}F_i(\omega_{\mathbf{k}})\right)^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{764}$$

$$= \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}\right)^*-g_{i\mathbf{k}}^*(1-F_i(\omega_{\mathbf{k}}))^*\frac{g_{1\mathbf{k}}F_1(\omega_{\mathbf{k}})-g_{0\mathbf{k}}F_0(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}}e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right), \tag{765}$$

$$Q\left(\omega\right) = \sqrt{J_i\left(\omega\right)}\left(1-F_i\left(\omega\right)\right)\left(\frac{\sqrt{J_1\left(\omega\right)}F_1\left(\omega\right)-\sqrt{J_0\left(\omega\right)}F_0\left(\omega\right)}{\omega}\right)^*, \tag{766}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_x}(0)\right\rangle_B \approx \mathrm{i}B_{10}^{\Im}\int_0^\infty\left(Q\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega, \tag{767}$$

$$\left\langle\widetilde{B_x}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(N_{\mathbf{k}}+1\right)\right) \tag{768}$$

$$= \mathrm{i}B_{10}^{\Im}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^*(1-F_i^*(\omega_{\mathbf{k}}))\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}-g_{i\mathbf{k}}(1-F_i(\omega))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}(N_{\mathbf{k}}+1)\right) \tag{769}$$

$$\approx \mathrm{i}B_{10}^{\Im}\int_0^\infty\left(Q^*\left(\omega\right)N\left(\omega\right)e^{\mathrm{i}\omega\tau}-Q\left(\omega\right)\left(N\left(\omega\right)+1\right)e^{-\mathrm{i}\omega\tau}\right)\mathrm{d}\omega, \tag{770}$$

$$\left\langle\widetilde{B_{iz}}(\tau)\widetilde{B_y}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(N_{\mathbf{k}}+1\right)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{771}$$

$$= \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}g_{i\mathbf{k}}^*(1-F_i^*(\omega_{\mathbf{k}}))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)(N_{\mathbf{k}}+1)-e^{\mathrm{i}\omega_{\mathbf{k}}\tau}g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*N_{\mathbf{k}}\right) \tag{772}$$

$$\approx \mathrm{i}B_{10}^{\Re}\int_0^\infty\left(e^{-\mathrm{i}\omega\tau}Q^*\left(\omega\right)\left(N\left(\omega\right)+1\right)-e^{\mathrm{i}\omega\tau}Q\left(\omega\right)N\left(\omega\right)\right)\mathrm{d}\omega, \tag{773}$$

$$\left\langle\widetilde{B_y}(\tau)\widetilde{B_{iz}}(0)\right\rangle_B = \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)^*N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-\left(g_{i\mathbf{k}}-v_{i\mathbf{k}}\right)(N_{\mathbf{k}}+1)\,e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{774}$$

$$= \mathrm{i}B_{10}^{\Re}\sum_{\mathbf{k}}\left(g_{i\mathbf{k}}^*(1-F_i^*(\omega_{\mathbf{k}}))N_{\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)-g_{i\mathbf{k}}(1-F_i(\omega_{\mathbf{k}}))(N_{\mathbf{k}}+1)e^{-\mathrm{i}\omega_{\mathbf{k}}\tau}\left(\frac{v_{1\mathbf{k}}-v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right)^*\right) \tag{775}$$

$$= \mathrm{i}B_{10}^{\Re}\int_0^\infty\left(e^{\mathrm{i}\omega\tau}Q^*\left(\omega\right)N\left(\omega\right)-e^{-\mathrm{i}\omega\tau}Q\left(\omega\right)\left(N\left(\omega\right)+1\right)\right)\mathrm{d}\omega. \tag{776}$$

The eigenvalues of the Hamiltonian $\overline{H_{\bar{S}}}$ are given by the solution of the following algebraic equation:

$$\lambda^2-\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right)\lambda+\mathrm{Det}\left(\overline{H_{\bar{S}}}\right)=0. \tag{777}$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_\pm = \frac{\xi\pm\eta}{2}$ and they satisfy $H_S\left|\pm\right\rangle = \lambda_\pm\left|\pm\right\rangle$. Using this notation is possible to write $H_{\bar{S}} = \lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|$.
The time-dependence of the system operators $\widetilde{A_i}\left(t\right)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_{\bar{S}}}$ we will obtain:

$$\begin{aligned}
\widetilde{A_i}(\tau) &= e^{\mathrm{i}\overline{H_{\bar{S}}}\tau} A_i e^{-\mathrm{i}\overline{H_{\bar{S}}}\tau} && (778)\\
&= \sum_{w} e^{-\mathrm{i}w\tau} \mathscr{A}_i(w)\,. && (779)
\end{aligned}$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.
In order to use the equation (779) to descompose the equation (355) we need to consider the time ordering operator $\mathcal{T}$, it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right)$ like:

$$\begin{aligned}
U(t) &\equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right) && (780)\\
&= \mathbb{I} + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n H(t_1) H(t_2) ... H(t_n)\,. && (781)
\end{aligned}$$

Here $0<t_1<t_2<...<t_{n-1}<t_n = t$ is a partition of the set $[0,t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right) \equiv \exp(-\mathrm{i}tH_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...$ so we can write:

$$U(t) = \exp\left(-\mathrm{i}t\left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + ...\right)\right). \tag{782}$$

The terms can be found expanding $\mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_{\bar{S}}}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$\begin{aligned}
H_E^{(0)}(t) &= \frac{1}{t}\int_0^t \overline{H_{\bar{S}}}(t')\,\mathrm{d}t', && (783)\\
H_E^{(1)}(t) &= -\frac{\mathrm{i}}{2t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], && (784)\\
H_E^{(2)}(t) &= \frac{1}{6t}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int_0^{t''} \mathrm{d}t''' \left(\left[\left[\overline{H_{\bar{S}}}(t'), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t''')\right] + \left[\left[\overline{H_{\bar{S}}}(t'''), \overline{H_{\bar{S}}}(t'')\right], \overline{H_{\bar{S}}}(t')\right]\right). && (785)
\end{aligned}$$

In this case the Fourier decomposition using the Magnus expansion is

$$\begin{aligned}
\widetilde{A_i}(t) &= U^{\dagger}(t) A_i(t) U(t) && (786)\\
&= e^{\mathrm{i}H_E(t)t} A_i(t) e^{-\mathrm{i}H_E(t)t} && (787)\\
&= \sum_{w(t)} e^{-\mathrm{i}w(t)t} \mathscr{A}_i(w(t))\,. && (788)
\end{aligned}$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.
Extending the Fourier decomposition to the matrix $\widetilde{A_j}(t-\tau, t)$ using the Magnus expansion generates:

$$\begin{aligned}
\widetilde{A_j}\left(t-\tau,t\right) &= U\left(t\right)U^{\dagger}\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right) && (789)\\
&= e^{-\mathrm{i}tH_E(t)}e^{\mathrm{i}(t-\tau)H_E(t-\tau)}A_j\left(t\right)e^{-\mathrm{i}(t-\tau)H_E(t-\tau)}e^{\mathrm{i}tH_E(t)} && (790)\\
&= e^{-\mathrm{i}tH_E(t)}\left(\sum_{w'(t-\tau)} e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j\left(w\left(t-\tau\right)\right)\right)e^{\mathrm{i}tH_E(t)} && (791)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) && (792)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}w'(t)t}e^{-\mathrm{i}(t-\tau)w(t-\tau)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) && (793)\\
&= \sum_{w(t),w'(t-\tau)} e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) && (794)
\end{aligned}$$

where $w'\left(t-\tau\right)$ and $w\left(t\right)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H_E}\left(t-\tau\right)$ and $\overline{H_E}\left(t\right)$ respectively.
In order to show the explicit form of the matrices present in the RHS of the equation (779) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $\mathrm{V}=\{|+\rangle,|-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha,\beta\in\mathrm{V}} \langle\alpha|A_i|\beta\rangle|\alpha\rangle\!\langle\beta|. \tag{795}$$

Given that $[|+\rangle\!\langle+|,|-\rangle\!\langle-|]=0$, then using the Zassenhaus formula we obtain:

$$\begin{aligned}
e^{\mathrm{i}\overline{H_E}\tau} &= e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)\tau} && (796)\\
&= e^{\mathrm{i}\lambda_+|+\rangle\langle+|\tau}e^{\mathrm{i}\lambda_-|-\rangle\langle-|\tau} && (797)\\
&= \left(|-\rangle\!\langle-|+e^{\mathrm{i}\lambda_+\tau}|+\rangle\!\langle+|\right)\left(|+\rangle\!\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\!\langle-|\right) && (798)\\
&= e^{\mathrm{i}\lambda_+\tau}|+\rangle\!\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\!\langle-|. && (799)
\end{aligned}$$

Calculating the transformation (779) directly using the previous relationship we find that:

$$\begin{aligned}
U^{\dagger}\left(\tau\right)A_i\left(\tau\right)U\left(\tau\right) &= \left(e^{\mathrm{i}\lambda_+\tau}|+\rangle\!\langle+|+e^{\mathrm{i}\lambda_-\tau}|-\rangle\!\langle-|\right)\left(\sum_{\alpha,\beta\in\mathrm{V}}\langle\alpha|A_i\left(\tau\right)|\beta\rangle|\alpha\rangle\!\langle\beta|\right)\left(e^{-\mathrm{i}\lambda_+\tau}|+\rangle\!\langle+|+e^{-\mathrm{i}\lambda_-\tau}|-\rangle\!\langle-|\right) && (800)\\
&= \langle+|A_i\left(\tau\right)|+\rangle|+\rangle\!\langle+|+e^{\mathrm{i}\eta\tau}\langle+|A_i\left(\tau\right)|-\rangle|+\rangle\!\langle-|+e^{-\mathrm{i}\eta\tau}\langle-|A_i\left(\tau\right)|+\rangle|-\rangle\!\langle+|+\langle-|A_i\left(\tau\right)|-\rangle|-\rangle\!\langle-|. && (801)\\
&= \mathscr{A}_i\left(0\right)+\mathscr{A}_i\left(-w\right)e^{\mathrm{i}w\tau}+\mathscr{A}_i\left(w\right)e^{-\mathrm{i}w\tau} && (802)
\end{aligned}$$

Here $w=\lambda_+-\lambda-$. Comparing the RHS of the equations (779) and the explicit expression for $\widetilde{A_i}\left(\tau\right)$ in (787), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$\begin{aligned}
\mathscr{A}_i\left(0\right) &= \langle+|A_i\left(\tau\right)|+\rangle|+\rangle\!\langle+|+\langle-|A_i\left(\tau\right)|-\rangle|-\rangle\!\langle-|, && (803)\\
\mathscr{A}_i\left(-w\right) &= \langle+|A_i\left(\tau\right)|-\rangle|+\rangle\!\langle-|, && (804)\\
\mathscr{A}_i\left(w\right) &= \langle-|A_i\left(\tau\right)|+\rangle|-\rangle\!\langle+|. && (805)
\end{aligned}$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A_i}\left(\tau\right)=\widetilde{A_i}^{\dagger}\left(\tau\right)$ and $\widetilde{B_i}\left(\tau\right)=\widetilde{B_i}^{\dagger}\left(\tau\right)$ we can use the equation (779) to write the master equation in the following neater form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ij}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i,\widetilde{A_j}(t-\tau,t)\,\overline{\rho_S}(t)\Big] + \mathscr{B}_{ji}(-\tau)\Big[\overline{\rho_S}(t)\widetilde{A_j}(t-\tau,t),A_i\Big]\Big) \quad (806)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] \quad (807)$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right)\right]\Big) \quad (808)$$

Given that $\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) = \mathscr{A}_j^{\dagger}\left(-w\left(t-\tau\right),-w'\left(t\right)\right)$ from the Fourier decomposition (779) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\mathscr{B}_{ij}\left(\tau\right) = \mathrm{Tr}_B\left(\widetilde{B_i}\left(t\right)\widetilde{B_j}\left(s\right)\rho_B\right) \quad (809)$$

$$= \mathrm{Tr}_B\left(\widetilde{B_i}\left(\tau\right)\widetilde{B_j}\left(0\right)\rho_B\right). \quad (810)$$

Let's define:

$$\mathscr{A}_j\left(w\left(t-\tau\right),w'\left(t\right)\right) = \mathscr{A}_{jww'}\left(t-\tau,t\right) \quad (811)$$

The master equation can be re-written in the following form:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \quad (812)$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \quad (813)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \quad (814)$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \quad (815)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \quad (816)$$

$$+\sum_{ijww'}\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right] \quad (817)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathscr{B}_{ij}(\tau)\left[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\overline{\rho_S}(t)\right] \quad (818)$$

$$-\mathscr{B}_{ji}\left(-\tau\right)\left[A_i,\overline{\rho_S}\left(t\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right]\Big) \quad (819)$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right] - \textstyle\sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\mathrm{Tr}_B\left(\left[A_i,\widetilde{B_i}(\tau)\widetilde{B_j}(0)\rho_B e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\overline{\rho_S}(t)\right]\right. \quad (820)$$

$$\left.-\left[A_i,\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}(t-\tau,t)\right]\right) \quad (821)$$

Given that if we define:

$$D_{ijww'}\left(t-\tau,t\right) = C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right) \quad (822)$$

then

$$D^{\dagger}_{ijww'}\left(t-\tau,t\right) = \left(C_i\left(t\right)C_j\left(t-\tau\right)\mathscr{B}_{ij}\left(\tau\right)e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_{jww'}\left(t-\tau,t\right)\right)^{\dagger} \quad (823)$$

$$= \mathscr{B}^{*}_{ij}\left(\tau\right)C_i\left(t\right)C_j\left(t-\tau\right)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}^{\dagger}_{jww'}\left(t-\tau,t\right) \quad (824)$$

We used the fact that $C_i(t), C_j(t-\tau)$ are real. Now let's consider the following trace recalling that $\mathrm{Tr}(A)^* = \mathrm{Tr}\left(A^\dagger\right)$ so:

$$\begin{aligned}
\mathrm{Tr}_B\left(\widetilde{B_j}(-\tau)\widetilde{B_i}(0)\rho_B\right) &= \mathrm{Tr}_B\left(e^{-\mathrm{i}\tau H_B(\tau)}B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B\right) && (825)\\
&= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i\rho_B e^{-\mathrm{i}\tau H_B(\tau)}\right) \text{ (by cyclic permutivity of trace)} && (826)\\
&= \mathrm{Tr}_B\left(B_j e^{\mathrm{i}\tau H_B(\tau)}B_i e^{-\mathrm{i}\tau H_B(\tau)}\rho_B\right) \text{ (by commutativity of } e^{-\mathrm{i}\tau H_B(\tau)} \text{ and } \rho_B) && (827)\\
&= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) \text{ (by definition of time evolution)} && (828)\\
&= \mathrm{Tr}_B\left(B_j\widetilde{B_i}(\tau)\rho_B\right) && (829)\\
&= \mathrm{Tr}_B\left(\rho_B B_j\widetilde{B_i}(\tau)\right) && (830)\\
&= \mathrm{Tr}_B\left(\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^\dagger\right) \text{ (by definition of adjoint)} && (831)\\
&= \mathrm{Tr}_B\left(\widetilde{B_i}(\tau)B_j\rho_B\right)^* && (832)\\
&= \mathscr{B}_{ij}^*(\tau) && (833)
\end{aligned}$$

So we can write the master equation like:

$$\begin{aligned}
\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} &= -\mathrm{i}\left[\overline{H_{\bar S}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau C_i(t)C_j(t-\tau)\Big(\mathscr{B}_{ij}(\tau)\Big[A_i, e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j(w(t-\tau),w'(t))\overline{\rho_S}(t)\Big] && (834)\\
&-\mathscr{B}_{ij}^*(\tau)\left[A_i,\overline{\rho_S}(t)e^{-\mathrm{i}\tau w(t-\tau)}e^{\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j^\dagger(w(t-\tau),w'(t))\right]\Big) && (835)\\
&= -\mathrm{i}\left[\overline{H_{\bar S}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\int_0^t \mathrm{d}\tau\left(\left[A_i, D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\right]\right) && (836)
\end{aligned}$$

Let's define the response matrix in the following way.

$$\mathscr{D}_{ijww'}(t) = \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau,t) \tag{837}$$

Then the master equation can be written as:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar S}}(t),\overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)\mathscr{D}_{ijww'}^\dagger(t)\right]\right) \tag{838}$$

If we extend the upper limit of integration to ∞ in the equation (837) then the system will be independent of any preparation at $t=0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$\begin{aligned}
e^{-V}\frac{\mathrm{d}\overline{\rho}_S(t)}{\mathrm{d}t}e^V &= \frac{\mathrm{d}\left(e^{-V}\overline{\rho}_S e^V\right)}{\mathrm{d}t} && (839)\\
&= \frac{\mathrm{d}\rho_S}{\mathrm{d}t} && (840)\\
&= -\mathrm{i}e^{-\mathrm{V}}\left[\overline{H_{\bar S}}(t),\overline{\rho_S}(t)\right]e^V - \sum_{ijww'}\int_0^t \mathrm{d}\tau\Big(e^{-V}\left[A_i,D_{ijww'}(t-\tau,t)\overline{\rho_S}(t)\right]e^V - e^{-V}\Big[A_i,\overline{\rho_S}(t)D_{ijww'}^\dagger(t-\tau,t)\Big]e^V\Big). && (841)
\end{aligned}$$

For a product we have the following:

$$e^{-V}\overline{AB}e^{V} = e^{-V}\overline{A\mathbb{I}B}e^{V} \tag{842}$$
$$= e^{-V}\overline{A}e^{V}e^{-V}\bar{B}e^{V} \tag{843}$$
$$= \left(e^{-V}\overline{A}e^{V}\right)\left(e^{-V}\bar{B}e^{V}\right) \tag{844}$$
$$= AB. \tag{845}$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V}\overline{[A,B]}e^{V} = e^{-V}\overline{(AB-BA)}e^{V} \tag{846}$$
$$= e^{-V}\overline{AB}e^{V} - e^{-V}\overline{BA}e^{V} \tag{847}$$
$$= AB - BA \tag{848}$$
$$= [A,B]. \tag{849}$$

So we will obtain that

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]e^{V} - e^{-V}\sum_{ijww'}\left(\left[A_i,\mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i,\overline{\rho_S}(t)\mathscr{D}^{\dagger}_{ijww'}(t)\right]\right)e^{V} \tag{850}$$
$$= -\mathrm{i}e^{-V}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]e^{V} - \sum_{ijww'}\left(e^{-V}\left[A_i,\mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right]e^{V} - e^{-V}\left[A_i,\overline{\rho_S}(t)\mathscr{D}^{\dagger}_{ijww'}(t)\right]e^{V}\right) \tag{851}$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}(t)\mathscr{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right) \tag{852}$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}(t)e^{V}e^{-V}\overline{\rho_S}(t)e^{V}\right] - \left[e^{-V}A_ie^{V},e^{-V}\overline{\rho_S}(t)e^{V}e^{-V}\mathscr{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right) \tag{853}$$
$$= -\mathrm{i}\left[H_{\bar{S}}(t),\rho_S(t)\right] - \sum_{ijww'}\left(\left[e^{-V}A_ie^{V},e^{-V}\mathscr{D}_{ijww'}(t)e^{V}\rho_S(t)\right] - \left[e^{-V}A_ie^{V},\rho_S(t)e^{-V}\mathscr{D}^{\dagger}_{ijww'}(t)e^{V}\right]\right). \tag{854}$$

V. LIMIT CASES

In order to show the plausibility of the master equation (838) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (838) reproduces the results of the reference [1], for the latter case we have that $i,j \in \{1,2,3\}$ and $\omega \in (0,\pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}}\left(b^{\dagger}_{\mathbf{k}} + b_{\mathbf{k}}\right)\right)|1\rangle\langle 1| + \frac{\Omega}{2}\sigma_x + \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{855}$$

After performing the transformation (24) on the Hamiltonian (855) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R)|1\rangle\langle 1| + \frac{\Omega_r}{2}\sigma_x, \tag{856}$$
$$\overline{H_I} = B_z|1\rangle\langle 1| + \frac{\Omega}{2}\left(B_x\sigma_x + B_y\sigma_y\right), \tag{857}$$
$$H_B = \sum_{\mathbf{k}}\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}. \tag{858}$$

The Hamiltonian (856) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R)\,|1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right)|1\rangle\langle 1| - \frac{\delta}{2}|0\rangle\langle 0| \tag{859}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \tag{860}$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \tag{861}$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_{\bar{S}}}$. Given that $\overline{H_{\bar{S}}} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\mathrm{Tr}\left(\overline{H_{\bar{S}}}\right) = R$ and $\mathrm{Det}\left(\overline{H_{\bar{S}}}\right) = \frac{R^2-\epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \tag{862}$$

$$\lambda_\pm = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2-\epsilon^2-\Omega_r^2}{4}\right)}}{2} \tag{863}$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \tag{864}$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \tag{865}$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \tag{866}$$

$$\lambda_\pm = \frac{R \pm \eta}{2}. \tag{867}$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \tag{868}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle$.
For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \tag{869}$$

so the eigenvector $|+\rangle = a\,|0\rangle + b\,|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}\,|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle$.
Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon+\eta}{2}, \tag{870}$$
$$\lambda_- = \frac{\epsilon-\eta}{2}, \tag{871}$$
$$|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle\,, \tag{872}$$
$$|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle\,, \tag{873}$$
$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}, \tag{874}$$
$$\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}. \tag{875}$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right). \tag{876}$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1}\left(\frac{\Omega_r}{\epsilon}\right)$.

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{\cos(x)+1}. \tag{877}$$

So the value of $\tan(\theta)$ using (877) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}+1} \tag{878}$$
$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}}{\frac{\epsilon+\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}} \tag{879}$$
$$= \frac{\Omega_r}{\epsilon+\eta}. \tag{880}$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (804)-(805) and the fact that $|+\rangle = \sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle = \begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}$ and $|-\rangle = \cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$ withsin $(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2+\Omega_r^2}}$:

$$
\begin{aligned}
\langle +|\sigma_x|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (881)\\
&= 2\sin(\theta)\cos(\theta) && (882)\\
&= \sin(2\theta)\,, && (883)\\
\langle -|\sigma_x|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (884)\\
&= -2\sin(\theta)\cos(\theta) && (885)\\
&= -\sin(2\theta)\,, && (886)\\
\langle -|\sigma_x|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (887)\\
&= \cos^2(\theta) - \sin^2(\theta) && (888)\\
&= \cos(2\theta)\,, && (889)\\
\langle +|\sigma_y|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (890)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (891)\\
&= 0, && (892)\\
\langle -|\sigma_y|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (893)\\
&= \mathrm{i}\sin(\theta)\cos(\theta) - \mathrm{i}\sin(\theta)\cos(\theta) && (894)\\
&= 0, && (895)\\
\langle -|\sigma_y|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (896)\\
&= \mathrm{i}\cos^2(\theta) + \mathrm{i}\sin^2(\theta) && (897)\\
&= \mathrm{i}. && (898)
\end{aligned}
$$

$$
\begin{aligned}
\langle +|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (899)\\
&= \cos(\theta)\cos(\theta) && (900)\\
&= \cos^2(\theta)\,, && (901)\\
\langle -|\frac{1+\sigma_z}{2}|-\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix} && (902)\\
&= \sin(\theta)\sin(\theta) && (903)\\
&= \sin^2(\theta)\,, && (904)\\
\langle -|\frac{1+\sigma_z}{2}|+\rangle &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix} && (905)\\
&= -\sin(\theta)\cos(\theta) && (906)\\
&= -\sin(\theta)\cos(\theta)\,. && (907)
\end{aligned}
$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta)\left(|+\rangle\langle+| - |-\rangle\langle-|\right), \tag{908}$$
$$A_1(\eta) = \cos(2\theta)\,|-\rangle\langle+|, \tag{909}$$
$$A_2(0) = 0, \tag{910}$$
$$A_2(\eta) = \mathrm{i}|-\rangle\langle+|, \tag{911}$$
$$A_3(0) = \cos^2(\theta)\,|+\rangle\langle+| + \sin^2(\theta)\,|-\rangle\langle-|, \tag{912}$$
$$A_3(\eta) = -\sin(\theta)\cos(\theta)\,|-\rangle\langle+|. \tag{913}$$

Now to prove the fact that the model of the "Time-independent variational quantum master equation" is a special case the master equation (841) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (822) is equivalent to:

$$\begin{aligned}
\mathscr{D}_{ijww'}(t) &= \int_0^t \mathrm{d}\tau D_{ijww'}(t-\tau,t) &(914)\\
&= \int_0^t \mathrm{d}\tau C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}\tau w(t-\tau)}e^{-\mathrm{i}t\left(w(t-\tau)-w'(t)\right)}\mathscr{A}_j\left(w(t-\tau),w'(t)\right) &(915)\\
&= \int_0^t \mathrm{d}\tau C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}\tau w}e^{-\mathrm{i}t\left(w-w'\right)}\mathscr{A}_j(w,w'). &(916)
\end{aligned}$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (392) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau). \tag{917}$$

So the response matrix can be rewritten as:

$$\mathscr{D}_{ijww'}(t) = \left(\int_0^t \mathrm{d}\tau\Lambda'_{ij}(\tau)\,e^{\mathrm{i}\tau w}e^{-\mathrm{i}t\left(w-w'\right)}\right)\mathscr{A}_j(w,w') \tag{918}$$

Let's define the response function like:

$$\begin{aligned}
K_{ij}(w,w',t) &= \int_0^t C_i(t)\,C_j(t-\tau)\,\Lambda_{ij}(\tau)\,e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau &(919)\\
&= \int_0^t \Lambda'_{ij}(\tau)\,e^{\mathrm{i}w\tau}e^{-\mathrm{i}t\left(w-w'\right)}\mathrm{d}\tau &(920)\\
&= K_{ijww'}(t). &(921)
\end{aligned}$$

Then we have the following equivalence:

$$\begin{aligned}
\mathscr{D}_{ijww'}(t) &= K_{ijww'}(t)\,\mathscr{A}_j(w,w') &(922)\\
&= K_{ijww'}(t)\,\mathscr{A}_{jww'} &(923)
\end{aligned}$$

We can proof that

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, \mathscr{D}_{ijww'}(t)\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)\mathscr{D}^{\dagger}_{ijww'}(t)\right]\right) \tag{924}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left[A_i, K_{ijww'}(t)\mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - \left[A_i, \overline{\rho_S}(t)K^{*}_{ijww'}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{925}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(K_{ijww'}(t)\left[A_i, \mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - K^{*}_{ijww'}(t)\left[A_i, \overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{926}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'}\left(\left(K^{\Re}_{ijww'}(t) + \mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i, \mathscr{A}_{jww'}\overline{\rho_S}(t)\right] - \left(K^{\Re}_{ijww'}(t) - \mathrm{i}K^{\Im}_{ijww'}(t)\right)\left[A_i, \overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right]\right) \tag{927}$$

$$= -\mathrm{i}\left[\overline{H_{\bar{S}}}(t), \overline{\rho_S}(t)\right] - \sum_{ijww'} K^{\Re}_{ijww'}(t)\left[A_i, \mathscr{A}_{jww'}\overline{\rho_S}(t) - \overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] - \mathrm{i}\sum_{ijww'} K^{\Im}_{ijww'}(t)\left[A_i, \mathscr{A}_{jww'}\overline{\rho_S}(t) + \overline{\rho_S}(t)\mathscr{A}^{\dagger}_{jww'}\right] \tag{928}$$

Using the notation of the master equation (838), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(728)-(738) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (392) are equal to:

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_1}(\tau)\widetilde{B_1}(0)\rho_B\right) \tag{929}$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)} - 2\right), \tag{930}$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \mathrm{Tr}_B\left(\widetilde{B_2}(\tau)\widetilde{B_2}(0)\rho_B\right) \tag{931}$$

$$= \frac{\Omega_r^2}{8}\left(\mathrm{e}^{\phi(\tau)} + \mathrm{e}^{-\phi(\tau)}\right), \tag{932}$$

$$\Lambda'_{33}(\tau) = \int_0^{\infty} \mathrm{d}\omega J(\omega)(1 - F(\omega))^2 G_+(\tau), \tag{933}$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2}\int_0^{\infty} \mathrm{d}\omega \frac{J(\omega)}{\omega} F(\omega)(1 - F(\omega))\mathrm{i}G_-(\tau), \tag{934}$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \tag{935}$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \tag{936}$$

$$= \Lambda'_{13}(\tau) \tag{937}$$

$$= \Lambda'_{31}(\tau) \tag{938}$$

$$= 0. \tag{939}$$

Finally taking the Hamiltonian (855) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\mathrm{Det}\left(\overline{H_S}\right) = -\frac{\Omega_r^2}{4}$, $\mathrm{Tr}\left(\overline{H_S}\right) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (336) we have that:

$$f_k = \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(\epsilon - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k}\right)} \tag{940}$$

$$= \frac{g_k\left(1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\right)}{1 - \frac{\epsilon\tanh\left(\frac{\beta\eta}{2}\right)}{\eta}\left(1 - \frac{\Omega_r^2\coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k}\right)}. \tag{941}$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (419) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)U^{\dagger}(t-\tau) = U(\tau)$. From the equation (838) and using the fact that

$$\widetilde{A_j}(t-\tau,t) = U(\tau)A_jU(-\tau) \tag{942}$$
$$= \sum_w e^{\mathrm{i}w\tau}\mathscr{A}_j(-w) \tag{943}$$
$$= \sum_w e^{-\mathrm{i}w\tau}\mathscr{A}_j(w). \tag{944}$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[H_S(t), \overline{\rho_S}(t)\right] - \frac{1}{2}\sum_{ij}\sum_w \gamma_{ij}(w,t)\left[A_i, \mathscr{A}_j(w)\overline{\rho}_S(t) - \overline{\rho}_S(t)\mathscr{A}_j^{\dagger}(w)\right] \tag{945}$$
$$-\sum_{ij}\sum_w S_{ij}(w,t)\left[A_i, \mathscr{A}_j(w)\overline{\rho}_S(t) + \overline{\rho}_S(t)\mathscr{A}_j^{\dagger}(w)\right]. \tag{946}$$

where $\mathscr{A}_j^{\dagger}(w) = \mathscr{A}_j(-w)$, as we can see the equation (946) contains the rates and energy shifts $\gamma_{ij}(w,t) = 2K_{ij}^{\Re}(w,t)$ and $S_{ij}(w,t) = K_{ij}^{\Im}(w,t)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}(w,t) = \int_0^t \Lambda'_{ij}(\tau)e^{\mathrm{i}w\tau}\mathrm{d}\tau.$$

The fact $\mathscr{A}_j^{\dagger}(w) = \mathscr{A}_j(-w)$ can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{\mathrm{i}\overline{H_S}(t-\tau)} = e^{\mathrm{i}(\lambda_+|+\rangle\langle+|+\lambda_-|-\rangle\langle-|)(t-\tau)} \tag{947}$$
$$= e^{\mathrm{i}\lambda_+|+\rangle\langle+|(t-\tau)}e^{\mathrm{i}\lambda_-|-\rangle\langle-|(t-\tau)} \tag{948}$$
$$= \left(|-\rangle\langle-| + e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|\right)\left(|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{949}$$
$$= e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|. \tag{950}$$

Where $\lambda_+, \lambda_-$are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (779) of (803)-(805) directly using the previous relationship we find that:

$$\widetilde{A_i(0)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|+\rangle|+\rangle\langle+|+\langle-|A_i|-\rangle|-\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+|+e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{951}$$
$$= \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|, \tag{952}$$
$$\widetilde{A_i(w)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle+|A_i|-\rangle|+\rangle\langle-|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{953}$$
$$= \langle+|A_i|-\rangle|+\rangle\langle-|e^{\mathrm{i}w(t-\tau)}, \tag{954}$$
$$\widetilde{A_i(-w)}(t-\tau) = \left(e^{\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|.\right)\left(\langle-|A_i|+\rangle|-\rangle\langle+|\right)\left(e^{-\mathrm{i}\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-\mathrm{i}\lambda_-(t-\tau)}|-\rangle\langle-|\right) \tag{955}$$
$$= \langle-|A_i|+\rangle|-\rangle\langle+|e^{-\mathrm{i}w(t-\tau)}. \tag{956}$$

Here $w = \lambda_+ - \lambda-$. So we can see that for the equation (789) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w \neq w'$ then $A'_j(w,w') = 0$ so:

$$\widetilde{A_j}\left(t-\tau,t\right)=U\left(t\right)U^{\dagger}\left(t-\tau\right)A_j\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right) \tag{957}$$

$$=U\left(t\right)\left(\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}A_j\left(w(t-\tau)\right)\right)U^{\dagger}\left(t\right) \tag{958}$$

$$=\sum_{w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}U\left(t\right)A_j\left(w(t-\tau)\right)U^{\dagger}\left(t\right) \tag{959}$$

$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w(t-\tau),w'\left(t\right)\right) \tag{960}$$

$$=\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_{jww'} \tag{961}$$

$$==\sum_{w'(t),w(t-\tau)}e^{-\mathrm{i}(t-\tau)w(t-\tau)}e^{\mathrm{i}tw'(t)}A_j\left(w\right)\delta_{ww'} \tag{962}$$

$$=\sum_{w}e^{-\mathrm{i}(t-\tau)w}e^{\mathrm{i}tw}A_j\left(w\right) \tag{963}$$

$$=\sum_{w}e^{\mathrm{i}\tau w}A_j\left(w\right) \tag{964}$$

$$=U^{\dagger}\left(-\tau\right)A_jU\left(-\tau\right) \tag{965}$$

So using now as reference the equation (928) and $A_j'\left(w,w'\right)=0$ we can deduce that:

$$\frac{\mathrm{d}\overline{\rho_S}(t)}{\mathrm{d}t}=-\mathrm{i}\left[\overline{H_{\bar{S}}}(t),\overline{\rho_S}(t)\right]-\sum_{ijw}K_{ij}^{\Re}(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t)-\overline{\rho_S}(t)A_j^{\dagger}(w)\right]-\mathrm{i}\sum_{ijw}K_{ij}^{\Im}(w,t)\left[A_i,A_j(w)\overline{\rho_S}(t)+\overline{\rho_S}(t)A_j^{\dagger}(w)\right] \tag{966}$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k\ll\omega_k$ then $f_k\approx g_k$ so we recover the full polaron transformation. It means from the equation (107) that $B_z=0$. The Hamiltonian studied is given by:

$$H=\left(\delta+\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right)\right)|1\rangle\langle 1|+\frac{\Omega\left(t\right)}{2}\sigma_x+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}. \tag{967}$$

If $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then $B\left(\tau\right)=B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (967) using the Hamiltonian of the equation (1) we can say that $\delta=\varepsilon_1\left(t\right)$, $\varepsilon_0\left(t\right)=0$, $V_{10}\left(t\right)=\frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ then, in this case and using the equation (225) and (236) we obtain the following transformed Hamiltonians:

$$\overline{H_S}=\left(\delta+R_1\right)|1\rangle\langle 1|+\frac{B\sigma_x}{2}\Omega\left(t\right), \tag{968}$$

$$\overline{H_{\mathrm{I}}}=\frac{\Omega\left(t\right)}{2}\left(B_x\sigma_x+B_y\sigma_y\right). \tag{969}$$

In this case $R_1=\sum_{\mathbf{k}}\left(\omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}-2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}g_{\mathbf{k}}\right)$ from (27) and given that $v_{\mathbf{k}}\approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}=g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}^{-1}|g_{\mathbf{k}}|^2\right)=\sum_{\mathbf{k}}\left(-\omega_{\mathbf{k}}|\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}|^2\right)$ as expected, take $\delta+R_1=\delta'$. If $F\left(\omega_{\mathbf{k}}\right)=1$ and using the equations (929)-(936) we can deduce that the only terms that survive are $\Lambda_{11}\left(\tau\right)$ and $\Lambda_{22}\left(\tau\right)$. The phonon propagator for this case is:

$$\phi\left(\tau\right)=\int_0^{\infty}\frac{J\left(\omega\right)}{\omega^2}G_+\left(\tau\right)\mathrm{d}\omega. \tag{970}$$

Writing $G_+\left(\tau\right)=\coth\left(\frac{\beta\omega}{2}\right)\cos\left(\omega\tau\right)-i\sin\left(\omega\tau\right)$ so (970) can be written as:

$$\phi\left(\tau\right)=\int_{0}^{\infty}\frac{J\left(\omega\right)}{\omega^{2}}\left(\coth\left(\frac{\beta\omega}{2}\right)\cos\left(\omega\tau\right)-i\sin\left(\omega\tau\right)\right)\mathrm{d}\omega. \tag{971}$$

Writing the interaction Hamiltonian (969) in the similar way to the equation (236) allow us to to write $A_1=\sigma_x$, $A_2=\sigma_y$, $B_1\left(t\right)=B_x$, $B_2\left(t\right)=B_y$ and $C_1\left(t\right)=\frac{\Omega(t)}{2}=C_2\left(t\right)$. Now taking the equation (225) with $\delta'|1\rangle\langle 1|=\frac{\delta'}{2}\sigma_z+\frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$is equal to:

$$\overline{H_S}=\frac{\delta'}{2}\sigma_z+\frac{B\sigma_x}{2}\Omega\left(t\right). \tag{972}$$

As we can see the function B is a time-independent function because we consider that $g_{\mathbf{k}}$ doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}\left(\tau\right)=\mathrm{Tr}_B\left(\widetilde{B_1}\left(\tau\right)\widetilde{B_1}\left(0\right)\rho_B\right) \tag{973}$$
$$=\frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}-2\right), \tag{974}$$
$$\Lambda_{22}\left(\tau\right)=\mathrm{Tr}_B\left(\widetilde{B_2}\left(\tau\right)\widetilde{B_2}\left(0\right)\rho_B\right) \tag{975}$$
$$=\frac{B^2}{2}\left(\mathrm{e}^{\phi(\tau)}+\mathrm{e}^{-\phi(\tau)}\right). \tag{976}$$

These functions match with the equations $\Lambda_x\left(\tau\right)$ and $\Lambda_y\left(\tau\right)$ of the reference [2] and $\Lambda_i\left(\tau\right)=\Lambda_i\left(-\tau\right)$ for $i\in\{x,y\}$ respectively. The master equation for this section based on the equation(419) is:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t}=-\mathrm{i}\left[\frac{\delta'}{2}\sigma_z+\frac{\Omega_r\left(t\right)\sigma_x}{2},\rho_S\left(t\right)\right]-\sum_{i=1}^{2}\int_0^t\mathrm{d}\tau\left(C_i\left(t\right)C_i\left(t-\tau\right)\Lambda_{ii}\left(\tau\right)\left[A_i,\widetilde{A_i}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\right. \tag{977}$$
$$\left.+C_i\left(t\right)C_i\left(t-\tau\right)\Lambda_{ii}\left(-\tau\right)\left[\rho_S\left(t\right)\widetilde{A_i}\left(t-\tau,t\right),A_i\right]\right). \tag{978}$$

Replacing $C_i\left(t\right)=\frac{\Omega(t)}{2}$ and $\widetilde{A_i}\left(t-\tau,t\right)=\widetilde{\sigma_i}\left(t-\tau,t\right)$, also using the equations (973) and (976) on the equation (978) we obtain that:

$$\frac{\mathrm{d}\rho_S\left(t\right)}{\mathrm{d}t}=-\frac{\mathrm{i}}{2}\left[\delta'\sigma_z+\Omega_r\left(t\right)\sigma_x,\rho_S\left(t\right)\right]-\frac{\Omega\left(t\right)}{4}\int_0^t\mathrm{d}\tau\Omega\left(t-\tau\right)\left(\left[\sigma_x,\widetilde{\sigma_x}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\Lambda_x\left(\tau\right)\right. \tag{979}$$
$$\left.+\left[\sigma_y,\widetilde{\sigma_y}\left(t-\tau,t\right)\rho_S\left(t\right)\right]\Lambda_y\left(\tau\right)+\left[\rho_S\left(t\right)\widetilde{\sigma_x}\left(t-\tau,t\right),\sigma_x\right]\Lambda_x\left(\tau\right)+\left[\rho_S\left(t\right)\widetilde{\sigma_y}\left(t-\tau,t\right),\sigma_y\right]\Lambda_y\left(\tau\right)\right). \tag{980}$$

As we can see $\left[A_j,\widetilde{A_i}\left(t-\tau,t\right)\rho_S\left(t\right)\right]^{\dagger}=\left[\rho_S\left(t\right)\widetilde{A_i}\left(t-\tau,t\right),A_j\right]$, $\Lambda_x\left(\tau\right)=\Lambda_x\left(-\tau\right)$ and $\Lambda_y\left(\tau\right)=\Lambda_y\left(-\tau\right)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation $(S17)$ of the reference [3] we will impose that $F\left(\omega\right)=0$, so there is no transformation in this case. As we can see from the definition (392) the only term that survives is $\Lambda_{33}\left(\tau\right)$. Taking $\hbar=1$ the Hamiltonian of the reference can be written in the form:

$$H=\Delta|1\rangle\langle 1|+\frac{\Omega\left(t\right)}{2}\left(|1\rangle\langle 0|+|0\rangle\langle 1|\right)+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}+|1\rangle\langle 1|\sum_{\mathbf{k}}\left(g_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}+g_{\mathbf{k}}^{*}b_{\mathbf{k}}\right). \tag{981}$$

Using the equation (838), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right)), \rho_S\left(t\right)\right] - \frac{1}{2}\sum_w \gamma_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) - \rho_S\left(t\right)A_3^\dagger\left(w\right)\right] \tag{982}$$

$$-\sum_w S_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right) + \rho_S\left(t\right)A_3^\dagger\left(w\right)\right]\Bigg). \tag{983}$$

The correlation functions are relevant if $F\left(\omega\right) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}\left(\tau\right) = \int_0^\infty \mathrm{d}\omega J\left(\omega\right) G_+\left(\tau\right), \tag{984}$$

$$\Lambda_{33}\left(-\tau\right) = \int_0^\infty \mathrm{d}\omega J\left(\omega\right) G_+\left(-\tau\right). \tag{985}$$

In our case $A_3 = \frac{\mathbb{I}+\sigma_z}{2}$, the equation (983) can be transformed in

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\mathrm{i}\left[H_S\left(t\right), \rho_S\left(t\right))\right] - \sum_w \left(K_{33}\left(w,t\right)\left[A_3, A_3\left(w\right)\rho_S\left(t\right)\right] + K_{33}^*\left(w,t\right)\left[\rho_S\left(t\right)A_3\left(w\right), A_3\right]\right). \tag{986}$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthemore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\widetilde{\Omega}$ To find the matrices $A_3\left(w\right)$, we have to remember that $H_S = \frac{\Omega(t)}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$, this Hamiltonian using the approximation $\widetilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\widetilde{\Omega}}{2}, \tag{987}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right), \tag{988}$$

$$\lambda_- = -\frac{\widetilde{\Omega}}{2}, \tag{989}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{990}$$

The elements of the decomposition matrices are:

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{991}$$

$$= \frac{1}{2}, \tag{992}$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{993}$$

$$= \frac{1}{2}, \tag{994}$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{995}$$

$$= -\frac{1}{2}. \tag{996}$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \tag{997}$$
$$= \frac{\mathbb{I}}{2}, \tag{998}$$
$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle+| \tag{999}$$
$$= \frac{1}{4}(\sigma_{\mathrm{z}} + \mathrm{i}\sigma_{\mathrm{y}}), \tag{1000}$$
$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle-| \tag{1001}$$
$$= \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y). \tag{1002}$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}[\sigma_x, \rho_S(t))] - K_{33}\left(\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y)\rho_S(t)\right] - K_{33}\left(-\widetilde{\Omega}, t\right)\left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y)\rho_S(t)\right] \tag{1003}$$
$$- K_{33}^*\left(\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z + \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right] - K_{33}^*\left(-\widetilde{\Omega}, t\right)\left[\rho_S(t)\frac{1}{4}(\sigma_z - \mathrm{i}\sigma_y), \frac{\sigma_z}{2}\right]. \tag{1004}$$

Calculating the response functions extending the upper limit of τ to ∞, we obtain:

$$K_{33}\left(\widetilde{\Omega}\right) = \int_0^\infty\int_0^\infty J(\omega)\,G_+(\tau)\,e^{\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1005}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\,e^{\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega)\,e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1006}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\,e^{\mathrm{i}\widetilde{\Omega}\tau}(n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1007}$$
$$= \int_0^\infty\int_0^\infty J(\omega)(n(\omega)+1)\,\mathrm{e}^{\mathrm{i}\widetilde{\Omega}\tau - \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1008}$$
$$= \int_0^\infty J(\omega)(n(\omega)+1)\,\pi\delta\left(\widetilde{\Omega} - \omega\right)\mathrm{d}\omega \tag{1009}$$
$$= \pi J\left(\widetilde{\Omega}\right)\left(n\left(\widetilde{\Omega}\right) + 1\right), \tag{1010}$$

$$K_{33}\left(-\widetilde{\Omega}\right) = \int_0^\infty\int_0^\infty J(\omega)\,G_+(\tau)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}\mathrm{d}\tau\mathrm{d}\omega \tag{1011}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}\left((n(\omega)+1)\,\mathrm{e}^{-\mathrm{i}\tau\omega} + n(\omega)\,e^{\mathrm{i}\tau\omega}\right)\mathrm{d}\tau\mathrm{d}\omega \tag{1012}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\,e^{-\mathrm{i}\widetilde{\Omega}\tau}n(\omega)\,\mathrm{e}^{\mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1013}$$
$$= \int_0^\infty\int_0^\infty J(\omega)\,n(\omega)\,\mathrm{e}^{-\mathrm{i}\widetilde{\Omega}\tau + \mathrm{i}\tau\omega}\mathrm{d}\tau\mathrm{d}\omega \tag{1014}$$
$$= \int_0^\infty J(\omega)\,n(\omega)\,\pi\delta\left(-\widetilde{\Omega} + \omega\right)\mathrm{d}\omega \tag{1015}$$
$$= \pi J\left(\widetilde{\Omega}\right)n\left(\widetilde{\Omega}\right). \tag{1016}$$

Here we have used $\int_0^\infty \mathrm{d}s\, e^{\pm i\varepsilon s} = \pi\delta(\varepsilon) \pm \mathrm{i}\frac{\mathrm{V.P.}}{\varepsilon}$, where V.P. denotes the Cauchy's principal value. Theses principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1003) lead us to obtain:

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\widetilde{\Omega}}{2}\left[\sigma_x, \rho_S(t)\right)] - \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S(t)\right] + n\left(\widetilde{\Omega}\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)\right]\right) \tag{1017}$$

$$- \frac{\pi}{8}J\left(\widetilde{\Omega}\right)\left(\left(n\left(\widetilde{\Omega}\right)+1\right)\left[\rho_S(t)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\widetilde{\Omega}\right)\left[\rho_S(t)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right). \tag{1018}$$

This is the same result than the equation $(S17)$, so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\mathrm{i}\frac{\Omega(\mathrm{t})}{2}\left[\sigma_x, \rho_S(t)\right)] - \frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right)+1\right)\left[\sigma_z, \left(\sigma_z + \mathrm{i}\sigma_y\right)\rho_S(t)\right] + n\left(\Omega(t)\right)\left[\sigma_z, \left(\sigma_z - \mathrm{i}\sigma_y\right)\rho_S(t)\right]\right) \tag{1019}$$

$$- \frac{\pi}{8}J\left(\Omega(t)\right)\left(\left(n\left(\Omega(t)\right)+1\right)\left[\rho_S(t)\left(\sigma_z + \mathrm{i}\sigma_y\right), \sigma_z\right] + n\left(\Omega(t)\right)\left[\rho_S(t)\left(\sigma_z - \mathrm{i}\sigma_y\right), \sigma_z\right]\right). \tag{1020}$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1021}$$

$$H_S(t) = \sum_n \varepsilon_n(t)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t)|n\rangle\langle m|, \tag{1022}$$

$$H_I = \sum_{nu\mathbf{k}} |n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} + g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right), \tag{1023}$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}. \tag{1024}$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{nu\mathbf{k}} |n\rangle\langle n|\omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right) \tag{1025}$$

At first let's obtain $e^{\pm V}$ under the transformation (1025), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)$, so the equation (1025) can be written as $V = \sum_n |n\rangle\langle n|\hat{\varphi}_n$, then we have:

$$
\begin{aligned}
e^{\pm V} &= e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} && (1026)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\left(\sum_n |n\rangle\langle n| \hat{\varphi}_n\right)^2}{2!} + ... && (1027)\\
&= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + ... && (1028)\\
&= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + ... && (1029)\\
&= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + ...\right) && (1030)\\
&= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} && (1031)
\end{aligned}
$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^{\dagger} - f_{nu'\mathbf{k}'}^{*} b_{u'\mathbf{k}'}\right] = 0$ for all $\mathbf{k}^{'}, \mathbf{k}$ and u, u' then we can proof using the Zassenhaus formula and defining $D\left(\pm\alpha_{nu\mathbf{k}}\right) = e^{\pm\left(\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - \alpha_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$
\begin{aligned}
e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)} &= \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)} && (1032)\\
&= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1}\left(f_{nu\mathbf{k}} b_{u\mathbf{k}}^{\dagger} - f_{nu\mathbf{k}}^{*} b_{u\mathbf{k}}\right)}\right) && (1033)\\
&= \prod_u \left(\prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right)\right) && (1034)\\
&= \prod_{u\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) && (1035)\\
&= \prod_u B_{nu\pm} && (1036)\\
B_{nu\pm} &\equiv \prod_{\mathbf{k}} D\left(\pm\alpha_{nu\mathbf{k}}\right) && (1037)
\end{aligned}
$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^{V} = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^{V} = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$
e^{V} A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right) \qquad (1038)
$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1021):

$$
\begin{aligned}
\overline{|0\rangle\langle 0|} &= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), && (1039)\\
&= \prod_u B_{0u+} |0\rangle\langle 0|0\rangle\langle 0|0\rangle\langle 0| \prod_u B_{0u-}, && (1040)\\
&= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, && (1041)\\
&= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} && (1042)\\
&= |0\rangle\langle 0| \prod_u \mathbb{I} && (1043)\\
&= |0\rangle\langle 0|. && (1044)\\
\overline{|m\rangle\langle n|} &= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), && (1045)\\
&= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n|n\rangle\langle n| \prod_u B_{nu-}, && (1046)\\
&= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, && (1047)\\
&= |m\rangle\langle n| \prod_u \left(B_{mu+} B_{nu-}\right),\ m \neq n, && (1048)\\
&= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D\left(\alpha_{mu\mathbf{k}}\right) \prod_{\mathbf{k}} D\left(-\alpha_{nu\mathbf{k}}\right)\right), && (1049)\\
&= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}}\right) D\left(-\alpha_{nu\mathbf{k}}\right)\right), && (1050)\\
&= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). && (1051)\\
\prod_u \left(B_{mu+} B_{nu-}\right) &= \prod_{u\mathbf{k}} \left(D\left(\alpha_{mu\mathbf{k}} - \alpha_{nu\mathbf{k}}\right) \exp\left(\frac{1}{2}\left(-\alpha_{mu\mathbf{k}}\alpha^*_{nu\mathbf{k}} + \alpha^*_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}\right)\right)\right). && (1052)\\
\overline{\textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}} &= \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu+}\right) \textstyle\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \left(\textstyle\sum_n |n\rangle\langle n| \prod_u B_{nu-}\right), && (1053)\\
&= \left(|0\rangle\langle 0| \textstyle\prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \ldots\right)\left(\textstyle\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}}\right)\left(|0\rangle\langle 0| \textstyle\prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \ldots\right), && (1054)\\
&= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} \prod_u B_{1u-} + \ldots, && (1055)\\
&= |0\rangle\langle 0| \textstyle\prod_u B_{0u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} + \ldots\right) \prod_u B_{1u-} + \ldots && (1056)\\
&= |0\rangle\langle 0| \left(\textstyle\prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{0u-} + \ldots\right) && (1057)\\
&+ |1\rangle\langle 1| \left(\textstyle\prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b^\dagger_{0\mathbf{k}} b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b^\dagger_{1\mathbf{k}} b_{1\mathbf{k}} \prod_u B_{1u-} + \ldots\right) + \ldots && (1058)\\
&= |0\rangle\langle 0| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) && (1059)\\
&+ |1\rangle\langle 1| \left(\textstyle\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b^\dagger_{0\mathbf{k}} - \frac{v^*_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}}\right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b^\dagger_{1\mathbf{k}} - \frac{v^*_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right)\left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}}\right) + \ldots\right) + \ldots && (1060)\\
&= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) + \ldots && (1061)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) && (1062)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\omega_{u\mathbf{k}} \left(b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b^\dagger_{u\mathbf{k}} - \frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} b_{u\mathbf{k}} + \left|\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right|^2\right)\right) && (1063)\\
&= \sum_{nu\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} - v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right) && (1064)\\
&= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b^\dagger_{u\mathbf{k}} b_{u\mathbf{k}} + \sum_{nu\mathbf{k}} |n\rangle\langle n| \left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}} - \left(v_{nu\mathbf{k}} b^\dagger_{u\mathbf{k}} + v^*_{nu\mathbf{k}} b_{u\mathbf{k}}\right)\right) && (1065)
\end{aligned}
$$

The transformed Hamiltonians of the equations (1022) to (1024) written in terms of (1039) to (1063) are:

$$\overline{H_S\left(t\right)} = \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} \tag{1066}$$

$$= \overline{\sum_n \varepsilon_n\left(t\right)|n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|} \tag{1067}$$

$$= \sum_n \varepsilon_n\left(t\right)|n\rangle\langle n| + \sum_{n\neq m} V_{nm}\left(t\right)|n\rangle\langle m|\prod_u\left(B_{mu+}B_{nu-}\right) \tag{1068}$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) \tag{1069}$$

$$= \left(\sum_n |n\rangle\langle n|\prod_u B_{nu+}\right)\left(\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)+...\right)\left(\sum_n |n\rangle\langle n|\prod_u B_{nu-}\right) \tag{1070}$$

$$= \prod_u B_{0u+}\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{0u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{0u-}+\prod_u B_{1u+}\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{1u\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\prod_u B_{1u-}+\ldots \tag{1071}$$

$$=\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\prod_u B_{0u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{0u-}+g_{0u\mathbf{k}}^{*}\prod_u B_{0u+}b_{u\mathbf{k}}\prod_u B_{0u-}\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\prod_u B_{1u+}b_{u\mathbf{k}}^{\dagger}\prod_u B_{1u-}+g_{1u\mathbf{k}}^{*}\prod_u B_{1u+}b_{u\mathbf{k}}\prod_u B_{1u-}\right)+\ldots \tag{1072}$$

$$=\sum_{u\mathbf{k}}|0\rangle\langle 0|\left(g_{0u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{0u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{0u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\sum_{u\mathbf{k}}|1\rangle\langle 1|\left(g_{1u\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{1u\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{1u\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)+\ldots \tag{1073}$$

$$= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}\left(b_{u\mathbf{k}}^{\dagger}-\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}\right)+g_{nu\mathbf{k}}^{*}\left(b_{u\mathbf{k}}-\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1074}$$

$$= \sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1075}$$

$$\overline{H_B} = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1076}$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H}=\sum_n \varepsilon_n(t)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_u(B_{mu+}B_{nu-})+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(v_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+v_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}\right)\right) \tag{1077}$$

$$+\sum_{nu\mathbf{k}}|n\rangle\langle n|\left(g_{nu\mathbf{k}}b_{u\mathbf{k}}^{\dagger}+g_{nu\mathbf{k}}^{*}b_{u\mathbf{k}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1078}$$

Let's define the following functions:

$$R_n\left(t\right) = \sum_{u\mathbf{k}}\left(\frac{\left|v_{nu\mathbf{k}}\right|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}^{*}}{\omega_{u\mathbf{k}}}+g_{nu\mathbf{k}}^{*}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1079}$$

$$B_{z,n}\left(t\right) = \sum_{u\mathbf{k}}\left(\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)b_{u\mathbf{k}}^{\dagger}+\left(g_{nu\mathbf{k}}-v_{nu\mathbf{k}}\right)^{*}b_{u\mathbf{k}}\right) \tag{1080}$$

Using the previous functions we have that (1077) can be re-written in the following way:

$$\overline{H}=\sum_n \varepsilon_n(t)|n\rangle\langle n|+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\prod_u(B_{mu+}B_{nu-})+\sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}}+\sum_n R_n(t)|n\rangle\langle n|+\sum_n B_{z,n}(t)|n\rangle\langle n| \tag{1081}$$

$$\tag{1082}$$

Now in order to separate the elements of the hamiltonian (1082) let's follow the references of the equations (225) and (236) to separate the hamiltonian, before proceding to do this we need to consider the term of the form:

$$\langle\prod_{u}(B_{mu+}B_{nu-})\rangle_{\overline{H_0}}=\langle\prod_{u\mathbf{k}}\left(D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*}+\alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right)\rangle_{\overline{H_0}} \tag{1083}$$

$$= \left(\prod_{u\mathbf{k}}\exp\left(\frac{1}{2}(-\alpha_{mu\mathbf{k}}\alpha_{nu\mathbf{k}}^{*}+\alpha_{mu\mathbf{k}}^{*}\alpha_{nu\mathbf{k}})\right)\right)\langle\prod_{u\mathbf{k}}D(\alpha_{mu\mathbf{k}}-\alpha_{nu\mathbf{k}})\rangle_{\overline{H_0}} \tag{1084}$$

$$= \left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{mu\mathbf{k}}^{*}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v_{nu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1085}$$

$$\equiv B_{nm} \tag{1086}$$

$$\langle\prod_{u}(B_{nu+}B_{mu-})\rangle_{\overline{H_0}}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v_{nu\mathbf{k}}^{*}v_{mu\mathbf{k}}-v_{nu\mathbf{k}}v_{mu\mathbf{k}}^{*})}{2\omega_{u\mathbf{k}}^{2}}\right)\right)\prod_{u}\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^{2}}{\omega_{u\mathbf{k}}^{2}}\coth\left(\frac{\beta\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1087}$$

$$= B_{nm}^{*} \tag{1088}$$

Following the reference [4] we define:

$$J_{nm} = \prod_{u}(B_{mu+}B_{nu-}) - B_{nm} \tag{1089}$$

As we can see:

$$J_{nm}^{\dagger} = \left(\prod_{u}(B_{mu+}B_{nu-}) - B_{nm}\right)^{\dagger} \tag{1090}$$

$$= \prod_{u}(B_{nu+}B_{mu-}) - B_{nm}^{*} \tag{1091}$$

$$= \prod_{u}(B_{nu+}B_{mu-}) - B_{mn} \tag{1092}$$

$$= J_{mn} \tag{1093}$$

We can separate the Hamiltonian (1082) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}(t)} = \sum_{n}(\varepsilon_{n}(t)+R_{n})|n\rangle\langle n| + \sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|B_{nm} \tag{1094}$$

$$\overline{H_{\bar{I}}} = \sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|J_{nm} + \sum_{n}B_{z,n}(t)|n\rangle\langle n|, \tag{1095}$$

$$\overline{H_{\bar{B}}} = \sum_{u\mathbf{k}}\omega_{u\mathbf{k}}b_{u\mathbf{k}}^{\dagger}b_{u\mathbf{k}} \tag{1096}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_{\bar{S}}(t)+H_{\bar{B}}}\right)}\right)\right) + \langle\overline{H_{\bar{I}}}\rangle_{\overline{H_{\bar{S}}(t)+H_{\bar{B}}}} + O\left(\left\langle\overline{H_{\bar{I}}^{2}}\right\rangle_{\overline{H_{\bar{S}}(t)+H_{B}}}\right). \tag{1097}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_{\bar{S}}(t)}}\right) = C\left(R_{0}, R_{1}, ..., R_{d-1}, B_{01}, ..., B_{0(d-1)}, ..., B_{(d-2)(d-1)}\right)$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nu\mathbf{k}}\}$. Given that the numbers $v_{nu\mathbf{k}}$ are complex then we can separate them as $v_{nu\mathbf{k}} = v_{nu\mathbf{k}}^{\Re} + \mathrm{i}v_{nu\mathbf{k}}^{\Im}$. So our approach will be based on the derivation respect to $v_{nu\mathbf{k}}^{\Re}$ and $v_{nu\mathbf{k}}^{\Im}$. The Hamiltonian $\overline{H_{\bar{S}}(t)}$ can be written like:

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)\right)|n\rangle\langle n| \tag{1098}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1099}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}-\frac{g_{nu\mathbf{k}}v^*_{nu\mathbf{k}}+g^*_{nu\mathbf{k}}v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1100}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1101}$$

$$=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1102}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1103}$$

$$v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}}=\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)-\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right)\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right) \tag{1104}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1105}$$

$$-\left(v^{\Re}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}+v^{\Im}_{mu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}\right) \tag{1106}$$

$$=2\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right) \tag{1107}$$

$$\overline{H_{\bar{S}}(t)}=\sum_n\left(\varepsilon_n(t)+\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega_{u\mathbf{k}}}-\frac{(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right)\right)|n\rangle\langle n| \tag{1108}$$

$$+\sum_{n\neq m}V_{nm}(t)|n\rangle\langle m|\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1109}$$

$$|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2=(v_{mu\mathbf{k}}-v_{nu\mathbf{k}})(v_{mu\mathbf{k}}-v_{nu\mathbf{k}})^* \tag{1110}$$

$$=|v_{mu\mathbf{k}}|^2+|v_{nu\mathbf{k}}|^2-(v_{nu\mathbf{k}}v^*_{mu\mathbf{k}}+v^*_{nu\mathbf{k}}v_{mu\mathbf{k}}) \tag{1111}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-\left(v^{\Re}_{nu\mathbf{k}}+\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}-\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1112}$$

$$-\left(v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}\right)\left(v^{\Re}_{mu\mathbf{k}}+\mathrm{i}v^{\Im}_{mu\mathbf{k}}\right) \tag{1113}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}\right)^2+\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2\left(v^{\Re}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}+v^{\Im}_{nu\mathbf{k}}v^{\Im}_{mu\mathbf{k}}\right) \tag{1114}$$

$$=\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2 \tag{1115}$$

$$R_n(t)=\sum_{u\mathbf{k}}\left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}}-\left(g_{nu\mathbf{k}}\frac{v^*_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}+g^*_{nu\mathbf{k}}\frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right)\right) \tag{1116}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-(g_{nu\mathbf{k}}+g^*_{nu\mathbf{k}})v^{\Re}_{nu\mathbf{k}}-\mathrm{i}v^{\Im}_{nu\mathbf{k}}(g^*_{nu\mathbf{k}}-g_{nu\mathbf{k}})}{\omega_{u\mathbf{k}}}\right) \tag{1117}$$

$$=\sum_{u\mathbf{k}}\left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{nu\mathbf{k}}\right)^2-2g^{\Re}_{nu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}-2g^{\Im}_{nu\mathbf{k}}v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) \tag{1118}$$

$$B_{mn}=\left(\prod_{u\mathbf{k}}\exp\left(\frac{(v^*_{mu\mathbf{k}}v_{nu\mathbf{k}}-v_{mu\mathbf{k}}v^*_{nu\mathbf{k}})}{2\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{|v_{mu\mathbf{k}}-v_{nu\mathbf{k}}|^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1119}$$

$$=\left(\prod_{u\mathbf{k}}\exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}}v^{\Re}_{mu\mathbf{k}}-v^{\Im}_{mu\mathbf{k}}v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)\prod_u\exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\frac{\left(v^{\Re}_{mu\mathbf{k}}-v^{\Re}_{nu\mathbf{k}}\right)^2+\left(v^{\Im}_{mu\mathbf{k}}-v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) \tag{1120}$$

Then we can obtain using the chain rule that:

$$
\begin{aligned}
\frac{\partial R_{n'}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Re}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1121) \\
&= \frac{2v^{\Re}_{nu\mathbf{k}} - 2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1122) \\
&= 2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1123) \\
\frac{\partial R_{n'}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\partial}{\partial v^{\Im}_{nu\mathbf{k}}} \sum_{u\mathbf{k}} \left(\frac{\left(v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{nu\mathbf{k}}\right)^2 - 2g^{\Re}_{nu\mathbf{k}} v^{\Re}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}} v^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) && (1124) \\
&= \frac{2v^{\Im}_{nu\mathbf{k}} - 2g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1125) \\
&= 2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \delta_{nn'} && (1126)
\end{aligned}
$$

Given that:

$$
\begin{aligned}
\ln B_{mn} &= \ln\left(\left(\prod_{u\mathbf{k}} \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \prod_u \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right)\right) && (1127) \\
&= \sum_{u\mathbf{k}} \ln \exp\left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_u \ln \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1128) \\
&= \sum_{u\mathbf{k}} \left(\frac{\mathrm{i}\left(v^{\Im}_{nu\mathbf{k}} v^{\Re}_{mu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}} v^{\Re}_{nu\mathbf{k}}\right)}{\omega^2_{u\mathbf{k}}}\right) + \sum_{u\mathbf{k}} \left(-\frac{1}{2} \frac{\left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)^2 + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)^2}{\omega^2_{u\mathbf{k}}} \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)\right) && (1129) \\
\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} &= \frac{-\mathrm{i} v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1130) \\
\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} &= \frac{\mathrm{i} v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} && (1131) \\
\frac{\partial \ln B_{mn}}{\partial a} &= \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} && (1132) \\
\frac{\partial B_{mn}}{\partial a} &= B_{mn} \frac{\partial \ln B_{mn}}{\partial a} && (1133) \\
\frac{\partial B_{mn}}{\partial a} &= \frac{\partial \left(B_{nm}\right)^{\dagger}}{\partial a} && (1134)
\end{aligned}
$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}} \tag{1135}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} - \left(v^{\Re}_{nu\mathbf{k}} - v^{\Re}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1136}$$

$$= B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1137}$$

$$\frac{\partial B_{nm}}{\partial v^{\Re}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Re}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1138}$$

$$= \left(B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)^{\dagger} \tag{1139}$$

$$= B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1140}$$

$$\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} = B_{mn}\frac{\partial \ln B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}} \tag{1141}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1142}$$

$$= B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1143}$$

$$\frac{\partial B_{nm}}{\partial v^{\Im}_{nu\mathbf{k}}} = \left(\frac{\partial B_{mn}}{\partial v^{\Im}_{nu\mathbf{k}}}\right)^{\dagger} \tag{1144}$$

$$= \left(B_{mn}\right)^{\dagger} \tag{1145}$$

$$= B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} + \left(v^{\Im}_{mu\mathbf{k}} - v^{\Im}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1146}$$

Introducing this derivates in the equation (1121) give us:

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Re}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Re}_{nu\mathbf{k}} - g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1147}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + \left(v^{\Re}_{mu\mathbf{k}} - v^{\Re}_{nu\mathbf{k}}\right)\coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1148}$$

$$= 0 \tag{1149}$$

We can obtain the variational parameters:

$$- 2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v^{\Re}_{nu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) \tag{1150}$$

$$= - \frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1151}$$

$$v^{\Re}_{nu\mathbf{k}} = \frac{\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{2g^{\Re}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}} + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)} \tag{1152}$$

$$= \frac{2g^{\Re}_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v^{\Im}_{mu\mathbf{k}} + v^{\Re}_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n} - \sum_{n\neq m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right) + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1153}$$

Let’s consider the imaginary part of the variation parameters

$$\frac{\partial A_{\mathrm{B}}}{\partial v^{\Im}_{nu\mathbf{k}}} = \frac{\partial A_{\mathrm{B}}}{\partial R_n}\left(2\frac{v^{\Im}_{nu\mathbf{k}} - g^{\Im}_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}\right) + \sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right. \tag{1154}$$

$$\left. + \frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v^{\Re}_{mu\mathbf{k}} - \left(v^{\Im}_{nu\mathbf{k}} - v^{\Im}_{mu\mathbf{k}}\right)\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega^2_{u\mathbf{k}}}\right)\right) \tag{1155}$$

$$= 0 \tag{1156}$$

Rearranging we obtain

$$- 2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{v_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{v_{nu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right) \tag{1157}$$

$$= -2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}+\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right) \tag{1158}$$

$$v_{nu\mathbf{k}}^{\Im}=\frac{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{g_{nu\mathbf{k}}^{\Im}}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\frac{-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\frac{\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)\right)}{2\frac{\partial A_{\mathrm{B}}}{\partial R_n}\frac{1}{\omega_{u\mathbf{k}}}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\frac{\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2}\right)} \tag{1159}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1160}$$

$$v_{nu\mathbf{k}}=v_{nu\mathbf{k}}^{\Re}+\mathrm{i}v_{nu\mathbf{k}}^{\Im} \tag{1161}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1162}$$

$$\mathrm{i}\frac{2g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1163}$$

$$=\frac{2g_{nu\mathbf{k}}^{\Re}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}+2\mathrm{i}g_{nu\mathbf{k}}^{\Im}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1164}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Im}+v_{mu\mathbf{k}}^{\Re}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1165}$$

$$-\mathrm{i}\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(-\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(\mathrm{i}v_{mu\mathbf{k}}^{\Re}+v_{mu\mathbf{k}}^{\Im}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1166}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1167}$$

$$-\frac{\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1168}$$

$$=\frac{2g_{nu\mathbf{k}}\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\left(v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\left(-v_{mu\mathbf{k}}+v_{mu\mathbf{k}}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)\right)}{2\omega_{u\mathbf{k}}\frac{\partial A_{\mathrm{B}}}{\partial R_n}-\sum_{n<m}\left(\frac{\partial A_{\mathrm{B}}}{\partial B_{nm}}B_{nm}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)+\frac{\partial A_{\mathrm{B}}}{\partial B_{mn}}B_{mn}\coth\left(\frac{\beta_u\omega_{u\mathbf{k}}}{2}\right)\right)} \tag{1169}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0)=|0\rangle\langle 0|\otimes\rho_B$, as we can see this state is independent of the variation transformation:

$$e^V\rho(0)\,e^{-V}=\left(\sum_n|n\rangle\langle n|B_{n+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(\sum_n|n\rangle\langle n|B_{n+}\right) \tag{1170}$$

$$0=\left(B_0^+|0\rangle\langle 0|B_0^-\right)\otimes\rho_B \tag{1171}$$

$$0=\rho(0) \tag{1172}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\, O U(t) \tag{1173}$$
$$U(t) \equiv \mathcal{T}\text{exp}\left(-\mathrm{i}\int_0^t \mathrm{d}t' \overline{H_S}(t')\right). \tag{1174}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\, U(t)\,, \text{ where} \tag{1175}$$
$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1176}$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \tag{1177}$$
$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \tag{1178}$$
$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \tag{1179}$$
$$B_{nm,x} = \frac{B_{nm} - B_{mn}}{2\mathrm{i}} \tag{1180}$$

We can proof that $B_{nm} = B^{\dagger}_{mn}$

$$B^{\dagger}_{mn} = \left(B_{m+}B_{n-} - B_m B_n\right)^{\dagger} \tag{1181}$$
$$= B^{\dagger}_{n-}B^{\dagger}_{m+} - B_n B_m \tag{1182}$$
$$= B_{n+}B_{m-} - B_n B_m \tag{1183}$$
$$= B_{nm} \tag{1184}$$

So we can say that the set of matrices (1177) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1177) give us.

$$\overline{H_I} = \sum_{n\neq m} V_{nm}(t)\,|n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t)\,|n\rangle\langle n|, \tag{1185}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(V_{nm}(t)\,|n\rangle\langle m| B_{nm} + V_{mn}(t)\,|m\rangle\langle n| B_{mn}\right) \tag{1186}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right) B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right) + \mathrm{i}V^{\Im}_{nm}(t)\, B_{nm}\left(\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2}\right)\right. \tag{1187}$$
$$\left. + \Re\left(V_{nm}(t)\right) B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) - \mathrm{i}V^{\Im}_{nm}(t)\, B_{mn}\left(\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right)\right) \tag{1188}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} + B_{mn}}{2}\right) + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}\frac{\mathrm{i}\left(B_{mn} - B_{nm}\right)}{2}\right. \tag{1189}$$
$$\left. + \mathrm{i}\Im\left(V_{nm}(t)\right)\sigma_{nm,x}\left(\frac{B_{nm} - B_{mn}}{2}\right) + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}\left(\frac{B_{nm} + B_{mn}}{2}\right)\right) \tag{1190}$$
$$= \sum_n B_{z,n}(t)\,|n\rangle\langle n| + \sum_{n<m}\left(\Re\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,x} - \Im\left(V_{nm}(t)\right)\sigma_{nm,x}B_{nm,y} + \Re\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,y}\right. \tag{1191}$$
$$\left. + \Im\left(V_{nm}(t)\right)\sigma_{nm,y}B_{nm,x}\right) \tag{1192}$$

Let's define the set

$$P = \left\{(n,m) \in \mathbb{N}^2 | 0 \le n,m \le d-1 \,\wedge\, (n = m \vee n < m)\right\} \tag{1193}$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1-\delta_{mn}) \tag{1194}$$
$$A_{2,nm}(t) = \sigma_{nm,y}(1-\delta_{mn}) \tag{1195}$$
$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \tag{1196}$$
$$A_{4,nm}(t) = A_{2,mn}(t) \tag{1197}$$
$$A_{5,nm}(t) = A_{1,nm}(t) \tag{1198}$$
$$B_{1,nm}(t) = B_{nm,x} \tag{1199}$$
$$B_{2,nm}(t) = B_{nm,y} \tag{1200}$$
$$B_{3,nm}(t) = B_{z,n}(t) \tag{1201}$$
$$B_{4,nm}(t) = B_{1,nm}(t) \tag{1202}$$
$$B_{5,nm}(t) = B_{2,nm}(t) \tag{1203}$$
$$C_{1,nm}(t) = \Re(V_{nm}(t)) \tag{1204}$$
$$C_{2,nm}(t) = C_{1,nm}(t) \tag{1205}$$
$$C_{3,nm}(t) = 1 \tag{1206}$$
$$C_{4,nm}(t) = \Im(V_{nm}(t)) \tag{1207}$$
$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \tag{1208}$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J, p\in P} C_{jp}(t)\left(A_{jp}\otimes B_{jp}(t)\right) \tag{1209}$$

Here $J=\{1,2,3,4,5\}$and P the set defined in (1193).
We write the interaction Hamiltonian transformed under (1173) as:

$$\widetilde{H_I}(t) = \sum_{j\in J, p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1210}$$
$$\widetilde{A_{jp}}(t) = U^{\dagger}(t)\,A_{jp}U(t) \tag{1211}$$
$$\widetilde{B_{jp}}(t) = \mathrm{e}^{\mathrm{i}H_B t}B_{jp}(t)\,(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} \tag{1212}$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1213}$$

Replacing the equation (1210) in (1213) we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1214}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \left[\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1215}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right), \sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1216}$$

$$\left.-\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\right]\mathrm{d}s \tag{1217}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1218}$$

$$-\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right) \tag{1219}$$

$$-\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right) \tag{1220}$$

$$\left.+\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J,p'\in P} C_{j'p'}(s)\left(\widetilde{A_{j'p'}}(s)\otimes\widetilde{B_{j'p'}}(s)\right)\sum_{j\in J,p\in P} C_{jp}(t)\left(\widetilde{A_{jp}}(t)\otimes\widetilde{B_{jp}}(t)\right)\right)\mathrm{d}s \tag{1221}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B \tag{1222}$$

$$= \left\langle\widetilde{B_{jp}}(\tau)\,\widetilde{B_{j'p'}}(0)\right\rangle_B \tag{1223}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right) = \left\langle\widetilde{B_{jp}}(t)\,\widetilde{B_{j'p'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jp}}(t)$ and elements related to the system given by $\widetilde{A_{jp}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jp}}(t)\,\widetilde{B_{j'p'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jp}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'p'}}(t)\right)$. The correlation functions relevant of the master equation (1221) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) = \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1224}$$

$$= \left\langle\widetilde{B_{jp}}\left(0\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1225}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1226}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\rho_B\widetilde{B_{j'p'}}\left(s\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1227}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1228}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1229}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1230}$$

$$\mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\rho_B\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\rho_B\right) \tag{1231}$$

$$= \left\langle\widetilde{B_{jp}}\left(t\right)\widetilde{B_{j'p'}}\left(s\right)\right\rangle_B \tag{1232}$$

$$= \left\langle\widetilde{B_{jp}}\left(\tau\right)\widetilde{B_{j'p'}}\left(0\right)\right\rangle_B \tag{1233}$$

$$= \Lambda_{jpj'p'}\left(\tau\right) \tag{1234}$$

$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right) = \mathrm{Tr}_B\left(\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\rho_B\right) \tag{1235}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(s\right)\widetilde{B_{jp}}\left(t\right)\right\rangle_B \tag{1236}$$

$$= \left\langle\widetilde{B_{j'p'}}\left(-\tau\right)\widetilde{B_{jp}}\left(0\right)\right\rangle_B \tag{1237}$$

$$= \Lambda_{j'p'jp}\left(-\tau\right) \tag{1238}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1214)and (1221) and using the equations (1224)-(1238) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}\left(t\right)}{\mathrm{d}t} = -\int_0^t\sum_{j,j',p,p'}\Big(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\Big(\Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right) - \Lambda_{j'p'jp}\left(-\tau\right)\widetilde{A_{jp}}\left(t\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\Big) \tag{1239}$$

$$+C_{jp}\left(t\right)C_{j'p'}\left(s\right)\Big(\Lambda_{j'p'jp}\left(-\tau\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{A_{jp}}\left(t\right) - \Lambda_{jpj'p'}\left(\tau\right)\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{jp}}\left(t\right)\Big)\Big)\,\mathrm{d}s \tag{1240}$$

$$= -\int_0^t\sum_{jj'pp'}\Big(C_{jp}\left(t\right)C_{j'p'}\left(s\right)\Big(\Lambda_{jpj'p'}\left(\tau\right)\Big[\widetilde{A_{jp}}\left(t\right),\widetilde{A_{j'p'}}\left(s\right)\widetilde{\widetilde{\rho_S}}\left(t\right)\Big] + \Lambda_{j'p'jp}\left(-\tau\right)\Big[\widetilde{\widetilde{\rho_S}}\left(t\right)\widetilde{A_{j'p'}}\left(s\right),\widetilde{A_{jp}}\left(t\right)\Big]\Big)\Big) \tag{1241}$$

Rearranging and identofying the commutators allow us to write a more simplified version

$$\frac{\mathrm{d}\,\overline{\rho_S}\left(t\right)}{\mathrm{d}t} = -\int_0^t\sum_{jj'pp'}\left(C_{jp}\left(t\right)C_{j'p'}\left(t-\tau\right)\left(\Lambda_{jpj'p'}\left(\tau\right)\left[A_{jp}\left(t\right),A_{j'p'}\left(t-\tau,t\right)\overline{\rho_S}\left(t\right)\right] + \Lambda_{j'p'jp}\left(-\tau\right)\left[\overline{\rho_S}\left(t\right)A_{j'p'}\left(t-\tau,t\right),A_{jp}\left(t\right)\right]\right)\right)\mathrm{d}\tau - \mathrm{i}\left[H_S\left(t\right),\overline{\rho_S}\left(t\right)\right] \tag{1242}$$

For this case we used that $A_{jp}\left(t-\tau,t\right) = U\left(t\right)U^{\dagger}\left(t-\tau\right)A_{jp}\left(t\right)U\left(t-\tau\right)U^{\dagger}\left(t\right)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \tag{1243}$$
$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|, \tag{1244}$$
$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right), \tag{1245}$$
$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}. \tag{1246}$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n|\right)\left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)\right) \tag{1247}$$

At first let's obtain e^V under the transformation (1247), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}} \tag{1248}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\left(\sum_{n=1} |n\rangle\langle n|\hat{\varphi}\right)^2}{2!} + ... \tag{1249}$$
$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n|\hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n|\hat{\varphi}^2}{2!} + ... \tag{1250}$$
$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n|\left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + ...\right) \tag{1251}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|e^{\hat{\varphi}} \tag{1252}$$
$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+} \tag{1253}$$

Given that $\left[b_{\mathbf{k}'}^{\dagger} - b_{\mathbf{k}'}, b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} = \prod_{\mathbf{k}} e^{\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\left(b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}\right)} \tag{1254}$$
$$= \prod_{\mathbf{k}} D\left(\pm\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1255}$$
$$= B_{\pm} \tag{1256}$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B_{-}$ because this form imposes that $e^{-V}e^{V} = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^{V}Ae^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right) \tag{1257}$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1243):

$$
\begin{aligned}
\overline{|0\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right), && (1258)\\
&= |0\rangle\langle 0|, && (1259)\\
\overline{|m\rangle\langle n|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right), && (1260)\\
&= |m\rangle\langle m|B^{+}|m\rangle\langle n|n\rangle\langle n|B^{-}, && (1261)\\
&= |m\rangle\langle n|,\ m \neq 0,\ n \neq 0, && (1262)\\
\overline{|0\rangle\langle m|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right), && (1263)\\
&= |0\rangle\langle m|B^{-}\ m \neq 0, && (1264)\\
\overline{|m\rangle\langle 0|} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right) && (1265)\\
&= |0\rangle\langle m|B^{+}\ m \neq 0, && (1266)\\
\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} &= \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{+}\right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n|B^{-}\right) && (1267)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B^{+} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} B^{-} && (1268)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B^{+} b_{\mathbf{k}}^{\dagger} B^{-}\right) \left(B^{+} b_{\mathbf{k}} B^{-}\right) && (1269)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1270)\\
&= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) && (1271)\\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\right) && (1272)\\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) && (1273)
\end{aligned}
$$

The transformed Hamiltonians of the equations (1244) to (1246) written in terms of (1258) to (1273) are:

$$\overline{H_{\bar{S}}(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1274}$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n\neq m} V_{nm}(t) |n\rangle\langle m|} \tag{1275}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \overline{\sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|\right)} + \overline{\sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n|} \tag{1276}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|}\right) + \sum_{m.n\neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \tag{1277}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) B^- |0\rangle\langle n| + V_{n0}(t) B^+ |n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1278}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1279}$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+\right) \left(\left(\sum_{n=0} \mu_n(t) |n\rangle\langle n|\right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right)\right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^-\right) \tag{1280}$$

$$= \left(\mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B^+\right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^-\right) \tag{1281}$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B^+ \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) B^- \tag{1282}$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1283}$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1284}$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1285}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1286}$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \tag{1287}$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} \left(V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+\right) + \sum_{m.n\neq 0} V_{mn}(t) |m\rangle\langle n| \tag{1288}$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1289}$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1290}$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{1291}$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}}\right) \tag{1292}$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \tag{1293}$$

Using the previous functions we have that (1290) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|B^- + V_{n0}(t)|n\rangle\langle 0|B^+\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| \quad (1294)$$

$$+\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} R_n|n\rangle\langle n| + \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) \quad (1295)$$

Now in order to separate the elements of the hamiltonian (1295) let's follow the references of the equations (236) and (225) to separate the hamiltonian like:

$$\overline{H_S(t)} = \sum_{n=0} \varepsilon_n(t)|n\rangle\langle n| + B\sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n| + V_{n0}(t)|n\rangle\langle 0|\right) + \sum_{m.n\neq 0} V_{mn}(t)|m\rangle\langle n| + \sum_{n=1} R_n|n\rangle\langle n| \quad (1296)$$

$$\overline{H_I} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)|n\rangle\langle 0|\left(B^+ - B\right)\right), \quad (1297)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (1298)$$

Here B is given by:

$$B = \left\langle B^+\right\rangle$$
$$= \left\langle B^-\right\rangle$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1299)$$

$$\sigma_{nm,y} = \mathrm{i}\left(|n\rangle\langle m| - |m\rangle\langle n|\right) \quad (1300)$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \quad (1301)$$

$$B_y = \frac{B^- - B^+}{2\mathrm{i}} \quad (1302)$$

Using this set of hermitian operators to write the Hamiltonians (1244)-(1246)

$$\overline{H_S(t)} = \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{m.n\neq 0} V_{mn}(t)\,|m\rangle\langle n| \tag{1303}$$

$$= \varepsilon_0(t)\,|0\rangle\langle 0| + B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{0<m<n}(V_{mn}(t)\,|m\rangle\langle n| + V_{nm}(t)\,|n\rangle\langle m|) \tag{1304}$$

$$+ \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1305}$$

$$= \sum_{0<m<n}\left(\left(\Re(V_{mn}(t)) + \mathrm{i}\Im(V_{mn}(t))\right)|m\rangle\langle n| + \left(\Re(V_{mn}(t)) - \mathrm{i}\Im(V_{mn}(t))\right)|n\rangle\langle m|\right) + \varepsilon_0(t)\,|0\rangle\langle 0| \tag{1306}$$

$$+ B\sum_{n=1}(V_{0n}(t)\,|0\rangle\langle n| + V_{n0}(t)\,|n\rangle\langle 0|) + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1307}$$

$$= \sum_{0<m<n}\left(\left(\Re(V_{nm}(t)) + \mathrm{i}\Im(V_{mn}(t))\right)\frac{\sigma_{nm,x} - \mathrm{i}\sigma_{nm,y}}{2} + \left(\Re(V_{nm}(t)) - \mathrm{i}\Im(V_{mn}(t))\right)\frac{\sigma_{nm,x} + \mathrm{i}\sigma_{nm,y}}{2}\right) \tag{1308}$$

$$+ B\sum_{n=1}\left(V_{0n}(t)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + V_{n0}(t)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) + \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1309}$$

$$= \sum_{0<m<n}\left(\Re(V_{nm}(t))\,\sigma_{nm,x} + \Im(V_{nm}(t))\,\sigma_{nm,y}\right) + B\sum_{n=1}\left(\Re(V_{0n}(t))\,\sigma_{0n,x} + \Im(V_{mn}(t))\,\sigma_{0n,y}\right) \tag{1310}$$

$$+ \varepsilon_0(t)\,|0\rangle\langle 0| + \sum_{n=1}(\varepsilon_n(t)+R_n)\,|n\rangle\langle n| \tag{1311}$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(V_{0n}(t)\,|0\rangle\langle n|\left(B^- - B\right) + V_{n0}(t)\,|n\rangle\langle 0|\left(B^+ - B\right)\right) \tag{1312}$$

$$= \sum_{n=1}\left(\left(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))\right)\left(B^- - B\right)\frac{\sigma_{0n,x} - \mathrm{i}\sigma_{0n,y}}{2} + \left(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))\right)\left(B^+ - B\right)\frac{\sigma_{0n,x} + \mathrm{i}\sigma_{0n,y}}{2}\right) \tag{1313}$$

$$+ \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1314}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(\left(B^- - B\right)\left(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))\right) + \left(B^+ - B\right)\left(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))\right)\right)\right. \tag{1315}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B\right)\left(\Re(V_{0n}(t)) - \mathrm{i}\Im(V_{0n}(t))\right) - \left(B^- - B\right)\left(\Re(V_{0n}(t)) + \mathrm{i}\Im(V_{0n}(t))\right)\right)\right) \tag{1316}$$

$$+ \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1317}$$

$$= \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\frac{\sigma_{0n,x}}{2}\left(B^+ + B^- - 2B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B^- - B - B^+ + B\right)\Im(V_{0n}(t))\right. \tag{1318}$$

$$\left. + \frac{\mathrm{i}\sigma_{0n,y}}{2}\left(\left(B^+ - B - B^- + B\right)\Re(V_{0n}(t)) + \mathrm{i}\left(B - B^- + B - B^+\right)\Im(V_{0n}(t))\right)\right) + \sum_{n=1} B_{z,n}|n\rangle\langle n| \tag{1319}$$

$$= \sum_{n=1} B_{z,n}|n\rangle\langle n| + \mu_0(t)\,|0\rangle\langle 0|\sum_{\mathbf{k}} g_{\mathbf{k}}\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) + \sum_{n=1}\left(\sigma_{0n,x}\left(B_x\Re(V_{0n}(t)) - B_y\Im(V_{0n}(t))\right)\right. \tag{1320}$$

$$\left. + \sigma_{0n,y}\left(B_y\Re(V_{0n}(t)) + B_x\Im(V_{0n}(t))\right)\right) \tag{1321}$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_{\mathrm{B}} \equiv -\frac{1}{\beta}\ln\left(\mathrm{Tr}\left(e^{-\beta\left(\overline{H_S+H_B}\right)}\right)\right)+\left\langle\overline{H_I}\right\rangle_{\overline{H_S+H_B}}+O\left(\left\langle\overline{H_I^2}\right\rangle_{\overline{H_S+H_B}}\right). \tag{1322}$$

Taking the equations (246)-(254) and given that $\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)=C\left(R_1,R_2,...,R_{d-1},B\right)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (254) and using the chain rule we obtain that:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\frac{\partial B}{\partial v_{\mathbf{k}}}+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\frac{\partial R_n}{\partial v_{\mathbf{k}}}, &(1323)\\ &= 0 &(1324)\end{aligned}$$

Let's recall the equations (1291) and (1293), we can write them in terms of the variational parameters

$$\begin{aligned}B &= \exp\left(-(1/2)\sum_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)\right) &(1325)\\ R_n &= \sum_{\mathbf{k}}\omega_{\mathbf{k}}^{-1}\left(v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}v_{\mathbf{k}}\right) &(1326)\end{aligned}$$

The derivates needed to obtain the set of variational parameter are given by:

$$\begin{aligned}\frac{\partial B}{\partial v_{\mathbf{k}}} &= -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B &(1327)\\ \frac{\partial R_n}{\partial v_{\mathbf{k}}} &= \omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1328)\end{aligned}$$

Introducing this derivates in the equation (1323) give us:

$$\begin{aligned}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial v_{\mathbf{k}}} &= \frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}\coth\left(\beta\omega_{\mathbf{k}}/2\right)B\right)+\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\omega_{\mathbf{k}}^{-1}\left(2v_{\mathbf{k}}-2\mu_n\left(t\right)g_{\mathbf{k}}\right) &(1329)\\ &= v_{\mathbf{k}}\left(\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth\left(\beta\omega_{\mathbf{k}}/2\right)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}\right)-\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right) &(1330)\end{aligned}$$

We can obtain the variational parameters:

$$\begin{aligned}v_{\mathbf{k}} &= \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{\frac{2}{\omega_{\mathbf{k}}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-\frac{\coth(\beta\omega_{\mathbf{k}}/2)B}{\omega_{\mathbf{k}}^2}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1331)\\ &= \frac{2g_{\mathbf{k}}\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}} &(1332)\end{aligned}$$

Now taking $v_{\mathbf{k}}=g_{\mathbf{k}}v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}\mu_n\left(t\right)}{2\omega_{\mathbf{k}}\sum_{n=1}\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial R_n}-B\coth\left(\beta\omega_{\mathbf{k}}/2\right)\frac{\partial\mathrm{Tr}\left(e^{-\beta\overline{H_S(t)}}\right)}{\partial B}}. \tag{1333}$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^{V}\rho(0)\,e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{+}\right)\left(|0\rangle\langle 0|\otimes\rho_B\right)\left(|0\rangle\langle 0| + \sum_{n=1}|n\rangle\langle n|B^{-}\right) \tag{1334}$$

$$0 = |0\rangle\langle 0|\otimes\rho_B \tag{1335}$$

$$0 = \rho(0) \tag{1336}$$

We transform any operator O into the interaction picture in the following way:

$$\widetilde{O} \equiv U^{\dagger}(t)\,OU(t) \tag{1337}$$

$$U(t) \equiv \mathcal{T}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\overline{H_S}(t')\right). \tag{1338}$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^{\dagger}(t)\,\overline{\rho_S}(t)\,U(t)\,,\ \text{where} \tag{1339}$$

$$\overline{\rho_S}(t) = \mathrm{Tr}_B\left(\bar{\rho}(t)\right) \tag{1340}$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I(t)} = B_{z,0}|0\rangle\langle 0| + \sum_{n=1}\left(\Re\left(V_{0n}(t)\right)B_x\sigma_{0n,x} + \Re\left(V_{0n}(t)\right)B_y\sigma_{0n,y} + B_{z,n}|n\rangle\langle n|\right. \tag{1341}$$

$$\left.+\Im\left(V_{0n}(t)\right)B_x\sigma_{0n,y} - \Im\left(V_{0n}(t)\right)B_y\sigma_{0n,x}\right) \tag{1342}$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}}\mu_0(t)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right) \tag{1343}$$

$$B_{z,n} = \sum_{\mathbf{k}}\left(g_{\mathbf{k}}\mu_n(t) - \omega_{\mathbf{k}}\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right)\left(b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}\right)\ \text{if} n \neq 0 \tag{1344}$$

Now consider the following set of operators:

$$
\begin{aligned}
A_{1n}(t) &= \sigma_{0n,x} && (1345)\\
A_{2n}(t) &= \sigma_{0n,y} && (1346)\\
A_{3n}(t) &= |n\rangle\langle n| && (1347)\\
A_{4n}(t) &= A_{2n}(t) && (1348)\\
A_{5n}(t) &= A_{1n}(t) && (1349)\\
B_{1n}(t) &= B_x && (1350)\\
B_{2n}(t) &= B_y && (1351)\\
B_{3n}(t) &= B_{z,n} && (1352)\\
B_{4n}(t) &= B_{1n}(t) && (1353)\\
B_{5n}(t) &= B_{2n}(t) && (1354)\\
C_{10}(t) &= 0 && (1355)\\
C_{20}(t) &= 0 && (1356)\\
C_{40}(t) &= 0 && (1357)\\
C_{50}(t) &= 0 && (1358)\\
C_{30}(t) &= 1 && (1359)\\
C_{1n}(t) &= \Re\left(V_{0n}(t)\right) && (1360)\\
C_{2n}(t) &= C_{1n}(t) && (1361)\\
C_{3n}(t) &= 1 && (1362)\\
C_{4n}(t) &= \Im\left(V_{0n}(t)\right) && (1363)\\
C_{5n}(t) &= -\Im\left(V_{0n}(t)\right) && (1364)
\end{aligned}
$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(A_{jn}\otimes B_{jn}(t)\right) \tag{1365}$$

Here $J = \{1,2,3,4,5\}$.

We write the interaction Hamiltonian transformed under (1337) as:

$$
\begin{aligned}
\widetilde{H_I}(t) &= \sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) && (1366)\\
\widetilde{A_i}(t) &= U^\dagger(t)\,A_i U(t) && (1367)\\
\widetilde{B_i}(t) &= \mathrm{e}^{\mathrm{i}H_B t}B_i(\mathrm{t})\,\mathrm{e}^{-\mathrm{i}H_B t} && (1368)
\end{aligned}
$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$), we obtain our master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho_S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t),\left[\widetilde{H_I}(s),\widetilde{\rho_S}(t)\,\rho_B\right]\right]\mathrm{d}s \tag{1369}$$

Replacing the equation (1366)in (1369)we can obtain:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{Tr}_B\left[\widetilde{H_I}(t), \left[\widetilde{H_I}(s), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1370}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \left[\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right), \widetilde{\widetilde{\rho_S}}(t)\rho_B\right]\right]\mathrm{d}s \tag{1371}$$

$$= -\int_0^t \mathrm{Tr}_B\left[\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right), \sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1372}$$

$$\left. -\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\right]\mathrm{d}s \tag{1373}$$

$$= -\int_0^t \mathrm{Tr}_B\left(\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\right. \tag{1374}$$

$$-\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right) \tag{1375}$$

$$-\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right) \tag{1376}$$

$$\left. +\widetilde{\widetilde{\rho_S}}(t)\rho_B\sum_{j'\in J}\sum_{n'=1} C_{j'n'}(s)\left(\widetilde{A_{j'n'}}(s)\otimes\widetilde{B_{j'n'}}(s)\right)\sum_{j\in J}\sum_{n=1} C_{jn}(t)\left(\widetilde{A_{jn}}(t)\otimes\widetilde{B_{jn}}(t)\right)\right)\mathrm{d}s \tag{1377}$$

In order to calculate the correlation functions we define:

$$\Lambda_{jnj'n'}(\tau) = \left\langle \widetilde{B_{jn}}(t)(t)\,\widetilde{B_{j'n'}}(t)(s)\right\rangle_B \tag{1378}$$

$$= \left\langle \widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1379}$$

Here $s \to t-\tau$ and $\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\rho_B\right) = \left\langle \widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B_{jn}}(t)$ and elements related to the system given by $\widetilde{A_{jn}}(t)$. The systems considered are in different Hilbert spaces so $\mathrm{Tr}\left(\widetilde{A_{jn}}(t)\,\widetilde{B_{j'n'}}(t)\right) = \mathrm{Tr}\left(\widetilde{A_{jn}}(t)\right)\mathrm{Tr}\left(\widetilde{B_{j'n'}}(t)\right)$. The correlation functions relevant of the master equation (1377) are:

$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right)=\left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1380}$$
$$=\left\langle\widetilde{B_{jn}}(0)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1381}$$
$$=\Lambda_{jnj'n'}(\tau) \tag{1382}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\rho_B\widetilde{B_{j'n'}}(s)\right)=\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1383}$$
$$=\left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1384}$$
$$=\left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1385}$$
$$=\Lambda_{j'n'jn}(-\tau) \tag{1386}$$
$$\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\rho_B\widetilde{B_{jn}}(t)\right)=\mathrm{Tr}_B\left(\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\,\rho_B\right) \tag{1387}$$
$$=\left\langle\widetilde{B_{jn}}(t)\,\widetilde{B_{j'n'}}(s)\right\rangle_B \tag{1388}$$
$$=\left\langle\widetilde{B_{jn}}(\tau)\,\widetilde{B_{j'n'}}(0)\right\rangle_B \tag{1389}$$
$$=\Lambda_{jnj'n'}(\tau) \tag{1390}$$
$$\mathrm{Tr}_B\left(\rho_B\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right)=\mathrm{Tr}_B\left(\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\,\rho_B\right) \tag{1391}$$
$$=\left\langle\widetilde{B_{j'n'}}(s)\,\widetilde{B_{jn}}(t)\right\rangle_B \tag{1392}$$
$$=\left\langle\widetilde{B_{j'n'}}(-\tau)\,\widetilde{B_{jn}}(0)\right\rangle_B \tag{1393}$$
$$=\Lambda_{j'n'jn}(-\tau) \tag{1394}$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1370)and (1377) and using the equations (1380)-(1394) we can re-write:

$$\frac{\mathrm{d}\widetilde{\widetilde{\rho_S}}(t)}{\mathrm{d}t}=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)-\Lambda_{j'n'jn}(-\tau)\,\widetilde{A_{jn}}(t)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\right)\right. \tag{1395}$$
$$\left.+C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{j'n'jn}(-\tau)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,\widetilde{A_{jn}}(t)-\Lambda_{jnj'n'}(\tau)\,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{jn}}(t)\right)\right)\mathrm{d}s \tag{1396}$$
$$=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(s)\left(\Lambda_{jnj'n'}(\tau)\left[\widetilde{A_{jn}}(t)\,,\widetilde{A_{j'n'}}(s)\,\widetilde{\widetilde{\rho_S}}(t)\right]+\Lambda_{j'n'jn}(-\tau)\left[\widetilde{\widetilde{\rho_S}}(t)\,\widetilde{A_{j'n'}}(s)\,,\widetilde{A_{jn}}(t)\right]\right)\right) \tag{1397}$$

$$\frac{\mathrm{d}\,\overline{\rho_S}(t)}{\mathrm{d}t}=-\int_0^t\sum_{j,j',n,n'}\left(C_{jn}(t)\,C_{j'n'}(t-\tau)\left(\Lambda_{jnj'n'}(\tau)\left[A_{jn}(t)\,,A_{j'n'}(t-\tau,t)\,\overline{\rho_S}(t)\right]+\Lambda_{j'n'jn}(-\tau)\left[\overline{\rho_S}(t)\,A_{j'n'}(t-\tau,t)\,,A_{jn}(t)\right]\right)\right)\mathrm{d}\tau-\mathrm{i}\left[H_S(t)\,,\overline{\rho_S}(t)\right] \tag{1398}$$

For this case we used that $A_{jn}(t-\tau,t)=U(t)\,U^{\dagger}(t-\tau)\,A_{jn}(t)\,U(t-\tau)\,U^{\dagger}(t)$. This is a non-Markovian equation and if we take $n=2$ (two sites), $\mu_0(t)=0$, $\mu_1(t)=1$ then we can reproduce a similar expression to (419) as expected.

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