

A general non-Markovian master equation for time-dependent Hamiltonians with coupling that is weak, strong, or anything in between

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I. THE HAMILTONIAN

We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1)$$

$$H_S(t) = \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| + V_{01}(t) |0\rangle\langle 1|, \quad (2)$$

$$H_I = |0\rangle\langle 0| \sum_{\mathbf{k}} \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (3)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (4)$$

For the states $|0\rangle, |1\rangle$ we have the orthonormal condition:

$$\langle i|j\rangle = \delta_{ij}. \quad (5)$$

II. UNITARY TRANSFORMATION INTO THE VARIATIONALLY OPTIMIZABLE FRAME

We will apply to $H(t)$, the unitary transformation defined by $e^{\pm V(t)}$ where is the variationally optimizable anti-Hermitian operator:

$$V(t) \equiv |0\rangle\langle 0| \sum_{\mathbf{k}} \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right). \quad (6)$$

in terms of the variational scalar parameters $v_{i\mathbf{k}}(t)$ defined as:

$$v_{i\mathbf{k}}(t) = \omega_{\mathbf{k}} \alpha_{i\mathbf{k}}(t). \quad (7)$$

which will soon be optimized in order to give the most accurate possible master equation for the system's dynamics in the presence of this bath. We define the following notation for the function (6):

$$\hat{\varphi}_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right), \quad (8)$$

$$V(t) = |0\rangle\langle 0| \hat{\varphi}_0(t) + |1\rangle\langle 1| \hat{\varphi}_1(t). \quad (9)$$

Here $*$ denotes the complex conjugate. Expanding $e^{\pm V(t)}$ using the notation (6) will give us the following result:

$$e^{\pm V(t)} = e^{\pm(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t))} \quad (10)$$

$$= \mathbb{I} \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{(\pm(|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)))^2}{2!} + \dots \quad (11)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \pm (|0\rangle\langle 0|\hat{\varphi}_0(t) + |1\rangle\langle 1|\hat{\varphi}_1(t)) + \frac{|0\rangle\langle 0|\hat{\varphi}_0^2(t)}{2!} + \frac{|1\rangle\langle 1|\hat{\varphi}_1^2(t)}{2!} + \dots \quad (12)$$

$$= |0\rangle\langle 0| \left(\mathbb{I} \pm \hat{\varphi}_0(t) + \frac{\hat{\varphi}_0^2(t)}{2!} \pm \dots \right) + |1\rangle\langle 1| \left(\mathbb{I} \pm \hat{\varphi}_1(t) + \frac{\hat{\varphi}_1^2(t)}{2!} \pm \dots \right) \quad (13)$$

$$= |0\rangle\langle 0| e^{\pm \hat{\varphi}_0(t)} + |1\rangle\langle 1| e^{\pm \hat{\varphi}_1(t)} \quad (14)$$

$$= |0\rangle\langle 0| e^{\pm \sum_{\mathbf{k}} (\alpha_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger - \alpha_{0\mathbf{k}}^*(t) b_{\mathbf{k}})} + |1\rangle\langle 1| e^{\pm \sum_{\mathbf{k}} (\alpha_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger - \alpha_{1\mathbf{k}}^*(t) b_{\mathbf{k}})} \quad (15)$$

$$= |0\rangle\langle 0| B_0^\pm(t) + |1\rangle\langle 1| B_1^\pm(t), \quad (16)$$

$$B_i^\pm(t) \equiv e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}. \quad (17)$$

Let's recall the Zassenhaus formula:

$$e^{r(X+Y)} = e^{rX} e^{rY} e^{-\frac{r^2}{2}[X,Y]} e^{\frac{r^3}{6}(2[Y,[X,Y]] + [X,[X,Y]])} e^{-\frac{r^4}{24}([[[X,Y],X],X] + 3[[[X,Y],X],Y] + 3[[[X,Y],Y],Y])} \dots \quad (18)$$

Since $\left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'} \right] = 0$ for all \mathbf{k}', \mathbf{k} and i, j we can show making $r = 1$ in (18) the following result:

$$e^{\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right) + \left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'} \right)} = e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}} e^{-\frac{1}{2} \left[\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}, \frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'} \right]} \dots \quad (19)$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}} e^{-\frac{1}{2} 0} \dots \quad (20)$$

$$= e^{\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}} e^{\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}^\dagger - \frac{v_{j\mathbf{k}'}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}'}}. \quad (21)$$

By induction of this result we can write an expression of $B_i^\pm(t)$ (shown in equation (17)) as a product of exponentials, which we will call "displacement" operators $D(\pm v_{i\mathbf{k}}(t))$:

$$D(\pm v_{i\mathbf{k}}(t)) \equiv e^{\pm \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}, \quad (22)$$

$$B_i^\pm(t) = \prod_{\mathbf{k}} D\left(\pm \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right). \quad (23)$$

this will help us to write operators $O(t)$ transformed in the variational frame as:

$$\overline{O(t)} \equiv e^{V(t)} O(t) e^{-V(t)}. \quad (24)$$

We will use the following identities:

$$\overline{|0\rangle\langle 0|}(t) = e^{V(t)}|0\rangle\langle 0|e^{-V(t)} \quad (25)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|0\rangle\langle 0|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (26)$$

$$= (|0\rangle\langle 0|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (27)$$

$$= |0\rangle\langle 0|B_0^+(t)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (28)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)B_1^-(t) \quad (29)$$

$$= |0\rangle\langle 0|, \quad (30)$$

$$\overline{|1\rangle\langle 1|}(t) = (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|1\rangle\langle 1|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (31)$$

$$= (|0\rangle\langle 0|1\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (32)$$

$$= |1\rangle\langle 1|B_1^+(t)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (33)$$

$$= |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)B_0^-(t) + B_1^+(t)|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) \quad (34)$$

$$= B_1^+(t)|1\rangle\langle 1|1\rangle\langle 1|B_1^-(t) \quad (35)$$

$$= |1\rangle\langle 1|, \quad (36)$$

$$\overline{|0\rangle\langle 1|}(t) = e^{V(t)}|0\rangle\langle 1|e^{-V(t)} \quad (37)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|0\rangle\langle 1|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (38)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)|0\rangle\langle 1|)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (39)$$

$$= (|0\rangle\langle 0|0\rangle\langle 1|B_0^+(t) + |1\rangle\langle 1|0\rangle\langle 1|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (40)$$

$$= |0\rangle\langle 1|B_0^+(t)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (41)$$

$$= |0\rangle\langle 1|0\rangle\langle 0|B_0^+(t)B_0^-(t) + |0\rangle\langle 1|1\rangle\langle 1|B_0^+(t)B_1^-(t) \quad (42)$$

$$= |0\rangle\langle 1|B_0^+(t)B_1^-(t), \quad (43)$$

$$\overline{|1\rangle\langle 0|}(t) = e^{V(t)}|1\rangle\langle 0|e^{-V(t)} \quad (44)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))|1\rangle\langle 0|(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (45)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t)|1\rangle\langle 0|)(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (46)$$

$$= (|0\rangle\langle 0|1\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|1\rangle\langle 0|B_1^+(t))(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (47)$$

$$= |1\rangle\langle 0|0\rangle\langle 0|B_1^+(t)B_0^-(t) + |1\rangle\langle 0|1\rangle\langle 1|B_1^+(t)B_1^-(t) \quad (48)$$

$$= |1\rangle\langle 0|B_1^+(t)B_0^-(t), \quad (49)$$

$$\overline{b_{\mathbf{k}}}(t) = e^{V(t)}b_{\mathbf{k}}e^{-V(t)} \quad (50)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))b_{\mathbf{k}}(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (51)$$

$$= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t)|1\rangle\langle 1| \quad (52)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}B_1^-(t) \quad (53)$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (54)$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|)b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (55)$$

$$= b_{\mathbf{k}} - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (56)$$

$$\overline{b_{\mathbf{k}}(t)}^\dagger = e^{V(t)}b_{\mathbf{k}}^\dagger e^{-V(t)} \quad (57)$$

$$= (|0\rangle\langle 0|B_0^+(t) + |1\rangle\langle 1|B_1^+(t))b_{\mathbf{k}}^\dagger(|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^-(t)) \quad (58)$$

$$= |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^\dagger B_0^-(t)|0\rangle\langle 0| + |0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^\dagger|1\rangle\langle 1|B_1^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^\dagger|0\rangle\langle 0|B_0^-(t) + |1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^\dagger B_1^-(t)|1\rangle\langle 1| \quad (59)$$

$$= |0\rangle\langle 0|0\rangle\langle 0|B_0^+(t)b_{\mathbf{k}}^\dagger B_0^-(t) + |0\rangle\langle 0|1\rangle\langle 1|B_0^+(t)b_{\mathbf{k}}^\dagger B_1^-(t) + |1\rangle\langle 1|0\rangle\langle 0|B_1^+(t)b_{\mathbf{k}}^\dagger B_0^-(t) + |1\rangle\langle 1|1\rangle\langle 1|B_1^+(t)b_{\mathbf{k}}^\dagger B_1^-(t) \quad (60)$$

$$= |0\rangle\langle 0|\left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) + |1\rangle\langle 1|\left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}\right) \quad (61)$$

$$= b_{\mathbf{k}}^\dagger - |1\rangle\langle 1|\frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - |0\rangle\langle 0|\frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. \quad (62)$$

We have used the following results as well to obtain the transformed $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$:

$$B_i^+(t) b_{\mathbf{k}} B_i^-(t) = b_{\mathbf{k}} - \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}, \quad (63)$$

$$B_i^+(t) b_{\mathbf{k}}^\dagger B_i^-(t) = b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}}. \quad (64)$$

We therefore have the following relationships:

$$\overline{\varepsilon_0(t) |0\rangle\langle 0| (t)} = \varepsilon_0(t) |0\rangle\langle 0|, \quad (65)$$

$$\overline{\varepsilon_1(t) |1\rangle\langle 1| (t)} = \varepsilon_1(t) |1\rangle\langle 1|, \quad (66)$$

$$\overline{V_{10}(t) |1\rangle\langle 0| (t)} = V_{10}(t) |1\rangle\langle 0| B_1^+(t) B_0^-(t), \quad (67)$$

$$\overline{V_{01}(t) |0\rangle\langle 1| (t)} = V_{01}(t) |0\rangle\langle 1| B_0^+(t) B_1^-(t), \quad (68)$$

$$\overline{(g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}})(t)} = g_{i\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) \right) + g_{i\mathbf{k}}^* \left(|0\rangle\langle 0| \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (69)$$

$$= g_{i\mathbf{k}} \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (70)$$

$$= g_{i\mathbf{k}} \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) + g_{i\mathbf{k}}^* \left((|0\rangle\langle 0| + |1\rangle\langle 1|) b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| \right) \quad (71)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |0\rangle\langle 0| - g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| - g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} |1\rangle\langle 1| \quad (72)$$

$$= g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) |0\rangle\langle 0| - \left(g_{i\mathbf{k}} \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) |1\rangle\langle 1|, \quad (73)$$

$$\overline{|0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}})(t)} = (|0\rangle\langle 0| B_0^+(t) + |1\rangle\langle 1| B_1^+(t)) |0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) (|0\rangle\langle 0| B_0^-(t) + |1\rangle\langle 1| B_1^-(t)) \quad (74)$$

$$= |0\rangle\langle 0| B_0^+(t) |0\rangle\langle 0| (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) |0\rangle\langle 0| B_0^-(t) \quad (75)$$

$$= |0\rangle\langle 0| B_0^+(t) (g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}}) B_0^-(t) \quad (76)$$

$$= |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (77)$$

$$\overline{|1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}})(t)} = (|0\rangle\langle 0| B_0^+(t) + |1\rangle\langle 1| B_1^+(t)) |1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) (|0\rangle\langle 0| B_0^-(t) + |1\rangle\langle 1| B_1^-(t)) \quad (78)$$

$$= |1\rangle\langle 1| B_1^+(t) |1\rangle\langle 1| (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) |1\rangle\langle 1| B_1^-(t) \quad (79)$$

$$= |1\rangle\langle 1| B_1^+(t) (g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}}) B_1^-(t) \quad (80)$$

$$= |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (81)$$

$$\overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}(t)} = \omega_{\mathbf{k}} (|0\rangle\langle 0| B_0^+(t) + |1\rangle\langle 1| B_1^+(t)) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} (|0\rangle\langle 0| B_0^-(t) + |1\rangle\langle 1| B_1^-(t)) \quad (82)$$

$$= \omega_{\mathbf{k}} (|0\rangle\langle 0| \Pi_{\mathbf{k}'} D\left(\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D\left(\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} (|0\rangle\langle 0| \Pi_{\mathbf{k}'} D\left(-\frac{v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) + |1\rangle\langle 1| \Pi_{\mathbf{k}'} D\left(-\frac{v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right)) \quad (83)$$

$$= \omega_{\mathbf{k}} (|0\rangle\langle 0| B_0^+(t) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_0^-(t) + |1\rangle\langle 1| B_1^+(t) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B_1^-(t)) \quad (84)$$

$$= \omega_{\mathbf{k}} \sum_j |j\rangle\langle j| D\left(\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}' \neq \mathbf{k}} \left(D\left(\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) D\left(-\frac{v_{j\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}\right) \right) \quad (85)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| D\left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \mathbb{I} + |1\rangle\langle 1| D\left(\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} D\left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \mathbb{I} \right) \quad (86)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (87)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger + \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) + |1\rangle\langle 1| \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger + \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) \right) \quad (88)$$

$$= \omega_{\mathbf{k}} \left(|0\rangle\langle 0| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (89)$$

$$= \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (90)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \omega_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger \right) \right) \quad (91)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} - v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \quad (92)$$

$$= \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right). \quad (93)$$

So all parts of $H(t)$ can be written in the variationally optimizable frame now:

$$\overline{H_S(t)} = \overline{\varepsilon_0(t)} |0\rangle\langle 0| + \overline{\varepsilon_1(t)} |1\rangle\langle 1| + \overline{V_{10}(t)} |1\rangle\langle 0| + \overline{V_{01}(t)} |0\rangle\langle 1| \quad (94)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + \varepsilon_1(t) |1\rangle\langle 1| + V_{10}(t) |1\rangle\langle 0| B_1^+(t) B_0^-(t) + V_{01}(t) |0\rangle\langle 1| B_0^+(t) B_1^-(t), \quad (95)$$

$$\overline{H_I} = \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (96)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{0\mathbf{k}}^* b_{\mathbf{k}} \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{1\mathbf{k}}^* b_{\mathbf{k}} \right) \quad (97)$$

$$= \sum_{\mathbf{k}} |0\rangle\langle 0| \left(g_{0\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{0\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) + \sum_{\mathbf{k}} |1\rangle\langle 1| \left(g_{1\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} \right) + g_{1\mathbf{k}}^* \left(b_{\mathbf{k}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (98)$$

$$= \sum_{\mathbf{k}, i} |i\rangle\langle i| \left(g_{i\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{i\mathbf{k}}^* b_{\mathbf{k}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (99)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \overline{\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} \quad (100)$$

$$= \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right) \quad (101)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \left(|1\rangle\langle 1| \left(\frac{|v_{1\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{1\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{1\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) + |0\rangle\langle 0| \left(\frac{|v_{0\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(v_{0\mathbf{k}}^*(t) b_{\mathbf{k}} + v_{0\mathbf{k}}(t) b_{\mathbf{k}}^\dagger \right) \right) \right). \quad (102)$$

Finally merging these expressions gives the transformed Hamiltonian:

$$\overline{H(t)} = \sum_j \varepsilon_j(t) |j\rangle\langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle\langle j| \left(\left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right) b_{\mathbf{k}}^\dagger + \left(g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) \right)^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right). \quad (103)$$

Also we may write this transformed Hamiltonian as a sum of the form:

$$\overline{H(t)} = \overline{H_S}(t) + \overline{H_I}(t) + \overline{H_B}. \quad (104)$$

Let's define:

$$R_i(t) \equiv \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right), \quad (105)$$

$$B_{iz}(t) \equiv \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right), \quad (106)$$

$$\chi_{ij}(t) \equiv \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t) v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t) v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right). \quad (107)$$

$\chi_{ij}(t)$ is an imaginary number so $e^{\chi_{ij}(t)}$ is the phase associated to $B_{ij}(t)$ as we will show. We can summarize these definitions with other that we will proof later and use from now in the following matrix:

$$\begin{pmatrix} B_{iz}(t) & B_i^\pm(t) \\ B_x(t) & B_i(t) \\ B_y(t) & B_{ij}(t) \end{pmatrix} \equiv \begin{pmatrix} \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) & e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \\ \frac{B_1^+(t)B_0^-(t) + B_0^+(t)B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth(\beta\omega_{\mathbf{k}}/2)} \\ \frac{B_0^+(t)B_1^-(t) - B_1^+(t)B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i} & e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t) - v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\chi_{ij}(t)} \end{pmatrix}, \quad (108)$$

$$(\cdot)^\Re \equiv \Re(\cdot), \quad (109)$$

$$(\cdot)^\Im \equiv \Im(\cdot). \quad (110)$$

We reduced the lenght of the expression for the real and imaginary part as shown before. We assume that the bath is at equilibrium with inverse temperature $\beta = \frac{1}{k_B T}$, considering the stationary bath state as reference written in the following way:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})}. \quad (111)$$

We can show using the coherence representation of the creation and annihilation operators that:

$$b^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (112)$$

$$b = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (113)$$

So the product of the matrix representation of b^\dagger and b with $-\beta$ is:

$$-\beta\omega b^\dagger b = -\beta\omega \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 2 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & 0 & \dots & n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (114)$$

$$= \sum_{j=0}^{\infty} -j\beta\omega |j\rangle\langle j|, \quad (115)$$

So the density matrix ρ_B written in the coherence representation can be obtained using the Zassenhaus formula and the fact that $[|j\rangle\langle j|, |i\rangle\langle i|] = 0$ for all i, j .

$$e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|, \quad (116)$$

$$e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|. \quad (117)$$

The value of $\text{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right)$ is:

$$\text{Tr}\left(e^{-\beta\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) = \text{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (118)$$

$$= \sum_{j_{\mathbf{k}}} (e^{-\beta\omega_{\mathbf{k}}})^{j_{\mathbf{k}}} \quad (119)$$

$$= \frac{1}{1 - e^{-\beta\omega_{\mathbf{k}}}} \quad (\text{by geometric series}) \quad (120)$$

$$\equiv f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}), \quad (121)$$

$$\text{Tr}\left(e^{-\beta\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}^\dagger b_{\mathbf{k}}}\right) = \text{Tr}\left(\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (122)$$

$$= \prod_{\mathbf{k}} \text{Tr}\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|\right) \quad (123)$$

$$= \prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}}). \quad (124)$$

So the density matrix of the bath is:

$$\rho_B = \frac{e^{-\beta H_B}}{\text{Tr}(e^{-\beta H_B})} \quad (125)$$

$$= \frac{\prod_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{\prod_{\mathbf{k}} f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})} \quad (126)$$

$$= \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}}\beta\omega_{\mathbf{k}}}|j_{\mathbf{k}}\rangle\langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta\omega_{\mathbf{k}})}. \quad (127)$$

Now, given that creation and annihilation satisfy:

$$b_{\mathbf{k}} |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}}|j_{\mathbf{k}}-1\rangle, \quad (128)$$

$$b_{\mathbf{k}}^\dagger |j_{\mathbf{k}}\rangle = \sqrt{j_{\mathbf{k}}+1}|j_{\mathbf{k}}+1\rangle. \quad (129)$$

Then we can prove that $\langle B_{iz} \rangle_{\overline{H_B}} = 0$ using the following property based on (128)-(129):

$$\langle B_{iz}(t) \rangle_{\overline{H_B}} = \text{Tr} (B_{iz}(t) \rho_B) \quad (130)$$

$$= \text{Tr} \left(\left(\sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right) \right) \rho_B \right) \quad (131)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \rho_B \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \rho_B \right) \quad (132)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \text{Tr} \left(b_{\mathbf{k}}^\dagger \rho_B \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \text{Tr} \left(b_{\mathbf{k}} \rho_B \right) \quad (133)$$

$$= \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} \text{Tr} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) \quad (134)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \text{Tr} \left(b_{\mathbf{k}}^\dagger \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right) + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \text{Tr} \left(b_{\mathbf{k}} \prod_{\mathbf{k}} \frac{\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}|}{f_{\text{Bose-Einstein}}(-\beta \omega_{\mathbf{k}})} \right), \quad (135)$$

$$\text{Tr} \left(b_{\mathbf{k}}^\dagger \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \right) b_{\mathbf{k}}^\dagger |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}^\dagger) \quad (136)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \right) \sqrt{j_{\mathbf{k}} + 1} |j_{\mathbf{k}} + 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (137)$$

$$= 0, \quad (138)$$

$$\text{Tr} \left(b_{\mathbf{k}} \sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) = \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \right) b_{\mathbf{k}} |j_{\mathbf{k}} \rangle \langle j_{\mathbf{k}}| \right) \quad (\text{by cyclic permutivity of trace, move } b_{\mathbf{k}}) \quad (139)$$

$$= \text{Tr} \left(\left(\sum_{j_{\mathbf{k}}} e^{-j_{\mathbf{k}} \beta \omega_{\mathbf{k}}} \right) \sqrt{j_{\mathbf{k}}} |j_{\mathbf{k}} - 1 \rangle \langle j_{\mathbf{k}}| \right) \quad (140)$$

$$= 0. \quad (141)$$

we therefore find that:

$$\langle B_{iz}(t) \rangle_{\overline{H_B}} = 0. \quad (142)$$

Another important expected value is $B(t) = \langle B^\pm(t) \rangle_{\overline{H_B}}$, where $B^\pm(t) = e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)}$ is given by:

$$\langle B^\pm(t) \rangle_{H_B} = \text{Tr} (\rho_B B^\pm(t)) = \text{Tr} (B^\pm(t) \rho_B) \quad (143)$$

$$= \text{Tr} \left(e^{\pm \sum_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} b_{\mathbf{k}} \right)} \rho_B \right) \quad (144)$$

$$= \prod_{\mathbf{k}} \text{Tr} (D(\pm \alpha_{\mathbf{k}}(t)) \rho_B) \quad (145)$$

$$= \prod_{\mathbf{k}} \langle D(\pm \alpha_{\mathbf{k}}(t)) \rangle. \quad (146)$$

Given that we can write a density operator as:

$$\rho = \int P(\alpha) |\alpha \rangle \langle \alpha| d^2 \alpha. \quad (147)$$

where $P(\alpha)$ satisfies $\int P(\alpha) d^2 \alpha = 1$ and describes the state. It follows that the expectation value of an operator A with respect to the density operator described by $P(\alpha)$ is given by:

$$\langle A \rangle = \text{Tr} (A \rho) \quad (148)$$

$$= \int P(\alpha) \langle \alpha | A | \alpha \rangle d^2 \alpha. \quad (149)$$

We are typically interested in thermal state density operators, for which it can be shown that $P(\alpha) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}}$ where $N = (e^{\beta\omega} - 1)^{-1}$ is the average number of excitations in an oscillator of frequency ω at inverse temperature $\beta = \frac{1}{k_B T}$.

Using the integral representation (149) we could obtain that the expected value for the displacement operator $D(h)$ with $h \in \mathbb{C}$ is equal to:

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle \alpha | D(h) | \alpha \rangle d^2 \alpha \quad (150)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(-\alpha) D(h) D(\alpha) | 0 \rangle d^2 \alpha, \quad (151)$$

$$D(h) D(\alpha) = D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^* \alpha)}, \quad (152)$$

$$D(-\alpha) (D(h) D(\alpha)) = D(-\alpha) D(h + \alpha) e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (153)$$

$$= D(h) e^{\frac{1}{2}(-\alpha(h+\alpha)^* + \alpha^*(h+\alpha))} e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (154)$$

$$= D(\alpha) e^{\frac{1}{2}(-\alpha h^* - |\alpha|^2 + \alpha^* h + |\alpha|^2)} e^{\frac{1}{2}(h\alpha^* - h^* \alpha)} \quad (155)$$

$$= D(\alpha) e^{(h\alpha^* - h^* \alpha)}, \quad (156)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} \langle 0 | D(h) e^{(h\alpha^* - h^* \alpha)} | 0 \rangle d^2 \alpha \quad (157)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^* \alpha)} \langle 0 | D(h) | 0 \rangle d^2 \alpha \quad (158)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^* \alpha)} \langle 0 | h \rangle d^2 \alpha, \quad (159)$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (160)$$

$$\langle D(h) \rangle = \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^* \alpha)} \langle 0 | e^{-\frac{|h|^2}{2}} \sum_{n=0}^{\infty} \frac{h^n}{\sqrt{n!}} |n\rangle d^2 \alpha \quad (161)$$

$$= \frac{1}{\pi N} \int e^{-\frac{|\alpha|^2}{N}} e^{(h\alpha^* - h^* \alpha)} e^{-\frac{|h|^2}{2}} d^2 \alpha \quad (162)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int e^{-\frac{|\alpha|^2}{N} + h\alpha^* - h^* \alpha} d^2 \alpha, \quad (163)$$

$$\alpha = x + iy, \quad (164)$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{N} + h(x-iy) - h^*(x+iy)} dx dy \quad (165)$$

$$= \frac{e^{-\frac{|h|^2}{2}}}{\pi N} \int_{-\infty}^{\infty} e^{-\frac{x^2}{N} + hx - h^* x} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{N} + hy - h^* y} dy, \quad (166)$$

$$-\frac{x^2}{N} + hx - h^* x = -\frac{1}{N} (x^2 - Nhx + Nh^* x) \quad (167)$$

$$= -\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 + \frac{N(h^* - h)^2}{4}, \quad (168)$$

$$-\frac{y^2}{N} - ihy - ih^* y = -\frac{1}{N} (y^2 + iNhy + iNh^* y) \quad (169)$$

$$= -\frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right)^2 - \frac{N(h + h^*)^2}{4}, \quad (170)$$

$$\sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx, \quad (171)$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}}}{\pi N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{N} \left(x + \frac{(Nh^* - Nh)}{2} \right)^2 - \frac{1}{N} \left(y^2 + \frac{iN(h + h^*)}{2} \right)} dx dy, \quad (172)$$

$$\langle D(h) \rangle = \frac{e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}}}{\pi N} \int_{-\infty}^{\infty} e^{-\frac{\left(x + \frac{(Nh^* - Nh)}{2} \right)^2}{2 \left(\sqrt{\frac{N}{2}} \right)^2}} dx \int_{-\infty}^{\infty} e^{-\frac{\left(y^2 + \frac{iN(h + h^*)}{2} \right)}{2 \left(\sqrt{\frac{N}{2}} \right)^2}} dy \quad (173)$$

$$= \frac{e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}}}{\pi N} \left(\sqrt{2\pi} \sqrt{\frac{N}{2}} \right)^2 \quad (174)$$

$$= e^{-\frac{|h|^2}{2} + \frac{N(h^* - h)^2}{4} - \frac{N(h + h^*)^2}{4}} \quad (175)$$

$$= e^{-\frac{|h|^2}{2} + \frac{N(h^{*2} - 2hh^* + h^2)}{4} - \frac{N(h^2 + 2hh^* + h^{*2})}{4}} \quad (176)$$

$$= e^{-|h|^2 \left(N + \frac{1}{2} \right)} \quad (177)$$

$$= e^{-|h|^2 \left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2} \right)} \quad (178)$$

$$= e^{-\frac{|h|^2}{2} \left(\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} \right)} \quad (179)$$

$$= e^{-\frac{|h|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)}. \quad (180)$$

In the last line we used $\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} = \coth\left(\frac{\beta\omega}{2}\right)$. So the value of (145) using (??) is given by:

$$B = e^{-\sum_{\mathbf{k}} \frac{|\alpha_{\mathbf{k}}|^2}{2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}. \quad (181)$$

We will now force $\langle \overline{H_I}(t) \rangle_{\overline{H_B}} = 0$. We will also introduce the bath renormalizing driving in $\overline{H_S}(t)$ to treat it non-perturbatively in the subsequent formalism, we associate the terms related with $B_i^+(t) \sigma^+$ and $B_i^-(t) \sigma^-$ with the interaction part of the Hamiltonian $\overline{H_I}(t)$ and we subtract their expected value in order to satisfy $\langle \overline{H_I}(t) \rangle_{\overline{H_B}} = 0$.

A final form of the terms of the Hamiltonian $\overline{H}(t)$ is:

$$\overline{H}(t) = \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| B_j^+(t) B_{j'}^-(t) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{j\mathbf{k}} |j\rangle \langle j| \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} + \frac{|v_{j\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{j\mathbf{k}} \frac{v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{j\mathbf{k}}^* \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (182)$$

$$= \sum_j \varepsilon_j(t) |j\rangle \langle j| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| B_{jj'}(t) + \sum_j |j\rangle \langle j| B_{jz}(t) + \sum_{j \neq j'} V_{jj'}(t) |j\rangle \langle j'| (B_j^+(t) B_{j'}^-(t) - B_{jj'}(t)) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (183)$$

$$\equiv \overline{H_S}(t) + \overline{H_I}(t) + \overline{H_B}. \quad (184)$$

The parts of the Hamiltonian splitted are obtained using the following expected value:

$$\langle B_i^+(t) B_j^-(t) \rangle = B_{ij}(t) \quad (185)$$

$$= \left\langle \prod_{\mathbf{k}} D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \prod_{\mathbf{k}} D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right\rangle \quad (186)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D\left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) D\left(-\frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}}\right) \right) \right\rangle \quad (187)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (188)$$

$$= \prod_{\mathbf{k}} \left\langle D \left(\frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \quad (189)$$

$$= \prod_{\mathbf{k}} e^{-\frac{1}{2} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \quad (190)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{j\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{i\mathbf{k}}^*(t)v_{j\mathbf{k}}(t) - v_{i\mathbf{k}}(t)v_{j\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \quad (191)$$

From the definition $B_{01}(t) = \langle B_0^+(t) B_1^-(t) \rangle$ using the displacement operator we have:

$$\langle B_0^+(t) B_1^-(t) \rangle = B_{01}(t) \quad (192)$$

$$= \left\langle \prod_{\mathbf{k}} D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \prod_{\mathbf{k}} D \left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle \quad (193)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) D \left(-\frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle \quad (194)$$

$$= \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \right\rangle \quad (195)$$

$$= \prod_{\mathbf{k}} \left(\left\langle D \left(\frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right\rangle e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (196)$$

$$= \prod_{\mathbf{k}} \left(e^{-\frac{1}{2} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \quad (197)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}. \quad (198)$$

We can check:

$$\langle B_0^+(t) B_1^-(t) \rangle = B_{01}(t) \quad (199)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t)v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \quad (200)$$

$$= e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t)v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)}^* \quad (201)$$

$$= \langle B_1^+(t) B_0^-(t) \rangle^* \quad (202)$$

$$= B_{10}^*(t). \quad (203)$$

The parts of the splitted Hamiltonian are:

$$\overline{H_{\bar{S}}}(t) \equiv (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + V_{10}(t) B_{10}\sigma^+ + V_{01}(t) B_{01}\sigma^-, \quad (204)$$

$$\overline{H_{\bar{I}}}(t) \equiv V_{10}(t) (B_1^+(t) B_0^-(t) - B_{10}(t)) \sigma^+ + V_{01}(t) (B_0^+(t) B_1^-(t) - B_{01}(t)) \sigma^- + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t), \quad (205)$$

$$\overline{H_B} \equiv \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \quad (206)$$

$$= H_B. \quad (207)$$

Note that $\overline{H_B}$, which is the bath acting on the effective “system” \bar{S} in the variational frame, is just the original bath, H_B , before transforming to the variational frame.

For the Hamiltonian (205) we can verify the condition $\langle \overline{H_I} \rangle_{\overline{H_B}} = 0$ in the following way:

$$\langle \overline{H_I} \rangle_{\overline{H_B}} = \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right) |n\rangle\langle n| + \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+(t) B_{j'}^-(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_B}} \quad (208)$$

$$= \left\langle \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right) |n\rangle\langle n| \right\rangle_{\overline{H_B}} + \left\langle \sum_{j \neq j'} V_{jj'}(t) |j\rangle\langle j'| \left(B_j^+(t) B_{j'}^-(t) - B_{jj'}(t) \right) \right\rangle_{\overline{H_B}} \quad (209)$$

$$= \sum_{n\mathbf{k}} \left(\left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + \left\langle (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| \left(\left\langle V_{jj'}(t) B_j^+(t) B_{j'}^-(t) \right\rangle_{\overline{H_B}} - \left\langle V_{jj'}(t) B_{jj'}(t) \right\rangle_{\overline{H_B}} \right) \quad (210)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) \left(\left\langle B_j^+(t) B_{j'}^-(t) \right\rangle_{\overline{H_B}} - \left\langle B_{jj'}(t) \right\rangle_{\overline{H_B}} \right) \quad (211)$$

$$= \sum_{n\mathbf{k}} \left((g_{n\mathbf{k}} - v_{n\mathbf{k}}(t)) \left\langle b_{\mathbf{k}}^\dagger \right\rangle_{\overline{H_B}} + (g_{n\mathbf{k}} - v_{n\mathbf{k}}(t))^* \left\langle b_{\mathbf{k}} \right\rangle_{\overline{H_B}} \right) |n\rangle\langle n| + \sum_{j \neq j'} |j\rangle\langle j'| V_{jj'}(t) (B_{jj'}(t) - B_{jj'}(t)) \quad (212)$$

$$= 0. \quad (213)$$

We used (142) and (??) to evaluate the expression. Let's consider the following Hermitian combinations:

$$B_x(t) = B_x^\dagger(t) \quad (214)$$

$$= \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2}, \quad (215)$$

$$B_y(t) = B_y^\dagger(t) \quad (216)$$

$$= \frac{B_0^+(t) B_1^-(t) - B_1^+(t) B_0^-(t) + B_{10}(t) - B_{01}(t)}{2i}, \quad (217)$$

$$B_{iz}(t) = B_{iz}^\dagger(t) \quad (218)$$

$$= \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} \right). \quad (219)$$

Writing the equations (204) and (205) using the previous combinations we obtain that:

$$\overline{H_S}(t) = \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \sigma^+ + V_{01}(t) B_{01}(t) \sigma^- \quad (220)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) B_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) B_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \quad (221)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + V_{10}(t) \left(B_{10}^{\Re}(t) + iB_{10}^{\Im}(t) \right) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \left(B_{10}^{\Re}(t) - iB_{10}^{\Im}(t) \right) \frac{\sigma_x - i\sigma_y}{2} \quad (222)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} + V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) + iB_{10}^{\Im}(t) \left(V_{10}(t) \frac{\sigma_x + i\sigma_y}{2} - V_{01}(t) \frac{\sigma_x - i\sigma_y}{2} \right) \quad (223)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x \frac{V_{10}(t) + V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) - V_{01}(t)}{2} \right) + iB_{10}^{\Im}(t) \left(\sigma_x \frac{V_{10}(t) - V_{01}(t)}{2} + i\sigma_y \frac{V_{10}(t) + V_{01}(t)}{2} \right) \quad (224)$$

$$= \sum_{j \in \{0,1\}} (\varepsilon_j(t) + R_j(t)) |j\rangle\langle j| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + iB_{10}^{\Im}(t) \left(i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t) \right) \quad (225)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + B_{10}^{\Re}(t) \left(\sigma_x V_{10}^{\Re}(t) - \sigma_y V_{10}^{\Im}(t) \right) + iB_{10}^{\Im}(t) \left(i\sigma_x V_{10}^{\Im}(t) + i\sigma_y V_{10}^{\Re}(t) \right) \quad (226)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \left(\sigma_x B_{10}^{\Re}(t) V_{10}^{\Re}(t) - \sigma_y B_{10}^{\Re}(t) V_{10}^{\Im}(t) \right) - \left(\sigma_x B_{10}^{\Im}(t) V_{10}^{\Im}(t) + \sigma_y B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (227)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right) \quad (228)$$

$$= (\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + \sigma_x \left(B_{10}^{\Re}(t) V_{10}^{\Re}(t) - B_{10}^{\Im}(t) V_{10}^{\Im}(t) \right) - \sigma_y \left(B_{10}^{\Re}(t) V_{10}^{\Im}(t) + B_{10}^{\Im}(t) V_{10}^{\Re}(t) \right), \quad (229)$$

$$\overline{H_I} = V_{10}(t) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) \right) + V_{01}(t) \left(\sigma^- B_0^+(t) B_1^-(t) - \sigma^- B_{01}(t) \right) + |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t) \quad (230)$$

$$= |0\rangle\langle 0| B_{0z}(t) + |1\rangle\langle 1| B_{1z}(t) + \left(V_{10}^{\Re}(t) + i V_{10}^{\Im}(t) \right) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) \right) + \left(V_{10}^{\Re}(t) - i V_{10}^{\Im}(t) \right) \left(\sigma^- B_0^+(t) B_1^-(t) - \sigma^- B_{01}(t) \right) \quad (231)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) + \sigma^- B_0^+(t) B_1^-(t) - \sigma^- B_{01}(t) \right) + i V_{10}^{\Im}(t) \left(\sigma^+ B_1^+(t) B_0^-(t) - \sigma^+ B_{10}(t) - \sigma^- B_0^+(t) B_1^-(t) + \sigma^- B_{01}(t) \right) \quad (232)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+(t) B_0^-(t) - \frac{\sigma_x + i\sigma_y}{2} B_{10}(t) + \frac{\sigma_x - i\sigma_y}{2} B_0^+(t) B_1^-(t) - \frac{\sigma_x - i\sigma_y}{2} B_{01}(t) \right) \quad (233)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+(t) B_0^-(t) - \frac{\sigma_x + i\sigma_y}{2} B_{10}(t) + \frac{\sigma_x - i\sigma_y}{2} B_0^+(t) B_1^-(t) - \frac{\sigma_x - i\sigma_y}{2} B_{01}(t) \right) \quad (234)$$

$$+ i V_{10}^{\Im}(t) \left(\frac{\sigma_x + i\sigma_y}{2} B_1^+(t) B_0^-(t) - \frac{\sigma_x + i\sigma_y}{2} B_{10}(t) - \frac{\sigma_x - i\sigma_y}{2} B_0^+(t) B_1^-(t) + \frac{\sigma_x - i\sigma_y}{2} B_{01}(t) \right) \quad (235)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) \left(\sigma_x \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} + i\sigma_y \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} \right) \quad (236)$$

$$+ i V_{10}^{\Im}(t) \left(\sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} + i\sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (237)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \left(i\sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2} - \sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (238)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \left(i^2 \sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2i} - \sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (239)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) \left(i^2 \sigma_x \frac{B_1^+(t) B_0^-(t) - B_0^+(t) B_1^-(t) - B_{10}(t) + B_{01}(t)}{2i} - \sigma_y \frac{B_1^+(t) B_0^-(t) + B_0^+(t) B_1^-(t) - B_{10}(t) - B_{01}(t)}{2} \right) \quad (240)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (i^2 \sigma_x (-B_y(t)) - \sigma_y B_x(t)) \quad (241)$$

$$= \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)). \quad (242)$$

III. FREE-ENERGY MINIMIZATION

The true free energy A is bounded by the Bogoliubov inequality:

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right) + \langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} + O \left(\langle \overline{H_I}^2 \rangle_{\overline{H_S}(t) + \overline{H_B}} \right). \quad (243)$$

We will optimize the set of variational parameters $\{v_{ik}(t)\}$ in order to minimize A_B (i.e. to make it as close to the true free energy A as possible). Neglecting the higher order terms and using $\langle \overline{H_I} \rangle_{\overline{H_S}(t) + \overline{H_B}} = 0$ we can obtain the following condition to obtain the set $\{v_{ik}(t)\}$:

$$\frac{\partial A_B}{\partial v_{ik}(t)} = 0. \quad (244)$$

Using this condition and given that $[\overline{H_S}(t), \overline{H_B}] = 0$, we have:

$$e^{-\beta(\overline{H_S}(t) + \overline{H_B})} = e^{-\beta \overline{H_S}(t)} e^{-\beta \overline{H_B}}. \quad (245)$$

Then using the fact that $\overline{H_S}(t)$ and $\overline{H_B}$ relate to different Hilbert spaces, we obtain:

$$\text{Tr} \left(e^{-\beta \overline{H_S}(t)} e^{-\beta \overline{H_B}} \right) = \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right). \quad (246)$$

So Eq. (244) becomes:

$$\frac{\partial A_B}{\partial v_{i\mathbf{k}}(t)} = -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_S}(t) + \overline{H_B})} \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (247)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (248)$$

$$= -\frac{1}{\beta} \frac{\partial \left(\ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right) + \ln \left(\text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (249)$$

$$= -\frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}(t)} - \frac{1}{\beta} \frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_B}} \right) \right)}{\partial v_{i\mathbf{k}}(t)} \quad (250)$$

$$= 0 \quad (\text{by Eq. (244)}). \quad (251)$$

But since $\overline{H_B} = H_B$ which doesn't contain any $v_{i\mathbf{k}}(t)$, a derivative of any function of H_B that does not introduce new $v_{i\mathbf{k}}(t)$ will be zero. We therefore require the following:

$$\frac{\partial \ln \left(\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) \right)}{\partial v_{i\mathbf{k}}(t)} = \frac{1}{e^{-\beta \overline{H_S}(t)}} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}(t)} \quad (252)$$

$$= 0. \quad (253)$$

This means we need to impose:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}(t)} = 0. \quad (254)$$

First we look at:

$$-\beta \overline{H_S}(t) = -\beta \left((\varepsilon_0(t) + R_0(t)) |0\rangle\langle 0| + (\varepsilon_1(t) + R_1(t)) |1\rangle\langle 1| + V_{10}(t) B_{10}(t) \sigma^+ + V_{01}(t) B_{01}(t) \sigma^- \right). \quad (255)$$

Then the eigenvalues of $-\beta \overline{H_S}(t)$ satisfy the following relationship deduced from the Caley-Hamilton theorem:

$$\lambda^2 - \text{Tr} \left(-\beta \overline{H_S}(t) \right) + \text{Det} \left(-\beta \overline{H_S}(t) \right) = 0. \quad (256)$$

Let's define:

$$\varepsilon(t) \equiv \text{Tr} \left(\overline{H_S}(t) \right), \quad (257)$$

$$\eta \equiv \sqrt{\left(\text{Tr} \left(\overline{H_S}(t) \right) \right)^2 - 4 \text{Det} \left(\overline{H_S}(t) \right)}. \quad (258)$$

The solutions of the equation (256) are:

$$\lambda = \beta \frac{-\text{Tr} \left(\overline{H_S}(t) \right) \pm \sqrt{\left(\text{Tr} \left(\overline{H_S}(t) \right) \right)^2 - 4 \text{Det} \left(\overline{H_S}(t) \right)}}{2} \quad (259)$$

$$= \beta \frac{-\varepsilon(t) \pm \eta(t)}{2} \quad (260)$$

$$= -\beta \frac{\varepsilon(t) \mp \eta(t)}{2}. \quad (261)$$

The value of $\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)$ can be written in terms of this eigenvalues as (since there's only 2 eigenvalues of a 2×2 matrix):

$$\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) = \exp \left(-\frac{\varepsilon(t)\beta}{2} \right) \exp \left(\frac{\eta(t)\beta}{2} \right) + \exp \left(-\frac{\varepsilon(t)\beta}{2} \right) \exp \left(-\frac{\eta(t)\beta}{2} \right) \quad (262)$$

$$= 2 \exp \left(-\frac{\varepsilon(t)\beta}{2} \right) \cosh \left(\frac{\eta(t)\beta}{2} \right). \quad (263)$$

Given that $v_{i\mathbf{k}}(t)$ is a complex number then we will optimize in the real and complex parts of this element, this can be seen in the following reasoning.

Using the chain rule on the function $\text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right) = A(\varepsilon(t), \eta(t))$ to calculate $\frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)}$ can lead to:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H_S}(t)} \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial \left(2 \exp \left(-\frac{\varepsilon(t)\beta}{2} \right) \cosh \left(\frac{\eta(t)\beta}{2} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (264)$$

$$= 2 \left(-\frac{\beta}{2} \frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \right) \exp \left(-\frac{\varepsilon(t)\beta}{2} \right) \cosh \left(\frac{\eta(t)\beta}{2} \right) + 2 \left(\frac{\beta}{2} \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \right) \exp \left(-\frac{\varepsilon(t)\beta}{2} \right) \sinh \left(\frac{\eta(t)\beta}{2} \right) \quad (265)$$

$$= -\beta \exp \left(-\frac{\varepsilon(t)\beta}{2} \right) \left(\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \cosh \left(\frac{\eta(t)\beta}{2} \right) - \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \sinh \left(\frac{\eta(t)\beta}{2} \right) \right). \quad (266)$$

Making the derivate equal to zero make us suitable to write:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \cosh \left(\frac{\eta(t)\beta}{2} \right) - \frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \sinh \left(\frac{\eta(t)\beta}{2} \right) = 0. \quad (267)$$

The derivatives included in the expression given are related to:

$$\langle B_0^+(t) B_1^-(t) \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (268)$$

$$= \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right)^* e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (269)$$

$$= \langle B_1^+(t) B_0^-(t) \rangle^*, \quad (270)$$

$$R_i(t) = \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (271)$$

$$= \sum_{\mathbf{k}} \left(\frac{|v_{i\mathbf{k}}(t)|^2}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}} - g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right), \quad (272)$$

$$\langle B_0^+(t) B_1^-(t) \rangle = \left(\prod_{\mathbf{k}} e^{\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)} \quad (273)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2} \right) \right) e^{-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}, \quad (274)$$

$$v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t) = (v_{0\mathbf{k}}^{\Re}(t) - i v_{0\mathbf{k}}^{\Im}(t)) (v_{1\mathbf{k}}^{\Re}(t) + i v_{1\mathbf{k}}^{\Im}(t)) - (v_{0\mathbf{k}}^{\Re}(t) + i v_{0\mathbf{k}}^{\Im}(t)) (v_{1\mathbf{k}}^{\Re}(t) - i v_{1\mathbf{k}}^{\Im}(t)) \quad (275)$$

$$= (v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Re}(t) + i v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) - i v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Im}(t)) \quad (276)$$

$$- (v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Re}(t) - i v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) + i v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t) + v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Im}(t)) \quad (277)$$

$$= 2i (v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t)), \quad (278)$$

$$|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|^2 = (v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t))^* \quad (279)$$

$$= |v_{1\mathbf{k}}(t)|^2 + |v_{0\mathbf{k}}(t)|^2 - (v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t) + v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t)) \quad (280)$$

$$= (v_{1\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t))^2 + (v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{0\mathbf{k}}^{\Im}(t))^2 - (v_{1\mathbf{k}}^{\Re}(t) + i v_{1\mathbf{k}}^{\Im}(t))(v_{0\mathbf{k}}^{\Re}(t) - i v_{0\mathbf{k}}^{\Im}(t)) \quad (281)$$

$$- (v_{1\mathbf{k}}^{\Re}(t) - i v_{1\mathbf{k}}^{\Im}(t))(v_{0\mathbf{k}}^{\Re}(t) + i v_{0\mathbf{k}}^{\Im}(t)) \quad (282)$$

$$= (v_{1\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t))^2 + (v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{0\mathbf{k}}^{\Im}(t))^2 - 2(v_{1\mathbf{k}}^{\Re}(t) v_{0\mathbf{k}}^{\Re}(t) + v_{1\mathbf{k}}^{\Im}(t) v_{0\mathbf{k}}^{\Im}(t)) \quad (283)$$

$$= (v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2. \quad (284)$$

Rewriting in terms of real and imaginary parts.

$$R_i(t) = \sum_{\mathbf{k}} \left(\frac{(v_{i\mathbf{k}}^{\Re}(t))^2 + (v_{i\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}} - \left(g_{i\mathbf{k}} \frac{v_{i\mathbf{k}}^{\Re}(t) - i v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} + g_{i\mathbf{k}}^* \frac{v_{i\mathbf{k}}^{\Re}(t) + i v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} \right) \right) \quad (285)$$

$$= \sum_{\mathbf{k}} \left(\frac{(v_{i\mathbf{k}}^{\Re}(t))^2 + (v_{i\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im}(t) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right), \quad (286)$$

$$\langle B_0^+(t) B_1^-(t) \rangle = \left(\prod_{\mathbf{k}} \exp \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (287)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{2i(v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t))}{2\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (288)$$

$$= \left(\prod_{\mathbf{k}} \exp \left(\frac{i(v_{0\mathbf{k}}^{\Re}(t) v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t) v_{1\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (289)$$

Calculating the derivates respect to $\alpha_{i\mathbf{k}}^{\Re}$ and $\alpha_{i\mathbf{k}}^{\Im}$ we have:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial (\varepsilon_1(t) + R_1 + \varepsilon_0(t) + R_0(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (290)$$

$$= \frac{\partial \left(\left(\frac{(v_{i\mathbf{k}}^{\Re}(t))^2 + (v_{i\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im}(t) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (291)$$

$$= \frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}}, \quad (292)$$

$$\frac{\partial |B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial \left(\exp \left(-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (293)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} \exp \left(-\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (294)$$

$$= -\frac{2(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} |B_{10}(t)|^2, \quad (295)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial \sqrt{(\text{Tr}(\overline{H_{\bar{S}}(t)}))^2 - 4\text{Det}(\overline{H_{\bar{S}}(t)})}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (296)$$

$$= \frac{2\text{Tr}(\overline{H_{\bar{S}}(t)}) \frac{\partial \text{Tr}(\overline{H_{\bar{S}}(t)})}{\partial v_{i\mathbf{k}}^{\Re}(t)} - 4 \frac{\partial \text{Det}(\overline{H_{\bar{S}}(t)})}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2\sqrt{(\text{Tr}(\overline{H_{\bar{S}}(t)}))^2 - 4\text{Det}(\overline{H_{\bar{S}}(t)})}} \quad (297)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1(t))(\varepsilon_0(t) + R_0(t)) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \quad (298)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left(\varepsilon(t) - \varepsilon_i(t) - R_i(t) \right) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \quad (299)$$

$$= \frac{\varepsilon(t) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) - 2 \left(\varepsilon(t) - \varepsilon_i(t) - R_i(t) \right) \left(\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) + \frac{2(v_{i\mathbf{k}}^{\Re}(t) - v_{i'\mathbf{k}}^{\Re}(t))}{\omega_{\mathbf{k}}^2} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \quad (300)$$

$$= \frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} \left(\frac{2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + \frac{1}{\eta(t)} \left(-\frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \varepsilon(t) + 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} \right) \quad (301)$$

$$+ 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^2} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \quad (302)$$

From the equation (267) and replacing the derivates obtained we have:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{\frac{\partial\varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\frac{\partial\eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)}} \quad (303)$$

$$= \frac{\frac{2v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} - \frac{2g_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}}}{\frac{v_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} \left(\frac{\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\eta(t)} \right) + 2 \frac{(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} + 2 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}^2} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) - \frac{g_{i\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} \varepsilon(t)}{\eta(t)}}} \quad (304)$$

Rearrannng this equation will lead to:

$$\tanh\left(\frac{\beta\eta(t)}{2}\right) = \frac{(2v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}} - g_{i\mathbf{k}}^*) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} + g_{i\mathbf{k}}^*)(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t))) + 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (305)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re}(t) - 2g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t))) + 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (306)$$

$$= \frac{(2v_{i\mathbf{k}}^{\Re}(t) - 2g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(2\varepsilon(t) - 4(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - 2g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 4 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (307)$$

$$= \frac{(v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{v_{i\mathbf{k}}^{\Re}(t) \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)} \quad (308)$$

Separating (307) such that the terms with $v_{i\mathbf{k}}$ are located at one side of the equation permit us to write

$$\frac{(v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re}(t)) \eta(t)}{\tanh\left(\frac{\beta\eta(t)}{2}\right)} = v_{i\mathbf{k}}^{\Re}(t) \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - g_{i\mathbf{k}}^{\Re}(2(\varepsilon_i(t) + R_i(t) - \varepsilon(t)) + 2 \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (309)$$

$$v_{i\mathbf{k}}^{\Re}(t) - g_{i\mathbf{k}}^{\Re}(t) = v_{i\mathbf{k}}^{\Re}(t) \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right) - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} g_{i\mathbf{k}}^{\Re}(2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) \quad (310)$$

$$+ 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \frac{v_{i'\mathbf{k}}^{\Re}(t)}{\omega_{\mathbf{k}}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right), \quad (311)$$

$$v_{i\mathbf{k}}^{\Re}(t) = \frac{g_{i\mathbf{k}}^{\Re} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (312)$$

$$v_{i\mathbf{k}}^{\Re}(t) = \frac{g_{i\mathbf{k}}^{\Re} \left(1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} (2\varepsilon_i(t) + 2R_i(t) - \varepsilon(t)) + 2 \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)\omega_{\mathbf{k}}} \frac{v_{i'\mathbf{k}}^{\Re}(t)}{g_{i\mathbf{k}}^{\Re}} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right)}{1 - \frac{\tanh\left(\frac{\beta\eta(t)}{2}\right)}{\eta(t)} \left(\varepsilon(t) - 2(\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{2|V_{10}(t)|^2 |B_{10}(t)|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)}{\omega_{\mathbf{k}}} \right)} \quad (313)$$

The imaginary part can be found in the following way:

$$\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)} = \frac{\partial (\varepsilon_1(t) + R_1(t) + \varepsilon_0(t) + R_0(t))}{\partial v_{i\mathbf{k}}^{\Im}(t)} \quad (314)$$

$$= \frac{\partial \left(\left(\frac{(v_{i\mathbf{k}}^{\Re}(t))^2 + (v_{i\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}} - v_{i\mathbf{k}}^{\Re}(t) \frac{g_{i\mathbf{k}} + g_{i\mathbf{k}}^*}{\omega_{\mathbf{k}}} - i v_{i\mathbf{k}}^{\Im}(t) \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} \quad (315)$$

$$= 2 \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (316)$$

$$\frac{\partial |B_{10}(t)|^2}{\partial v_{i\mathbf{k}}^{\Im}(t)} = \frac{\partial \left(\exp \left(- \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right)}{\partial v_{i\mathbf{k}}^{\Im}(t)} \quad (317)$$

$$= - \frac{2 (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))}{\partial v_{i\mathbf{k}}^{\Im}(t)} \exp \left(- \sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}^{\Re}(t) - v_{0\mathbf{k}}^{\Re}(t))^2 + (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (318)$$

$$= - \frac{2 (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))}{\partial v_{i\mathbf{k}}^{\Im}(t)} |B_{10}(t)|^2, \quad (319)$$

$$\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Re}(t)} = \frac{\partial \sqrt{(\text{Tr}(\overline{H_{\bar{S}}(t)}))^2 - 4 \text{Det}(\overline{H_{\bar{S}}(t)})}}{\partial v_{i\mathbf{k}}^{\Re}(t)} \quad (320)$$

$$= \frac{2 \text{Tr}(\overline{H_{\bar{S}}(t)}) \frac{\partial \text{Tr}(\overline{H_{\bar{S}}(t)})}{\partial v_{i\mathbf{k}}^{\Re}(t)} - 4 \frac{\partial \text{Det}(\overline{H_{\bar{S}}(t)})}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{2 \sqrt{(\text{Tr}(\overline{H_{\bar{S}}(t)}))^2 - 4 \text{Det}(\overline{H_{\bar{S}}(t)})}} \quad (321)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \frac{\partial ((\varepsilon_1(t) + R_1(t))(\varepsilon_0(t) + R_0(t)) - |V_{10}(t)|^2 |B_{10}(t)|^2)}{\partial v_{i\mathbf{k}}^{\Re}(t)}}{\eta(t)} \quad (322)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \left(2 \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \frac{2 (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))}{\omega_{\mathbf{k}}^2} \frac{\partial (v_{1\mathbf{k}}^{\Im}(t) - v_{0\mathbf{k}}^{\Im}(t))}{\partial v_{i\mathbf{k}}^{\Re}(t)} |B_{10}(t)|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}{\eta(t)} \quad (323)$$

$$= \frac{\varepsilon(t) \left(2 \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) - 2 \left((\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \left(2 \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \frac{2 (v_{i\mathbf{k}}^{\Im}(t) - v_{i'\mathbf{k}}^{\Im}(t)) |B_{10}(t)|^2 |V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}^2} \right)}{\eta(t)} \quad (324)$$

$$= \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} \frac{4 (\varepsilon_i(t) + R_i(t)) - 2 \varepsilon(t) - \frac{4 |B_{10}(t) V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}}}{\eta(t)} + \frac{1}{\eta(t)} \left(2 \frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} \varepsilon(t) - 4 (\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} + 4 \frac{v_{i'\mathbf{k}}^{\Im}(t) |B_{10}(t) V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}^2} \right). \quad (325)$$

From the equation (267) and replacing the derivatives obtained we have:

$$\frac{\frac{\partial \varepsilon(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}}{\frac{\partial \eta(t)}{\partial v_{i\mathbf{k}}^{\Im}(t)}} = \tanh \left(\frac{\beta \eta(t)}{2} \right) \quad (326)$$

$$= \frac{2 \frac{v_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} - i \frac{g_{i\mathbf{k}}^* - g_{i\mathbf{k}}}{\omega_{\mathbf{k}}}}{v_{i\mathbf{k}}^{\Im}(t) \left(\frac{2 \varepsilon(t) - 4 (\varepsilon(t) - \varepsilon_i(t) - R_i(t)) - \frac{4 |B_{10}(t) V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right)}{\omega_{\mathbf{k}}}}{\eta(t) \omega_{\mathbf{k}}} \right) + \frac{2}{\eta(t)} \left(\frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} \varepsilon(t) - 2 (\varepsilon(t) - \varepsilon_i(t) - R_i(t)) \frac{g_{i\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}} + 2 \frac{v_{i'\mathbf{k}}^{\Im}(t)}{\omega_{\mathbf{k}}^2} |B_{10}(t) V_{10}(t)|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right)}. \quad (327)$$

Rearranging this equation will lead to:

$\rho_B^{\text{Thermal}} \equiv \rho_B(0) \equiv \rho_B$, so the transformed density operator is equal to:

$$\overline{\rho_T(0)} \equiv e^{V(0)} \rho_T(0) e^{-V(0)} \quad (341)$$

$$= (|0\rangle\langle 0| B_0^+(0) + |1\rangle\langle 1| B_1^+(0)) (\rho_S(0) \otimes \rho_B) (|0\rangle\langle 0| B_0^-(0) + |1\rangle\langle 1| B_1^-(0)), \quad (342)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 0| : |0\rangle\langle 0| B_0^+(0) \langle 0| \rho_B |0\rangle\langle 0| B_0^-(0) \quad (343)$$

$$= |0\rangle\langle 0| B_0^+(0) \langle 0| \rho_B |0\rangle\langle 0| B_0^-(0) \quad (344)$$

$$= |0\rangle\langle 0| \otimes B_0^+(0) \rho_B B_0^-(0), \quad (345)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 1| : |1\rangle\langle 1| B_1^+(0) |1\rangle\langle 1| \rho_B |1\rangle\langle 1| B_1^-(0) \quad (346)$$

$$= |1\rangle\langle 1| B_1^+(0) \rho_B B_1^-(0) \quad (347)$$

$$= |1\rangle\langle 1| \otimes B_1^+(0) \rho_B B_1^-(0), \quad (348)$$

$$\text{for } \rho_S(0) = |0\rangle\langle 1| : |0\rangle\langle 0| B_0^+(0) |0\rangle\langle 1| \rho_B |1\rangle\langle 1| B_1^-(0) \quad (349)$$

$$= |0\rangle\langle 1| B_0^+(0) \rho_B |1\rangle\langle 1| B_1^-(0) \quad (350)$$

$$= |0\rangle\langle 1| |1\rangle\langle 1| B_0^+(0) \rho_B B_1^-(0) \quad (351)$$

$$= |0\rangle\langle 1| \otimes B_0^+(0) \rho_B B_1^-(0), \quad (352)$$

$$\text{for } \rho_S(0) = |1\rangle\langle 0| : |1\rangle\langle 1| B_1^+(0) |1\rangle\langle 0| \rho_B |0\rangle\langle 0| B_0^-(0) \quad (353)$$

$$= |1\rangle\langle 0| \otimes B_1^+(0) \rho_B B_0^-(0). \quad (354)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O}(t) \equiv U^\dagger(t) O(t) U(t), \quad (355)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (356)$$

Here \mathcal{T} denotes a time ordering operator. Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (357)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}_T(t)). \quad (358)$$

. In order to separate the Hamiltonian we define the matrix $\Lambda(t)$ such that $\Lambda_{1i}(t) = A_i$, $\Lambda_{2i}(t) = B_i$ and $\Lambda_{3i}(t) = C_i(t)$ written as:

$$\begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} \sigma_x & \sigma_y & \frac{I-\sigma_z}{2} & \sigma_x & \sigma_y & \frac{I+\sigma_z}{2} \\ B_x(t) & B_y(t) & B_{1z}(t) & B_y(t) & B_x(t) & B_{0z}(t) \\ V_{10}^{\Re}(t) & V_{10}^{\Re}(t) & 1 & V_{10}^{\Im}(t) & -V_{10}^{\Im}(t) & 1 \end{pmatrix}. \quad (359)$$

In this case $|1\rangle\langle 1| = \frac{I-\sigma_z}{2}$ and $|0\rangle\langle 0| = \frac{I+\sigma_z}{2}$ with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

The previous notation allows us to write the interaction Hamiltonian $\overline{H_I}(t)$ as pointed in the equation (??):

$$\overline{H_I}(t) = \sum_i B_{iz}(t) |i\rangle\langle i| + V_{10}^{\Re}(t) (\sigma_x B_x(t) + \sigma_y B_y(t)) + V_{10}^{\Im}(t) (\sigma_x B_y(t) - \sigma_y B_x(t)) \quad (360)$$

$$= B_{0z}(t) |0\rangle\langle 0| + B_{1z}(t) |1\rangle\langle 1| + V_{10}^{\Re}(t) \sigma_x B_x(t) + V_{10}^{\Re}(t) \sigma_y B_y(t) + V_{10}^{\Im}(t) \sigma_x B_y(t) - V_{10}^{\Im}(t) \sigma_y B_x(t) \quad (361)$$

$$= \sum_i C_i(t) (A_i \otimes B_i(t)). \quad (362)$$

As the combined system and environment is closed, within the interaction picture the system-environment density operator evolves according to:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \widetilde{\rho_T}(t)]. \quad (363)$$

This equation has the formal solution

$$\widetilde{\rho_T}(t) = \overline{\rho_T}(0) - i \int_0^t [\widetilde{H_I}(s), \widetilde{\rho_T}(s)] ds. \quad (364)$$

Replacing the equation (364) in the equation (363) gives us:

$$\frac{d\widetilde{\rho_T}(t)}{dt} = -i[\widetilde{H_I}(t), \overline{\rho_T}(0)] - \int_0^t [\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_T}(s)]] ds. \quad (365)$$

This equation allow us to iterate and write in terms of a series expansion with $\overline{\rho_T}(0)$ the solution as:

$$\widetilde{\rho_T}(t) = \overline{\rho_T}(0) + \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \overline{\rho_T}(0)]] \dots]. \quad (366)$$

Taking the trace over the environmental degrees of freedom, we find

$$\widetilde{\rho_S}(t) = \overline{\rho_S}(0) + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), \overline{\rho_S}(0) \rho_B]] \dots]. \quad (367)$$

here we have assumed that $\overline{\rho_T}(0) = \overline{\rho_S}(0) \otimes \rho_B$. Consider the following notation:

$$\widetilde{\rho_S}(t) = (1 + W_1(t) + W_2(t) + \dots) \overline{\rho_S}(0) \quad (368)$$

$$= W(t) \overline{\rho_S}(0). \quad (369)$$

in this case

$$W_n(t) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_B [\widetilde{H_I}(t_1), [\widetilde{H_I}(t_2), \dots [\widetilde{H_I}(t_n), (\cdot) \rho_B]] \dots]. \quad (370)$$

are superoperators acting on the initial system density operator. Differentiating with respect to time, we have:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = (\dot{W}_1(t) + \dot{W}_2(t) + \dots) \overline{\rho_S}(0) \quad (371)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} W(t) \overline{\rho_S}(0) \quad (372)$$

$$= (\dot{W}_1(t) + \dot{W}_2(t) + \dots) W(t)^{-1} \widetilde{\rho_S}(t). \quad (373)$$

where we assumed that $W(t)$ is invertible. Usually, it is convenient (and possible) to define the interaction Hamiltonian such that $\text{Tr}_B [\widetilde{H_I}(t) \rho_B] = 0$ so $W_1(t) = 0$. Thus, to second order and approximating $W(t) \approx \mathbb{I}$ then the equation (371) becomes:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = \dot{W}_2(t) \widetilde{\rho_S}(t) \quad (374)$$

$$= - \int_0^t dt_1 \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(t_1), \widetilde{\rho_S}(t) \rho_B]]. \quad (375)$$

Replacing $t_1 \rightarrow t - \tau$

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \int_0^t d\tau \text{Tr}_B [\overline{H_I}(t), [\widetilde{\overline{H_I}}(-\tau), \overline{\rho_S}(t) \rho_B]] . \quad (376)$$

From the interaction picture applied on $\overline{H_I}(t)$ we find:

$$\widetilde{\overline{H_I}}(t) = U^\dagger(t) e^{iH_B t} \overline{H_I}(t) e^{-iH_B t} U(t) . \quad (377)$$

we use the time-ordering operator \mathcal{T} because in general $\overline{H_S}(t)$ doesn't commute with itself at two different times. We write the interaction Hamiltonian as:

$$\widetilde{\overline{H_I}}(t) = \sum_i C_i(t) (\widetilde{A_i}(t) \otimes \widetilde{B_i}(t)) , \quad (378)$$

$$\widetilde{A_i}(t) = U^\dagger(t) e^{iH_B t} A_i e^{-iH_B t} U(t) \quad (379)$$

$$= U^\dagger(t) A_i U(t) e^{iH_B t} e^{-iH_B t} \quad (380)$$

$$= U^\dagger(t) A_i U(t) \mathbb{I} \quad (381)$$

$$= U^\dagger(t) A_i U(t) , \quad (382)$$

$$\widetilde{B_i}(t) = U^\dagger(t) e^{iH_B t} B_i(t) e^{-iH_B t} U(t) \quad (383)$$

$$= U^\dagger(t) U(t) e^{iH_B t} B_i(t) e^{-iH_B t} \quad (384)$$

$$= \mathbb{I} e^{iH_B t} B_i(t) e^{-iH_B t} \quad (385)$$

$$= e^{iH_B t} B_i(t) e^{-iH_B t} . \quad (386)$$

Here we have used the fact that $[\overline{H_S}(t), H_B] = 0$ because these operators belong to different Hilbert spaces, so $[U(t), e^{iH_B t}] = 0$.

Using the expression (378) to replace it in the equation (375)

$$\frac{d\widetilde{\overline{\rho_S}}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{\overline{H_I}}(t), [\widetilde{\overline{H_I}}(s), \widetilde{\overline{\rho_S}}(t) \rho_B]] ds \quad (387)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \left[\sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) , \widetilde{\overline{\rho_S}}(t) \rho_B \right] \right] ds \quad (388)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) , \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B - \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \right] ds \quad (389)$$

$$= - \int_0^t \text{Tr}_B (\sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B - \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s))) \quad (390)$$

$$- \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \widetilde{\overline{\rho_S}}(t) \rho_B \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t)) + \widetilde{\overline{\rho_S}}(t) \rho_B \sum_i C_i(s) (\widetilde{A_i}(s) \otimes \widetilde{B_i}(s)) \sum_j C_j(t) (\widetilde{A_j}(t) \otimes \widetilde{B_j}(t))) ds. \quad (391)$$

In order to calculate the correlation functions we define:

$$\mathcal{B}_{ij}(t, s) = \text{Tr}_B (\widetilde{B_i}(t) \widetilde{B_j}(s) \rho_B) \quad (392)$$

An useful property is

$$\mathcal{B}_{ji}^*(t, s) = \text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B \right)^\dagger \quad (393)$$

$$= \text{Tr}_B \left(\rho_B^\dagger \widetilde{B}_i^\dagger(s) \widetilde{B}_j^\dagger(t) \right) \quad (394)$$

$$= \text{Tr}_B \left(\rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) \right) \quad (395)$$

$$= \text{Tr}_B \left(\widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B \right) \quad (396)$$

$$= \mathcal{B}_{ij}(s, t) \quad (397)$$

The correlation functions relevant that appear in the equation (391) are:

$$\text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B \right) = \left\langle \widetilde{B}_j(t) \widetilde{B}_i(s) \right\rangle_B \quad (398)$$

$$= \mathcal{B}_{ji}(t, s) \quad (399)$$

$$= \mathcal{B}_{ij}^*(s, t) \quad (400)$$

$$\text{Tr}_B \left(\widetilde{B}_j(t) \rho_B \widetilde{B}_i(s) \right) = \text{Tr}_B \left(\widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B \right) \quad (401)$$

$$= \mathcal{B}_{ij}(s, t) \quad (402)$$

$$\text{Tr}_B \left(\widetilde{B}_i(s) \rho_B \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B \right) \quad (403)$$

$$= \mathcal{B}_{ij}^*(s, t) \quad (404)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) \right) = \text{Tr}_B \left(\widetilde{B}_i(s) \widetilde{B}_j(t) \rho_B \right) \quad (405)$$

$$= \mathcal{B}_{ij}(s, t) \quad (406)$$

The cyclic property of the trace was use widely in the development of equations (398) and (406). Replacing in (391)

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left(\sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \widetilde{\rho_S}(t) \rho_B - \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \widetilde{\rho_S}(t) \rho_B \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \right) \quad (407)$$

$$- \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \widetilde{\rho_S}(t) \rho_B \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) + \widetilde{\rho_S}(t) \rho_B \sum_i C_i(s) (\widetilde{A}_i(s) \otimes \widetilde{B}_i(s)) \sum_j C_j(t) (\widetilde{A}_j(t) \otimes \widetilde{B}_j(t)) \Big) ds. \quad (408)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ji} C_j(t) C_i(s) (\widetilde{A}_j(t) \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{B}_j(t) \rho_B \widetilde{B}_i(s)) \right) \quad (409)$$

$$+ \sum_{ij} C_i(s) C_j(t) (\widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{A}_j(t) \rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) - \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(s) \rho_B \widetilde{B}_j(t)) ds \quad (410)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ji} C_j(t) C_i(s) (\widetilde{A}_j(t) \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{B}_j(t) \rho_B \widetilde{B}_i(s)) \right) \quad (411)$$

$$+ \sum_{ij} C_i(s) C_j(t) (\widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{A}_j(t) \rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) - \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(s) \rho_B \widetilde{B}_j(t)) ds \quad (412)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ij} C_j(t) C_i(s) (\widetilde{A}_j(t) \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{B}_j(t) \rho_B \widetilde{B}_i(s)) \right) \text{ (by permuting i and j because } i, j \in \mathbb{I}) \quad (413)$$

$$+ \sum_{ij} C_i(s) C_j(t) (\widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{A}_j(t) \rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) - \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(s) \rho_B \widetilde{B}_j(t)) ds \quad (414)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{ij} C_j(t) C_i(s) (\widetilde{A}_j(t) \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{B}_j(t) \widetilde{B}_i(s) \rho_B - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{B}_j(t) \rho_B \widetilde{B}_i(s)) \right) \quad (415)$$

$$+ \widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{A}_j(t) \rho_B \widetilde{B}_i(s) \widetilde{B}_j(t) - \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{A}_j(t) \widetilde{B}_i(s) \rho_B \widetilde{B}_j(t) \Big) ds \quad (416)$$

$$= - \int_0^t \left(\sum_{ij} C_j(t) C_i(s) (\widetilde{A}_j(t) \widetilde{A}_i(s) \widetilde{\rho_S}(t) \mathcal{B}_{ji}(t, s) - \widetilde{A}_j(t) \widetilde{\rho_S}(t) \widetilde{A}_i(s) \mathcal{B}_{ij}(s, t)) \right. \quad (417)$$

$$\left. + \widetilde{\rho_S}(t) \widetilde{A}_i(s) \widetilde{A}_j(t) \mathcal{B}_{ij}(s, t) - \widetilde{A}_i(s) \widetilde{\rho_S}(t) \widetilde{A}_j(t) \mathcal{B}_{ji}(t, s) \right) ds \quad (418)$$

$$= - \int_0^t \left(\sum_{ij} C_j(t) C_i(s) \left(\mathcal{B}_{ji}(t, s) \left[\widetilde{A}_j(t), \widetilde{A}_i(s) \widetilde{\rho_S}(t) \right] + \mathcal{B}_{ij}(s, t) \left[\widetilde{\rho_S}(t) \widetilde{A}_i(s), \widetilde{A}_j(t) \right] \right) \right) ds \quad (419)$$

$$= - \int_0^t \left(\sum_{ij} C_i(t) C_j(s) \left(\mathcal{B}_{ij}(t, s) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] + \mathcal{B}_{ji}(s, t) \left[\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t) \right] \right) \right) ds \text{ (exchanging i and j)} \quad (420)$$

$$= - \int_0^t \left(\sum_{ij} C_i(t) C_j(s) \left(\mathcal{B}_{ij}(t, s) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] + \mathcal{B}_{ij}^*(t, s) \left[\widetilde{\rho_S}(t) \widetilde{A}_j(s), \widetilde{A}_i(t) \right] \right) \right) ds \quad (421)$$

$$= - \int_0^t \left(\sum_{ij} C_i(t) C_j(s) \left(\mathcal{B}_{ij}(t, s) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] - \mathcal{B}_{ij}^*(t, s) \left[\widetilde{A}_i(t), \widetilde{\rho_S}(t) \widetilde{A}_j(s) \right] \right) \right) ds \quad (422)$$

We could identify the following commutators in the equation deduced:

$$\mathcal{B}_{ij}(t, s) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) - \mathcal{B}_{ij}(t, s) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) = \mathcal{B}_{ij}(t, s) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right], \quad (423)$$

$$\mathcal{B}_{ij}^*(t, s) \widetilde{\rho_S}(t) \widetilde{A}_j(s) \widetilde{A}_i(t) - \mathcal{B}_{ij}^*(t, s) \widetilde{A}_i(t) \widetilde{\rho_S}(t) \widetilde{A}_j(s) = \mathcal{B}_{ij}^*(t, s) \left[\widetilde{\rho_S}(t) \widetilde{A}_j, \widetilde{A}_i(t) \right]. \quad (424)$$

Returning to the Schroedinger picture we have:

$$U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) = U(t) \widetilde{A}_i(t) U^\dagger(t) U(t) \widetilde{A}_j(s) U^\dagger(t) U(t) \widetilde{\rho_S}(t) U^\dagger(t), \quad (425)$$

$$= \left(U(t) \widetilde{A}_i(t) U^\dagger(t) \right) \left(U(t) \widetilde{A}_j(s) U^\dagger(t) \right) \left(U(t) \widetilde{\rho_S}(t) U^\dagger(t) \right), \quad (426)$$

$$= A_i(t) \widetilde{A}_j(s, t) \overline{\rho_S}(t). \quad (427)$$

This procedure applying to the relevant commutators give us:

$$U(t) \left[\widetilde{A}_i(t), \widetilde{A}_j(s) \widetilde{\rho_S}(t) \right] U^\dagger(t) = \left(U(t) \widetilde{A}_i(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) U^\dagger(t) - U(t) \widetilde{A}_j(s) \widetilde{\rho_S}(t) \widetilde{A}_i(t) U^\dagger(t) \right) \quad (428)$$

$$= A_i(t) \widetilde{A}_j(s, t) \overline{\rho_S}(t) - \widetilde{A}_j(s, t) \overline{\rho_S}(t) A_i(t) \quad (429)$$

$$= \left[A_i(t), \widetilde{A}_j(s, t) \overline{\rho_S}(t) \right]. \quad (430)$$

Introducing this transformed commutators in the equation (422) allow us to obtain the master equation of the system written as an integro-differential equation with the correlation functions $\mathcal{B}_{ij}(\tau)$ as defined before, this equations has the following form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t ds C_i(t) C_j(s) \left(\mathcal{B}_{ij}(t, s) \left[A_i(t), \widetilde{A}_j(s, t) \overline{\rho_S}(t) \right] + \mathcal{B}_{ij}^*(t, s) \left[\overline{\rho_S}(t) \widetilde{A}_j(s, t), A_i(t) \right] \right) \quad (431)$$

$$s = t - \tau \text{ (Change of variables in the integration process)} \quad (432)$$

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau C_i(t) C_j(t - \tau) \left(\mathcal{B}_{ij}(t, t - \tau) \left[A_i(t), \widetilde{A}_j(t - \tau, t) \overline{\rho_S}(t) \right] + \mathcal{B}_{ij}^*(t, t - \tau) \left[\overline{\rho_S}(t) \widetilde{A}_j(t - \tau, t), A_i(t) \right] \right) \quad (433)$$

where $i, j \in \{1, 2, 3, 4, 5, 6\}$.

Here $\widetilde{A}_j(t - \tau, t) = U(t) U^\dagger(t - \tau) A_j(t) U(t - \tau) U^\dagger(t)$ where $U(t)$ is given by (356). The equation obtained is a non-Markovian master equation which describes the QD exciton dynamics in the variational frame with a general time-dependent Hamiltonian, and valid at second order in $H_I(t)$. In order to write in a simplified way we define the following notation:

$$\mathcal{B}_{ij}(t, s) = \text{Tr}_B \left(\widetilde{B}_i(t) \widetilde{B}_j(s) \rho_B \right) \quad (434)$$

$$= \text{Tr}_B \left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B s} B_j(s) e^{-iH_B s} \rho_B \right) \quad (435)$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad (436)$$

$$e^{-iH_B s} e^{-\beta H_B} = \sum_{m=0}^{\infty} \frac{(-iH_B s)^m}{m!} \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \quad (437)$$

$$= \sum_{m,n} \frac{(-iH_B s)^m}{m!} \frac{(-\beta H_B)^n}{n!} \quad (438)$$

$$= \sum_{m,n} \frac{(-is)^m}{m!} \frac{(-\beta)^n}{n!} H_B^m H_B^n \quad (439)$$

$$= \sum_{m,n} \frac{(-is)^m}{m!} \frac{(-\beta)^n}{n!} H_B^n H_B^m \text{ (because the powers of a matrix commute)} \quad (440)$$

$$= \sum_{m,n} \frac{(-\beta)^n}{n!} H_B^n \frac{(-is)^m}{m!} H_B^m \quad (441)$$

$$= \sum_{m,n} \frac{(-\beta H_B)^n}{n!} \frac{(-is H_B)^m}{m!} \quad (442)$$

$$= \sum_{n=0}^{\infty} \frac{(-\beta H_B)^n}{n!} \sum_{m=0}^{\infty} \frac{(-iH_B s)^m}{m!} \quad (443)$$

$$= e^{-\beta H_B} e^{-iH_B s} \quad (444)$$

$$0 = e^{-iH_B s} e^{-\beta H_B} - e^{-\beta H_B} e^{-iH_B s} \text{ (then } e^{-iH_B s} \text{ and } \rho_B \text{ commute)} \quad (445)$$

$$\mathcal{B}_{ij}(t, s) = \text{Tr}_B \left(e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B s} B_j(s) \rho_B e^{-iH_B s} \right) \text{ (by permuting } e^{-iH_B s} \text{ and } \rho_B \text{ because they commute)} \quad (446)$$

$$= \text{Tr}_B \left((e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B s} B_j(s)) \rho_B e^{-iH_B s} \right) \text{ (by associative property)} \quad (447)$$

$$= \text{Tr}_B \left(e^{-iH_B s} (e^{iH_B t} B_i(t) e^{-iH_B t} e^{iH_B s} B_j(s)) \rho_B \right) \text{ (by cyclic property of the trace)} \quad (448)$$

$$= \text{Tr}_B \left((e^{-iH_B s} e^{iH_B t}) B_i(t) (e^{-iH_B t} e^{iH_B s}) B_j(s) \rho_B \right) \text{ (by associative property)} \quad (449)$$

$$[iH_B t, -iH_B s] = iH_B t (-iH_B s) - (-iH_B s) iH_B t \quad (450)$$

$$= ts H_B^2 - ts H_B^2 \quad (451)$$

$$= 0 \text{ (so } iH_B t \text{ and } -iH_B s \text{ commute)} \quad (452)$$

$$e^{-iH_B s} e^{iH_B t} = e^{iH_B t - iH_B s} \text{ (by the Zassenhaus formula because } iH_B t \text{ and } -iH_B s \text{ commute)} \quad (453)$$

$$= e^{iH_B(t-s)} \quad (454)$$

$$= e^{iH_B \tau} \quad (455)$$

$$e^{iH_B s} e^{-iH_B t} = e^{-iH_B t + iH_B s} \text{ (by the Zassenhaus formula because } -iH_B t \text{ and } iH_B s \text{ commute)} \quad (456)$$

$$= e^{iH_B(-t+s)} \quad (457)$$

$$= e^{-iH_B \tau} \quad (458)$$

$$\mathcal{B}_{ij}(t, s) = \text{Tr}_B \left(e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(s) \rho_B \right) \quad (459)$$

$$B_i(t, \tau) \equiv e^{iH_B \tau} B_i(t) e^{-iH_B \tau} \quad (460)$$

$$\mathcal{B}_{ij}(t, s) = \text{Tr}_B \left(e^{iH_B(t-s)} B_i(t) e^{-iH_B(t-s)} B_j(s) \rho_B \right) \quad (461)$$

$$s = t - \tau \quad (462)$$

$$\mathcal{B}_{ij}(t, s) = \text{Tr}_B \left(e^{iH_B \tau} B_i(t) e^{-iH_B \tau} B_j(s) \rho_B \right) \quad (463)$$

$$= \text{Tr}_B \left(B_i(t, \tau) B_j(s, 0) \rho_B \right) \quad (464)$$

Calculating the correlation functions allow us to obtain:

$$\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \rangle_B = \text{Tr}_B (B_{jz}(t, \tau) B_{jz}(s, 0) \rho_B) \quad (465)$$

$$= \int d^2\alpha P(\alpha) \langle \alpha | B_{jz}(t, \tau) B_{jz}(s, 0) | \alpha \rangle \quad (466)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | B_{jz}(t, \tau) B_{jz}(s, 0) | \alpha \rangle d^2\alpha \quad (467)$$

$$= \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha|^2}{N}\right) \langle \alpha | B_{jz}(t, \tau) B_{jz}(s, 0) | \alpha \rangle d^2\alpha, \quad (468)$$

$$B_{jz}(t, \tau) = \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right), \quad (469)$$

$$B_{jz}(s, 0) = \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(s))^* b_{\mathbf{k}'} \right), \quad (470)$$

$$\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \rangle_B = \text{Tr}_B (B_{jz}(t, \tau) B_{jz}(s, 0) \rho_B) \quad (471)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(s))^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (472)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger + (g_{j\mathbf{k}'} - v_{j\mathbf{k}'}(s))^* b_{\mathbf{k}'} \right) \rho_B \right) \quad (473)$$

$$+ \text{Tr}_B \left(\sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(s)) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))^* b_{\mathbf{k}} \right) \rho_B \right), \quad (474)$$

$$g_{j\mathbf{k}} - v_{j\mathbf{k}}(t) = q_{j\mathbf{k}}(t) \quad (475)$$

$$\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \rangle_B = \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}'}(s) b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(s) b_{\mathbf{k}'} \right) \rho_B \right) \quad (476)$$

$$+ \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}}(s) b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*(s) b_{\mathbf{k}} \right) \rho_B \right) \quad (477)$$

$$0 = \text{Tr}_B \left(\sum_{\mathbf{k} \neq \mathbf{k}'} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}'}(s) b_{\mathbf{k}'}^\dagger + q_{j\mathbf{k}'}^*(s) b_{\mathbf{k}'} \right) \rho_B \right) \quad (478)$$

$$\langle \widetilde{B_{jz}}(t) \widetilde{B_{jz}}(s) \rangle_B = 0 + \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left(q_{j\mathbf{k}}(s) b_{\mathbf{k}}^\dagger + q_{j\mathbf{k}}^*(s) b_{\mathbf{k}} \right) \rho_B \right) \quad (479)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} \left(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(s) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(s) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} + q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}^*(s) b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \rho_B \right) \quad (480)$$

$$= (\sum_{\mathbf{k}} \text{Tr}_B(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}(s) b_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} \rho_B) + \text{Tr}_B(q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(s) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B) + \text{Tr}_B(q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(s) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B) + \text{Tr}_B(q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}^*(s) b_{\mathbf{k}} b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \rho_B)) \quad (481)$$

$$= \text{Tr}_B \left(\sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(s) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \rho_B \right) + \text{Tr}_B \left(\sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(s) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \rho_B \right) \quad (482)$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(s) e^{i\omega_{\mathbf{k}}\tau} \text{Tr}_B (b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rho_B) + \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(s) e^{-i\omega_{\mathbf{k}}\tau} \text{Tr}_B (b_{\mathbf{k}} b_{\mathbf{k}}^\dagger \rho_B) \quad (483)$$

$$= \sum_{\mathbf{k}} q_{j\mathbf{k}}(t) q_{j\mathbf{k}}^*(s) e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}} \quad (484)$$

$$+ \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j\mathbf{k}}(s) e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{N}\right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2\alpha_{\mathbf{k}} \quad (485)$$

$$\left\langle \widetilde{B_{jz}}(t) \widetilde{B_{j'z}}(s) \right\rangle_B = \text{Tr}_B (B_{jz}(t, \tau) B_{j'z}(s, 0) \rho_B) \quad (507)$$

$$= \int d^2 \alpha P(\alpha) \langle \alpha | B_{jz}(t, \tau) B_{j'z}(s, 0) | \alpha \rangle \quad (508)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha|^2}{N} \right) \langle \alpha | B_{jz}(t, \tau) B_{j'z}(s, 0) | \alpha \rangle d^2 \alpha \quad (509)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}'} \left((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}(s))^* b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (510)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k} \neq \mathbf{k}'} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger + (g_{j'\mathbf{k}'} - v_{j'\mathbf{k}'}(s))^* b_{\mathbf{k}'} \right) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (511)$$

$$+ \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s)) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* b_{\mathbf{k}} \right) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (512)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \left((g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s)) b_{\mathbf{k}}^\dagger + (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* b_{\mathbf{k}} \right) | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (513)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \left\langle \alpha_{\mathbf{k}} \left| \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* b_{\mathbf{k}}^\dagger b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \right| \alpha_{\mathbf{k}} \right\rangle d^2 \alpha_{\mathbf{k}} \quad (514)$$

$$+ \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \left\langle \alpha_{\mathbf{k}} \left| \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s)) b_{\mathbf{k}} b_{\mathbf{k}}^\dagger e^{-i\omega_{\mathbf{k}}\tau} \right| \alpha_{\mathbf{k}} \right\rangle d^2 \alpha_{\mathbf{k}} \quad (515)$$

$$= \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t)) (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* e^{i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} \quad (516)$$

$$+ \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s)) e^{-i\omega_{\mathbf{k}}\tau} \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}}, \quad (517)$$

$$\frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (518)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (519)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (520)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) |\alpha_{\mathbf{k}}|^2 d^2 \alpha_{\mathbf{k}} \quad (521)$$

$$= N, \quad (522)$$

$$\frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle \alpha_{\mathbf{k}} | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | \alpha_{\mathbf{k}} \rangle d^2 \alpha_{\mathbf{k}} = \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}} D(\alpha_{\mathbf{k}}) D(-\alpha_{\mathbf{k}}) b_{\mathbf{k}}^\dagger D(\alpha_{\mathbf{k}}) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (523)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | (b_{\mathbf{k}} + \alpha_{\mathbf{k}}) (b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}}^*) | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (524)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + \alpha_{\mathbf{k}} b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} \alpha_{\mathbf{k}}^* + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (525)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger + |\alpha_{\mathbf{k}}|^2 | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (526)$$

$$= \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | \alpha_{\mathbf{k}} | 0 \rangle d^2 \alpha_{\mathbf{k}} + \frac{1}{\pi N} \int \exp \left(-\frac{|\alpha_{\mathbf{k}}|^2}{N} \right) \langle 0 | b_{\mathbf{k}} b_{\mathbf{k}}^\dagger | 0 \rangle d^2 \alpha_{\mathbf{k}} \quad (527)$$

$$= N + 1, \quad (528)$$

$$\langle \widetilde{B_{jz}}(t) \widetilde{B_{j'z}}(s) \rangle_B = \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* e^{i\omega_{\mathbf{k}}\tau} N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s)) e^{-i\omega_{\mathbf{k}}\tau} (N + 1) \quad (529)$$

$$= \sum_{\mathbf{k}} ((g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))(g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s))^* e^{i\omega_{\mathbf{k}}\tau} + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s)) e^{-i\omega_{\mathbf{k}}\tau}) N + \sum_{\mathbf{k}} (g_{j\mathbf{k}} - v_{j\mathbf{k}}(t))^* (g_{j'\mathbf{k}} - v_{j'\mathbf{k}}(s)) e^{-i\omega_{\mathbf{k}}\tau} \quad (530)$$

$$= \sum_{\mathbf{k}} 2N (q_{j\mathbf{k}}(t) q_{j'\mathbf{k}}^*(s) e^{i\omega_{\mathbf{k}}\tau})^{\Re} + \sum_{\mathbf{k}} q_{j\mathbf{k}}^*(t) q_{j'\mathbf{k}}(s) e^{-i\omega_{\mathbf{k}}\tau} \quad (531)$$

$$D(h') D(h) = \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) D(h' + h), \quad (532)$$

$$\langle D(h') D(h) \rangle_B = \text{Tr}_B \left(\exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) D(h' + h) \rho_B \right) \quad (533)$$

$$= \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \text{Tr}_B (D(h' + h) \rho_B) \quad (534)$$

$$= \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \frac{1}{\pi N} \int d^2\alpha P(\alpha) \langle \alpha | D(h' + h) | \alpha \rangle \quad (535)$$

$$= \exp\left(\frac{1}{2}(h'h^* - h'^*h)\right) \exp\left(-\frac{|h + h'|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \quad (536)$$

$$h' = h \exp(i\omega\tau), \quad (537)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp\left(\frac{1}{2}(h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau))\right) \exp\left(-\frac{|h + h \exp(i\omega\tau)|^2}{2} \coth\left(\frac{\beta\omega}{2}\right)\right), \quad (538)$$

$$\frac{1}{2}|h|^2(\exp(i\omega\tau) - \exp(-i\omega\tau)) = \frac{1}{2}(h h^* \exp(i\omega\tau) - h^* h \exp(-i\omega\tau)) \quad (539)$$

$$= \frac{1}{2}|h|^2(\cos(\omega\tau) + i \sin(\omega\tau) - \cos(\omega\tau) + i \sin(\omega\tau)) \quad (540)$$

$$= \frac{1}{2}|h|^2(2i \sin(\omega\tau)) \quad (541)$$

$$= i|h|^2 \sin(\omega\tau), \quad (542)$$

$$-\frac{|h + h \exp(i\omega\tau)|^2}{2} = -|h|^2 \frac{|1 + \exp(i\omega\tau)|^2}{2} \quad (543)$$

$$= -|h|^2 \frac{(1 + 2 \cos(\omega\tau) + \cos^2(\omega\tau)) + \sin^2(\omega\tau)}{2} \quad (544)$$

$$= -|h|^2 \frac{2 + 2 \cos(\omega\tau)}{2} \quad (545)$$

$$= -|h|^2 (1 + \cos(\omega\tau)), \quad (546)$$

$$\langle D(h \exp(i\omega\tau)) D(h) \rangle_B = \exp(i|h|^2 \sin(\omega\tau)) \exp(-|h|^2 (1 + \cos(\omega\tau)) \coth\left(\frac{\beta\omega}{2}\right)) \quad (547)$$

$$= \exp \left(i |h|^2 \sin(\omega\tau) - |h|^2 (1 + \cos(\omega\tau)) \coth \left(\frac{\beta\omega}{2} \right) \right) \quad (548)$$

$$= \exp \left(- |h|^2 \left(-i \sin(\omega\tau) + \cos(\omega\tau) \coth \left(\frac{\beta\omega}{2} \right) \right) \right) \exp \left(- |h|^2 \coth \left(\frac{\beta\omega}{2} \right) \right) \quad (549)$$

$$= \langle D(h) \rangle_B \exp(-\phi(\tau)), \quad (550)$$

$$\exp(-\phi(\tau)) = \exp \left(- |h|^2 \left(\cos(\omega\tau) \coth \left(\frac{\beta\omega}{2} \right) - i \sin(\omega\tau) \right) \right), \quad (551)$$

$$\phi(\tau) = |h|^2 \left(\cos(\omega\tau) \coth \left(\frac{\beta\omega}{2} \right) - i \sin(\omega\tau) \right), \quad (552)$$

$$\langle D(h') D(h) \rangle_B = \exp \left(\frac{1}{2} (h' h^* - h'^* h) \right) \exp \left(- \frac{|h + h'|^2}{2} \coth \left(\frac{\beta\omega}{2} \right) \right), \quad (553)$$

$$h' = v \exp(i\omega\tau), \quad (554)$$

$$\left\langle \widetilde{B_1^+ B_0^-}(t) \widetilde{B_1^+ B_0^-}(s) \right\rangle_B = \langle B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(s, 0) \rangle_B \quad (555)$$

$$= \langle B_{10}(t, \tau) B_{10}(s, 0) \rangle_B \quad (556)$$

$$= \text{Tr}_B (B_{10}(t, \tau) B_{10}(s, 0) \rho_B) \quad (557)$$

$$= \text{Tr}_B \left(\prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)} \right) \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) e^{\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(s) v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s) v_{0\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2} \right)} \right) \right) \quad (558)$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s)) \text{Tr}_B \left(\prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) D \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right) \rho_B \right) \quad (559)$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s)) \prod_{\mathbf{k}} \text{Tr}_B \left(\left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) D \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right) \rho_B \right) \quad (560)$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s)) \prod_{\mathbf{k}} \left(\exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right)^* - \frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right) \right) \quad (561)$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s)) \prod_{\mathbf{k}} \left(\exp \left(i \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right)^* \right)^{\Im} \right) \exp \left(- \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right) \quad (562)$$

$$= \exp(\chi_{10}(t) + \chi_{10}(s)) \prod_{\mathbf{k}} \left(\exp \left(i \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right)^* \right)^{\Im} \right) \exp \left(- \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right) \quad (563)$$

$$\left\langle \widetilde{B_0^+ B_1^-}(t) \widetilde{B_0^+ B_1^-}(s) \right\rangle_B = \exp(\chi_{01}(t) + \chi_{01}(s)) \prod_{\mathbf{k}} \left(\exp \left(i \left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right)^* \right)^{\Im} \right) \exp \left(- \frac{|v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)|}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right) \quad (564)$$

$$\langle D(h) b \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(- \frac{|\alpha|^2}{2} \right) \langle \alpha | D(h) b | \alpha \rangle \quad (565)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | D(-\alpha) D(h) b D(\alpha) | \alpha \rangle \quad (566)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b D(\alpha) | 0 \rangle \quad (567)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b D(\alpha) | 0 \rangle \quad (568)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b + \alpha) | 0 \rangle \quad (569)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b + \alpha) | 0 \rangle \quad (570)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (571)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha | 0 \rangle \quad (572)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \exp \left(-\frac{|h|^2}{2} \right) d^2 \alpha \quad (573)$$

$$= hN \langle D(h) \rangle_B, \quad (574)$$

$$\langle D(h) b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle \alpha | D(h) b^\dagger | \alpha \rangle \quad (575)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (576)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) b^\dagger D(\alpha) | 0 \rangle \quad (577)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) D(-\alpha) b^\dagger D(\alpha) | 0 \rangle \quad (578)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \langle 0 | D(-\alpha) D(h) D(\alpha) (b^\dagger + \alpha^*) | 0 \rangle \quad (579)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) (b^\dagger + \alpha^*) | 0 \rangle \quad (580)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) b^\dagger | 0 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) \alpha^* | 0 \rangle \quad (581)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \langle 0 | D(h) | 1 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp \left(-\frac{|\alpha|^2}{2} \right) \exp(h\alpha^* - h^* \alpha) \alpha^* \langle 0 | D(h) | 0 \rangle \quad (582)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \langle -h|1 \rangle + \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \alpha^* \langle 0|D(h)|0 \rangle, \quad (583)$$

$$\langle -h| = \exp\left(-\frac{|-h^*|^2}{2}\right) \sum_n \frac{(-h^*)^n}{\sqrt{n!}} \langle n|, \quad (584)$$

$$\langle -h|1 \rangle = \exp\left(-\frac{|-h^*|^2}{2}\right) (-h^*), \quad (585)$$

$$\langle D(h)b^\dagger \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \exp\left(-\frac{|-h^*|^2}{2}\right) (-h^*) + \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \alpha^* \exp\left(-\frac{|-h^*|^2}{2}\right) \quad (586)$$

$$= -h^* \langle D(h) \rangle_B (N+1), \quad (587)$$

$$\langle bD(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \langle \alpha | bD(h) | \alpha \rangle \quad (588)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \exp\left(-\frac{|h|^2}{2}\right) h + \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \alpha \exp\left(-\frac{|h|^2}{2}\right) \quad (589)$$

$$= h \langle D(h) \rangle_B (N+1), \quad (590)$$

$$\langle b^\dagger D(h) \rangle_B = \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \langle \alpha | b^\dagger D(h) | \alpha \rangle \quad (591)$$

$$= \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \exp\left(-\frac{|h|^2}{2}\right) h + \frac{1}{\pi N} \int d^2 \alpha \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(h\alpha^* - h^* \alpha) \alpha \exp\left(-\frac{|h|^2}{2}\right) \quad (592)$$

$$= -h^* \langle D(h) \rangle_B N, \quad (593)$$

The correlation functions can be found readily as:

$$B_1^+ B_0^- (t, \tau) = \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right), \quad (594)$$

$$B_0^+ B_1^- (t, \tau) = \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right), \quad (595)$$

$$B_{10}(t) = \left(\prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (596)$$

$$B_x(t, \tau) = \frac{B_1^+ B_0^- (t, \tau) + B_0^+ B_1^- (t, \tau) - B_{10}(t) - B_{01}(t)}{2}, \quad (597)$$

$$B_y(t, \tau) = \frac{B_0^+ B_1^- (t, \tau) - B_1^+ B_0^- (t, \tau) + B_{10}(t) - B_{01}(t)}{2i}, \quad (598)$$

$$B_{iz}(t, \tau) = \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right), \quad (599)$$

$$\langle \widetilde{B}_{iz}(t) \widetilde{B}_{jz}(s) \rangle_B = \langle B_{iz}(t, \tau) B_{jz}(s, 0) \rangle_B \quad (600)$$

$$= \left\langle \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) b_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}}\tau} + (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* b_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}\tau} \right) \sum_{\mathbf{k}} \left((g_{j\mathbf{k}} - v_{j\mathbf{k}}(s)) b_{\mathbf{k}}^\dagger + (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))^* b_{\mathbf{k}} \right) \right\rangle_B \quad (601)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s)) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \quad (602)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_x(s) \rangle_B = \langle B_x(t, \tau) B_x(s, 0) \rangle_B \quad (603)$$

$$= \left\langle \left(\frac{B_1^+ B_0^- (t, \tau) + B_0^+ B_1^- (t, \tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left(\frac{B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (s, 0) - B_{10}(s) - B_{01}(s)}{2} \right) \right\rangle_B \quad (604)$$

$$= \frac{1}{4} \langle (B_1^+ B_0^- (t, \tau) + B_0^+ B_1^- (t, \tau) - B_{10}(t) - B_{01}(t)) (B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (s, 0) - B_{10}(s) - B_{01}(s)) \rangle_B \quad (605)$$

$$= \frac{1}{4} \langle B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) + B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) - B_1^+ B_0^- (t, \tau) B_{10}(s) - B_1^+ B_0^- (t, \tau) B_{01}(s) + B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) - B_0^+ B_1^- (t, \tau) B_{10}(s) - B_0^+ B_1^- (t, \tau) B_{01}(s) \rangle_B \quad (606)$$

$$- B_0^+ B_1^- (t, \tau) B_{01}(s) - B_{10}(t) B_1^+ B_0^- (s, 0) - B_{10}(t) B_0^+ B_1^- (s, 0) + B_{10}(t) B_{10}(s) + B_{10}(t) B_{01}(s) - B_{01}(t) B_1^+ B_0^- (s, 0) - B_{01}(t) B_0^+ B_1^- (s, 0) + B_{01}(t) B_{10}(s) + B_{01}(t) B_{01}(s) \rangle_B \quad (607)$$

$$= \frac{1}{4} \langle B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) + B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) + B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) \rangle_B \quad (608)$$

$$+ B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) \rangle - \frac{(B_{01}(t) + B_{10}(t)) (B_{01}(s) + B_{10}(s))}{4}, \quad (609)$$

$$U_{10}(t, s) = \prod_{\mathbf{k}} \exp \left(i \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \quad (610)$$

$$\langle B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) \rangle_B = \exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (611)$$

$$\langle B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) \rangle_B = \exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (612)$$

$$\langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) \rangle_B = \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \exp \left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(s) v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s) v_{1\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \right\rangle_B \quad (613)$$

$$= \exp(\chi_{10}(t) + \chi_{01}(s)) \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right) \prod_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_B \quad (614)$$

$$= \exp(\chi_{10}(t) + \chi_{01}(s)) \prod_{\mathbf{k}} \left\langle \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) D \left(\frac{v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right) \right\rangle_B \quad (615)$$

$$= \exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (616)$$

$$\langle B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(s, 0) \rangle_B = \left\langle \Pi_{\mathbf{k}} \left(D \left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \exp \left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(s) v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s) v_{0\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \right\rangle_B \quad (617)$$

$$= \Pi_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{0\mathbf{k}}^*(t) v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t) v_{1\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(s) v_{0\mathbf{k}}(s) - v_{1\mathbf{k}}(s) v_{0\mathbf{k}}^*(s)}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle D \left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) D \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right\rangle_B \quad (618)$$

$$= \exp(\chi_{01}(t) + \chi_{10}(s)) \Pi_{\mathbf{k}} \left\langle D \left(\frac{v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) D \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) \right\rangle_B \quad (619)$$

$$= \exp(\chi_{01}(t) + \chi_{10}(s)) U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp \left(- \frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (620)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_x(s) \rangle_B = \frac{1}{4} \langle B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(s, 0) + B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(s, 0) + B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(s, 0) + B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(s, 0) \rangle_B \quad (621)$$

$$+ B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(s, 0) \rangle - \frac{(B_{01}(t) + B_{10}(t))(B_{01}(s) + B_{10}(s))}{4}, \quad (622)$$

$$= \frac{1}{4} \left(\exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t, s) \Pi_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (623)$$

$$+ \exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (624)$$

$$+ \exp(\chi_{01}(t) + \chi_{10}(s)) U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp \left(- \frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (625)$$

$$+ \exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t, s) \Pi_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (626)$$

$$- \left(\exp(\chi_{10}(t)) \exp \left(- \frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \left(\exp(\chi_{01}(s)) \exp \left(- \frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \quad (627)$$

$$= \frac{1}{2} \left((\exp(\chi_{10}(t) + \chi_{10}(s)))^{\Re} U_{10}(t, s) \Pi_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (628)$$

$$+ (\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Re} U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (629)$$

$$- \left(\exp(\chi_{10}(t)) \exp \left(- \frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \left(\exp(\chi_{01}(s)) \exp \left(- \frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \right) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right)^{\Re} \quad (630)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_y(s) \rangle_B = \langle B_y(t, \tau) B_y(s, 0) \rangle_B \quad (631)$$

$$= \left\langle \left(\frac{B_0^+ B_1^-(t, \tau) - B_1^+ B_0^-(t, \tau) + B_{10}(t) - B_{01}(t)}{2i} \right) \left(\frac{B_0^+ B_1^-(s, 0) - B_1^+ B_0^-(s, 0) + B_{10}(s) - B_{01}(s)}{2i} \right) \right\rangle_B \quad (632)$$

$$= -\frac{1}{4} \langle (B_0^+ B_1^-(t, \tau) - B_1^+ B_0^-(t, \tau) + B_{10}(t) - B_{01}(t)) (B_0^+ B_1^-(s, 0) - B_1^+ B_0^-(s, 0) + B_{10}(s) - B_{01}(s)) \rangle_B \quad (633)$$

$$(634)$$

$$= -\frac{1}{4} \langle B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(s, 0) - B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(s, 0) + B_0^+ B_1^-(t, \tau) B_{10}(s) - B_0^+ B_1^-(\tau) B_{01}(s) - B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(s, 0) \rangle \quad (635)$$

$$+ B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(s, 0) - B_1^+ B_0^-(t, \tau) B_{10}(s) + B_1^+ B_0^-(t, \tau) B_{01}(s) + B_{10}(t) B_0^+ B_1^-(s, 0) - B_{10}(t) B_1^+ B_0^-(s, 0) + B_{10}(t) B_{10}(s) \rangle \quad (636)$$

$$- B_{10}(t) B_{01}(s) - B_{01}(t) B_0^+ B_1^-(s, 0) + B_{01}(t) B_1^+ B_0^-(s, 0) - B_{01}(t) B_{10}(s) + B_{01}(t) B_{01}(s) \rangle \quad (637)$$

$$= -\frac{1}{4} \langle B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(s, 0) - B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(s, 0) - B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(s, 0) + B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(s, 0) \rangle \quad (638)$$

$$+ (B_{01}(t))^{\Im} (B_{10}(s))^{\Im} \quad (639)$$

$$= -\frac{1}{4} \left(\exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \quad (640)$$

$$- \exp(\chi_{01}(t) + \chi_{10}(s)) U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (641)$$

$$- \exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (642)$$

$$+ \exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (643)$$

$$+ \left(\exp(\chi_{01}(t)) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \right)^{\Im} \left(\exp(\chi_{10}(s)) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \right)^{\Im} \quad (644)$$

$$= -\frac{1}{2} \left((\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Re} U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \quad (645)$$

$$- (\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Re} U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \quad (646)$$

$$+ \left(\exp(\chi_{01}(t)) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \right)^{\Im} \left(\exp(\chi_{10}(s)) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right) \right)^{\Im} \quad (647)$$

$$(648)$$

$$(649)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_y(s) \rangle_B = \langle B_x(t, \tau) B_y(s, 0) \rangle_B \quad (650)$$

$$= \left\langle \left(\frac{B_1^+ B_0^-(t, \tau) + B_0^+ B_1^-(t, \tau) - B_{10}(t) - B_{01}(t)}{2} \right) \left(\frac{B_0^+ B_1^-(s, 0) - B_1^+ B_0^-(s, 0) + B_{10}(s) - B_{01}(s)}{2i} \right) \right\rangle_B \quad (651)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^-(t, \tau) B_0^+ B_1^-(s, 0) - B_1^+ B_0^-(t, \tau) B_1^+ B_0^-(s, 0) + B_1^+ B_0^-(t, \tau) B_{10}(s) - B_1^+ B_0^-(t, \tau) B_{01}(s) \rangle_B \quad (652)$$

$$+ B_0^+ B_1^-(t, \tau) B_0^+ B_1^-(s, 0) - B_0^+ B_1^-(t, \tau) B_1^+ B_0^-(s, 0) + B_0^+ B_1^-(t, \tau) B_{10}(s) - B_0^+ B_1^-(t, \tau) B_{01}(s) \rangle_B \quad (653)$$

$$- B_{10}(t) B_0^+ B_1^-(s, 0) + B_{10}(t) B_1^+ B_0^-(s, 0) - B_{10}(t) B_{10}(s) + B_{10}(t) B_{01}(s) \rangle_B \quad (654)$$

$$- B_{01}(t) B_0^+ B_1^-(s, 0) + B_{01}(t) B_1^+ B_0^-(s, 0) - B_{01}(t) B_{10}(s) + B_{01}(t) B_{01}(s) \rangle_B \quad (655)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) + B_1^+ B_0^- (t, \tau) B_{10} (s) - B_1^+ B_0^- (t, \tau) B_{01} (s) \quad (656)$$

$$+ B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) - B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (t, \tau) B_{10} (s) - B_0^+ B_1^- (t, \tau) B_{01} (s) \quad (657)$$

$$- B_{10} (t) B_0^+ B_1^- (s, 0) + B_{10} (t) B_1^+ B_0^- (s, 0) - B_{10} (t) B_{10} (s) + B_{10} (t) B_{01} (s) \quad (658)$$

$$- B_{01} (t) B_0^+ B_1^- (s, 0) + B_{01} (t) B_1^+ B_0^- (s, 0) - B_{01} (t) B_{10} (s) + B_{01} (t) B_{01} (s) \rangle_B \quad (659)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) \quad (660)$$

$$- B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) \rangle + \frac{1}{4i} (B_{10} (t) + B_{01} (t)) (B_{10} (s) - B_{01} (s)) \quad (661)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) \quad (662)$$

$$- B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) \rangle + \frac{1}{4i} (B_{10} (t) + B_{01} (t)) (B_{10} (s) - B_{01} (s)) \quad (663)$$

$$= \frac{1}{4i} \langle B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) \quad (664)$$

$$- B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) \rangle + (B_{10} (t))^{\Re} (B_{10} (s))^{\Im} \quad (665)$$

$$= \frac{1}{4i} \left(\exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (666)$$

$$- \exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (667)$$

$$+ \exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (668)$$

$$- \exp(\chi_{01}(t) + \chi_{10}(s)) U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \rangle + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \quad (669)$$

$$= \frac{1}{4i} \left(2i (\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Im} U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (670)$$

$$+ 2i (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \rangle + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \quad (671)$$

$$= \frac{1}{2} \left((\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Im} U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (672)$$

$$+ (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}} \tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \rangle + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \quad (673)$$

$$\langle \widetilde{B}_y(t) \widetilde{B}_x(s) \rangle_B = \left\langle \left(\frac{B_0^+ B_1^- (t, \tau) - B_1^+ B_0^- (t, \tau) + B_{10}(t) - B_{01}(t)}{2i} \right) \left(\frac{B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (s, 0) - B_{10}(s) - B_{01}(s)}{2} \right) \right\rangle_B \quad (674)$$

$$= \frac{1}{4i} \langle (B_0^+ B_1^- (t, \tau) - B_1^+ B_0^- (t, \tau) + B_{10}(t) - B_{01}(t)) (B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (s, 0) - B_{10}(s) - B_{01}(s)) \rangle_B \quad (675)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) - B_0^+ B_1^- (t, \tau) B_{10}(s) - B_0^+ B_1^- (t, \tau) B_{01}(s) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) \rangle_B \quad (676)$$

$$- B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) + B_1^+ B_0^- (t, \tau) B_{10}(s) + B_1^+ B_0^- (t, \tau) B_{01}(s) + B_{10}(t) B_1^+ B_0^- (s, 0) + B_{10}(t) B_0^+ B_1^- (s, 0) \rangle_B \quad (677)$$

$$- B_{10}(t) B_{10}(s) - B_{10}(t) B_{01}(s) - B_{01}(t) B_1^+ B_0^- (s, 0) - B_{01}(t) B_0^+ B_1^- (s, 0) + B_{01}(t) B_{10}(s) + B_{01}(t) B_{01}(s) \rangle_B \quad (678)$$

$$= \frac{1}{4i} \langle B_0^+ B_1^- (t, \tau) B_1^+ B_0^- (s, 0) + B_0^+ B_1^- (t, \tau) B_0^+ B_1^- (s, 0) - B_1^+ B_0^- (t, \tau) B_1^+ B_0^- (s, 0) - B_1^+ B_0^- (t, \tau) B_0^+ B_1^- (s, 0) \rangle + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \rangle_B \quad (679)$$

$$= \frac{1}{4i} \left(\exp(\chi_{01}(t) + \chi_{10}(s)) U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (680)$$

$$+ \exp(\chi_{01}(t) + \chi_{01}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (681)$$

$$- \exp(\chi_{10}(t) + \chi_{10}(s)) U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (682)$$

$$- \exp(\chi_{10}(t) + \chi_{01}(s)) U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \rangle + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \rangle_B \quad (683)$$

$$= \frac{1}{4i} \left(2i (\exp(\chi_{01}(t) + \chi_{10}(s)))^{\Im} U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (684)$$

$$+ 2i (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \rangle + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \rangle_B \quad (685)$$

$$= \frac{1}{2} \left((\exp(\chi_{01}(t) + \chi_{10}(s)))^{\Im} U_{10}^*(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (686)$$

$$+ (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \prod_{\mathbf{k}} \exp \left(- \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \rangle + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \rangle_B \quad (687)$$

$$\langle b^\dagger D(h) \rangle_B = -h^* \langle D(h) \rangle_B N \quad (688)$$

$$\langle b D(h) \rangle_B = h \langle D(h) \rangle_B (N + 1) \quad (689)$$

$$\langle D(h) b^\dagger \rangle_B = -h^* \langle D(h) \rangle_B (N + 1) \quad (690)$$

$$\langle D(h) b \rangle_B = h \langle D(h) \rangle_B N \quad (691)$$

$$\left\langle B_1^+ B_0^- (t, \tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger \right\rangle_B = \prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \right) \right\rangle_B \quad (692)$$

$$= \exp(\chi_{10}(t)) \left\langle D \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) (g_{i\mathbf{k}} - v_{i\mathbf{k}}(s)) b_{\mathbf{k}}^\dagger \right) \right\rangle_B \quad (693)$$

$$= \exp(\chi_{10}(t)) q_{i\mathbf{k}'}(s) \left\langle D \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) b_{\mathbf{k}'}^\dagger \right\rangle_B \left\langle \prod_{\mathbf{k} \neq \mathbf{k}'} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right. \right. \quad (694)$$

$$= \exp(\chi_{10}(t)) q_{i\mathbf{k}'}(s) \left(-\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right. \right. \quad (695)$$

$$= q_{i\mathbf{k}'}(s) \left(-\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) \exp(\chi_{10}(t)) \left\langle \prod_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}}\tau} \right) \right. \right. \quad (696)$$

$$= -q_{i\mathbf{k}'}(s) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) \quad (697)$$

$$\langle B_1^+ B_0^-(t, \tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* b_{\mathbf{k}'} \rangle_B = (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* \Pi_{\mathbf{k}} \exp\left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right)\right) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right) N_{\mathbf{k}'} \langle \Pi_{\mathbf{k}} \left(D \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} \tau} \right) \right. \quad (698)$$

$$= q_{i\mathbf{k}'}^*(s) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{10}(t), \quad (699)$$

$$\langle B_0^+ B_1^-(t, \tau) (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) b_{\mathbf{k}'}^\dagger \rangle_B = -q_{i\mathbf{k}'}(s) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'} \tau} \right)^* (N_{\mathbf{k}'} + 1) B_{01}(t), \quad (700)$$

$$\langle B_0^+ B_1(t, \tau) g_{i\mathbf{k}'-v_{i\mathbf{k}'}(s)}^* b_{\mathbf{k}'} \rangle_B = q_{i\mathbf{k}'}^*(s) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right) N_{\mathbf{k}'} B_{01}(t), \quad (701)$$

$$\langle B_x(t, \tau) B_{iz}(s, 0) \rangle_B = \left\langle \left(\frac{B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{10}^*}{2} \right) \sum_{\mathbf{k}'} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \quad (702)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle (B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau) - B_{10} - B_{10}^*) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \quad (703)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left\langle (B_{1+} B_{0-}(\tau) + B_{0+} B_{1-}(\tau)) \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}) b_{\mathbf{k}'}^\dagger + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^* b_{\mathbf{k}'} \right) \right\rangle_B \quad (704)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \langle B_{1+B_0-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger + B_{0+B_1-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})b_{\mathbf{k}'}^\dagger + B_{1+B_0-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*b_{\mathbf{k}'} + B_{0+B_1-}(\tau)(g_{i\mathbf{k}'} - v_{i\mathbf{k}'})^*b_{\mathbf{k}'} \rangle \quad (705)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* N_{\mathbf{k}'} \right) \quad (706)$$

$$+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s))^*\left(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t)+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s))^*\left(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t) \quad (707)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) B_{10}(t) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* (N_{\mathbf{k}'} + 1) \right) \quad (708)$$

$$+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s))^*\left(\frac{v_{1\mathbf{k}'}(t)-v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{10}(t)+(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(s))^*\left(\frac{v_{0\mathbf{k}'}(t)-v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}}e^{i\omega_{\mathbf{k}'}\tau}\right)N_{\mathbf{k}'}B_{01}(t) \quad (709)$$

$$= \frac{1}{2} \sum_{\mathbf{k}'} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s)) (N_{\mathbf{k}'} + 1) \left(\left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} e^{i\omega_{\mathbf{k}'}\tau} \right)^* B_{10}(t) + \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) \right. \right. \quad (710)$$

$$+ (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(s))^* N_{\mathbf{k}'} \left(\left(\frac{v_{1\mathbf{k}'}(t) - v_{0\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) e^{i\omega_{\mathbf{k}'}\tau} \right) B_{10}(t) + \left(\frac{v_{0\mathbf{k}'}(t) - v_{1\mathbf{k}'}(t)}{\omega_{\mathbf{k}'}} \right) e^{i\omega_{\mathbf{k}'}\tau} B_{01}(t)$$

$$+e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*\left(\left(\frac{v_{0\mathbf{k}'}(s)-v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{01}(s)-e^{-i\omega_{\mathbf{k}'}\tau}(g_{i\mathbf{k}'}-v_{i\mathbf{k}'}(t))^*\left(\frac{v_{1\mathbf{k}'}(s)-v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}}\right)(N_{\mathbf{k}'}+1)B_{10}(s)\right)\right) \quad (784)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} \left(-(g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left(\frac{v_{0\mathbf{k}'}(s) - v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right)^* B_{01}(s) N_{\mathbf{k}'} + (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right)^* B_{10}(s) N_{\mathbf{k}'} \right) \right. \quad (785)$$

$$\left. + e^{-i\omega_{\mathbf{k}'}\tau} \left((g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \left(\frac{v_{0\mathbf{k}'}(s) - v_{1\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{01}(s) - (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) B_{10}(s) \right) \right) \quad (786)$$

$$= \frac{1}{2i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right)^* (B_{10}(s) + B_{01}(s)) N_{\mathbf{k}'} \right. \quad (787)$$

$$\left. - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right) (B_{10}(s) + B_{01}(s)) (N_{\mathbf{k}'} + 1) \right) \quad (788)$$

$$= \frac{1}{i} \sum_{\mathbf{k}'} \left(e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re}(s) N_{\mathbf{k}'} - e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re}(s) (N_{\mathbf{k}'} + 1) \right) \quad (789)$$

$$= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re}(s) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re}(s) N_{\mathbf{k}'} \right) \quad (790)$$

$$= i \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right) B_{10}^{\Re}(s) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right)^* B_{10}^{\Re}(s) N_{\mathbf{k}'} \right) \quad (791)$$

$$= i B_{10}^{\Re}(s) \sum_{\mathbf{k}'} \left(e^{-i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t))^* \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right) (N_{\mathbf{k}'} + 1) - e^{i\omega_{\mathbf{k}'}\tau} (g_{i\mathbf{k}'} - v_{i\mathbf{k}'}(t)) \left(\frac{v_{1\mathbf{k}'}(s) - v_{0\mathbf{k}'}(s)}{\omega_{\mathbf{k}'}} \right)^* N_{\mathbf{k}'} \right). \quad (792)$$

The correlation functions are equal to:

$$\langle \widetilde{B_{iz}}(t) \widetilde{B_{jz}}(s) \rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s)) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \quad (793)$$

$$\langle \widetilde{B_x}(t) \widetilde{B_x}(s) \rangle_B = \frac{1}{2} \left((\exp(\chi_{10}(t) + \chi_{10}(s)))^{\Re} U_{10}(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \quad (794)$$

$$\left. + (\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Re} U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right) \quad (795)$$

$$- \left(\exp(\chi_{10}(t)) \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)^{\Re} \left(\exp(\chi_{01}(s)) \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)^{\Re} \quad (796)$$

$$\langle \widetilde{B_y}(t) \widetilde{B_y}(s) \rangle_B = -\frac{1}{2} \left((\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Re} U_{10}(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \quad (797)$$

$$\left. - (\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Re} U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right) \quad (798)$$

$$+ \left(\exp(\chi_{01}(t)) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)^{\Im} \left(\exp(\chi_{10}(s)) \left(\exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(s)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right|^2 \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right) \right) \right)^{\Im} \right) \quad (799)$$

$$\langle \widetilde{B_x}(t) \widetilde{B_y}(s) \rangle_B = \frac{1}{2} \left((\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Im} U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \quad (800)$$

$$\left. + (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Im} \quad (801)$$

$$\langle \widetilde{B_y}(t) \widetilde{B_x}(s) \rangle_B = \frac{1}{2} \left((\exp(\chi_{01}(t) + \chi_{10}(s)))^{\Im} U_{10}^*(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) - (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right. \quad (802)$$

$$\left. + (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \Pi_{\mathbf{k}} \exp\left(-\frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right) \right) + (B_{10}(t))^{\Im} (B_{10}(s))^{\Re} \quad (803)$$

$$\langle \widetilde{B_{iz}}(t) \widetilde{B_x}(s) \rangle_B = i \sum_{\mathbf{k}} B_{01}^{\Im}(s) \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right)^* - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right), \quad (804)$$

$$\langle \widetilde{B_x}(t) \widetilde{B_{iz}}(s) \rangle_B = i B_{10}^{\Im}(t) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(s))^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(s)) \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1) \right) \quad (805)$$

$$\langle \widetilde{B_{iz}}(t) \widetilde{B_y}(s) \rangle_B = i B_{10}^{\Re}(s) \sum_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right) (N_{\mathbf{k}} + 1) - e^{i\omega_{\mathbf{k}}\tau} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) \left(\frac{v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{\omega_{\mathbf{k}}} \right)^* N_{\mathbf{k}} \right) \quad (806)$$

$$\langle \widetilde{B_y}(t) \widetilde{B_{iz}}(s) \rangle_B = i B_{10}^{\Re}(t) \sum_{\mathbf{k}} \left((g_{i\mathbf{k}} - v_{i\mathbf{k}}(s))^* N_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right) - (g_{i\mathbf{k}} - v_{i\mathbf{k}}(s)) (N_{\mathbf{k}} + 1) e^{-i\omega_{\mathbf{k}}\tau} \left(\frac{v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right)^* \right) \quad (807)$$

$$(808)$$

$$(809)$$

$$(810)$$

$$(811)$$

$$(812)$$

$$(813)$$

$$(814)$$

$$(815)$$

$$(816)$$

$$(817)$$

The spectral density is defined in the usual way:

$$J_i(\omega) \equiv \sum_{\mathbf{k}} |g_{i\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}}), \quad (818)$$

$$v_{i\mathbf{k}}(t) = g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t). \quad (819)$$

it takes account of the density of states, dispersion relation and interaction mechanism with the environment. In the continuous case a way to measure the strenght of the system-environment coupling is:

$$\lambda_i = \int_0^\infty \frac{J_i(\omega)}{\omega} d\omega. \quad (820)$$

The integral version of the correlation functions are given by:

$$\begin{aligned} \chi_{10}(t) &= \int_0^\infty \frac{\sqrt{J_1^*(\omega)} J_0(\omega) F_1^*(\omega, t) F_0(\omega, t) - \sqrt{J_1(\omega)} J_0^*(\omega) F_1(\omega, t) F_0^*(\omega, t)}{2\omega^2} d\omega \\ U_{10}(t, s) &= \exp \left(i \left(\int_0^\infty \frac{(\sqrt{J_1(\omega)} F_1(\omega, t) - \sqrt{J_0(\omega)} F_0(\omega, t)) (\sqrt{J_1(\omega)} F_1(\omega, s) - \sqrt{J_0(\omega)} F_0(\omega, s))^* \exp(i\omega\tau)}{\omega^2} d\omega \right) \right)^{\Im} \\ B_{10}(t) &= \exp(\chi_{10}(t)) \exp \left(-\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1(\omega)} F_1(\omega, t) - \sqrt{J_0(\omega)} F_0(\omega, t)}{\omega} \right|^2 \coth \left(\frac{\beta\omega}{2} \right) d\omega \right), \\ \xi^+(t, s) &= \exp \left(-\int_0^\infty \frac{|\left(\sqrt{J_1(\omega)} F_1(\omega, t) - \sqrt{J_0(\omega)} F_0(\omega, t) \right) \exp(i\omega\tau) + \sqrt{J_1(\omega)} F_1(\omega, s) - \sqrt{J_0(\omega)} F_0(\omega, s)|^2}{2\omega^2} \coth \right. \\ \xi^-(t, s) &= \exp \left(-\int_0^\infty \frac{|\left(\sqrt{J_1(\omega)} F_1(\omega, t) - \sqrt{J_0(\omega)} F_0(\omega, t) \right) \exp(i\omega\tau) - \left(\sqrt{J_1(\omega)} F_1(\omega, s) - \sqrt{J_0(\omega)} F_0(\omega, s) \right)|^2}{2\omega^2} \coth \right. \\ Q(\omega, t) &= \frac{\sqrt{J_1(\omega)} F_1(\omega, t) - \sqrt{J_0(\omega)} F_0(\omega, t)}{\omega} \\ \langle \widetilde{B}_x(t) \widetilde{B}_x(s) \rangle_B &= \frac{1}{2} \left((\exp(\chi_{10}(t) + \chi_{10}(s)))^{\Re} U_{10}(t, s) \xi^+(t, s) + (\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Re} U_{10}^*(t, s) \xi^-(t, s) \right) - (B_{10}(t))^{\Re} (B_{01}(s))^{\Re} \\ \langle \widetilde{B}_y(t) \widetilde{B}_y(s) \rangle_B &= -\frac{1}{2} \left((\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Re} U_{10}(t, s) \xi^+(t, s) - (\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Re} U_{10}^*(t, s) \xi^-(t, s) \right) + (B_{01}(t))^{\Im} (B_{10}(s))^{\Im} \\ \langle \widetilde{B}_x(t) \widetilde{B}_y(s) \rangle_B &= \frac{1}{2} \left((\exp(\chi_{10}(t) + \chi_{01}(s)))^{\Im} U_{10}^*(t, s) \xi^-(t, s) + (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \xi^+(t, s) \right) + (B_{10}(t))^{\Re} (B_{10}(s))^{\Re} \\ \langle \widetilde{B}_y(t) \widetilde{B}_x(s) \rangle_B &= \frac{1}{2} \left((\exp(\chi_{01}(t) + \chi_{10}(s)))^{\Im} U_{10}^*(t, s) \xi^-(t, s) + (\exp(\chi_{01}(t) + \chi_{01}(s)))^{\Im} U_{10}(t, s) \xi^+(t, s) \right) + (B_{10}(t))^{\Im} (B_{10}(s))^{\Im} \\ \langle \widetilde{B}_{iz}(t) \widetilde{B}_{jz}(s) \rangle_B &= \int_0^\infty (\sqrt{J_i(\omega)} J_j^*(\omega) (1 - F_i(\omega, t)) (1 - F_j^*(\omega, s)) e^{i\omega\tau} N(\omega) + \sqrt{J_i^*(\omega)} J_j(\omega) (1 - F_i^*(\omega, t)) (1 - F_j(\omega, s)) e^{-i\omega\tau} (N(\omega) + 1)) d\omega, \\ \langle \widetilde{B}_{iz}(t) \widetilde{B}_x(s) \rangle_B &= i B_{01}^{\Im}(s) \int_0^\infty \left(\sqrt{J_i(\omega)} (1 - F_i(\omega, t)) Q^*(\omega, s) N(\omega) e^{i\omega\tau} - \sqrt{J_i^*(\omega)} (1 - F_i^*(\omega, t)) Q(\omega, s) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega \\ \langle \widetilde{B}_x(t) \widetilde{B}_{iz}(s) \rangle_B &= i B_{01}^{\Im}(t) \int_0^\infty \left(\sqrt{J_i^*(\omega)} (1 - F_i^*(\omega, s)) Q(\omega, t) N(\omega) e^{i\omega\tau} - \sqrt{J_i(\omega)} (1 - F_i(\omega, s)) Q^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega \\ \langle \widetilde{B}_{iz}(t) \widetilde{B}_y(s) \rangle_B &= i B_{10}^{\Re}(s) \int_0^\infty \left(\sqrt{J_i^*(\omega)} (1 - F_i^*(\omega, s)) Q(\omega, s) (N(\omega) + 1) e^{-i\omega\tau} - \sqrt{J_i(\omega)} (1 - F_i(\omega, s)) Q^*(\omega, s) e^{i\omega\tau} N(\omega) \right) d\omega \\ \langle \widetilde{B}_y(t) \widetilde{B}_{iz}(s) \rangle_B &= i B_{10}^{\Re}(t) \int_0^\infty \left(\sqrt{J_i^*(\omega)} (1 - F_i^*(\omega, s)) Q(\omega, t) N(\omega) e^{i\omega\tau} - \sqrt{J_i(\omega)} (1 - F_i(\omega, s)) Q^*(\omega, t) e^{-i\omega\tau} (N(\omega) + 1) \right) d\omega \end{aligned}$$

$$\left\langle \widetilde{B_{iz}}(t) \widetilde{B_{jz}}(s) \right\rangle_B = \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t)) (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + \sum_{\mathbf{k}} (g_{i\mathbf{k}} - v_{i\mathbf{k}}(t))^* (g_{j\mathbf{k}} - v_{j\mathbf{k}}(s)) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1), \quad (821)$$

$$= \sum_{\mathbf{k}} ((g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t)) (g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}}, s))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{i\mathbf{k}} - g_{i\mathbf{k}} F_i(\omega_{\mathbf{k}}, t))^* (g_{j\mathbf{k}} - g_{j\mathbf{k}} F_j(\omega_{\mathbf{k}}, s)) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (822)$$

$$= \sum_{\mathbf{k}} (g_{i\mathbf{k}}(1 - F_i(\omega_{\mathbf{k}}, t)) g_{j\mathbf{k}}^*(1 - F_j(\omega_{\mathbf{k}}, s))^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + g_{i\mathbf{k}}^*(1 - F_i(\omega_{\mathbf{k}}, t))^* g_{j\mathbf{k}}(1 - F_j(\omega_{\mathbf{k}}, s)) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (823)$$

$$\approx \int_0^\infty (\sqrt{J_i(\omega) J_j^*(\omega)} (1 - F_i(\omega, t)) (1 - F_j^*(\omega, s)) e^{i\omega\tau} N(\omega) + \sqrt{J_i^*(\omega) J_j(\omega)} (1 - F_i^*(\omega, t)) (1 - F_j(\omega, s)) e^{-i\omega\tau} (N(\omega) + 1)) d\omega, \quad (824)$$

$$\chi_{10}(t) = \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \quad (825)$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g_{1\mathbf{k}}^* F_1^*(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) g_{0\mathbf{k}}^* F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \quad (826)$$

$$= \sum_{\mathbf{k}} \frac{1}{2} \left(\frac{g_{1\mathbf{k}}^* g_{0\mathbf{k}} F_1^*(\omega_{\mathbf{k}}, t) F_0(\omega_{\mathbf{k}}, t) - g_{1\mathbf{k}} g_{0\mathbf{k}}^* F_1(\omega_{\mathbf{k}}, t) F_0^*(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}^2} \right) \quad (827)$$

$$\approx \int_0^\infty \frac{\sqrt{J_1^*(\omega) J_0(\omega)} F_1^*(\omega, t) F_0(\omega, t) - \sqrt{J_1(\omega) J_0^*(\omega)} F_1(\omega, t) F_0^*(\omega, t)}{2\omega^2} d\omega, \quad (828)$$

$$U_{10}(t, s) = \prod_{\mathbf{k}} \exp \left(i \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \quad (829)$$

$$= \exp \left(i \sum_{\mathbf{k}} \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \quad (830)$$

$$= \exp \left(i \left(\sum_{\mathbf{k}} \frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \quad (831)$$

$$= \exp \left(i \left(\sum_{\mathbf{k}} \frac{(g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t)) (g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, s) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \quad (832)$$

$$\approx \exp \left(i \left(\int_0^\infty \frac{(\sqrt{J_1(\omega)} F_1(\omega, t) - \sqrt{J_0(\omega)} F_0(\omega, t)) (\sqrt{J_1(\omega)} F_1(\omega, s) - \sqrt{J_0(\omega)} F_0(\omega, s))^* \exp(i\omega\tau)}{\omega^2} d\omega \right)^{\Im} \right) \quad (833)$$

$$(834)$$

$$B_{10}(t) = \left(\prod_{\mathbf{k}} \exp \left(\frac{1}{2} \left(\frac{v_{1\mathbf{k}}^*(t) v_{0\mathbf{k}}(t) - v_{1\mathbf{k}}(t) v_{0\mathbf{k}}^*(t)}{\omega_{\mathbf{k}}^2} \right) \right) \right) \left(\exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{v_{1\mathbf{k}}(t)}{\omega_{\mathbf{k}}} - \frac{v_{0\mathbf{k}}(t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \right), \quad (835)$$

$$= \exp(\chi_{10}(t)) \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \left| \frac{g_{1\mathbf{k}} F_1(\omega_{\mathbf{k}}, t) - g_{0\mathbf{k}} F_0(\omega_{\mathbf{k}}, t)}{\omega_{\mathbf{k}}} \right|^2 \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (836)$$

$$\approx \exp(\chi_{10}(t)) \exp \left(-\frac{1}{2} \int_0^\infty \left| \frac{\sqrt{J_1(\omega)} F_1(\omega, t) - \sqrt{J_0(\omega)} F_0(\omega, t)}{\omega} \right|^2 \coth \left(\frac{\beta \omega}{2} \right) d\omega \right) \quad (837)$$

$$(838)$$

$$\xi^+(t, s) = \prod_{\mathbf{k}} \exp \left(-\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (839)$$

$$= \exp \left(-\sum_{\mathbf{k}} \frac{|(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s)|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta \omega_{\mathbf{k}}}{2} \right) \right) \quad (840)$$

The eigenvalues of the Hamiltonian $\overline{H_S}$ are given by the solution of the following algebraic equation:

$$\lambda^2 - \text{Tr}(\overline{H_S}) \lambda + \text{Det}(\overline{H_S}) = 0. \quad (879)$$

The solutions of this equation written in terms of η and ξ as defined in the previous section are given by $\lambda_{\pm} = \frac{\xi \pm \eta}{2}$ and they satisfy $H_S |\pm\rangle = \lambda_{\pm} |\pm\rangle$. Using this notation is possible to write $H_S = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|$.

The time-dependence of the system operators $\widetilde{A}_i(t)$ may be made explicit using the Fourier decomposition, in the case for time-independent $\overline{H_S}$ we will obtain:

$$\widetilde{A}_i(\tau) = e^{i\overline{H_S}\tau} A_i e^{-i\overline{H_S}\tau} \quad (880)$$

$$= \sum_w e^{-i w \tau} \mathcal{A}_i(w). \quad (881)$$

Where the sum is defined on the set of all the differences between the eigenvalues of the system, in our case $w \in \{0, \pm\eta\}$.

In order to use the equation (881) to descompose the equation (356) we need to consider the time ordering operator \mathcal{T} , it's possible to write using the Dyson series or the expansion of the operator of the form $U(t) \equiv \mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right)$ like:

$$U(t) \equiv \mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right) \quad (882)$$

$$= \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n H(t_1) H(t_2) \dots H(t_n). \quad (883)$$

Here $0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t$ is a partition of the set $[0, t]$. We will use a perturbative solution to the exponential of a time-varying operator, this can be done if we write an effective hamiltonian $H_E(t)$ such that $\mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right) \equiv \exp(-it H_E(t))$. The effective Hamiltonian is expanded in a series of terms of increasing order in time $H_E(t) = H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots$ so we can write:

$$U(t) = \exp\left(-it \left(H_E^{(0)}(t) + H_E^{(1)}(t) + H_E^{(2)}(t) + \dots\right)\right). \quad (884)$$

The terms can be found expanding $\mathcal{T} \exp\left(-i \int_0^t dt' \overline{H_S}(t')\right)$ and $U(t)$ then equating the terms of the same power. The lowest terms are:

$$H_E^{(0)}(t) = \frac{1}{t} \int_0^t \overline{H_S}(t') dt', \quad (885)$$

$$H_E^{(1)}(t) = -\frac{i}{2t} \int_0^t dt' \int_0^{t'} dt'' [\overline{H_S}(t'), \overline{H_S}(t'')], \quad (886)$$

$$H_E^{(2)}(t) = \frac{1}{6t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' ([\overline{H_S}(t'), \overline{H_S}(t'')], \overline{H_S}(t''')) + [[\overline{H_S}(t'''), \overline{H_S}(t'')], \overline{H_S}(t')]. \quad (887)$$

In this case the Fourier decomposition using the expansion of $H_E(t)$ is:

$$\widetilde{A}_i(t) = U^\dagger(t) A_i(t) U(t) \quad (888)$$

$$\widetilde{A}_i(t) = e^{iH_E(t)t} A_i(t) e^{-iH_E(t)t} \quad (889)$$

$$= \sum_{w(t)} e^{-i w(t)t} \mathcal{A}_i(t, w(t)). \quad (890)$$

$w(t)$ belongs to the set of differences of eigenvalues of $H_E(t)$ that depends of the time. As we can see the decomposition matrices are time-dependent as well.

Extending the Fourier decomposition to the matrix $\widetilde{A}_j(t - \tau, t)$ using the Magnus expansion generates:

$$\widetilde{A}_j(t - \tau, t) = U(t) U^\dagger(t - \tau) A_j(t) U(t - \tau) U^\dagger(t) \quad (891)$$

$$= e^{-itH_E(t)} e^{i(t-\tau)H_E(t-\tau)} A_j(t) e^{-i(t-\tau)H_E(t-\tau)} e^{itH_E(t)} \quad (892)$$

$$= e^{-itH_E(t)} \left(\sum_{w'(t-\tau)} e^{-i(t-\tau)w(t-\tau)} \mathcal{A}_j(t, w(t-\tau)) \right) e^{itH_E(t)} \quad (893)$$

$$= \sum_{w(t), w'(t-\tau)} e^{iw'(t)t} e^{-i(t-\tau)w(t-\tau)} \mathcal{A}_j(t, w(t-\tau), w'(t)) \quad (894)$$

$$= \sum_{w(t), w'(t-\tau)} e^{iw'(t)t} e^{-i(t-\tau)w(t-\tau)} \mathcal{A}_j(t, w(t-\tau), w'(t)) \quad (895)$$

$$= \sum_{w(t), w'(t-\tau)} e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_j(t, w(t-\tau), w'(t)) \quad (896)$$

where $w'(t - \tau)$ and $w(t)$ belongs to the set of the differences of the eigenvalues of the Hamiltonian $\overline{H}_E(t - \tau)$ and $\overline{H}_E(t)$ respectively.

In order to show the explicit form of the matrices present in the RHS of the equation (881) for a general 2×2 matrix in a given time let's write the matrix A_i in the base $V = \{|+\rangle, |-\rangle\}$ in the following way:

$$A_i = \sum_{\alpha, \beta \in V} \langle \alpha | A_i | \beta \rangle | \alpha \rangle \langle \beta |. \quad (897)$$

Given that $[|+\rangle\langle+|, |-\rangle\langle-|] = 0$, then using the Zassenhaus formula we obtain:

$$e^{i\overline{H}_E\tau} = e^{i(\lambda_+|+\rangle\langle+| + \lambda_-|-\rangle\langle-|)\tau} \quad (898)$$

$$= e^{i\lambda_+|+\rangle\langle+|\tau} e^{i\lambda_-|-\rangle\langle-|\tau} \quad (899)$$

$$= (|-\rangle\langle-| + e^{i\lambda_+\tau}|+\rangle\langle+|) (|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|) \quad (900)$$

$$= e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|. \quad (901)$$

Calculating the transformation (881) directly using the previous relationship we find that:

$$U^\dagger(\tau) A_i(\tau) U(\tau) = (e^{i\lambda_+\tau}|+\rangle\langle+| + e^{i\lambda_-\tau}|-\rangle\langle-|) \left(\sum_{\alpha, \beta \in V} \langle \alpha | A_i(\tau) | \beta \rangle | \alpha \rangle \langle \beta | \right) (e^{-i\lambda_+\tau}|+\rangle\langle+| + e^{-i\lambda_-\tau}|-\rangle\langle-|) \quad (902)$$

$$= \langle + | A_i(\tau) | + \rangle | + \rangle \langle + | + e^{i\eta\tau} \langle + | A_i(\tau) | - \rangle | + \rangle \langle - | + e^{-i\eta\tau} \langle - | A_i(\tau) | + \rangle | - \rangle \langle + | + \langle - | A_i(\tau) | - \rangle | - \rangle \langle - |. \quad (903)$$

$$= \mathcal{A}_i(0) + \mathcal{A}_i(-w) e^{iw\tau} + \mathcal{A}_i(w) e^{-iw\tau} \quad (904)$$

Here $w = \lambda_+ - \lambda_-$. Comparing the RHS of the equations (881) and the explicit expression for $\widetilde{A}_i(\tau)$ in (889), we obtain the form of the expansion matrices of the Fourier decomposition for a general 2×2 matrix:

$$\mathcal{A}_i(0) = \langle + | A_i(\tau) | + \rangle | + \rangle \langle + | + \langle - | A_i(\tau) | - \rangle | - \rangle \langle - |, \quad (905)$$

$$\mathcal{A}_i(-w) = \langle + | A_i(\tau) | - \rangle | + \rangle \langle - |, \quad (906)$$

$$\mathcal{A}_i(w) = \langle - | A_i(\tau) | + \rangle | - \rangle \langle + |. \quad (907)$$

For a decomposition of the interaction Hamiltonian in terms of Hermitian operators, i.e. $\widetilde{A}_i(\tau) = \widetilde{A}_i^\dagger(\tau)$ and $\widetilde{B}_i(\tau) = \widetilde{B}_i^\dagger(\tau)$ we can use the equation (881) to write the master equation in the following neater form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ij} \int_0^t d\tau C_i(t) C_j(t-\tau) \left(\mathcal{B}_{ij}(\tau) \left[A_i, \widetilde{A}_j(t-\tau, t) \overline{\rho_S}(t) \right] + \mathcal{B}_{ji}(-\tau) \left[\overline{\rho_S}(t) \widetilde{A}_j(t-\tau, t), A_i \right] \right) \quad (908)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \left(\mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_j(w(t-\tau), w'(t)) \overline{\rho_S}(t) \right] \right. \quad (909)$$

$$\left. - \mathcal{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_j(w(t-\tau), w'(t)) \right] \right) \quad (910)$$

Given that $\mathcal{A}_j(w(t-\tau), w'(t)) = \mathcal{A}_j^\dagger(-w(t-\tau), -w'(t))$ from the Fourier decomposition (881) then we can re-arrange the precedent sum in the following way with the trace respect to the bath:

$$\mathcal{B}_{ij}(\tau) = \text{Tr}_B \left(\widetilde{B}_i(t) \widetilde{B}_j(s) \rho_B \right) \quad (911)$$

$$= \text{Tr}_B \left(\widetilde{B}_i(\tau) \widetilde{B}_j(0) \rho_B \right). \quad (912)$$

Let's define:

$$\mathcal{A}_j(w(t-\tau), w'(t)) = \mathcal{A}_{jww'}(t-\tau, t) \quad (913)$$

The master equation can be re-written in the following form:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (914)$$

$$+ \sum_{ijww'} \mathcal{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (915)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (916)$$

$$+ \sum_{ijww'} \mathcal{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (917)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (918)$$

$$+ \sum_{ijww'} \mathcal{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (919)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right] \quad (920)$$

$$- \mathcal{B}_{ji}(-\tau) \left[A_i, \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (921)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t-\tau) \text{Tr}_B \left(\left[A_i, \widetilde{B}_i(\tau) \widetilde{B}_j(0) \rho_B \right] e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \overline{\rho_S}(t) \right) \quad (922)$$

$$- \left[A_i, \widetilde{B}_j(-\tau) \widetilde{B}_i(0) \rho_B \overline{\rho_S}(t) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right] \quad (923)$$

Given that if we define:

$$D_{ijww'}(t-\tau, t) = C_i(t) C_j(t-\tau) \mathcal{B}_{ij}(\tau) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \quad (924)$$

then

$$D_{ijww'}^\dagger(t-\tau, t) = \left(C_i(t) C_j(t-\tau) \mathcal{B}_{ij}(\tau) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}(t-\tau, t) \right)^\dagger \quad (925)$$

$$= \mathcal{B}_{ij}^*(\tau) C_i(t) C_j(t-\tau) e^{-i\tau w(t-\tau)} e^{it(w(t-\tau)-w'(t))} \mathcal{A}_{jww'}^\dagger(t-\tau, t) \quad (926)$$

We used the fact that $C_i(t), C_j(t - \tau)$ are real. Now let's consider the following trace recalling that $\text{Tr}(A)^* = \text{Tr}(A^\dagger)$ so:

$$\text{Tr}_B \left(\widetilde{B}_j(-\tau) \widetilde{B}_i(0) \rho_B \right) = \text{Tr}_B \left(e^{-i\tau H_B(\tau)} B_j e^{i\tau H_B(\tau)} B_i \rho_B \right) \quad (927)$$

$$= \text{Tr}_B \left(B_j e^{i\tau H_B(\tau)} B_i \rho_B e^{-i\tau H_B(\tau)} \right) \text{ (by cyclic permutivity of trace)} \quad (928)$$

$$= \text{Tr}_B \left(B_j e^{i\tau H_B(\tau)} B_i e^{-i\tau H_B(\tau)} \rho_B \right) \text{ (by commutativity of } e^{-i\tau H_B(\tau)} \text{ and } \rho_B) \quad (929)$$

$$= \text{Tr}_B \left(B_j \widetilde{B}_i(\tau) \rho_B \right) \text{ (by definition of time evolution)} \quad (930)$$

$$= \text{Tr}_B \left(B_j \widetilde{B}_i(\tau) \rho_B \right) \quad (931)$$

$$= \text{Tr}_B \left(\rho_B B_j \widetilde{B}_i(\tau) \right) \quad (932)$$

$$= \text{Tr}_B \left(\left(\widetilde{B}_i(\tau) B_j \rho_B \right)^\dagger \right) \text{ (by definition of adjoint)} \quad (933)$$

$$= \text{Tr}_B \left(\widetilde{B}_i(\tau) B_j \rho_B \right)^* \quad (934)$$

$$= \mathcal{B}_{ij}^*(\tau) \quad (935)$$

So we can write the master equation like:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau C_i(t) C_j(t - \tau) \left(\mathcal{B}_{ij}(\tau) \left[A_i, e^{i\tau w(t - \tau)} e^{-it(w(t - \tau) - w'(t))} \mathcal{A}_j(w(t - \tau), w'(t)) \overline{\rho_S}(t) \right] \right. \quad (936)$$

$$\left. - \mathcal{B}_{ij}^*(\tau) \left[A_i, \overline{\rho_S}(t) e^{-i\tau w(t - \tau)} e^{it(w(t - \tau) - w'(t))} \mathcal{A}_j^\dagger(w(t - \tau), w'(t)) \right] \right) \quad (937)$$

$$= -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \int_0^t d\tau \left([A_i, D_{ijww'}(t - \tau, t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) D_{ijww'}^\dagger(t - \tau, t)] \right) \quad (938)$$

Let's define the response matrix in the following way.

$$\mathcal{D}_{ijww'}(t) = \int_0^t d\tau D_{ijww'}(t - \tau, t) \quad (939)$$

Then the master equation can be written as:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left([A_i, \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t)] \right) \quad (940)$$

If we extend the upper limit of integration to ∞ in the equation (939) then the system will be independent of any preparation at $t = 0$, so the evolution of the system will depend only on its present state as expected in the Markovian approximation.

Applying the inverse transformation we will obtain that:

$$e^{-V} \frac{d\overline{\rho_S}(t)}{dt} e^V = \frac{d(e^{-V} \overline{\rho_S} e^V)}{dt} \quad (941)$$

$$= \frac{d\rho_S}{dt} \quad (942)$$

$$= -ie^{-V} [\overline{H_S}(t), \overline{\rho_S}(t)] e^V - \sum_{ijww'} \int_0^t d\tau \left(e^{-V} [A_i, D_{ijww'}(t - \tau, t) \overline{\rho_S}(t)] e^V - e^{-V} [A_i, \overline{\rho_S}(t) D_{ijww'}^\dagger(t - \tau, t)] e^V \right). \quad (943)$$

For a product we have the following:

$$e^{-V} \overline{AB} e^V = e^{-V} \overline{A} \overline{B} e^V \quad (944)$$

$$= e^{-V} \overline{A} e^V e^{-V} \overline{B} e^V \quad (945)$$

$$= (e^{-V} \overline{A} e^V) (e^{-V} \overline{B} e^V) \quad (946)$$

$$= \overline{AB}. \quad (947)$$

We can use this to prove the following property for the inverse transformation of a commutator:

$$e^{-V} [\overline{A}, \overline{B}] e^V = e^{-V} \overline{(AB - BA)} e^V \quad (948)$$

$$= e^{-V} \overline{AB} e^V - e^{-V} \overline{BA} e^V \quad (949)$$

$$= \overline{AB} - \overline{BA} \quad (950)$$

$$= [\overline{A}, \overline{B}]. \quad (951)$$

So we will obtain that

$$\frac{d\rho_S}{dt} = -ie^{-V} [\overline{H_S}(t), \overline{\rho_S}(t)] e^V - e^{-V} \sum_{ijww'} \left([A_i, \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t)] \right) e^V \quad (952)$$

$$= -ie^{-V} [\overline{H_S}(t), \overline{\rho_S}(t)] e^V - \sum_{ijww'} \left(e^{-V} [A_i, \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t)] e^V - e^{-V} [A_i, \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t)] e^V \right) \quad (953)$$

$$= -i [H_S(t), \rho_S(t)] - \sum_{ijww'} \left([e^{-V} A_i e^V, e^{-V} \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t) e^V] - [e^{-V} A_i e^V, e^{-V} \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t) e^V] \right) \quad (954)$$

$$= -i [H_S(t), \rho_S(t)] - \sum_{ijww'} \left([e^{-V} A_i e^V, e^{-V} \mathcal{D}_{ijww'}(t) e^V e^{-V} \overline{\rho_S}(t) e^V] - [e^{-V} A_i e^V, e^{-V} \overline{\rho_S}(t) e^V e^{-V} \mathcal{D}_{ijww'}^\dagger(t) e^V] \right) \quad (955)$$

$$= -i [H_S(t), \rho_S(t)] - \sum_{ijww'} \left([e^{-V} A_i e^V, e^{-V} \mathcal{D}_{ijww'}(t) e^V \rho_S(t)] - [e^{-V} A_i e^V, \rho_S(t) e^{-V} \mathcal{D}_{ijww'}^\dagger(t) e^V] \right). \quad (956)$$

V. LIMIT CASES

In order to show the plausibility of the master equation (940) for a time-dependent Hamiltonian we will show that this equation reproduces the following cases under certain limits conditions that will be pointed in each subsection.

A. Time-independent variational quantum master equation

At first let's show that the master equation (940) reproduces the results of the reference [1], for the latter case we have that $i, j \in \{1, 2, 3\}$ and $\omega \in (0, \pm\eta)$. The Hamiltonian of the system considered in this reference written in the same basis than the Hamiltonian (1) is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) |1\rangle\langle 1| + \frac{\Omega}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (957)$$

After performing the transformation (24) on the Hamiltonian (957) it's possible to split that result in the following set of Hamiltonians:

$$\overline{H_S} = (\delta + R) |1\rangle\langle 1| + \frac{\Omega_r}{2} \sigma_x, \quad (958)$$

$$\overline{H_I} = B_z |1\rangle\langle 1| + \frac{\Omega}{2} (B_x \sigma_x + B_y \sigma_y), \quad (959)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (960)$$

The Hamiltonian (958) differs from the transformed Hamiltonian H_S of the reference written like $H_S = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ by a term proportional to the identity, this can be seen in the following way taking $\epsilon = \delta + R$

$$(\delta + R) |1\rangle\langle 1| - \frac{\delta}{2}\mathbb{I} = \left(\frac{\delta}{2} + R\right) |1\rangle\langle 1| - \frac{\delta}{2} |0\rangle\langle 0| \quad (961)$$

$$= \frac{R}{2}\mathbb{I} + \frac{\delta + R}{2}\sigma_z \quad (962)$$

$$= \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z. \quad (963)$$

In this Hamiltonian we can write $A_i = \sigma_x$, $A_2 = \sigma_y$ and $A_3 = \frac{I+\sigma_z}{2} = |1\rangle\langle 1|$ with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. In order to find the decomposition matrices of the Fourier decomposition let's obtain the eigenvalues and eigenvectors of the matrix $\overline{H_S}$. Given that $\overline{H_S} = \frac{R}{2}\mathbb{I} + \frac{\epsilon}{2}\sigma_z + \frac{\Omega_r}{2}\sigma_x$ then $\text{Tr}(\overline{H_S}) = R$ and $\text{Det}(\overline{H_S}) = \frac{R^2 - \epsilon^2}{4} - \frac{\Omega_r^2}{4}$ then by the Caley-Hamilton theorem then we will have that the equations of the eigenvalues and it's values are given by::

$$0 = \lambda^2 - R\lambda + \frac{R^2 - \epsilon^2 - \Omega_r^2}{4}, \quad (964)$$

$$\lambda_{\pm} = \frac{R \pm \sqrt{(-R)^2 - 4\left(\frac{R^2 - \epsilon^2 - \Omega_r^2}{4}\right)}}{2} \quad (965)$$

$$= \frac{R \pm \sqrt{R^2 - (R^2 - \epsilon^2 - \Omega_r^2)}}{2} \quad (966)$$

$$= \frac{R \pm \sqrt{\epsilon^2 + \Omega_r^2}}{2} \quad (967)$$

$$\eta = \sqrt{\epsilon^2 + \Omega_r^2}, \quad (968)$$

$$\lambda_{\pm} = \frac{R \pm \eta}{2}. \quad (969)$$

For $\lambda_+ = \frac{R+\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R+\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R+\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} - \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} - \frac{\eta}{2} \end{pmatrix}. \quad (970)$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $-\frac{\epsilon+\eta}{2}a + \frac{\Omega_r}{2}b = 0$, so $a = \frac{\Omega_r}{\epsilon+\eta}b$ then the normalized eigenvector is $|+\rangle = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|0\rangle + \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|1\rangle$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}$. The vector is written in reduced way like $|+\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$.

For $\lambda_- = \frac{R-\eta}{2}$ we will obtain the associated eigenvector like:

$$\begin{pmatrix} \frac{R}{2} - \frac{\epsilon}{2} - \frac{R-\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{R}{2} + \frac{\epsilon}{2} - \frac{R-\eta}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon}{2} + \frac{\eta}{2} & \frac{\Omega_r}{2} \\ \frac{\Omega_r}{2} & \frac{\epsilon}{2} + \frac{\eta}{2} \end{pmatrix}. \quad (971)$$

so the eigenvector $|+\rangle = a|0\rangle + b|1\rangle$ satisfies $\frac{\Omega_r}{2}a + \frac{\epsilon+\eta}{2}b = 0$, so $a = -\frac{\epsilon+\eta}{\Omega_r}b$ then the normalized eigenvector is $|-\rangle = \frac{\epsilon+\eta}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|0\rangle - \frac{\Omega_r}{\sqrt{(\epsilon+\eta)^2 + \Omega_r^2}}|1\rangle$. The vector is written in reduced way like $|-\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$. Summarizing these results we can write:

$$\lambda_+ = \frac{\epsilon + \eta}{2}, \quad (972)$$

$$\lambda_- = \frac{\epsilon - \eta}{2}, \quad (973)$$

$$|+\rangle = \sin(\theta) |0\rangle + \cos(\theta) |1\rangle, \quad (974)$$

$$|-\rangle = \cos(\theta) |0\rangle - \sin(\theta) |1\rangle, \quad (975)$$

$$\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}, \quad (976)$$

$$\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}. \quad (977)$$

This result is plausible because in the paper [1] we have that:

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{\Omega_r}{\epsilon} \right). \quad (978)$$

We can obtain the value of $\tan(\theta)$ through the following trigonometry identity for $x = \tan^{-1} \left(\frac{\Omega_r}{\epsilon} \right)$.

$$\tan \left(\frac{x}{2} \right) = \frac{\sin(x)}{\cos(x) + 1}. \quad (979)$$

So the value of $\tan(\theta)$ using (979) is equal to:

$$\tan(\theta) = \frac{\frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}}{\frac{\epsilon}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}} + 1} \quad (980)$$

$$= \frac{\frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}}{\frac{\epsilon + \sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}} \quad (981)$$

$$= \frac{\Omega_r}{\epsilon + \eta}. \quad (982)$$

This proves our assertion.

Using this basis we can find the decomposition matrices using the equations (906)-(907) and the fact that $|+\rangle = \sin(\theta) |0\rangle + \cos(\theta) |1\rangle = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$ and $|-\rangle = \cos(\theta) |0\rangle - \sin(\theta) |1\rangle = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ with $\sin(\theta) = \frac{\Omega_r}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$ and $\cos(\theta) = \frac{\epsilon + \eta}{\sqrt{(\epsilon + \eta)^2 + \Omega_r^2}}$:

$$\langle +|\sigma_x|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (983)$$

$$= 2 \sin(\theta) \cos(\theta) \quad (984)$$

$$= \sin(2\theta), \quad (985)$$

$$\langle -|\sigma_x|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (986)$$

$$= -2 \sin(\theta) \cos(\theta) \quad (987)$$

$$= -\sin(2\theta), \quad (988)$$

$$\langle -|\sigma_x|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (989)$$

$$= \cos^2(\theta) - \sin^2(\theta) \quad (990)$$

$$= \cos(2\theta), \quad (991)$$

$$\langle +|\sigma_y|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (992)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (993)$$

$$= 0, \quad (994)$$

$$\langle -|\sigma_y|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (995)$$

$$= i \sin(\theta) \cos(\theta) - i \sin(\theta) \cos(\theta) \quad (996)$$

$$= 0, \quad (997)$$

$$\langle -|\sigma_y|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (998)$$

$$= i \cos^2(\theta) + i \sin^2(\theta) \quad (999)$$

$$= i. \quad (1000)$$

$$\langle +|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1001)$$

$$= \cos(\theta) \cos(\theta) \quad (1002)$$

$$= \cos^2(\theta), \quad (1003)$$

$$\langle -|\frac{1+\sigma_z}{2}|-\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \quad (1004)$$

$$= \sin(\theta) \sin(\theta) \quad (1005)$$

$$= \sin^2(\theta), \quad (1006)$$

$$\langle -|\frac{1+\sigma_z}{2}|+\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (1007)$$

$$= -\sin(\theta) \cos(\theta) \quad (1008)$$

$$= -\sin(\theta) \cos(\theta). \quad (1009)$$

Composing the parts shown give us the Fourier decomposition matrices for this case:

$$A_1(0) = \sin(2\theta) (|+\rangle\langle+| - |-\rangle\langle-|), \quad (1010)$$

$$A_1(\eta) = \cos(2\theta) |-\rangle\langle+|, \quad (1011)$$

$$A_2(0) = 0, \quad (1012)$$

$$A_2(\eta) = i|-\rangle\langle+|, \quad (1013)$$

$$A_3(0) = \cos^2(\theta) |+\rangle\langle+| + \sin^2(\theta) |-\rangle\langle-|, \quad (1014)$$

$$A_3(\eta) = -\sin(\theta) \cos(\theta) |-\rangle\langle+|. \quad (1015)$$

Now to prove the fact that the model of the “Time-independent variational quantum master equation” is a special case the master equation (943) we need to take account of the time-independence of the hamiltonian of this system. From this perspective is possible to show that for the equation (924) is equivalent to:

$$\mathcal{D}_{ijww'}(t) = \int_0^t d\tau D_{ijww'}(t-\tau, t) \quad (1016)$$

$$= \int_0^t d\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{i\tau w(t-\tau)} e^{-it(w(t-\tau)-w'(t))} \mathcal{A}_j(w(t-\tau), w'(t)) \quad (1017)$$

$$= \int_0^t d\tau C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{i\tau w} e^{-it(w-w')} \mathcal{A}_j(w, w'). \quad (1018)$$

Now to make comparisons between the model obtained and the model of the system under discussion we will define that the correlation functions of the reference [1] denoted by $\Lambda'_{ij}(\tau)$ relate with the correlation functions defined in the equation (397) in the following way:

$$\Lambda'_{ij}(\tau) = C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau). \quad (1019)$$

So the response matrix can be rewritten as:

$$\mathcal{D}_{ijww'}(t) = \left(\int_0^t d\tau \Lambda'_{ij}(\tau) e^{i\tau w} e^{-it(w-w')} \right) \mathcal{A}_j(w, w') \quad (1020)$$

Let's define the response function like:

$$K_{ij}(w, w', t) = \int_0^t C_i(t) C_j(t-\tau) \Lambda_{ij}(\tau) e^{i\tau w} e^{-it(w-w')} d\tau \quad (1021)$$

$$= \int_0^t \Lambda'_{ij}(\tau) e^{i\tau w} e^{-it(w-w')} d\tau \quad (1022)$$

$$= K_{ijww'}(t). \quad (1023)$$

Then we have the following equivalence:

$$\mathcal{D}_{ijww'}(t) = K_{ijww'}(t) \mathcal{A}_j(w, w') \quad (1024)$$

$$= K_{ijww'}(t) \mathcal{A}_{jww'} \quad (1025)$$

We can proof that

$$\frac{d\overline{\rho_S}(t)}{dt} = -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left([A_i, \mathcal{D}_{ijww'}(t) \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) \mathcal{D}_{ijww'}^\dagger(t)] \right) \quad (1026)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left([A_i, K_{ijww'}(t) \mathcal{A}_{jww'} \overline{\rho_S}(t)] - [A_i, \overline{\rho_S}(t) K_{ijww'}^*(t) \mathcal{A}_{jww'}^\dagger] \right) \quad (1027)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left(K_{ijww'}(t) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t)] - K_{ijww'}^*(t) [A_i, \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] \right) \quad (1028)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} \left((K_{ijww'}^{\Re}(t) + i K_{ijww'}^{\Im}(t)) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t)] - (K_{ijww'}^{\Re}(t) - i K_{ijww'}^{\Im}(t)) [A_i, \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] \right) \quad (1029)$$

$$= -i [\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijww'} K_{ijww'}^{\Re}(t) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t) - \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] - i \sum_{ijww'} K_{ijww'}^{\Im}(t) [A_i, \mathcal{A}_{jww'} \overline{\rho_S}(t) + \overline{\rho_S}(t) \mathcal{A}_{jww'}^\dagger] \quad (1030)$$

Using the notation of the master equation (940), we can say that $C_1(t) = \frac{\Omega}{2} = C_2(t)$ and $C_3(t) = 1$, being Ω a constant. Furthermore given that $\overline{H_S}$ is time-independent then $B(t) = B$. Taking the equations(793)-(817) we find that the correlation functions of the reference [1] written in terms of the RHS of the equation (397) are equal to:

$$\langle \widetilde{B}_{1z}(t) \widetilde{B}_{1z}(s) \rangle_B = \sum_{\mathbf{k}} ((g_{1\mathbf{k}} - v_{1\mathbf{k}}) (g_{1\mathbf{k}} - v_{1\mathbf{k}})^* e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + (g_{1\mathbf{k}} - v_{1\mathbf{k}})^* (g_{1\mathbf{k}} - v_{1\mathbf{k}}) e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (1031)$$

$$= \sum_{\mathbf{k}} |g_{1\mathbf{k}} - v_{1\mathbf{k}}|^2 (e^{i\omega_{\mathbf{k}}\tau} N_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}\tau} (N_{\mathbf{k}} + 1)) \quad (1032)$$

$$\approx \int_0^\infty J_1(\omega) (1 - F_1(\omega))^2 (e^{i\omega\tau} N(\omega) + e^{-i\omega\tau} (N(\omega) + 1)) d\omega \quad (1033)$$

$$G_{\pm}(\omega, \tau) = e^{i\omega\tau} N(\omega) + e^{-i\omega\tau} (N(\omega) + 1) \quad (1034)$$

$$\langle \widetilde{B}_{1z}(t) \widetilde{B}_{1z}(s) \rangle_B \approx \int_0^\infty J_1(\omega) (1 - F_1(\omega))^2 G_+(\omega, t) d\omega \quad (1035)$$

$$\chi_{10}(t) = 0 \text{ (because } v_{0\mathbf{k}}(t) = 0 \text{ for all } \mathbf{k}) \quad (1036)$$

$$U_{10}(t, s) = \prod_{\mathbf{k}} \exp \left(i \left(\frac{(v_{1\mathbf{k}}(t) - v_{0\mathbf{k}}(t)) (v_{1\mathbf{k}}(s) - v_{0\mathbf{k}}(s))^* \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \quad (1037)$$

$$= \prod_{\mathbf{k}} \exp \left(i \left(\frac{v_{1\mathbf{k}}^2(t) \exp(i\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right)^{\Im} \right) \quad (1038)$$

$$= \prod_{\mathbf{k}} \exp \left(i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right) \quad (1039)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_x(s) \rangle_B = \frac{1}{2} \left(\prod_{\mathbf{k}} \exp \left(i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right) \prod_{\mathbf{k}} \exp \left(- \frac{|v_{1\mathbf{k}} \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) + \prod_{\mathbf{k}} \exp \left(- i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \right) \prod_{\mathbf{k}} \exp \left(- \frac{|v_{1\mathbf{k}} \exp(i\omega_{\mathbf{k}}\tau) - v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (1040)$$

$$- \left(\exp \left(- \frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \left(\exp \left(- \frac{1}{2} \sum_{\mathbf{k}} \left(\left| \frac{v_{1\mathbf{k}}}{\omega_{\mathbf{k}}} \right|^2 \right) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (1041)$$

$$= \frac{1}{2} \left(\prod_{\mathbf{k}} \exp \left(i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}} \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) + \prod_{\mathbf{k}} \exp \left(- i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}} \exp(i\omega_{\mathbf{k}}\tau) - v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (1042)$$

$$- \left(\exp \left(- \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (1043)$$

$$|v_{1\mathbf{k}} \exp(i\omega_{\mathbf{k}}\tau) \pm v_{1\mathbf{k}}|^2 = v_{1\mathbf{k}}^2 |\exp(i\omega_{\mathbf{k}}\tau) \pm 1|^2 \quad (1044)$$

$$= v_{1\mathbf{k}}^2 |\cos(\omega_{\mathbf{k}}\tau) + i \sin(\omega_{\mathbf{k}}\tau) \pm 1|^2 \quad (1045)$$

$$= v_{1\mathbf{k}}^2 \left((1 \pm \cos(\omega_{\mathbf{k}}\tau))^2 + \sin^2(\omega_{\mathbf{k}}\tau) \right) \quad (1046)$$

$$= 2v_{1\mathbf{k}}^2 (1 \pm \cos(\omega_{\mathbf{k}}\tau)) \quad (1047)$$

$$B \equiv \exp \left(- \frac{1}{2} \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (1048)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_x(s) \rangle_B = \frac{1}{2} \left(\exp \left(\sum_{\mathbf{k}} i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}} \exp(i\omega_{\mathbf{k}}\tau) + v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) + \exp \left(\sum_{\mathbf{k}} - i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{|v_{1\mathbf{k}} \exp(i\omega_{\mathbf{k}}\tau) - v_{1\mathbf{k}}|^2}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (1049)$$

$$- \left(\exp \left(- \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (1050)$$

$$\phi(\tau) = \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \left(-i \sin(\omega_{\mathbf{k}}\tau) + \cos(\omega_{\mathbf{k}}\tau) \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \quad (1051)$$

$$\approx \int_0^\infty \frac{J_1(\omega) F_1^2(\omega)}{\omega^2} \left(-i \sin(\omega\tau) + \cos(\omega\tau) \coth \left(\frac{\beta\omega}{2} \right) \right) d\omega \quad (1052)$$

$$= \int_0^\infty \frac{J_1(\omega) F_1^2(\omega)}{\omega^2} G_+(\omega, \tau) d\omega \quad (1053)$$

$$\langle \widetilde{B}_x(t) \widetilde{B}_x(s) \rangle_B = \frac{1}{2} \left(\exp \left(\sum_{\mathbf{k}} i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2 (1 + \cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) + \exp \left(\sum_{\mathbf{k}} - i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2 (1 - \cos(\omega_{\mathbf{k}}\tau))}{2\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) - B^2 \quad (1054)$$

$$= \frac{1}{2} \left(\exp \left(\sum_{\mathbf{k}} i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2 (1 + \cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) + \exp \left(\sum_{\mathbf{k}} - i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2 (1 - \cos(\omega_{\mathbf{k}}\tau))}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) - B^2 \quad (1055)$$

$$= \frac{1}{2} \left(\exp \left(- \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left(\sum_{\mathbf{k}} i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2 \cos(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) + \exp \left(- \sum_{\mathbf{k}} \frac{v_{1\mathbf{k}}^2}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \exp \left(\sum_{\mathbf{k}} - i \frac{v_{1\mathbf{k}}^2 \sin(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} - \frac{v_{1\mathbf{k}}^2 \cos(\omega_{\mathbf{k}}\tau)}{\omega_{\mathbf{k}}^2} \coth \left(\frac{\beta\omega_{\mathbf{k}}}{2} \right) \right) \right) \quad (1056)$$

$$= \frac{B^2}{2} (e^{-\phi(\tau)} + e^{\phi(\tau)} - 2) \quad (1057)$$

$$\Lambda'_{11}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left(\widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (1098)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right) \quad (1099)$$

$$= \frac{\Omega_r^2}{4} (\cosh(\phi(\tau)) - 1) \quad (1100)$$

$$\Lambda'_{22}(\tau) = \left(\frac{\Omega}{2}\right)^2 \text{Tr}_B \left(\widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (1101)$$

$$= \frac{\Omega_r^2}{8} \left(e^{\phi(\tau)} - e^{-\phi(\tau)} \right), \quad (1102)$$

$$\Lambda'_{33}(\tau) = \int_0^\infty d\omega J(\omega) (1 - F(\omega))^2 G_+(\tau), \quad (1103)$$

$$\Lambda'_{32}(\tau) = \frac{\Omega_r}{2} \int_0^\infty d\omega \frac{J(\omega)}{\omega} F(\omega) (1 - F(\omega)) iG_-(\tau), \quad (1104)$$

$$\Lambda'_{32}(\tau) = -\Lambda'_{23}(\tau), \quad (1105)$$

$$\Lambda'_{12}(\tau) = \Lambda'_{21}(\tau) \quad (1106)$$

$$= \Lambda'_{13}(\tau) \quad (1107)$$

$$= \Lambda'_{31}(\tau) \quad (1108)$$

$$= 0. \quad (1109)$$

Finally taking the Hamiltonian (957) and given that to reproduce this Hamiltonian we need to impose in (5) that $V_{10}(t) = \frac{\Omega}{2}$, $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = \delta$, then we obtain that $\text{Det}(\overline{H}_S) = -\frac{\Omega_r^2}{4}$, $\text{Tr}(\overline{H}_S) = \epsilon$. Now $\eta = \sqrt{\epsilon^2 + \Omega_r^2}$ and using the equation (337) we have that:

$$f_k = \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(\epsilon - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\omega_k} \right)} \quad (1110)$$

$$= \frac{g_k \left(1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \right)}{1 - \frac{\epsilon \tanh\left(\frac{\beta\eta}{2}\right)}{\eta} \left(1 - \frac{\Omega_r^2 \coth\left(\frac{\beta\omega_k}{2}\right)}{2\epsilon\omega_k} \right)}. \quad (1111)$$

This shows that the expression obtained reproduces the variational parameters of the time-independent model of the reference. In general we can see that the time-independent model studied can be reproduced using the master equation (433) under a time-independent approach providing similar results.

Given that the Hamiltonian of this system is time-independent, then $U(t)U^\dagger(t-\tau) = U(\tau)$. From the equation (940) and using the fact that

$$\widetilde{A}_j(t-\tau, t) = U(\tau) A_j U(-\tau) \quad (1112)$$

$$= \sum_w e^{iw\tau} \mathcal{A}_j(-w) \quad (1113)$$

$$= \sum_w e^{-iw\tau} \mathcal{A}_j(w). \quad (1114)$$

because the matrices $U(t)$ and $U(t-\tau)$ commute from the fact that $H_S(t)$ and $H_S(t-\tau)$ commute as well for time independent Hamiltonians. The master equation is equal to:

$$\frac{d\bar{\rho}_S(t)}{dt} = -i[H_S(t), \bar{\rho}_S(t)] - \frac{1}{2} \sum_{ij} \sum_w \gamma_{ij}(w, t) \left[A_i, \mathcal{A}_j(w) \bar{\rho}_S(t) - \bar{\rho}_S(t) \mathcal{A}_j^\dagger(w) \right] \quad (1115)$$

$$- \sum_{ij} \sum_w S_{ij}(w, t) \left[A_i, \mathcal{A}_j(w) \bar{\rho}_S(t) + \bar{\rho}_S(t) \mathcal{A}_j^\dagger(w) \right]. \quad (1116)$$

where $\mathcal{A}_j^\dagger(w) = \mathcal{A}_j(-w)$, as we can see the equation (1116) contains the rates and energy shifts $\gamma_{ij}(w, t) = 2K_{ij}^{\Re}(w, t)$ and $S_{ij}(w, t) = K_{ij}^{\Im}(w, t)$, respectively, defined in terms of the response functions

$$K_{ij}^{\Im}(w, t) = \int_0^t \Lambda'_{ij}(\tau) e^{iw\tau} d\tau.$$

The fact $\mathcal{A}_j^\dagger(w) = \mathcal{A}_j(-w)$ can be verified directly for a 2×2 matrix. given that $\overline{H_S}$ is independent of time then we have that:

$$e^{i\overline{H_S}(t-\tau)} = e^{i(\lambda_+|+\rangle\langle+| + \lambda_-|-\rangle\langle-|)(t-\tau)} \quad (1117)$$

$$= e^{i\lambda_+|+\rangle\langle+|(t-\tau)} e^{i\lambda_-|-\rangle\langle-|(t-\tau)} \quad (1118)$$

$$= \left(|-\rangle\langle-| + e^{i\lambda_+(t-\tau)}|+\rangle\langle+| \right) \left(|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-| \right) \quad (1119)$$

$$= e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|. \quad (1120)$$

Where λ_+, λ_- are the eigenvalues associated to the eigenvectors $|+\rangle\langle+|, |-\rangle\langle-|$ of $\overline{H_S}$. Calculating the transformation (881) of (905)-(907) directly using the previous relationship we find that:

$$\widetilde{A_i(0)(t-\tau)} = (e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|)(\langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|)(e^{-i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-i\lambda_-(t-\tau)}|-\rangle\langle-|) \quad (1121)$$

$$= \langle+|A_i|+\rangle|+\rangle\langle+| + \langle-|A_i|-\rangle|-\rangle\langle-|, \quad (1122)$$

$$\widetilde{A_i(w)(t-\tau)} = (e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|)(\langle+|A_i|-\rangle|+\rangle\langle-|)(e^{-i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-i\lambda_-(t-\tau)}|-\rangle\langle-|) \quad (1123)$$

$$= \langle+|A_i|-\rangle|+\rangle\langle-|e^{iw(t-\tau)}, \quad (1124)$$

$$\widetilde{A_i(-w)(t-\tau)} = (e^{i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{i\lambda_-(t-\tau)}|-\rangle\langle-|)(\langle-|A_i|+\rangle|-\rangle\langle+|)(e^{-i\lambda_+(t-\tau)}|+\rangle\langle+| + e^{-i\lambda_-(t-\tau)}|-\rangle\langle-|) \quad (1125)$$

$$= \langle-|A_i|+\rangle|-\rangle\langle+|e^{-iw(t-\tau)}. \quad (1126)$$

Here $w = \lambda_+ - \lambda_-$. So we can see that for the equation (891) it's possible to deduce for this case of time-independent matrix $\overline{H_S}$ if $w \neq w'$ then $A'_j(w, w') = 0$ so:

$$\widetilde{A}_j(t - \tau, t) = U(t) U^\dagger(t - \tau) A_j(t) U(t - \tau) U^\dagger(t) \quad (1127)$$

$$= U(t) \left(\sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} A_j(w(t-\tau)) \right) U^\dagger(t) \quad (1128)$$

$$= \sum_{w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} U(t) A_j(w(t-\tau)) U^\dagger(t) \quad (1129)$$

$$= \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_j(w(t-\tau), w'(t)) \quad (1130)$$

$$= \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_{jww'} \quad (1131)$$

$$== \sum_{w'(t), w(t-\tau)} e^{-i(t-\tau)w(t-\tau)} e^{itw'(t)} A_j(w) \delta_{ww'} \quad (1132)$$

$$= \sum_w e^{-i(t-\tau)w} e^{itw} A_j(w) \quad (1133)$$

$$= \sum_w e^{i\tau w} A_j(w) \quad (1134)$$

$$= U^\dagger(-\tau) A_j U(-\tau) \quad (1135)$$

So using now as reference the equation (1030) and $A'_j(w, w') = 0$ we can deduce that:

$$\frac{d\overline{\rho_S}(t)}{dt} = -i[\overline{H_S}(t), \overline{\rho_S}(t)] - \sum_{ijw} K_{ij}^{\Re}(w, t) [A_i, A_j(w) \overline{\rho_S}(t) - \overline{\rho_S}(t) A_j^\dagger(w)] - i \sum_{ijw} K_{ij}^{\Im}(w, t) [A_i, A_j(w) \overline{\rho_S}(t) + \overline{\rho_S}(t) A_j^\dagger(w)] \quad (1136)$$

B. Time-dependent polaron quantum master equation

Following the reference [1], when $\Omega_k \ll \omega_k$ then $f_k \approx g_k$ so we recover the full polaron transformation. It means from the equation (106) that $B_z = 0$. The Hamiltonian studied is given by:

$$H = \left(\delta + \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right) \right) |1\rangle\langle 1| + \frac{\Omega(t)}{2} \sigma_x + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1137)$$

If $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then $B(\tau) = B$, so B is independent of the time. In order to reproduce the Hamiltonian of the equation (1137) using the Hamiltonian of the equation (1) we can say that $\delta = \varepsilon_1(t)$, $\varepsilon_0(t) = 0$, $V_{10}(t) = \frac{\Omega(t)}{2}$. Now given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ then, in this case and using the equation (??) and (??) we obtain the following transformed Hamiltonians:

$$\overline{H_S} = (\delta + R_1) |1\rangle\langle 1| + \frac{B\sigma_x}{2} \Omega(t), \quad (1138)$$

$$\overline{H_I} = \frac{\Omega(t)}{2} (B_x \sigma_x + B_y \sigma_y). \quad (1139)$$

In this case $R_1 = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} g_{\mathbf{k}} \right)$ from (27) and given that $v_{\mathbf{k}} \approx g_{\mathbf{k}}$ and $\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} = g_{\mathbf{k}}/\omega_{\mathbf{k}}$ then $R_1 = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}}^{-1} |g_{\mathbf{k}}|^2) = \sum_{\mathbf{k}} (-\omega_{\mathbf{k}} | \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} |^2)$ as expected, take $\delta + R_1 = \delta'$. If $F(\omega_{\mathbf{k}}) = 1$ and using the equations (1098)-(1106) we can deduce that the only terms that survive are $\Lambda_{11}(\tau)$ and $\Lambda_{22}(\tau)$. The phonon propagator for this case is:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} G_+(\tau) d\omega. \quad (1140)$$

Writing $G_+(\tau) = \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau)$ so (1140) can be written as:

$$\phi(\tau) = \int_0^\infty \frac{J(\omega)}{\omega^2} \left(\coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) - i \sin(\omega\tau) \right) d\omega. \quad (1141)$$

Writing the interaction Hamiltonian (1139) in the similar way to the equation (??) allow us to write $A_1 = \sigma_x$, $A_2 = \sigma_y$, $B_1(t) = B_x$, $B_2(t) = B_y$ and $C_1(t) = \frac{\Omega(t)}{2} = C_2(t)$. Now taking the equation (??) with $\delta'|1\rangle\langle 1| = \frac{\delta'}{2}\sigma_z + \frac{\delta'}{2}\mathbb{I}$ help us to reproduce the hamiltonian of the reference [2]. Then $\overline{H_S}$ is equal to:

$$\overline{H_S} = \frac{\delta'}{2}\sigma_z + \frac{B\sigma_x}{2}\Omega(t). \quad (1142)$$

As we can see the function B is a time-independent function because we consider that g_k doesn't depend of the time. In this case the relevant correlation functions are given by:

$$\Lambda_{11}(\tau) = \text{Tr}_B \left(\widetilde{B}_1(\tau) \widetilde{B}_1(0) \rho_B \right) \quad (1143)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} - 2 \right), \quad (1144)$$

$$\Lambda_{22}(\tau) = \text{Tr}_B \left(\widetilde{B}_2(\tau) \widetilde{B}_2(0) \rho_B \right) \quad (1145)$$

$$= \frac{B^2}{2} \left(e^{\phi(\tau)} + e^{-\phi(\tau)} \right). \quad (1146)$$

These functions match with the equations $\Lambda_x(\tau)$ and $\Lambda_y(\tau)$ of the reference [2] and $\Lambda_i(\tau) = \Lambda_i(-\tau)$ for $i \in \{x, y\}$ respectively. The master equation for this section based on the equation(433) is:

$$\frac{d\rho_S(t)}{dt} = -i \left[\frac{\delta'}{2}\sigma_z + \frac{\Omega_r(t)\sigma_x}{2}, \rho_S(t) \right] - \sum_{i=1}^2 \int_0^t d\tau \left(C_i(t) C_i(t-\tau) \Lambda_{ii}(\tau) \left[A_i, \widetilde{A}_i(t-\tau, t) \rho_S(t) \right] \right) \quad (1147)$$

$$+ C_i(t) C_i(t-\tau) \Lambda_{ii}(-\tau) \left[\rho_S(t) \widetilde{A}_i(t-\tau, t), A_i \right] \Big). \quad (1148)$$

Replacing $C_i(t) = \frac{\Omega(t)}{2}$ and $\widetilde{A}_i(t-\tau, t) = \widetilde{\sigma}_i(t-\tau, t)$, also using the equations (1143) and (1146) on the equation (1148) we obtain that:

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{2} [\delta'\sigma_z + \Omega_r(t)\sigma_x, \rho_S(t)] - \frac{\Omega(t)}{4} \int_0^t d\tau \Omega(t-\tau) ([\sigma_x, \widetilde{\sigma}_x(t-\tau, t) \rho_S(t)] \Lambda_x(\tau) \quad (1149)$$

$$+ [\sigma_y, \widetilde{\sigma}_y(t-\tau, t) \rho_S(t)] \Lambda_y(\tau) + [\rho_S(t) \widetilde{\sigma}_x(t-\tau, t), \sigma_x] \Lambda_x(\tau) + [\rho_S(t) \widetilde{\sigma}_y(t-\tau, t), \sigma_y] \Lambda_y(\tau)). \quad (1150)$$

As we can see $[A_j, \widetilde{A}_i(t-\tau, t) \rho_S(t)]^\dagger = [\rho_S(t) \widetilde{A}_i(t-\tau, t), A_j]$, $\Lambda_x(\tau) = \Lambda_x(-\tau)$ and $\Lambda_y(\tau) = \Lambda_y(-\tau)$, so the result obtained is the same master equation (21) of the reference [2] extended in the hermitian conjugate.

C. Time-Dependent Weak-Coupling Limit

In order to prove that the master equation deduced reproduces the equation (S17) of the reference [3] we will impose that $F(\omega) = 0$, so there is no transformation in this case. As we can see from the definition (397) the only term that survives is $\Lambda_{33}(\tau)$. Taking $\hbar = 1$ the Hamiltonian of the reference can be written in the form:

$$H = \Delta|1\rangle\langle 1| + \frac{\Omega(t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + |1\rangle\langle 1| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{\mathbf{k}}^* b_{\mathbf{k}} \right). \quad (1151)$$

Using the equation (940), from the fact that the Hamiltonian is time-independent in the evolution time allow us to write:

$$\frac{d\rho_S}{dt} = -i [H_S(t), \rho_S(t)] - \frac{1}{2} \sum_w \gamma_{33}(w, t) \left[A_3, A_3(w) \rho_S(t) - \rho_S(t) A_3^\dagger(w) \right] \quad (1152)$$

$$- \sum_w S_{33}(w, t) \left[A_3, A_3(w) \rho_S(t) + \rho_S(t) A_3^\dagger(w) \right] \Bigg). \quad (1153)$$

The correlation functions are relevant if $F(\omega) = 0$ for the weak-coupling approximation are:

$$\Lambda_{33}(\tau) = \int_0^\infty d\omega J(\omega) G_+(\tau), \quad (1154)$$

$$\Lambda_{33}(-\tau) = \int_0^\infty d\omega J(\omega) G_+(-\tau). \quad (1155)$$

In our case $A_3 = \frac{\mathbb{I} + \sigma_z}{2}$, the equation (1153) can be transformed in

$$\frac{d\rho_S}{dt} = -i [H_S(t), \rho_S(t)] - \sum_w (K_{33}(w, t) [A_3, A_3(w) \rho_S(t)] + K_{33}^*(w, t) [\rho_S(t) A_3(w), A_3]). \quad (1156)$$

As the paper suggest we will consider that the quantum system is in resonance, so $\Delta = 0$ and furthermore, the relaxation time of the bath is less than the evolution time to be considered, so the frequency of the Rabi frequency of the laser can be taken as constant and equal to $\tilde{\Omega}$. To find the matrices $A_3(w)$, we have to remember that $H_S = \frac{\Omega(t)}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|)$, this Hamiltonian using the approximation $\tilde{\Omega}$ have the following eigenvalues and eigenvectors:

$$\lambda_+ = \frac{\tilde{\Omega}}{2}, \quad (1157)$$

$$|+\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |0\rangle), \quad (1158)$$

$$\lambda_- = -\frac{\tilde{\Omega}}{2}, \quad (1159)$$

$$|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \quad (1160)$$

The elements of the decomposition matrices are:

$$\langle + | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1161)$$

$$= \frac{1}{2}, \quad (1162)$$

$$\langle - | \frac{1 + \sigma_z}{2} | - \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1163)$$

$$= \frac{1}{2}, \quad (1164)$$

$$\langle - | \frac{1 + \sigma_z}{2} | + \rangle = \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1165)$$

$$= -\frac{1}{2}. \quad (1166)$$

The decomposition matrices are

$$A_3(0) = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| \quad (1167)$$

$$= \frac{\mathbb{I}}{2}, \quad (1168)$$

$$A_3(\eta) = -\frac{1}{2}|-\rangle\langle+| \quad (1169)$$

$$= \frac{1}{4}(\sigma_z + i\sigma_y), \quad (1170)$$

$$A_3(-\eta) = -\frac{1}{2}|+\rangle\langle-| \quad (1171)$$

$$= \frac{1}{4}(\sigma_z - i\sigma_y). \quad (1172)$$

Neglecting the term proportional to the identity in the Hamiltonian we obtain that:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2}[\sigma_x, \rho_S(t)] - K_{33}(\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z + i\sigma_y)\rho_S(t) \right] - K_{33}(-\tilde{\Omega}, t) \left[\frac{\sigma_z}{2}, \frac{1}{4}(\sigma_z - i\sigma_y)\rho_S(t) \right] \quad (1173)$$

$$- K_{33}^*(\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4}(\sigma_z + i\sigma_y), \frac{\sigma_z}{2} \right] - K_{33}^*(-\tilde{\Omega}, t) \left[\rho_S(t) \frac{1}{4}(\sigma_z - i\sigma_y), \frac{\sigma_z}{2} \right]. \quad (1174)$$

Calculating the response functions extending the upper limit of τ to ∞ , we obtain:

$$K_{33}(\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{i\tilde{\Omega}\tau} d\tau d\omega \quad (1175)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} ((n(\omega) + 1)e^{-i\tau\omega} + n(\omega)e^{i\tau\omega}) d\tau d\omega \quad (1176)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{i\tilde{\Omega}\tau} (n(\omega) + 1)e^{-i\tau\omega} d\tau d\omega \quad (1177)$$

$$= \int_0^\infty \int_0^\infty J(\omega) (n(\omega) + 1) e^{i\tilde{\Omega}\tau - i\tau\omega} d\tau d\omega \quad (1178)$$

$$= \int_0^\infty J(\omega) (n(\omega) + 1) \pi \delta(\tilde{\Omega} - \omega) d\omega \quad (1179)$$

$$= \pi J(\tilde{\Omega}) (n(\tilde{\Omega}) + 1), \quad (1180)$$

$$K_{33}(-\tilde{\Omega}) = \int_0^\infty \int_0^\infty J(\omega) G_+(\tau) e^{-i\tilde{\Omega}\tau} d\tau d\omega \quad (1181)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} ((n(\omega) + 1)e^{-i\tau\omega} + n(\omega)e^{i\tau\omega}) d\tau d\omega \quad (1182)$$

$$= \int_0^\infty \int_0^\infty J(\omega) e^{-i\tilde{\Omega}\tau} n(\omega) e^{i\tau\omega} d\tau d\omega \quad (1183)$$

$$= \int_0^\infty \int_0^\infty J(\omega) n(\omega) e^{-i\tilde{\Omega}\tau + i\tau\omega} d\tau d\omega \quad (1184)$$

$$= \int_0^\infty J(\omega) n(\omega) \pi \delta(-\tilde{\Omega} + \omega) d\omega \quad (1185)$$

$$= \pi J(\tilde{\Omega}) n(\tilde{\Omega}). \quad (1186)$$

Here we have used $\int_0^\infty ds e^{\pm i\epsilon s} = \pi \delta(\epsilon) \pm i \frac{\text{V.P.}}{\epsilon}$, where V.P. denotes the Cauchy's principal value. These principal values are ignored because they lead to small renormalizations of the Hamiltonian. Furthermore we don't take

account of value associated to the matrix $A_3(0)$ because the spectral density $J(\omega)$ is equal to zero when $\omega = 0$. Replacing in the equation (1173) lead us to obtain:

$$\frac{d\rho_S(t)}{dt} = -i\frac{\tilde{\Omega}}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\tilde{\Omega}) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (1187)$$

$$- \frac{\pi}{8} J(\tilde{\Omega}) \left((n(\tilde{\Omega}) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\tilde{\Omega}) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right). \quad (1188)$$

This is the same result than the equation (S17), so we have proved that our general master equation allows to reproduce the results of the weak-coupling time-dependent. Now the master equation in the evolution time is given by

$$\frac{d\rho_S(t)}{dt} = -i\frac{\Omega(t)}{2} [\sigma_x, \rho_S(t)] - \frac{\pi}{8} J(\Omega(t)) \left((n(\Omega(t)) + 1) [\sigma_z, (\sigma_z + i\sigma_y) \rho_S(t)] + n(\Omega(t)) [\sigma_z, (\sigma_z - i\sigma_y) \rho_S(t)] \right) \quad (1189)$$

$$- \frac{\pi}{8} J(\Omega(t)) \left((n(\Omega(t)) + 1) [\rho_S(t) (\sigma_z + i\sigma_y), \sigma_z] + n(\Omega(t)) [\rho_S(t) (\sigma_z - i\sigma_y), \sigma_z] \right). \quad (1190)$$

VI. TIME-DEPENDENT MULTI-SITE MODEL WITH V BATHS COUPLING

Let's consider the following Hamiltonian for a system of m-level system coupled to v-baths. We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1191)$$

$$H_S(t) = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1192)$$

$$H_I = \sum_{n\mathbf{k}} |n\rangle\langle n| \left(g_{n\mathbf{k}} b_{\mathbf{k}}^\dagger + g_{n\mathbf{k}}^* b_{\mathbf{k}} \right), \quad (1193)$$

$$H_B = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}. \quad (1194)$$

A. Variational Transformation

We consider the following operator:

$$V = \sum_{n\mathbf{k}} |n\rangle\langle n| \omega_{u\mathbf{k}}^{-1} \left(f_{n\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{n\mathbf{k}}^* b_{u\mathbf{k}} \right) \quad (1195)$$

At first let's obtain $e^{\pm V}$ under the transformation (1195), consider $\hat{\varphi}_n = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} \left(f_{n\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{n\mathbf{k}}^* b_{u\mathbf{k}} \right)$, so the equation (1195) can be written as $V = \sum_n |n\rangle\langle n| \hat{\varphi}_n$, then we have:

$$e^{\pm V} = e^{\pm \sum_n |n\rangle\langle n| \hat{\varphi}_n} \quad (1196)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{(\sum_n |n\rangle\langle n| \hat{\varphi}_n)^2}{2!} + \dots \quad (1197)$$

$$= \mathbb{I} \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1198)$$

$$= \sum_n |n\rangle\langle n| \pm \sum_n |n\rangle\langle n| \hat{\varphi}_n + \frac{\sum_n |n\rangle\langle n| \hat{\varphi}_n^2}{2!} + \dots \quad (1199)$$

$$= \sum_n |n\rangle\langle n| \left(\mathbb{I} \pm \hat{\varphi}_n + \frac{\hat{\varphi}_n^2}{2!} + \dots \right) \quad (1200)$$

$$= \sum_n |n\rangle\langle n| e^{\pm \hat{\varphi}_n} \quad (1201)$$

Given that $\left[f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}}, f_{nu'\mathbf{k}'} b_{u'\mathbf{k}'}^\dagger - f_{nu'\mathbf{k}'}^* b_{u'\mathbf{k}'} \right] = 0$ for all \mathbf{k}', \mathbf{k} and u, u' then we can proof using the Zassenhaus formula and defining $D(\pm \alpha_{nu\mathbf{k}}) = e^{\pm (\alpha_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - \alpha_{nu\mathbf{k}}^* b_{u\mathbf{k}})}$ in the same way than (23) with $\alpha_{nu\mathbf{k}} = \frac{f_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}}$:

$$e^{\pm \sum_{u\mathbf{k}} \omega_{u\mathbf{k}}^{-1} (f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}})} = \prod_u e^{\pm \sum_{\mathbf{k}} \omega_{u\mathbf{k}}^{-1} (f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}})} \quad (1202)$$

$$= \prod_u \left(\prod_{\mathbf{k}} e^{\pm \omega_{u\mathbf{k}}^{-1} (f_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger - f_{nu\mathbf{k}}^* b_{u\mathbf{k}})} \right) \quad (1203)$$

$$= \prod_u \left(\prod_{\mathbf{k}} D(\pm \alpha_{nu\mathbf{k}}) \right) \quad (1204)$$

$$= \prod_{u\mathbf{k}} D(\pm \alpha_{nu\mathbf{k}}) \quad (1205)$$

$$= \prod_u B_{nu\pm} \quad (1206)$$

$$B_{nu\pm} \equiv \prod_{\mathbf{k}} D(\pm \alpha_{nu\mathbf{k}}) \quad (1207)$$

As we can see $e^{-V} = \sum_n |n\rangle\langle n| \prod_u B_{nu-}$ and $e^V = \sum_n |n\rangle\langle n| \prod_u B_{nu+}$ this implies that $e^{-V} e^V = \mathbb{I}$. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) A \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1208)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1191):

$$\overline{|0\rangle\langle 0|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |0\rangle\langle 0| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1209)$$

$$= \prod_u B_{0u+} |0\rangle\langle 0| \prod_u B_{0u-}, \quad (1210)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} \prod_u B_{0u-}, \quad (1211)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} B_{0u-} \quad (1212)$$

$$= |0\rangle\langle 0| \prod_u \mathbb{I} \quad (1213)$$

$$= |0\rangle\langle 0|. \quad (1214)$$

$$\overline{|m\rangle\langle n|} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) |m\rangle\langle n| \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right), \quad (1215)$$

$$= |m\rangle\langle m| \prod_u B_{mu+} |m\rangle\langle n| \prod_u B_{nu-}, \quad (1216)$$

$$= |m\rangle\langle n| \prod_u B_{mu+} \prod_u B_{nu-}, \quad (1217)$$

$$= |m\rangle\langle n| \prod_u (B_{mu+} B_{nu-}), \quad m \neq n, \quad (1218)$$

$$= |m\rangle\langle n| \prod_u \left(\prod_{\mathbf{k}} D(\alpha_{muk}) \prod_{\mathbf{k}} D(-\alpha_{nuk}) \right), \quad (1219)$$

$$= |m\rangle\langle n| \prod_u \prod_{\mathbf{k}} (D(\alpha_{muk}) D(-\alpha_{nuk})), \quad (1220)$$

$$= |m\rangle\langle n| \prod_{u\mathbf{k}} \left(D(\alpha_{muk} - \alpha_{nuk}) \exp \left(\frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (1221)$$

$$\Pi_u(B_{mu+} B_{nu-}) = \prod_{u\mathbf{k}} \left(D(\alpha_{muk} - \alpha_{nuk}) \exp \left(\frac{1}{2} (-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}) \right) \right). \quad (1222)$$

$$\overline{\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}} = (\sum_n |n\rangle\langle n| \prod_u B_{nu+}) \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} (\sum_n |n\rangle\langle n| \prod_u B_{nu-}), \quad (1223)$$

$$= (|0\rangle\langle 0| \prod_u B_{0u+} + |1\rangle\langle 1| \prod_u B_{1u+} + \dots) (\sum_n |n\rangle\langle n| \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}}) (|0\rangle\langle 0| \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u-} + \dots), \quad (1224)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} \prod_u B_{1u-} + \dots, \quad (1225)$$

$$= |0\rangle\langle 0| \prod_u B_{0u+} + (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{0u-} + |1\rangle\langle 1| \prod_u B_{1u+} + (\sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} + \dots) \prod_u B_{1u-} + \dots \quad (1226)$$

$$= |0\rangle\langle 0| (\prod_u B_{0u+} + \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{0u-} + \prod_u B_{0u+} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{0u-} + \dots) \quad (1227)$$

$$+ |1\rangle\langle 1| (\prod_u B_{1u+} + \sum_{\mathbf{k}} \omega_{0\mathbf{k}} b_{0\mathbf{k}}^\dagger b_{0\mathbf{k}} \prod_u B_{1u-} + \prod_u B_{1u+} + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} b_{1\mathbf{k}}^\dagger b_{1\mathbf{k}} \prod_u B_{1u-} + \dots) + \dots \quad (1228)$$

$$= |0\rangle\langle 0| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{00\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{01\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) \quad (1229)$$

$$+ |1\rangle\langle 1| \left(\sum_{\mathbf{k}} \omega_{0\mathbf{k}} \left(b_{0\mathbf{k}}^\dagger - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{10\mathbf{k}}}{\omega_{0\mathbf{k}}} \right) + \sum_{\mathbf{k}} \omega_{1\mathbf{k}} \left(b_{1\mathbf{k}}^\dagger - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) \left(b_{0\mathbf{k}} - \frac{v_{11\mathbf{k}}}{\omega_{1\mathbf{k}}} \right) + \dots \right) + \dots \quad (1230)$$

$$= |0\rangle\langle 0| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + |1\rangle\langle 1| \left(\sum_{u\mathbf{k}} \omega_{u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \dots \quad (1231)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\omega_{n\mathbf{k}} \left(b_{n\mathbf{k}}^\dagger - \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} \right) \left(b_{n\mathbf{k}} - \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} \right) \right) \quad (1232)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\omega_{n\mathbf{k}} \left(b_{n\mathbf{k}}^\dagger b_{n\mathbf{k}} - \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} b_{n\mathbf{k}}^\dagger - \frac{v_{n\mathbf{k}}^*}{\omega_{n\mathbf{k}}} b_{n\mathbf{k}} + \left| \frac{v_{n\mathbf{k}}}{\omega_{n\mathbf{k}}} \right|^2 \right) \right) \quad (1233)$$

$$= \sum_{n\mathbf{k}} |n\rangle\langle n| \omega_{n\mathbf{k}} b_{n\mathbf{k}}^\dagger b_{n\mathbf{k}} + \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{n\mathbf{k}}|^2}{\omega_{n\mathbf{k}}} - v_{n\mathbf{k}} b_{n\mathbf{k}}^\dagger - v_{n\mathbf{k}}^* b_{n\mathbf{k}} \right) \quad (1234)$$

$$= \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{n\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{n\mathbf{k}}|^2}{\omega_{n\mathbf{k}}} - (v_{n\mathbf{k}} b_{n\mathbf{k}}^\dagger + v_{n\mathbf{k}}^* b_{n\mathbf{k}}) \right) \quad (1235)$$

The transformed Hamiltonians of the equations (1192) to (1194) written in terms of (1209) to (1233) are:

$$\overline{H_S(t)} = \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1236)$$

$$= \overline{\sum_n \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1237)$$

$$= \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) \quad (1238)$$

$$\overline{H_I} = \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{n \neq m} |n\rangle\langle m| (g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}}) \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1239)$$

$$= \left(\sum_n |n\rangle\langle n| \prod_u B_{nu+} \right) \left(\sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}) + \dots \right) \left(\sum_n |n\rangle\langle n| \prod_u B_{nu-} \right) \quad (1240)$$

$$= \prod_u B_{0u+} \sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{0u\mathbf{k}}^* b_{u\mathbf{k}}) \prod_u B_{0u-} + \prod_u B_{1u+} \sum_{u\mathbf{k}} |1\rangle\langle 1| (g_{1u\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{1u\mathbf{k}}^* b_{u\mathbf{k}}) \prod_u B_{1u-} + \dots \quad (1241)$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| (g_{0u\mathbf{k}} \prod_u B_{0u+} b_{u\mathbf{k}}^\dagger \prod_u B_{0u-} + g_{0u\mathbf{k}}^* \prod_u B_{0u+} b_{u\mathbf{k}} \prod_u B_{0u-}) + \sum_{u\mathbf{k}} |1\rangle\langle 1| (g_{1u\mathbf{k}} \prod_u B_{1u+} b_{u\mathbf{k}}^\dagger \prod_u B_{1u-} + g_{1u\mathbf{k}}^* \prod_u B_{1u+} b_{u\mathbf{k}} \prod_u B_{1u-}) + \dots \quad (1242)$$

$$= \sum_{u\mathbf{k}} |0\rangle\langle 0| \left(g_{0u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{0u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{0u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{0u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \sum_{u\mathbf{k}} |1\rangle\langle 1| \left(g_{1u\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{1u\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{1u\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{1u\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) + \dots \quad (1243)$$

$$= \sum_{n \neq u\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} \left(b_{u\mathbf{k}}^\dagger - \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} \right) + g_{nu\mathbf{k}}^* \left(b_{u\mathbf{k}} - \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1244)$$

$$= \sum_{n \neq u\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1245)$$

$$\overline{H_B} = \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{n \neq u\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - (v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}) \right) \quad (1246)$$

Joining this terms allow us to write the transformed Hamiltonian as:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_{n \neq u\mathbf{k}} |n\rangle\langle n| \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - (v_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + v_{nu\mathbf{k}}^* b_{u\mathbf{k}}) \right) \quad (1247)$$

$$+ \sum_{n \neq u\mathbf{k}} |n\rangle\langle n| \left(g_{nu\mathbf{k}} b_{u\mathbf{k}}^\dagger + g_{nu\mathbf{k}}^* b_{u\mathbf{k}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1248)$$

Let's define the following functions:

$$R_n(t) = \sum_{u\mathbf{k}} \left(\frac{|v_{nu\mathbf{k}}|^2}{\omega_{u\mathbf{k}}} - \left(g_{nu\mathbf{k}} \frac{v_{nu\mathbf{k}}^*}{\omega_{u\mathbf{k}}} + g_{nu\mathbf{k}}^* \frac{v_{nu\mathbf{k}}}{\omega_{u\mathbf{k}}} \right) \right) \quad (1249)$$

$$B_{z,n}(t) = \sum_{u\mathbf{k}} \left((g_{nu\mathbf{k}} - v_{nu\mathbf{k}}) b_{u\mathbf{k}}^\dagger + (g_{nu\mathbf{k}} - v_{nu\mathbf{k}})^* b_{u\mathbf{k}} \right) \quad (1250)$$

Using the previous functions we have that (1247) can be re-written in the following way:

$$\overline{H} = \sum_n \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \prod_u (B_{mu+} B_{nu-}) + \sum_{u\mathbf{k}} \omega_{u\mathbf{k}} b_{u\mathbf{k}}^\dagger b_{u\mathbf{k}} + \sum_n R_n(t) |n\rangle\langle n| + \sum_n B_{z,n}(t) |n\rangle\langle n| \quad (1251)$$

$$(1252)$$

Now in order to separate the elements of the hamiltonian (1252) let's follow the references of the equations (??) and (??) to separate the hamiltonian, before proceeding to do this we need to consider the term of the form:

$$\langle \Pi_u (B_{mu+} B_{nu-}) \rangle_{\overline{H_0}} = \langle \Pi_{uk} (D(\alpha_{muk} - \alpha_{nuk}) \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk})) \rangle_{\overline{H_0}} \quad (1253)$$

$$= (\Pi_{uk} \exp(\frac{1}{2}(-\alpha_{muk} \alpha_{nuk}^* + \alpha_{muk}^* \alpha_{nuk}))) \langle \Pi_{uk} D(\alpha_{muk} - \alpha_{nuk}) \rangle_{\overline{H_0}} \quad (1254)$$

$$= \left(\Pi_{uk} \exp\left(\frac{(v_{muk}^* v_{nuk} - v_{nuk} v_{muk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (1255)$$

$$\equiv B_{nm} \quad (1256)$$

$$\langle \Pi_u (B_{nu+} B_{mu-}) \rangle_{\overline{H_0}} = \left(\Pi_{uk} \exp\left(\frac{(v_{nuk}^* v_{muk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2}\right) \right) \Pi_u \exp\left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth\left(\frac{\beta\omega_{uk}}{2}\right)\right) \quad (1257)$$

$$= B_{nm}^* \quad (1258)$$

Following the reference [4] we define:

$$J_{nm} = \prod_u (B_{mu+} B_{nu-}) - B_{nm} \quad (1259)$$

As we can see:

$$J_{nm}^\dagger = \left(\prod_u (B_{mu+} B_{nu-}) - B_{nm} \right)^\dagger \quad (1260)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{nm}^* \quad (1261)$$

$$= \prod_u (B_{nu+} B_{mu-}) - B_{mn} \quad (1262)$$

$$= J_{mn} \quad (1263)$$

We can separate the Hamiltonian (1252) on the following way using similar arguments to the precedent sections to obtain:

$$\overline{H_{\bar{S}}}(t) = \sum_n (\varepsilon_n(t) + R_n) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} \quad (1264)$$

$$\overline{H_{\bar{I}}}(t) = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| J_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1265)$$

$$\overline{H_{\bar{B}}}(t) = \sum_{uk} \omega_{uk} b_{uk}^\dagger b_{uk} \quad (1266)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}})} \right) \right) + \langle \overline{H_{\bar{I}}} \rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} + O \left(\left\langle \overline{H_{\bar{I}}}^2 \right\rangle_{\overline{H_{\bar{S}}}(t) + \overline{H_{\bar{B}}}} \right). \quad (1267)$$

Taking the equations (244)-(252) and given that $\text{Tr} \left(e^{-\beta \overline{H_{\bar{S}}}(t)} \right) = C(R_0, R_1, \dots, R_{d-1}, B_{01}, \dots, B_{0(d-1)}, \dots, B_{(d-2)(d-1)})$, where each R_i and B_{kj} depend of the set of variational parameters $\{v_{nuk}\}$. Given that the numbers v_{nuk} are complex then we can separate them as $v_{nuk} = v_{nuk}^{\Re} + i v_{nuk}^{\Im}$. So our approach will be based on the derivation respect to v_{nuk}^{\Re} and v_{nuk}^{\Im} . The Hamiltonian $\overline{H_{\bar{S}}}(t)$ can be written like:

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \right) |n\rangle\langle n| \quad (1268)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1269)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \frac{g_{nuk} v_{nuk}^* + g_{nuk}^* v_{nuk}}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1270)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1271)$$

$$= \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2}{\omega_{uk}} - \frac{(g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} + i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1272)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1273)$$

$$v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^* = (v_{muk}^{\Re} - i v_{muk}^{\Im})(v_{nuk}^{\Re} + i v_{nuk}^{\Im}) - (v_{muk}^{\Re} + i v_{muk}^{\Im})(v_{nuk}^{\Re} - i v_{nuk}^{\Im}) \quad (1274)$$

$$= (v_{muk}^{\Re} v_{nuk}^{\Re} + i v_{nuk}^{\Im} v_{muk}^{\Re} - i v_{muk}^{\Re} v_{nuk}^{\Im} + v_{muk}^{\Im} v_{nuk}^{\Im}) \quad (1275)$$

$$- (v_{muk}^{\Re} v_{nuk}^{\Re} - i v_{nuk}^{\Im} v_{muk}^{\Re} + i v_{muk}^{\Re} v_{nuk}^{\Im} + v_{muk}^{\Im} v_{nuk}^{\Im}) \quad (1276)$$

$$= 2i (v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re}) \quad (1277)$$

$$\overline{H_S(t)} = \sum_n \left(\varepsilon_n(t) + \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2}{\omega_{uk}} - \frac{(g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} + i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \right) |n\rangle\langle n| \quad (1278)$$

$$+ \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| \left(\Pi_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1279)$$

$$|v_{muk} - v_{nuk}|^2 = (v_{muk} - v_{nuk})(v_{muk} - v_{nuk})^* \quad (1280)$$

$$= |v_{muk}|^2 + |v_{nuk}|^2 - (v_{nuk} v_{muk}^* + v_{muk} v_{nuk}^*) \quad (1281)$$

$$= (v_{muk}^{\Re})^2 + (v_{muk}^{\Im})^2 + (v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - (v_{nuk}^{\Re} + i v_{nuk}^{\Im})(v_{muk}^{\Re} - i v_{muk}^{\Im}) \quad (1282)$$

$$- (v_{muk}^{\Re} - i v_{muk}^{\Im})(v_{nuk}^{\Re} + i v_{nuk}^{\Im}) \quad (1283)$$

$$= (v_{muk}^{\Re})^2 + (v_{muk}^{\Im})^2 + (v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2(v_{nuk}^{\Re} v_{muk}^{\Re} + v_{nuk}^{\Im} v_{muk}^{\Im}) \quad (1284)$$

$$= (v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2 \quad (1285)$$

$$R_n(t) = \sum_{uk} \left(\frac{|v_{nuk}|^2}{\omega_{uk}} - \left(g_{nuk} \frac{v_{nuk}^*}{\omega_{uk}} + g_{nuk}^* \frac{v_{nuk}}{\omega_{uk}} \right) \right) \quad (1286)$$

$$= \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - (g_{nuk} + g_{nuk}^*) v_{nuk}^{\Re} - i v_{nuk}^{\Im} (g_{nuk}^* - g_{nuk})}{\omega_{uk}} \right) \quad (1287)$$

$$= \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1288)$$

$$B_{mn} = \left(\Pi_{uk} \exp \left(\frac{(v_{muk}^* v_{nuk} - v_{muk} v_{nuk}^*)}{2\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{|v_{muk} - v_{nuk}|^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1289)$$

$$= \left(\Pi_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \Pi_u \exp \left(-\frac{1}{2} \sum_k \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1290)$$

Then we can obtain using the chain rule that:

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Re}} = \frac{\partial}{\partial v_{nuk}^{\Re}} \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1291)$$

$$= \frac{2v_{nuk}^{\Re} - 2g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1292)$$

$$= 2 \frac{v_{nuk}^{\Re} - g_{nuk}^{\Re}}{\omega_{uk}} \delta_{nn'} \quad (1293)$$

$$\frac{\partial R_{n'}}{\partial v_{nuk}^{\Im}} = \frac{\partial}{\partial v_{nuk}^{\Im}} \sum_{uk} \left(\frac{(v_{nuk}^{\Re})^2 + (v_{nuk}^{\Im})^2 - 2g_{nuk}^{\Re} v_{nuk}^{\Re} - 2g_{nuk}^{\Im} v_{nuk}^{\Im}}{\omega_{uk}} \right) \quad (1294)$$

$$= \frac{2v_{nuk}^{\Im} - 2g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1295)$$

$$= 2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \delta_{nn'} \quad (1296)$$

Given that:

$$\ln B_{mn} = \ln \left(\left(\prod_{uk} \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) \right) \prod_u \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \right) \quad (1297)$$

$$= \sum_{uk} \ln \exp \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_u \ln \exp \left(-\frac{1}{2} \sum_{\mathbf{k}} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1298)$$

$$= \sum_{uk} \left(\frac{i(v_{nuk}^{\Im} v_{muk}^{\Re} - v_{muk}^{\Im} v_{nuk}^{\Re})}{\omega_{uk}^2} \right) + \sum_{uk} \left(-\frac{1}{2} \frac{(v_{muk}^{\Re} - v_{nuk}^{\Re})^2 + (v_{muk}^{\Im} - v_{nuk}^{\Im})^2}{\omega_{uk}^2} \coth \left(\frac{\beta_u \omega_{uk}}{2} \right) \right) \quad (1299)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Re}} = \frac{-i v_{muk}^{\Im} - (v_{nuk}^{\Re} - v_{muk}^{\Re}) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1300)$$

$$\frac{\partial \ln B_{mn}}{\partial v_{nuk}^{\Im}} = \frac{i v_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth \left(\frac{\beta_u \omega_{uk}}{2} \right)}{\omega_{uk}^2} \quad (1301)$$

$$\frac{\partial \ln B_{mn}}{\partial a} = \frac{1}{B_{mn}} \frac{\partial B_{mn}}{\partial a} \quad (1302)$$

$$\frac{\partial B_{mn}}{\partial a} = B_{mn} \frac{\partial \ln B_{mn}}{\partial a} \quad (1303)$$

$$\frac{\partial B_{mn}}{\partial a} = \frac{\partial (B_{nm})^{\dagger}}{\partial a} \quad (1304)$$

Then the principal derivates are given by:

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \quad (1305)$$

$$= B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} - (v_{n\mathbf{k}}^{\Re} - v_{m\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1306)$$

$$= B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1307)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Re}} = \left(\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Re}} \right)^{\dagger} \quad (1308)$$

$$= \left(B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right)^{\dagger} \quad (1309)$$

$$= B_{nm} \left(\frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1310)$$

$$\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} = B_{mn} \frac{\partial \ln B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \quad (1311)$$

$$= B_{mn} \left(\frac{iv_{m\mathbf{k}}^{\Re} - (v_{n\mathbf{k}}^{\Im} - v_{m\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1312)$$

$$= B_{mn} \left(\frac{iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1313)$$

$$\frac{\partial B_{nm}}{\partial v_{n\mathbf{k}}^{\Im}} = \left(\frac{\partial B_{mn}}{\partial v_{n\mathbf{k}}^{\Im}} \right)^{\dagger} \quad (1314)$$

$$= (B_{mn})^{\dagger} \quad (1315)$$

$$= B_{nm} \left(\frac{-iv_{m\mathbf{k}}^{\Re} + (v_{m\mathbf{k}}^{\Im} - v_{n\mathbf{k}}^{\Im}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1316)$$

Introducing this derivatives in the equation (1291) give us:

$$\frac{\partial A_{\mathbf{B}}}{\partial v_{n\mathbf{k}}^{\Re}} = \frac{\partial A_{\mathbf{B}}}{\partial R_n} \left(2 \frac{v_{n\mathbf{k}}^{\Re} - g_{u\mathbf{k}}^{\Re}}{\omega_{u\mathbf{k}}} \right) + \sum_{n < m} \left(\frac{\partial A_{\mathbf{B}}}{\partial B_{nm}} B_{nm} \left(\frac{iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \right) \quad (1317)$$

$$+ \frac{\partial A_{\mathbf{B}}}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{m\mathbf{k}}^{\Im} + (v_{m\mathbf{k}}^{\Re} - v_{n\mathbf{k}}^{\Re}) \coth\left(\frac{\beta_u \omega_{u\mathbf{k}}}{2}\right)}{\omega_{u\mathbf{k}}^2} \right) \quad (1318)$$

$$= 0 \quad (1319)$$

We can obtain the variational parameters:

$$-2 \frac{\partial A_B}{\partial R_n} \frac{v_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{v_{nuk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1320)$$

$$= -\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1321)$$

$$v_{nuk}^{\Re} = \frac{\frac{\partial A_B}{\partial R_n} \frac{2g_{nuk}^{\Re}}{\omega_{uk}} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right)}{2 \frac{\partial A_B}{\partial R_n} \frac{1}{\omega_{uk}} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \frac{\coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right)} \quad (1322)$$

$$= \frac{2g_{nuk}^{\Re} \omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} (iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} (-iv_{muk}^{\Im} + v_{muk}^{\Re} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)) \right)}{2\omega_{uk} \frac{\partial A_B}{\partial R_n} - \sum_{n \neq m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) + \frac{\partial A_B}{\partial B_{mn}} B_{mn} \coth\left(\frac{\beta_u \omega_{uk}}{2}\right) \right)} \quad (1323)$$

Let's consider the imaginary part of the variation parameters

$$\frac{\partial A_B}{\partial v_{nuk}^{\Im}} = \frac{\partial A_B}{\partial R_n} \left(2 \frac{v_{nuk}^{\Im} - g_{nuk}^{\Im}}{\omega_{uk}} \right) + \sum_{n < m} \left(\frac{\partial A_B}{\partial B_{nm}} B_{nm} \left(\frac{-iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \right) \quad (1324)$$

$$+ \frac{\partial A_B}{\partial B_{mn}} B_{mn} \left(\frac{iv_{muk}^{\Re} - (v_{nuk}^{\Im} - v_{muk}^{\Im}) \coth\left(\frac{\beta_u \omega_{uk}}{2}\right)}{\omega_{uk}^2} \right) \quad (1325)$$

$$= 0 \quad (1326)$$

Rearranging we obtain

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1343)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1344)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1345)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\bar{\rho}(t)) \quad (1346)$$

We can re-write the transformed interaction Hamiltonian operator using the following matrices:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1347)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1348)$$

$$B_{nm,x} = \frac{B_{nm} + B_{mn}}{2} \quad (1349)$$

$$B_{nm,y} = \frac{B_{nm} - B_{mn}}{2i} \quad (1350)$$

We can proof that $B_{nm} = B_{mn}^\dagger$

$$B_{mn}^\dagger = (B_{m+}B_{n-} - B_m B_n)^\dagger \quad (1351)$$

$$= B_{n-}^\dagger B_{m+}^\dagger - B_n B_m \quad (1352)$$

$$= B_{n+} B_{m-} - B_n B_m \quad (1353)$$

$$= B_{nm} \quad (1354)$$

So we can say that the set of matrices (1347) are hermetic. Re-writing the transformed interaction Hamiltonian using the set (1347) give us.

$$\overline{H_I} = \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m| B_{nm} + \sum_n B_{z,n}(t) |n\rangle\langle n|, \quad (1355)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (V_{nm}(t) |n\rangle\langle m| B_{nm} + V_{mn}(t) |m\rangle\langle n| B_{mn}) \quad (1356)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) + iV_{nm}^\Im(t) B_{nm} \left(\frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} \right) \right) \quad (1357)$$

$$+ \Re(V_{nm}(t)) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) - iV_{nm}^\Im(t) B_{mn} \left(\frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1358)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} \left(\Re(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} + B_{mn}}{2} \right) + \Re(V_{nm}(t)) \sigma_{nm,y} \frac{i(B_{mn} - B_{nm})}{2} \right) \quad (1359)$$

$$+ i\Im(V_{nm}(t)) \sigma_{nm,x} \left(\frac{B_{nm} - B_{mn}}{2} \right) + \Im(V_{nm}(t)) \sigma_{nm,y} \left(\frac{B_{nm} + B_{mn}}{2} \right) \quad (1360)$$

$$= \sum_n B_{z,n}(t) |n\rangle\langle n| + \sum_{n < m} (\Re(V_{nm}(t)) \sigma_{nm,x} B_{nm,x} - \Im(V_{nm}(t)) \sigma_{nm,x} B_{nm,y} + \Re(V_{nm}(t)) \sigma_{nm,y} B_{nm,y} \quad (1361)$$

$$+ \Im(V_{nm}(t)) \sigma_{nm,y} B_{nm,x}) \quad (1362)$$

Let's define the set

$$P = \{(n, m) \in \mathbb{N}^2 | 0 \leq n, m \leq d-1 \wedge (n = m \vee n < m)\} \quad (1363)$$

Now consider the following set of operators,

$$A_{1,nm}(t) = \sigma_{nm,x}(1 - \delta_{mn}) \quad (1364)$$

$$A_{2,nm}(t) = \sigma_{nm,y}(1 - \delta_{mn}) \quad (1365)$$

$$A_{3,nm}(t) = \delta_{mn}|n\rangle\langle m| \quad (1366)$$

$$A_{4,nm}(t) = A_{2,mn}(t) \quad (1367)$$

$$A_{5,nm}(t) = A_{1,nm}(t) \quad (1368)$$

$$B_{1,nm}(t) = B_{nm,x} \quad (1369)$$

$$B_{2,nm}(t) = B_{nm,y} \quad (1370)$$

$$B_{3,nm}(t) = B_{z,n}(t) \quad (1371)$$

$$B_{4,nm}(t) = B_{1,nm}(t) \quad (1372)$$

$$B_{5,nm}(t) = B_{2,nm}(t) \quad (1373)$$

$$C_{1,nm}(t) = \Re(V_{nm}(t)) \quad (1374)$$

$$C_{2,nm}(t) = C_{1,nm}(t) \quad (1375)$$

$$C_{3,nm}(t) = 1 \quad (1376)$$

$$C_{4,nm}(t) = \Im(V_{nm}(t)) \quad (1377)$$

$$C_{5,nm}(t) = -\Im(V_{nm}(t)) \quad (1378)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H_I}(t)$ as:

$$\overline{H_I} = \sum_{j \in J, p \in P} C_{jp}(t) (A_{jp} \otimes B_{jp}(t)) \quad (1379)$$

Here $J = \{1, 2, 3, 4, 5\}$ and P the set defined in (1363).

We write the interaction Hamiltonian transformed under (1343) as:

$$\widetilde{H_I}(t) = \sum_{j \in J, p \in P} C_{jp}(t) (\widetilde{A_{jp}}(t) \otimes \widetilde{B_{jp}}(t)) \quad (1380)$$

$$\widetilde{A_{jp}}(t) = U^\dagger(t) A_{jp} U(t) \quad (1381)$$

$$\widetilde{B_{jp}}(t) = e^{iH_B t} B_{jp}(t) e^{-iH_B t} \quad (1382)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B [\widetilde{H_I}(t), [\widetilde{H_I}(s), \widetilde{\rho_S}(t) \rho_B]] ds \quad (1383)$$

Replacing the equation (1380) in (1383) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1384)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \left[\sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1385)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right), \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1386)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \Big] ds \quad (1387)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right. \quad (1388)$$

$$\left. - \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \right) \quad (1389)$$

$$- \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \quad (1390)$$

$$\left. + \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J, p' \in P} C_{j'p'}(s) \left(\widetilde{A}_{j'p'}(s) \otimes \widetilde{B}_{j'p'}(s) \right) \sum_{j \in J, p \in P} C_{jp}(t) \left(\widetilde{A}_{jp}(t) \otimes \widetilde{B}_{jp}(t) \right) \right) ds \quad (1391)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jpj'p'}(\tau) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B \quad (1392)$$

$$= \left\langle \widetilde{B}_{jp}(\tau) \widetilde{B}_{j'p'}(0) \right\rangle_B \quad (1393)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right) = \left\langle \widetilde{B}_{jp}(t) \widetilde{B}_{j'p'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jp}(t)$ and elements related to the system given by $\widetilde{A}_{jp}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jp}(t) \widetilde{B}_{j'p'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jp}(t) \right) \text{Tr} \left(\widetilde{B}_{j'p'}(t) \right)$. The correlation functions relevant of the master equation (1391) are:

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1394)$$

$$= \left\langle \widetilde{B_{jp}}(0) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1395)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1396)$$

$$\text{Tr}_B \left(\widetilde{B_{jp}}(t) \rho_B \widetilde{B_{j'p'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1397)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1398)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1399)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1400)$$

$$\text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \rho_B \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \rho_B \right) \quad (1401)$$

$$= \left\langle \widetilde{B_{jp}}(t) \widetilde{B_{j'p'}}(s) \right\rangle_B \quad (1402)$$

$$= \left\langle \widetilde{B_{jp}}(\tau) \widetilde{B_{j'p'}}(0) \right\rangle_B \quad (1403)$$

$$= \Lambda_{jpj'p'}(\tau) \quad (1404)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \rho_B \right) \quad (1405)$$

$$= \left\langle \widetilde{B_{j'p'}}(s) \widetilde{B_{jp}}(t) \right\rangle_B \quad (1406)$$

$$= \left\langle \widetilde{B_{j'p'}}(-\tau) \widetilde{B_{jp}}(0) \right\rangle_B \quad (1407)$$

$$= \Lambda_{j'p'jp}(-\tau) \quad (1408)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1384) and (1391) and using the equations (1394)-(1408) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',p,p'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \widetilde{A_{jp}}(t) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'p'jp}(-\tau) \widetilde{A_{jp}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \right) \right. \quad (1409)$$

$$\left. + C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{j'p'jp}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s) \widetilde{A_{jp}}(t) - \Lambda_{jpj'p'}(\tau) \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jp}}(t) \right) \right) ds \quad (1410)$$

$$= - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(s) \left(\Lambda_{jpj'p'}(\tau) \left[\widetilde{A_{jp}}(t), \widetilde{A_{j'p'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'p'}}(s), \widetilde{A_{jp}}(t) \right] \right) \right) \quad (1411)$$

Rearranging and identifying the commutators allow us to write a more simplified version

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{jj'pp'} \left(C_{jp}(t) C_{j'p'}(t-\tau) \left(\Lambda_{jpj'p'}(\tau) \left[A_{jp}(t), A_{j'p'}(t-\tau, t) \widetilde{\rho_S}(t) \right] + \Lambda_{j'p'jp}(-\tau) \left[\widetilde{\rho_S}(t) A_{j'p'}(t-\tau, t), A_{jp}(t) \right] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1412)$$

For this case we used that $A_{jp}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jp}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation.

VII. TIME-DEPENDENT MULTI-SITE MODEL WITH ONE BATH COUPLING

Let's consider the following Hamiltonian for a system of d-levels (qudit). We start with a time-dependent Hamiltonian of the form:

$$H(t) = H_S(t) + H_I + H_B, \quad (1413)$$

$$H_S(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|, \quad (1414)$$

$$H_I = \left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right), \quad (1415)$$

$$H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}. \quad (1416)$$

We will start with a system-bath coupling operator of the form $\sum_{n=0} \mu_n(t) |n\rangle\langle n|$.

A. Variational Transformation

We consider the following operator:

$$V = \left(\sum_{n=1} |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}) \right) \quad (1417)$$

At first let's obtain e^V under the transformation (1417), consider $\hat{\varphi} = \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})$:

$$e^V = e^{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}} \quad (1418)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{(\sum_{n=1} |n\rangle\langle n| \hat{\varphi})^2}{2!} + \dots \quad (1419)$$

$$= \mathbb{I} + \sum_{n=1} |n\rangle\langle n| \hat{\varphi} + \frac{\sum_{n=1} |n\rangle\langle n| \hat{\varphi}^2}{2!} + \dots \quad (1420)$$

$$= \mathbb{I} - \sum_{n=1} |n\rangle\langle n| + \sum_{n=1} |n\rangle\langle n| \left(\mathbb{I} + \hat{\varphi} + \frac{\hat{\varphi}^2}{2!} + \dots \right) \quad (1421)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| e^{\hat{\varphi}} \quad (1422)$$

$$= |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \quad (1423)$$

Given that $[b_{\mathbf{k}'}^\dagger - b_{\mathbf{k}'}^\dagger, b_{\mathbf{k}}^\dagger - b_{\mathbf{k}}] = 0$ if $\mathbf{k}' \neq \mathbf{k}$ then we can proof using the Zassenhaus formula and defining $D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) = e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})}$ in the same way than (23):

$$e^{\sum_{\mathbf{k}} \pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} = \prod_{\mathbf{k}} e^{\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger - b_{\mathbf{k}})} \quad (1424)$$

$$= \prod_{\mathbf{k}} D\left(\pm \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \quad (1425)$$

$$= B_{\pm} \quad (1426)$$

As we can see $e^{-V} = |0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B$. because this form imposes that $e^{-V} e^V = \mathbb{I}$ and the inverse of a operator is unique. This allows us to write the canonical transformation in the following explicit way:

$$e^V A e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) A \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1427)$$

Now let's obtain the canonical transformation of the principal elements of the Hamiltonian (1413):

$$\overline{|0\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1428)$$

$$= |0\rangle\langle 0|, \quad (1429)$$

$$\overline{|m\rangle\langle n|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle n| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1430)$$

$$= |m\rangle\langle m| B^+ |m\rangle\langle n| n\rangle\langle n| B^-, \quad (1431)$$

$$= |m\rangle\langle n|, \quad m \neq 0, \quad n \neq 0, \quad (1432)$$

$$\overline{|0\rangle\langle m|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |0\rangle\langle m| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right), \quad (1433)$$

$$= |0\rangle\langle m| B^- \quad m \neq 0, \quad (1434)$$

$$\overline{|m\rangle\langle 0|} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) |m\rangle\langle 0| \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1435)$$

$$= |0\rangle\langle m| B^+ \quad m \neq 0, \quad (1436)$$

$$\overline{\sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}}} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1437)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} B^+ b_{\mathbf{k}}^\dagger b_{\mathbf{k}} B^- \quad (1438)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B^+ b_{\mathbf{k}}^\dagger B^- \right) \left(B^+ b_{\mathbf{k}} B^- \right) \quad (1439)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \left(b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1440)$$

$$= |0\rangle\langle 0| \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1441)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \quad (1442)$$

$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1443)$$

The transformed Hamiltonians of the equations (1414) to (1416) written in terms of (1428) to (1443) are:

$$\overline{H_S(t)} = \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1444)$$

$$= \overline{\sum_{n=0} \varepsilon_n(t) |n\rangle\langle n|} + \overline{\sum_{n \neq m} V_{nm}(t) |n\rangle\langle m|} \quad (1445)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1446)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) \overline{|0\rangle\langle n|} + V_{n0}(t) \overline{|n\rangle\langle 0|}) + \sum_{m,n \neq 0} V_{mn}(t) \overline{|m\rangle\langle n|} \quad (1447)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) B^- |0\rangle\langle n| + V_{n0}(t) B^+ |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1448)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1449)$$

$$\overline{H_I} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) \left(\left(\sum_{n=0} \mu_n(t) |n\rangle\langle n| \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1450)$$

$$= \left(\mu_0(t) |0\rangle\langle 0| + \sum_{n=1} \mu_n(t) |n\rangle\langle n| B^+ \right) \left(\sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \right) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1451)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} B^+ (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) B^- \quad (1452)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} - 2 \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1453)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1454)$$

Joining this terms allow us to write:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1455)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - \sum_{n=1} |n\rangle\langle n| \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1456)$$

$$+ \sum_{n=0} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) - \sum_{n=1} \mu_n(t) |n\rangle\langle n| \sum_{\mathbf{k}} 2g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \quad (1457)$$

$$= \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1458)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1459)$$

$$+ \sum_{n=1} |n\rangle\langle n| \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1460)$$

Let's define the following functions:

$$R_n(t) = \sum_{\mathbf{k}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \quad (1461)$$

$$= \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} - 2\mu_n(t) g_{\mathbf{k}} \right) \quad (1462)$$

$$B_{z,n}(t) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1463)$$

Using the previous functions we have that (1460) can be re-written in the following way:

$$\overline{H} = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| B^- + V_{n0}(t) |n\rangle\langle 0| B^+) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1464)$$

$$+ \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + \sum_{n=1} R_n |n\rangle\langle n| + \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) \quad (1465)$$

Now in order to separate the elements of the hamiltonian (1465) let's follow the references of the equations (??) and (??) to separate the hamiltonian like:

$$\overline{H_S}(t) = \sum_{n=0} \varepsilon_n(t) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| + \sum_{n=1} R_n |n\rangle\langle n| \quad (1466)$$

$$\overline{H_I} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)), \quad (1467)$$

$$\overline{H_B} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \quad (1468)$$

Here B is given by:

$$\begin{aligned} B &= \langle B^+ \rangle \\ &= \langle B^- \rangle \end{aligned}$$

The transformed Hamiltonian can be written in function of the following set of hermitian operators:

$$\sigma_{nm,x} = |n\rangle\langle m| + |m\rangle\langle n| \quad (1469)$$

$$\sigma_{nm,y} = i(|n\rangle\langle m| - |m\rangle\langle n|) \quad (1470)$$

$$B_x = \frac{B^+ + B^- - 2B}{2} \quad (1471)$$

$$B_y = \frac{B^- - B^+}{2i} \quad (1472)$$

Using this set of hermitian operators to write the Hamiltonians (1414)-(1416)

$$\overline{H_S(t)} = \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{m,n \neq 0} V_{mn}(t) |m\rangle\langle n| \quad (1473)$$

$$= \varepsilon_0(t) |0\rangle\langle 0| + B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{0 < m < n} (V_{mn}(t) |m\rangle\langle n| + V_{nm}(t) |n\rangle\langle m|) \quad (1474)$$

$$+ \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1475)$$

$$= \sum_{0 < m < n} ((\Re(V_{mn}(t)) + i\Im(V_{mn}(t))) |m\rangle\langle n| + (\Re(V_{mn}(t)) - i\Im(V_{mn}(t))) |n\rangle\langle m|) + \varepsilon_0(t) |0\rangle\langle 0| \quad (1476)$$

$$+ B \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| + V_{n0}(t) |n\rangle\langle 0|) + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1477)$$

$$= \sum_{0 < m < n} \left((\Re(V_{nm}(t)) + i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} - i\sigma_{nm,y}}{2} + (\Re(V_{nm}(t)) - i\Im(V_{mn}(t))) \frac{\sigma_{nm,x} + i\sigma_{nm,y}}{2} \right) \quad (1478)$$

$$+ B \sum_{n=1} \left(V_{0n}(t) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + V_{n0}(t) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) + \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1479)$$

$$= \sum_{0 < m < n} (\Re(V_{nm}(t)) \sigma_{nm,x} + \Im(V_{nm}(t)) \sigma_{nm,y}) + B \sum_{n=1} (\Re(V_{0n}(t)) \sigma_{0n,x} + \Im(V_{mn}(t)) \sigma_{0n,y}) \quad (1480)$$

$$+ \varepsilon_0(t) |0\rangle\langle 0| + \sum_{n=1} (\varepsilon_n(t) + R_n) |n\rangle\langle n| \quad (1481)$$

$$\overline{H_I(t)} = \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (V_{0n}(t) |0\rangle\langle n| (B^- - B) + V_{n0}(t) |n\rangle\langle 0| (B^+ - B)) \quad (1482)$$

$$= \sum_{n=1} \left((\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) (B^- - B) \frac{\sigma_{0n,x} - i\sigma_{0n,y}}{2} + (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) (B^+ - B) \frac{\sigma_{0n,x} + i\sigma_{0n,y}}{2} \right) \quad (1483)$$

$$+ \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1484)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} ((B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t))) + (B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t)))) \right) \quad (1485)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B) (\Re(V_{0n}(t)) - i\Im(V_{0n}(t))) - (B^- - B) (\Re(V_{0n}(t)) + i\Im(V_{0n}(t)))) \quad (1486)$$

$$+ \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1487)$$

$$= \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} \left(\frac{\sigma_{0n,x}}{2} (B^+ + B^- - 2B) \Re(V_{0n}(t)) + i(B^- - B - B^+ + B) \Im(V_{0n}(t)) \right) \quad (1488)$$

$$+ \frac{i\sigma_{0n,y}}{2} ((B^+ - B - B^- + B) \Re(V_{0n}(t)) + i(B - B^- + B - B^+) \Im(V_{0n}(t))) + \sum_{n=1} B_{z,n} |n\rangle\langle n| \quad (1489)$$

$$= \sum_{n=1} B_{z,n} |n\rangle\langle n| + \mu_0(t) |0\rangle\langle 0| \sum_{\mathbf{k}} g_{\mathbf{k}} (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) + \sum_{n=1} (\sigma_{0n,x} (B_x \Re(V_{0n}(t)) - B_y \Im(V_{0n}(t))) \quad (1490)$$

$$+ \sigma_{0n,y} (B_y \Re(V_{0n}(t)) + B_x \Im(V_{0n}(t)))) \quad (1491)$$

B. Free-energy minimization

As first approach let's consider the minimization of the free-energy through the Feynman-Bogoliubov inequality

$$A \leq A_B \equiv -\frac{1}{\beta} \ln \left(\text{Tr} \left(e^{-\beta(\overline{H}_S + \overline{H}_B)} \right) \right) + \langle \overline{H}_I \rangle_{\overline{H}_S + \overline{H}_B} + O \left(\langle \overline{H}_I^2 \rangle_{\overline{H}_S + \overline{H}_B} \right). \quad (1492)$$

Taking the equations (244)-(252) and given that $\text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right) = C(R_1, R_2, \dots, R_{d-1}, B)$, where each R_i and B depend of the set of variational parameters $\{v_{\mathbf{k}}\}$. From (252) and using the chain rule we obtain that:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \frac{\partial B}{\partial v_{\mathbf{k}}} + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \frac{\partial R_n}{\partial v_{\mathbf{k}}}, \quad (1493)$$

$$= 0 \quad (1494)$$

Let's recall the equations (1461) and (1463), we can write them in terms of the variational parameters

$$B = \exp \left(- (1/2) \sum_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) \right) \quad (1495)$$

$$R_n = \sum_{\mathbf{k}} \omega_{\mathbf{k}}^{-1} (v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}} v_{\mathbf{k}}) \quad (1496)$$

The derivates needed to obtain the set of variational parameter are given by:

$$\frac{\partial B}{\partial v_{\mathbf{k}}} = -\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \quad (1497)$$

$$\frac{\partial R_n}{\partial v_{\mathbf{k}}} = \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1498)$$

Introducing this derivates in the equation (1493) give us:

$$\frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial v_{\mathbf{k}}} = \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \left(-\frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \coth(\beta \omega_{\mathbf{k}}/2) B \right) + \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \omega_{\mathbf{k}}^{-1} (2v_{\mathbf{k}} - 2\mu_n(t) g_{\mathbf{k}}) \quad (1499)$$

$$= v_{\mathbf{k}} \left(\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B} \right) - \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t) \quad (1500)$$

We can obtain the variational parameters:

$$v_{\mathbf{k}} = \frac{\frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{\frac{2}{\omega_{\mathbf{k}}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - \frac{\coth(\beta \omega_{\mathbf{k}}/2) B}{\omega_{\mathbf{k}}^2} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1501)$$

$$= \frac{2g_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}} \quad (1502)$$

Now taking $v_{\mathbf{k}} = g_{\mathbf{k}} v_{\mathbf{k}}$ then we can obtain $v_{\mathbf{k}}$ like:

$$v_{\mathbf{k}} = \frac{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} \mu_n(t)}{2\omega_{\mathbf{k}} \sum_{n=1} \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial R_n} - B \coth(\beta \omega_{\mathbf{k}}/2) \frac{\partial \text{Tr} \left(e^{-\beta \overline{H}_S(t)} \right)}{\partial B}}. \quad (1503)$$

C. Master Equation

Let's consider that the initial state of the system is given by $\rho(0) = |0\rangle\langle 0| \otimes \rho_B$, as we can see this state is independent of the variational transformation:

$$e^V \rho(0) e^{-V} = \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^+ \right) (|0\rangle\langle 0| \otimes \rho_B) \left(|0\rangle\langle 0| + \sum_{n=1} |n\rangle\langle n| B^- \right) \quad (1504)$$

$$0 = |0\rangle\langle 0| \otimes \rho_B \quad (1505)$$

$$0 = \rho(0) \quad (1506)$$

We transform any operator O into the interaction picture in the following way:

$$\tilde{O} \equiv U^\dagger(t) O U(t) \quad (1507)$$

$$U(t) \equiv \mathcal{T} \exp \left(-i \int_0^t dt' \overline{H_S}(t') \right). \quad (1508)$$

Therefore:

$$\widetilde{\overline{\rho_S}}(t) = U^\dagger(t) \overline{\rho_S}(t) U(t), \text{ where} \quad (1509)$$

$$\overline{\rho_S}(t) = \text{Tr}_B(\tilde{\rho}(t)) \quad (1510)$$

We can re-write the transformed interaction Hamiltonian operator like:

$$\overline{H_I}(t) = B_{z,0} |0\rangle\langle 0| + \sum_{n=1} (\Re(V_{0n}(t)) B_x \sigma_{0n,x} + \Re(V_{0n}(t)) B_y \sigma_{0n,y} + B_{z,n} |n\rangle\langle n|) \quad (1511)$$

$$+ \Im(V_{0n}(t)) B_x \sigma_{0n,y} - \Im(V_{0n}(t)) B_y \sigma_{0n,x} \quad (1512)$$

where

$$B_{z,0} = \sum_{\mathbf{k}} g_{\mathbf{k}} \mu_0(t) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \quad (1513)$$

$$B_{z,n} = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} \mu_n(t) - \omega_{\mathbf{k}} \frac{v_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) (b_{\mathbf{k}}^\dagger + b_{\mathbf{k}}) \text{ if } n \neq 0 \quad (1514)$$

Now consider the following set of operators:

$$A_{1n}(t) = \sigma_{0n,x} \quad (1515)$$

$$A_{2n}(t) = \sigma_{0n,y} \quad (1516)$$

$$A_{3n}(t) = |n\rangle\langle n| \quad (1517)$$

$$A_{4n}(t) = A_{2n}(t) \quad (1518)$$

$$A_{5n}(t) = A_{1n}(t) \quad (1519)$$

$$B_{1n}(t) = B_x \quad (1520)$$

$$B_{2n}(t) = B_y \quad (1521)$$

$$B_{3n}(t) = B_{z,n} \quad (1522)$$

$$B_{4n}(t) = B_{1n}(t) \quad (1523)$$

$$B_{5n}(t) = B_{2n}(t) \quad (1524)$$

$$C_{10}(t) = 0 \quad (1525)$$

$$C_{20}(t) = 0 \quad (1526)$$

$$C_{40}(t) = 0 \quad (1527)$$

$$C_{50}(t) = 0 \quad (1528)$$

$$C_{30}(t) = 1 \quad (1529)$$

$$C_{1n}(t) = \Re(V_{0n}(t)) \quad (1530)$$

$$C_{2n}(t) = C_{1n}(t) \quad (1531)$$

$$C_{3n}(t) = 1 \quad (1532)$$

$$C_{4n}(t) = \Im(V_{0n}(t)) \quad (1533)$$

$$C_{5n}(t) = -\Im(V_{0n}(t)) \quad (1534)$$

The previous notation allows us to write the interaction Hamiltonian in $\overline{H}_I(t)$ as:

$$\overline{H}_I = \sum_{j \in J} \sum_{n=1} C_{jn}(t) (A_{jn} \otimes B_{jn}(t)) \quad (1535)$$

Here $J = \{1, 2, 3, 4, 5\}$.

We write the interaction Hamiltonian transformed under (1507) as:

$$\widetilde{H}_I(t) = \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1536)$$

$$\widetilde{A}_i(t) = U^\dagger(t) A_i U(t) \quad (1537)$$

$$\widetilde{B}_i(t) = e^{iH_B t} B_i(t) e^{-iH_B t} \quad (1538)$$

Taking as reference state ρ_B and truncating at second order in $H_I(t)$, we obtain our master equation in the interaction picture:

$$\frac{d\widetilde{\rho}_S(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho}_S(t) \rho_B \right] \right] ds \quad (1539)$$

Replacing the equation (1536) in (1539) we can obtain:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \text{Tr}_B \left[\widetilde{H}_I(t), \left[\widetilde{H}_I(s), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1540)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \left[\sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right), \widetilde{\rho_S}(t) \rho_B \right] \right] ds \quad (1541)$$

$$= - \int_0^t \text{Tr}_B \left[\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right), \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right] ds \quad (1542)$$

$$- \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \Big] ds \quad (1543)$$

$$= - \int_0^t \text{Tr}_B \left(\sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \right) ds \quad (1544)$$

$$- \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \quad (1545)$$

$$- \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \widetilde{\rho_S}(t) \rho_B \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \quad (1546)$$

$$+ \widetilde{\rho_S}(t) \rho_B \sum_{j' \in J} \sum_{n'=1} C_{j'n'}(s) \left(\widetilde{A}_{j'n'}(s) \otimes \widetilde{B}_{j'n'}(s) \right) \sum_{j \in J} \sum_{n=1} C_{jn}(t) \left(\widetilde{A}_{jn}(t) \otimes \widetilde{B}_{jn}(t) \right) \Big) ds \quad (1547)$$

In order to calculate the correlation functions we define:

$$\Lambda_{jn j'n'}(\tau) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B \quad (1548)$$

$$= \left\langle \widetilde{B}_{jn}(\tau) \widetilde{B}_{j'n'}(0) \right\rangle_B \quad (1549)$$

Here $s \rightarrow t - \tau$ and $\text{Tr}_B \left(\widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \rho_B \right) = \left\langle \widetilde{B}_{jn}(t) \widetilde{B}_{j'n'}(s) \right\rangle_B$. To evaluate the trace respect to the bath we need to recall that our master equation depends of elements related to the bath and represented by the operators $\widetilde{B}_{jn}(t)$ and elements related to the system given by $\widetilde{A}_{jn}(t)$. The systems considered are in different Hilbert spaces so $\text{Tr} \left(\widetilde{A}_{jn}(t) \widetilde{B}_{j'n'}(t) \right) = \text{Tr} \left(\widetilde{A}_{jn}(t) \right) \text{Tr} \left(\widetilde{B}_{j'n'}(t) \right)$. The correlation functions relevant of the master equation (1547) are:

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) = \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1550)$$

$$= \left\langle \widetilde{B_{jn}}(0) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1551)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1552)$$

$$\text{Tr}_B \left(\widetilde{B_{jn}}(t) \rho_B \widetilde{B_{j'n'}}(s) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1553)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1554)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1555)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1556)$$

$$\text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \rho_B \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \rho_B \right) \quad (1557)$$

$$= \left\langle \widetilde{B_{jn}}(t) \widetilde{B_{j'n'}}(s) \right\rangle_B \quad (1558)$$

$$= \left\langle \widetilde{B_{jn}}(\tau) \widetilde{B_{j'n'}}(0) \right\rangle_B \quad (1559)$$

$$= \Lambda_{jn j'n'}(\tau) \quad (1560)$$

$$\text{Tr}_B \left(\rho_B \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right) = \text{Tr}_B \left(\widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \rho_B \right) \quad (1561)$$

$$= \left\langle \widetilde{B_{j'n'}}(s) \widetilde{B_{jn}}(t) \right\rangle_B \quad (1562)$$

$$= \left\langle \widetilde{B_{j'n'}}(-\tau) \widetilde{B_{jn}}(0) \right\rangle_B \quad (1563)$$

$$= \Lambda_{j'n' jn}(-\tau) \quad (1564)$$

We made use of the cyclic property for the trace to evaluate the correlation functions, from the equations obtained in (1540) and (1547) and using the equations (1550)-(1564) we can re-write:

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jn j'n'}(\tau) \widetilde{A_{jn}}(t) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) - \Lambda_{j'n' jn}(-\tau) \widetilde{A_{jn}}(t) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \right) \right. \quad (1565)$$

$$\left. + C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{j'n' jn}(-\tau) \widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s) \widetilde{A_{jn}}(t) - \Lambda_{jn j'n'}(\tau) \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \widetilde{A_{jn}}(t) \right) \right) ds \quad (1566)$$

$$= - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(s) \left(\Lambda_{jn j'n'}(\tau) \left[\widetilde{A_{jn}}(t), \widetilde{A_{j'n'}}(s) \widetilde{\rho_S}(t) \right] + \Lambda_{j'n' jn}(-\tau) \left[\widetilde{\rho_S}(t) \widetilde{A_{j'n'}}(s), \widetilde{A_{jn}}(t) \right] \right) \right) \quad (1567)$$

$$\frac{d\widetilde{\rho_S}(t)}{dt} = - \int_0^t \sum_{j,j',n,n'} \left(C_{jn}(t) C_{j'n'}(t-\tau) \left(\Lambda_{jn j'n'}(\tau) [A_{jn}(t), A_{j'n'}(t-\tau, t) \widetilde{\rho_S}(t)] + \Lambda_{j'n' jn}(-\tau) [\widetilde{\rho_S}(t) A_{j'n'}(t-\tau, t), A_{jn}(t)] \right) \right) d\tau - i [H_S(t), \widetilde{\rho_S}(t)] \quad (1568)$$

For this case we used that $A_{jn}(t-\tau, t) = U(t) U^\dagger(t-\tau) A_{jn}(t) U(t-\tau) U^\dagger(t)$. This is a non-Markovian equation and if we take $n = 2$ (two sites), $\mu_0(t) = 0$, $\mu_1(t) = 1$ then we can reproduce a similar expression to (433) as expected.

VIII. BIBLIOGRAPHY

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