SFU MATH 260 D100
Intro to ODEs

Instructor Question #5

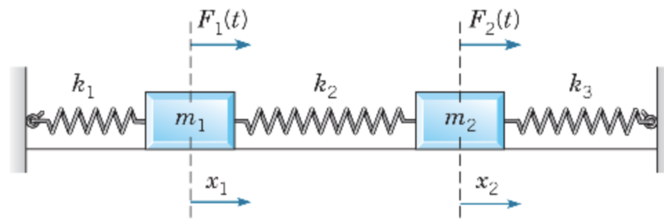
Due: Friday March 22, 2024 at 11:59pm

- Write up a complete and detailed solution to the problem below, being sure to justify all steps in your solution.
- Before submitting your assignment, carefully review the Crowdmark submission guidelines posted on Canvas.
- Scan your written solution and upload it to Crowdmark using the URL sent to you by e-mail. Check on Crowdmark that your uploaded solution is clearly readable. Whatever your grader cannot read, will not be marked.

IQ5. This question is based on the two-mass/three-spring problem pictured in the diagram below, which I presented as a "motivating example" in Section 7.1 of the lecture notes. Assuming that there is no external forcing, $F_1(t) = F_2(t) = 0$, this system can be described by two second-order ODEs:

$$m_1 \frac{d^2 x_1}{dt^2} = -\left(k_1 + k_2\right) x_1 + k_2 x_2$$

$$m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - \left(k_2 + k_3\right) x_2$$



Suppose the masses are $m_1 = 1$, $m_2 = \frac{3}{2}$, and the spring constants are $k_1 = \frac{2}{3}$, $k_2 = 2$, $k_3 = 1$. The same problem (with different constants) appears in the text as the first example in Chapter 7 and in Example 7.6.3.

(a) Convert this second-order system into a linear system of four first-order ODEs having the form $\vec{u}' = \mathbf{A}\vec{u}$ with $u_1 = x_1$, $u_2 = x_2$, etc.

(b) Find the eigenvalues and eigenvectors of **A**. Notice that the eigenvalues consist of two complex conjugate pairs, the first two which I'll denote as r_1 and $r_2 = \overline{r_1}$, and the second two as r_3 and $r_4 = \overline{r_3}$.

(c) Write down the general solution $\vec{u}(t)$ for this system.

Hint: The first two fundamental solutions can be found by computing the real and imaginary parts of $e^{r_1 t}\vec{z}^{(1)}$. *Then the remaining two fundamental solutions come from* $e^{r_3 t}\vec{z}^{(3)}$.

(d) Observe that your general solution can be viewed as the sum of two "fundamental modes" that oscillate at two different frequencies. Consider these two modes separately, and for each mode include plots of the following:

- A single solution plot displaying all four components $u_i(t)$ versus t. Choose a time interval that shows 2–3 complete periods, and distinguish your four u-curves using different colors or line types.
- A phase portrait that shows solution trajectories in the $u_1 u_3$ and/or $u_2 u_4$ planes.

(e) Find the solution that satisfies the initial condition $\vec{u}(0) = [1, 2, -1, 2]^T$, and plot solution trajectories in the $u_1 u_3$ phase plane. Generate 3 separate plots over time intervals $[0, T]$ with the end times $T = 10, 40, 200$. Compare your phase portraits with those from Figure 7.6.5 in the textbook. Describe the main difference(s) you see and try to explain the reason for this behaviour.

For fun: Try plotting trajectories in the $u_1 u_2$ *plane, which corresponds to the two spring displacements* x_1 *and* x_2.