MONTE CARLO PROJECT NUMBER 9

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TABLE DES MATIÈRES

1
INTRODUCTION

Rough volatility models are among the important topics in financial mathematics because, they take into account the implied volatility surface and, the irregularity of volatility time series in the modelling of financial asset prices. These models have been the subject of research that has produced results of varying degrees of interest.

This article discusses an example of rough volatility models, which have the advantage of better expressing the roughness of volatility in financial markets, in order to explain the dynamics. This model is essentially based on the famous stochastic Volterra equation. In particular, it will allow efficient calculation of European call and put prices in the context of e.g. (affine) rough volatility models if the studies prove successful : it is the Heston rough volatility model. For this model to be useful, one must be able to solve the stochastic differential equations involved using good approximation methods in order to make relevant analyses. This is one of the reasons why the authors focus on a natural and simple method, where more sophisticated but complex methods seem to fail. The discrete time Euler scheme is such a good solving approach used in this framework.

The main objective of the paper is to study the discrete time Euler scheme for solving the stochastic differential equation of the rough Heston volatility model, bringing out convergence results that use already existing techniques as first attempts and that reassure about the implementation of such a procedure. Then, thanks to a Monte Carlo method, this Euler scheme method will allow the evaluation of option prices depending on the path of the underlying asset.

2 MODELS AND METHODS

The aim of the problem is to solve the stochastic differential equation given in the Heston model ; a simple and natural method is the use of the discrete time Euler scheme to resolve it as recommended in the article, but before implementing the method let us see the different formulations of the Heston model.

2.1 Heston Model

2.1.1 • first formulation : rough model

The Heston volatility model is given by the following equation :

$$S_t = S_0 + \int_0^t S_s\sqrt{V_s}dW_s^1, \quad V_t = V_0 + \int_0^t K(t-s)((\theta - \lambda V_s ds) + \nu\sqrt{V_s}dW_s^2)$$

were (W^1, W^2) are two Brownian motions with $Cov(W^1, W^2) = \rho \in [-1, 1]$, $K(t) = Ct^{H-\frac{1}{2}}$ is the kernel function and $H \in (0, \frac{1}{2})$ Hurst parameter. Here S can represent the price of the risky asset under the risk neutral probability and V is the volatility.

Using Cholesky decomposition we can rewrite the equation with an uncorrelated Brownian pair $(W, W^{'})$ by introducing the correlation in another form :

$$S_t = S_0 + \int_0^t S_s\sqrt{V_s}d(\rho W_s + \sqrt{1-\rho^2}W_s^{'}),$$

$$V_t = V_0 + \int_0^t K(t-s)((\theta - \lambda V_s ds) + \nu\sqrt{V_s}dW_s)$$

Now we can look at the definition,

Definition : The equation has a weak solution if there exists a filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, P)$, equipped with two independent Brownian motion $W, W^{'}$, and a pair of R_+-valued adapted continuous processes (V, S) such that it is satisfied for all $t \geq 0$.

2.1.2 • Second formulation : integrated-rough Heston model

The first formulation of the Heston model can be reformulated without S_t depending visibly on $\sqrt{V_t}$, we pose

$$dX_t = \int_0^t V_s ds, \quad t \geq 0$$

we get a reformulation :

$$S_t = S_0 + \int_0^t S_s d(\rho M_s + \sqrt{1-\rho^2} M_s^{'}),$$

$$X_t = V_0 t + \int_0^t K(t-s)(\theta s - \lambda X_s + \nu M_s) ds,$$

where $M_t = \int_0^t \sqrt{V_s} dW_s$, $M_t^{'} = \int_0^t \sqrt{V_s} dW_s^{'}$ are two orthogonal continuous martingales with quadratic variation equal to X and $M_0 = M_0^{'} = 0$. The definition of weak solution of the equation is given by ;

Definition : The Equation has a weak solution if there exists a filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, P)$ supporting a pair of orthogonal continuous martingales $(M, M^{'})$, a non-decreasing, non-negative, continuous and adapted process X and a non-negative continuous and adapted process S, such that (the equation) holds.

2.1.3 • Existence and uniqueness

In this part we just state the result which is used to prove the existence and the uniqueness of a weak solution in those previous equations. Also the result will be useful to provide the convergence result.

Let's consider a sequence $(\pi_n)_{n\geq 1}$ of discrete-time grid on $[0, T]$, with $\pi_n := \left\{ 0 = t_0^n < t_1^n < t_2^n < ... < t_n^n = T \right\}$ for each $n > 0$. We define $\eta_n(s) := t_k^n$ for $s \in [t_k^n,\, t_{k+1}^n)$, $k = 0, \cdot \cdot \cdot, n-1$, and $\eta_n(T) := T$.

The following assumption is about the Kernel K, we define first the resolvent of the first kind of K as a finite measure L on [0, T] such that : for $t \in (0, T]$,

$$(K * L) = \int_{[0,t]} K(t-s) L(ds) = 1.$$

Assumption : The function K in $L^2([0, T])$ is non-negative, not identically 0, non-increasing and continuous. Its resolvent of the first kind L is non-negative and such that $s \longrightarrow L([s, s+t])$ is non-increasing for all $t \geq 0$. Moreover, there exist constants $C > 0$ and $H > 0$ such that, for all $0 \leq t \leq T$, and $n \geq 1$, $\delta \in (0, T-t]$, one has

$$\int_t^{t+\delta} |K(t+\delta - \eta_n(s))|^2 ds \leq C\delta^{2H}$$

and

$$\int_0^t |K(t+\delta - \eta_n(s)) - K(t - \eta_n(s))|^2 \leq C\delta^{2H}.$$

2.2 Euler scheme method

2.2.1 • Description

For the implementation of the Euler scheme as suggested in the article we use instead of S_t the $Y_t = log(S_t)$ defined by :

$$Y_t = Y_0 - \int_0^t \frac{1}{2} V_s ds + \int_0^t \sqrt{V_s} d(\rho W_s + \sqrt{1-\rho^2} W_s^{'})$$

for the first formulation, and

$$Y_t = Y_0 - \frac{1}{2} X_t + \rho M_t + \sqrt{1-\rho^2} M_t^{'}$$

for the second formulation.
Let's consider a sequence $(\pi_n)_{n>=1}$ of discrete-time grid on [0, T], with $\pi_n := (0 = t_0^n < t_1^n < t_2^n < ... < t_n^n = T)$ for each $n > 0$ with $t_k^n = k\Delta t$ and $\Delta t = \frac{T}{n}$.

For the first formulation for removing the problem of negativity of V_t we use $(V_{t_i}^n)_+ = Max(0, V_{t_i}^n)$ $(H1)$ instead of V_t then we get following scheme :

$$Y_{t_k}^n = Y_0 + \sum_{i=0}^{k-1} (-\frac{1}{2}(V_{t_i}^n)_+ + \Delta t + \rho \sqrt{(V_{t_i}^n)_+}(W_{t_{i+1}} - W_{t_i}) + \sqrt{1-\rho^2}\sqrt{(V_{t_i}^n)_+}(W_{t_{i+1}}^{'} - W_{t_i}^{'}),$$

$$V_{t_k}^n = V_0 + \sum_{i=0}^{k-1} (K(t_k - t_i)(\theta - \lambda (V_{t_i}^n)_+)\Delta t + K(t_k - t_i)\nu \sqrt{(V_{t_i}^n)_+}(W_{t_{i+1}} - W_{t_i})).$$

For the second formulation for removing the problem of non-increasing of X_t we use $Xb_{t_i}^n = Max_{0 \le j \le i}(X_{t_j})$ $(H2)$ instead of X_t and we get the scheme :

$$Y_{t_k}^n = Y_0 - \frac{1}{2} Xb_{t_i}^n + \rho M_{t_k}^n + \sqrt{1-\rho^2} M_{t_k}^{'n},$$

$$X_{t_k}^n = V_0 t_i + \sum_{i=0}^{k-1} K(t_k - t_i)(\theta t_i - \lambda Xb_{t_i}^n + \nu M_{t_i}^n)\Delta t,$$

$$M_{t_i}^n = \sum_{i=1}^{k} \sqrt{Xb_{t_i}^n - Xb_{t_{i-1}}^n} Z_i, \quad M_{t_i}^{'n} = \sum_{i=1}^{k} \sqrt{Xb_{t_i}^n - Xb_{t_{i-1}}^n} Z_i^{'},$$

Where $(Z_i, Z_i^{'})_i$ is a i.i.d sequence of Gaussian N(0,I).

2.2.2 • Convergence result

With proposition 2-1 and relying on several lemmas, a convergence result to a weak solution has been found, thus proving to set up the Euler scheme of discrete times.

3
APPLICATION WITH MONTE CARLO METHOD

3.1 Description

Our application is taken directly from the article, and consists in using the model for the pricing of three categories of options defined by a payoff that depends on the price of the underlying. The interest in obtaining a fairly accurate estimate of the real value of the option lies in the principle of no-arbitrage. In effect, one is supposed to set up a model that coincides point for point with the historical data at the risk of arbitraging. The data used for the implementation of the method is the one used in the article and allows to reflect the realities of the financial markets in the sense that we are sure to have a good approximation of the price of the derivative assets which will allow to predict by making interpolations. $V_0 = 0.02$, $\theta = 0.02$, $\lambda = 0.3$, $\nu = 0.3$, $\rho = -0.7$ and Hurst index H $= 0.1$ with C $= \frac{1}{\Gamma(H+\frac{1}{2})}$ and $M = 10^5$ samples.

3.2 results

If $f(S_T)$ is the payoff function then the option price is $E(f(S_T))$ Thus we take the estimation :

$$\frac{1}{M}\sum_{m=1}^{M} f(S_T^{n,m})$$

Then we get this kind of result
The difference between the two graphs lies in the speed of convergence, which depends on the assumptions $(H1)$ and $(H2)$ made for the simulations. Other results are presented in the attached notebook.

We do not mention here but keep in mind that one of the precision measures of the Monte Carlo method used for the estimation is the 95 percent confidence interval, which allows us to know the precision of our estimation according to the number of samples

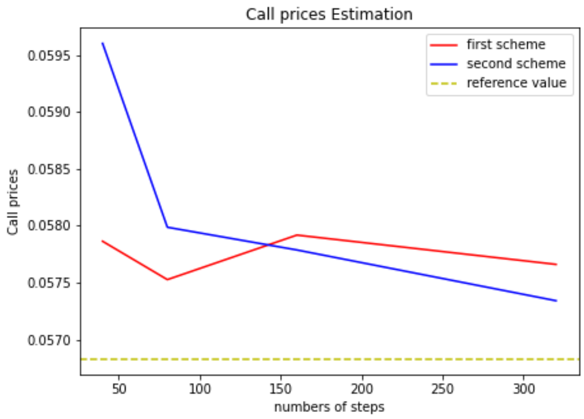

Figure 1 – Call price with our result

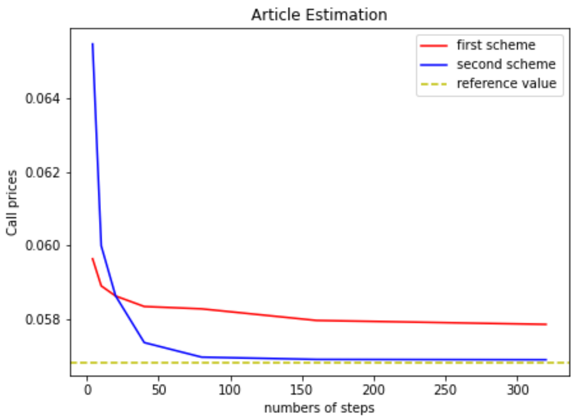

Figure 2 – Call price with article result

3.3 Discussion

Another technique for numerically solving the stochastic differential equations from the rough volatility Heston model is the Milstein method. The principle is quite similar to that of the Euler scheme method, but it is sufficient to add the second order term in the Taylor approximation. Let consider for example the autonomous Itô stochastic differential equation :

$$dX_t = a(X_t)dt + b(X_t)dW_t.$$

If we want to find the solution on a specific interval $\left[0, T\right]$ The Milstein approximation to the true solution X is the Markov chain Y define as follows : partition the interval $[0, T]$ into n sub-intervals $0 = t_O^n < t_1^n < ... < t_n^n = T$ $\Delta_k^n = t_k^n - t_{k-1}^n$ set $Y_0 = X_0$ and recursively define Y_n

for $1 < k < n$ such that

$$Y_{k+1} = Y_k + a(Y_n)\Delta_k^n + b(X_k)\Delta W_k^n + \frac{1}{2}b(X_k)b^{'}(X_k)((\Delta W_k^n)^2 - \Delta_k^n)$$

in our case we have :
$a(t, V_t) = K(. - t)(\theta - \lambda V_t)$
$b(t, V_t) = \nu\sqrt{V_t}$
$a(t, S_t) = 0$
$b(t, S_t) = S_t\sqrt{V_t}\Big[\rho, \sqrt{1-\rho^2}\Big]$
The Milstein scheme has a weak and strong order of convergence, Δt_k^n, which is superior to the Euler-Maruyama method, which in turn has the same weak order of convergence, Δt_k^n, but a lower strong order of convergence, $\sqrt{(\Delta t_k^n)}$.

RÉFÉRENCES

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[2] Fan Yang Alexandre Richard, Xiaolu Tan. *On the discrete-time simulation of the rough Heston model.* March, 8 2022.

[3] All the articles in [2].