Sample Exam Solutions

May 28, 2019

Question 1

The demand and supply functions for two goods X and Y are given below.

$$\begin{aligned} \text{Demand:} \quad & Q_X = 120 - 2P_X + 3P_Y, \quad Q_Y = 150 + 6P_X - 4P_Y \\ \text{Supply:} \quad & Q_X = -240 + 6P_X, \quad Q_Y = -150 + 6P_Y \end{aligned}$$

(a) Determine the matrix $\mathbf{A}$ in the equation of the form

$$\mathbf{A} \begin{bmatrix} P_X \\ P_Y \end{bmatrix} = \begin{bmatrix} 360 \\ 300 \end{bmatrix}$$

which determine market equilibrium.

Answer:

$$A = \begin{bmatrix} 8 & -3 \\ -6 & 10 \end{bmatrix}$$

□

(b) Show that $\mathbf{A}$ is invertible, by verifying that $\det(A) \neq 0$.

Answer: $\mathbf{A}$ is a 2×2 matrix with

$$\det(\mathbf{A}) = 8 \cdot 10 - (-3) \cdot (-6) = 62 \neq 0.$$

So $\mathbf{A}$ is invertible. □

(c) Using $\mathbf{A}^{-1}$, obtain the prices of the two goods at market equilibrium. Show your working and provide answers correct to 2 decimal places.

Answer:

$$\mathbf{A}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 10 & 3 \\ 6 & 8 \end{bmatrix} = \frac{1}{62} \begin{bmatrix} 10 & 3 \\ 6 & 8 \end{bmatrix}$$

So

$$\begin{bmatrix} P_X \\ P_Y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 360 \\ 300 \end{bmatrix} = \frac{1}{62} \begin{bmatrix} 10 & 3 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 360 \\ 300 \end{bmatrix} = \begin{bmatrix} (10 * 360 + 3 * 300)/62 \\ (6 * 360 + 8 * 300)/62 \end{bmatrix} \approx \begin{bmatrix} 72.58 \\ 73.55 \end{bmatrix}$$

At market equilibrium, the prices of good X and Y are, to 2 decimal place, 72.58 and 73.55 respectively. $\square$

Question 2

A monopolist faces a demand function $P = 173 - 2Q$, with P denoting the market price and Q denoting the quantity demanded. There is a fixed cost of 200, the total variable cost function is $TVC(Q) = \frac{1}{3}Q^3 - 10Q^2 + 188Q$.

(a) Obtain an expression for the profit function $f(Q)$ defined for $Q > 0$.

Answer:

$$\begin{aligned} f(Q) =& \text{Total Revenue} - \text{Total Cost} \\ =& Q(173 - 2Q) - \left(200 + \frac{1}{3}Q^3 - 10Q^2 + 188Q\right) \\ =& -\frac{1}{3}Q^3 + 8Q^2 - 15Q - 200 \end{aligned}$$

$\square$

(b) Determine the stationary point(s) of the profit function.

Answer: The stationary point(s) is(are) the solution(s) of the equation

$$f'(Q) = -3 \cdot \frac{1}{3}Q^2 + 2 \cdot 8Q - 15 = -Q^2 + 16Q - 15 = 0.$$

Using abc-formula with $a = -1$, $b = 16$, $c = -15$, we have

$$\Delta = b^2 - 4ac = 16^2 - 4 \cdot (-1) \cdot (-15) = 196 > 0$$

and the equation have two roots

$$\begin{aligned} Q_1 &= \frac{-b + \sqrt{\Delta}}{2a} = \frac{-16 + \sqrt{196}}{2 \cdot (-1)} = 1, \\ Q_2 &= \frac{-b - \sqrt{\Delta}}{2a} = \frac{-16 - \sqrt{196}}{2 \cdot (-1)} = 15. \end{aligned}$$

So the stationary points are $Q = 1$ and $Q = 15$. $\square$

(c) Assume the maximal point of the profit function exist. What is the maximal value of the profit function?

Answer: The maximal point is one of the stationary points from (b). Compare

$$f(1) = -\frac{622}{3}$$
$$f(15) = 250$$

So the maximal value of the profit function is 250. $\square$

(d) Compute the elasticity $El_Q f(Q)$ at point $Q = 10$, to 2 decimal places. Provide a brief interpretation of the computed value of the price elasticity, in the context of this example.

Answer:

$$El_Q f(Q) = \frac{f'(Q)Q}{f(Q)} = \frac{(-Q^2 + 16Q - 15)Q}{-\frac{1}{3}Q^3 + 8Q^2 - 15Q - 200}$$

Plug in $Q = 10$ yields $El_Q f(10) \approx 3.86$.

If the demand quantity Q increases by 1% from 10 units, the profit increases by 3.86% approximately. $\square$

Question 3

Suppose the market price P can be written as a function of the market demand quantity $Q \in [0, 5)$

$$P(Q) = 100 - 10Q.$$

The current demand quantity is $Q_0 = 2$. Showing all steps of your working, evaluate the consumer surplus in this market

$$\text{CS} = \int_0^{Q_0} P(Q)dQ - P_0Q_0$$

where Q_0 is the current market demand quantity.

Answer: The current price is $P_0 = P(Q_0) = 100 - 10 \cdot 2 = 80$.

$$\begin{aligned}
\text{CS} &= \int_0^{Q_0} P(Q)dQ - P_0Q_0 \\
&= \int_0^{2} (100 - 10Q)dQ - 80 \cdot 2 \\
&= \left[100Q - 5Q^2\right]_{Q=0}^{Q=2} - 160 \\
&= 180 - 0 - 160 = 20.
\end{aligned}$$

$\square$

Question 4

Energy company GreenVolt (GV) owns a property at the wind-swept, sunny location of Ocean Heads. GV is evaluating two projects: a wind farm and a solar energy plant. The wind farm requires an initial investment of \$10m, and a \$5m loss is expected for the first year. For the following 3 years, GV expects annual returns of \$8m from electricity sales. The solar plant requires an initial investment of \$15m. GV expects a loss of \$2m for the first year and annual returns of \$9m for the following 3 years.

Assume a discount (interest) rate of 8% compounded annually.

(a) Calculate, in \$m to 3 decimal places, the present value of the two projects.

Answer: The present value of the cash flows stream for the wind farm and the solar energy plant project, respectively, are

$$PV_1 = -10 + \frac{-5}{1.08} + \frac{8}{(1+0.08)^2} + \frac{8}{(1+0.08)^3} + \frac{8}{(1+0.08)^4} \approx \$4.460m$$
$$PV_2 = -15 + \frac{-2}{1.08} + \frac{9}{(1+0.08)^2} + \frac{9}{(1+0.08)^3} + \frac{9}{(1+0.08)^4} \approx \$4.624m$$

□

(b) Based on the present values that you have carried out in (a), explain which project you think is preferable.

Answer: The Solar energy plant project is preferable because it has a higher present value.

□

(c) Calculate, in percentage to 3 decimal places, the internal rate of return of the two projects.

Answer: Solve the equation

$$-10 + \frac{-5}{1+r} + \frac{8}{(1+r)^2} + \frac{8}{(1+r)^3} + \frac{8}{(1+r)^4} = 0$$

yields the internal rate of return for the wind farm project $IRR_1 \approx 19.594\%$

Solving the equation

$$-15 + \frac{-2}{1+r} + \frac{9}{(1+r)^2} + \frac{9}{(1+r)^3} + \frac{9}{(1+r)^4} = 0$$

yields the internal rate of return for the solar energy plant project $IRR_2 \approx 17.717\%$

□

(d) On the basis of the internal rate of return of the two projects, which project do you think is preferable?

Answer: The wind farm project is preferable because it has a higher internal rate of return.

□

Question 5

Given the Cobb-Douglas production function

$$Q = 100L^{0.3}K, \quad L, K > 0.$$

(a) Write down the equation of the isoquant for $Q = 800$ in the form $K = f(L)$

Answer: Rewrite $100L^{0.3}K = 800$ to get

$$K = 8L^{-0.3}$$

□

(b) Show by differentiation that $f(L)$ is convex.

Answer:

$$f'(L) = 8L^{-0.3-1} \cdot (-0.3) = -2.4L^{-1.3}$$
$$f''(L) = -2.4L^{-1.3-1} \cdot (-1.3) = 3.12L^{-2/3}$$

So $f''(L) > 0$ for all $L > 0$, and therefore f is a convex function. □

(c) Find the values of L and K, to 2 decimal places, for which the production is maximised under the budget restriction $L + 2K = 30$ using Lagrange method.

Answer: Construct the Lagrange function

$$g(L, K, \lambda) = Q(L, K) + \lambda(30 - L - 2K) = 100L^{0.3}K + \lambda(30 - L - 2K)$$

The stationary points are the solutions to the system of equations

$$g'_L(L, K, \lambda) = 100 \cdot 0.3 \cdot L^{0.3-1}K - \lambda = 30L^{-0.7}K - \lambda = 0$$
$$g'_K(L, K, \lambda) = 100L^{0.3} \cdot 1 - 2\lambda = 100L^{0.3} - 2\lambda = 0$$
$$g'_\lambda(L, K, \lambda) = 30 - L - 2K = 0$$

From the first two equations, we have $100L^{0.3} = 2\lambda = 60L^{-0.7}K$ that is

$$L = \frac{60}{100}K = 0.6K.$$

Substituting back to the last equation (the budget constraint) in the system yields

$$30 - 0.6K - 2K = 0 \Rightarrow K = \frac{30}{2.6} = \frac{300}{26} \approx 11.54$$

Hence, from above, $L = 0.6K = \frac{180}{26} \approx 6.92$. (Alternatively, $L = 0.6K \approx 0.6 \cdot 11.54 \approx 6.92$ or $L = 30 - 2K \approx 30 - 2 \cdot 11.54 = 6.92$)

The production is maximised with $L \approx 6.92$ and $K \approx 11.54$ under the required budget constraint. □

(d) If the budget increases by 1 (that is, increases from 30 to 31), compute the resulting change (rounded off to 2 decimal places) in the maximal level of production, using the Lagrange multiplier method.

Answer (preferred): The corresponding value of Lagrange multiplier at the optimal point in question (c) is $\lambda^* = 30 \cdot (L^*)^{-0.7} \cdot K^* = 30 \cdot \left(\frac{180}{26}\right)^{-0.7} \cdot \frac{300}{26} \approx 89.3428$. If the budget increases by 1 (from 30), the maximal level of production will increase by approximately 89.3428 units. □

Remark 1. You may solve the question with the constraint $L + 2K = 31$, as in question (c), and calculate the true change in the maximal level of production. Note that the final outcome will be different. Please try on your own, and here I do not provide the details.

Question 6

The following difference equation models the salary scale for part-time staff

$$Y_t = 20 + 1.2Y_{t-1}$$

where Y_t denotes the salary (in dollars) in year $t = 0, 1, 2, \ldots$.

(a) If $Y_0 = 2,300$, deduce Y_1, Y_2 and Y_3 directly from the difference equation.

Answer:

$$Y_1 = 20 + 1.2Y_0 = 20 + 1.2 \cdot 2300 = 2780$$
$$Y_2 = 20 + 1.2Y_1 = 20 + 1.2 \cdot 2780 = 3356$$
$$Y_3 = 20 + 1.2Y_2 = 20 + 1.2 \cdot 3356 = 4047.2$$

□

(b) Solve the difference equation. In other words, determine Y_t for all $t = 0, 1, \ldots$.

Answer: The equilibrium state is $x^* = \frac{20}{1-1.2} = -100$, and we can rewrite the difference equation as

$$Y_t - x^* = 1.2 \cdot (Y_{t-1} - x^*), \text{ that is } Y_t + 100 = 1.2 \cdot (Y_{t-1} + 100)$$

So $Y_0 + 100, Y_1 + 100, Y_2 + 100, \ldots$ is a geometric sequence with common ratio $r = 1.2$. Hence,

$$Y_t + 100 = r^t(Y_0 + 100) = 2400 \cdot 1.2^t$$

and it follows $Y_t = 2400 \cdot 1.2^t - 100$, for all $t \geq 1$. □

(c) Calculate the number of year until the salary first exceeds \$15,000.

Answer: Solve $Y_t = 2400 \cdot 1.2^t - 100 > 15000$, that is, $1.2^t > \frac{15100}{2400}$, or

$$e^{\ln(1.2)\cdot t} > \frac{15100}{2400}$$

Using the fact that $f(x) = e^x$ is an increasing function on $(-\infty, \infty)$, we have

$$\ln(1.2) \cdot t > \ln\left(\frac{15100}{2400}\right) \quad \Leftrightarrow \quad t > \frac{\ln\left(\frac{15100}{2400}\right)}{\ln(1.2)} \approx 10.1,$$

where the inequality sign did not change since $\ln(1.2) \approx 0.18 > 0$. Noting that the time index t is an integer, so the number of year until the salary first exceeds \$15,000 is 11. $\square$

Question 7

Suppose that a firm's capital stock $K(t)$ satisfies the differential equation

$$\frac{dK}{dt} = I - \delta K(t)$$

where investment I is constant, and $\delta K(t)$ denotes depreciation, with δ a positive constant

(a) Find the solution of the equation if the capital stock at time $t = 0$ is K_0.

Answer: We have $K'(t) = I - \delta K(t)$ so

$$\left(K(t) - \frac{I}{\delta}\right)' = -\delta\left(K(t) - \frac{I}{\delta}\right).$$

In other words, defining $\widetilde{K}(t) := K(t) - \frac{I}{\delta}$, then it satisfies the differential equation

$$\widetilde{K}'(t) = -\delta \widetilde{K}(t).$$

It follows that $\widetilde{K}(t) = Ae^{-\delta t}$ for some constant A, and therefore

$$K(t) = \widetilde{K}(t) + \frac{I}{\delta} = Ae^{-\delta t} + \frac{I}{\delta}$$

Plug-in $K(0) = K_0$ we have

$$A + \frac{I}{\delta} = K_0 \quad \Leftrightarrow \quad A = K_0 - \frac{I}{\delta}$$

So the firm's capital stock

$$K(t) = \left(K_0 - \frac{I}{\delta}\right)e^{-\delta t} + \frac{I}{\delta}$$

$\square$

(b) Let $\delta = 0.05$ and $I = 10$. Explain what happens as $t \to \infty$ when: (i) $K_0 = 150$; (ii) $K_0 = 250$.

Answer: Plugging in $\delta = 0.05$ and $I = 10$ we have

$$K(t) = (K_0 - 200)\, e^{-0.05t} + 200$$

As $t \to \infty$, for both the cases of (i) $K_0 = 150$; (ii) $K_0 = 250$ we have

$$K(t) \to 0 + 200 = 200$$

since $e^{-0.05t} = \left(e^{-0.05}\right)^t \to 0$ where $e^{-0.05} \approx 0.95 < 1$. □

Question 8

A daily diet mix requires a minimum of: 160 mg of Vitamin K and 1000 mg of Vitamin D. Two foods A and B contain these vitamins:

	Vitamin K	Vitamin D	Cost per 1 kg
Food A	10 mg	100 mg	40
Food B	8 mg	40 mg	20

(a) Write down the inequality constraints for each vitamin. Denote the consumption (in kg) of food A and food B as x and y.

Answer:

- Vitamin K: $10x + 8y \geq 160$;
- Vitamin D: $100x + 40y \geq 1000$;

□

(b) Using extreme point theorem, determine the number of units of food A and B which fulfil the daily requirements at a minimum cost. You may assume the optimal point exist.

Answer: The question is a linear programming problem as follows:

Min $40x + 20y$, subject to the following (five) constraints

$$10x + 8y \geq 160$$
$$100x + 40y \geq 1000$$
$$x \geq 0, \quad y \geq 0$$

Setting any two of the constraint active, we have candidates of the corner points as follows:

(1) $10x + 8y = 160$ and $100x + 40y = 1000$: by Cramer's rule we have

$$x = \frac{\begin{vmatrix} 160 & 8 \\ 1000 & 40 \end{vmatrix}}{\begin{vmatrix} 10 & 8 \\ 100 & 40 \end{vmatrix}} = \frac{160 \cdot 40 - 8 \cdot 1000}{10 \cdot 40 - 8 \cdot 100} = \frac{-1600}{-400} = 4$$

$$y = \frac{\begin{vmatrix} 10 & 160 \\ 100 & 1000 \end{vmatrix}}{\begin{vmatrix} 10 & 8 \\ 100 & 40 \end{vmatrix}} = \frac{10 \cdot 10000 - 160 \cdot 100}{-400} = \frac{-6000}{-400} = 15$$

It satisfies the other constraints $x \geq 0$ and $y \geq 0$

(2) $10x + 8y = 160$, $x = 0$: $y = 20$. It **does not** satisfy $100x + 40y \geq 1000$.

(3) $10x + 8y = 160$, $y = 0$: $x = 16$. It satisfies $100x + 40y \geq 1000$ and $x \geq 0$.

(4) $100x + 40y = 1000$, $x = 0$: $y = 25$. It satisfies $10x + 8y \geq 160$ and $x \geq 0$.

(5) $100x + 40y = 1000$, $y = 0$: $x = 10$. It **does not** satisfy $10x + 8y \geq 160$.

(6) $x = 0$, $y = 0$. It **does not** satisfy $10x + 8y \geq 160$, nor $100x + 40y \geq 1000$.

We have found in total 6 candidate points, but only 3 corner points

$$(4, 15), (16, 0), (0, 25)$$

corresponding to the costs $40x + 20y$ equal to, respectively

$$460, 640, 500.$$

Hence, the minimal cost is 460, which can be obtained by consuming 4 units of food A and 15 units of food B. $\square$

END OF EXAMINATION

Formulae are provided in the next two pages.