Depolarization index and the average degree of polarization

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Two single number metrics for depolarization of samples are contrasted: (1) the average degree of polarization of the exiting light averaged over the Poincaré sphere and (2) the depolarization index of Gill and Berbenau [Opt. Acta **32,** 259–261 (1985); **33,** 185–189 (1986)]. The depolarization index is a geometric measure that varies from 0 for the ideal depolarizer to 1 for nondepolarizing Mueller matrices. The average degree of polarization also varies from 0 to 1 and characterizes the typical level of depolarization. Although the depolarization index is very often close to the average degree of polarization, these two metrics can differ by more than 0.5 for certain Mueller matrices. © 2005 Optical Society of America

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1. Introduction

The term depolarization, as used in polarization optics, refers to the coupling of polarized light into unpolarized light and is associated with a reduction in the degree of polarization.

The depolarization properties of Mueller matrices cannot be fully described by a single number. Some samples depolarize all polarization states equally. Other depolarizing samples partially depolarize most polarization states but may not depolarize one or two particular incident states. A single-number metric for depolarization provides a useful summary of the quantity of depolarization of a sample.

In this paper I explore single-number metrics for the depolarization of Mueller matrices. In particular it addresses a common misunderstanding that the depolarization index is the average degree of polarization of the exiting light. The depolarization index introduced by Gill and Berbenau in 1985 has a well-defined and useful geometric meaning in the configuration space of Mueller matrices.[1,2] However, the depolarization index can produce misleading results when used as a metric for the magnitude of depolarization.

In general most optical elements barely depolarize or they would not be useful or competitive optical elements. Conversely, most scattering interactions involve significant depolarization.

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2. Mueller Matrices

We assume a linear interaction of incident light with a sample where its polarization-transformation properties are described by a Mueller matrix $\mathbf{M}$ that relates the incident Stokes vector $\mathbf{S}_{\text{incident}}$ with the exiting Stokes vector $\mathbf{S}_{\text{Exiting}}$ by

$$\mathbf{S}_{\text{Exiting}} = \mathbf{M} \cdot \mathbf{S}_{\text{Incident}} = \begin{bmatrix} \mathbf{M}_{00} & \mathbf{M}_{01} & \mathbf{M}_{02} & \mathbf{M}_{03} \\ \mathbf{M}_{10} & \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{20} & \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{30} & \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{S}_0 \\ \mathbf{S}_1 \\ \mathbf{S}_2 \\ \mathbf{S}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_0' \\ \mathbf{S}_1' \\ \mathbf{S}_2' \\ \mathbf{S}_3' \end{bmatrix}. \tag{1}$$

The Mueller matrix contains 16 degrees of freedom (DOFs). Seven DOFs are associated with nondepolarizing processes: diattenuation, retardance, and polarization-independent loss such as absorption. The other nine DOFs are associated with depolarization and describe how different states of polarization are depolarized. These depolarizing DOFs arise from variations in the polarization properties within a measurement interval, for example, spatial inhomogeneities in diattenuation, retardance, diattenuation orientation, and retardance orientation occurring within the field of view of a pixel of a polarimeter. The

Mueller matrix may be used to represent completely general linear light–sample interactions including a beam transmitted through a surface, through an optical element, and through a sequence of elements; a reflected beam; a scattered beam; a diffracted beam; and other linear light–sample interactions.

Mueller matrices are geometrically represented within a Mueller matrix hyperspace, which is a 16-dimensional configuration space where the axes correspond to the 16 elements of the Mueller matrix. Each Mueller matrix occupies a single point within the hyperspace. The two most important points in our analysis of this hyperspace are the identity matrix **I** and the ideal depolarizer **ID** (Ref. 3):

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{ID} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

I is the point approached by all families of retarder Mueller matrices as the retardance approaches zero. Similarly **I** is the limit approached by all diattenuator Mueller matrices as the diattenuation approaches zero.

The ideal depolarizer Mueller matrix completely depolarizes all incident polarization states so that only unpolarized light exits. Integrating spheres have Mueller matrices very close to **ID**.[4,5]

The degree of polarization (**DoP**) of a Stokes vector is defined as

$$\mathrm{DoP}[\mathbf{S}] = \frac{(\mathbf{S}_1{}^2 + \mathbf{S}_2{}^2 + \mathbf{S}_3{}^2)^{1/2}}{\mathbf{S}_0}. \tag{3}$$

When the **DoP** of the exiting beam is less than the **DoP** of the incident beam, depolarization occurs. In particular, when the incident beam is completely polarized (**DoP** = 1), the reduction in **DoP** indicates the capacity for depolarizing that particular state.

While depolarization reduces the **DoP** of light beams, diattenuation[6,7] (a property of polarizers and partial polarizers) increases the **DoP** of the light beams by the attenuation of one polarization component. Depolarization and diattenuation can operate together in the same device, simultaneously increasing and decreasing the **DoP**. For example, a sheet polarizer with the protective plastic film still attached is polarizing in the sheet polarizer and depolarizing in the random birefringent film.

For almost all useful optical elements, such as lenses, mirrors, and filters, the reduction in the degree of polarization of the light is very small, usually much less than 1%. Such depolarization in optical elements is usually due to defects such as bubbles in the glass, stria, roughness, fingerprints, or coating defects. Typically the **DoP** must be reduced by much less than 1% for an optical element to be a competitive and useful optical element. This is the optical characteristic the depolarization metrics characterize in the polarization testing of optical elements.

In scattering interactions the reduction of the degree of polarization assumes the entire range from 0% (no depolarization) to 100% (complete depolarization), depending on the type of scatterer. The scattering Mueller matrix is defined for a particular incident wavelength, illuminated area, illuminating solid angle, and collected solid angle.

It is useful for both optical elements and scattering objects that the magnitude of depolarization be expressed simply as a fraction of the polarized incident light which is depolarized. This is complicated because for most elements the depolarization varies with the incident polarization states. Thus an averaged depolarization metric summarizes the depolarization. First, nondepolarizing Mueller matrices, matrices that do not reduce the **DoP**, are reviewed.

3. Nondepolarizing Mueller Matrices

Nondepolarizing Mueller matrices act on all polarized Stokes vectors (**DoP** = 1), yielding fully polarized exiting Stokes vectors. The light's degree of polarization is never reduced as it interacts with the sample. Many conditions between the elements of nondepolarizing Mueller matrices are found in the literature.[1–3,8–11] Nondepolarizing Mueller matrices have 7 degrees of freedom corresponding to the 8 DOFs in the Jones matrix minus the absolute phase DOF. All ideal polarizers, ideal retarders, and their arbitrary combinations in sequence (cascades) are nondepolarizing. One constraint on Mueller matrix elements $\mathbf{M}_{i,j}$ so that the Mueller matrix is nondepolarizing is

$$4\mathbf{M}_{00}{}^2 = \sum_{i=0}^{3}\sum_{j=0}^{3} \mathbf{M}_{i,j}{}^2. \tag{4}$$

Rearranging Eq. (4), we readily show that for matrices normalized to $M_{00} = 1$, all nondepolarizing Mueller matrices lie on a 15-dimensional hypersphere of radius $\sqrt{3}$ centered on the ideal depolarizer. Thus the ideal depolarizer plays a special geometric role in the structure of the Mueller matrix hyperspace.

4. Average Degree of Polarization

It is not obvious from a simple examination of a Mueller matrix how much depolarization is introduced because the change in the degree of polarization is a function of the incident polarization state. For most depolarizing Mueller matrices, some incident polarization states are depolarized more and other states less.[12] Paints depolarize circular polarized light much more than they depolarize linearly polarized light. Scattering surfaces show considerable variability in the depolarization with the incident state. Similarly liquid-crystal cells exhibit **DoP** variations with the incident state.

The average degree of polarization **AverageDoP** is defined as the arithmetic mean of the degree of polarization of the exiting light when averaged equally over all incident polarization states. The **AverageDoP** is calculated by integrating the **DoP** as the incident state **S** varies over the Poincaré sphere and normalizing with $1/4\pi$ for the area of the sphere:

$$\mathbf{AverageDoP}[\mathbf{M}] = \frac{\displaystyle\int_0^{\pi}\int_{\pi/2}^{\pi/2} \mathbf{DoP}[\mathbf{M}\cdot\mathbf{S}(\theta,\ \phi)]\cos(\phi)\mathrm{d}\theta\mathrm{d}\phi}{4\pi}. \quad (5)$$

The terms $\cos(\phi)\mathrm{d}\theta\mathrm{d}\phi$ integrate the incident polarization state over the Poincaré sphere, with the latitude ϕ relating to its degree of circular polarization and its polarization ellipse major axis represented by θ. The Stokes vector $\mathbf{S}(\theta, \phi)$ parameterized by orientation of polarization θ and latitude ϕ on the Poincaré sphere is

$$\mathbf{S}(\theta,\ \phi) = \begin{bmatrix} 1 \\ \cos(2\theta)\cos(\phi) \\ \sin(2\theta)\cos(\phi) \\ \sin(\phi) \end{bmatrix}. \quad (6)$$

The **AverageDoP**, which varies from zero to one, provides a summary of the depolarizing property in a single number. When **AverageDoP** is equal to one, the exiting light is always completely polarized, indicating a nondepolarizing Mueller matrix. Values near one indicate little depolarization. When **AverageDoP** equals zero, the exiting light is completely depolarized; only unpolarized light exits the interaction.

A closed-form expression for **AverageDoP** is readily obtained in terms of the individual Mueller matrix elements, but the resulting expression is lengthy and does not simplify into a more useful form. Hence **AverageDoP** is most readily calculated with numerical integration routines.

A related metric **WeightedDoP** is obtained by weighting the **AverageDoP** calculation by the exiting flux:

$$\mathbf{WeightedDoP}[\mathbf{M}] = \frac{\displaystyle\int_0^{\pi}\int_{\pi/2}^{\pi/2} (\mathbf{M}[0]\cdot\mathbf{S}[0,\phi])\mathbf{DoP}[\mathbf{M}\cdot\mathbf{S}[\theta,\phi]\cos[\phi]\mathrm{d}\theta\mathrm{d}\varphi}{\displaystyle\int_0^{\pi}\int_{\pi/2}^{\pi/2} (\mathbf{M}[0]\cdot\mathbf{S}[\theta,\phi]\cos[\phi]\mathrm{d}\theta\mathrm{d}\varphi}. \quad (7)$$

M[0] indicates the top row of the Mueller matrix; the dot product of **M**[0] with the incident polarization state yields the exiting flux. The numerator weights the exiting **DoP** by the exiting flux while the denominator calculates the average exiting flux. So the **WeightedDoP** weights the **DoP** of bright exiting states more than dim exiting states. This simulates averaging the **DoP** over the exiting light versus the **AverageDoP**, which weights all input states equally. For a Mueller matrix constructed from a sum of retarder Mueller matrices the transmission is the same for all incident polarization states so that the **WeightedDoP** equals the **AverageDoP**. But for Mueller matrices involving polarizers the transmission varies considerably with the incident polarization state, and significant differences between these two metrics occur. **WeightedDoP** can be usefully applied, for example, to beams exiting polarizers with some depolarization where some incident states generate little exiting light, and those dim exiting states are to be weighted less.

5. Depolarization Index

In 1985 Gil and Bernabeu[1,2] proposed the depolarization index **DI** as a single number metric for characterizing the depolarization of a Mueller matrix. The **DI** is defined as

$$\mathbf{Di}[\mathbf{M}] = \frac{\left(\sum_{i,j=0}^{3} \mathbf{M}_{i,j}{}^2 - \mathbf{M}_{0,0}{}^2\right)^{1/2}}{\sqrt{3}\,\mathbf{M}_{0,0}}. \quad (8)$$

The **DI** equals one for nondepolarizing Mueller matrices and equals zero for **ID**. The numerator is the Euclidean distance from the ideal depolarizer to the Mueller matrix. The denominator is the radius of the hyperspherical surface of nondepolarizing Mueller matrices for that $\mathbf{M}_{0,0}$. So the **DI** is the fractional distance of a Mueller matrix along a line segment from **DI** to the hyperspherical surface for nondepolarizing Mueller matrices. Note that Eq. (7) has a structure analogous to Eq. (3) for the **DoP** of a Stokes vector; the numerator is the square root of the sum of the squares of all the elements except the first; the denominator involves only the first element.

A **DI** greater than one or less than zero represents a nonphysical Mueller matrix; in practice such Mueller matrices occasionally result from measurement error, calibration error, and noise.

6. Comparisons between the Depolarization Index and the Average Degree of Polarization

The **DI** and the **AverageDoP** are often equal and usually quite close. However, for certain classes of Mueller matrices they diverge. A series of Mueller matrices is presented to highlight the similarities and differences between these metrics.

First consider the Mueller matrices for uniform depolarizers **UD**[a] and diagonal nonuniform depolarizers **NDD**[a, b, c]:

$$\mathbf{UD}[a] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix},$$

$$\mathbf{NDD}[a,\ b,\ c] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix}. \tag{9}$$

Parametrically the uniform depolarizers lie along a line between the ideal depolarizer and the identity matrix. Along this line the **DI** and the **AverageDoP** are both equal to a for all a. For the diagonal nonuniform depolarizers the **DI** is

$$\mathbf{DI}[\mathbf{NDD}[a,\ b,\ c]] = \frac{(a^2+b^2+c^2)^{1/2}}{\sqrt{3}}. \tag{10}$$

The average degree of depolarization varies from this value by as much as $1/\sqrt{3} - 1/2 \approx 0.07735$, a small but not negligible difference for quantities that vary from zero to one.

Next consider depolarizing retarders constructed by filling an aperture with aperture fraction a of retarder and fraction $(1-a)$ of an ideal depolarizer. For elements such as these constructed from polarization elements configured in parallel, the Mueller matrix is the sum of the component Mueller matrices weighted by the aperture fraction. In the configuration space these depolarizing retarders lie on the line between the ideal depolarizer and the corresponding retarder Mueller matrix. For example, a depolarizing quarter-wave horizontal linear retarder (**DQWHLR**) with a horizontal fast axis has the Mueller matrix

$$\mathbf{DQWHLR}[\delta, a] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a\cos[\delta] & -a\sin[\delta] \\ 0 & 0 & a\sin[\delta] & a\cos[\delta] \end{bmatrix}. \tag{11}$$

Other divided aperture depolarizing retarders are related by a unitary change of basis and show the same properties. For divided aperture depolarizing retarders, including **DQWHLR**, the two metrics are equal:

$$\mathbf{DI}[\mathbf{DQWHLR}[\delta, a]] = \mathbf{AverageDoP}[\mathbf{DQWHLR}[\delta, a]] = a. \tag{12}$$

Depolarizing polarizers can be constructed by dividing an aperture between an ideal polarizer and an ideal depolarizer; then small differences between the two metrics result. An example of such a Mueller matrix is a depolarizing horizontal linear polarizer **DHLP**$[a]$ with an aperture fraction a filled by a horizontal polarizer and the balance filled with **ID**:

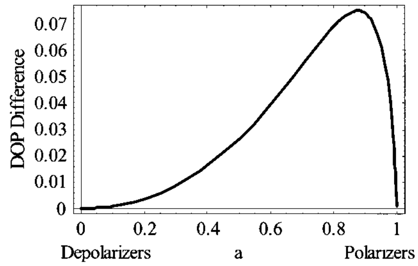

Fig. 1. Difference between the depolarization index and the average degree of polarization for an aperture divided between an ideal polarizer (aperture fraction a) and an ideal depolarizer.

$$\mathbf{DHLP}[a] = \begin{bmatrix} 1 & a & 0 & 0 \\ a & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

When **DHLP** depolarizes horizontal polarized light, there is the least depolarization $\mathbf{DoP}[\mathbf{H}] = 2a/(1+a)$; with vertical polarized light, there is the most depolarization $\mathbf{DoP}[\mathbf{V}] = 2a/(1-a)$, and with all other incident polarization states, there are intermediate values.

For all depolarizing polarizers of this class located along lines between ideal polarizers and **ID**, the **DI** and the **AverageDoP** are close with the difference as a function of parameter a as shown in Fig. 1, and the maximum difference is 0.075.

7. Large Differences between the Depolarization Index and the Average Degree of Polarization

The greater differences between **DI** and the **AverageDoP** are found for cascades of diattenuators followed by depolarizers and for depolarizers followed by diattenuators. These cascades differ from previous examples in several ways. First, in the configuration space they do not lie along lines from **ID**, which terminate on nondepolarizing Mueller matrices. In the previous examples the depolarization acted in parallel with the diattenuation or retardance in a shared aperture model. Second, in these cascades the depolarization acts nonsymmetrically. This nonsymmetry manifests as a difference between the diattenuation **D** and the polarizance **P** of the Mueller matrix:

$$\mathbf{D}[\mathbf{M}] = \frac{(\mathbf{M}_{0,1}{}^2 + \mathbf{M}_{0,2}{}^2 + \mathbf{M}_{0,3}{}^2)^{1/2}}{\mathbf{M}_{0,0}},$$

$$\mathbf{P}[\mathbf{M}] = \frac{(\mathbf{M}_{1,0}{}^2 + \mathbf{M}_{2,0}{}^2 + \mathbf{M}_{3,0}{}^2)^{1/2}}{\mathbf{M}_{0,0}}, \tag{14}$$

where **P** is smaller when the depolarization follows the diattenuation and **D** is smaller for the reverse orientation. For example, Mueller matrices are (1) **DD**1, a horizontal diattenuator with diattenuation d followed by a uniform depolarizer with coefficient b, and (2) **DD**2, a uniform depolarizer with coefficient b followed by a horizontal diattenuator. For comparison a parallel case **DD**3 with equal diattenuation and polarizance is included where **DD**3 has a horizontal diattenuator filling aperture fraction $(1 - a)$ with the remainder filled with an ideal depolarizer:

the difference is even greater, 0.577! This is the greatest difference between the **DI** and the **AverageDoP** for all physically realizable Mueller matrices.

$$\mathbf{DD}1[d,\ b] = \frac{1}{1+d}\begin{bmatrix} 1 & bd & 0 & 0 \\ d & b & 0 & 0 \\ 0 & 0 & b(1-d^2)^{1/2} & 0 \\ 0 & 0 & 0 & b(1-d^2)^{1/2} \end{bmatrix}, \tag{15}$$

$$\mathbf{DD}2[d,\ b] = \frac{1}{1+d}\begin{bmatrix} 1 & d & 0 & 0 \\ bd & b & 0 & 0 \\ 0 & 0 & b(1-d^2)^{1/2} & 0 \\ 0 & 0 & 0 & b(1-d^2)^{1/2} \end{bmatrix}, \tag{16}$$

$$\mathbf{DD}3[d,\ a] = \frac{1}{1+d}\begin{bmatrix} 1 & d(1-a) & 0 & 0 \\ d(1-a) & (1-a) & 0 & 0 \\ 0 & 0 & (1-a)(1-d^2)^{1/2} & 0 \\ 0 & 0 & 0 & (1-a)(1-d^2)^{1/2} \end{bmatrix}. \tag{17}$$

Comparing the first row and first column, we see how the diattenuator followed by a depolarizer **DD**2 reduces the polarizance while a depolarizer before a diattenuator **DD**1 reduces the diattenuation. The divided aperture or shuffled diattenuator–depolarizer combinations maintain the diattenuation–polarizance equality. Other diattenuation orientations and elliptical diattenuator–depolarizer combinations are obtained by orthogonal transformations of the above and exhibit the same **DI** and **AverageDoP**.

Comparing **DI** and **AverageDoP** for **DD**1 and **DD**2, we found that the difference increases with increasing d and/or decreasing a. For an ideal depolarizer ($b = 0$) followed by a polarizer ($d = 1$), such as can be constructed from an integrating sphere with a polarizer over the output port,

$$\mathbf{DI}[\mathbf{DD}1[1, 0]] = 1/\sqrt{3} \approx 0.577,$$

$$\mathbf{AverageDOP}[\mathbf{DD}1[1, 0]] = 1, \tag{18}$$

which results in a difference of 0.423, a very great difference. For a linear polarizer followed by an ideal depolarizer, such as a polarizer over the entrance port of an integrating sphere,

$$\mathbf{DI}[\mathbf{DD}3[1, 0]] = 1/\sqrt{3} \approx 0.577,$$

$$\mathbf{AverageDOP}[\mathbf{DD}3[1, 0]] = 0, \tag{19}$$

8. Conclusion

Three methods for representing the depolarization of a Mueller matrix as a single metric have been compared. Frequently researchers desire a single measure of depolarization. The **AverageDoP** is the easier metric to understand; it provides the mean **DoP** of the exiting light averaged over the Poincaré sphere, the expected value.

The **DI** does not in general equal the **AverageDoP**. For most Mueller matrices the **DI** and **AverageDoP** are very close. But when strong diattenuation occurs before the depolarization (and vice versa), great differences are observed. These cases are also associated with great differences between the magnitudes of the diattenuation and the polarizance. The **DI** has a clear geometric meaning in the Mueller matrix configuration space, being the fractional distance of a Mueller matrix along a line segment from the ideal depolarizer to the hypersphere of nondepolarizing Mueller matrices, so it remains a useful and meaningful parameter, but more useful for theoretical studies of the Mueller calculus than for representing the depolarization of an optical element. The **DI** is proportional to the distance of a normalized Mueller matrix from the ideal depolarizer.

The **DI** is a far simpler calculation than **AverageDoP**, requiring a small number of calculations rather than involving a complex numerical subroutine. In Mathematica the **DI** evaluates approximately 1070 times faster than when **AverageDoP** is evaluated by using the built-in numerical integration routine. Applying **AverageDoP** to an image takes minutes.

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